# Stochastic Processes

## Homework 6

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**Problem 1**

- $X_n$ is our Markov chain with state space $\mathbb{S} = \{0, 1\}$
- Each time $X_n$ is in a state, it sends one message
- The probability of a good message sent is $\{p_o, p_1\}$ when $X_n$ is in state $\{0, 1\}$ respectively
- The probability of a bad message sent is $\{1 - p_o, 1 - p_1\}$ when $X_n$ is in state $\{0, 1\}$ respectively

- Using the law of total probability and Markovian property, we can write out the probability that a good message is sent on Tuesday given the process is in state 0 on Monday

$$\sum_{i=0}^{1} \Pr(X_n = i, \text{"good message in state i"} \mid X_{n-1} = 0)$$
$$\sum_{i=0}^{1} (\Pr(X_n = i \mid X_{n-1} = 0) * \Pr(\text{"good message in state i"}))$$
$$\sum_{i=0}^{1} P_{0i} * \Pr(\text{"good message in state i"})$$
$$P_{00}p_0 + P_{01}p_1$$
$$0.4p_0 + 0.6p_1$$

- The proportion of good messages in the long run is computed with the limiting distribution of $X_n$
- We can compute the limiting distribution of $X_n$ by solving the following system of equations:

$$\pi_j = \sum_{k \in \mathbb{S}} \pi_k P_{kj} \quad \forall\, j \in \mathbb{S} \text{ ; where } \pi_j \text{ is the initial probability of state } j \in \mathbb{S}$$
$$1 = \sum_{k \in \mathbb{S}} \pi_k$$

- Applying the above equations, we have the following system of equations:

$$\pi_0 = 0.4\pi_0 + 0.2\pi_1$$
$$\pi_1 = 0.6\pi_0 + 0.8\pi_1$$
$$1 = \pi_0 + \pi_1$$

- Solving the above system will yield: $(\pi_0, \pi_1) = (1/4, 3/4)$
- The proportion of good messages in the long run is:

$$\pi_0 p_0 + \pi_1 p_1 = \frac{1}{4}p_0 + \frac{3}{4}p_1$$

- If $Y_n$ were a Markov chain, then it would have the state space $\mathbb{S} = \{1, 2\}$ corresponding to "good message" and "bad message" respectively
- Furthermore, it would have to follow Markovian property where:

$$\Pr(Y_{n+1} = j \mid Y_0 = i_0, \dots, Y_{n-1} = i_{n-1}, Y_n = i) = \Pr(Y_{n+1} = j \mid Y_n = i)$$

- But according to the problem definition, a message being good or bad only depends on the current state of $X_n$ and mentions no dependence on the previous state, $X_{n-1}$
- Because the current message has no dependence on $X_{n-1}$ it also has no dependence on the previous message in $X_{n-1}$
- Therefore, by contradiction of the Markovian property, $Y_n$ is not a Markov chain

**Problem 2**

- $X_n$ is our Markov chain with state space $\mathbb{S} = \{0, 1, 2, 3\}$ with the following transition probability matrix:

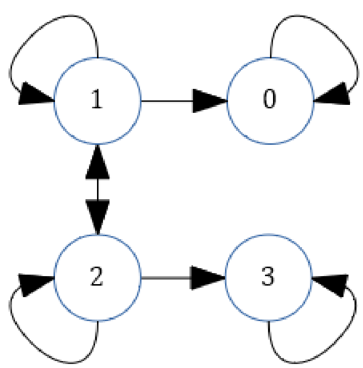
$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1.00 & 0.00 & 0.00 & 0.00 \\ 0.50 & 0.25 & 0.25 & 0.00 \\ 0.00 & 0.50 & 0.25 & 0.25 \\ 0.00 & 0.00 & 0.00 & 1.00 \end{bmatrix} \end{array}$$

- Let A represent all absorbing states where $A = \{0, 3\}$
- Let $A^C$ represent all non-absorbing states where $A^C = \{1, 2\}$
- Let's rewrite P in terms of A and $A^C$

$$P = \begin{array}{c} \\ A^C \\ A \end{array} \begin{array}{c} \begin{array}{cc} A^C & A \end{array} \\ \begin{bmatrix} Q & B \\ 0 & I \end{bmatrix} \end{array}$$

