Stochastic Processes

Homework 7

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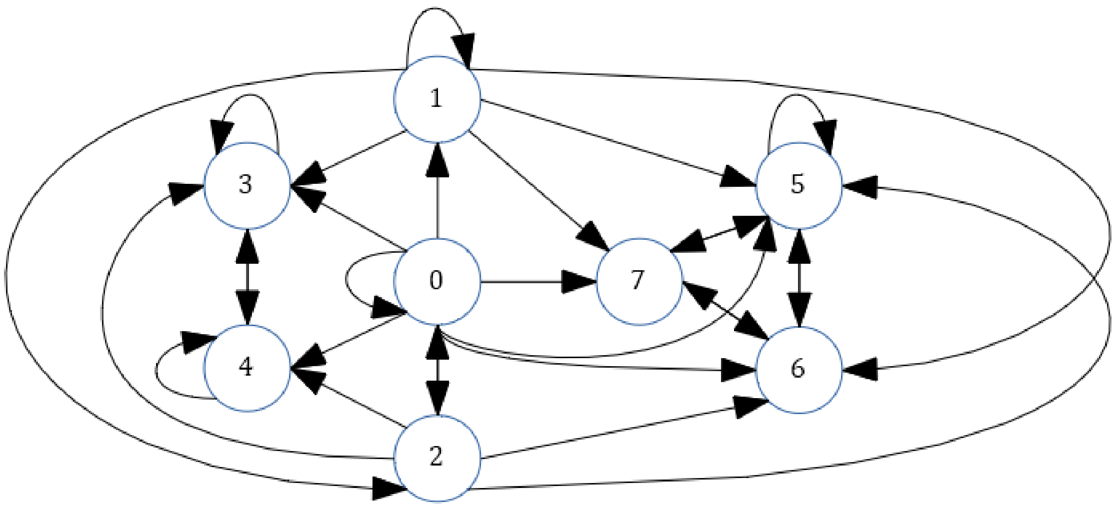
Problem 1

- X_n is our Markov chain with state space $\mathbb{S} = \{0, 1, 2, 3, 4, 5, 6, 7\}$
- The state transition diagram of the given transition probability matrix P is shown below:



- We will apply the following rule to determine the communicating classes:
  - Two states i and j belong to the same communicating class if and only if $i \leftrightarrow j$
- We will apply the following rule to determine the transient and recurrent states:
  - For any state i, denote $f_i = \Pr(\text{"ever reenter i"} \mid X_0 = i)$, where a state i is recurrent if $f_i = 1$, and is transient if $f_i < 1$
- We will apply the following rule to determine if a recurrent state is positive recurrent:
  - If there is only a finite number of states, then there are no null recurrent states, and not all states can be transient

- We can see that there are three communicating classes:
  - $A = \{0, 1, 2\}$ which is transient because each of those states are reachable from another, but there is no guarantee that a state $i \in A$ will return to back to state i given that the process started in state i
  - $B = \{3, 4\}$ which is positive recurrent because each of those states are reachable from another, it is guaranteed that a state $i \in B$ will eventually return to back to state i given that the process started in state i, and there are only a finite number of states in this chain
  - $C = \{5, 6, 7\}$ which is positive recurrent because each of those states are reachable from another, it is guaranteed that a state $i \in C$ will eventually return to back to state i given that the process started in state i, and there are only a finite number of states in this chain

- Let's write out the transition probability sub-matrices P_B and P_C corresponding to classes B and C respectively:

$$P_B = \begin{array}{c c} & \begin{array}{cc} 3 & 4 \end{array} \\ \begin{array}{c} 3 \\ 4 \end{array} & \begin{bmatrix} 0.3 & 0.7 \\ 0.7 & 0.3 \end{bmatrix} \end{array} \qquad P_C = \begin{array}{c c} & \begin{array}{ccc} 5 & 6 & 7 \end{array} \\ \begin{array}{c} 5 \\ 6 \\ 7 \end{array} & \begin{bmatrix} 0.3 & 0.4 & 0.3 \\ 0.1 & 0.0 & 0.9 \\ 0.8 & 0.2 & 0.0 \end{bmatrix} \end{array}$$

- We can compute the stationary distributions of the states in classes B and C by solving the following system of equations:
  - For states in class B:

$$\pi_j = \sum_{k \in B} \pi_k P_{kj} \quad \forall\, j \in B \text{ ; where } \pi_j \text{ is the initial probability of state } j \in B$$
$$1 = \sum_{k \in B} \pi_k$$

