# Bipartite graphs

## Bipartite check, MCBM, MVC, MIS

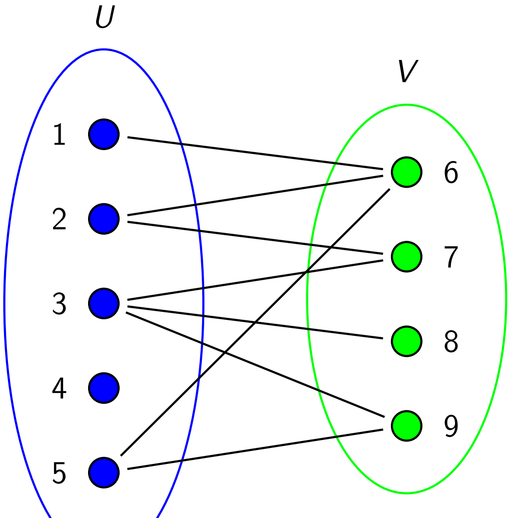
beOI Training

## Reminder about graphs

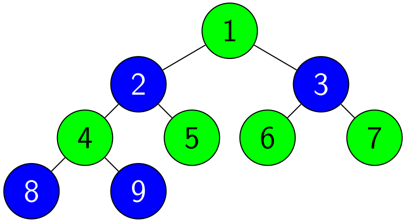
See 06-graph-basics

## Bipartite graphs

A graph is bipartite if it can be separated into two sets $U$ and $V$ such that nodes in $U$ are only connected to nodes in $V$, and conversely.



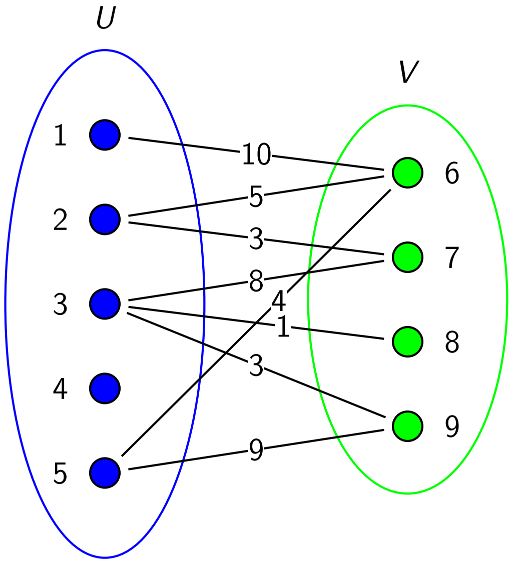
## A tree is also a bipartite graph



Here $U = \{1, 4, 5, 6, 7\}$ and $V = \{2, 3, 8, 9\}$

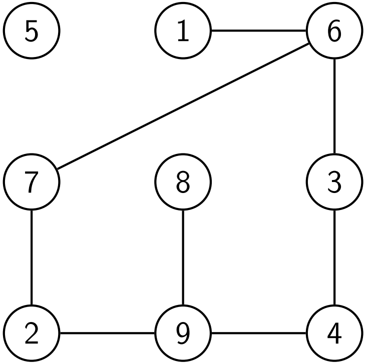
## Weighted bipartite graphs

The same, but with weights:



## Bipartite check

How to test if a particular graph is a bipartite one?



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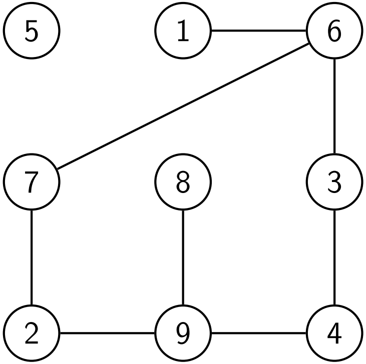
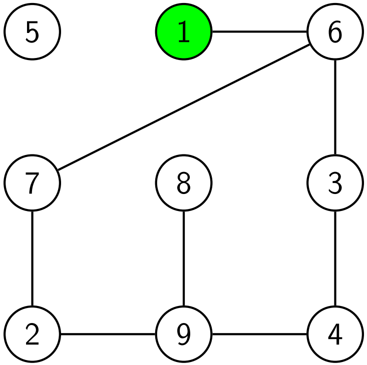


DFS with coloration!

