Specialized DFS

cycles, bridges, articulation points

beOI Training
(slides by François Aubry)

OLYMPIADE BELGE D'INFORMATIQUE
BELGISCHE INFORMATICA-OLYMPIADE

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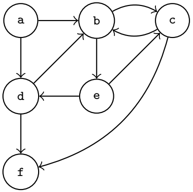
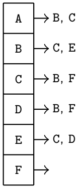
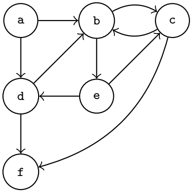
```
enum state {UNVISITED, OPENED, CLOSED};
vector<int> adj[N];
state st[N];

void dfs(int u) {
    st[v] = OPENED;
    for (int v : adj[u])
        if (st[u] == UNVISITED)
            dfs(v);
    st[v] = CLOSED;
}

// in main()
fill(st, st+n, UNVISITED);
for (int u = 0; u < n; i++)
    if (st[u] == UNVISITED)
        dfs(u);
```

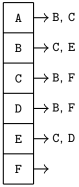
An extended version of DFS: remember if node is finished or not





DFS execution from node a





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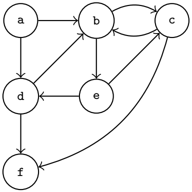

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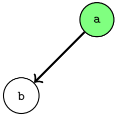

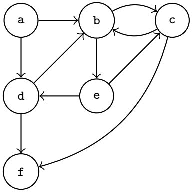

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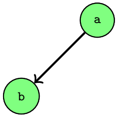

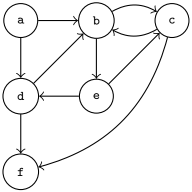

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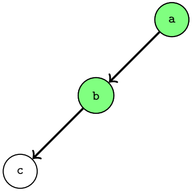

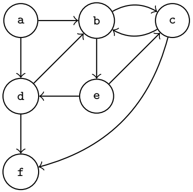

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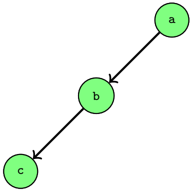

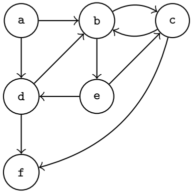

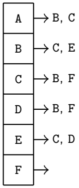

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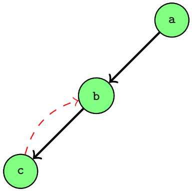

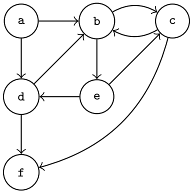

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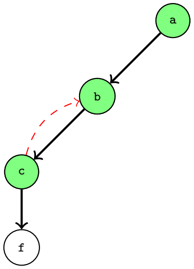

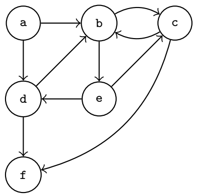

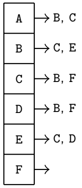

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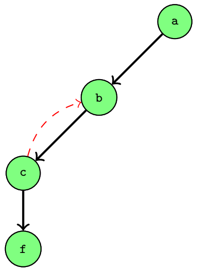

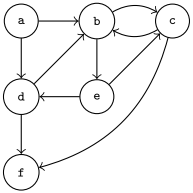

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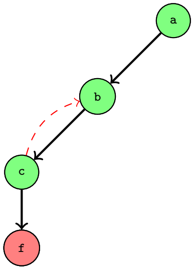

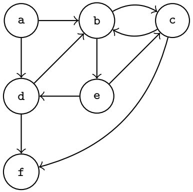

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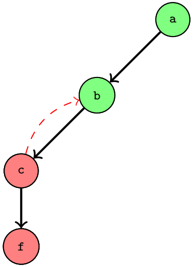

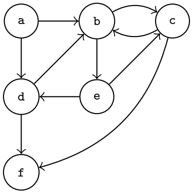

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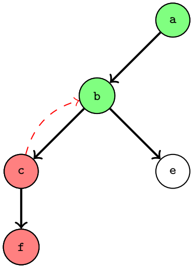

