Stochastic Processes

Midterm Exam

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Problem 1

- Let X_n be a Markov chain with state space $\begin{cases} 0, \text{if Coin 1 lands on heads at step n} \\ 1, \text{if Coin 1 lands on tails at step n} \end{cases}$
- X_n has the following transition probability matrix:

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 0.6 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

- Given that we start with Coin 1, the probability that Coin 2 is used on the fifth flip corresponds to the following probability: $P(X_0 = 0, X_1 = 0, X_2 = 0, X_3 = 1)$; where $P(X_0 = 0) = 0.6$
- Using conditional probability:

$$\begin{aligned} &P(X_0 = 0, X_1 = 0, X_2 = 0, X_3 = 1) \\ &= P(X_3 = 1 \mid X_0 = 0, X_1 = 0, X_2 = 0) * P(X_0 = 0, X_1 = 0, X_2 = 0) \\ &= P(X_3 = 1 \mid X_0 = 0, X_1 = 0, X_2 = 0) * P(X_2 = 0 \mid X_0 = 0, X_1 = 0) * P(X_0 = 0, X_1 = 0) \\ &= P(X_3 = 1 \mid X_0 = 0, X_1 = 0, X_2 = 0) * P(X_2 = 0 \mid X_0 = 0, X_1 = 0) * P(X_1 = 0 \mid X_0 = 0) * P(X_0 = 0) \end{aligned}$$

- Using Markovian property:

$$\begin{aligned} &P(X_0 = 0, X_1 = 0, X_2 = 0, X_3 = 1) \\ &= P(X_3 = 1 \mid X_0 = 0, X_1 = 0, X_2 = 0) * P(X_2 = 0 \mid X_0 = 0, X_1 = 0) * P(X_1 = 0 \mid X_0 = 0) * P(X_0 = 0) \\ &= P(X_3 = 1 \mid X_2 = 0) * P(X_2 = 0 \mid X_1 = 0) * P(X_1 = 0 \mid X_0 = 0) * P(X_0 = 0) \\ &= P_{01} * P_{00} * P_{00} * (0.6) \\ &= (0.4) * (0.6) * (0.6) * (0.6) \\ &= 0.0864 \end{aligned}$$

Problem 2

- Using n-step transition probability for a Markov chain, the PMF of X is the probability of the Markov process going from state 0 to state 0 in k transitions:

$$P(X = k) = P_{00}^{(k)}$$

- Lets compute $P_{00}^{(k)}$ (See Appendix for computation):

$$P^k = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} \frac{\alpha*(1-\alpha-\beta)^k}{\alpha+\beta} + \frac{-\alpha}{\alpha+\beta} + 1 & \frac{-\alpha*(1-\alpha-\beta)^k}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta} \\ \frac{-\alpha}{\alpha+\beta} + \frac{-\beta*(1-\alpha-\beta)^k}{\alpha+\beta} + 1 & \frac{\alpha}{\alpha+\beta} + \frac{\beta*(1-\alpha-\beta)^k}{\alpha+\beta} \end{bmatrix} \end{array}$$

$$P_{00}^{(k)} = \frac{\alpha*(1-\alpha-\beta)^k}{\alpha+\beta} + \frac{-\alpha}{\alpha+\beta} + 1 = P(X = k)$$

Problem 3

Part A

- Using conditional probability:

$P(X_5 = 2, X_6 = 0, X_7 = 2, X_8 = 2 \mid X_4 = 1)$
$= P(X_8 = 2 \mid X_4 = 1, X_5 = 2, X_6 = 0, X_7 = 2) * P(X_7 = 2 \mid X_4 = 1, X_5 = 2, X_6 = 0) * P(X_6 = 0 \mid X_4 = 1, X_5 = 2) * P(X_5 = 2 \mid X_4 = 1)$

- Using Markovian property:

$P(X_5 = 2, X_6 = 0, X_7 = 2, X_8 = 2 \mid X_4 = 1)$
$= P(X_8 = 2 \mid X_7 = 2) * P(X_7 = 2 \mid X_6 = 0) * P(X_6 = 0 \mid X_5 = 2) * P(X_5 = 2 \mid X_4 = 1)$
$= P_{22} * P_{02} * P_{20} * P_{12}$
$= (0.2) * (1) * (0.8) * (0.6)$
$= 0.096$

Part B

$E[f(X_5) * f(X_6) \mid X_4 = 3]$
$= \sum_{j=0}^{3} \sum_{k=0}^{3} f(k) * P(X_5 = k \mid X_4 = 3) * f(j) * P(X_6 = j \mid X_5 = k)$
$= \sum_{j=0}^{3} \sum_{k=0}^{3} f(k) * P_{3k} * f(j) * P_{kj}$

