Stochastic Processes

Homework 5

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03/29/2017

Problem 1

- X_n is our Markov chain with state space $\mathbb{S} = \{(S,S), (S,C), (C,S), (C,C)\}$
- X_0 corresponds to day 0 and 1 where it is given that $X_0 = (S,S)$
- X_1 corresponds to day 2 and 3, and X_2 corresponds to day 4 and 5
- We are interested in day 5 being sunny which means day 4 can be sunny or cloudy, therefore by using the law of total probability, this state of day 5 being sunny can be computed as:

$$\Pr(X_2 = (S,S) \mid X_0 = (S,S)) + \Pr(X_2 = (C,S) \mid X_0 = (S,S))$$

- Using n-step transition probability for a Markov chain, we can rewrite the above probability as:

$$P^{(2)}_{(S,S),(S,S)} + P^{(2)}_{(S,S),(C,S)} \text{ ; where P is the given transition probability matrix}$$

- $P^{(2)}$ is computed by $P * P$:

$$P^{(2)} = \begin{array}{c} \\ (S,S) \\ (S,C) \\ (C,S) \\ (C,C) \end{array} \begin{array}{c} \begin{array}{cccc} (S,S) & (S,C) & (C,S) & (C,C) \end{array} \\ \begin{bmatrix} 0.49 & 0.21 & 0.12 & 0.18 \\ 0.20 & 0.20 & 0.12 & 0.48 \\ 0.35 & 0.15 & 0.20 & 0.30 \\ 0.10 & 0.10 & 0.16 & 0.64 \end{bmatrix} \end{array}$$

- Finally, the probability that it is sunny on day 5 is:

$$P^{(2)}_{(S,S),(S,S)} + P^{(2)}_{(S,S),(C,S)} = 0.49 + 0.12 = 0.61$$

- The long run fraction of sunny days is computed by the limiting distribution for sunny days, where sunny days correspond to the states (S,S) and (C,S)
- So, we are looking for $\pi_{(S,S)} + \pi_{(C,S)}$, where π_j is the initial probability of state $j \in \mathbb{S}$
- We can compute the entire limiting distribution of X_n by solving the following system of equations:

$$\pi_j = \textstyle\sum_{k \in \mathbb{S}} \pi_k P_{kj} \quad \forall\, j \in \mathbb{S}$$
$$1 = \textstyle\sum_{k \in \mathbb{S}} \pi_k$$

- Applying the above equations, we have the following system of equations:

$$\begin{aligned}
\pi_{(S,S)} &= 0.7\pi_{(S,S)} + 0.0\pi_{(S,C)} + 0.5\pi_{(C,S)} + 0.0\pi_{(C,C)} \\
\pi_{(S,C)} &= 0.3\pi_{(S,S)} + 0.0\pi_{(S,C)} + 0.5\pi_{(C,S)} + 0.0\pi_{(C,C)} \\
\pi_{(C,S)} &= 0.0\pi_{(S,S)} + 0.4\pi_{(S,C)} + 0.0\pi_{(C,S)} + 0.2\pi_{(C,C)} \\
\pi_{(C,C)} &= 0.0\pi_{(S,S)} + 0.6\pi_{(S,C)} + 0.0\pi_{(C,S)} + 0.8\pi_{(C,C)} \\
1 &= \pi_{(S,S)} + \pi_{(S,C)} + \pi_{(C,S)} + \pi_{(C,C)}
\end{aligned}$$

- Solving the above system will yield: $(\pi_{(S,S)}, \pi_{(S,C)}, \pi_{(C,S)}, \pi_{(C,C)}) = (0.25, 0.15, 0.15, 0.45)$
- Therefore, the long run fraction of sunny days is $\pi_{(S,S)} + \pi_{(C,S)} = 0.25 + 0.15 = 0.40$

Problem 2

- X_n is our Markov chain with state space $\mathbb{S} = \{\text{exam 1, exam 2, exam 3}\} = \{1, 2, 3\}$ where X_n represents the type of the n^{th} exam.
  - X_n is a Markov chain because the probability of X_n taking a value in $\mathbb{S}$ is dependent only on what the previous exam was, X_{n-1}, and how the class performed on X_{n-1}
- We can build the transition probability matrix P by using conditional probability:

$$P_{ij} = \Pr(X_n = j \mid X_{n-1} = i) \ \forall \ i,j \in \mathbb{S}$$

- By using the law of total probability and conditioning on the class' performance, we can rewrite P_{ij}:

$P_{ij} = \Pr(X_n = j \mid X_{n-1} = i)$
$= \Pr(X_n = j \mid \text{class did well on } X_{n-1} = i) + \Pr(X_n = j \mid \text{class did bad on } X_{n-1} = i)$
$= \Pr(X_n = j \mid \text{class did well on } X_{n-1} = i) * \Pr(\text{class did well on } X_{n-1} = i) +$
$\Pr(X_n = j \mid \text{class did bad on } X_{n-1} = i) * \Pr(\text{class did bad on } X_{n-1} = i)$

$P_{11} = \Pr(X_n = 1 \mid \text{class did well on } X_{n-1} = 1) * \Pr(\text{class did well on } X_{n-1} = 1) +$
$\Pr(X_n = 1 \mid \text{class did bad on } X_{n-1} = 1) * \Pr(\text{class did bad on } X_{n-1} = 1)$
$= (1/3)*(0.3) + (1)*(1 - 0.3) = 0.8$

$P_{12} = \Pr(X_n = 2 \mid \text{class did well on } X_{n-1} = 1) * \Pr(\text{class did well on } X_{n-1} = 1) +$
$\Pr(X_n = 2 \mid \text{class did bad on } X_{n-1} = 1) * \Pr(\text{class did bad on } X_{n-1} = 1)$
$= (1/3)*(0.3) + (0)*(1 - 0.3) = 0.1$

$P_{13} = \Pr(X_n = 3 \mid \text{class did well on } X_{n-1} = 1) * \Pr(\text{class did well on } X_{n-1} = 1) +$
$\Pr(X_n = 3 \mid \text{class did bad on } X_{n-1} = 1) * \Pr(\text{class did bad on } X_{n-1} = 1)$
$= (1/3)*(0.3) + (0)*(1 - 0.3) = 0.1$

$P_{21} = \Pr(X_n = 1 \mid \text{class did well on } X_{n-1} = 2) * \Pr(\text{class did well on } X_{n-1} = 2) +$
$\Pr(X_n = 1 \mid \text{class did bad on } X_{n-1} = 2) * \Pr(\text{class did bad on } X_{n-1} = 2)$
$= (1/3)*(0.6) + (1)*(1 - 0.6) = 0.6$

$P_{22} = \Pr(X_n = 2 \mid \text{class did well on } X_{n-1} = 2) * \Pr(\text{class did well on } X_{n-1} = 2) +$
$\Pr(X_n = 2 \mid \text{class did bad on } X_{n-1} = 2) * \Pr(\text{class did bad on } X_{n-1} = 2)$
$= (1/3)*(0.6) + (0)*(1 - 0.6) = 0.2$

$P_{23} = \Pr(X_n = 3 \mid \text{class did well on } X_{n-1} = 2) * \Pr(\text{class did well on } X_{n-1} = 2) +$
$\Pr(X_n = 3 \mid \text{class did bad on } X_{n-1} = 2) * \Pr(\text{class did bad on } X_{n-1} = 2)$
$= (1/3)*(0.6) + (0)*(1 - 0.6) = 0.2$

$P_{31} = \Pr(X_n = 1 \mid \text{class did well on } X_{n-1} = 3) * \Pr(\text{class did well on } X_{n-1} = 3) +$
$\Pr(X_n = 1 \mid \text{class did bad on } X_{n-1} = 3) * \Pr(\text{class did bad on } X_{n-1} = 3)$
$= (1/3)*(0.9) + (1)*(1 - 0.9) = 0.4$

$P_{32} = Pr(X_n = 2 \mid \text{class did well on } X_{n-1} = 3) * Pr(\text{class did well on } X_{n-1} = 3) +$
$\quad Pr(X_n = 2 \mid \text{class did bad on } X_{n-1} = 3) * Pr(\text{class did bad on } X_{n-1} = 3)$
$= (1/3)*(0.9) + (0)*(1 - 0.9) = 0.3$

