# Dynamic Programming I

## Top-Down, Bottom-Up and Classical Problems

beOI Training
(many thanks to François Aubry)

OLYMPIADE BELGE D'INFORMATIQUE
BELGISCHE INFORMATICA-OLYMPIADE

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# Let's start with an example

**Partition Problem**:

Given a set of $n \leq 50$ goodies each with value $v[i] \leq 10$, is it possible to divide them between 3 persons evenly?

# Strategy

Put yourself in the shoes of the one who divides the goodies.

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2. At the end we need to know if the division is fair. Do we need to know the items given to each person? **No**.

   $\Rightarrow$ Just keep track of how much was given to each person.

# Brute-Force solution

```cpp
bool complete_search(int i, int given1, int given2, int given3) {
    if(i == n) // all goodies have been considered
        return given1 == given2 && given2 == given3;
    else {
        bool giveTo1 = complete_search(i+1, given1 + v[i], given2, given3);
        bool giveTo2 = complete_search(i+1, given1, given2 + v[i], given3);
        bool giveTo3 = complete_search(i+1, given1, given2, given3 + v[i]);
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Complexity? $\mathcal{O}(3^n)$ TLE

We can do better...

# State space and state graph

We call one tuple $(i, given1, given2, given3)$ a **state**.

We can now define the **state graph**: its vertices are the states, and there is an edge from $s_1$ to $s_2$ if $s_1$ calls $s_2$ recursively.

How many nodes does the state graph of the previous algorithm have?

$\mathcal{O}\left(n \cdot S^3\right)$ where $S$ is the sum of the goodie values.

But the algorithm is $\mathcal{O}\left(3^n\right)$. What's going on?

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**Each state is visited several times $\Rightarrow$ waste of time!**

What we want to do is to traverse the state graph (DFS like).

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```
int canDo[n][S][S][S];
// undefined: -1
// we cannot divide evenly from that state: 0
// we can    divide evenly from that state: 1

bool solve(int i, int given1, int given2, int given3) {
    if(i == n) // all goodies have been considered
        return given1 == given2 && given2 == given3;

    if(canDo[i][given1][given2][given3] != -1) {
        // the state (i, given1, given2, given3) has already been visited
        return canDo[i][given1][given2][given3];
    } else {
        bool giveTo1 = solve(i+1, given1 + v[i], given2, given3);
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        // save the value for later
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        return canDo[i][given1][given2][given3];
    }
}
```

Is this enough to get AC? **No**.
The graph has $\approx n \cdot S^3 = 50 \cdot 500^3 = 6250000000$ nodes $\Rightarrow$ MLE.


Or can we make it work?

# State space reduction

Observe that at the end $given3 = S - given1 - given2$.

Thus we can drop one parameter and reduce the state space to $n \cdot S^2$.

```
int canDo[n][S][S];
// undefined: -1
// we cannot divide evenly from that state: 0
// we can    divide evenly from that state: 1
int S; // sum of the v[i]

bool solve(int i, int given1, int given2) {
    if(i == n) { // all goodies have been considered
        int given3 = S - given1 - given2;
        return given1 == given2 && given2 == given3;
    }
    if(canDo[i][given1][given2] != -1) {
        // the state (i, given1, given2) has already been visited
        return canDo[i][given1][given2];
    } else {
        bool giveTo1 = solve(i+1, given1 + v[i], given2);
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        // save the value for later
        canDo[i][given1][given2] = giveTo1 || giveTo2 || giveTo3;
        return canDo[i][given1][given2];
    }
}
```

# What we learned so far

1. View the problem as a **sequence of choices**.
2. Represent the problem with the smallest state space possible.
3. Perform a DFS on the state graph (remembering visited states).

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Let's see another example!

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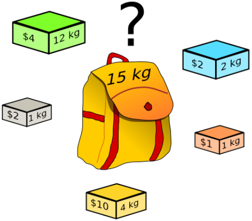
# Knapsack Problem

Given a set of $n$ objects each with value $v[i]$ and weight $w[i]$, and a knapsack that can hold a total capacity of $C$.

Choose a subset of objects that fits into the knapsack and has maximum total value.



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- Succession of choices: for each item, take it or leave it.
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State space? $(i, wtaken)$

- $i = item\ we\ are\ considering$
- $wtaken = total\ weight\ of\ the\ items\ we\ selected\ so\ far$

Size of the state space?

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- Order does not matter.

State space? $(i, wtaken)$

- $i$ = *item we are considering*
- $wtaken$ = *total weight of the items we selected so far*

Size of the state space? $\mathcal{O}(n \cdot C)$

Successors of state $(i, wtaken)$?

