

# Specialized DFS

cycles, bridges, articulation points

beOI Training  
(slides by François Aubry)



October 6, 2016

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Adding one state

Cycle detection

Bridges and articulation points

```

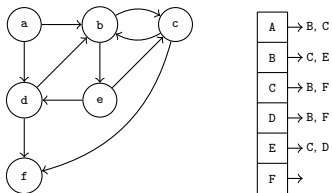
enum state {UNVISITED, OPENED, CLOSED};
vector<int> adj[N];
state st[N];

void dfs(int u) {
    st[v] = OPENED;
    for (int v : adj[u])
        if (st[u] == UNVISITED)
            dfs(v);
    st[v] = CLOSED;
}

// in main()
fill(st, st+n, UNVISITED);
for (int u = 0; u < n; i++)
    if (st[u] == UNVISITED)
        dfs(u);

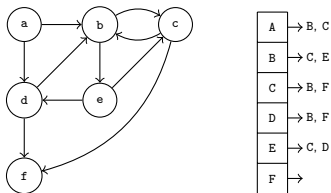
```

An extended version of DFS: remember if node is finished or not



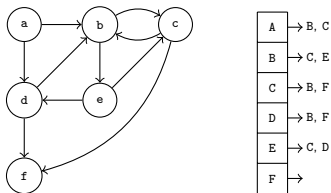
DFS execution from node a



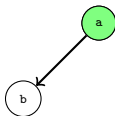


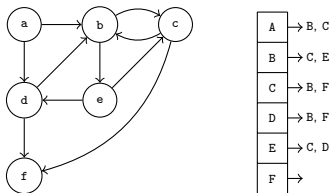
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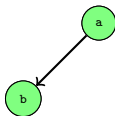


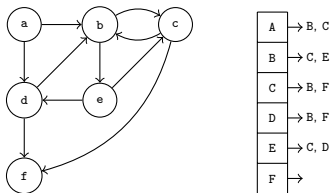
DFS execution from node a



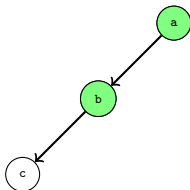


DFS execution from node a

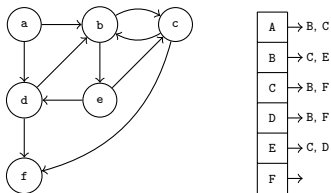




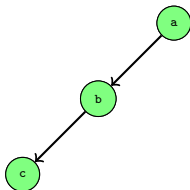
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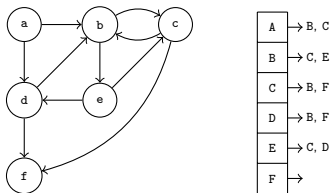




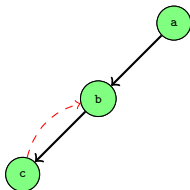


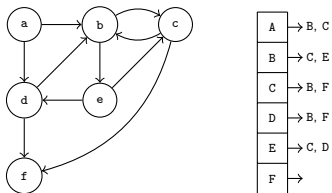
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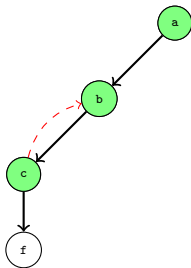


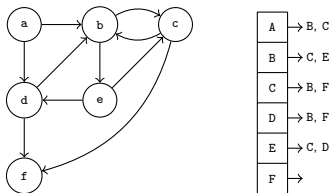
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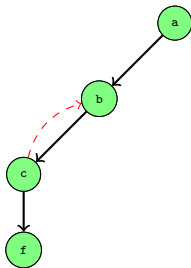


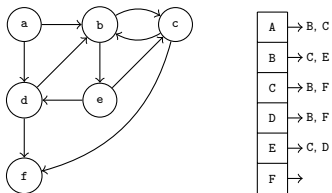
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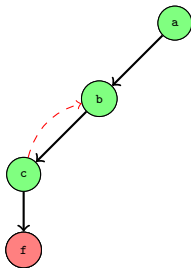