## Bipartite check



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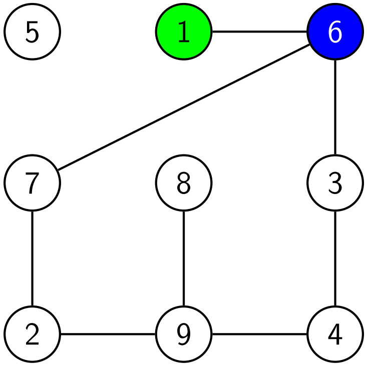

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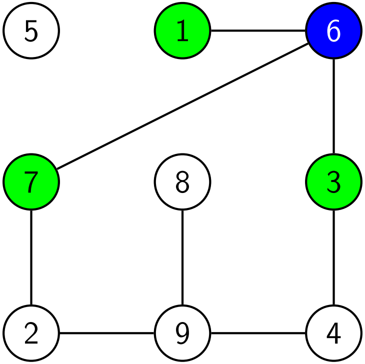

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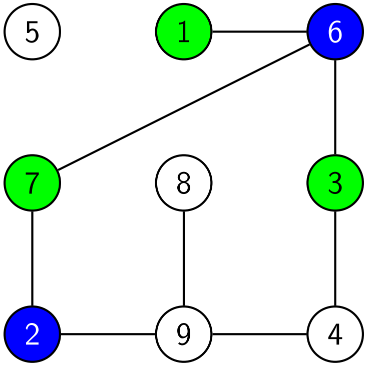

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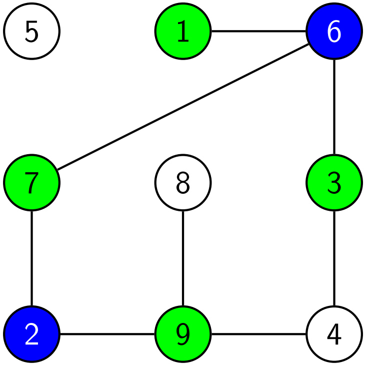

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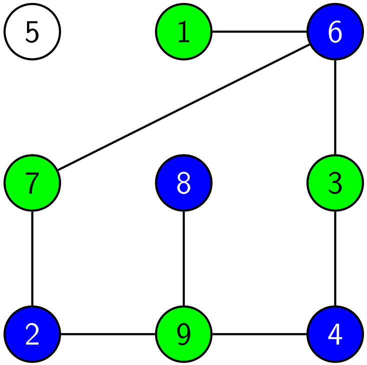

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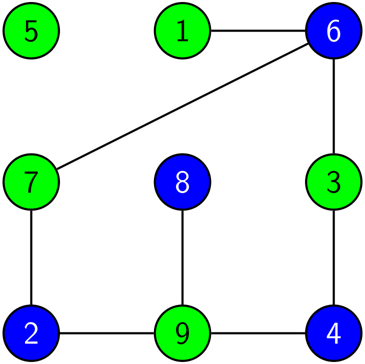


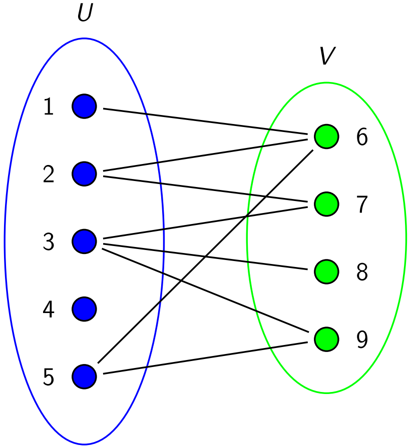
Ok!