Successors of state $(i, wtaken)$?

$$
\begin{tikzcd}
 & (i+1, wtaken) \\
(i, wtaken) \arrow[ur, "\text{do not take item } i"] \arrow[dr, "\text{take item } i", "\text{if } wtaken + w[i] \leq C"'] & \\
 & (i+1, wtaken + w[i])
\end{tikzcd}
$$

Recurrence relation?

Recurrence relation?

$$f(i, wtaken) = \max\Big(f(i{+}1, wtaken), v[i]{+}f(i{+}1, wtaken{+}w[i])\Big)$$

Beware of the knapsack constraint! If $wtaken > C$, the knapsack has no value.

$$wtaken > C \Rightarrow f(i, wtaken) = -\infty$$

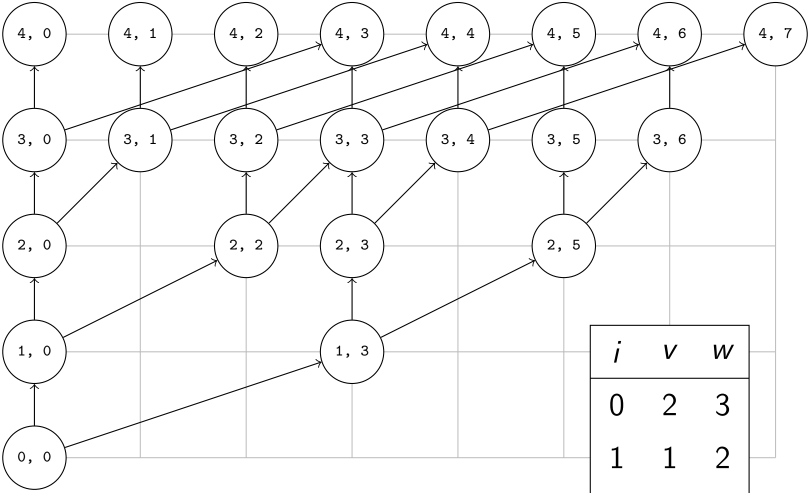
# Knapsack solution

```
int C, n;
int w[n], v[n];
int memo[n][C];

int solve(int i, int wtaken) {
    if(i == n) {
        // we cannot take any object anymore
        return 0;
    } else if(wtaken > C) {
        // knapsack is invalid
        return -INF;
    }

    if(memo[i][wtaken] != -1)
        return memo[i][wtaken];
    else {
        memo[i][wtaken] = max(solve(i+1, wtaken), v[i] + solve(i+1, wtaken - w[i]));
        return memo[i][wtaken];
    }
}
```

# Example of a Knapsack state space.




| $i$ | $v$ | $w$ |
|---|---|---|
| 0 | 2 | 3 |
| 1 | 1 | 2 |
| 2 | 2 | 1 |
| 3 | 3 | 3 |

# Memory optimization

Observe that we don't need all the entries from the `memo` table.

Sometimes the table is too big and causes MLE.

In this situation an alternative is to use a `HashMap` (or `unordered_map`).

This way we only use the memory we need.

# Another approach

This is one view of Dynamic Programming. Usually refered to as **memoization**.

Another view is to decompose the problem into **sub-problems**.

Define an order on the sub-problems: The bigger the harder.

Express the solution of large sub-problems as a function of smaller sub-problems.

Solve from smaller to larger using the function.

# Let's redo the Knapsack this way

Decomposition into sub-problems:

$$\begin{aligned} best[i][c] = &\text{ best way to take objects } 0, 1, \cdots, i \\ &\text{ on a knapsack of capacity } c \end{aligned}$$

Observe that the real problem we want to solve is $best[n-1][C]$.

The idea is that maybe $best[i][c]$ relates to $best[i-1][c']$.

# Case 1: item $i$ **does not** belong to Knapsack

Suppose we know ***magically*** that item $i$ **does not belong** to the optimal solution of $best[i][c]$.

Then we **forget** about $i$ and take items $0, 1, \cdots, i-1$ in the best possible way in the same knapsack.

That is, $best[i][c] = best[i-1][c]$.

# Case 2: item $i$ **does** belong to Knapsack

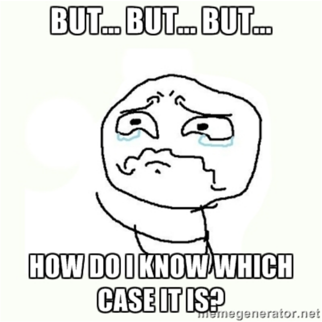
But what if item $i$ belongs to the optimal solution of $best[i][c]$?

In this case we simply start by putting item $i$ into the knapsack.

Then we put items $0, 1, \cdots, i-1$ in the best possible way in a new knapsack of capacity $c - w[i]$.

Thus, $best[i][c] = v[i] + best[i-1][c - w[i]]$.


