

Fast pow, Fibonacci, tortoise and hare

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Powers

Definition:

- ▶ Chain multiplication
- ▶ “ n -th power of b ”
- ▶ b is the base, n is the exponent
- ▶ $b^n = \underbrace{b \times \cdots \times b}_{n \text{ times}}$

Examples:

- ▶ $3^0 = 1$ (by definition)
- ▶ $3^1 = 3$
- ▶ $3^2 = 3 \times 3 = 9$ (square)
- ▶ $3^3 = 3 \times 3 \times 3 = 27$ (cube)

Power computation: linear

Problem: compute the power the n -th power of b , for given b and n .

Solution 1: Simple loop

```
int nthPower(int b, int n)
{
    int power = 1;
    for (int i = 0; i < n; i++)
        power *= b;
    return power;
}
```

Complexity: $O(n)$

Power computation: logarithmic (1)

Can we do it faster? Yes, because associativity!

For example, to compute 3^{10} , we can compute 3^5 then square it:

- ▶ $3^2 = 3 \times 3 = 9$
- ▶ $3^5 = 3^2 \times 3^2 \times 3 = 9 \times 9 \times 3 = 243$
- ▶ $3^{10} = 3^5 \times 3^5 = 243 \times 243 = 59049$

Only 4 multiplications instead of 9.

Power computation: logarithmic (2)

Solution 2: Recursive function

```
int nthPower(int b, int n)
{
    // Initial case
    if (n == 0)
        return 1;

    // Recursive case
    int power = powerOfThree(b, n/2);
    power *= power;
    if (n % 2 == 1)
        power *= b;
    return power;
}
```

We divide n by 2 on every call $\Rightarrow O(\log n)$

Fast pow: usage

When to use it:

- ▶ When linear time is too slow
- ▶ Typically when computing a number of possibilities

Limits:

- ▶ Exponent $\leq 10^{18}$ if using `long long` (or more!)
- ▶ Many powers with the same base \Rightarrow store in an array
- ▶ Be careful with overflows! Often, the statement asks for the result *modulo* some number.

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Matrices

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \\ a_{30} & a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} 0 & 4 & 1 \\ 2 & 0 & 0 \\ 1 & 1 & 2 \\ 8 & 0 & 1 \end{pmatrix}$$

- ▶ 4×3 matrix = table with 4 lines and 3 columns
- ▶ $a_{ij} = a[i][j]$ = element at line i , column j
- ▶ Mathematicians usually start at 1, but that's a bad idea

Matrix product

$$\begin{pmatrix} 0 & 4 & 1 \\ 2 & 0 & 0 \\ 1 & 1 & 2 \\ 8 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 8 & 7 \\ 2 & 0 \\ 3 & 7 \\ 8 & 3 \end{pmatrix}$$

- ▶ Take line on left, column on right
- ▶ Add the products (they must have the same length!)
- ▶ $0 \times 1 + 4 \times 2 + 1 \times 0 = 8$ (top left element)
- ▶ Dimensions: $n \times m \cdot m \times p = n \times p$
- ▶ Associative: $(AB)C = A(BC)$

Matrix power

$$A^2 = \begin{pmatrix} 0 & 4 & 1 \\ 2 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 4 & 1 \\ 2 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 1 & 2 \\ 0 & 8 & 2 \\ 4 & 6 & 5 \end{pmatrix}$$

- ▶ Only works on square matrices
- ▶ Associative, so we can use fast pow!
- ▶ Complexity for A^n : cost of product $\times O(\log n)$
- ▶ With $m \times m$ matrix: $O(m^3 \log n)$

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Definition of Fibonacci

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

$$F_0 = 0, F_1 = 1, F_2 = 1, \dots$$

- ▶ Modelling rabbit reproduction (terribly)
- ▶ At 2 months old, rabbits start having 1 child every month
- ▶ At month 1, we introduce one newborn rabbit
- ▶ F_n = population at month n
- ▶ $F_n = F_{n-1} + F_{n-2}$
- ▶ Example: $F_2 = F_1 + F_0$, $F_3 = F_2 + F_1$, ...

Fibonacci as a matrix product

- ▶ We want to compute Fibonacci as a matrix product
- ▶ We always need to know at least two numbers
- ▶ Which square matrix to choose?

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix}$$

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Fibonacci as a matrix product

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Logarithmic Fibonacci

From the formula, we deduce:

$$\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} F_0 \\ F_1 \end{pmatrix}$$

- ▶ So we can compute the power in $O(\log n)$
- ▶ And then get F_n immediately
- ▶ Because of overflows, we often need the result *modulo* some number

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Cycle finding problem

- ▶ Finite set of states S , for example $\{1, \dots, n\}$
- ▶ Function $f : S \rightarrow S$ of transitions $x \mapsto f(x)$
- ▶ Starting value: $x_0 \in S$

What is the first value repeated in this sequence?

$$x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \dots$$

Cycle finding example

Table of f :

x		1	2	3	4	5	6	7	8	9
<hr/>										
$f(x)$		8	6	1	7	1	4	5	5	2

Sequence for $x_0 = 9$:

$$\underbrace{9, 2, 6, 4, 7}_{\text{tail}}, \underbrace{5, 1, 8}_{\text{cycle}}, \underbrace{5, 1, 8}_{\text{cycle}}, 5, 1, \dots$$

Problem: find the tail size and the cycle size

Cycle finding with map

```
pair<int, int> findCycle(int x)
{
    map<int, int> pos;

    // While x is a new value
    int i;
    for (i = 0; pos.find(x) == pos.end(); i++)
    {
        pos[x] = i; // Remember the position
        x = f(x);   // Move one step
    }
    return make_pair(pos[x], i - pos[x]);
}
```

- ▶ Time: $O(n \log n)$, or $O(n)$ with hashmap
- ▶ Space: $O(n)$, that's a lot...

Tortoise and hare 1: find match

Keep two pointers: tortoise (slow) and hare ($2 \times$ faster)

Iterate until match:

9	2	6	4	7	5	1	8	5	1	8	5	1	8	5
<hr/>														
	T	H												
		T		H										
			T			H								
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The gap is a multiple of the cycle length

Tortoise and hare 2: find tail

Hare jumps to beginning, then they move at the same speed (the hare is tired by the jump)

Iterate until match:

9	2	6	4	7	5	1	8	5	1	8	5	1	8	5
H						T								
	H						T							
		H						T						
			H						T					
				H						T				
					H						T			
						H						T		

The number of steps is the tail length

Tortoise and hare 3: find cycle

Hare stops (*really* tired) and tortoise continues

Iterate until next match:

9	2	6	4	7	5	1	8	5	1	8	5	1	8	5
					H						T			
					H							T		
					H								T	
					H									T

The number of steps is the cycle length

Tortoise and hare: implementation

```
pair<int ,int> findCycle(int x0)
{
    int t = x0, h = x0, tail = 0, cycle = 0;
    // Part 1: find a match
    do { t = f(t); h = f(f(h)); } while (t != h);
    // Part 2: find tail
    h = x0; // Rabbit jump
    while (t != h) { t = f(t); h = f(h); tail++; }
    // Part 3: find cycle
    do { t = f(t); cycle++; } while (t != h);
    return make_pair(tail, cycle);
}
```