  - For states in class C:

$$\pi_j = \sum_{k \in C} \pi_k P_{kj} \quad \forall\, j \in C \text{ ; where } \pi_j \text{ is the initial probability of state } j \in C$$
$$1 = \sum_{k \in C} \pi_k$$

- Applying the above equations, we have the following system of equations:
  - For states in B:

$$\pi_3 = 0.3\pi_3 + 0.7\pi_4$$
$$\pi_4 = 0.7\pi_3 + 0.3\pi_4$$
$$1 = \pi_3 + \pi_4$$

Solving the above system will yield: $(\pi_3, \pi_4) = (\frac{1}{2}, \frac{1}{2})$

  - For states in C:

$$\pi_5 = 0.3\pi_5 + 0.1\pi_6 + 0.8\pi_7$$
$$\pi_6 = 0.4\pi_5 + 0.0\pi_6 + 0.2\pi_7$$
$$\pi_7 = 0.3\pi_5 + 0.9\pi_6 + 0.0\pi_7$$
$$1 = \pi_5 + \pi_6 + \pi_7$$

Solving the above system will yield: $(\pi_5, \pi_6, \pi_7) = (\frac{41}{97}, \frac{23}{97}, \frac{33}{97})$

- We can compute the hitting probabilities from the transient states in A to the positive recurrent states in $\{B, C\}$ with first step analysis.
- Let $u_i^B \;\forall\, i \in A$ be the probability of ultimate absorption in class B from state i
- Let $u_i^C \;\forall\, i \in A$ be the probability of ultimate absorption in class C from state i
  - Where $u_i^B + u_i^C = 1 \;\forall\, i \in A$ because the Markov process will eventually end up in a recurrent class given it started in a transient state
- Using first step analysis we have the following system of equations:

$$u_0^B = 0.1u_0^B + 0.1u_1^B + 0.2u_2^B + 0.2(1) + 0.1(1) + 0.1(0) + 0.1(0) + 0.1(0)$$
$$u_1^B = 0.0u_0^B + 0.1u_1^B + 0.1u_2^B + 0.1(1) + 0.0(1) + 0.3(0) + 0.2(0) + 0.2(0)$$
$$u_2^B = 0.6u_0^B + 0.0u_1^B + 0.0u_2^B + 0.1(1) + 0.1(1) + 0.1(0) + 0.1(0) + 0.0(0)$$

- Solving the above system will yield: $(u_0^B, u_1^B, u_2^B) = (\frac{53}{116}, \frac{19}{116}, \frac{55}{116})$
- We know that $u_i^B + u_i^C = 1 \;\forall\, i \in A$, therefore: $(u_0^C, u_1^C, u_2^C) = (\frac{63}{116}, \frac{97}{116}, \frac{61}{116})$

- Using the initial probabilities and hitting probabilities we can now determine the limiting behavior of the Markov chain with the following limiting matrix of P:

$$P^{\infty} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & u_0^B\pi_3 & u_0^B\pi_4 & u_0^C\pi_5 & u_0^C\pi_6 & u_0^C\pi_7 \\ 0 & 0 & 0 & u_1^B\pi_3 & u_1^B\pi_4 & u_1^C\pi_5 & u_1^C\pi_6 & u_1^C\pi_7 \\ 0 & 0 & 0 & u_2^B\pi_3 & u_2^B\pi_4 & u_2^C\pi_5 & u_2^C\pi_6 & u_2^C\pi_7 \\ 0 & 0 & 0 & \pi_3 & \pi_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi_3 & \pi_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi_5 & \pi_6 & \pi_7 \\ 0 & 0 & 0 & 0 & 0 & \pi_5 & \pi_6 & \pi_7 \\ 0 & 0 & 0 & 0 & 0 & \pi_5 & \pi_6 & \pi_7 \end{bmatrix} \end{array}$$

$$P^{\infty} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 53/232 & 53/232 & 2583/11252 & 1449/11252 & 2079/11252 \\ 0 & 0 & 0 & 19/232 & 19/232 & 41/116 & 23/116 & 33/116 \\ 0 & 0 & 0 & 55/232 & 55/232 & 2501/11252 & 1403/11252 & 2013/11252 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 41/97 & 23/97 & 33/97 \\ 0 & 0 & 0 & 0 & 0 & 41/97 & 23/97 & 33/97 \\ 0 & 0 & 0 & 0 & 0 & 41/97 & 23/97 & 33/97 \end{bmatrix} \end{array}$$

