

Specialized DFS

cycles, bridges, articulation points

beOI Training
(slides by François Aubry)



OLYMPIADE BELGE D'INFORMATIQUE
BELGISCHE INFORMATICA-OLYMPIADE

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Adding one state

Cycle detection

Bridges and articulation points

```

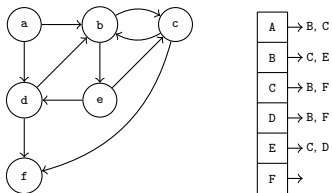
enum state {UNVISITED, OPENED, CLOSED};
vector<int> adj[N];
state st[N];

void dfs(int u) {
    st[v] = OPENED;
    for (int v : adj[u])
        if (st[u] == UNVISITED)
            dfs(v);
    st[v] = CLOSED;
}

// in main()
fill(st, st+n, UNVISITED);
for (int u = 0; u < n; i++)
    if (st[u] == UNVISITED)
        dfs(u);

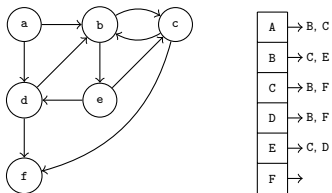
```

An extended version of DFS: remember if node is finished or not



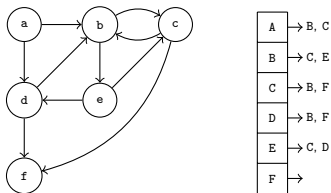
DFS execution from node a



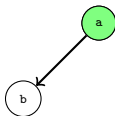


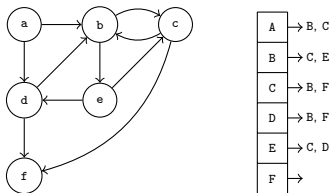
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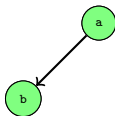


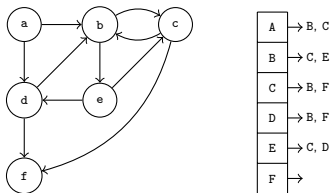
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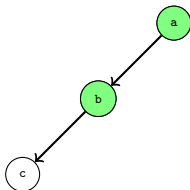


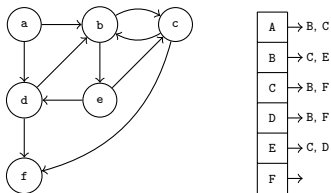
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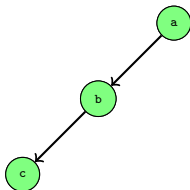


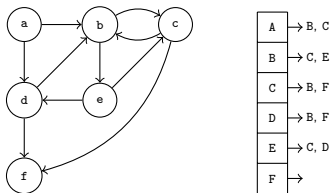
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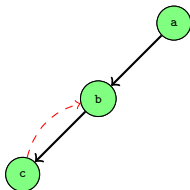


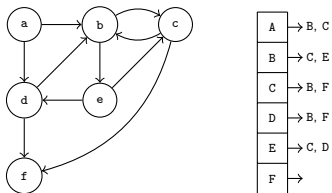
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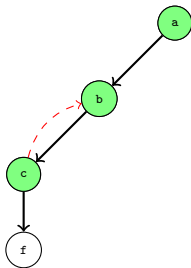


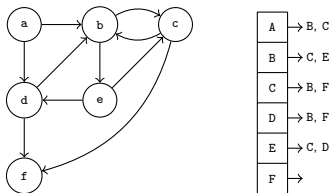
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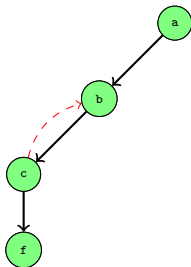


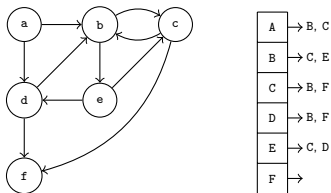
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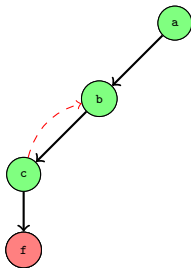