$P_{33} = Pr(X_n = 3 \mid \text{class did well on } X_{n-1} = 3) * Pr(\text{class did well on } X_{n-1} = 3) +$
$\quad Pr(X_n = 3 \mid \text{class did bad on } X_{n-1} = 3) * Pr(\text{class did bad on } X_{n-1} = 3)$
$= (1/3)*(0.9) + (0)*(1 - 0.9) = 0.3$

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.6 & 0.2 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{bmatrix} \end{array}$$

- The proportion of exams being type i in the long run, where $i \in \mathbb{S}$, is represented by the entire limiting distribution of X_n
- We can compute the entire limiting distribution of X_n by solving the following system of equations:

$$\pi_j = \sum_{k \in \mathbb{S}} \pi_k P_{kj} \quad \forall\, j \in \mathbb{S} \text{ ; where } \pi_j \text{ is the initial probability of state } j \in \mathbb{S}$$
$$1 = \sum_{k \in \mathbb{S}} \pi_k$$

- Applying the above equations, we have the following system of equations:

$$\begin{aligned} \pi_1 &= 0.8\pi_1 + 0.6\pi_2 + 0.4\pi_3 \\ \pi_2 &= 0.1\pi_1 + 0.2\pi_2 + 0.3\pi_3 \\ \pi_3 &= 0.1\pi_1 + 0.2\pi_2 + 0.3\pi_3 \\ 1 &= \pi_1 + \pi_2 + \pi_3 \end{aligned}$$

- Solving the above system will yield: $(\pi_1, \pi_2, \pi_3) = (5/7, 1/7, 1/7)$
- Therefore, the proportion of exams being type 1, 2, and 3 in the long run is 5/7, 1/7, and 1/7 respectively

Problem 3

- X_n is our Markov chain with state space $\mathbb{S} = \{\text{sunny, cloudy, rainy}\} = \{0, 1, 2\}$ where X_n represents the weather on day n.
  - X_n is a Markov chain because the probability of X_n taking a value in $\mathbb{S}$ is dependent only on the value that was taken by X_{n-1}, the weather of the previous day.
- We can build the transition probability matrix P by using conditional probability:

$$P_{ij} = \Pr(X_n = j \mid X_{n-1} = i) \ \forall\, i,j \in \mathbb{S}$$

$P_{00} = \Pr(X_n = 0 \mid X_{n-1} = 0) = 0$
$P_{01} = \Pr(X_n = 1 \mid X_{n-1} = 0) = 1/2$
$P_{02} = \Pr(X_n = 2 \mid X_{n-1} = 0) = 1/2$
$P_{10} = \Pr(X_n = 0 \mid X_{n-1} = 1) = 1/4$
$P_{11} = \Pr(X_n = 1 \mid X_{n-1} = 1) = 1/2$
$P_{12} = \Pr(X_n = 2 \mid X_{n-1} = 1) = 1/4$
$P_{20} = \Pr(X_n = 0 \mid X_{n-1} = 2) = 1/4$
$P_{21} = \Pr(X_n = 1 \mid X_{n-1} = 2) = 1/4$
$P_{22} = \Pr(X_n = 2 \mid X_{n-1} = 2) = 1/2$

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} \end{array}$$

- The proportion of days being sunny in the long run, and the proportion of days being cloudy in the long run can both be found in entire limiting distribution of X_n
- We are interested in the values of π_0 and π_1 to find the proportions of interest for sunny days and cloudy days respectively, where π_j is the initial probability of state $j \in \mathbb{S}$
- We can compute the entire limiting distribution of X_n by solving the following system of equations:

$$\pi_j = \textstyle\sum_{k \in \mathbb{S}} \pi_k P_{kj} \quad \forall\, j \in \mathbb{S}$$
$$1 = \textstyle\sum_{k \in \mathbb{S}} \pi_k$$

- Applying the above equations, we have the following system of equations:

$$\pi_0 = 0\pi_0 + \frac{1}{4}\pi_1 + \frac{1}{4}\pi_2$$
$$\pi_1 = \frac{1}{2}\pi_0 + \frac{1}{2}\pi_1 + \frac{1}{4}\pi_2$$
$$\pi_2 = \frac{1}{2}\pi_0 + \frac{1}{4}\pi_1 + \frac{1}{2}\pi_2$$
$$1 = \pi_1 + \pi_2 + \pi_3$$

