По Рэйкару (model A):

$$\ln Pr(D|\theta) = \sum_{i=1}^{N} \ln(a_i p_i + b_i(1-p_i))$$

, где

- $a_i = \prod_{j=1}^{R} \alpha_j^{y_i^j} (1-\alpha_j)^{1-y_i^j}$
- $b_i = \prod_{j=1}^{R} (\beta_j)^{1-y_i^j} (1-\beta_j)^{y_i^j}$
- $\alpha_j = Pr[y^j = 1|y = 1]$
- $\beta_j = Pr[y^j = 0|y = 0]$
- $p_i = Pr(y = 1|x, w, H)$

$$Pr(D|\theta) = \prod_{i=1}^{N} (a_i p_i + b_i(1-p_i))$$

$$E_y \ln Pr(D|\theta) = \sum_{i=1}^{N} \mu_i \ln(a_i p_i) + (1-\mu_i)\ln(b_i(1-p_i))$$

$$\mu_i = Pr[y_i = 1|y_i^j, x_i, \theta]$$

По DS (model B):

$$lnPr(D) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{g=1}^{k} [z_{ij} == g] \ln(c_{iy_jg}) = \sum_{j=1}^{n}\sum_{i=1}^{m} \ln(c_{iy_jz_{ij}})$$

, где i – работник, j – задание, y_j – настоящая метка, z_{ij} – метка работника.

По обозначениям, $\alpha_j = c_{j11}$, $(1-\alpha_j) = c_{j10}$, $\beta_j = c_{j00}$, $(1-\beta_j) = c_{j01}$.

Таким образом,

$$Pr(D, y|\theta) = P(D|y, \theta)P(y) = \prod_{j=1}^{n}\prod_{i=1}^{m} ((\alpha_j)^{y_i^j}(1-\alpha_j)^{1-y_i^j} P(y_j == 1))^{[y_j==1]}((\beta_j)^{1-y_i^j}(1-\beta_j)^{y_i^j}) Pr(y_j == 0))^{[y_j==0]}$$

$$\ln Pr(D, y) = \sum_{j=1}^{n}\sum_{i=1}^{m} \ln(((\alpha_j)^{y_i^j}(1-\alpha_j)^{1-y_i^j} P(y_j == 1))^{[y_j==1]}((\beta_j)^{1-y_i^j}(1-\beta_j)^{y_i^j} P(y_j == 0))^{[y_j==0]})$$

$$E_y \ln Pr(D, y) = \sum_{j=1}^{n}\sum_{i=1}^{m} \mu_j \ln((\alpha_j)^{y_i^j}(1-\alpha_j)^{1-y_i^j} P(y_j == 1)) + (1-\mu_j)\ln((\beta_j)^{1-y_i^j}(1-\beta_j)^{y_i^j} P(y_j == 0)) =$$

$$= \sum_{j=1}^{n} \mu_j \ln a_j + \mu_j \ln P(y_j == 1) + (1-\mu_j)\ln b_j + (1-\mu_j)P(y_j == 0) =$$

$$= \sum_{i=1}^{N} \mu_i \ln a_i + \mu_i \ln P(y_i == 1) + (1-\mu_i)\ln b_i + (1-\mu_i)P(y_i == 0)$$

- $q_i = P(y_i == 1)$

Смесь:

$$P(\theta|D) = \prod_{i=1}^{N} P(\theta|X_i) = \prod_{i=1}^{N} (P(\theta|A, X_i)P(A|X_i) + P(\theta|B, X_i)P(B|X_i)) =$$

$$= \prod_{i=1}^{N} \left(\frac{P_A(X_i|\theta)P_A(\theta)}{P_A(X_i)} P(A|X_i) + \frac{P_B(X_i|\theta)P_B(\theta)}{P_B(X_i)} P(B|X_i) \right)$$