Problem 2

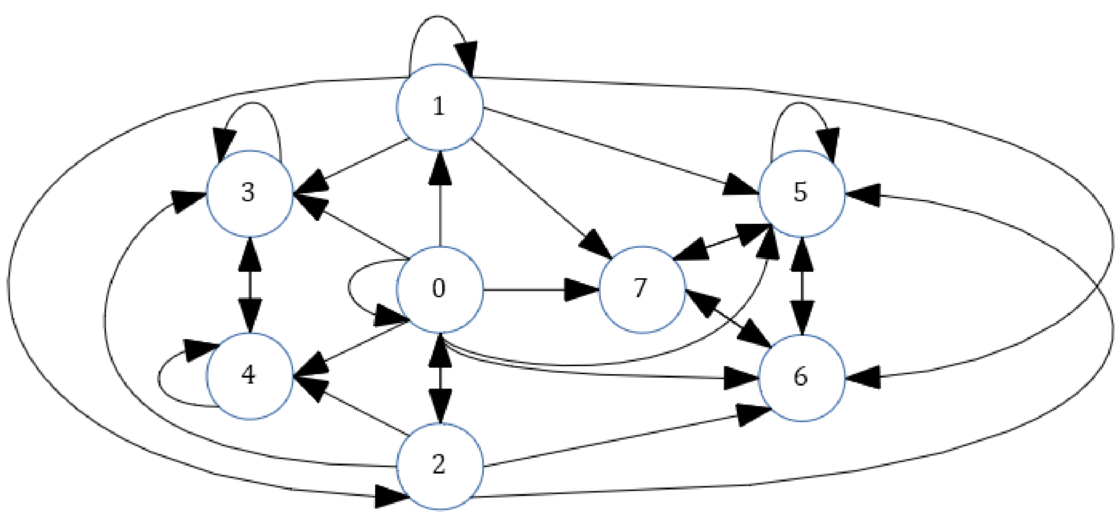
- Let us define a matrix R to denote the expected number of returns to a state j given that the initial state was i

$$R_{ij} = \begin{cases} 0\ ;\ \text{when i is recurrent and j is transient} \\ 0\ ;\ \text{when i is recurrent and j is in a different irreducible set than i} \\ 0\ ;\ \text{when i is transient and j is recurrent and } i \nrightarrow j \\ \infty\ ;\ \text{when i, j are in the same irreducible set} \\ \infty\ ;\ \text{when i is transient and j is recurrent and } i \rightarrow j \\ (I - Q)^{-1}_{ij}\ ;\ \text{when i, j are transient} \end{cases}$$

- Where Q is the following matrix, corresponding to the transient states of P from Problem 1 (class A)

$$Q = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0.1 & 0.1 & 0.2 \\ 0.0 & 0.1 & 0.1 \\ 0.6 & 0.0 & 0.0 \end{bmatrix} \end{array}$$

- Recall the state transition diagram from Problem 1:



- In Problem 1 we defined state 1 as a transient state and state 6 as a positive recurrent state
- The expected number of visits to state 1 given the initial state was 6, would be the entry R_{61}
- By the definition of R_{ij} we can see that the expected number of visits from state 6 to state 1 is 0 because state 6 is recurrent and state 1 is transient

- In the state transition diagram above we can see that state 6 is reachable from state 1 via states 2, 5, or 7
- By the definition of R_{ij} we can see that the expected number of visits from state 1 to state 6 is ∞ because state 1 is transient, state 6 is recurrent, and $1 \rightarrow 6$

- In Problem 1 we defined states 0 and 2 as transient states
- By the definition of R_{ij} we can see that the expected number of visits from state 0 to state 2 is the entry $(I-Q)^{-1}_{02}$ because states 0 and 2 are transient
- The value of entry $(I-Q)^{-1}_{02}$ is computed below:

$$(I - Q)^{-1} = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 75/58 & 25/174 & 95/348 \\ 5/58 & 65/58 & 15/116 \\ 45/58 & 5/58 & 135/116 \end{bmatrix} \end{array}$$

$$(I-Q)^{-1}_{02} = 95/348 \cong 0.273 \text{ visits}$$

- In Problem 1 we defined states 2 and 1 as transient states
- By the definition of R_{ij} we can see that the expected number of visits from state 2 to state 1 is the entry $(I-Q)^{-1}_{21}$ because states 2 and 1 are transient
- The value of entry $(I-Q)^{-1}_{21} = 5/58 \cong 0.086$ visits

