# Segment Tree and Lazy Propagation

beOI Training


July 13, 2016

# Table of Contents

# Motivating problem

You are given an integer array $A$ of size $n$ ($n < 10^6$).
Given two integers $a$ and $b$, can you give the minimum value of $A$ between indices $a$ and $b$ ?

$$\min_{i=a\ldots b} A[i]$$

Well that's easy, just iterate over the interval and return the minimum!

# Motivating problem

You are given an integer array $A$ of size $n$ ($n < 10^6$).
Given two integers $a$ and $b$, can you give give the minimum value of $A$ between indices $a$ and $b$ ?

$$\min_{i=a \ldots b} A[i]$$

# 100000 times?

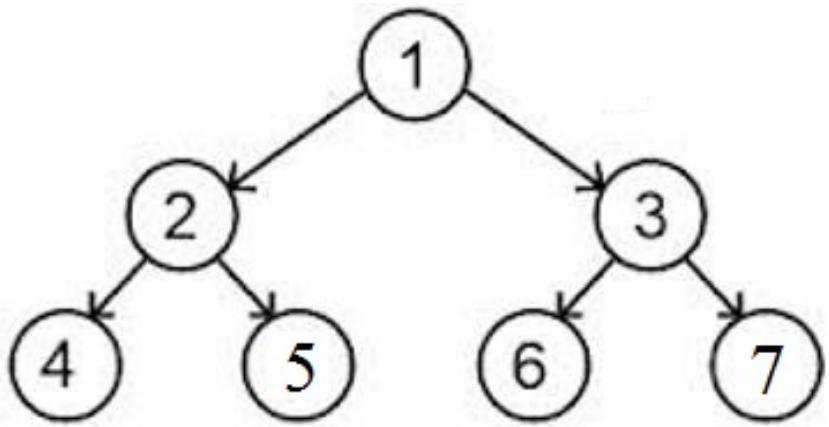
This is called the **range minimum query** (RMQ) problem.

# Naive solution

For each query, iterate over the corresponding range and return the minimum.
If $k$ is the number of queries, time complexity is $\mathcal{O}(nk)$.
TLE

# Array representation of a binary tree

- 1-based array, index 1 = root
- For each node of index $p$,
  - left child has index $2p$
  - right child has index $2p + 1$



# Segment Tree

Each node is responsible of one segment
Root represents the whole array $[0 \cdots n-1]$
Given a node representing segment $[l \cdots r]$

- left child represents the segment's first half $\left[l \cdots \frac{l+r}{2}\right]$
- right child represents the segment's second half $\left[\frac{l+r}{2}+1 \cdots r\right]$

The value of a node will be the **index of the minimum element in segment** $[l \cdots r]$.

# Querying

When we query the minimum value in an interval, we look for **big segments that are contained within the query range**, and among those segments, take the minimum value. Recursively,

- segment is within query range $\Rightarrow$ return value of the node;
- segment and query range are disjoint $\Rightarrow$ do nothing;
- otherwise, return minimum among both children.

# Querying implementation

```
// p is array index of current node,
// [L..R] is current segment,
// [i..j] is search interval
int query(int p, int L, int R, int i, int j) {
    // inside  query range
    if(L >= i && R <= j) return st[p];
    // outside query range
    if(i >  R || j <  L) return -1;

    // compute the min position in the left
    // and right part of the interval
    int p1 = query(2*p, L, (L+R)/2, i, j);
    int p2 = query(2*p+1, (L+R)/2+1, R, i, j);

    // if we try to access segment outside query
    if(p1 == -1) return p2;
    if(p2 == -1) return p1;
    return (A[p1] <= A[p2]) ? p1 : p2;
}
```

# Querying complexity

At each level, at most 4 nodes are visited (see coach for proof).
There are exactly $\lceil \log_2 n \rceil$ levels.

$$\mathcal{O}(4 \times \lceil \log_2 n \rceil) = \mathcal{O}(\log n)$$

Overall complexity $\mathcal{O}(k \log n)$ is now reasonable!
AC

# Building

Building the Segment Tree is also done recursively.
For each node,

- if no child, store current index;
- otherwise,
  - build left child;
  - build right child;
  - store minimum child.

