

Dynamic Programming I

Top-Down, Bottom-Up and Classical Problems

beOI Training

(many thanks to François Aubry)



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Let's start with an example

Partition Problem:

Given a set of $n \leq 50$ goodies each with value $v[i] \leq 10$, is it possible to divide them between 3 persons evenly?

Strategy

Put yourself in the shoes of the one who divides the goodies.



For each goodie, what **choices** can you make?

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Brute-Force?

Suppose we make a brute force algorithm that tries all choices.

What should we keep track of?

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2. At the end we need to know if the division is fair. Do we need to know the items given to each person?

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Does the order in which we consider the items matter?

No.

⇒ Just keep an integer i such that we have made choices for items $< i$.

2. At the end we need to know if the division is fair. Do we need to know the items given to each person? **No.**

⇒ Just keep track of how much was given to each person.

Brute-Force solution

```
1 bool complete_search(int i, int given1, int given2, int given3) {
2     if(i == n) // all goodies have been considered
3         return given1 == given2 && given2 == given3;
4     else {
5         bool giveTo1 = complete_search(i+1, given1 + v[i], given2, given3);
6         bool giveTo2 = complete_search(i+1, given1, given2 + v[i], given3);
7         bool giveTo3 = complete_search(i+1, given1, given2, given3 + v[i]);
8         return giveTo1 || giveTo2 || giveTo3;
9     }
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Complexity?

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Complexity? $\mathcal{O}(3^n)$ TLE

We can do better...

State space and state graph

We call one tuple $(i, given1, given2, given3)$ a **state**.

We can now define the **state graph**: its vertices are the states, and there is an edge from s_1 to s_2 if s_1 calls s_2 recursively.

How many nodes does the state graph of the previous algorithm have?

$\mathcal{O}(n \cdot S^3)$ where S is the sum of the goodie values.

But the algorithm is $\mathcal{O}(3^n)$. What's going on?

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Each state is visited several times \Rightarrow waste of time!

What we want to do is to traverse the state graph (DFS like).

How can we achieve that?

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How can we achieve that?

```
1 int canDo[n][S][S][S];
2 // undefined: -1
3 // we cannot divide evenly from that state: 0
4 // we can divide evenly from that state: 1
5
6 bool solve(int i, int given1, int given2, int given3) {
7     if(i == n) // all goodies have been considered
8         return given1 == given2 && given2 == given3;
9
10    if(canDo[i][given1][given2][given3] != -1) {
11        // the state (i, given1, given2, given3) has already been visited
12        return canDo[i][given1][given2][given3];
13    } else {
14        bool giveTo1 = solve(i+1, given1 + v[i], given2, given3);
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Is this enough to get AC?

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19        return canDo[i][given1][given2][given3];
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```

Is this enough to get AC? **No.**

The graph has $\approx n \cdot S^3 = 50 \cdot 500^3 = 6250000000$ nodes \Rightarrow

MLE.



Or can we make it work?

State space reduction

Observe that at the end $given3 = S - given1 - given2$.