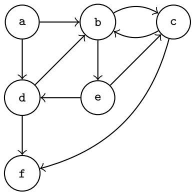

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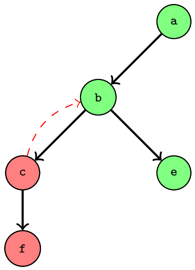

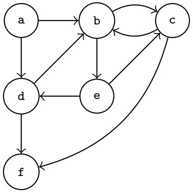

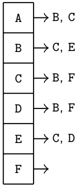

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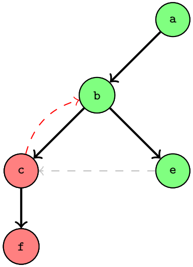

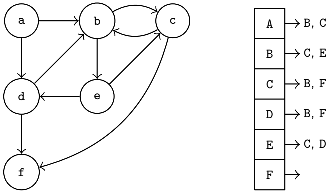

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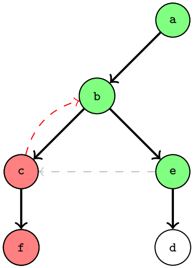

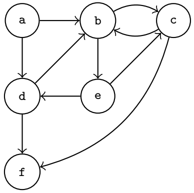

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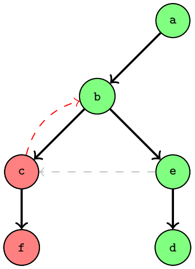

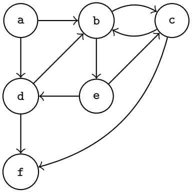

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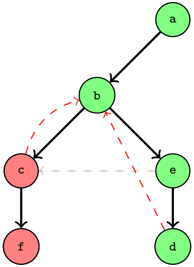

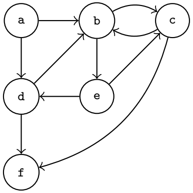

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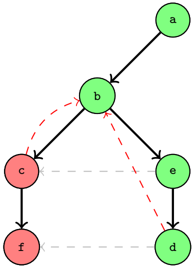

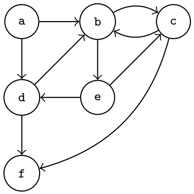

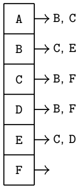

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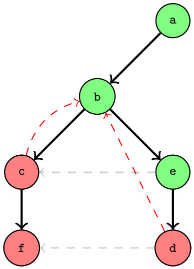

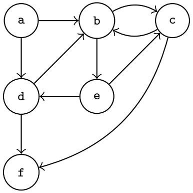

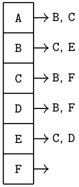

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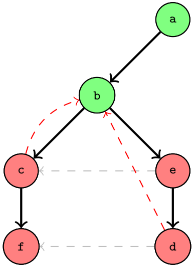

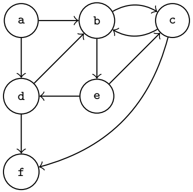

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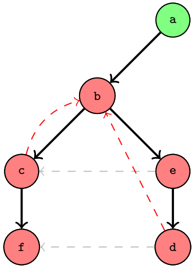

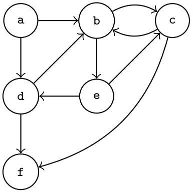

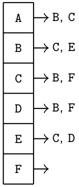

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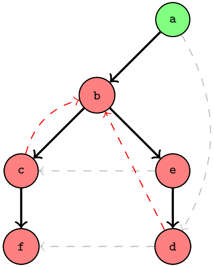

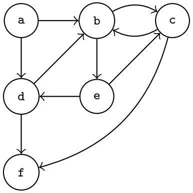

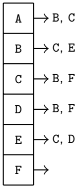

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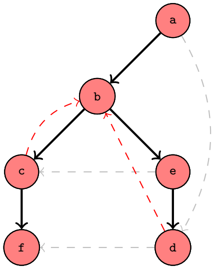

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How can we detect with DFS whether a graph is acyclic or not?

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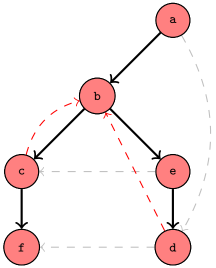

The red edges belong to cycles, they are called **back edges**.

