Determinant of a Matrix

Introduction

The determinant is a scalar value associated with a square matrix that provides important information about the matrix. It is a useful concept in linear algebra with applications in areas such as solving linear systems, evaluating matrix invertibility, and understanding geometric properties of transformations represented by matrices. The determinant is denoted as $\det(\mathbf{A})$ or $|\mathbf{A}|$.

Definition

For a square matrix $\mathbf{A}$ of order $n \times n$, the determinant is a single number that is calculated from the elements of the matrix. The determinant of a matrix $\mathbf{A}$ can be interpreted geometrically as the scaling factor of the linear transformation described by $\mathbf{A}$.

Calculating Determinants

The method for calculating the determinant depends on the size of the matrix. We will start with simple 2×2 and 3×3 matrices and then generalize to larger matrices.

Determinant of a 2×2 Matrix

For a 2×2 matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant is calculated as:

$$\det(\mathbf{A}) = ad - bc$$

Example:

Consider the matrix:

$$\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

The determinant is:

$$\det(\mathbf{A}) = (4 \cdot 1) - (3 \cdot 2) = 4 - 6 = -2$$

Determinant of a 3×3 Matrix

For a 3×3 matrix $\mathbf{B}$:

$$\mathbf{B} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The determinant is calculated using the rule of Sarrus or the cofactor expansion method. Using the rule of Sarrus:

$$\det(\mathbf{B}) = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Example:

Consider the matrix:

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$$

The determinant is:

$$\det(\mathbf{B}) = 1(4 \cdot 6 - 5 \cdot 0) - 2(0 \cdot 6 - 5 \cdot 1) + 3(0 \cdot 0 - 4 \cdot 1)$$

$$\det(\mathbf{B}) = 1(24 - 0) - 2(0 - 5) + 3(0 - 4)$$

$$\det(\mathbf{B}) = 24 + 10 - 12 = 22$$

Generalization to Larger Matrices

For larger $n \times n$ matrices, the determinant can be calculated using cofactor expansion (Laplace expansion). The expansion can be done along any row or column, but is typically done along the first row for simplicity.

Determinant by Cofactor Expansion

1. Select a row or column. For simplicity, let's use the first row.
2. For each element a_{1j} in the selected row or column, calculate the cofactor:

- The cofactor C_{1j} is given by $(-1)^{1+j} M_{1j}$, where M_{1j} is the determinant of the $(n-1) \times (n-1)$ submatrix obtained by deleting the 1-th row and j-th column.

3. Multiply each element a_{1j} by its corresponding cofactor and sum the results:

$$\det(\mathbf{A}) = \sum_{j=1}^{n} a_{1j} C_{1j}$$

Properties of Determinants

1. **Determinant of Identity Matrix**: The determinant of an identity matrix $\mathbf{I}$ is always 1, regardless of its size.

$$\det(\mathbf{I}) = 1$$

2. **Determinant of a Diagonal Matrix**: The determinant of a diagonal matrix is the product of its diagonal elements.

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

$$\det(\mathbf{D}) = d_1 \cdot d_2 \cdot \ldots \cdot d_n$$

3. **Determinant of a Triangular Matrix**: The determinant of an upper or lower triangular matrix is also the product of its diagonal elements.
4. **Row and Column Operations**:
   - Swapping two rows or columns of a matrix changes the sign of the determinant.
   - Multiplying a row or column by a scalar k multiplies the determinant by k.
   - Adding a multiple of one row or column to another row or column does not change the determinant.
5. **Determinant of a Product**: The determinant of the product of two matrices equals the product of their determinants.

$$\det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$$

6. **Inverse and Determinant**: A square matrix is invertible if and only if its determinant is non-zero. For an invertible matrix $\mathbf{A}$, the determinant of its inverse is the reciprocal of the determinant of the matrix.

$$\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$$

7. **Determinant and Transpose**: The determinant of a matrix is equal to the determinant of its transpose.

$$\det(\mathbf{A}) = \det(\mathbf{A}^T)$$

Geometric Interpretation

The determinant of a matrix can be interpreted as a scaling factor for the volume of a geometric shape. For example:

- In two dimensions, the determinant of a 2×2 matrix representing a transformation of a unit square gives the area of the parallelogram formed.
- In three dimensions, the determinant of a 3×3 matrix representing a transformation of a unit cube gives the volume of the parallelepiped formed.

If the determinant is zero, it indicates that the matrix transforms the space into a lower dimension, implying linear dependence among the rows or columns.

Applications in Data Science

1. **Solving Linear Systems**: The determinant is used in Cramer's Rule to solve linear systems. If the determinant of the coefficient matrix is zero, the system does not have a unique solution.
2. **Invertibility**: The determinant helps in determining if a matrix is invertible, which is crucial for algorithms that require matrix inversion, such as certain machine learning models.
3. **Eigenvalues and Eigenvectors**: The characteristic polynomial, which is derived from the determinant, is used to find the eigenvalues of a matrix. Eigenvalues and eigenvectors are essential in techniques like PCA.
4. **Volume and Area Calculations**: In multivariate statistics, the determinant of the covariance matrix is related to the volume of the confidence ellipsoid.