Problem 3

- Let us define a matrix F to denote the first passage probability, the probability of eventually reaching a state j given that the initial state was i

$$F_{ij} = \begin{cases} ((I-Q)^{-1}D)_i\,; \text{ when i is transient and j is recurrent} \\ 1 - \dfrac{1}{R_{jj}}\,; \text{when i, j are transient and } i = j \\ \dfrac{R_{ij}}{R_{jj}}\,; \text{when i, j are transient and } i \neq j\,; \text{ where } \dfrac{0}{0} = 0 \end{cases}$$

- Where D is the following matrix, corresponding to the transient states transitioning into the recurrent states of P from Problem 1

$$D = \begin{array}{c c} & \begin{array}{ccccc} 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0.2 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.0 & 0.3 & 0.2 & 0.2 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.0 \end{bmatrix} \end{array}$$

- In Problem 1 we defined states 0, 1, and 2 as transient states
- The probability of ever visiting states 0, 1, and 2 from states 0, 1, and 2 would correspond to a sub-matrix F_1, where F_1 is a square matrix corresponding to all transient states
- We build this sub-matrix F_1 by using the matrix $(I-Q)^{-1}$ and the last two conditions that define F_{ij}

$$F_1 = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 17/75 & 5/39 & 19/81 \\ 1/15 & 7/65 & 1/9 \\ 3/5 & 1/13 & 19/135 \end{bmatrix} \end{array}$$

- In Problem 1 we defined states 5 and 6 as positive recurrent states, and being in the same class
- The probability of ever visiting state 6 from 5 is 1 because they are both recurrent and in the same class, they share the same irreducible set

- In Problem 1 we defined state 2 as transient and state 4 as positive recurrent
- By the definition of F_{ij} we can see that the probability of visiting state 4 from state 2 is the entry $((I-Q)^{-1}D)_{24}$ because state 2 is transient and state 4 is recurrent
- The value of entry $((I-Q)^{-1}D)_{24}$ is computed below:

$$(I-Q)^{-1}D = \begin{array}{c c} & \begin{array}{ccccc} 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 209/696 & 109/696 & 139/696 & 43/232 & 55/348 \\ 33/232 & 5/232 & 83/232 & 57/232 & 27/116 \\ 65/232 & 45/232 & 51/232 & 49/232 & 11/116 \end{bmatrix} \end{array}$$

$$((I-Q)^{-1}D)_{24} = 45/232 \cong 0.194$$

Problem 4

- The conditional distribution of X_t given that $X_{t+s} = n$ is shown below:

$$\Pr(X_t = k \mid X_{t+s} = n)$$

- Using conditional probability

$$= \frac{\Pr(X_t = k, X_{t+s} = n)}{\Pr(X_{t+s} = n)}$$

$$= \frac{\Pr(X_t = k, X_{t+s} - X_t = n-k)}{\Pr(X_{t+s} = n)}$$

$$= \frac{\Pr(X_t = k, X_s = n-k)}{\Pr(X_{t+s} = n)}$$

- Since X_t and X_s are independent

$$= \frac{\Pr(X_t = k)\Pr(X_s = n-k)}{\Pr(X_{t+s} = n)}$$

- Substituting the PDF of the Poisson distribution

$$= \frac{\frac{(\mu t)^k e^{-\mu t}}{k!} \frac{(\mu s)^{n-k} e^{-\mu s}}{(n-k)!}}{\frac{(\mu(t+s))^n e^{-\mu(t+s)}}{n!}}$$

- Simplifying the above fractions

$$= \binom{n}{k}\left(\frac{t}{t+s}\right)^k\left(\frac{s}{t+s}\right)^{n-k}$$

- Therefore, X_t conditioned on $X_{t+s} = n$ means X_t is a binomial random variable: binom(n, t / (t + s))

Problem 5

- The first conditional moment is computed below:

$$E[X_T|T = t] = E[X_t] = \lambda t = 4t$$

- The conditional variance is computed below:

$$Var[X_T|T = t] = Var[X_t] = \lambda t = 4t$$

- The second conditional moment is computed below:

$$E[X_T^2|T = t] = Var[X_T|T = t] + (E[X_T|T = t])^2 = \lambda t + (\lambda t)^2 = 4t + 16t^2 = 4t(1 + 4t)$$

- The unconditional mean is computed below:

$$E[X_T] = \int_0^1 4t * 1\, dt = 2(1)^2 - 0 = 2$$

- The unconditional second moment is computed below:

$$E[X_T^2] = \int_0^1 (4t + 16t^2) * 1\, dt = 2(1)^2 + \frac{16}{3}(1)^3 = \frac{22}{3}$$

- The unconditional variance is computed below:

$$Var[X_T] = E[X_T^2] - E[X_T]^2 = \frac{22}{3} - 2^2 = \frac{10}{3}$$