A graph is acyclic if and only if DFS does not yield back edges.

How to distiguish these edges from the others?

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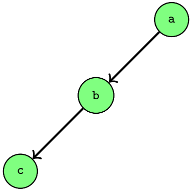

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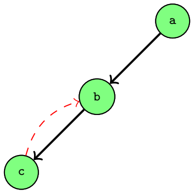

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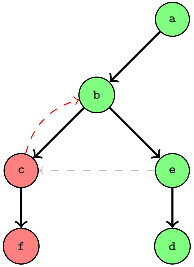

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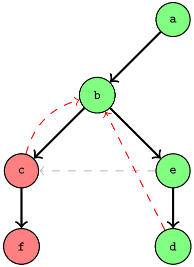

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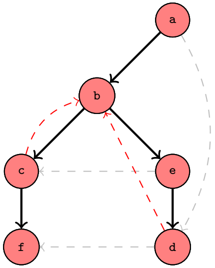

They are edges from a green (OPENED) node to another green (OPENED) node!

To implement this we simply add a check while listing neighbours.

Checks if node *u* (which is OPENED) points to another OPENED node.

```
bool hasCycle = false;

void dfs(int u) {
    st[v] = OPENED;
    for (int v : adj[u]) {
        if (st[u] == UNVISITED)
            dfs(v);
        else if (st[u] == OPENED)
            hasCycle = true;
    }
    st[v] = CLOSED;
}
```

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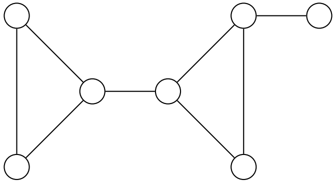
A **bridge** in an undirected graph is an edge such that its removal disconnects the graph.



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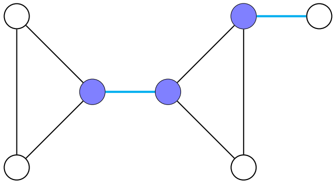

— bridges

● articulation points

How would you compute this?

How would you compute this?

Naive algorithm (for bridges):

For every edge (x, y), remove it and check with BFS or DFS whether x and y remain connected.

Complexity: $O(E \cdot (V + E)) = O(E \cdot V + E^2)$ TLE in big graphs.

We will solve it in **linear time** with a single DFS.

Observation:

Bridges can never belong to cycles.

Is this true for articulation points?

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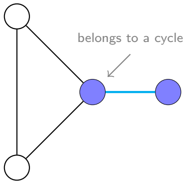
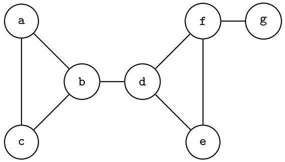
Bridges can never belong to cycles.

Is this true for articulation points? **NO**.



So we essentially need to find which edges belong to cycles.

We already say that DFS allows to find cycles.



A	→ B, C
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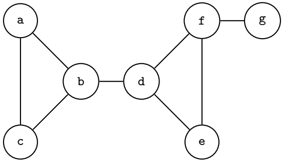
C	→ A, B
D	→ B, E, F
E	→ D, F
F	→ D, E, G
G	→ F

DFS execution from node a



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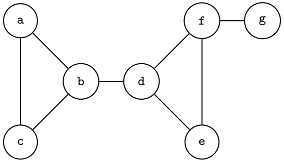

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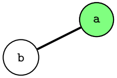
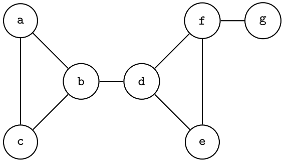
DFS execution from node a





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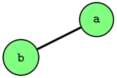
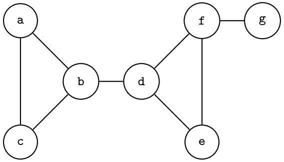
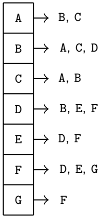
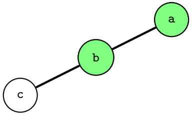
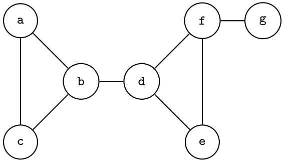






