# ISEE-720: Safety Stock Final Project

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**Abstract**

This paper proposes an approach for computing safety stock inventory levels for multiple bundles, where each bundle consists of one or more products and each product can appear across different bundles. The demand of each product is normally distributed and the lead time of each bundle is normally distributed with an interruption that is exponentially distributed. Each bundle has a respective risk of lead time interruption occurring. A numerical example is created to implement this approach of computing safety stock inventory levels. In this example, it is shown how the safety stock and reorder point of each bundle changes with respect to the stochastic inputs that make up a bundle's demand and lead time. This approach relies on knowing the mean and variance of a bundle's demand and lead time, so it does not rely on the probability distribution for a bundle's demand and lead time.


**Literature**

The first step of the analysis consisted of conducting research on 2 main topics: multi-product Q-R models and aggregation of probability distributions. For the multi-product Q-R model, a multi-product continuous review system model[2] and the Stochastic Inventory System slides[3] are used to generate the necessary equations for solving the safety stock. For the aggregation of probability distributions, chapter 9 on aggregating probability distributions from Advances in decision analysis[1] is used to combine the demand distributions of products into the demand distribution of each bundle. The combination of lead time and interruption distributions to achieve the bundle lead time is performed using a simulation.

**Bundle Demand**

Linear pooling is acceptable for combining the probability density functions of independent random variables using an axiomatic approach.[1] The product demands are given to be independent of each other, thus the demand of a bundle can be estimated using a convex linear combination (ie. linear pooling) of its product demands.

First, the demand per product is translated into demand per bundle. This is achieved by weighting the product demands by the number of bundles in which each product appears. For example, if the bundles consist of {A}, {A, B}, and {B, C}, the weighted demand of product A in each bundle that it appears in would be 50% of the original demand for A (See Assumption 2). Therefore, the weighted probability density function of the demand of product A would be the following:

$$wpdf_A = \frac{1}{n_A} N(\mu_A, \sigma_A) \qquad (1)$$

*where $n_A$ is the number of bundles product A appears in*

Next, the demand distribution of bundles are calculated. The convex coefficient for each product's demand distribution is computed based on the average of its weighted demand (See Assumption 3), where the sum of all coefficients equals one. For example, in a bundle {A, B} the average of the weighted demands of A and B are given by:

$$m_A = mean(wpdf_A)$$
$$m_B = mean(wpdf_B)$$

Then the convex coefficients of A and B are given by:

$$c_A = \frac{m_A}{m_A + m_B}$$
$$c_B = \frac{m_B}{m_A + m_B}$$

Therefore, the probability density function for this bundle's demand is the following:

$$pdf_{\{A,\,B\}} = c_A wpdf_A + c_B wpdf_B \qquad (2)$$

**Bundle Lead Time**

The lead times per bundle are calculated by setting up the normal distribution of lead time and adding a random binary variable (*r*) that will simulate the risk of supply shortage. If an interruption occurs according to (*r*), then the exponential interruption is added to the original lead time; otherwise, the original lead time remains the same. Therefore, a bundle's lead time is the following:

$$L_i = N(\mu_i,\ \sigma_i) + r * exp(\lambda_i) \qquad (3)$$

*where i refers to each bundle*
*where r has a likelihood of taking a value of* 1 *based on a given risk probability p, and a value of* 0 *otherwise*

**Safety Stock & Reorder Point**

The first step in computing the safety stock and reorder point for a bundle is determined by an equation provided from multi-product continuous review system.[2] That paper defined an equation for the expected value of the demand shortage during lead time, which is defined as the expected value of demand during lead time ($X_i$) minus the reorder point ($R_i$). Refer to Figure 1, Equation 1, and Lemma 1[2] for the resulting Equation (4) below:

$$E(X_i - R_i) = \tfrac{1}{2}\left(\sqrt{1 + k_i^2} - k_i\right)\sqrt{m_L \sigma_D^2 + m_D^2 \sigma_L^2} \qquad [2]\ (4)$$

*where* $k_i$ *is the safety stock weight for bundle i* (*larger values increase the value of safety stock*)
$m_L$ *is the mean lead time for bundle i*
$m_D$ *is the mean demand for bundle i*
$\sigma_L$ *is the lead time standard deviation for bundle i*
$\sigma_D$ *is the demand standard deviation for bundle i*

Next, Equation (4) is used to determine an equation for the expected service level. It is known that the expected proportion of lost demand during lead time would be the expected value of the demand shortage during lead time ($E(X_i - R_i)$) divided by the expected value of demand during lead time $(m_L m_D)$. One minus this proportion would be the expected proportion of satisfied demand during lead time, which is identified as the expected service level. The expected service level must equal a desired level $\alpha$, therefore by subtracting expected service level by $\alpha$, Equation (5) is formed for the expected service level:

$$1 - \frac{\frac{1}{2}\left(\sqrt{1+k_i^2} - k_i\right)\sqrt{m_L \sigma_D^2 + m_D^2 \sigma_L^2}}{m_L m_D} - \alpha = 0 \qquad (5)$$

*where* $k_i$ *is the only variable*

In Equation (5), $k_i$ is the only variable, so solving for the root of Equation (5) yields a value for $k_i$ that satisfies Equation (5). Newton's Method can be used to find a root of Equation (5). Before using Newton's Method, the derivative of Equation (5) is computed, which is given below:

$$-\frac{\frac{1}{2}\left(\frac{k_i}{\sqrt{1+k_i^2}}-1\right)\sqrt{m_L\sigma_D^2+m_D^2\sigma_L^2}}{m_L m_D} \qquad (6)$$

The derivative is necessary for applying Newton's Method[4] to find a root (ie. $k_i$) of Equation (5). Newton's Method is the following:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad [4]\ (7)$$
$$\textit{where } n+1 \textit{ iterations yield } x_{n+1} = k_i \textit{ for bundle } i$$

Newton's Method iterates through the Taylor series expansion until converging on a root. Newton's method requires a starting point such that it can travel to the nearest root: for this analysis, $k_i = 1$ is used as the starting point. Given that we are only interested in satisfying Equation (5) with one value of $k_i$, we only use Newton's Method once for each bundle ($i$) to solve for one root despite the possibility of multiple roots that may satisfy Equation (5).