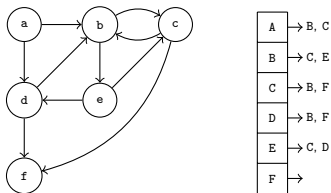
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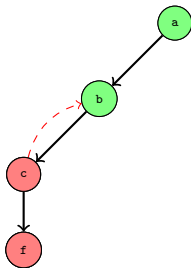


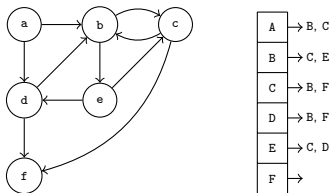
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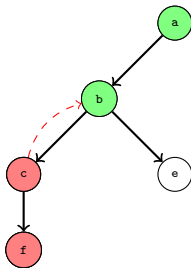


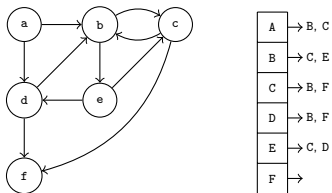
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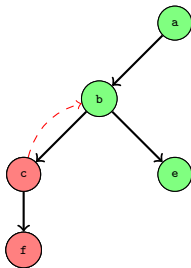


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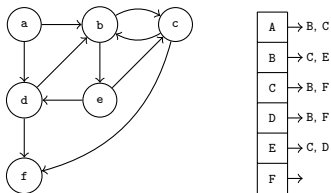




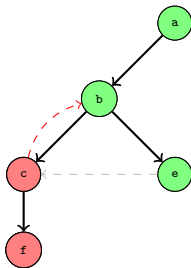
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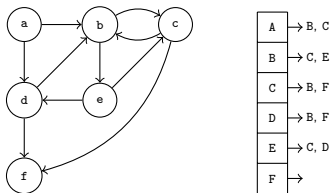




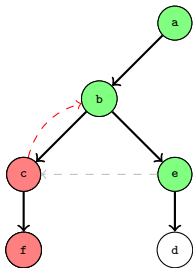


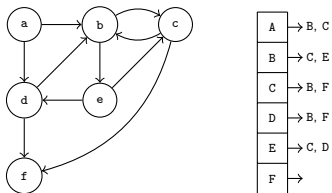
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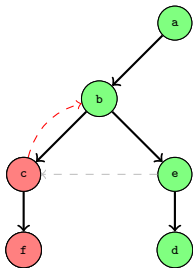


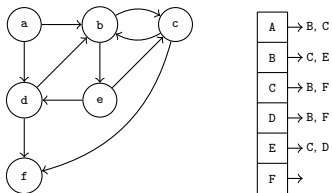
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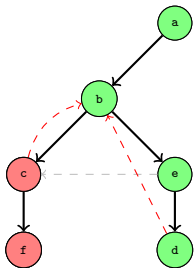


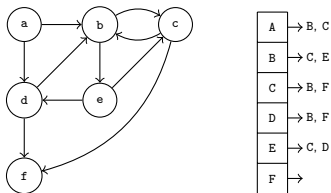
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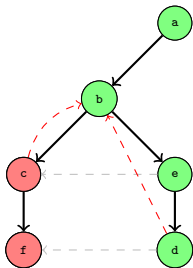


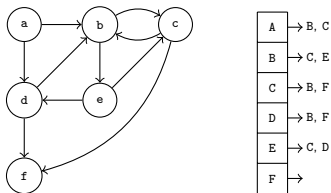
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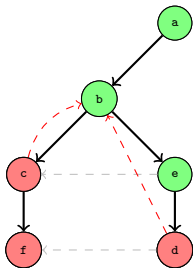


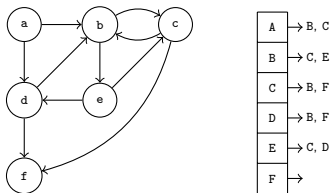
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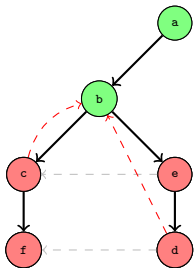