DFS execution from node a





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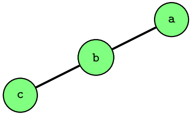

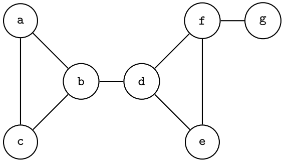

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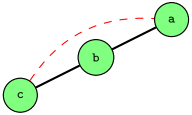

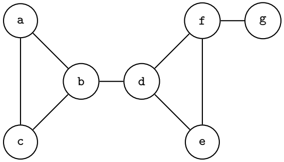

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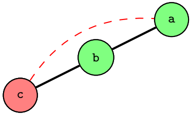

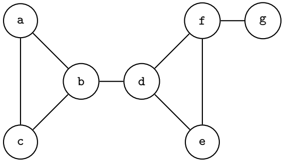

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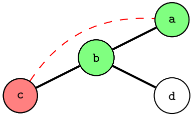
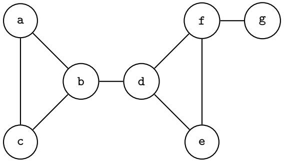
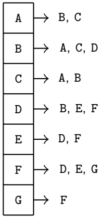
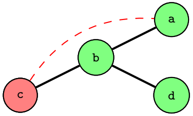
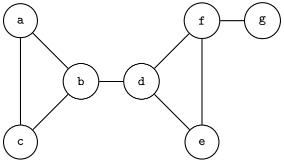


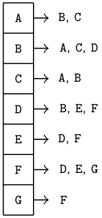




DFS execution from node a







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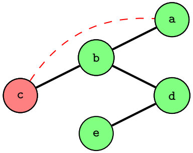

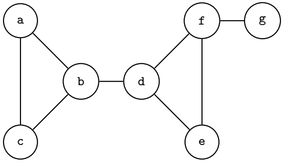

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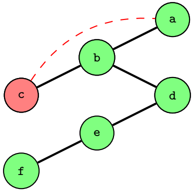

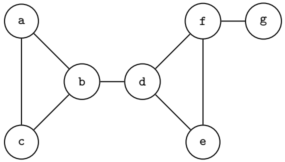

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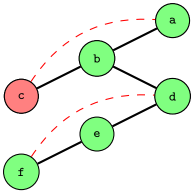

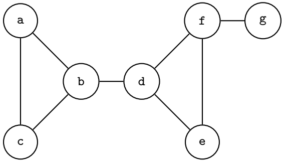

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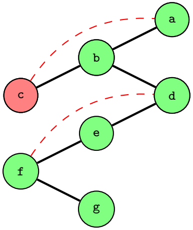

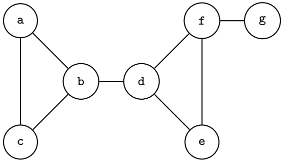

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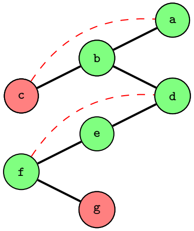

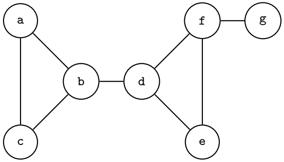

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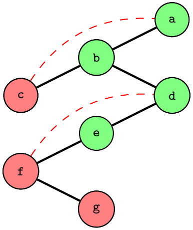

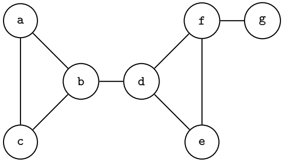

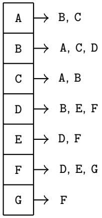

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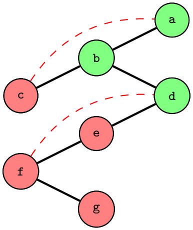