The value of $k_i$ is then used to compute the reorder point ($R_i$) [2] and the safety stock ($ss_i$) [3] below:

$$R_i = m_L m_D + k_i\sqrt{m_L\sigma_D^2 + m_D^2\sigma_L^2} \qquad [2]\ (8)$$
$$ss_i = R_i - m_L m_D = k_i\sqrt{m_L\sigma_D^2 + m_D^2\sigma_L^2} \qquad [3]\ (9)$$

Equation (9) follows the expected convention of the QR model[3], in which the safety stock is given by the following equation when there is normally distributed lead time and normally distributed demand:

$$ss = Z_\alpha\sqrt{m_L\sigma_D^2 + m_D^2\sigma_L^2} \qquad [3]\ (10)$$

**Numerical Example**

The first step to the analysis was creating baseline values for all demand and lead time parameters. The bundles are defined in Table 1:

**Table 1:** Bundles with corresponding product

| Bundle | Product |
|---|---|
| 1 | {A} |
| 2 | {B} |
| 3 | {C} |
| 4 | {A, B} |
| 5 | {A, C} |
| 6 | {B, C} |
| 7 | {A, B, C} |

Product demands and variance are assigned such that A is the high runner and C is the low runner and B is between these two values, as shown in Table 2:

**Table 2:** Product demands and variance

| **Product** | $\mu$ | $\sigma$ |
|---|---|---|
| A | 100 | 20 |
| B | 80 | 10 |
| C | 50 | 5 |

Suppliers are assigned depending on the bundle size, as shown in Table 3. The mean ($\mu$) and standard deviation ($\sigma$) increases as the bundle size increases. The supply risk ($p$) decreases as the bundle size increases, however the interruption length rate ($\lambda$) is assigned to be more severe for a smaller risk value. This assumes that a supplier with a long interruption length would mitigate the likelihood of interruption.

**Table 3:** Supplier lead time and shortage delay values

| **Supplier** | **Bundle** | $\mu$ | $\sigma$ | $p$ | $\lambda$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 0.1 | 0.3 | 1 |
| 1 | 2 | 1.25 | 0.2 | 0.3 | 1 |
| 1 | 3 | 1.33 | 0.2 | 0.3 | 1 |
| 2 | 4 | 2 | 0.3 | 0.2 | 0.75 |
| 2 | 5 | 2.5 | 0.2 | 0.2 | 0.75 |
| 2 | 6 | 2.25 | 0.3 | 0.2 | 0.75 |
| 3 | 7 | 3 | 0.5 | 0.1 | 0.5 |

Service level, shown in Table 4, is picked arbitrarily as a good service level:

**Table 4:** Service level

| $\alpha$ |
|---|
| 0.9 |

The baseline values from Tables 1–4 are used to compute safety stock weight ($k$), reorder point (R), and safety stock (SS) for each bundle using the equations outlined in this paper. This implementation was scripted in the R programming language. When computing the demand distribution of a bundle, a sample size of 1,000,000 was used to randomly sample product demand values to aggregate into 1,000,000 values from the demand distribution of a bundle. When computing the lead time distribution of a bundle, a sample size of 1,000,000 was used to randomly sample lead time values, randomly sample occurrences of supply interruption based on ($p$), and randomly sample interruption length values to aggregate into 1,000,000 values from the lead time distribution of a bundle. Using these two large samples that represent a bundles demand and lead time, the mean and variance of the demand and of the lead time were computed to then compute the safety stock weight, reorder point, and safety stock of a bundle. Table 5 summarizes the safety stock policy for the baseline values:

**Table 5:** Baseline Safety Stock Policy

| Supplier | Bundle | $\alpha$ | R | SS | $k$ |
|---|---|---|---|---|---|
| 1 | A | 0.9 | 56.7 | 24.2 | 1.281 |
| 1 | B | 0.9 | 46.4 | 15.4 | 1.018 |
| 1 | C | 0.8999997 | 29.2 | 8.8 | 0.937 |
| 2 | AB | 0.8999998 | 65.9 | 14.3 | 0.712 |
| 2 | AC | 0.9 | 66.0 | 8.4 | 0.464 |
| 2 | BC | 0.9 | 51.6 | 8.5 | 0.574 |
| 3 | ABC | 0.9 | 76.1 | 10.4 | 0.492 |

The second step of this analysis is to increase the product demand mean, product demand standard deviation, bundle lead time mean, bundle lead time standard deviation, bundle supply risk likelihood, and bundle mean interruption length. After one of these stochastic inputs is increased, a safety stock policy is computed and then compared to the baseline safety stock policy in Table 5. Only one stochastic input is increased at a time, so this analysis does not consider simultaneous increases in multiple stochastic inputs.

The increase of a stochastic input was chosen to be a random value ($u$) in the range of 25%–75%, where all values are equally likely of being chosen. A replication size of 30 was chosen to ensure that the range of 25%–75% was covered without any bias. Table 6 below outlines how each stochastic input was increased.

**Table 6:** Increasing Stochastic Inputs

| Input | Baseline Value | Increased Value |
|---|---|---|
| Product Demand Mean | $\mu_{Product}$ | $\mu_{Product}(1 + u_{random})$ |
| Product Demand Std. Dev. | $\sigma_{Product}$ | $\sigma_{Product}(1 + u_{random})$ |
| Bundle Lead Time Mean | $\mu_{Bundle}$ | $\mu_{Bundle}(1 + u_{random})$ |
| Bundle Lead Time Std. Dev. | $\sigma_{Bundle}$ | $\sigma_{Bundle}(1 + u_{random})$ |
| Bundle Supply Risk | $p_{Bundle}$ | $p_{Bundle}(1 + u_{random})$ |
| Bundle Mean Interruption Length | $(1 / \lambda_{Bundle})$ | $(1 / \lambda_{Bundle})(1 + u_{random})$ |

In this section, we will cover how the reorder point and safety stock change in value when Bundle Supply Risk increases, and when Bundle Mean Interruption Length increases. The Appendix contains the remainder of the analysis for how reorder point, safety stock, and safety stock weight change due to increases in the stochastic inputs.

The change in reorder point when supply risk (p) increases is shown below in Figure 1. There are four plots below to show what happens to reorder point when supply risk is increased for each of the suppliers individually and then all at once. This shows that increasing the supply risk of one bundle has no influence on the reorder point of other bundles. This makes sense as each bundle's safety stock inventory level is computed independently of the other bundles. This independence between bundles is consistent across the remaining Figures in this section. Overall, as the supply risk increases by 25%–75% the reorder point increases by about 5%–25% as seen in the bottom right plot of Figure 1. This plot also shows that a bundle with a higher baseline supply risk yields a larger increase relative to its baseline reorder point.

