Queueing Theory: Assignment 1, Simulation in discrete time

EBB074A05

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1 General info

This file contains the code and the results that go with these youtube movies:

- `https://youtu.be/DfYxayoQmjYc`
- `https://youtu.be/D8BIAoBICnw`
- `https://youtu.be/_BoagRyH5c0`

There are a number of exercises you have to address in your report. Keep your answers short. You don't have to win the Nobel prize on literature.

2 Simulation in Discrete time

2.1 one period, demand, service capacity, and queue

There is one server, jobs enter a queue in front of the server, and the server serves batches of customers, every hour say.

```
L = 10
a = 5
d = 8
L = L + a -d
print(L)
```

```
7
```

```
L = 3
a = 5
c = 7
d = min(c, L)
L += a -d
print(d, L)
```

```
3 5
```

2.2 two periods

```
L = 3
a = 5
c = 7
d = min(c, L)
L += a - d

a = 6
d = min(c, L)
L += a - d
print(d, L)
```

```
5 6
```

Ex 2.1. Add a third period, and report your result.

2.3 simulate many periods, make vectors

```
import numpy as np

a = np.random.uniform(5, 8, size=5)
print(a)
```

```
[7.66616096 5.46469981 5.28343426 7.84771106 5.80769232]
```

2.4 Set seed

```
import numpy as np

np.random.seed(3)

a = np.random.uniform(5, 8, size=5)
print(a)
```

```
[6.65239371 7.12444347 5.87271422 6.53248282 7.67884086]
```

2.5 update with a for loop

```
num = 5

#a = np.random.uniform(5, 8, size=num)
#c = np.random.uniform(5, 8, size=num)
a = 9*np.ones(num)
c = 10*np.ones(num)
L = np.zeros_like(a)

L[0] = 20
for i in range(1, num):
    d = min(c[i], L[i-1])
    L[i] = L[i-1] + a[i] - d

print(L)
```

```
[20. 19. 18. 17. 16.]
```

Ex 2.2. Run the code for 10 periods and report your result.

2.6 Compute mean and std of simulated queue length for $\rho \approx 1$

We discuss the concept of load more formally in the course at a later point in time. Conceptually the load is the rate at which work arrives. For instance, if $\lambda = 5$ jobs arrive per hour, and each requires 20 minutes of work (on average), then we say that the load is $5 \times 20/60 = 5/3$. Since one server can do only 1 hour of work per hour, we need at least two servers to deal with this load. We define the utilization ρ as the load divided by the number of servers present.

In discrete time, we define ρ as the average number of jobs arriving per period divided by the average number of jobs we can serve per period. Slightly more formally, for discrete time,

$$\rho \approx \frac{\sum_{k=1}^{n} a_k}{\sum_{k=1}^{c_k}}. \tag{1}$$

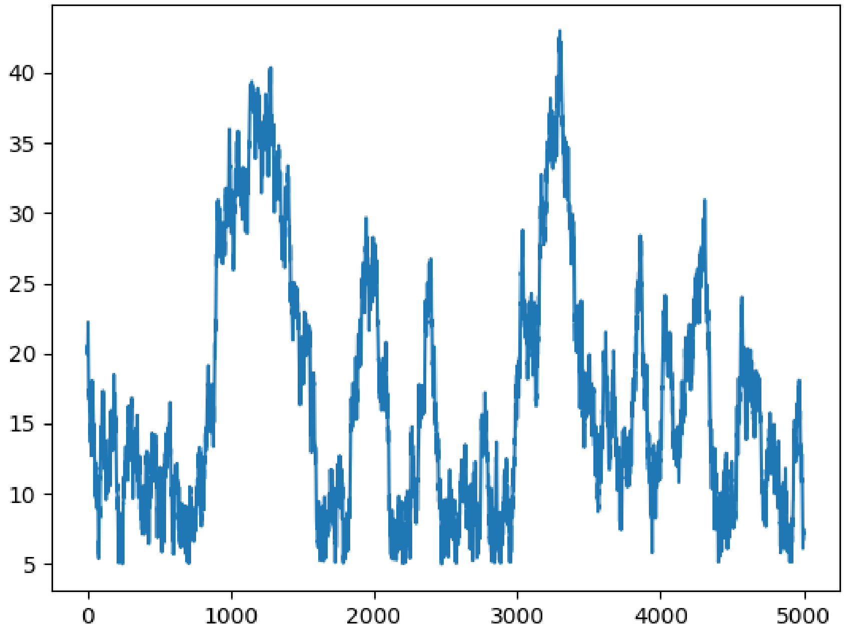
And formally, we should take the limit $n \to \infty$ (but such limits are a bit hard to obtain in simulation).

```
num = 5_000

np.random.seed(3)
a = np.random.uniform(5, 8, size=num)
c = (5+8)/1.99 * np.ones(num)
L = np.zeros_like(a)  # queue length at the end of a period

L[0] = 20
for i in range(1, num):
    d = min(c[i], L[i-1])
    L[i] = L[i-1] + a[i] - d

print(L.mean(), L.std())
```

```
16.78550665013682 8.695855000533511
```

