

Computational geometry

Guillaume Derval

April 17, 2016

Table of Contents

Basics of 2D geometry

Computational geometry

Polygons

Convex hull

Table of Contents

Basics of 2D geometry

Computational geometry

Polygons

Convex hull

Points

- ▶ Points in 2D: (x, y)
- ▶ Distance between two points $a = (x_1, y_1)$, $b = (x_2, y_2)$:

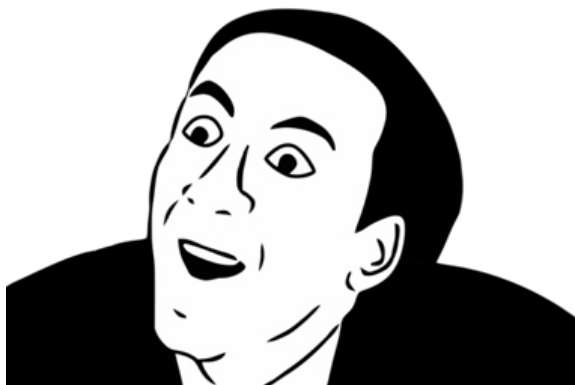
$$\text{dist}(a, b) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Points

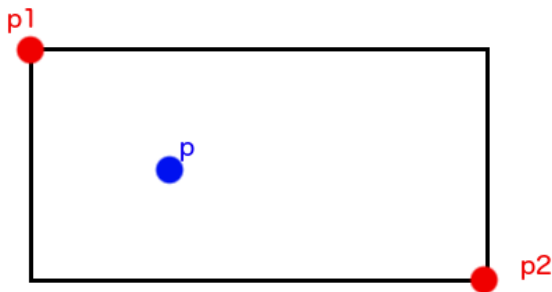
- ▶ Points in 2D: (x, y)
- ▶ Distance between two points $a = (x_1, y_1)$, $b = (x_2, y_2)$:

$$\text{dist}(a, b) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

YOU DON'T SAY?



Check that a point is in a box



```
boolean inBox(Point p1, Point p2, Point p) {  
    return  
        Math.min(p1.x, p2.x) <= p.x &&  
        p.x <= Math.max(p1.x, p2.x) &&  
        Math.min(p1.y, p2.y) <= p.y &&  
        p.y <= Math.max(p1.y, p2.y);  
}
```

Lines

General formula:

$$Ax + By = C$$

The line through $(x_1, y_1), (x_2, y_2)$ is given by:

$$A = y_2 - y_1$$

$$B = x_1 - x_2$$

$$C = Ax_1 + By_1$$

Distance from a point to a line

Euclidian distance from a point (x_0, y_0) to a line $Ax + By = C$ is

$$\frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

Line intersections

Two lines $A_1x + B_1y = C_1$ and $A_2x + B_2y = C_2$ intersects iff

$$d := \det \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \neq 0$$

Line intersections

Two lines $A_1x + B_1y = C_1$ and $A_2x + B_2y = C_2$ intersects iff

$$d := \det \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \neq 0$$

remember that:

$$\det \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} = A_1B_2 - A_2B_1 = d$$

Line intersections

Two lines $A_1x + B_1y = C_1$ and $A_2x + B_2y = C_2$ intersects iff

$$d := \det \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \neq 0$$

remember that:

$$\det \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} = A_1B_2 - A_2B_1 = d$$

The intersection is then given by:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}^{-1} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{d} \begin{pmatrix} B_2 & -B_1 \\ -A_2 & A_1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

For those who don't like matrices

$$\begin{aligned}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}^{-1} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{d} \begin{pmatrix} B_2 & -B_1 \\ -A_2 & A_1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \\ &\rightarrow \\ x &= \frac{B_2 C_1 - B_1 C_2}{d}, \\ y &= \frac{A_1 C_2 - A_2 C_1}{d}\end{aligned}$$

Perpendicular line in 2D

The line perpendicular to $Ax + By = C$ are

$$-Bx + Ay = D \text{ for } D \in \mathbb{R}$$

If you want that the line goes through (x_0, y_0) :

$$D = -Bx_0 + Ay_0$$

Orthogonal symmetry

For a line a , and a point x , find its orthogonal symmetry point x' :