$= f(0) * P_{30} * f(0) * P_{00} + f(1) * P_{31} * f(0) * P_{10} + f(2) * P_{32} * f(0) * P_{20} + f(3) * P_{33} * f(0) * P_{30} +$
$f(0) * P_{30} * f(1) * P_{01} + f(1) * P_{31} * f(1) * P_{11} + f(2) * P_{32} * f(1) * P_{21} + f(3) * P_{33} * f(1) * P_{31} +$
$f(0) * P_{30} * f(2) * P_{02} + f(1) * P_{31} * f(2) * P_{12} + f(2) * P_{32} * f(2) * P_{22} + f(3) * P_{33} * f(2) * P_{32} +$
$f(0) * P_{30} * f(3) * P_{03} + f(1) * P_{31} * f(3) * P_{13} + f(2) * P_{32} * f(3) * P_{23} + f(3) * P_{33} * f(3) * P_{33}$

$= 2 * 0.2 * 2 * 0 + 4 * 0.3 * 2 * 0 + 7 * 0 * 2 * 0.8 + 3 * 0.5 * 2 * 0.2 +$
$2 * 0.2 * 4 * 0 + 4 * 0.3 * 4 * 0.4 + 7 * 0 * 4 * 0 + 3 * 0.5 * 4 * 0.3 +$
$2 * 0.2 * 7 * 1 + 4 * 0.3 * 7 * 0.6 + 7 * 0 * 7 * 0.2 + 3 * 0.5 * 7 * 0 +$
$2 * 0.2 * 3 * 0 + 4 * 0.3 * 3 * 0 + 7 * 0 * 3 * 0 + 3 * 0.5 * 3 * 0.5$

$= 14.41$

Problem 4

- Let X be the number of winter storms in a season
- $X \sim Pois(\lambda)$; where $\lambda \sim Unif(0, 5)$ (discrete)
- Using conditional probability:

$P(X \geq 2)$
$= 1 - P(X < 2)$
$= 1 - (P(X < 2 \mid \lambda = 0) * P(\lambda = 0) + P(X < 2 \mid \lambda = 1) * P(\lambda = 1) + P(X < 2 \mid \lambda = 2) * P(\lambda = 2) +$
$P(X < 2 \mid \lambda = 3) * P(\lambda = 3) + P(X < 2 \mid \lambda = 4) * P(\lambda = 4) + P(X < 2 \mid \lambda = 5) * P(\lambda = 5))$
$= 1 - \sum_{j=0}^{5} \frac{1}{6} P(X < 2 \mid \lambda = j)$

- Using the CDF of the Poisson distribution:

$P(X \geq 2)$
$= 1 - \sum_{j=0}^{5} \frac{1}{6} P(X < 2 \mid \lambda = j)$
$= 1 - \frac{1}{6} \sum_{\lambda=0}^{5} \sum_{k=0}^{1} \frac{\lambda^{k} * e^{-\lambda}}{k!}$
$= 1 - \frac{1}{6} (1 + 0.73575888 + 0.40600585 + 0.19914827 + 0.09157819 + 0.04042768)$
$= 0.5878469$

Problem 5

- Let N be the number of machines that breakdown on a factory floor in a day
- $N \sim Pois(\lambda_N = 10)$
- Let x_i be the amount of time (minutes) that was spent repairing machine i
- $x_i \sim Exp(\lambda_x = \frac{1}{15})$
- The expected value and variance of N and x_i are the following because of their distributions:

$E(N) = \lambda_N = 10$
$V(N) = \lambda_N = 10$
$E(x_i) = \frac{1}{\lambda_x} = 15$
$V(x_i) = \frac{1}{\lambda_x^2} = 225$

- Let X be the amount of time spent repairing the machines in a day
- We can see that X is a random sum of a random variable: $X = \sum_{i=1}^{N} x_i$
- The expected value and variance of X is the following because X is a random sum of a random variable:

$E(X) = E(N) * E(x_i) = 10 * 15 = 150$
$V(X) = E(x_i)^2 * V(N) + E(N) * V(x_i) = 15^2 * 10 + 10 * 225 = 4500$

Problem 6

- Let x_n be a Markov chain with state space $\begin{cases} 0, \text{if the coin lands on heads at step n} \\ 1, \text{if the coin lands on tails at step n} \end{cases}$
- x_n has the following transition probability matrix:

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} p & q \\ p & q \end{bmatrix} \end{array} \text{; where } q = 1 - p$$