2.7 plot the queue length process

```
import matplotlib.pyplot as plt

plt.clf()
plt.plot(L)
plt.savefig('queue-discrete_1.png')
'queue-discrete_1.png'
```



2.8 Compute mean and std of simulated queue length for $\rho = 1/2$

```
num = 5_000

np.random.seed(3)
a = np.random.uniform(5, 8, size=num)
c = (5+8) * np.ones(num)
L = np.zeros_like(a)   # queue length at the end of a period

L[0] = 20
for i in range(1, num):
    d = min(c[i], L[i-1])
    L[i] = L[i-1] + a[i] - d

print(L.mean(), L.std())
```

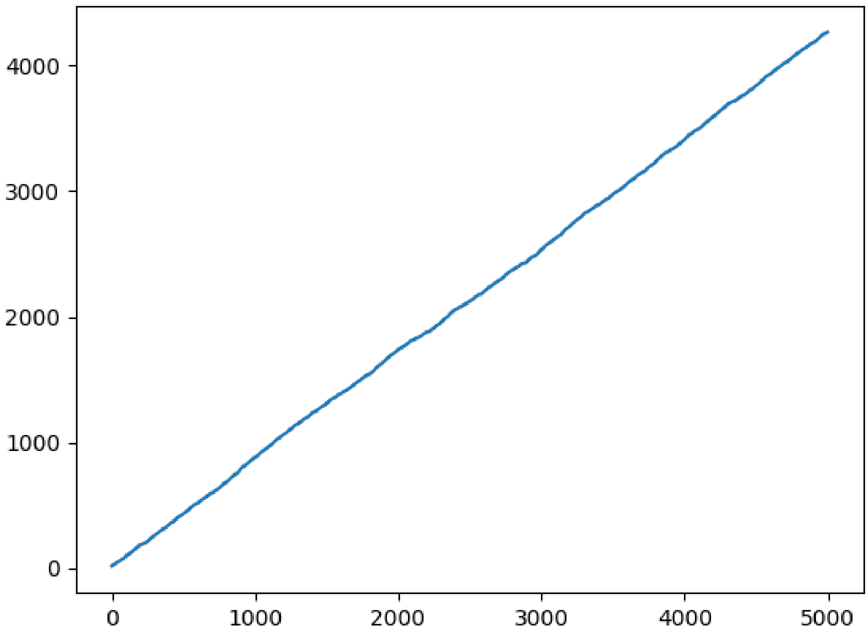
```
6.5051538887388 0.890452952271703
```

Ex 2.3. Change the code such that the arrivals that occur in period i can also be served in period i. Explain how this works, make a graph and compare your results with the results of the simulation we do here (i.e. arrivals cannot be served in the periods in which they arrive).