1. Compute the perpendicular b of a that goes through x
2. find the intersection y of a and b
3. $x' = y - (x - y)$

Orientation

$$\text{orient}(p, q, r) = \begin{vmatrix} 1 & p_x & p_y \\ 1 & q_x & q_y \\ 1 & r_x & r_y \end{vmatrix}$$

$$\text{orient}(p, q, r) \begin{cases} = 0 & p, q, r \text{ are collinear} \\ < 0 & p \rightarrow q \rightarrow r \text{ is clockwise} \\ > 0 & p \rightarrow q \rightarrow r \text{ is counterclockwise} \end{cases}$$

$$|\text{orient}(p, q, r)| = 2 \cdot \text{area } \triangle(p, q, r)$$

```
double orient(Point p, Point q, Point r) {  
    return q.x * r.y - r.x * q.y - p.x * (r.y - q.y) +  
        p.y * (r.x - q.x);  
}
```

Angle visibility

x lies strictly inside the angle formed by p, q, r iff

$$\text{sgn}(\text{orient}(p, q, x)) = \text{sgn}(\text{orient}(p, x, r))$$

$$\text{sgn}(\text{orient}(p, r, x)) = \text{sgn}(\text{orient}(p, x, q))$$

To allow it to lie on the border simply check if

$$\text{sgn}(\text{orient}(p, q, x)) = 0 \text{ or } \text{sgn}(\text{orient}(p, r, x)) = 0$$

Triangles

Notations and definitions:

- ▶ sides a, b, c
- ▶ angles α, β, γ
- ▶ perimeter $p = a + b + c$
- ▶ semi-perimeter $s = \frac{p}{2}$
- ▶ Area

$$A = \frac{\text{base} * \text{height}}{2}$$

- ▶ Heron's formula

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

Triangles (2)

- ▶ Inscribed circle radius

$$r = \frac{A}{s}$$

- ▶ Center of inscribed circle: intersection of the bisectors of the angles
- ▶ Circumscribed circle radius:

$$R = \frac{abc}{4A}$$

- ▶ Center of circumscribed circle: intersection of the heights

Triangles (3)

- ▶ Law of sines

$$\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)} = 2R$$

- ▶ Law of cosines

$$c^2 = a^2 + b^2 - 2ab \cos(\gamma)$$

Circles

Definitions and notations

- ▶ Radius r and center $c = (a, b)$
- ▶ Equation that point on the circle respects:

$$(x - a)^2 + (y - b)^2 = r^2$$

- ▶ Diameter

$$D = 2r$$

- ▶ Area

$$A = \pi r^2$$

- ▶ Perimeter

$$p = 2\pi r$$

(also called circumference)

Circles (2)

- ▶ Arc: connected section of the circumference of the circle. Given the angle α made, the size of an arc is

$$\frac{\rho}{\alpha} 2\pi$$

- ▶ Chord: ligne segment whose endpoints are on a circle. Given the angle α made, the size of the chord is:

$$\sqrt{2r^2(1 - \cos(\alpha))}$$

- ▶ Sector: area between two radiuses and the arc between these two radiuses. Given the angle α made:

$$A \frac{\alpha}{2\pi}$$

- ▶ Segment: sector minus the triangle made by the chord

Table of Contents

Basics of 2D geometry

Computational geometry

Polygons

Convex hull

Computational geometry, in one slide

Four things to remember

Computational geometry, in one slide

Four things to remember

- ▶ floating point operations make computation errors

Computational geometry, in one slide

Four things to remember

- ▶ floating point operations make computation errors
- ▶ computing on floating point lead to precision errors

Computational geometry, in one slide

Four things to remember

- ▶ floating point operations make computation errors
- ▶ computing on floating point lead to precision errors
- ▶ computations on floats and doubles are nearly always approximative

Computational geometry, in one slide

Four things to remember

- ▶ floating point operations make computation errors
- ▶ computing on floating point lead to precision errors
- ▶ computations on floats and doubles are nearly always approximative
- ▶ I think you have understood

Example of error

```
printf (" %.20f \n", 3.6);
```

Output:

```
3.600000000000000008882
```

Handling errors

Define an "error threshold" ϵ :

```
boolean eq(double a, double b){return Math.abs(a - b) <=
    E;}
boolean le(double a, double b){return a < b - E;}
boolean leq(double a, double b){return a <= b + E;}
```

Table of Contents

Basics of 2D geometry

Computational geometry

Polygons

Convex hull

Storing a polygon

Polygon = serie of points. Store them in an ordered way, for example, clockwise.

```
Point [] polygon ;
```

