MATH 401 STOCHASTIC PROCESSES
HOMEWORK # 5
Due: Wednesday, March 29

Note: Define all the variables clearly and completely. show all the work to get credit. Answers without proper and properly written explanations and supporting arguments will not get any credit. Work independently.

1. Suppose that the weather on any day depends on the weather conditions during the previous 2 days. We form a Markov chain with the states being described by a pair (Y,T), where T is the current status and Y is the previous day's status. The weather status could be either "sunny (S)" or "cloudy (C)." Suppose the transition matrix of the Markov chain is given by

$$\begin{array}{c} \\ (S,S) \\ (S,C) \\ (C,S) \\ (C,C) \end{array} \begin{array}{c} \begin{array}{cccc} (S,S) & (S,C) & (C,S) & (C,C) \end{array} \\ \begin{bmatrix} 0.7 & 0.3 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \end{bmatrix} \end{array}$$

(a) Given that it is sunny on days 0 and 1, what is the probability that it is sunny on day 5?

(b) In the long run what fraction of days are sunny?

2. A professor continually gives exams to her students. she can give three possible types of exams, and her class is graded as either having done well or badly. Let p_i denote the probability that the class does well on a type i exam, and suppose that $p_1 = 0.3, p_2 = 0.6, p_3 = 0.9$. If the class does well on an exam, then the next exam is equally likely to be any of the three types. If the class does badly, then the next exam is always type 1. Let X_n be the type of the nth exam with $X_n = 1, 2, 3$.

(a) Show that $\{X_n\}$ is a Markov chain and find its transition probability matrix.

(b) What proportion of exams are type $i, i = 1, 2, 3$, in the long run?

3. A certain town (not Rochester) never has two sunny days in a row. Each day is classified as being either sunny, cloudy (but dry), or rainy. If it is sunny one day, then it is equally likely to be either cloudy or rainy the next day. If it is rainy or cloudy one day, then there is one chance in two that it will be the same the next day, and if it changes, then it is equally likely to be either of the other two possiblities.Let $X_n, n \geq 0$ be the state of the weather on the nth day with the states labeled as 0 for sunny day, 1 for cloudy day and 2 for rainy day.

(a) Show that $\{X_n\}$ is a Markov chain and find its transition probability matrix.

(b) In the long run, what proportion of days are suny? What proportion of days are cloudy?

4. A Markov chain on state space $\{0, 1, 2\}$ has the transition probability matrix

$$P = \begin{bmatrix} 0.4 & 0.4 & 0.2 \\ 0.6 & 0.2 & 0.2 \\ 0.4 & 0.2 & 0.4 \end{bmatrix}.$$

After a long period of time, the chain is found to be in state 1. What is the probability that the previous state was 2. We want to find $\lim_{n\to\infty} P\Big(X_{n-1} = 2 | X_n = 1\Big)$.

5. Suppose $\{X_n\}$ is a regular Markov chain on the state space $\mathcal{S} = \Big\{1, 2, \ldots, M\Big\}$.with a transition probabiilty matrix, P, whose column sums are also equal to 1 (i.e) the sum of entries in each column is also 1, for all columns. Find the limiting probability vector $\underline{\pi} = (\pi_1, \pi_2, \ldots, \pi_M)$.