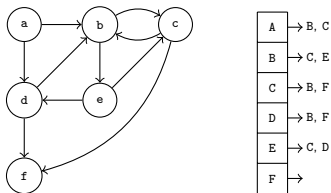
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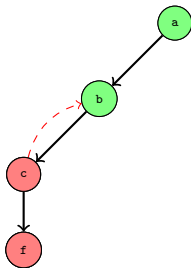


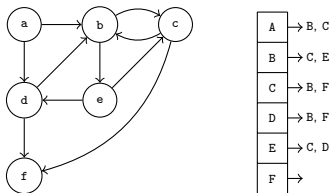
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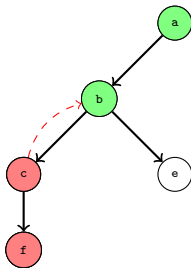


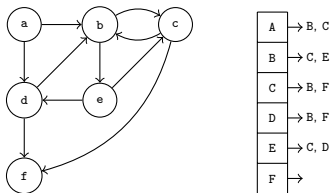
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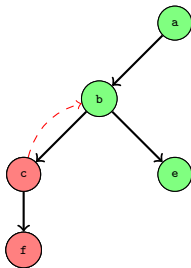


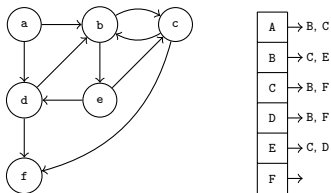
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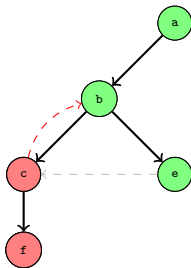


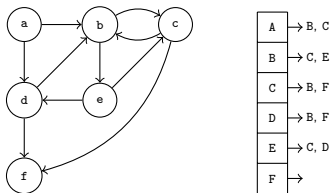
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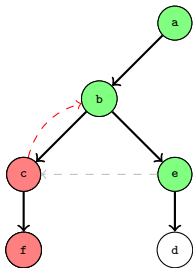


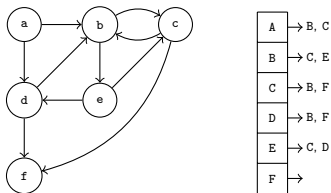
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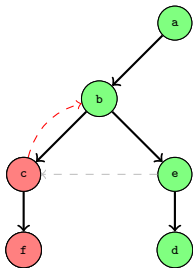


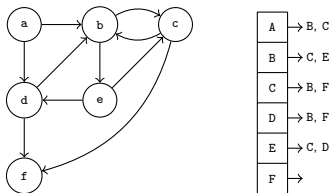
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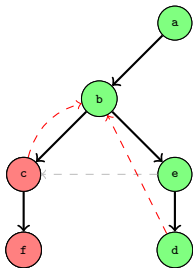


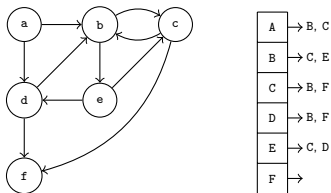
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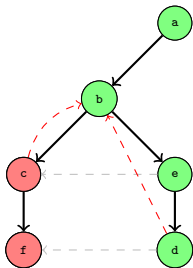


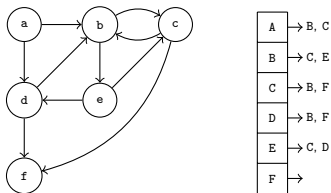
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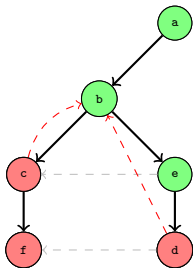


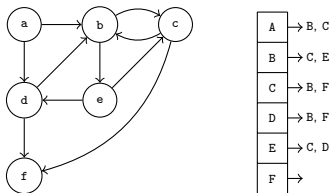
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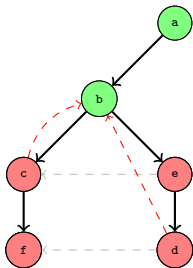