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 0 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 0 & 3 \end{array} \\ \begin{bmatrix} 0.25 & 0.25 & 0.50 & 0.00 \\ 0.50 & 0.25 & 0.00 & 0.25 \\ 0.00 & 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 1.00 \end{bmatrix} \end{array}$$

- Let's draw out the state transition diagram of P to identify the communicating classes, transient states, and recurrent states
- We will apply the following rule to determine the communicating classes:
  - Two states i and j belong to the same communicating class if and only if $i \leftrightarrow j$
- We will apply the following rule to determine the transient and recurrent states:
  - For any state i, denote $f_i = \Pr(\text{"ever reenter i"} \mid X_0 = i)$, where a state i is recurrent if $f_i = 1$, and is transient if $f_i < 1$




- We can see that for P the communicating classes are {1, 2}, {0}, and {3} making it not an irreducible chain
- We can see that for P the recurrent states are {0, 3} and the transient states are {1, 2}

- Let us define a matrix R to denote the expected number of returns to a state j given that the initial state was i

$$R_{ij} = \begin{cases} 0\ ;\ \text{when i is recurrent and j is transient} \\ 0\ ;\ \text{when i is recurrent and j is in a different irreducible set than i} \\ 0\ ;\ \text{when i is transient and j is recurrent and } i \nrightarrow j \\ \infty\ ;\ \text{when i, j are in the same irreducible set} \\ \infty\ ;\ \text{when i is transient and j is recurrent and } i \rightarrow j \\ (I-Q)^{-1}_{ij}\ ;\ \text{when i, j are transient} \end{cases}$$

$$R = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \infty & 0.000 & 0.000 & 0.000 \\ \infty & 1.714 & 0.571 & \infty \\ \infty & 1.143 & 1.714 & \infty \\ 0.000 & 0.000 & 0.000 & \infty \end{bmatrix} \end{array}$$

- Let us define a matrix F to denote the first passage probability, the probability of eventually reaching a state j at least once given that the initial state was i

$$F_{ij} = \begin{cases} ((I-Q)^{-1}B)_i\ ;\ \text{when i is transient and j is recurrent} \\ 1 - \dfrac{1}{R_{jj}}\ ;\ \text{when i, j are transient and } i = j \\ \dfrac{R_{ij}}{R_{jj}}\ ;\ \text{when i, j are transient and } i \neq j\ ;\ \text{where } \dfrac{0}{0} = 0 \end{cases}$$

$$F = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1.000 & 0.000 & 0.000 & 0.000 \\ 0.857 & 0.417 & 0.333 & 0.143 \\ 0.571 & 0.667 & 0.417 & 0.429 \\ 0.000 & 0.000 & 0.000 & 1.000 \end{bmatrix} \end{array}$$

- The probability of the rocket eventually getting on course (ie. state 0) given that the initial state was 2 corresponds to $F_{20} = 0.571$
- The probability of the rocket eventually getting destroyed (ie. state 3) given that the initial state was 2 corresponds to $F_{23} = 0.429$
- The expected number of times a minor correction is made (ie. state 1) given that the initial state was 2 corresponds to $R_{21} = 1.143$
- The expected number of times a major correction is made (ie. state 2) given that the initial state was 2 corresponds to $R_{22} = 1.714$
- When the rocket is launched, 50000 lb. of are used
- When a minor correction is made, 1000 lb. of fuel are used
- When a major correction is made, 5000 lb. of fuel are used
- Therefore, the expected amount of fuel needed for the mission given that the initial state was 2 is:

$$50000 + 1000(1.143) + 5000(1.714) = 59713 \text{ lb.}$$

**Problem 3**

- $X_n$ is our Markov chain with state space $\mathbb{S} = \{0, 1\}$ corresponding to, the car is where the man is and the car is not where the man is, respectively
- Given that it rains, independent of time, during successive mornings and afternoons with probability p, we can write out the transition probability matrix:

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} p & 1-p \\ 1 & 0 \end{bmatrix} \end{array}$$

- The long run fraction of days the man walks in the rain is computed with the limiting distribution of $X_n$
- We can compute the limiting distribution of $X_n$ by solving the following system of equations:

$$\pi_j = \sum_{k \in \mathbb{S}} \pi_k P_{kj} \quad \forall\, j \in \mathbb{S}\text{ ; where } \pi_j \text{ is the initial probability of state } j \in \mathbb{S}$$
$$1 = \sum_{k \in \mathbb{S}} \pi_k$$