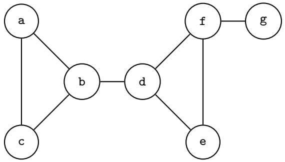

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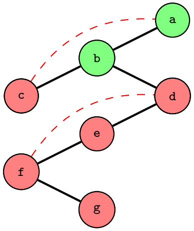
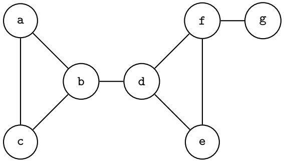
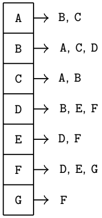
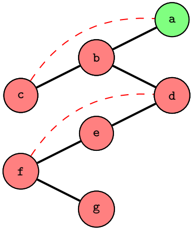
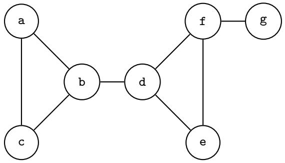






DFS execution from node a





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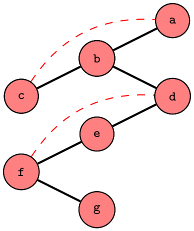

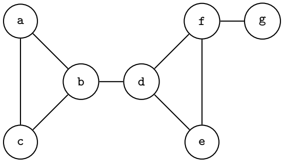

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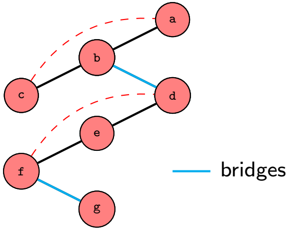
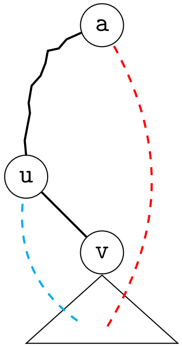


We observe that an edge (u, v) is a bridge if and only if:

No node in the sub-tree of v has a link to u **or** one of its ancestors other than (u, v).



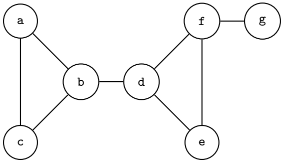
What should we add to our DFS to know this information?

We will **timestamp** the nodes as they are visited: *num*

Keep track of the node with minimum timestamp that we see: *low*

(u, v) is a bridge if and only if when we finish v, $low[v] > num[u]$

This is so because this means that in the sub-tree of v we did not see any ancestor of u.



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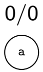
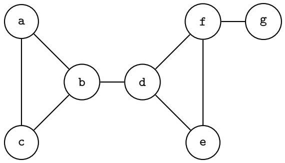


— bridges



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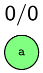
DFS execution from node a



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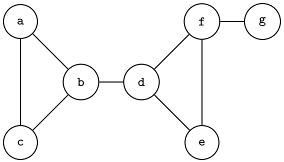

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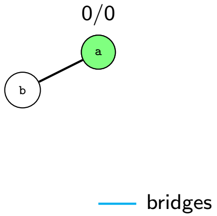

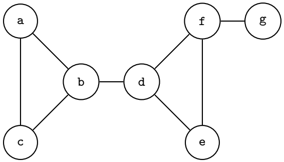

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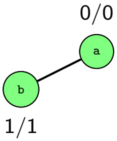
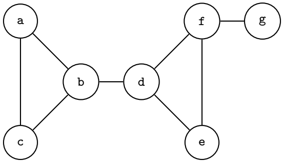


─ bridges



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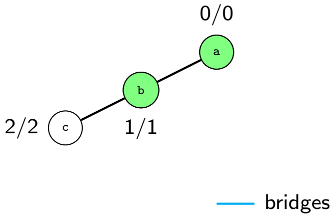

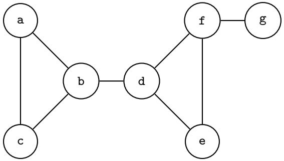

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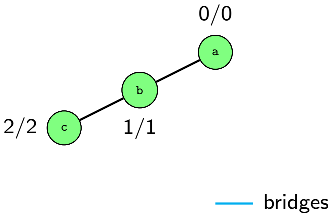