Thus we can drop one parameter and reduce the state space to $n \cdot S^2$.

```
1 int canDo[n][S][S];
2 // undefined: -1
3 // we cannot divide evenly from that state: 0
4 // we can divide evenly from that state: 1
5 int S; // sum of the v[i]
6
7 bool solve(int i, int given1, int given2) {
8     if(i == n) { // all goodies have been considered
9         int given3 = S - given1 - given2;
10        return given1 == given2 && given2 == given3;
11    }
12    if(canDo[i][given1][given2] != -1) {
13        // the state (i, given1, given2) has already been visited
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```

What we learned so far

1. View the problem as a **sequence of choices**.
2. Represent the problem with the smallest state space possible.
3. Perform a DFS on the state graph (remembering visited states).

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Let's see another example!

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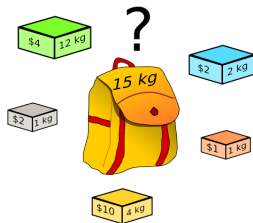
Motivating Problem III: Shortest Path Problem

Exercises

Knapsack Problem

Given a set of n objects each with value $v[i]$ and weight $w[i]$, and a knapsack that can hold a total capacity of C .

Choose a subset of objects that fits into the knapsack and has maximum total value.



In what way is this problem similar to the previous one?

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- ▶ Succession of choices: for each item, take it or leave it.
- ▶ Order does not matter.

State space?

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State space? $(i, wtaken)$

- ▶ $i = \text{item we are considering}$
- ▶ $wtaken = \text{total weight of the items we selected so far}$

Size of the state space?

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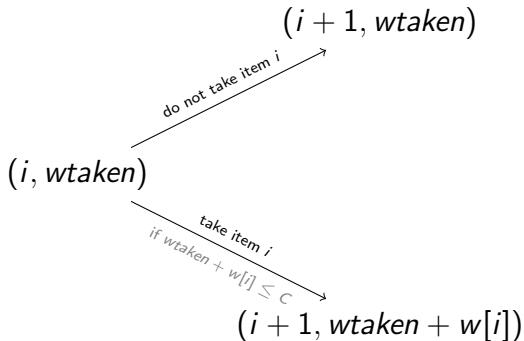
State space? $(i, wtaken)$

- ▶ $i = \text{item we are considering}$
- ▶ $wtaken = \text{total weight of the items we selected so far}$

Size of the state space? $\mathcal{O}(n \cdot C)$

Successors of state (i, w_{taken}) ?

Successors of state $(i, wtaken)$?



Recurrence relation?

Recurrence relation?

$$f(i, wtaken) = \max \left(f(i+1, wtaken), v[i] + f(i+1, wtaken + w[i]) \right)$$

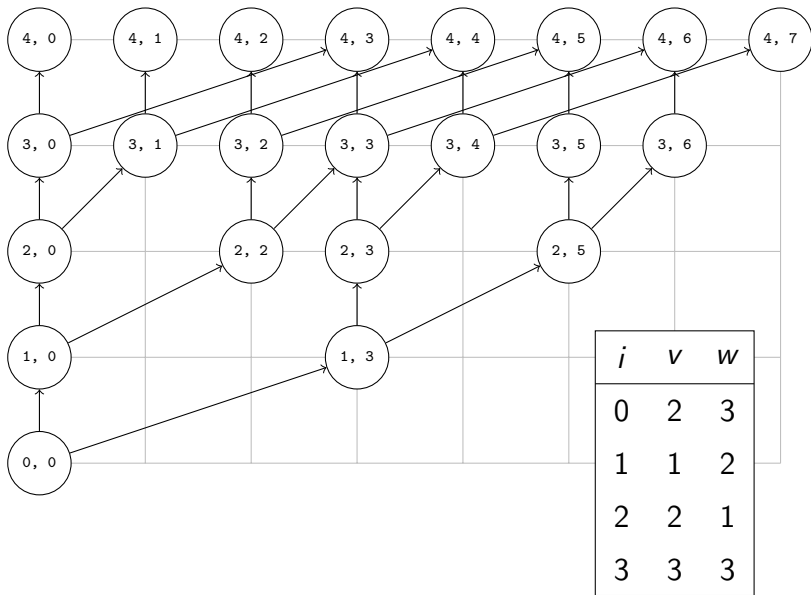
Beware of the knapsack constraint! If $wtaken > C$, the knapsack has no value.

$$wtaken > C \Rightarrow f(i, wtaken) = -\infty$$

Knapsack solution

```
1 int C, n;  
2 int w[n], v[n];  
3 int memo[n][C];  
4  
5 int solve(int i, int wtaken) {  
6     if(wtaken > C)  
7         return -INF;  
8     else if(i == n)  
9         return 0;  
10  
11     if(memo[i][wtaken] != -1)  
12         return memo[i][wtaken];  
13     else {  
14         memo[i][wtaken] = max(solve(i+1, wtaken), v[i] + solve(i+1, wtaken + w[i]  
15         ));  
16         return memo[i][wtaken];  
17     }
```

Example of a Knapsack state space.



Memory optimization

Observe that we don't need all the entries from the `memo` table.

Sometimes the table is too big and causes **MLE**.

In this situation an alternative is to use a `HashMap` (or `unordered_map`).

This way we only use the memory we need.

Another approach

This is one view of Dynamic Programming. Usually referred to as **memoization**.

Another view is to decompose the problem into **sub-problems**.

Define an order on the sub-problems: The bigger the harder.

Express the solution of large sub-problems as a function of smaller sub-problems.

Solve from smaller to larger using the function.

Let's redo the Knapsack this way

Decomposition into sub-problems:

$best[i][c]$ = best way to take objects $0, 1, \dots, i$
on a knapsack of capacity c

Observe that the real problem we want to solve is
 $best[n - 1][C]$.

The idea is that maybe $best[i][c]$ relates to $best[i - 1][c']$.

Case 1: item i **does not** belong to Knapsack

Suppose we know ***magically*** that item i **does not belong** to the optimal solution of $best[i][c]$.

Then we **forget** about i and take items $0, 1, \dots, i - 1$ in the best possible way in the same knapsack.

That is, $best[i][c] = best[i - 1][c]$.

Case 2: item i **does** belong to Knapsack

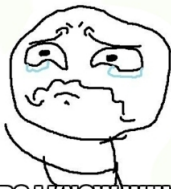
But what if item i belongs to the optimal solution of $best[i][c]$?

In this case we simply start by putting item i into the knapsack.

Then we put items $0, 1, \dots, i - 1$ in the best possible way in a new knapsack of capacity $c - w[i]$.

Thus, $best[i][c] = v[i] + best[i - 1][c - w[i]]$.

BUT... BUT... BUT...



**HOW DO I KNOW WHICH
CASE IT IS?**

memegenerator.net



Well... who cares? We know that item i either is or isn't in the knapsack.

So... just take the best of the two possibilities!

$$best[i][c] = \max(best[i-1][c], v[i] + best[i-1][c - w[i]])$$

Note that $best[i-1][c - w[i]]$ might not be defined if $c < w[i]$.

It remains to solve the easiest sub-problems, when $i = 0$.

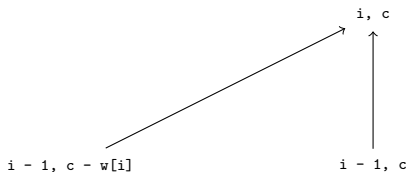
It remains to solve the easiest sub-problems, when $i = 0$.

$$\begin{aligned} best[0][c] &= v[0] && \text{if } c \geq w[0] \\ &0 && \text{otherwise} \end{aligned}$$

It all comes down to graphs

We can also think about sub-problems as nodes in a graph.

The edges link sub-problems as nodes in a graph.



We need to solve the sub-problems in a **topological order** of this graph.

$best[i - 1][c]$ and $best[i - 1][c - w[i]]$ need to be solved before $best[i][c]$.

Implementation