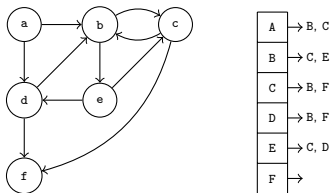
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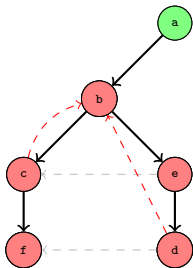


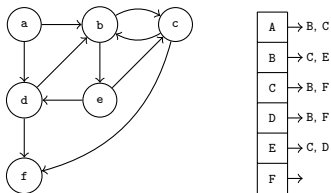
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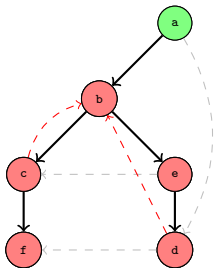


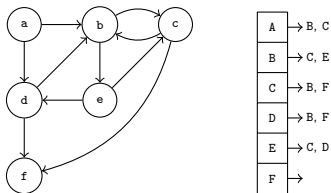
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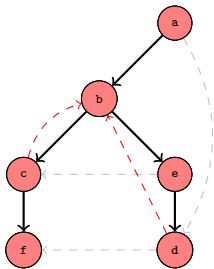


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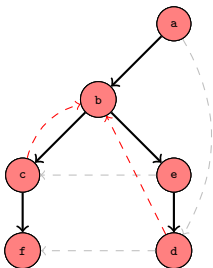
Bridges and articulation points

Let's start with **cycles**.

How can we detect with DFS whether a graph is acyclic or not?

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How can we detect with DFS whether a graph is acyclic or not?

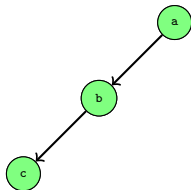


The red edges belong to cycles, they are called **back edges**.

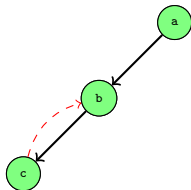
A graph is acyclic if and only if DFS does not yield back edges.

How to distinguish these edges from the others?

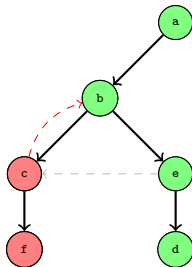
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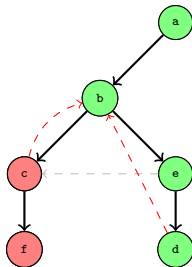
How to distinguish these edges from the others?



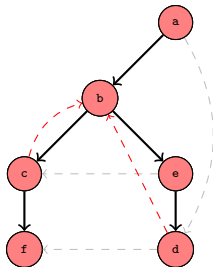
How to distinguish these edges from the others?



How to distinguish these edges from the others?



How to distinguish these edges from the others?



They are edges from a green (OPENED) node to another green (OPENED) node!

To implement this we simply add a check while listing neighbours.

Checks if node u (which is OPENED) points to another OPENED node.

```
bool hasCycle = false;

void dfs(int u) {
    st[v] = OPENED;
    for (int v : adj[u]) {
        if (st[u] == UNVISITED)
            dfs(v);
        else if (st[u] == OPENED)
            hasCycle = true;
    }
    st[v] = CLOSED;
}
```

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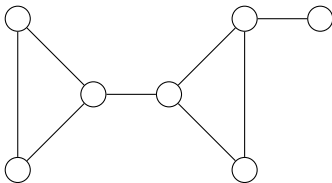
Adding one state

Cycle detection

Bridges and articulation points

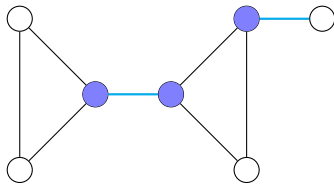
An **articulation point** in an undirected graph is a node such that its removal disconnects the graph.

A **bridge** in an undirected graph is an edge such that its removal disconnects the graph.