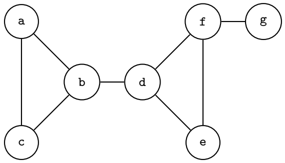

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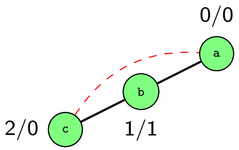
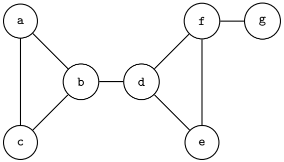
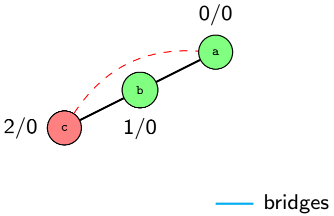


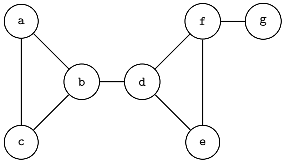
━ bridges



A	B, C
B	A, C, D
C	A, B
D	B, E, F
E	D, F
F	D, E, G
G	F

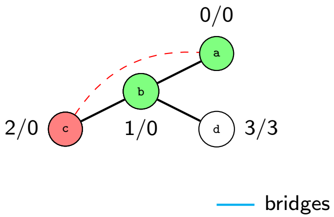
DFS execution from node a

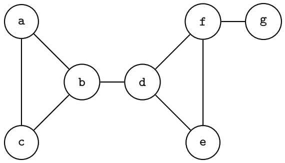




A	$\rightarrow$ B, C
B	$\rightarrow$ A, C, D
C	$\rightarrow$ A, B
D	$\rightarrow$ B, E, F
E	$\rightarrow$ D, F
F	$\rightarrow$ D, E, G
G	$\rightarrow$ F

DFS execution from node a





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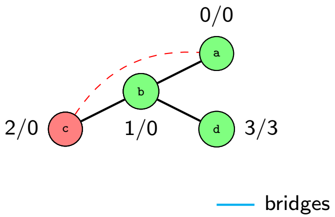
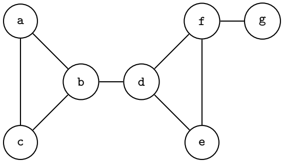
DFS execution from node a





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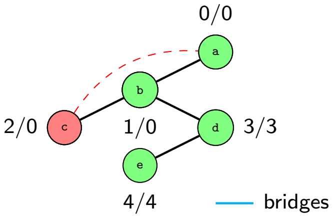
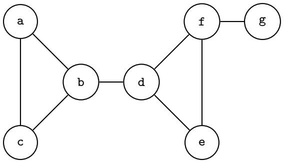
DFS execution from node a





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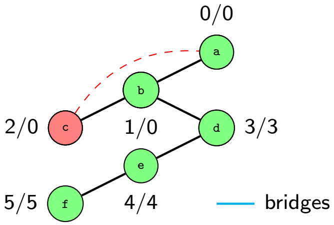
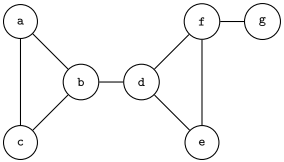
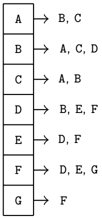
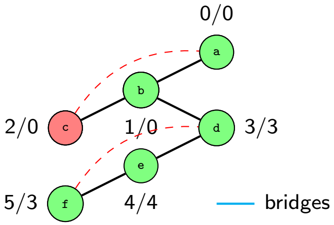
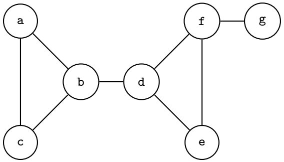






DFS execution from node a





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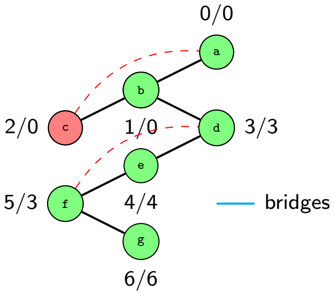
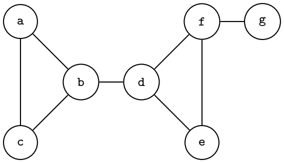
DFS execution from node a