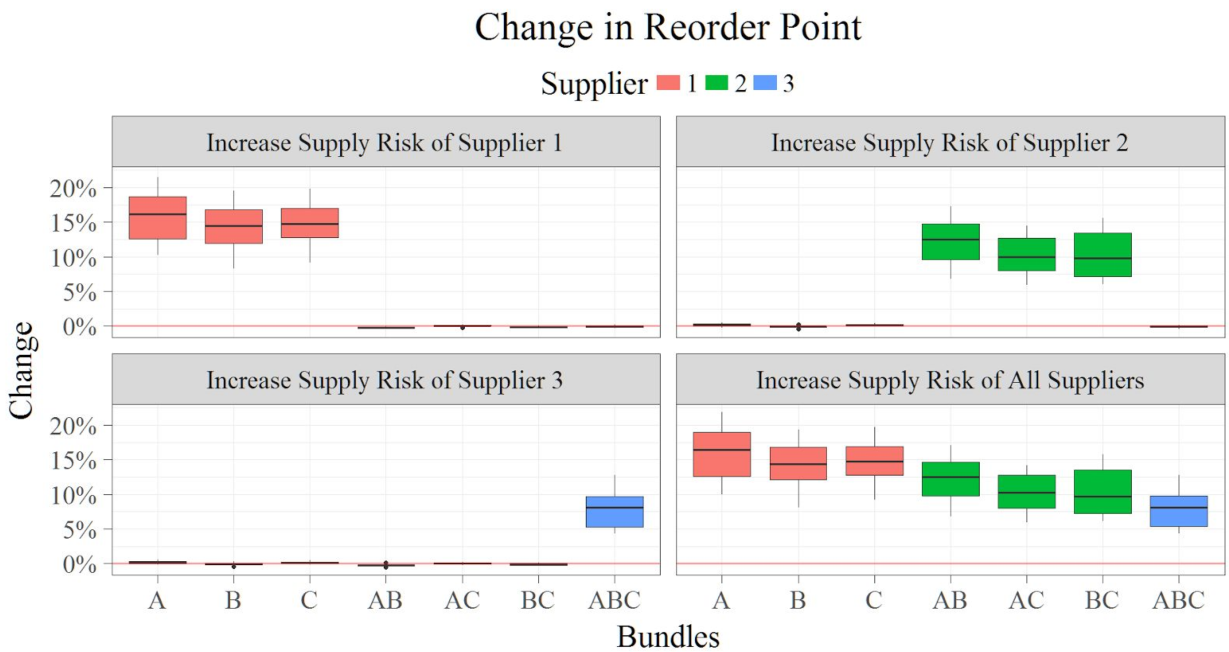


**Figure 1:** Change in Reorder Point v. Supply Risk Increase of 25%–75%

The change in safety stock when supply risk (p) increases is shown below in Figure 2. Overall, as the supply risk increases by 25%–75% the safety stock increases by about 15%–65% as seen in the bottom right plot of Figure 2. This plot also shows that a bundle with a lower baseline supply risk yields a larger increase relative to its baseline safety stock.

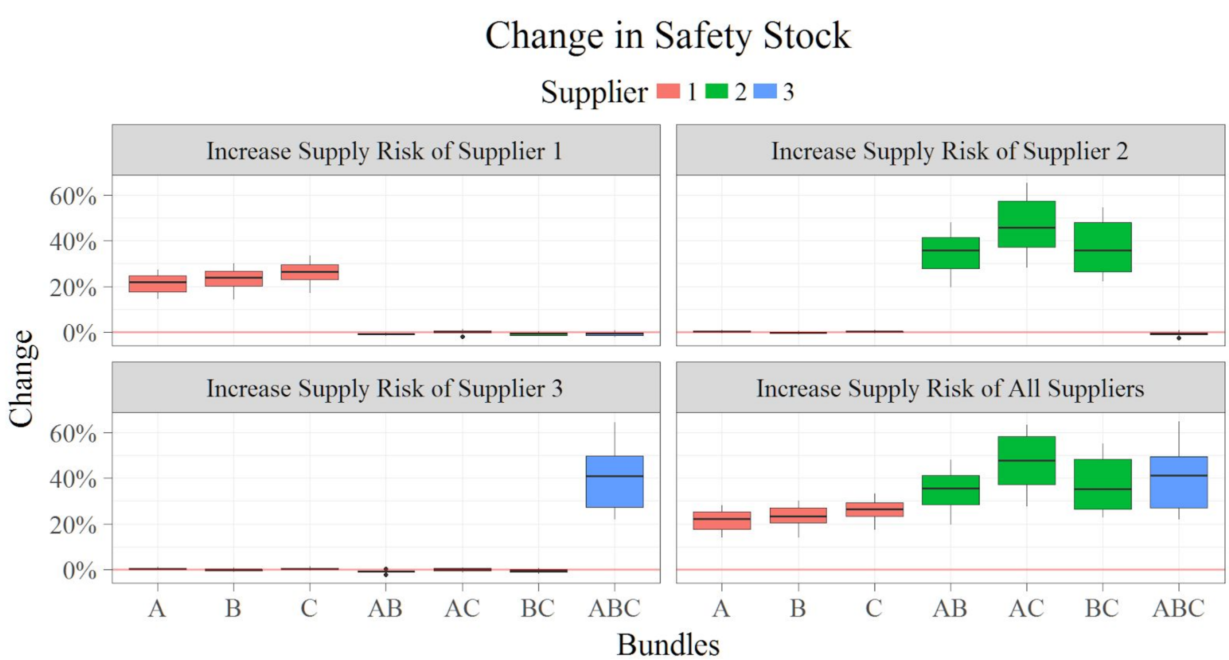


**Figure 2:** Change in Safety Stock v. Supply Risk Increase of 25%–75%

The change in reorder point when mean interruption length ($1 / \lambda$) increases is shown below in Figure 3. Overall, as the mean interruption length increases by 25%–75%, the reorder point increases by about 10%–80% as seen in the bottom right plot of Figure 3. This plot also shows that a bundle with a lower baseline mean interruption length yields a larger increase relative to its baseline reorder point.

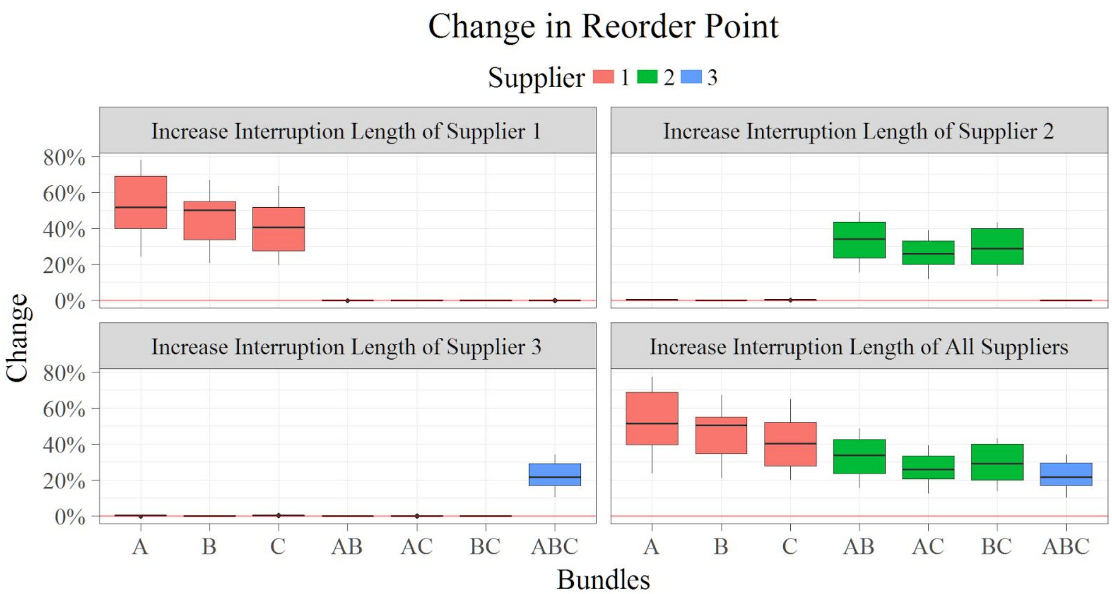


**Figure 3:** Change in Reorder Point v. Interruption Length Increase of 25%–75%

The change in safety stock when mean interruption length ($1 / \lambda$) increases is shown below in Figure 4. Overall, as the mean interruption length increases by 25%–75% the safety stock increases by about 50%–250% as seen in the bottom right plot of Figure 4. This plot also shows that a bundle with a higher baseline mean interruption length yields a larger increase relative to its baseline safety stock.