Если $P_A(\theta) = P_B(\theta) \sim Uni$ и $P_A(X_i) = P_B(X_i)$, то:

$$\theta_{MAP} = \arg\max_{\theta} \prod_{i=1}^{N} \left(\frac{P_A(X_i|\theta)P_A(\theta)}{P_A(X_i)} P(A|X_i) + \frac{P_B(X_i|\theta)P_B(\theta)}{P_B(X_i)} P(B|X_i) \right) =$$

$$= \arg\max_{\theta} \prod_{i=1}^{N} \left(P_A(X_i|\theta)P(A|X_i) + P_B(X_i|\theta)P(B|X_i) \right)$$

let for any i $P(A|X_i) = \lambda$ и $P(B|X_i) = 1 - \lambda$.

$$\theta_{MAP} = \arg\max_{\theta} \prod_{i=1}^{N} \left(\lambda P_A(X_i|\theta) + (1-\lambda) P_B(X_i|\theta) \right)$$

$$P(\theta, y|D) \propto \prod_{i=1}^{N} \left(P_A(X_i, y_i|\theta)P(A, y_i|X_i) + P(B, y_i|X_i)P_B(X_i, y_i|\theta) \right) =$$

$$= \prod_{i=1}^{N} ((a_i p_i)^{[y_i==1]}(b_i(1-p_i))^{[y_i==0]} P(A, y_i|X_i) + (a_i q_i)^{[y_i==1]}(b_i(1-q_i))^{[y_i==0]} P(B, y_i|X_i)) =$$

$$\log P(\theta, y|D) = \sum_{i=1}^{N} \log((a_i p_i)^{[y_i==1]}(b_i(1-p_i))^{[y_i==0]} P(A, y_i|X_i) + (a_i q_i)^{[y_i==1]}(b_i(1-q_i))^{[y_i==0]} P(B, y_i|X_i))$$

$$E_y \log P(\theta, y|D) = \sum_{i=1}^{N} \mu_i \log(a_i p_i P(A|X_i) + a_i q_i P(B|X_i)) + (1-\mu_i)\log(b_i(1-p_i)P(A|X_i) + b_i(1-q_i)P(B|X_i)) =$$

$$= \sum_{i=1}^{N} \mu_i \log(a_i p_i \lambda + a_i q_i (1-\lambda)) + (1-\mu_i)\log(b_i(1-p_i)\lambda + b_i(1-q_i)(1-\lambda)) =$$

$$= \sum_{i=1}^{N} \mu_i \log a_i + \mu_i \log(p_i\lambda + q_i(1-\lambda)) + (1-\mu_i)\log(b_i) + (1-\mu_i)\log((1-p_i)\lambda + (1-q_i)(1-\lambda)) =$$

$$= \sum_{i=1}^{N} \mu_i \log a_i + \mu_i \log(p_i\lambda + q_i(1-\lambda)) + (1-\mu_i)\log(b_i) + (1-\mu_i)\log(1 - p_i\lambda - q_i(1-\lambda))$$

- $p_i = \sigma(w^T x_i)$
- q_i – количество единичек в массиве/априорное распределение

$$\frac{dE_y \log P(\theta, y|D)}{d\alpha^j} = (\sum_{i=1}^{N} \mu_i (y_i^j \log \alpha^j + (1-y_i^j)\log(1-\alpha^j)))'_{\alpha^j} = \sum_{i=1}^{N} \mu_i \left(\frac{y_i^j}{\alpha^j} - \frac{1-y_i^j}{1-\alpha^j} \right) =$$

$$= \sum_{i=1}^{N} \mu_i \left(\frac{y_i^j - \alpha^j}{\alpha^j(1-\alpha^j)} \right)$$

$$\frac{dE_y \log P(\theta, y|D)}{d\alpha^j} = 0 \Rightarrow \alpha^j = \frac{\sum\limits_{i=1}^{N} \mu_i y_i^j}{\sum\limits_{i=1}^{N} \mu_i}$$

$$\beta_j = \frac{\sum\limits_{i=1}^{N} (1-\mu_i)(1-y_i^j)}{\sum\limits_{i=1}^{N} (1-\mu_i)}$$