- Solving the above system will yield: $(\pi_0, \pi_1, \pi_2) = (1/5, 2/5, 2/5)$
- Therefore, the proportion of days being sunny and cloudy in the long run is 1/5 and 2/5 respectively

Problem 4

- Using conditional probability, we can rewrite the given limit as:

$$\lim_{n\to\infty} \Pr(X_{n-1} = 2 \mid X_n = 1) = \lim_{n\to\infty} \frac{\Pr(X_n=1 \mid X_{n-1}=2) * \Pr(X_{n-1}=2)}{\Pr(X_n=1)}$$

- Since we are taking the limit of $n \to \infty$ this means we want to know the long run value of those probabilities, where:

$$\lim_{n\to\infty} \Pr(X_n = j \mid X_{n-1} = i) = P_{ij} \text{ ; where P is the given transition probability matrix}$$
$$\lim_{n\to\infty} \Pr(X_n = j) = \pi_j \text{ ; where } \pi_j \text{ is the initial probability of state } j \in \mathbb{S} = \{0, 1, 2\}$$

- The value of $\pi_j \ \forall\, j \in \mathbb{S}$ is the entire limiting distribution of the Markov Chain, X_n
- We can compute the entire limiting distribution of X_n by solving the following system of equations:

$$\pi_j = \sum_{k \in \mathbb{S}} \pi_k P_{kj} \quad \forall\, j \in \mathbb{S}$$
$$1 = \sum_{k \in \mathbb{S}} \pi_k$$

- Applying the above equations, we have the following system of equations:

$$\pi_0 = 0.4\pi_0 + 0.6\pi_1 + 0.4\pi_2$$
$$\pi_1 = 0.4\pi_0 + 0.2\pi_1 + 0.2\pi_2$$
$$\pi_2 = 0.2\pi_0 + 0.2\pi_1 + 0.4\pi_2$$
$$1 = \pi_1 + \pi_2 + \pi_3$$

- Solving the above system will yield: $(\pi_0, \pi_1, \pi_2) = (11/24, 7/24, 1/4)$
- Using the given matrix P and the computed values of (π_0, π_1, π_2) we can compute the given limit:

$$\lim_{n\to\infty} \Pr(X_{n-1} = 2 \mid X_n = 1) = \lim_{n\to\infty} \frac{\Pr(X_n=1 \mid X_{n-1}=2) * \Pr(X_{n-1}=2)}{\Pr(X_n=1)} = \frac{P_{21} * \pi_2}{\pi_1} = \frac{(0.2)*(1/4)}{(7/24)} = 6/35$$

Problem 5

- When the transition probability matrix has row sums and column sum both equal to one, then this is known as a doubly stochastic transition probability matrix. This means that the given P satisfies the following conditions:

$$P_{ij} \geq 0 \ \forall \ i,j \in \mathbb{S} = \{1, 2, \dots, M\}$$
$$\sum_{k \in \mathbb{S}} P_{ik} = \sum_{k \in \mathbb{S}} P_{kj} = 1 \ \forall \ i,j \in \mathbb{S} = \{1, 2, \dots, M\}$$

- When a transition probability matrix P satisfies the above conditions (ie. P is doubly stochastic), then when computing the entire limiting distribution of X_n there is only one solution to the following system of equations:

$$\pi_j = \sum_{k \in \mathbb{S}} \pi_k P_{kj} \quad \forall \ j \in \mathbb{S}$$
$$1 = \sum_{k \in \mathbb{S}} \pi_k$$

- The solution to the above system when P is doubly stochastic is the following:

$$\pi_j = \sum_{k \in \mathbb{S}} \frac{1}{M} P_{kj} = \frac{1}{M} \ \forall \ j \in \mathbb{S} \text{ ; where M is the size of the state space}$$

- Therefore, the limiting probability vector $\underline{\pi} = (\pi_1, \pi_2, \dots, \pi_M) = (1/M, 1/M, \dots, 1/M)$