An **articulation point** in an undirected graph is a node such that its removal disconnects the graph.

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— bridges

● articulation points

How would you compute this?

How would you compute this?

Naive algorithm (for bridges):

For every edge (x, y) , remove it and check with BFS or DFS whether x and y remain connected.

Complexity: $O(E \cdot (V + E)) = O(E \cdot V + E^2)$ **TLE** in big graphs.

We will solve it in **linear time** with a single DFS.

Observation:

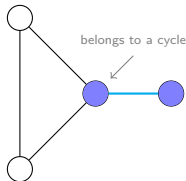
Bridges can never belong to cycles.

Is this true for articulation points?

Observation:

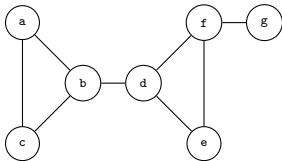
Bridges can never belong to cycles.

Is this true for articulation points? **NO**.



So we essentially need to find which edges belong to cycles.

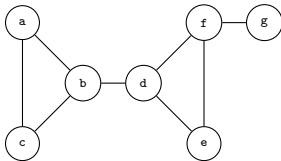
We already say that DFS allows to find cycles.



A	→ B, C
B	→ A, C, D
C	→ A, B
D	→ B, E, F
E	→ D, F
F	→ D, E, G
G	→ F

DFS execution from node a

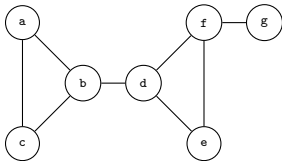




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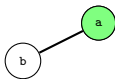
DFS execution from node a

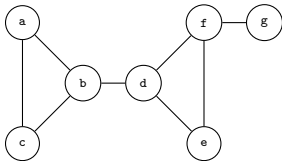




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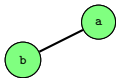
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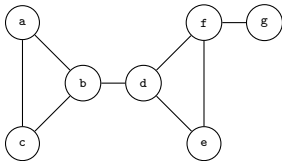




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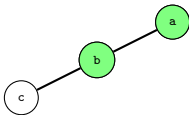
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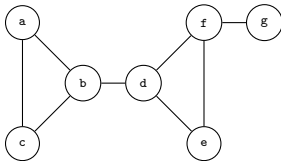




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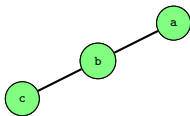
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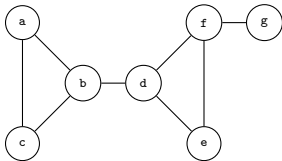




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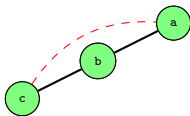
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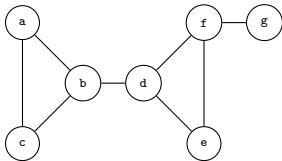




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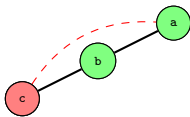
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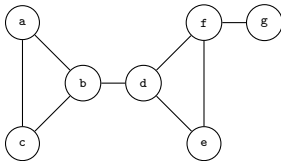




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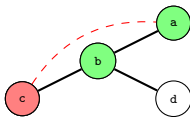
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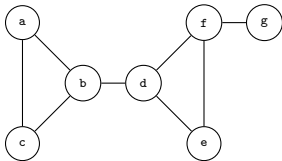




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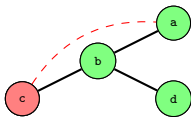
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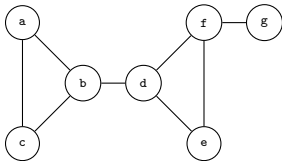




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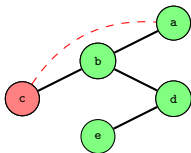
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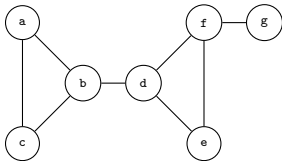