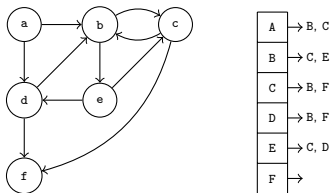
DFS execution from node a



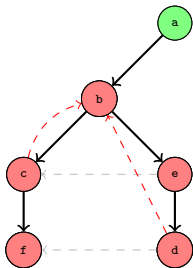


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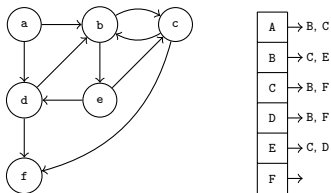




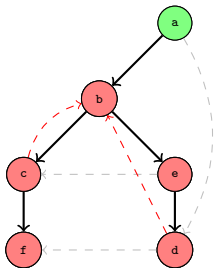
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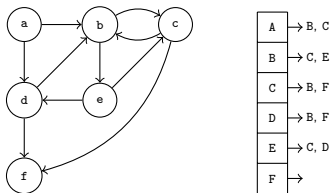




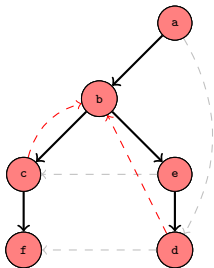


DFS execution from node a





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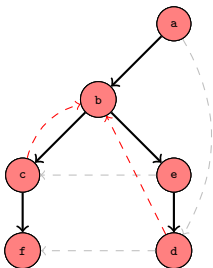
Bridges and articulation points

Let's start with **cycles**.

How can we detect with DFS whether a graph is acyclic or not?

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How can we detect with DFS whether a graph is acyclic or not?

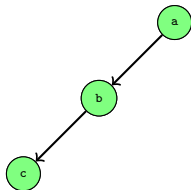


The red edges belong to cycles, they are called **back edges**.

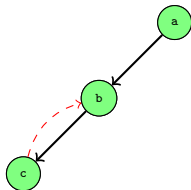
A graph is acyclic if and only if DFS does not yield back edges.

How to distinguish these edges from the others?

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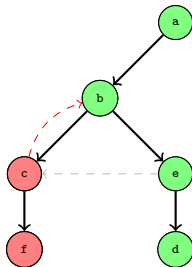


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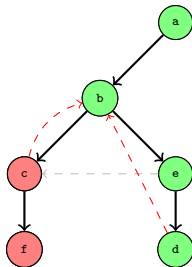




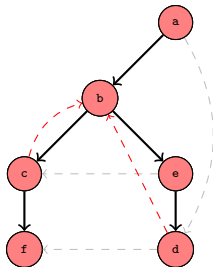
How to distinguish these edges from the others?



How to distinguish these edges from the others?



How to distinguish these edges from the others?



They are edges from a green (OPENED) node to another green (OPENED) node!

To implement this we simply add a check while listing neighbours.

Checks if node  $u$  (which is OPENED) points to another OPENED node.

```
bool hasCycle = false;

void dfs(int u) {
    st[v] = OPENED;
    for (int v : adj[u]) {
        if (st[u] == UNVISITED)
            dfs(v);
        else if (st[u] == OPENED)
            hasCycle = true;
    }
    st[v] = CLOSED;
}
```

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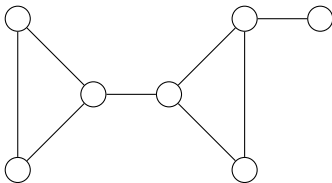
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Bridges and articulation points

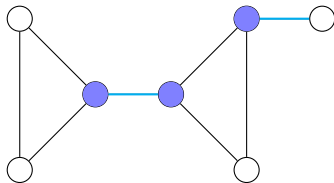
An **articulation point** in an undirected graph is a node such that its removal disconnects the graph.