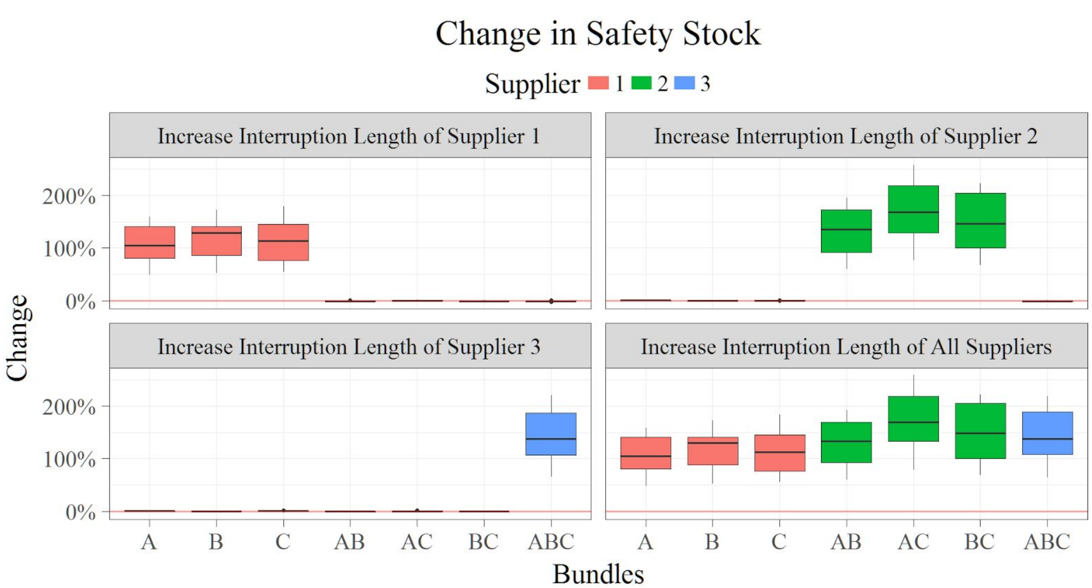


**Figure 4:** Change in Safety Stock v. Interruption Length Increase of 25%–75%

Refer to Figures 5–18 in the Appendix to gain a complete understanding of how the safety stock inventory policy behaves in the face of increasing the values of the stochastic inputs.

**Assumptions**

1. Lead time is independent from demand.
2. All bundles are used equally to satisfy product demands.
3. A product with a larger expected demand will have a larger influence on the demand of the bundle it belongs to based on the weighted average.

**Weaknesses**

1. Subjectivity of the weights for how much of a product's demand is satisfied by a bundle (Assumption 2).
2. Subjectivity of the weights for combining product demands into bundle demands (Assumption 3).
3. Each bundle has its own safety stock inventory policy, so the reorder points aren't coordinated to take advantage of the opportunity to place one large order to satisfy to the replenishment of multiple bundles.
4. If a bundle has multiple products with a large difference in expected demand, then the combined demand to make a bundles demand has high variance and may result in a large buildup of inventory to ensure a desired service level.

**Strengths**

1. The particular statistical distributions of product demand, bundle leadtime, and supply interruption don't matter. Our procedure can work with any type of distribution as long as the mean and variance can be calculated.
2. By linearly pooling product demands into bundle demands, we are able to treat each bundle as if it was just a product. Thus the procedure can provide safety stock for individual products or a mix of products and bundles.
3. Equation (5) is differentiable, therefore a real value of $k$ can always be solved for such that it can be expected that the desired service level is met.

**Conclusions**

This formulation proved to be adequate in determining safety stock for bundles but can also be used for single products. The computation of the resulting safety stock inventory policy only relied on the expected value and variance of the stochastic inputs so any type of stochastic behavior (i.e. normal, exponential, empirical, etc.) is allowed. The effects of bundling products and designating suppliers also provides more realistic circumstances and flexibility for the use of our procedure. The results of the numerical example showed that the resulting safety stock inventory policy was more sensitive to increases in interruption length than increases in supplier risk, thus compensating for unpredictability in internal operations. Our numerical example only considered increases in the stochastic inputs because these increases correspond to a more difficult problem to solve. Overall, our procedure is simple to implement in any programming language, works for bundles and/or products, and works for any type of stochastic behavior.

**Sources**


1. Edwards, W., Miles, R. F., & Winterfeldt, D. V. (2007). Ch. 9 Aggregating Probability Distributions. (pp. 157). *Advances in decision analysis: from foundations to applications*. Cambridge: Cambridge University Press.
2. Kundu, A., & Chakrabarti, T. (2010). A multi-product continuous review inventory system in stochastic environment with budget constraint. Optimization Letters, 6(2), 299-313. doi:10.1007/s11590-010-0245-3.
3. Proaño, R. (2017). Stochastic Inventory Systems. [PowerPoint slides]. Retrieved from http://mycourses.rit.edu.
4. Weisstein, Eric W. "Newton's Method." MathWorld--A Wolfram Web Resource. Retrieved from http://mathworld.wolfram.com/NewtonsMethod.html.