2.9 show the drift when $\rho > 1$

```
num = 5_000

np.random.seed(3)
```

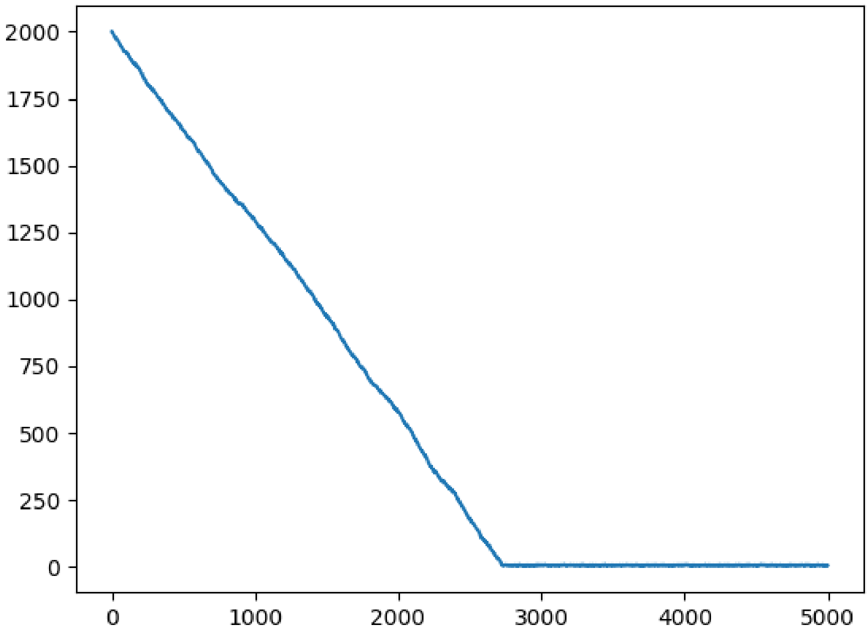
```
a = np.random.uniform(5, 8, size=num)
c = (5+8)/2.3 * np.ones(num)
L = np.zeros_like(a)  # queue length at the end of a period

L[0] = 20
for i in range(1, num):
    d = min(c[i], L[i-1])
    L[i] = L[i-1] + a[i] - d


plt.clf()
plt.plot(L)
plt.savefig('queue-discrete_2.png')
'queue-discrete_2.png'
```



2.10 Start with a large queue, take ρ< 1, show the drift

```
num = 5_000

np.random.seed(3)
a = np.random.uniform(5, 8, size=num)
c = (5+8)/1.8 * np.ones(num)
L = np.zeros_like(a)  # queue length at the end of a period

L[0] = 2_000
for i in range(1, num):
    d = min(c[i], L[i-1])
    L[i] = L[i-1] + a[i] - d
```

```


plt.clf()
plt.plot(L)
plt.savefig('queue-discrete_3.png')
'queue-discrete_3.png'
```



Things to memorize:

- if the capacity is equal or less than the arrival rate, the queue lenght will explode.
- If the capacity is larger than the arrival rate, the queue length will stay around 0 (between quotes).
- If we start with a huge queue, but the service capacity is larger than the arrival rate, then the queue will drain rather fast, in fact, about linear.

Ex 2.4. When $\rho < 1$ and L_0 is some large number, such as here. Why is the normal distribution reasonable to model the time τ until the queue hits zero for the first time? Use simulation to estimate the expected time $\mathsf{E}[\tau]$ and variance $\mathsf{V}[\tau]$. (You can also try to use EVE's law here, if you're fanatic; but if you're not, just skip the analytic part.)

Repeat the above simulation a number of times with different seeds (why?) to estimate $\mathsf{E}[\tau]$ and $\mathsf{V}[\tau]$.

Spoiler alert, hint: When $L_0 \gg 1$, then a_k jobs arrive in period k and c_k jobs leave. For ease, write $h_k = c_k - a_k$. Then the time τ to hit 0, i.e., the time until the queue is empty, is such that $\sum_{k=1}^{\tau} h_k \geq L_0$. But then, $\mathsf{E}[\tau]\ \mathsf{E}[h_k] = L_0$. Moreover, consider any sum of random variables, then with $\mu = \mathsf{E}[X]$, σ the std of X, and N the normal distribution, and by the central limit theorem,

$$\frac{1}{n}\sum_{i=1}^{n} X_i \sim N(\mu, \sigma^2) \implies \sum_{i=1}^{n} X_i \sim N(n\mu, n\sigma^2). \tag{2}$$