A **bridge** in an undirected graph is an edge such that its removal disconnects the graph.



An **articulation point** in an undirected graph is a node such that its removal disconnects the graph.

A **bridge** in an undirected graph is an edge such that its removal disconnects the graph.



— bridges

● articulation points

How would you compute this?



How would you compute this?

### **Naive algorithm (for bridges):**

For every edge  $(x, y)$ , remove it and check with BFS or DFS whether  $x$  and  $y$  remain connected.

Complexity:  $O(E \cdot (V + E)) = O(E \cdot V + E^2)$  **TLE** in big graphs.

We will solve it in **linear time** with a single DFS.

## **Observation:**

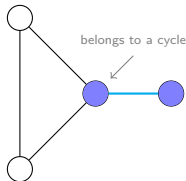
Bridges can never belong to cycles.

Is this true for articulation points?

## Observation:

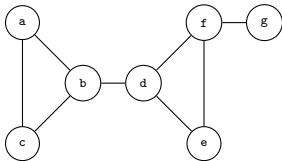
Bridges can never belong to cycles.

Is this true for articulation points? **NO**.



So we essentially need to find which edges belong to cycles.

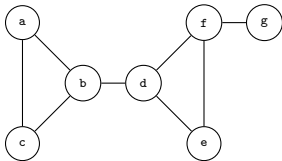
We already say that DFS allows to find cycles.



A	→ B, C
B	→ A, C, D
C	→ A, B
D	→ B, E, F
E	→ D, F
F	→ D, E, G
G	→ F

DFS execution from node a

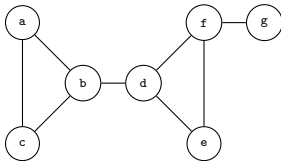




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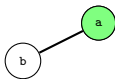
DFS execution from node a

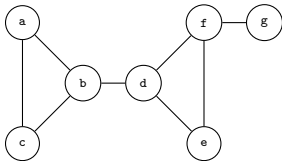




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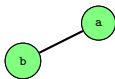
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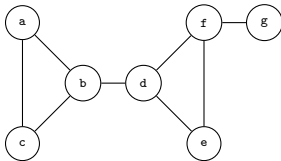




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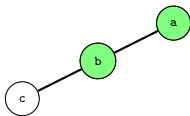
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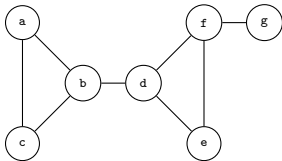


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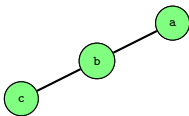


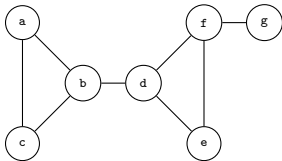




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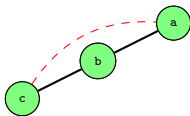
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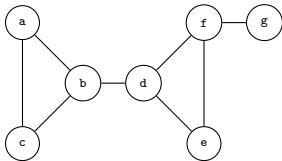




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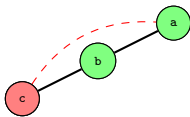
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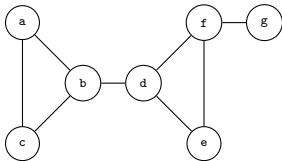




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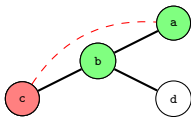
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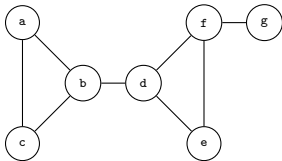




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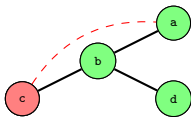
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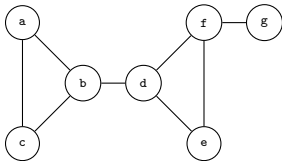