- The probability of observing the pattern: tails-tails-heads, corresponds to the following probability: $P(x_k = 1, x_{k+1} = 1, x_{k+2} = 0)$ for $k \geq 0$; where $P(x_k = 1) = q$
- Using conditional probability:

$P(x_k = 1, x_{k+1} = 1, x_{k+2} = 0)$
$= P(x_{k+2} = 0 \mid x_k = 1, x_{k+1} = 1) * P(x_k = 1, x_{k+1} = 1)$
$= P(x_{k+2} = 0 \mid x_k = 1, x_{k+1} = 1) * P(x_{k+1} = 1 \mid x_k = 1) * P(x_k = 1)$

- Using Markovian property:

$P(x_k = 1, x_{k+1} = 1, x_{k+2} = 0)$
$= P(x_{k+2} = 0 \mid x_k = 1, x_{k+1} = 1) * P(x_{k+1} = 1 \mid x_k = 1) * P(x_k = 1)$
$= P(x_{k+2} = 0 \mid x_{k+1} = 1) * P(x_{k+1} = 1 \mid x_k = 1) * P(x_k = 1)$
$= P_{10} * P_{11} * q$
$= p * q * q$
$= p * q^2$

- X corresponds to the total number of flips made until the pattern: tails-tails-heads, has occurred
- We can see that this definition follows the definition of a Geometric random variable, where a success is defined as tails-tails-heads occurring with a probability of $p * q^2$
- $X \sim \text{Geom}(\text{probability} = p * q^2)$
- The expected value of X is the following because of its distribution:

$E(X) = \frac{1}{p * q^2}$

Problem 7

- Let N be the total number of times the target is hit
- Let $x_k = \begin{cases} 0, \text{if shot k was a miss} \\ 1, \text{if shot k was a hit} \end{cases}$
- $\mu_1 = E(N \mid \text{player 1 shoots first})$
- $\mu_2 = E(N \mid \text{player 2 shoots first})$
- Let $q_i = 1 - p_i$ for $i = 1, 2$
- Using conditional probability:

$$\begin{aligned}\mu_1 &= E(N \mid \text{player 1 shoots first}) \\ &= E(N \mid x_1 = 1) * p_1 + E(N \mid x_1 = 0) * q_1 \\ &= (E(N \mid x_1 = 1, x_2 = 1) * p_1 * p_2 + E(N \mid x_1 = 1, x_2 = 0) * p_1 * q_2) + E(N \mid x_1 = 0) * q_1\end{aligned}$$

- We can rewrite each E(N | …) as the following:

$E(N \mid x_1 = 1, x_2 = 1) = 2$; first two shots were hits, so the game ends
$E(N \mid x_1 = 1, x_2 = 0) = 2 + \mu_1$; first two shots weren't hits, the game "restarts" with player 1 shooting first
$E(N \mid x_1 = 0) = 1 + \mu_2$; first shot wasn't a hit, the game "restarts" with player 2 shooting first

- Substitution:

$$\begin{aligned}\mu_1 &= E(N \mid x_1 = 1, x_2 = 1) * p_1 * p_2 + E(N \mid x_1 = 1, x_2 = 0) * p_1 * q_2 + E(N \mid x_1 = 0) * q_1 \\ &= 2 * p_1 * p_2 + (2 + \mu_1) * p_1 * q_2 + (1 + \mu_2) * q_1\end{aligned}$$

- Using the same process, we can write out the following as well:

$$\begin{aligned}\mu_2 &= E(N \mid x_1 = 1, x_2 = 1) * p_2 * p_1 + E(N \mid x_1 = 1, x_2 = 0) * p_2 * q_1 + E(N \mid x_1 = 0) * q_2 \\ &= 2 * p_2 * p_1 + (2 + \mu_2) * p_2 * q_1 + (1 + \mu_1) * q_2\end{aligned}$$

- Using the equations for μ_1 and μ_2, we can write out the following system of equations:

$$2 * p_1 * p_2 + (2 + \mu_1) * p_1 * q_2 + (1 + \mu_2) * q_1 - \mu_1 = 0$$
$$2 * p_2 * p_1 + (2 + \mu_2) * p_2 * q_1 + (1 + \mu_1) * q_2 - \mu_2 = 0$$

