Section 1: Fundamentals of Linear Algebra

Introduction to Linear Algebra

Linear algebra is a branch of mathematics that deals with vectors, matrices, and linear transformations. It provides the foundational tools for analyzing and manipulating data, which are essential for many data science tasks. Linear algebra concepts are used in various areas such as machine learning, optimization, computer graphics, and more.

Importance of Linear Algebra in Data Science

1. **Data Representation**: Data in data science is often represented as vectors and matrices. For instance, a dataset with m samples and n features can be represented as an $m \times n$ matrix.
2. **Dimensionality Reduction**: Techniques like Principal Component Analysis (PCA) and Singular Value Decomposition (SVD) use linear algebra to reduce the number of dimensions in a dataset while preserving as much information as possible.
3. **Optimization**: Many machine learning algorithms involve optimization problems that can be solved efficiently using linear algebra techniques.
4. **Transformations and Projections**: Linear transformations and projections are used to manipulate and understand data, making it easier to visualize and interpret.

Real-world Applications

1. **Machine Learning**: Linear algebra is used in algorithms like linear regression, logistic regression, support vector machines, and neural networks.
2. **Computer Vision**: Image processing tasks often involve operations on matrices representing pixel values.
3. **Natural Language Processing (NLP)**: Text data can be represented as vectors using techniques like word embeddings.
4. **Recommendation Systems**: Matrix factorization techniques decompose user-item interaction matrices to make recommendations.

Basic Concepts and Definitions

Scalars, Vectors, and Matrices

- **Scalar**: A single number. Example: $a = 3$.
- **Vector**: An ordered list of numbers. Can be represented as a column vector or row vector.
  - Column vector: $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

- Row vector: $\mathbf{w} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}$

- **Matrix**: A rectangular array of numbers arranged in rows and columns.
  - Example: $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Notation and Terminology

- **Dimension**: The size of a vector or matrix. A vector with n elements is called an n-dimensional vector. A matrix with m rows and n columns is called an $m \times n$ matrix.
- **Element**: Each number in a vector or matrix. In matrix $\mathbf{A}$, a_{ij} denotes the element in the i-th row and j-th column.

Vector Operations

Addition and Subtraction

- **Addition**: Adding corresponding elements of two vectors of the same dimension.
  - Example: $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1+4 \\ 2+5 \\ 3+6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

- **Subtraction**: Subtracting corresponding elements of two vectors of the same dimension.
  - Example: $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

$$\mathbf{u} - \mathbf{v} = \begin{bmatrix} 1-4 \\ 2-5 \\ 3-6 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix}$$

Scalar Multiplication

- Multiplying each element of a vector by a scalar.
  - Example: $c = 2$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$c\mathbf{u} = 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

Dot Product

- The dot product of two vectors is the sum of the products of their corresponding elements.
  - Example: $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

$$\mathbf{u} \cdot \mathbf{v} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 4 + 10 + 18 = 32$$

Cross Product

- The cross product of two 3-dimensional vectors results in another 3-dimensional vector perpendicular to both.
  - Example: $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 6 - 3 \cdot 5 \\ 3 \cdot 4 - 1 \cdot 6 \\ 1 \cdot 5 - 2 \cdot 4 \end{bmatrix} = \begin{bmatrix} 12 - 15 \\ 12 - 6 \\ 5 - 8 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$$

Matrix Operations

Matrix Addition and Subtraction

- **Addition**: Adding corresponding elements of two matrices of the same dimension.
  - Example: $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

- **Subtraction**: Subtracting corresponding elements of two matrices of the same dimension.
  - Example: $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} 1-5 & 2-6 \\ 3-7 & 4-8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

Matrix Multiplication

- Multiplying two matrices involves taking the dot product of rows of the first matrix with columns of the second matrix.
  - Example: $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

$$\mathbf{AB} = \begin{bmatrix} 1\cdot 5 + 2\cdot 7 & 1\cdot 6 + 2\cdot 8 \\ 3\cdot 5 + 4\cdot 7 & 3\cdot 6 + 4\cdot 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Transpose of a Matrix

- The transpose of a

matrix is obtained by swapping its rows and columns.

- Example: $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\mathbf{A}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Inverse of a Matrix

- The inverse of a matrix $\mathbf{A}$, denoted as $\mathbf{A}^{-1}$, is such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.
  - Example: For $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse is given by

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  - Condition: The determinant of $\mathbf{A}$ (denoted as $\det(\mathbf{A}) = ad - bc$) must be non-zero.

Determinant of a Matrix

- The determinant is a scalar value that can be computed from the elements of a square matrix and encapsulates certain properties of the matrix.
  - Example: For $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\det(\mathbf{A}) = ad - bc$$

  - For a 3×3 matrix $\mathbf{B} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$,

$$\det(\mathbf{B}) = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Special Types of Matrices

Identity Matrix

- An identity matrix $\mathbf{I}$ is a square matrix with ones on the diagonal and zeros elsewhere.

- Example: $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- Property: $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for any matrix $\mathbf{A}$ of appropriate dimensions.

Diagonal Matrix

- A diagonal matrix has non-zero elements only on the diagonal, and all off-diagonal elements are zero.
  - Example: $\mathbf{D} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$
  - Property: Diagonal matrices are easy to invert if all diagonal elements are non-zero, and their inverse is also a diagonal matrix with reciprocal diagonal elements.

Symmetric Matrix

- A symmetric matrix is equal to its transpose, $\mathbf{A} = \mathbf{A}^T$.
  - Example: $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$
  - Property: Symmetric matrices have real eigenvalues and orthogonal eigenvectors.

Orthogonal Matrix

- An orthogonal matrix $\mathbf{Q}$ has the property that $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I}$.
  - Example: $\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
  - Property: The rows and columns of an orthogonal matrix are orthonormal vectors, meaning they are both orthogonal and of unit length.