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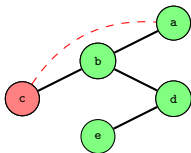
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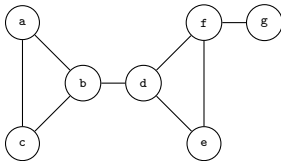




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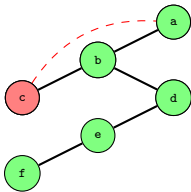
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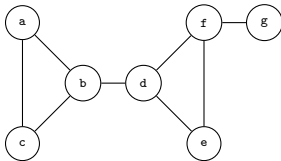




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E	→ D, F
F	→ D, E, G
G	→ F

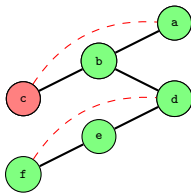
DFS execution from node a



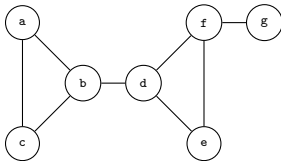


A	→ B, C
B	→ A, C, D
C	→ A, B
D	→ B, E, F
E	→ D, F
F	→ D, E, G
G	→ F

DFS execution from node a

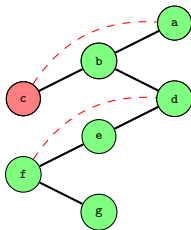


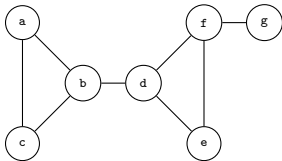




A	→ B, C
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C	→ A, B
D	→ B, E, F
E	→ D, F
F	→ D, E, G
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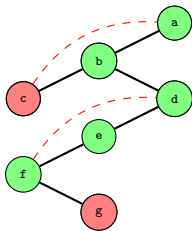
DFS execution from node a

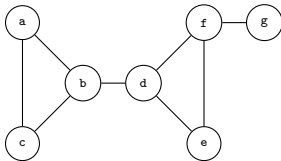




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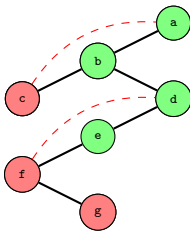
DFS execution from node a

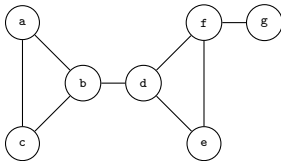




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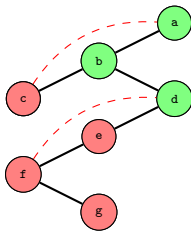
DFS execution from node a

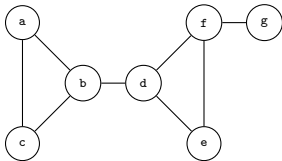




A	→ B, C
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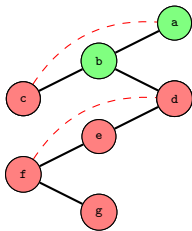
DFS execution from node a

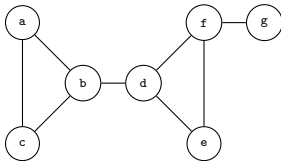




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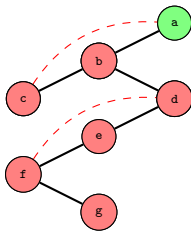
DFS execution from node a

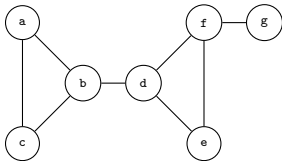




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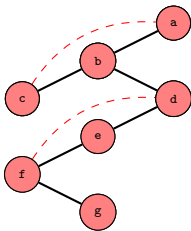
DFS execution from node a

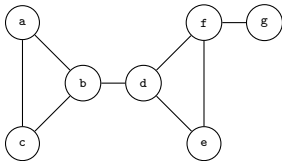




A	→ B, C
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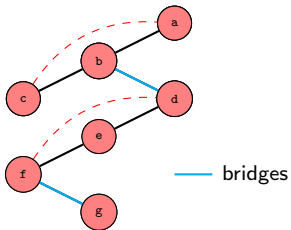
DFS execution from node a





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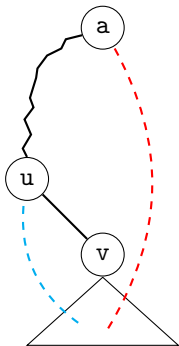
DFS execution from node a





We observe that an edge  $(u, v)$  is a bridge if and only if:

No node in the sub-tree of  $v$  has **a link to  $u$**  **or** **one of its ancestors other than  $(u, v)$** .