# Building implementation

```
void build(int p, int L, int R) {
    if(L == R)
        // leaf node
        st[p] = L;
    else {
        // build children
        build(2*p, L, (L+R)/2);
        build(2*p+1, (L+R)/2+1, R);
        // take minimum among them
        int p1 = st[2*p], p2 = st[2*p+1];
        st[p] = (A[p1] <= A[p2]) ? p1 : p2;
    }
}
```

# Building complexity

We visit every node once.
In general, the number of nodes is
$N + \frac{N}{2} + \frac{N}{4} + \cdots + 2 + 1 \approx 2N$, so time complexity is

$$\mathcal{O}(2 \times N) = \mathcal{O}(N)$$

This also proves memory is $\mathcal{O}(N)$ (in practice one always takes an array of $4 \times N$ for safety).

# Segment Trees are extremely powerful!

We saw how to solve the range **minimum** query problem.
But we can do much more than that!

- Range maximum query
- Range sum query
- Range *insert any function here* query

# One last operation

Suppose that, between queries, the array is being **updated**.
Naive solution: re-build the Segment Tree in $\mathcal{O}(N)$.
TLE
Segment Trees allow efficient **updating**!

# Updating

To update $p$, we only need to update the segments that contain $p$.
Update the leaf to root path in $\mathcal{O}(\log N)$!

# Updating implementation

```
// i is the node that is to be updated
void update(int p, int L, int R, int i) {
    // if leaf node
    if(L == R) return;
    // if i is in segment
    if(i >= L && i <= R) {
        // if new value is smaller, update
        if(A[i] <= A[st[p]])
            st[p] = i;
        update(2*p, L, (L+R)/2, i);
        update(2*p+1, (L+R)/2+1, R, i);
    }
}
```

# Table of Contents

# Motivating problem

In the Range Sum Query (RSQ) problem, we add one operation: range update.
We want to update a range (e.g. increment every value in range by $k$) efficiently.

# Naive solution

At each range update query, re-build tree in $\mathcal{O}(N)$.
TLE

# Let's be lazy!

Key idea behind lazy Segment Tree: don't update everything at once; put a flag on segments that need to be updated, and leave it for another traversal.

# Propagation

Keep an array lazy that stores for each segment by how much each value needs to be incremented.
Every time we visit a node $p$ (in query or update) where lazy[p] != 0,

- increment current segment by lazy[p] times size of segment;
- if node is not leaf,
  - increment lazy[2*p] by lazy[p]
  - increment lazy[2*p+1] by lazy[p]
- reset lazy[p].

That is called **propagation**.
Obviously, complexity is $\mathcal{O}(1)$.

# Propagation implementation

```
void propagate(int p, int L, int R) {
    if(lazy[p] != 0) {
        st[p] += (R-L+1)*lazy[p];

        if(L != R) {
            lazy[2*p] += lazy[p];
            lazy[2*p+1] += lazy[p];
        }

        lazy[p] = 0;
    }
}
```

# Querying

We do exactly the same, but we propagate at each node!
Complexity $\mathcal{O}(\log N)$.

# Querying implementation

```
int query(int p, int L, int R, int i, int j) {
    propagate(p, L, R);

    if(i > R || j < L) return 0;
    if(L >= i && R <= j) return st[p];

    return query(2*p, L, (L+R)/2, i, j)
         + query(2*p+1, (L+R)/2+1, i, j);
}
```

# Updating

For each node,

- propagate
- if outside of range, return
- if inside of range, set the lazy flag, and return
- otherwise
  - update left child
  - update right child
- merge both children (add them up)

Complexity $\mathcal{O}(\log N)$.

# Updating implementation

```
void update(int p, int L, int R, int i, int j, int k) {
    propagate(p, L, R);

    if(i > R || j < L) return;
    if(L >= i && R <= j) {
        lazy[p] = k;
        return;
    }

    update(2*p, L, (L+R)/2, i, j, k);
    update(2*p+1, (L+R)/2+1, R, i, j, k);

    st[p] = st[2*p] + st[2*p+1];
}
```