# Algorithm

```java
boolean isBipartite(int n, LinkedList<Integer>[] successors) {
  //successors[i] contains the successors id of node i
  int[] color = new int[n];
  for(int i = 0; i < n; i++) color[i] = -1; //init to -1

  for(int i = 0; i < n; i++)
    if(color[i] == -1)  //not visited yet
      if(!visit(i, successors, color, 1))
        return false;
  return true;
}

boolean visit(int node, LinkedList<Integer>[] successors, int[] color, int
      parentColor) {
  if(color[node] == parentColor) //fail!
    return false;
  color[node] = (parentColor+1)%2;
  for(int next: successors[node])
    if(!visit(next, successors, color, color[node]))
      return false;
  return true;
}

```

## Maximum Cardinality Bipartite Matching

Given a set $U$ of men and a set $V$ of women, and a list of "compatibilities" between men and women, we obtain this:



Can you create a maximum number of couples?

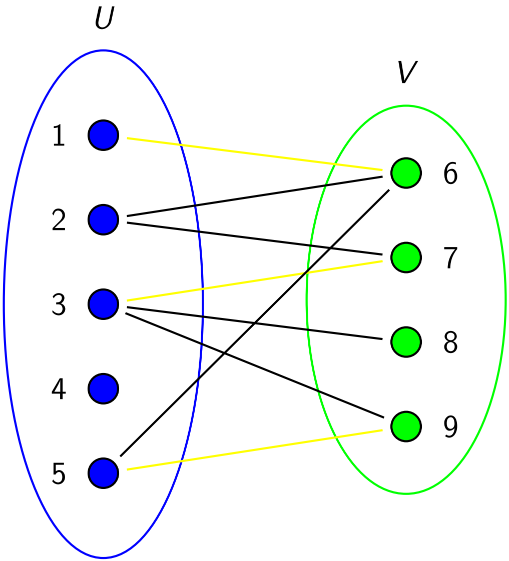
## A bit of theory

$M \in E$ is a **matching** if each node is used at most once by the edges in $M$.

A **maximum cardinality matching** is a matching that has the maximal number of edge/node possible.

A node which is not used by any edge in a matching is said **free**. The others are said **non-free**.

## An example of matching



It is maximal?

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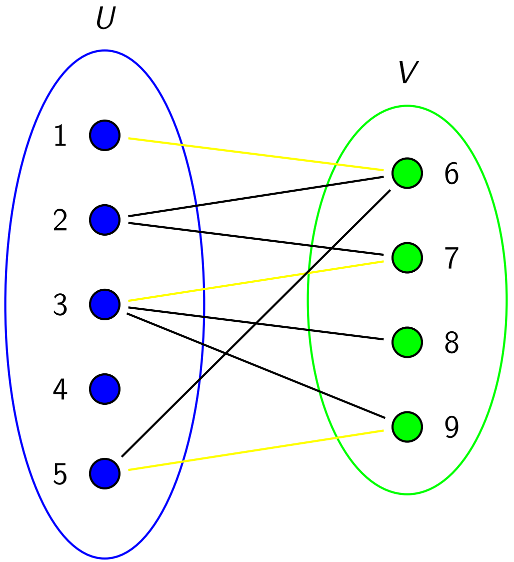
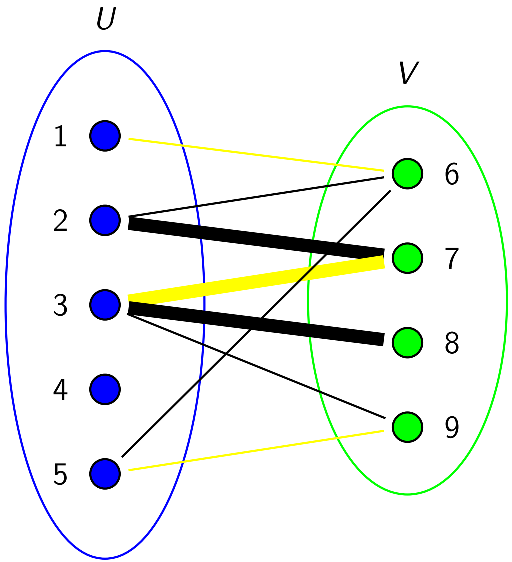


It is maximal? How to improve it?