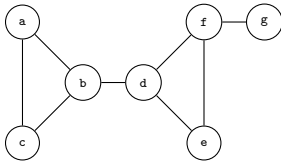
What should we add to our DFS to know this information?

We will **timestamp** the nodes as they are visited:  $num$

Keep track of the node with minimum timestamp that we see:  $low$

$(u, v)$  is a bridge if and only if when we finish  $v$ ,  $low[v] > num[u]$

This is so because this means that in the sub-tree of  $v$  we did not see any ancestor of  $u$ .



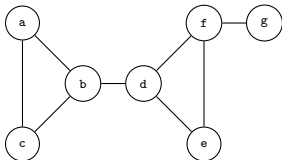
A	→ B, C
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DFS execution from node a

0/0



— bridges



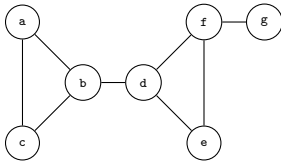
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DFS execution from node a

0/0

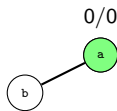


— bridges

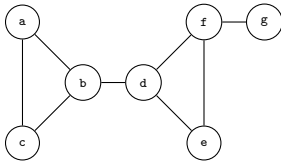


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DFS execution from node a

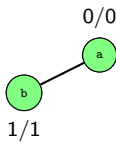


— bridges

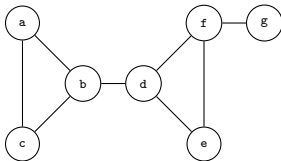


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DFS execution from node a

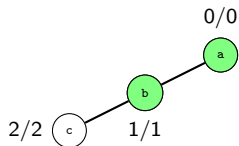


— bridges

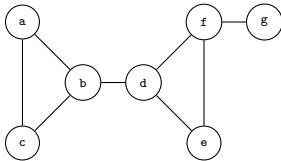


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DFS execution from node a

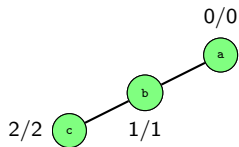


— bridges



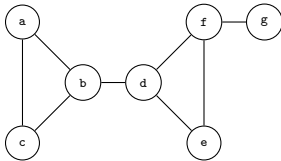
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DFS execution from node a



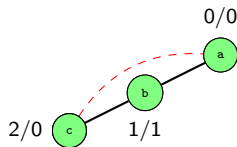
— bridges



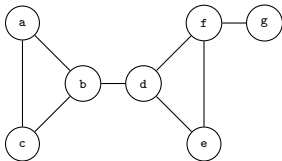


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F	→ D, E, G
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DFS execution from node a