Let $\lambda = \sigma(l)$.

$$\frac{dE_y \log P(\theta, y|D)}{dl} = \sum_{i=1}^{N} \mu_i \frac{(p_i - q_i)\lambda(1-\lambda)}{p_i\lambda + q_i(1-\lambda)} + (1-\mu_i)\frac{(q_i - p_i)\lambda(1-\lambda)}{1 - p_i\lambda - q_i(1-\lambda)}$$

$$\frac{dE_y \log P(\theta, y|D)}{dw_k} = \sum_{i=1}^{N} \mu_i \frac{\lambda_i}{p_i\lambda_i + q_i(1-\lambda_i)} (p_i)'_{w_k} + (1-\mu_i)\frac{-\lambda_i}{1 - p_i\lambda_i - q_i(1-\lambda_i)} (p_i)'_{w_k}$$

$$\frac{dE_y \log P(\theta, y|D)}{d^2 w_k w_j} = \sum_{i=1}^{N} (\mu_i \frac{\lambda_i}{p_i\lambda_i + q_i(1-\lambda_i)} (p_i)'_{w_k})'_{w_j} + ((1-\mu_i)\frac{-\lambda_i}{1 - p_i\lambda_i - q_i(1-\lambda_i)} (p_i)'_{w_k})'_{w_j} =$$

$$= \sum_{i=1}^{N} \mu_i\lambda_i \frac{-1}{(p_i\lambda_i + q_i(1-\lambda_i))^2} \lambda_i (p_i)'_{w_j}(p_i)'_{w_k} + \mu_i\lambda_i \frac{1}{p_i\lambda_i + q_i(1-\lambda_i)} (p_i)''_{w_k,w_j} +$$

$$+ (1-\mu_i)(-\lambda_i)\frac{-1}{(1 - p_i\lambda_i - q_i(1-\lambda_i))^2} (-\lambda_i)(p_i)'_{w_j}(p_i)'_{w_k} + (1-\mu_i)(-\lambda_i)\frac{1}{1 - p_i\lambda_i - q_i(1-\lambda_i)} (p_i)''_{w_k,w_j}$$

$$\mu_i = \frac{a_i p_i}{a_i p_i + b_i(1-p_i)}\lambda_i + \frac{a_i q_i}{a_i q_i + b_i(1-q_i)}(1-\lambda_i)$$

- Если $p_i = \sigma(w^T x_i)$, то

$$(p_i)'_w = p_i(1-p_i)x_i$$
$$(p_i)''_w = p_i(1-p_i)(1-2p_i)x_i x_i^T$$

$$\frac{dE_y \log P(\theta, y|D)}{dw} = \sum_{i=1}^{N} \mu_i \frac{\lambda_i}{p_i\lambda_i + q_i(1-\lambda_i)} p_i(1-p_i)x_i + (1-\mu_i)\frac{-\lambda_i}{1 - p_i\lambda_i - q_i(1-\lambda_i)} p_i(1-p_i)x_i$$

$$\frac{dE_y \log P(\theta, y|D)}{d^2 w} = \sum_{i=1}^{N} \frac{\mu_i\lambda_i p_i(1-p_i)(1-2p_i)}{p_i\lambda_i + q_i(1-\lambda_i)} x_i x_i^T - \frac{\mu_i\lambda_i^2 p_i^2(1-p_i)^2}{(p_i\lambda_i + q_i(1-\lambda_i))^2} x_i x_i^T -$$

$$- \frac{(1-\mu_i)\lambda_i p_i(1-p_i)(1-2p_i)}{1 - p_i\lambda_i - q_i(1-\lambda_i)} x_i x_i^T - \frac{(1-\mu_i)\lambda_i^2 p_i^2(1-p_i)^2}{(1 - p_i\lambda_i - q_i(1-\lambda_i))^2} x_i x_i^T$$

- Если $p_i = VGG[w]$