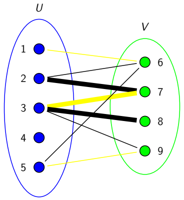
## More theory

An augmenting path for a matching $M$ is a path starting and ending at free nodes, and alternating between matched and unmatched edges.

## An example of augmenting path



## Let's think about augmenting paths



How can we use such paths? Let's find some useful properties

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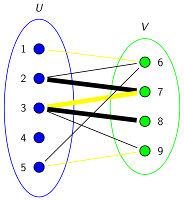

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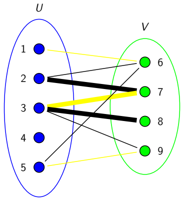

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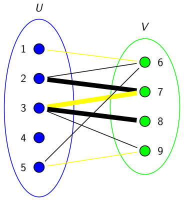

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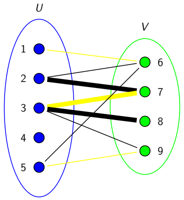

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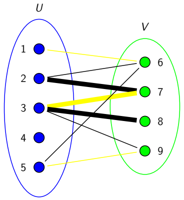

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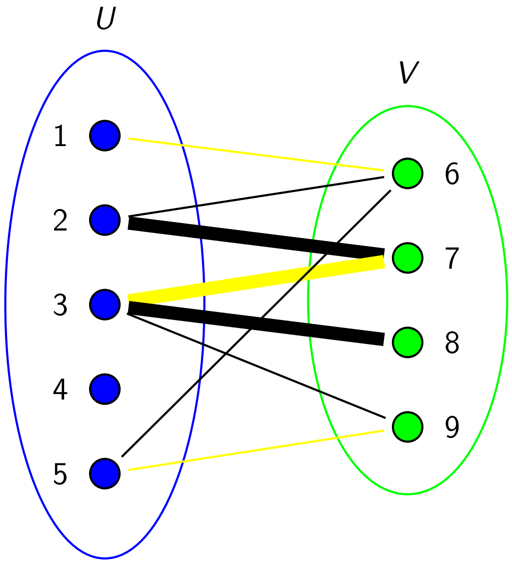
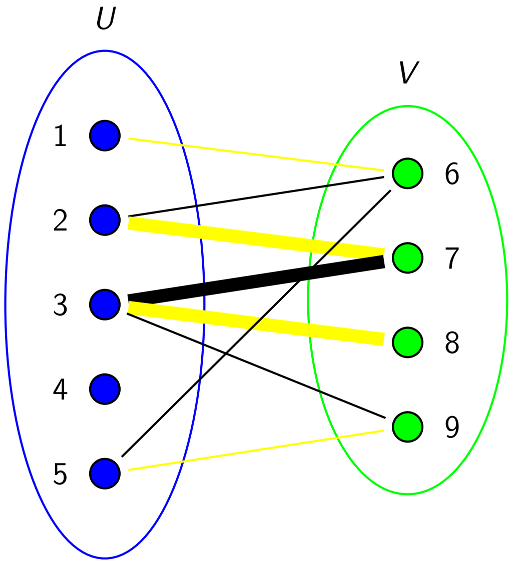
- The size of an aug. path is always odd
- There is always one "free edge" more than the number of "taken edges" in an aug. path
- What if we inverse the edges?

## An augmenting path



## Inversing the augmenting path



## Augmenting path $=$ good?

- It is always possible to inverse an augmenting path
- It always increase the size of the matching by 1!