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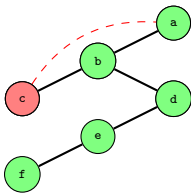
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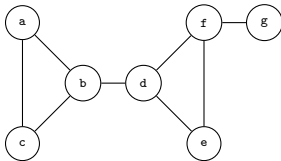




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F	→ D, E, G
G	→ F

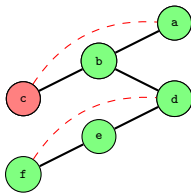
DFS execution from node a

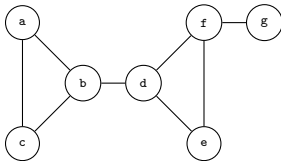




A	→ B, C
B	→ A, C, D
C	→ A, B
D	→ B, E, F
E	→ D, F
F	→ D, E, G
G	→ F

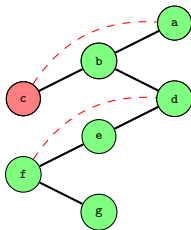
DFS execution from node a

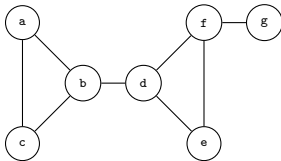




A	→ B, C
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D	→ B, E, F
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F	→ D, E, G
G	→ F

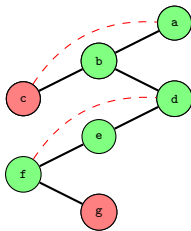
DFS execution from node a

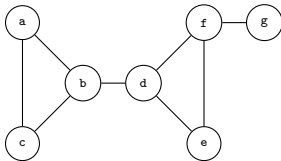




A	→ B, C
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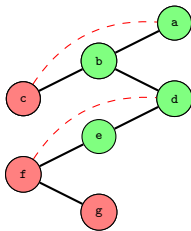
DFS execution from node a

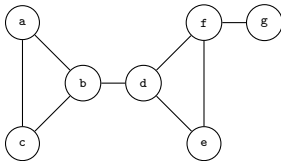




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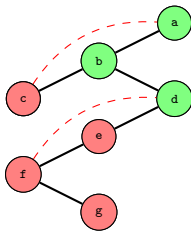
DFS execution from node a

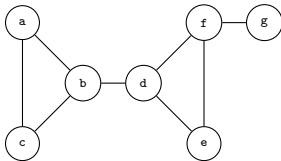




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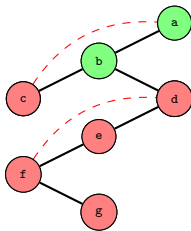
DFS execution from node a

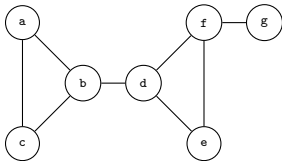




A	→ B, C
B	→ A, C, D
C	→ A, B
D	→ B, E, F
E	→ D, F
F	→ D, E, G
G	→ F

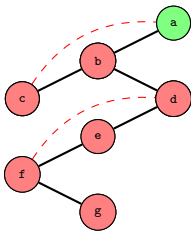
DFS execution from node a

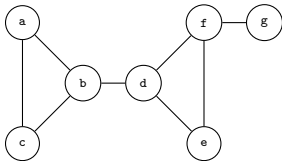




A	→ B, C
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C	→ A, B
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E	→ D, F
F	→ D, E, G
G	→ F

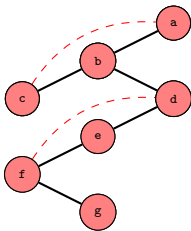
DFS execution from node a

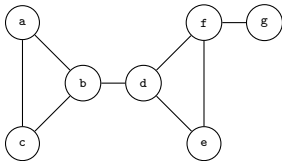




A	→ B, C
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E	→ D, F
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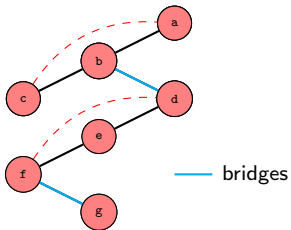
DFS execution from node a





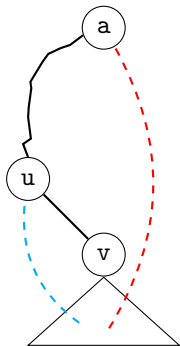
A	→ B, C
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DFS execution from node a



We observe that an edge (u, v) is a bridge if and only if:

No node in the sub-tree of v has a link to u other than (u, v) or a link to one of its ancestors.