## Appendix

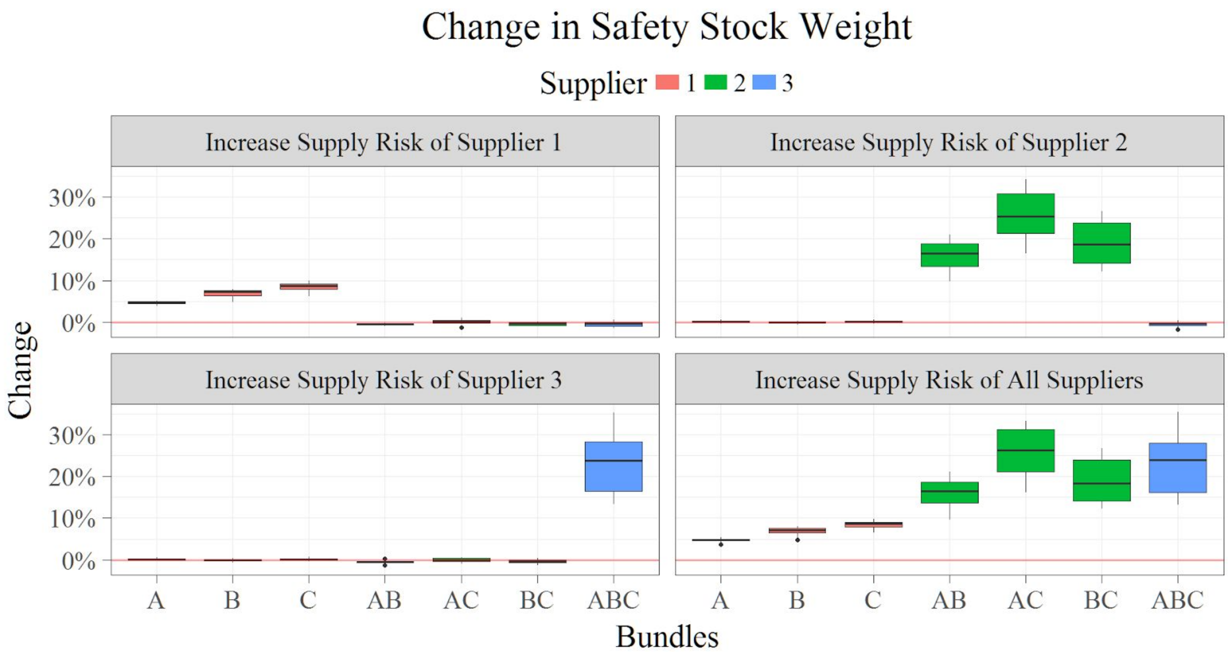


**Figure 5:** Change in Safety Stock Weight v. Supply Risk Increase of 25%–75%


**Figure 6:** Change in Safety Stock Weight v. Interruption Length Increase of 25%–75%

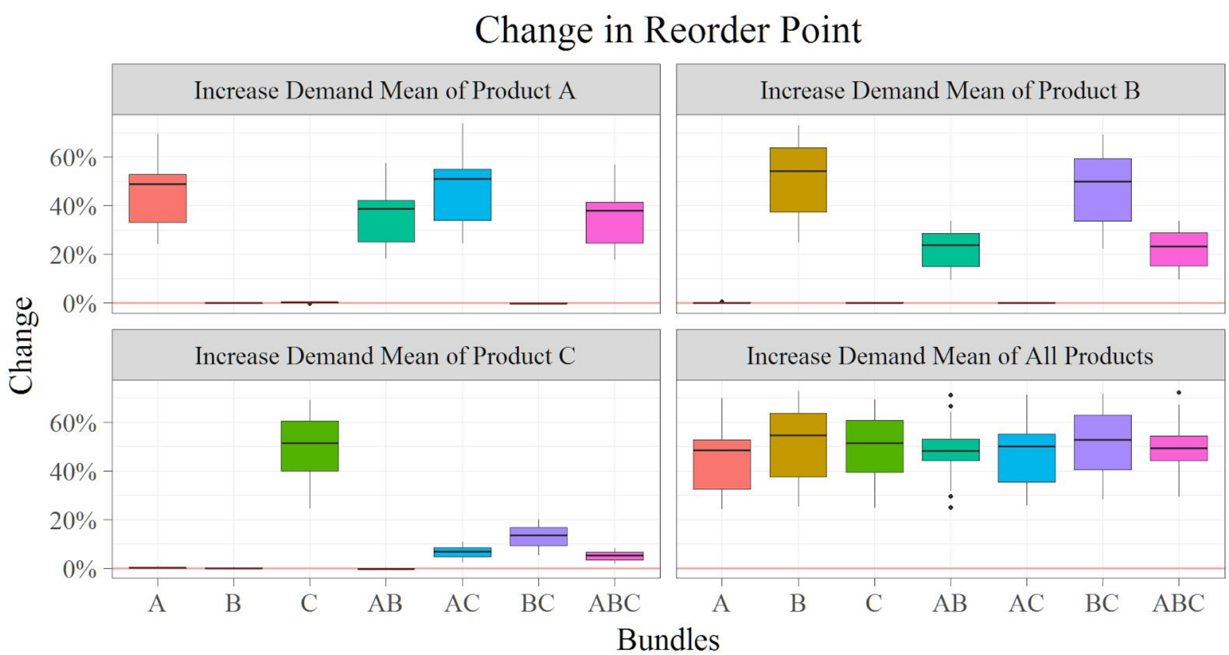


**Figure 7:** Change in Reorder Point v. Demand Mean Increase of 25%–75%

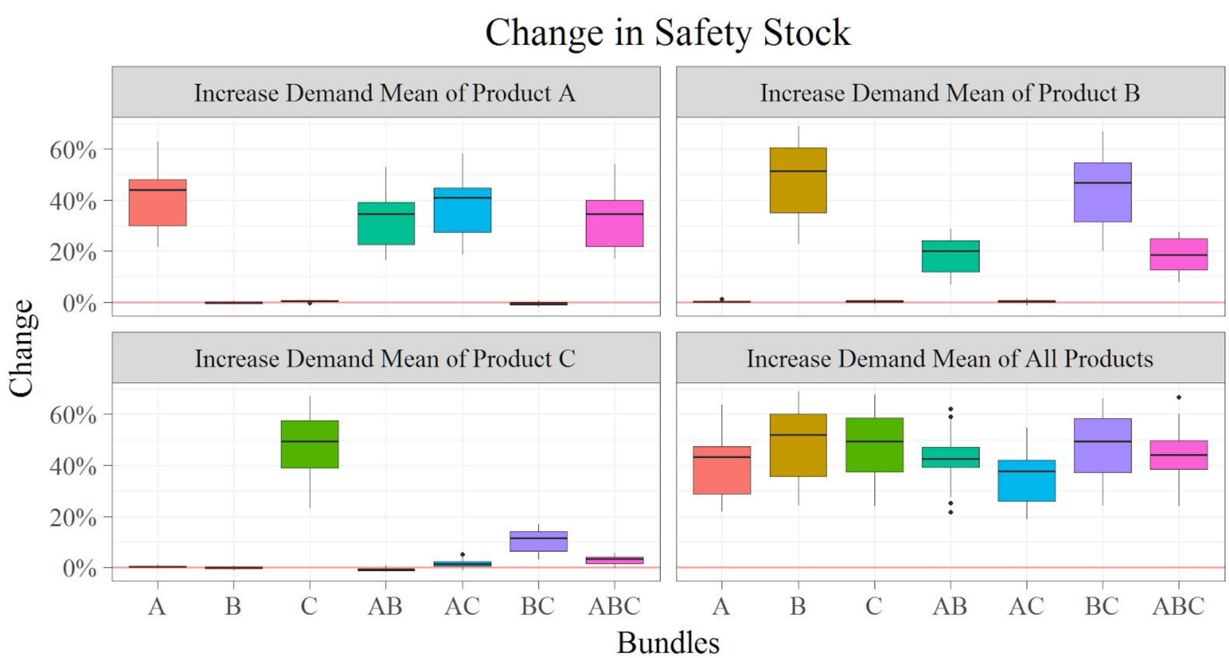