BUT... BUT... BUT...
HOW DO I KNOW WHICH CASE IT IS?
memegenerator.net




Well... who cares? We know that item $i$ either is or isn't in the knapsack.

So... just take the best of the two possibilities!

$$best[i][c] = \max\left(best[i-1][c], v[i] + best[i-1][c-w[i]]\right)$$

Note that $best[i-1][c-w[i]]$ might not be defined if $c < w[i]$.

It remains to solve the easiest sub-problems, when $i = 0$.

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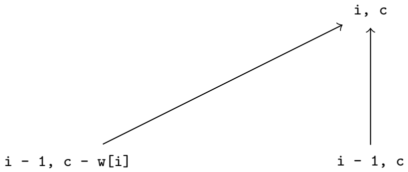
$$
best[0][c] = \begin{array}{ll} v[0] & \text{if } c \geq w[0] \\ 0 & \text{otherwise} \end{array}
$$

# It all comes down to graphs

We can also think about sub-problems as nodes in a graph.

The edges link sub-problems as nodes in a graph.




We need to solve the sob-problems in a **topological order** of this graph.

$best[i-1][c]$ and $best[i-1][c-w[i]]$ need to be solved before $best[i][c]$.

# Implementation

```
int n, C;
int w[n], v[n];

int knapsack() {
    int best[n][C+1];

    // solve the base case (easier sub-problems)
    for(int c = 0; c <= C; ++c)
        if(c < w[0])
            best[0][c] = 0;
        else
            best[0][c] = v[0];

    // iterate in the right order and solve all sub-problems
    for(int i = 1; i < n; ++i)
        for(int c = 0; c <= C; ++c)
            if(c < w[i])
                best[i][c] = best[i-1][c];
            else
                best[i][c] = max(best[i-1][c], v[i] + best[i-1][c - w[i]]);

    // return biggest problem
    return best[n-1][C];
}
```

For most DP problems, a topological order can be achieved simply with the proper sequencing of some (nested) loops.

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# Two approaches

The first approach starts from the hardest sub-problem (the pair $(0, 0)$) and breaks it down into easier sub-problems.

We call that a **Top-Down DP**.

The second approach starts from the easy sub-problems and builds up harder sub-problems upon it.

We call that a **Bottom-Up DP**

Generally, Top-Down is implemented **recursively** and Bottom-Up **iteratively**.

How do you know which one you should use?

# Top-Down DP vs Bottom-Up DP

Top-Down:

+ Only computes relevant states.

+ Easier to come up with.

Bottom-Up:

+ Usually possible to reduce the memory.

✎Implement Knapsack with $\mathcal{O}(C)$ memory.

– Solves all sub-problems.

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## Let's solve a problem together

# UVa 562: Dividing coins

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# Shortest path problem

Given a directed, weighted graph $G$ and a vertex $s$, compute the length of the shortest path from $s$ to all other vertices.

Let's solve this problem using **DP**.

What could be our sub-problems?

Let's try

$$sp[v] = \text{length of the shortest path from } s \text{ to } v$$

How does $sp[v]$ relate with other problems?

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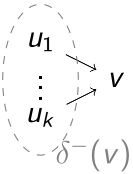
$$sp[s] = 0$$

$$sp[v] = \min_{u \in \delta^-(v)} sp[u] + w(u, v)$$

where $\delta^-(u)$ is the set of in-neighbours of $v$.

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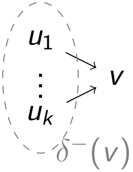


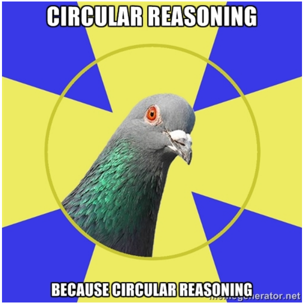


Does this work? **No!**

If $v$ belongs to a **cycle**, $sp[v]$ depends on $sp[v]$...

For **DP** to work, we need the underlying sub-problem graph to be **acyclic**.

Observe that the **sub-problem** graph is actually...the input graph $G$.

The above recurrence works for acyclic graphs.

# Shortest path for acyclic graphs

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We must have computed $sp[u]$ for all $u \in \delta^-(v)$ before we compute $sp[v]$.

$\Rightarrow$ we need to do it in the **topological order** of $G$.

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$\Rightarrow$ we need to do it in the **topological order** of $G$.

That is not surprising, we said we **always** evaluate DP states in the topological order of the sub-problem graph.
And in this case it is $G$.

# Sub-problem graph must be acyclic

This just to say that

**DP only works if your state space is acyclic!**

Be careful defining your state space.

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# Exercises

- UVa 10130
- UVa 562
- UVa 624

- UVa 990
- UVa 10261

- UVa 10032