- Applying the above equations, we have the following system of equations:

$$\pi_0 = p\pi_0 + \pi_1$$
$$\pi_1 = (1-p)\pi_0 + 0\pi_1$$
$$1 = \pi_0 + \pi_1$$

- Solving the above system will yield: $(\pi_0, \pi_1) = (\frac{1}{2-p}, \frac{1-p}{2-p})$
- The fraction of days that the man walks in the rain is when either of these two scenarios occur:
  - he doesn't have his car and it rains
  - he has his car and it doesn't rain but then it rains on his way back
- We can compute, in the long run, the fraction of days that the man walks in the rain using the limiting distribution, the transition probability matrix, and the two previous scenarios:

Pr((“no car” ⋂ “rain at time i”) ⋃ (“has car” ⋂ “no rain at time i” ⋂ “rain at time i+1”))
= Pr(“no car”) * Pr( “rain at time i”) + Pr( “has car”) * Pr( “no rain at time i”) * Pr( “rain at time i+1”)

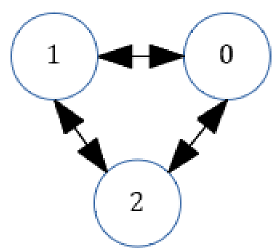
$$= \pi_1 * p + \pi_0 * (1-p) * p$$
$$= \frac{1-p}{2-p} * p + \frac{1}{2-p} * (1-p) * p$$
$$= 2\,\frac{(1-p)p}{2-p}$$

**Problem 4**

- Let $P_1$ have the following state space $\mathbb{S}_1 = \{0, 1, 2\}$ where the rows and columns of $P_1$ are labeled in the same order as the elements of $\mathbb{S}_1$
- Let $P_2$ have the following state space $\mathbb{S}_2 = \{0, 1, 2, 3\}$ where the rows and columns of $P_2$ are labeled in the same order as the elements of $\mathbb{S}_2$
- Let $P_3$ have the following state space $\mathbb{S}_3 = \{0, 1, 2, 3 ,4\}$ where the rows and columns of $P_3$ are labeled in the same order as the elements of $\mathbb{S}_3$
- Let $P_4$ have the following state space $\mathbb{S}_4 = \{0, 1, 2, 3, 4\}$ where the rows and columns of $P_4$ are labeled in the same order as the elements of $\mathbb{S}_4$

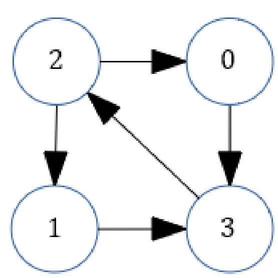
- We will apply the following rule to determine the communicating classes:
    - Two states i and j belong to the same communicating class if and only if $i \leftrightarrow j$
- We will apply the following rule to determine the transient and recurrent states:
    - For any state i, denote $f_i = \Pr(\text{"ever reenter i"} \mid X_0 = i)$, where a state i is recurrent if $f_i = 1$, and is transient if $f_i < 1$

- Let's draw out the state transition diagram of $P_1$ to identify the communicating classes, transient states, and recurrent states



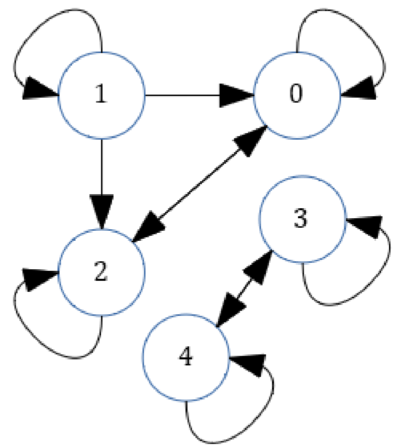

- We can see that for $P_1$ the one communicating class is $\{0, 1, 2\}$ making it an irreducible chain
- We can see that for $P_1$ the recurrent states are $\{0, 1, 2\}$ and there are no transient states

- Let's draw out the state transition diagram of $P_2$ to identify the communicating classes, transient states, and recurrent states