**Figure 8:** Change in Safety Stock v. Demand Mean Increase of 25%–75%

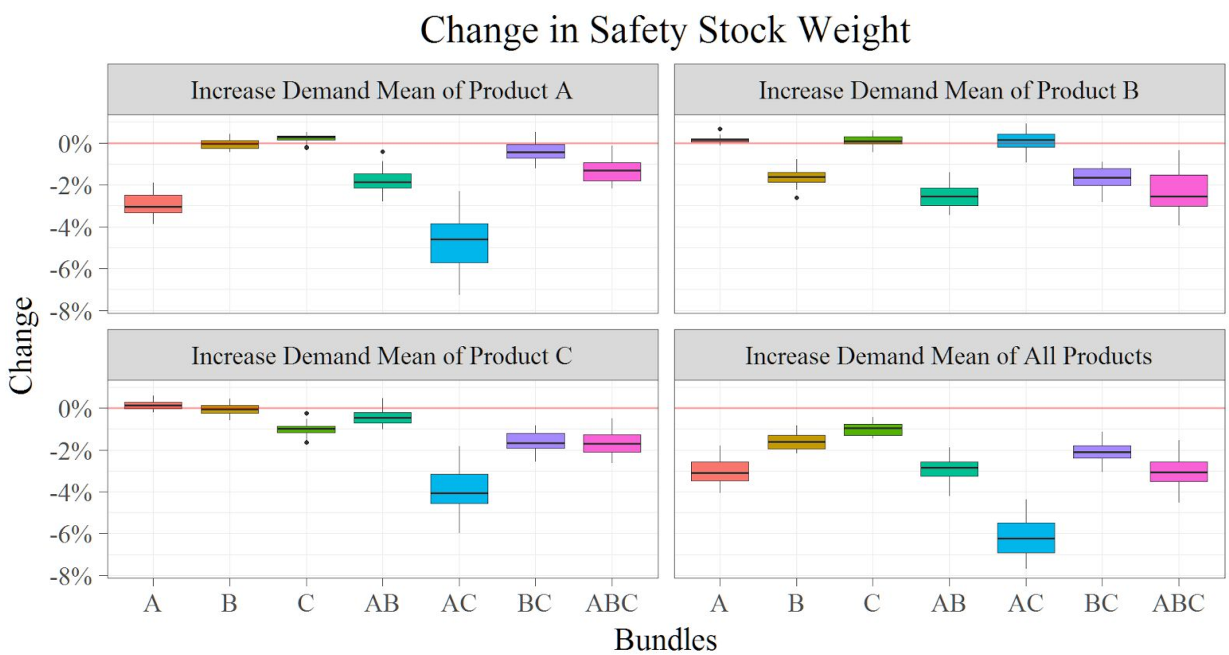


**Figure 9:** Change in Safety Stock Weight v. Demand Mean Increase of 25%–75%

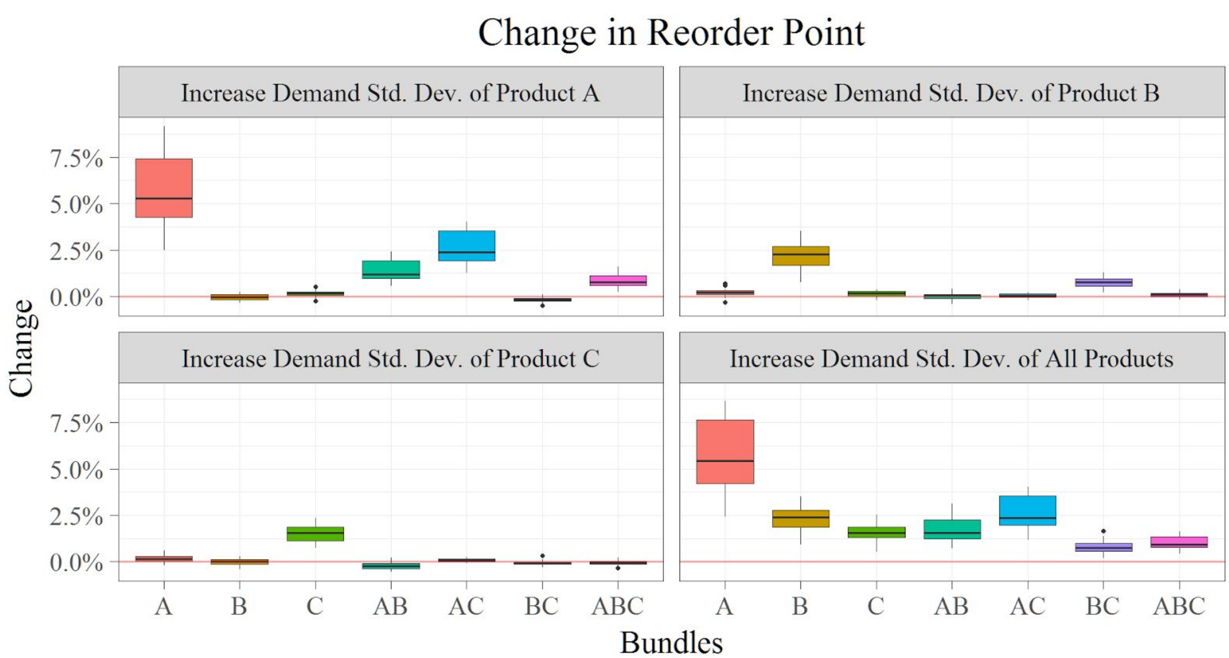


**Figure 10:** Change in Reorder Point v. Demand Std. Dev. Increase of 25%–75%

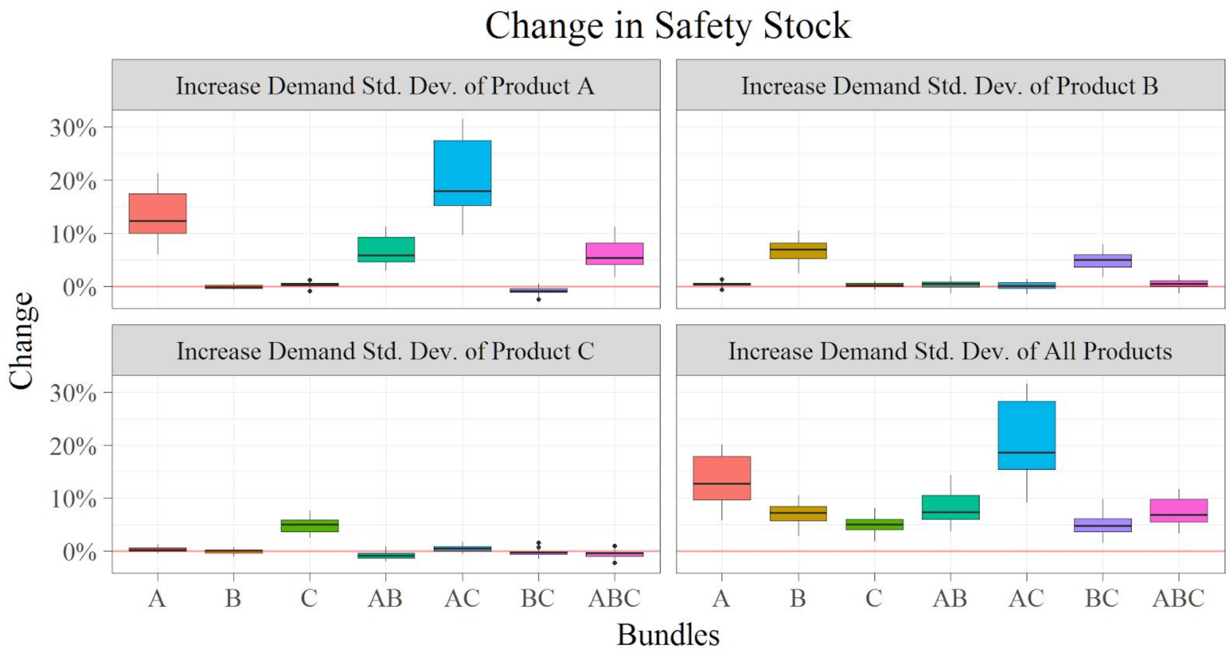


**Figure 11:** Change in Safety Stock v. Demand Std. Dev. Increase of 25%–75%

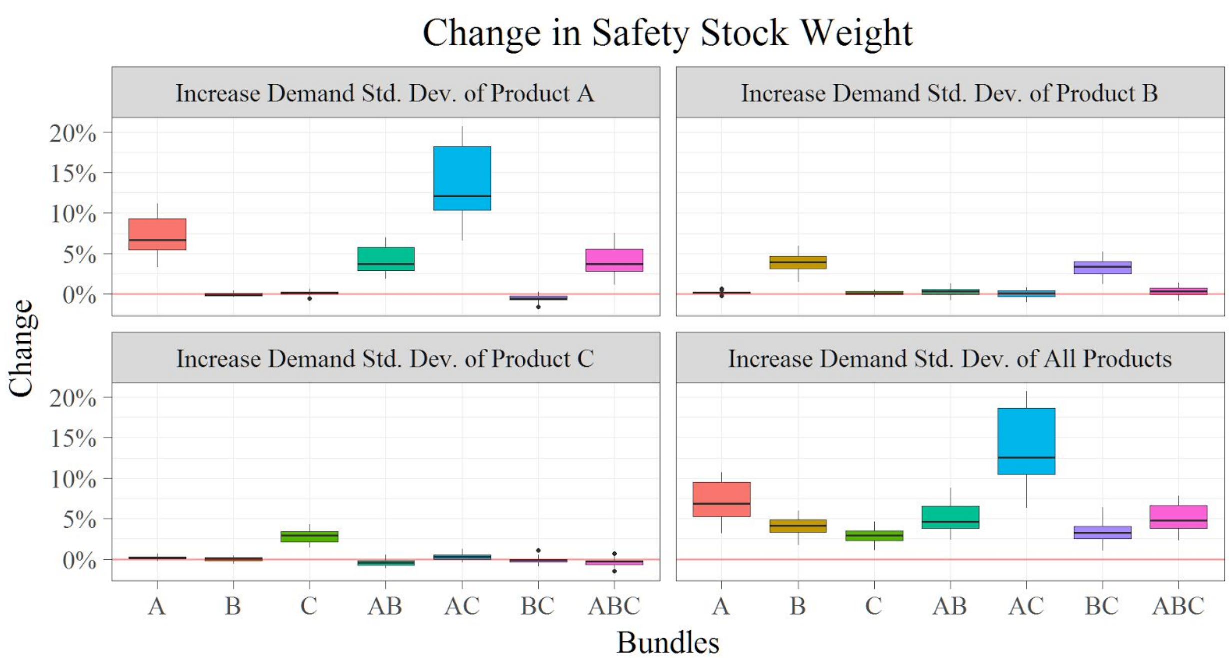