```
1 int n, C;
2 int w[n], v[n];
3
4 int knapsack() {
5     int best[n][C+1];
6
7     // solve the base case (easier sub-problems)
8     for(int c = 0; c <= C; ++c)
9         if(c < w[0])
10             best[0][c] = 0;
11         else
12             best[0][c] = v[0];
13
14     // iterate in the right order and solve all sub-problems
15     for(int i = 1; i < n; ++i)
16         for(int c = 0; c <= C; ++c)
17             if(c < w[i])
18                 best[i][c] = best[i-1][c];
19             else
20                 best[i][c] = max(best[i-1][c], v[i] + best[i-1][c - w[i]]);
21
22     // return biggest problem
23     return best[n-1][C];
24 }
```

For most DP problems, a topological order can be achieved simply with the proper sequencing of some (nested) loops.

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Two approaches

The first approach starts from the hardest sub-problem (the pair $(0, 0)$) and breaks it down into easier sub-problems.

We call that a **Top-Down DP**.

The second approach starts from the easy sub-problems and builds up harder sub-problems upon it.

We call that a **Bottom-Up DP**

Generally, Top-Down is implemented **recursively** and Bottom-Up **iteratively**.

How do you know which one you should use?

Top-Down DP vs Bottom-Up DP

Top-Down:

- + Only computes relevant states.
- + Easier to come up with.

Bottom-Up:


- + Usually possible to reduce the memory.
-  Implement Knapsack with $\mathcal{O}(C)$ memory.
- Solves all sub-problems.

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Let's solve a problem together

UVa 562: Dividing coins

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Shortest path problem

Given a directed, weighted graph G and a vertex s , compute the length of the shortest path from s to all other vertices.

Let's solve this problem using **DP**.

What could be our sub-problems?

Let's try

$sp[v]$ = length of the shortest path from s to v

How does $sp[v]$ relate with other problems?

Let's try

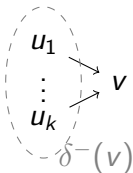
$sp[v]$ = length of the shortest path from s to v

How does $sp[v]$ relate with other problems?

$$sp[s] = 0$$

$$sp[v] = \min_{u \in \delta^-(v)} sp[u] + w(u, v)$$

where $\delta^-(u)$ is the set of in-neighbours of v .



Does this work? **No!**

Let's try

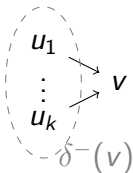
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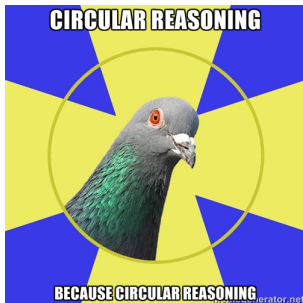
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Does this work? **No!**



If v belongs to a **cycle**, $sp[v]$ depends on $sp[v]$...

For **DP** to work, we need the underlying sub-problem graph to be **acyclic**.

Observe that the **sub-problem** graph is actually... the input graph G .

The above recurrence works for acyclic graphs.

Shortest path for acyclic graphs

$sp[v]$ = length of the shortest path from s to v

$$sp[s] = 0$$

$$sp[v] = \min_{u \in \delta^-(v)} sp[u] + w(u, v) \quad \forall v \in V \setminus \{s\}$$

In what order should we compute the problems?

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We must have computed $sp[u]$ for all $u \in \delta^-(v)$ before we compute $sp[v]$.

\Rightarrow we need to do it in the **topological order** of G .

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\Rightarrow we need to do it in the **topological order** of G .

That is not surprising, we said we **always** evaluate DP states in the topological order of the sub-problem graph.

And in this case it is G .

Sub-problem graph must be acyclic

This just to say that

DP only works if your state space is acyclic!

Be careful defining your state space.

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- ▶ UVa 990
- ▶ UVa 10261

- ▶ UVa 10032