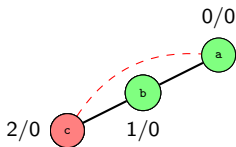


— bridges

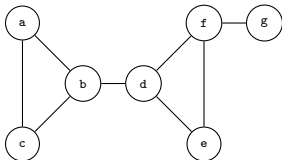


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DFS execution from node a

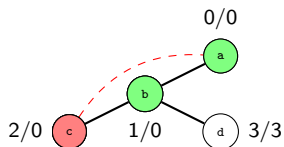


— bridges

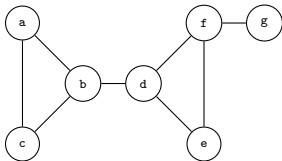


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DFS execution from node a

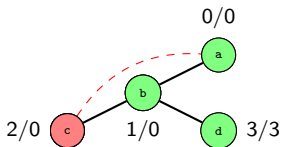


— bridges

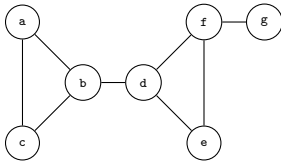


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DFS execution from node a

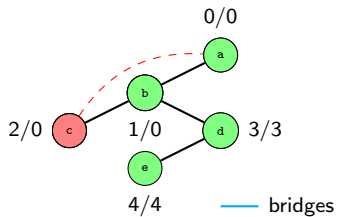


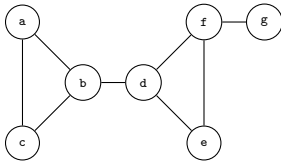
— bridges



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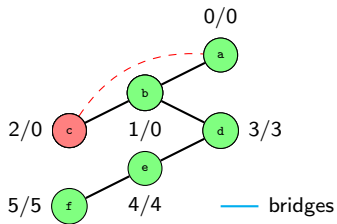
DFS execution from node a

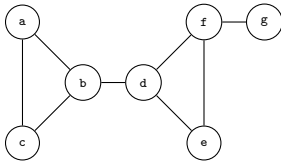




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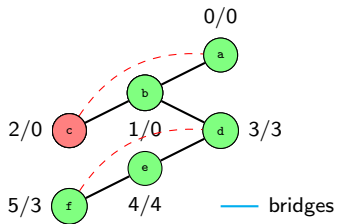
DFS execution from node a

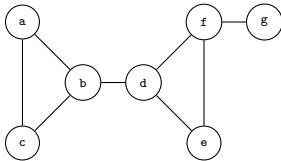




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F	→ D, E, G
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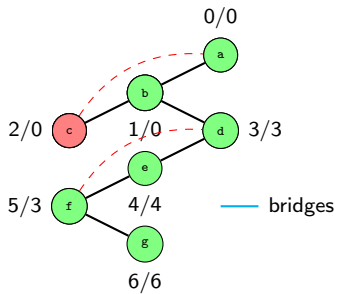
DFS execution from node a



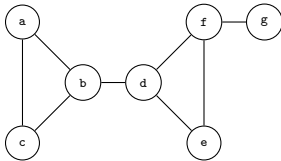


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DFS execution from node a

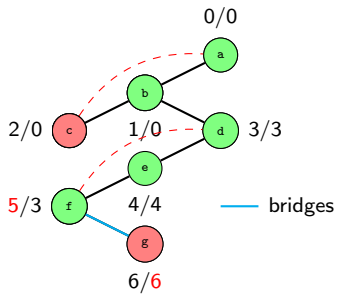


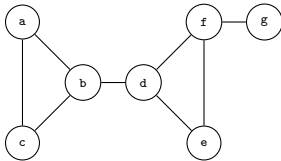




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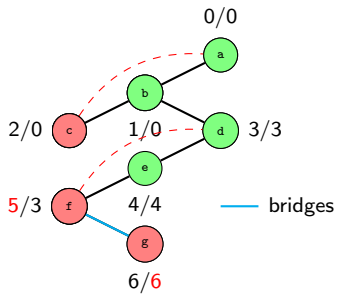
DFS execution from node a

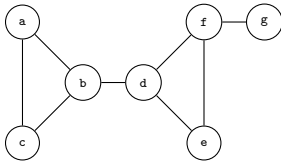




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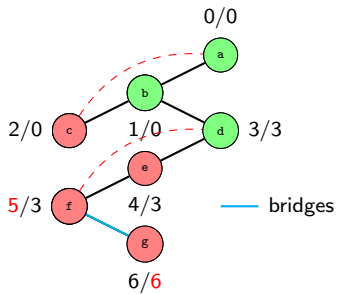
DFS execution from node a