Bigger hint. Below is the code that I use. Let me explain how it works. I want to find the smallest τ such that $\sum_{i=1}^{\tau}(a_k - c_k) \geq L_0$. Since I know that $L_0 \gg 0$, I don't have to compute $d_k = \min\{L_{k-1}, c_k\}$, because $L_{k-1} > c_k$ as long as there jobs in the system. So, I can skip the computation of d_k; it saves also computer time. Then, I just guess that 5000 periods is enough to always clear the queue. However, this is nothing but a guess. Here I don't check this, but formally speaking I should do so. (It is not a problem to live in a state of sin; as long as you know you are sinning, it's ok :-)). Then I change the seed to get different runs; here I do `N=10` runs. Finally, I assemble the hitting times for the different runs in the vector `tau`. And, `numpy` offers the functions `mean` and `std` that I can call on vectors, so I don't have to compute that myself.

```python
import numpy as np

num = 5_000


def hitting_time(seed):
    np.random.seed(seed)
    a = np.random.uniform(5, 8, size=num)
    c = (5 + 8) / 1.8 * np.ones(num)

    L = 2_000
    for i in range(1, num):
        L += +a[i] - c[i]
        if L <= 0:
            return i


N = 10
tau = np.zeros(N)
for n in range(N):
    tau[n] = hitting_time(n)

print(tau.mean(), tau.std())
```

```
2766.7 63.94380345271932
```

2.11 Queues with blocking.

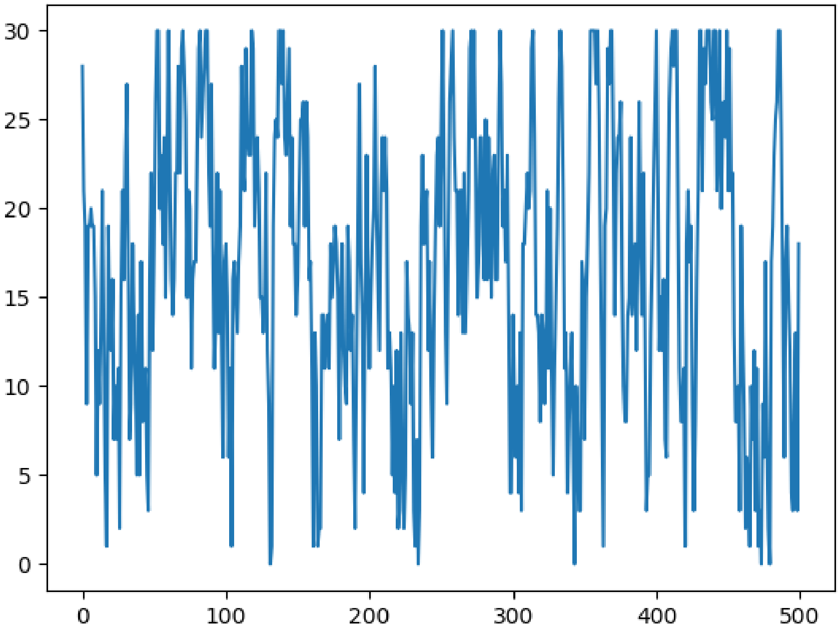
We have a queue subject to blocking. When the queue exceeds K, say, then whatever of the batch of items coming in exceeds K is rejected. This is the so-called complete reject rule. Two more assumptions: service occurs before arrival, and jobs arriving in a period cannot be served.

```python
num = 500

np.random.seed(3)
a = np.random.randint(0, 20, size=num)
c = 10*np.ones(num)
L = np.zeros_like(a)  # queue length at the end of a period
loss = np.zeros_like(a)  # queue length at the end of a period

K = 30 # max people in queue, otherwise they leave

```

```python
L[0] = 28
for i in range(1, num):
    d = min(c[i], L[i-1])
    loss[i] = max(L[i-1] + a[i] - d - K, 0) # service before arrivals.
    L[i] = L[i-1] + a[i] - d - loss[i]

lost_fraction = sum(loss)/sum(a)
print(lost_fraction)
```

```
0.026064291920069503
```

```python
plt.clf()
plt.plot(L)
plt.savefig('queue-discrete_loss.png')
'queue-discrete_loss.png'
```



If we would assume that departures occur at the end of a period (hence, after the arrivals), then the code has to be as follows:

```python
for i in range(1, num):
    d = min(c[i], L[i-1])
    loss[i] = max(L[i-1] + a[i] - K, 0) # service at end of period
    L[i] = L[i-1] + a[i] - d - loss[i]

lost_fraction = sum(loss)/sum(a)
print(lost_fraction)
```

Ex 2.5. When $\rho = 1.51$, simulate a number of times to estimate the distribution $\pi_i = \mathsf{P}[L = i]$.

Ex 2.6. When $\rho = 10$, use a simple argument to show that $\pi_{K-2} \approx 0$.