Perimeter of a polygon

Simply iterate on all the points

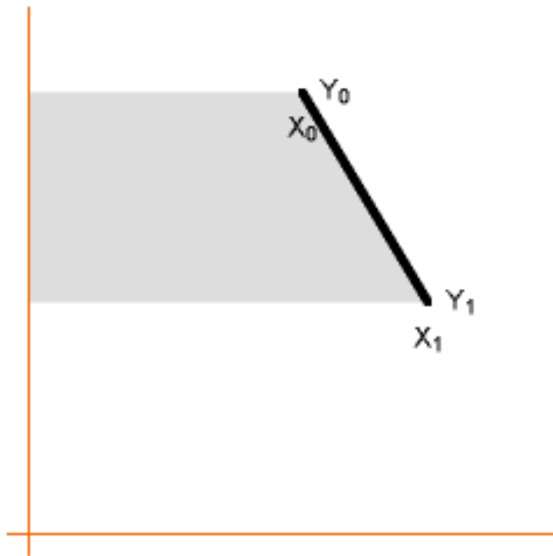
```
double p = 0;
for(int i = 0; i < polygon.length; i++) {
    p += dist(polygon[i], polygon[(i+1)%polygon.length]);
}
```


Area of a polygon

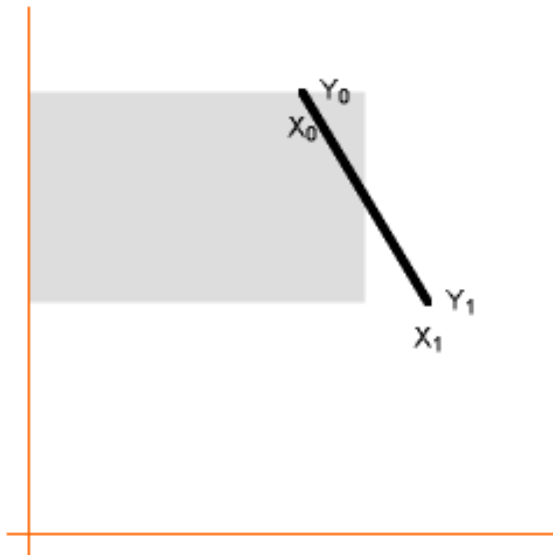
Area of a polygon

Idea: get the area below all successive pair of points, and add/subtract it depending on orientation

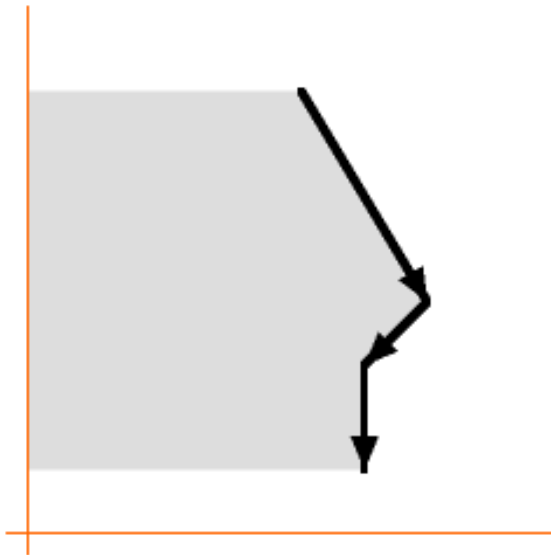
Area of a polygon (2)



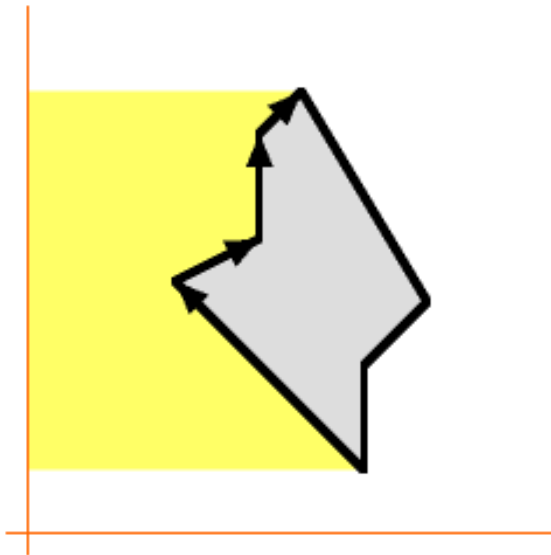
Area of a polygon (3)



Area of a polygon (4)



Area of a polygon (5)



Area of a polygon (6)

$$A = \frac{1}{2} \begin{vmatrix} x_0 & y_0 \\ x_1 & y_1 \\ x_2 & y_2 \\ \dots & \dots \\ x_n & y_n \end{vmatrix} = \frac{1}{2} * (x_0 y_1 + x_1 y_2 + \dots + x_n y_0 - x_1 y_0 - x_2 y_1 - \dots - x_0 y_n)$$

Area of a polygon (7)

```
double result = 0;
for(int i = 0; i < polygon.length() - 1; i++) {
    result += (P[i].x * P[i+1].y - P[i+1].x * P[i].y);
}
```


Checking if a polygon is convex

→ check that $\text{orient}(p,q,r)$ is always positive for all successive triplet of points

Check if point is inside a polygon

→ compute the angle made by all the successive pairs of points and the point we want to check. If it is 360 degrees, the point is inside the polygon.