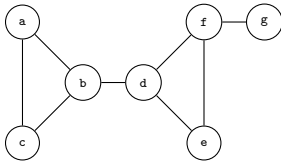
What should we add to our DFS to know this information?

We will **timestamp** the nodes as they are visited: num

Keep track of the node with minimum timestamp that we see. We define $low[v]$ as the **lowest timestamp to which any node in the subtree of v has a direct link**.

(u, v) is a bridge if and only if when we finish v , $low[v] > num[u]$

This is so because this means that in the sub-tree of v we did not see any ancestor of u .



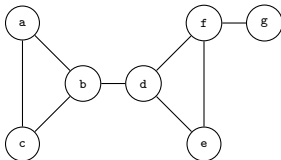
A	→	B, C
B	→	A, C, D
C	→	A, B
D	→	B, E, F
E	→	D, F
F	→	D, E, G
G	→	F

DFS execution from node a

0/0



— bridges



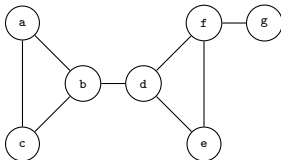
A	→ B, C
B	→ A, C, D
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DFS execution from node a

0/0

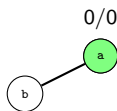


— bridges

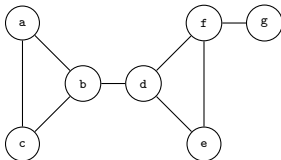


A	→ B, C
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DFS execution from node a

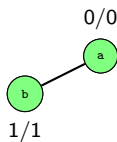


— bridges

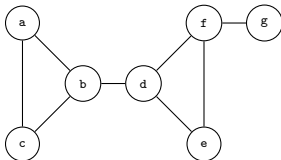


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DFS execution from node a

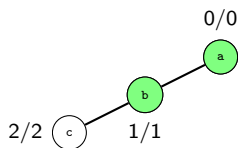


— bridges

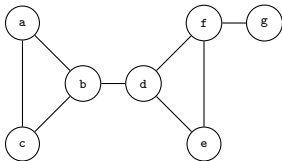


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DFS execution from node a

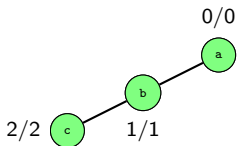


— bridges

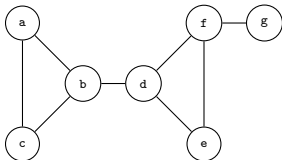


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DFS execution from node a

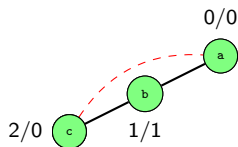


— bridges

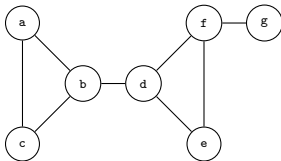


A	→ B, C
B	→ A, C, D
C	→ A, B
D	→ B, E, F
E	→ D, F
F	→ D, E, G
G	→ F

DFS execution from node a

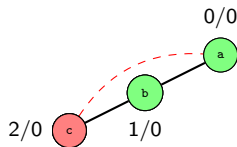


— bridges

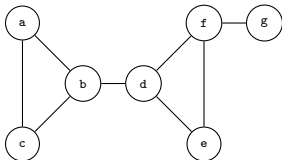


A	→ B, C
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DFS execution from node a