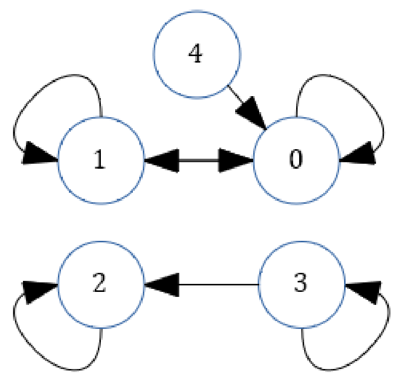

- We can see that for $P_2$ the one communicating class is $\{0, 1, 2, 3\}$ making it an irreducible chain
- We can see that for $P_2$ the recurrent states are $\{0, 1, 2, 3\}$ and there are no transient states

- Let's draw out the state transition diagram of $P_3$ to identify the communicating classes, transient states, and recurrent states




- We can see that for $P_3$ the communicating classes are {0, 2}, {3, 4}, and {1} making it not an irreducible chain
- We can see that for $P_3$ the recurrent states are {0, 2, 3, 4} and the transient states are {1}

- Let's draw out the state transition diagram of $P_4$ to identify the communicating classes, transient states, and recurrent states



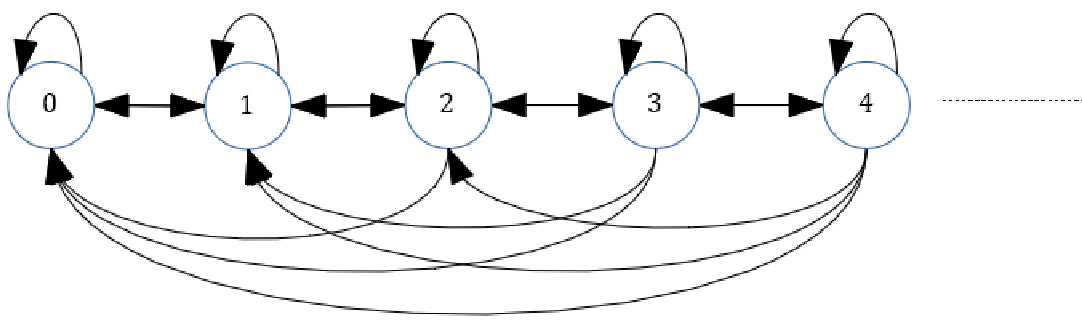

- We can see that for $P_4$ the communicating classes are {0, 1}, {2}, {3}, and {4}
- We can see that for $P_4$ the recurrent states are {0, 1, 2} and the transient states are {3, 4}

**Problem 5**

- The transition probability matrix of this Markov chain is built with the following pattern:

$$P = \begin{array}{c c} & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & \cdots \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} & \begin{bmatrix} 1/2 & 1/2 & 0/2 & 0/2 & 0/2 & 0/2 & \cdots \\ 1/3 & 1/3 & 1/3 & 0/3 & 0/3 & 0/3 & \cdots \\ 1/4 & 1/4 & 1/4 & 1/4 & 0/4 & 0/4 & \cdots \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0/5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \end{array}$$

- The state transition diagram of P is built with the following pattern:




- We can see that there is one communicating class of all states, making it an irreducible chain
- We can see that all states are recurrent
- We can compute the stationary distribution of this Markov chain $X_n$ by solving the following system of equations:

$$\pi_j = \textstyle\sum_{k \in \mathbb{S}} \pi_k P_{kj} \quad \forall\, j \in \mathbb{S}\text{ ; where } \pi_j \text{ is the initial probability of state } j \in \mathbb{S}$$
$$1 = \textstyle\sum_{k \in \mathbb{S}} \pi_k$$

- Applying the above equations, we can build a system of equations with the following pattern:

$$1 = \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4$$
$$\pi_0 = \pi_1 = \frac{1}{2}\pi_0 + \frac{1}{3}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{5}\pi_3$$
$$\pi_2 = \frac{1}{3}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{5}\pi_3$$
$$\pi_3 = \frac{1}{4}\pi_2 + \frac{1}{5}\pi_3$$
$$\pi_4 = \frac{1}{5}\pi_3$$

- The above system has no solution, this is because the first two columns of P are identical and therefore linearly dependent
- This means that there is no stationary distribution for this Markov chain