**Figure 12:** Change in Safety Stock Weight v. Demand Std. Dev. Increase of 25%–75%

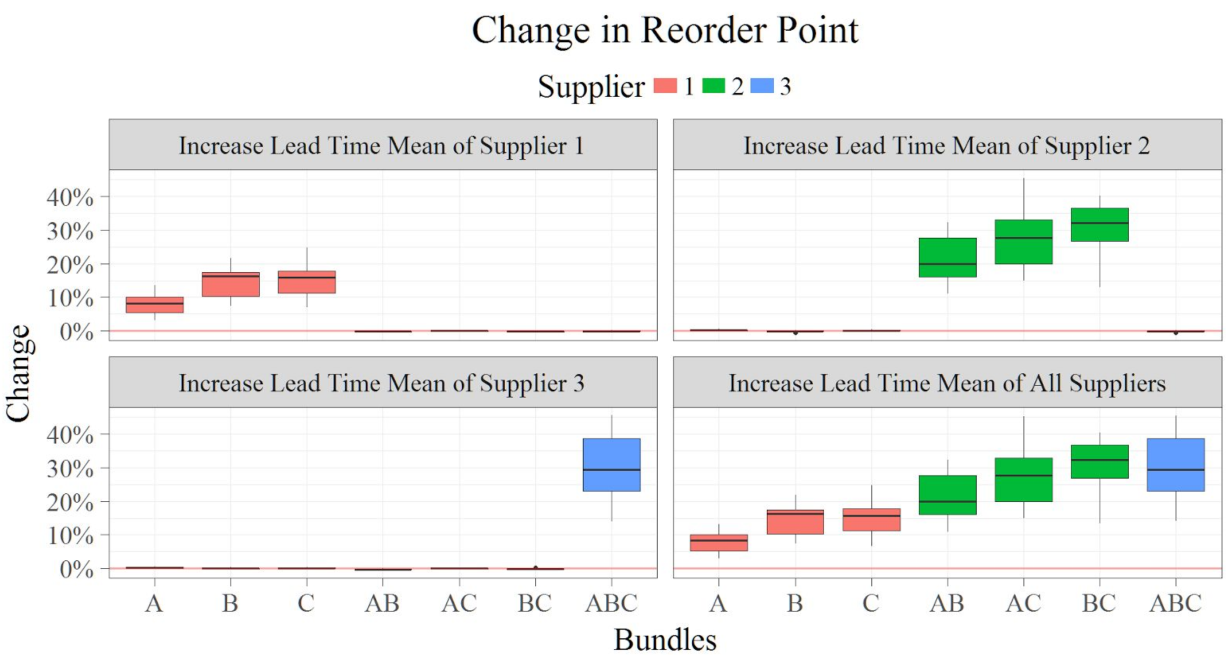


**Figure 13:** Change in Reorder Point v. Lead Time Mean Increase of 25%–75%

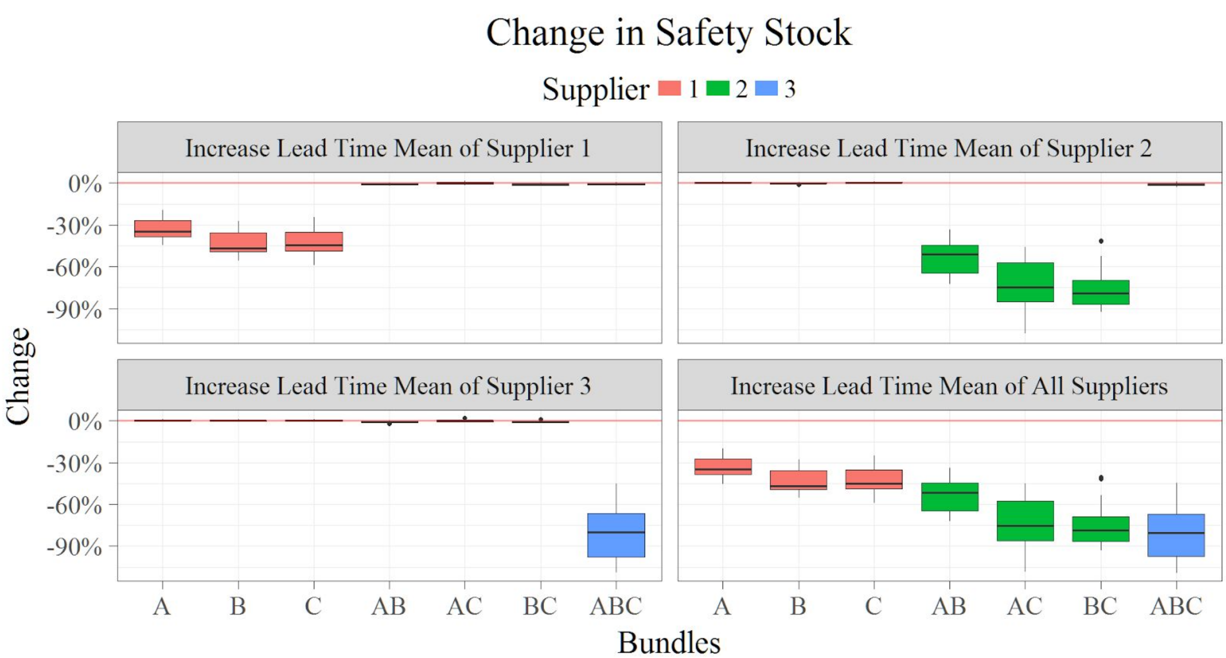


**Figure 14:** Change in Safety Stock v. Lead Time Mean Increase of 25%–75%


**Figure 15:** Change