MATH 401- Stochastic Processes

Spring 2165

Mid-term

Instructions

(a) Show all the work to get full credit. Define all variables clearly and justify your solutions.

(b) This test has eight (8) questions.

(c) **Solutions are due no later than 2 p.m. on Tuesday, March 14.**

(d) You may drop off an electronic copy in the dropbox in MyCourses, send me an email with your solutions attached, or slide a paper copy under my office door. Make sure that your solutions are clear and readable.

(e) You may only refer to your text and notes.

(f) Work independently. You are not to discuss the questions, solutions or methods with any one in class or outside.

(g) Any appearance of collusion or external assistance will result in a score of zero.

1. We have two coins. Coin 1 comes up heads with probabilty 0.6 and coin 2 comes up heads with probability 0.5. A coin is continually flipped until it comes up tails, at which time that coin is put aside and we start flipping the other one and the process is repeated. If we start the process with coin 1 find, **using an appropriate Markov chain** the probability that coin 2 is used on the fifth flip? [Note: Answers using any other method will not be accepted.] (12)

2. Consider the Markov chain on state space $\{0, 1\}$ whose transition probability matrix is

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} \end{array}$$

   Let X be the number of steps required to return to state 0. That is

$$X = \min\Big\{n : X_n = 0 | X_0 = 0\Big\}.$$

   So, the random variable X takes values $1, 2, \ldots$. Find the probability mass function of X. That is, find $P(X = k), k \geq 1$. (12)

3. Let $\{X_n\}$ be a Markov chain with state space $\{0, 1, 2, 3\}$ and transition probability matrix

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0.4 & 0.6 & 0 \\ 0.8 & 0 & 0.2 & 0 \\ 0.2 & 0.3 & 0 & 0.5 \end{bmatrix}$$

(a) Compute $P(X_5 = 2, X_6 = 0, X_7 = 2, X_8 = 2 | X_4 = 1)$ (8)

(b) Suppose $f(\cdot)$ is a function such that $f(0) = 2, f(1) = 4, f(2) = 7, f(3) = 3$. Compute $E\Big[f(X_5)f(X_6)|X_4 = 3\Big]$ (8)

4. The number of winter storms in a season is a Poisson random variable whose parameter is uniformly distributed over (0,5). Find the probability that there are at least two storms in a season. (12)

5. The number of machines that break down in a day on a factory floor is a Poisson random variable with a mean of 10 machines. If the amount of time(min) to repair a machine is exponentially distributed with a mean of 15 minutes, find the mean and variance of the amount of time spent repairing the machines on a given day. (12)

6. A coin that comes up heads with probabiliy p is continually flipped until the pattern T, T, H appears so that you stop flipping when the most recent flip lands heads and the two immediately preceding flips land tails. Let X denote the number of flips made. **Use an appropriate Markov chain** to find $E(X)$. [Note: Answers using any other method will not be accepted.] (12)

7. Two players take turns shooting at a target, with each shot by player i hitting the target with probability $p_i, i = 1, 2$. Shooting ends when two consecutive shots hit the target. Let μ_i denote the mean number of times the target is hit when player i shoots first, $i = 1, 2$. Find μ_1 and μ_2. (12)

8. Suppose you start a game with \$5. If at the end of the nth game you have \$i, at the (n+1)st game you will lose either \$1, \$2, ... \$ i with equal probability. Once you have lost all your money, the game is over. **Use an appropriate Markov chain** to determine the expected duration of the game? (12)