# Interesting Problems

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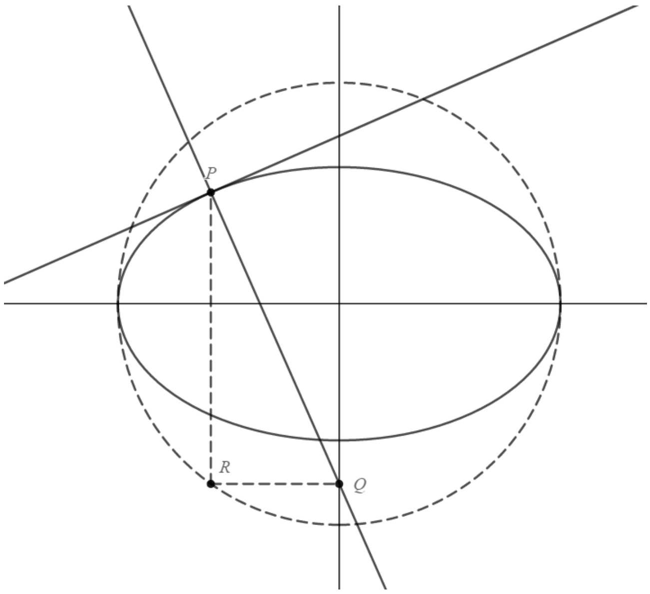


1. The diagram above shows an ellipse $E_1$, with equation $x^2+ay^2=1$, for some $a \in \mathbb{R}, a>1$.

   $P=(x_{\mathrm{P}}, y_{\mathrm{P}})$ is a point on the circumference of the ellipse, with $x_{\mathrm{P}}=p \in [-1,1]$.

   $Q=(0,q)$ is the $y$-intercept of the normal to the ellipse at the point $P$.

   $R$ is the point with coördinates $(p,q)$.

   As $P$ travels around the circumference of the ellipse $E_1$, $R$ travels around the circumference of another ellipse, $E_2$.

   (a) Write an expression for $q$ in terms of $a$ and $p$ for the case where $y_{\mathrm{P}} \geq 0$

   (b) When $a=2$, use your expression for $q$ to show that $E_1=E_2$.

   (c) Find the value of $a$ that results in $R$ maintaining a constant distance from the origin, i.e. for which $E_2$ is a circle.

2. For a curve $C$, a **strophoid**, $\Sigma_A(C)$, is defined as follows ($O = (0, 0)$ is the origin):

   *For every point $Q$ on $C$, there are two points $P$ and $P'$ on $\Sigma_A(C)$ which correspond to the points of intersection between the line through $O$ and $Q$, and the circle through $A$ centred at $Q$.*

   Let $A = (-1, 0)$, and let $L$ be the line with equation $x = 1$. Consider a point $P = (x, y)$ on $\Sigma_A(L)$.

   (a) Using the definition of a strophoid, and by rewriting $L$ parametrically, show that $x$ and $y$ satisfy the parametric equations:

   $$\begin{aligned}(1+t^2)(1-x)^2 &= 4+t^2 \\ y &= tx \qquad\qquad \text{for } t \in \mathbb{R}\end{aligned}$$

   (b) Hence show that $\Sigma_A(L)$ has the implicit Cartesian equation:

   $$y^2(a-x) = x(x+b)(x+c)$$

   where $a, b, c \in \mathbb{Z}$ are constants to be found.

   (c) Find the point $B \neq A$ for which $\Sigma_A(L) = \Sigma_B(L)$. (*Hint: sketch a diagram*)

3. For a real number $x$:

   - $\lfloor x \rfloor$ is the largest integer smaller than or equal to $x$, e.g: $\lfloor \pi \rfloor = 3$.
   - $\{x\}$ is the fractional part of $x$, e.g: $\{\pi\} = 0.14159...$

   (a) Find the smallest positive solution to the equation:

   $$x\lfloor x \rfloor \{x\} = 2$$

   (b) Are there any **rational** solutions to the equation?
   Either find one, or prove there are none.

4. For each point $X = (p, q)$ in the plane excluding the origin, we associate $X$ to the line $\ell_X$, the line with equation $px + qy = -1$.

   (a) Given two points $P = (a, b)$ and $Q = (c, d)$, describe a condition that must be satisfied for the corresponding lines $\ell_P$ and $\ell_Q$ to intersect.

   Formulate the condition both *geometrically*, in terms of the positions of $P$ and $Q$ in the plane; and *algebraically* in terms of $a, b, c$, and $d$.

   (b) Prove that if $\ell_P$ and $\ell_Q$ do intersect, say at $R$, then $\ell_R$ is the line through $P$ and $Q$.

   (c) If $P$ lies on the unit circle, what is the geometric interpretation of $\ell_P$?

   (d) Given two points $(\sin\theta, \cos\theta)$ and $(\sin\phi, \cos\phi)$ on the circumference of the unit circle, show that the point of intersection of the tangents at the two points is:

   $$\left(\frac{\sin\phi - \sin\theta}{\sin(\theta-\phi)}, \frac{\cos\theta - \cos\phi}{\sin(\theta-\phi)}\right)$$

5. There is a row of 9 parking spaces in front of a shop, labelled consecutively from 1 to 9.

   9 cars arrive at once. Each driver has a preference for a particular number space. We'll represent the *preference number* of the $i^{\text{th}}$ driver by $p_i$.

   Each driver in turn will drive along the row, starting at space 1, until they reach their preferred space. If it is empty, they will park in it. If another car is already parked there, they will park in the next available space. The car park is one way, so they cannot turn back.

   If none of the later spaces are available, the driver drives off and decides to shop elsewhere.

   The sequence $(p_1, p_2, \cdots, p_8, p_9)$ is called a *parking function* if this sequence of preference numbers allows each driver a space to park.

   How many different parking functions are there?

6. The function $g(x)$ is given by:

$$g(x) = \frac{x - a}{ax + 1}$$

   for some $a \in \mathbb{R}$, $a > 0$.

   Given that $g$ also satisfies $g(g(g(x))) = x$, find $a$.

   The function $f(x)$ satisfies the following relation involving $g$:

$$f(x) + f(g(x)) = x$$

   Find an explicit formula for $f(x)$, and hence show that $f(a) = 0$.

7. A cake is being shared between 100 guests at a party. The guests are instructed to line up to receive their cake.

   The first guest receives 1% of the cake. The second guess receives 2% of what's left after the first guest's slice was removed, the third guest receives 3% of what's left after that, and so on, up to the 100th guest who receives 100% of what's left after the 99th guest's slice was removed.

   Which guest receives the largest slice of cake?

8. An **Egyptian fraction** is any (positive) fraction whose numerator is 1.

   (a) Show that any fraction of the form $\frac{2}{n}$ can be written as the sum of no more than two different Egyptian fractions.

   (b) Find all pairs of Egyptian fractions which sum to make $\frac{2}{15}$.

   (c) Let $\frac{m}{p}$ be a proper fraction with $p$ prime. Find a condition for this fraction to be expressible as the sum of two different Egyptian fractions.