## Augmenting path = good!

Given a matching $M$ in a bipartite graph such that there exist no augmenting path, then $M$ is of **maximal cardinality**.

## Solving the MCBM

1. Find an aug. path. If no such path exists, return the matching, it is maximal
2. Inverse the path found
3. Repeat from 1

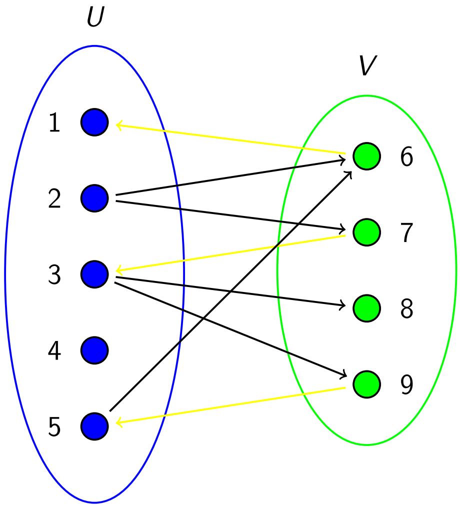
Simple, isn't it?

## Let's see two algorithms

- The first one is based on the "flow" representation of the graph, and is very similar to a max flow. Simple to understand, but longer.
- The second is recursive so more difficult to understand but shorter.

## Finding an aug. path

An alternative representation:



In this representation, an aug. path is a path starting at a free node in $U$ and ending at a free node in $V$.

# Finding an aug. path

```java
// succ[i] contains the nodes that can be reached from i
// initially, all the edges are from U->V
// inU[i] is true iff i is in U
boolean findAndReverse(int n, List<Integer>[] succ, boolean[] inU) {
    int[] pred = new int[n];
    boolean[] visited = new boolean[n];
    Stack<Integer> todo = new Stack<>();

    // Find free nodes in U
    boolean[] isFree = new boolean[n];
    Arrays.fill(isFree, true);
    for(int i = 0; i < n; i++)
        if(!inU[i])
            for(int s: succ[i])
                isFree[s] = false;

    for(int i = 0; i < n; i++) {
        if(inU[i] && isFree[i]) {
            todo.add(i);
            pred[i] = -1.
        }
    }
```

# Finding an aug. path (cont.)

```
    // Run the DFS
    int found = -1;
    while(!todo.isEmpty()) {
        int node = todo.pop();
        if(visited[node])
            continue;
        visited[node] = true;
        //if we are at a free node in V
        if(!inU[node] && succ[node].size() == 0) {
            found = node;
            break;
        }
        else {
            for(int next: successors[node]) {
                if(!visited[next]) {
                    pred[next] = node;
                    todo.add(next);
                }
            }
        }
    }

    // Reverse the nodes
    if(found != -1) {
        while(predecessors[found] != -1) {
            succ[pred[found]].remove(found);
            succ[found].add(pred[found]);
            found = pred[found];
        }
        return true;
    }
    return false;
}
```

## Final algorithm

```java
void getMCBM(int n, List<Integer>[] succ, boolean[] inU) {
    while(findAndReverse(n, succ, inU)) {}
    //MCBM == edges from nodes in V (in succ)
}
```