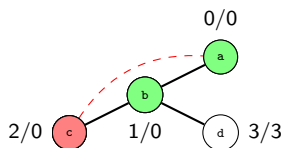


— bridges

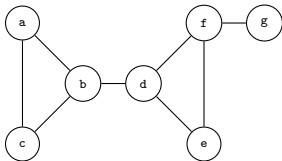


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DFS execution from node a

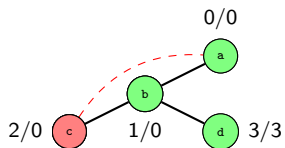


— bridges

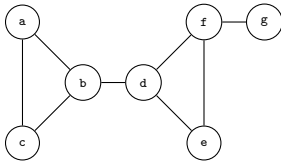


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DFS execution from node a

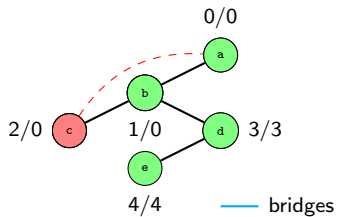


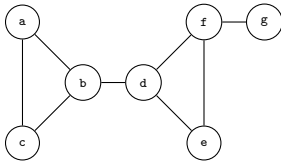
— bridges



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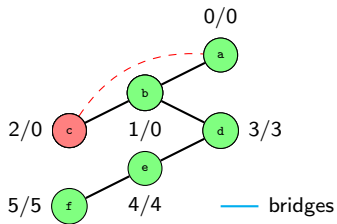
DFS execution from node a

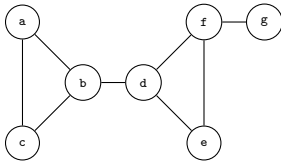




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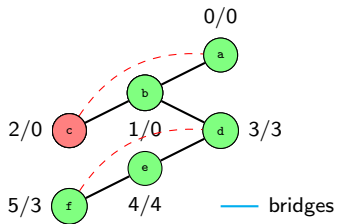
DFS execution from node a

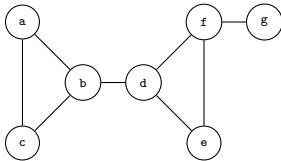




A	→ B, C
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D	→ B, E, F
E	→ D, F
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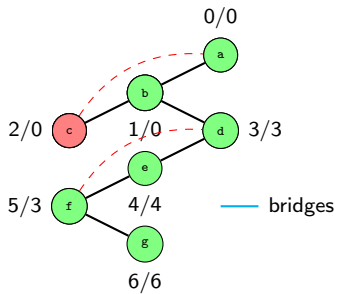
DFS execution from node a

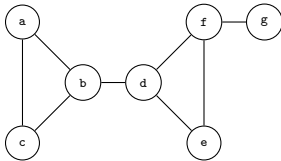




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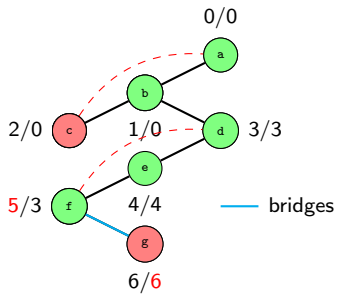
DFS execution from node a

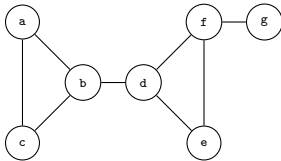




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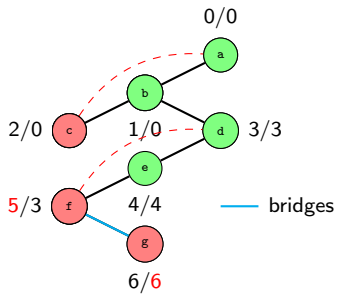
DFS execution from node a

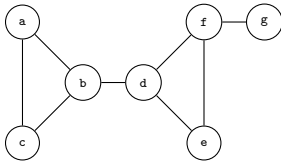




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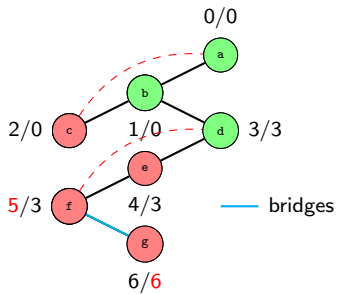
DFS execution from node a