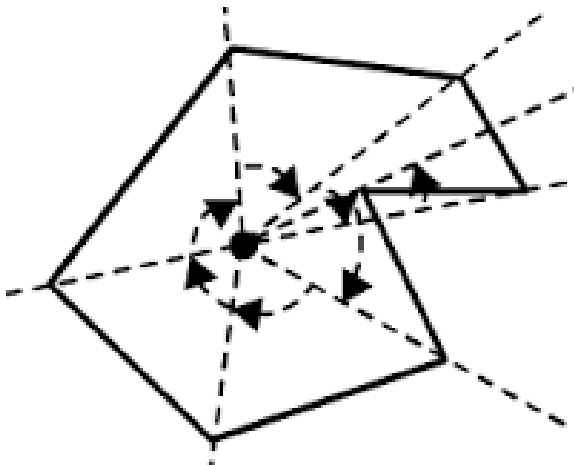


Table of Contents

Basics of 2D geometry

Computational geometry

Polygons

Convex hull

Convex Hull

Given a set of points in the plane, compute the smallest convex polygon in the plane that contains all the points.

Graham Scan

Method of computing the convex hull of a finite set of points in the plane with time complexity $O(n \log(n))$. The algorithm finds all vertices of the convex hull ordered along its boundary.

Graham Scan

Idea

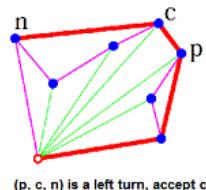
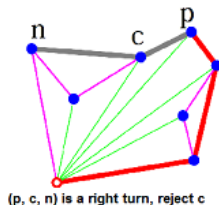
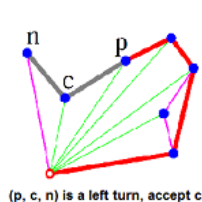
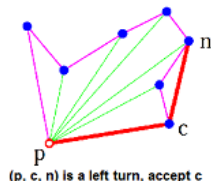
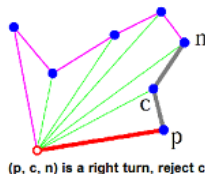
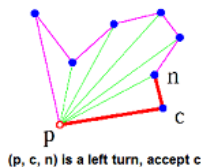
- ▶ find the point with the lowest y-coordinate. if there are multiple points with the lowest y-coordinate, the pick that one of them with the lowest x-coordinate. call this point P
- ▶ now sort all the points in increasing order of the angle they and point P make with the x-axis
- ▶ consider each point in the sorted array in sequence. For each point determine whether coming from the 2 previous points it makes a left of a right turn.
- ▶ if it makes a left turn, proceed with the next point
- ▶ if it makes a right turn, the second-to-last point is not part of the convex hull and should be removed from the convex hull, continue this removing for as long as the last 3 points make up a right turn

Graham Scan

p: previous

c: current

n:next



In the above algorithm and below code, a stack of points is used to store convex hull points. With reference to the code, p is next-to-top in stack, c is top of stack and n is points[i].

Graham Scan

Direction of the turn

To determine whether 3 points constitute a left or a right turn we do not have to compute the actual angles but we can use a cross product.

Consider the 3 points (x_1, y_1) , (x_2, y_2) and (x_3, y_3) , which we will call P_1 , P_2 and P_3 .

Now compute the z-component of the cross product of the vectors P_1P_2 and P_1P_3 . Which is given by the expression

$$(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1).$$

If the result is 0, the points are collinear. If the result is positive, the 3 points constitute a left turn. If the result is negative, the 3 points constitute a right turn.

Sorting points by angle

Here is a comparator (in C++)

```
point pivot(0, 0);
bool angleCmp(point a, point b) {
    if(orient(pivot, a, b) == 0) //collinear: return
the closer one
    return dist(pivot, a) < dist(pivot, b);
    double d1x = a.x - pivot.x, d1y = a.y - pivot.y;
    double d2x = b.x - pivot.x, d2y = b.y - pivot.y;
    return (atan2(d1y, d1x) - atan2(d2y, d2x)) < 0;
//compare angles
}
```

Exercices

Do them in this order!

- ▶ 634
- ▶ 10060
- ▶ 478
- ▶ 681
- ▶ 109