- Solving for μ_1 and μ_2 in the above system, we get the following (See Appendix for computation):

$$\mu_1 = \frac{-q1*(2*p1*p2 + 2*p2*q1 + q2) + (p2*q1 - 1)*(2*p1*p2 + 2*p1*q2 + q1)}{q1*q2 - (p1*q2 - 1)*(p2*q1 - 1)}$$

$$\mu_2 = \frac{-q2*(2*p1*p2 + 2*p1*q2 + q1) + (p1*q2 - 1)*(2*p1*p2 + 2*p2*q1 + q2)}{q1*q2 - (p1*q2 - 1)*(p2*q1 - 1)}$$

Problem 8

- Let X_n be a Markov chain with state space {0, 1, …, i} representing the amount of money owned after game n
- The amount of money owned after game n+1 can only reduce by \$1, \$2, …, or \$i
- X_n has the following transition probability matrix:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ i-1 \\ i \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & i-1 & i \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 1/2 & 1/2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1/(i-1) & 1/(i-1) & 1/(i-1) & \cdots & 0 & 0 \\ 1/i & 1/i & 1/i & \cdots & 1/i & 0 \end{bmatrix} \end{array}$$

- Let T be the time until absorption into state 0; $T = \min\{k \geq n : X_k = 0\}$
- Let $v_i = E(T \mid X_n = i)$
- v_i is the expected time until absorption, given there is \$i owned after game n
- Using the special case of the General Absorbing Markov Chain where $g(i) = 1$ for every transient state i, v_i can be solved for in the following system of equations:

$v_1 = 1$

$v_2 = 1 + \sum_{j=1}^{1} \frac{1}{2} * v_j$

$v_3 = 1 + \sum_{j=1}^{2} \frac{1}{3} * v_j$

$\vdots$

$v_i = 1 + \sum_{j=1}^{i-1} \frac{1}{i} * v_j$

- $v_1 = 1$ because only one more game can be played after arriving in state 1. This due to the condition of losing at least \$1 after game n
- Solving the above linear system of equations for v_i will return the expected time to reach state 0 from state i
- Substituting v_1 into v_2 and continuing this manner of substitution would yield:

$v_2 = 1 + \frac{1}{2} * (1) = 1.5$

$v_3 = 1 + \frac{1}{3} * (1.5) + \frac{1}{3} * (1) = 1.8\overline{3}$

$v_4 = 1 + \frac{1}{4} * (1.8\overline{3}) + \frac{1}{4} * (1.5) + \frac{1}{4} * (1) = 2.08\overline{3}$

$v_5 = 1 + \frac{1}{5} * (2.08\overline{3}) + \frac{1}{5} * (1.8\overline{3}) + \frac{1}{5} * (1.5) + \frac{1}{5} * (1) = 2.28\overline{3}$

$\vdots$

$v_i = 1 + \sum_{j=1}^{i-1} \frac{1}{i} * v_j$

- Given that $X_n = i$, there have already been n games played
- So the expected value of the total game duration:

$$n + v_i$$

Appendix

Problem 2: Solving for P^k using Python

```
# the following is a script that raises a matrix to a power

# import sympy and some functions
import sympy as sp
from sympy import MatrixSymbol, Matrix

# define alpha (a)
# define beta (b)
# define step constant (k)
# define transition probability matrix (P)

a = sp.Symbol('a')
b = sp.Symbol('b')
k = sp.Symbol('k')
P = Matrix(MatrixSymbol('p', 2, 2))

# update P

P[0,0] = 1 - a
P[0,1] = a
P[1,0] = b
P[1,1] = 1 - b

# compute the k step transition probability matrix

P**k
```

Problem 7: Solving for μ_1 and μ_2 using Python

```
# the following is a script to solve a system of equations

# import sympy
import sympy as sp

# define the mean number of times the target is hit when player 1 shoots first (m1)
# define the mean number of times the target is hit when player 2 shoots first (m2)
# define the probability of player 1 hitting the target (p1)
# define the probability of player 2 hitting the target (p2)
# define the probability of player 1 missing the target (q1)
# define the probability of player 2 missing the target (q2)

m1 = sp.Symbol('m1')
m2 = sp.Symbol('m2')
p1 = sp.Symbol('p1')
p2 = sp.Symbol('p2')
q1 = sp.Symbol('q1')
q2 = sp.Symbol('q2')

# define the two equations for m1 and m2

eqns = [sp.Eq((2 * p1 * p2) + ((2 + m1) * p1 * q2) + ((1 + m2) * q1) - m1),
        sp.Eq((2 * p2 * p1) + ((2 + m2) * p2 * q1) + ((1 + m1) * q2) - m2)]

# define the two variables that we are solving for (ie. m1 and m2)

varlist = [m1, m2]

# solve the system of two equations

sol = sp.solve(eqns, varlist)
sol
```