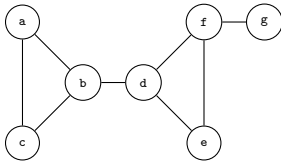




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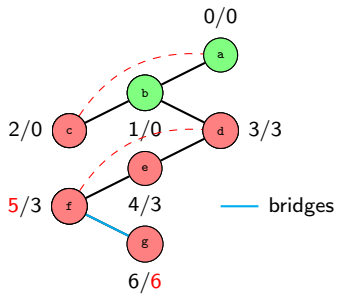
DFS execution from node a

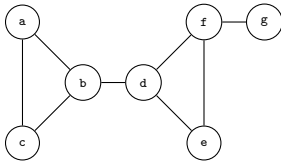




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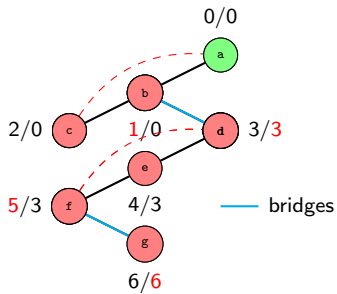
DFS execution from node a

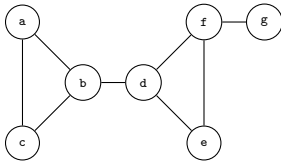




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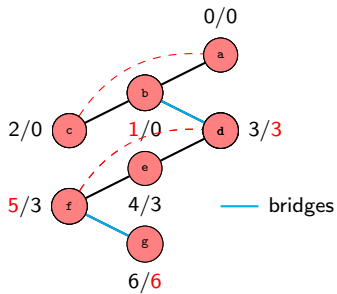
DFS execution from node a





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DFS execution from node a

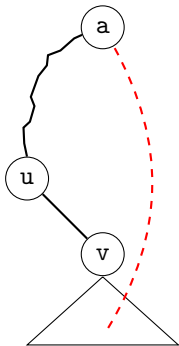


So what about **articulation points**?

So what about **articulation points**?

We observe that a node u is an articulation point if and only if:

No node in the sub-tree of v has a link to a (strict) ancestor of u .



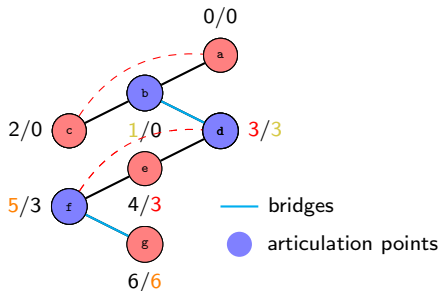
Note that a link to u does not help here.

What does this mean in terms of *num* and *low*?

¹Note that this might be detected several times with different children v , so don't output it directly!

What does this mean in terms of *num* and *low*?

A non-root node u is an articulation point if and only if when we finish v , $low[v] \geq num[u]$.¹

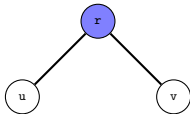


¹Note that this might be detected several times with different children v , so don't output it directly!

What about the root?

What about the root?

Root is articulation if and only if it has more than one child.



Removing node r disconnects the graph.

If there was another path from u to v , v would not be a child of r .

```

int num[N], low[N], root, rootChildren, cnt = 0;

void dfs(int u, int parent) {
    num[u] = low[u] = cnt++;
    vis[u] = true;
    for (int v : adj[u]) {
        if (!vis[v]) {
            dfs(v, u);
            if (u == root) rootChildren++;
            if (low[v] >= num[u] && u != root)
                // u is an articulation point
            if (low[v] > num[u])
                // (u,v) is a bridge
            low[u] = min(low[u], low[v]);
        } else if (v != parent)
            low[u] = min(low[u], num[v]);
    }
}

// loop in main():
if (!vis[u]) {
    root = u, rootChildren = 0;
    dfs(u, -1);
    if (rootChildren > 1)
        // u is an articulation point
}

```