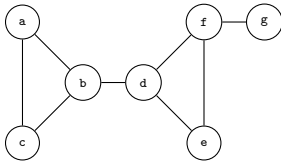




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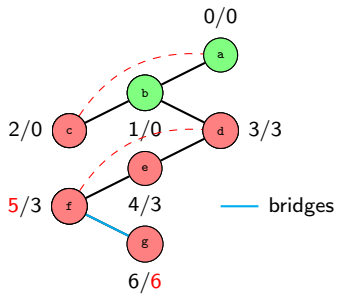
DFS execution from node a

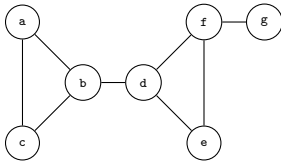




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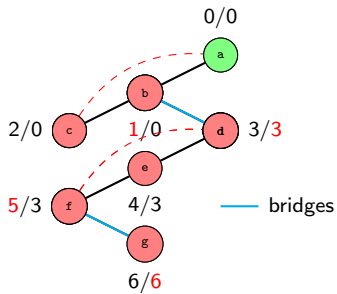
DFS execution from node a

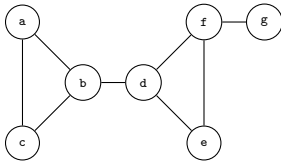




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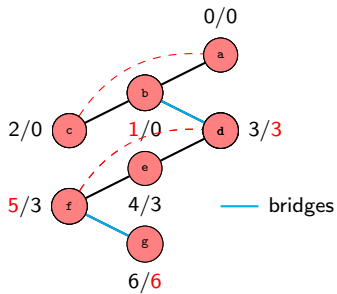
DFS execution from node a





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DFS execution from node a

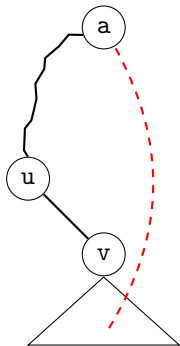


So what about **articulation points**?

So what about **articulation points**?

We observe that a node  $u$  is an articulation point if and only if:

No node in the sub-tree of  $v$  has a link to **one ancestor of  $u$  other than  $(u, v)$** .



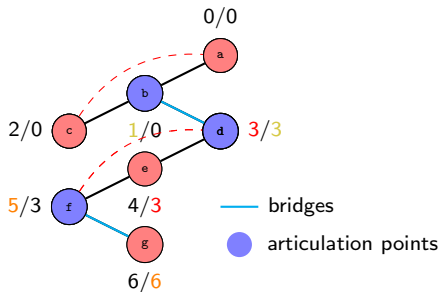
Note that a link to  $u$  does not help here.



What does this mean in terms of *num* and *low*?

What does this mean in terms of *num* and *low*?

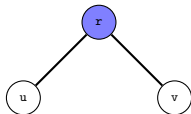
A non-root node  $u$  is an articulation point if and only if when we finish  $v$ ,  $low[v] \geq num[u]$



What about the root?

What about the root?

Root is articulation if and only if it has more than one child.



Removing node  $r$  disconnects the graph.

If there was another path from  $u$  to  $v$ ,  $v$  would not be a child of  $r$ .

```

int num[N], low[N], parent[N], root, rootChildren, cnt = 0;

void dfs(int u) {
    num[u] = low[u] = cnt++;
    st[v] = OPENED;
    for (int v : adj[u]) {
        if (st[u] == UNVISITED) {
            parent[v] = u; dfs(v);
            if (u == root) rootChildren++;
            if (low[v] >= num[u] && v != root)
                // u is an articulation point
            if (low[v] > num[u])
                // (u,v) is a bridge
            low[u] = min(low[u], low[v]);
        } else if (v != parent[u])
            low[u] = min(low[u], num[v]);
    }
    st[v] = CLOSED;
}

// loop in main():
if (st[u] == UNVISITED) {
    root = u, rootChildren = 0;
    dfs(u);
    if (rootChildren > 1)
        // u is an articulation point
}

```