in Safety Stock Weight v. Lead Time Mean Increase of 25%–75%

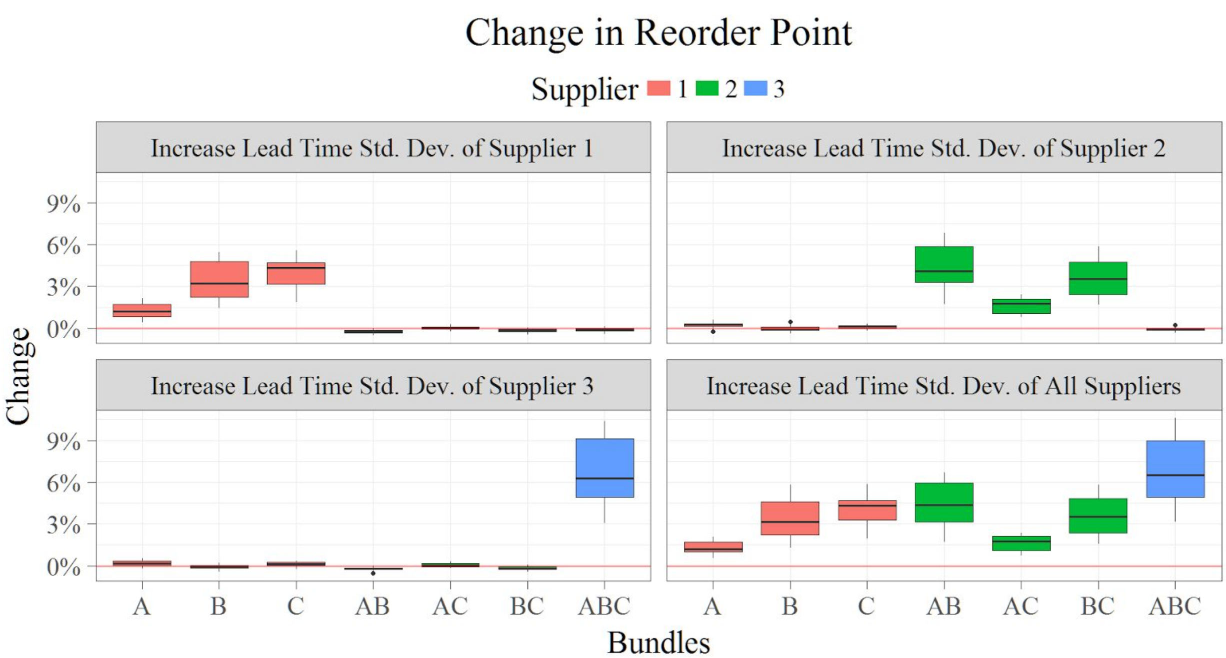


**Figure 16:** Change in Reorder Point v. Lead Time Std. Dev. Increase of 25%–75%

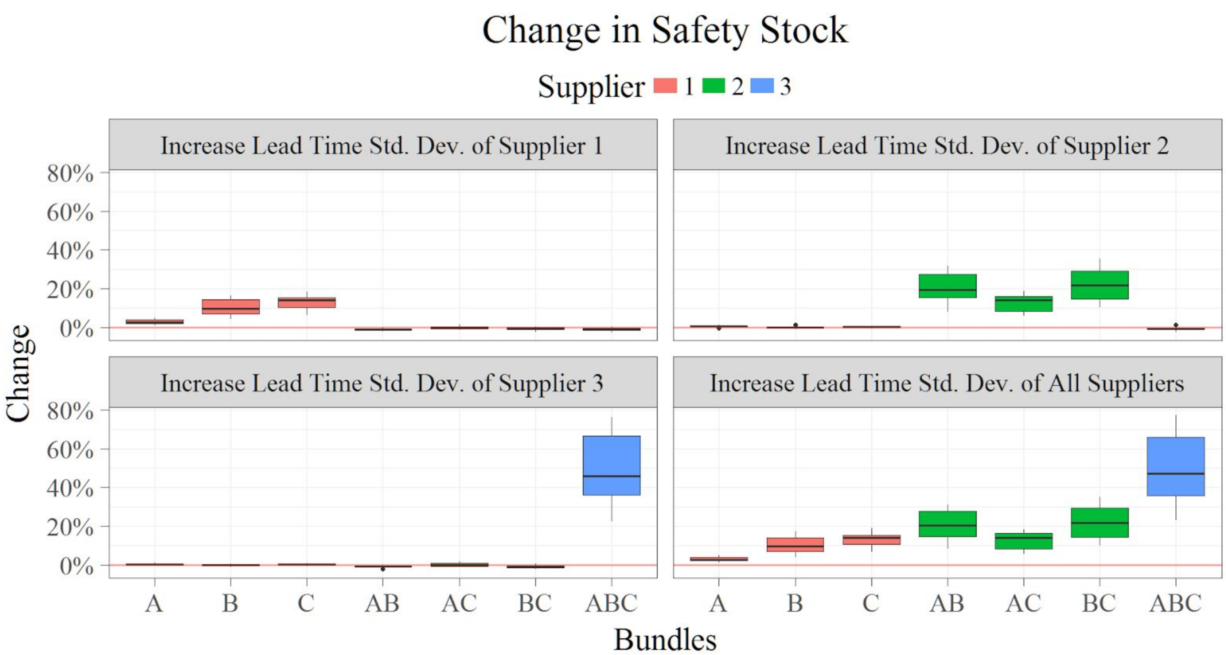


**Figure 17:** Change in Safety Stock v. Lead Time Std. Dev. Increase of 25%–75%


**Figure 18:** Change in Safety Stock Weight v. Lead Time Std. Dev. Increase of 25%–75%