# Another (shorter) algorithm

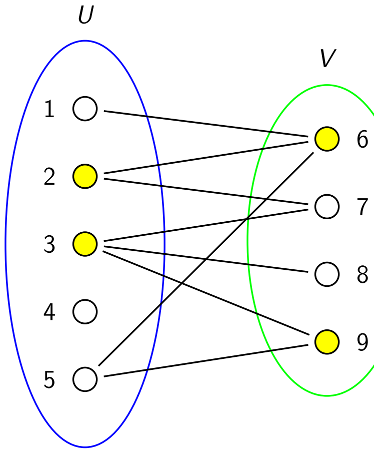
```
vector<vi> AdjList;
vi match, vis;

int Aug(int l) { // return 1 if an augmenting path is found
    if (vis[l]) return 0; // return 0 otherwise
    vis[l] = 1;
    for (int j = 0; j < (int)AdjList[l].size(); j++) {
        int r = AdjList[l][j];
        if (match[r] == -1 || Aug(match[r])) {
            match[r] = l;
            return 1; // found 1 matching
        }
    }
    return 0; // no matching
}

int MCBM = 0;
match.assign(V, -1); //V = total number of vertices
for(int l = 0; l < n; l++) { //n = size of the left set
    vis.assign(n, 0);
    MCBM += Aug(l);
}
```

## Minimum vertex cover (in bipartite graph)

A **Vertex cover** $K$ is a set of nodes from $G$ such that each edge of $G$ is incident to at least one node of $K$.
A **Minimal vertex cover** is a vertex cover of minimal size.



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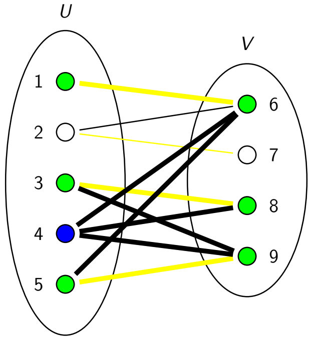

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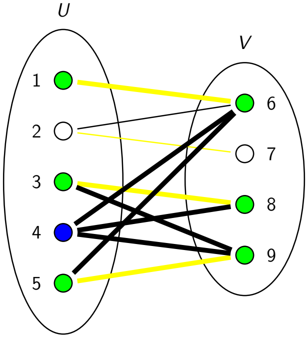

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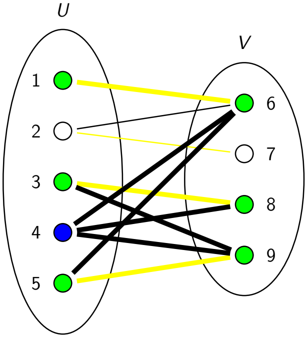
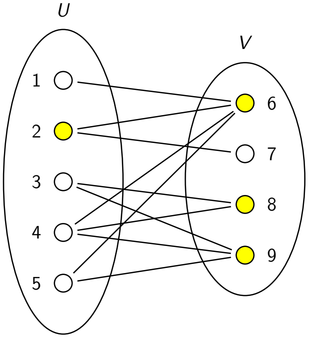




Then $K = (U \backslash Z) \cup (V \cap Z)$ is a minimum vertex cover

## Maximum independant set (in bipartite graph)

An **independant set** of a bipartite graph $G =< V, E >$ is a set of nodes $MIS$ such that there is no edges between nodes in $MIS$.

Said in another way, $\nexists v_1 \in MIS, v_2 \in MIS$ s.t. $(v_1, v_2) \in E$.

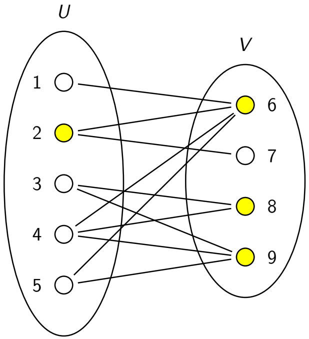
A **maximum independant set** is an independant set whose size is maximal (...)

# Maximum independant set (2)

Given a Minimum vertex cover $K$ on a graph with the set of nodes $S = U + V$, then

$$MIS = S - K$$

is a maximum independant set.



U
V
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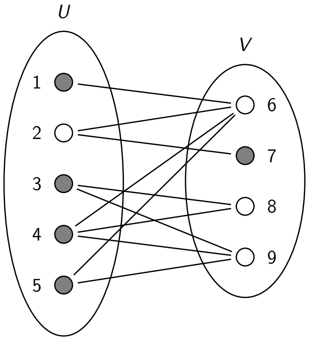

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