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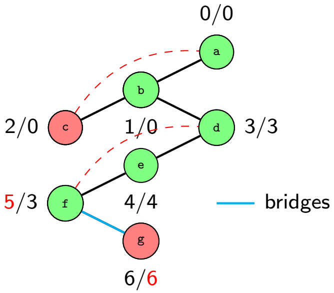
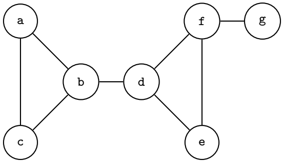
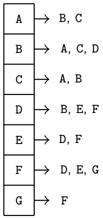
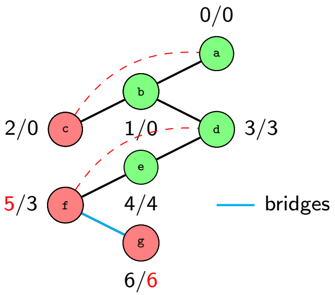
DFS execution from node a

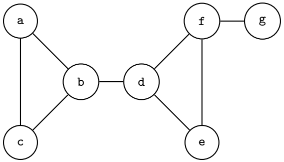






DFS execution from node a





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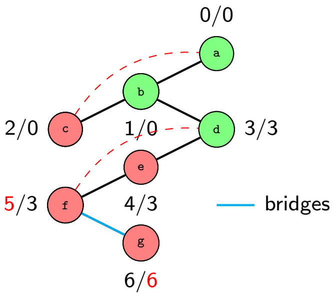

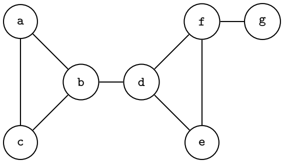

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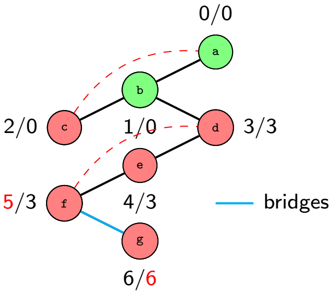
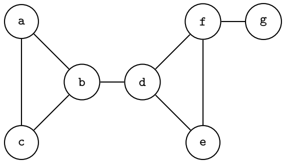
DFS execution from node a





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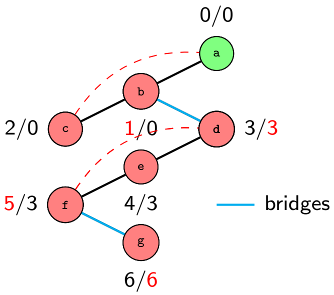

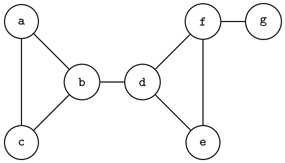

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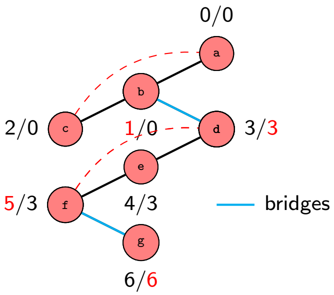
DFS execution from node a

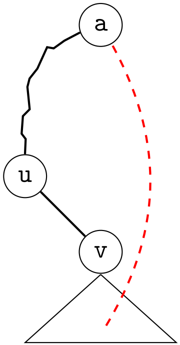


So what about **articulation points**?

So what about **articulation points**?

We observe that a node u is an articulation point if and only if:

No node in the sub-tree of v has a link to one ancestor of u other than (u, v).

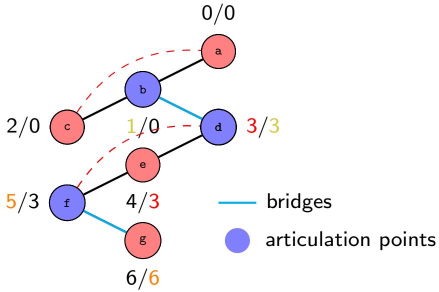


Note that a link to u does not help here.

What does this mean in terms of *num* and *low*?

What does this mean in terms of *num* and *low*?

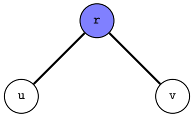
A non-root node u is an articulation point if and only if when we finish v, $low[v] \geq num[u]$



What about the root?

What about the root?

Root is articulation if and only if it has more than one child.



Removing node r disconnects the graph.

If there was another path from u to v, v would not be a child of r.

```
int num[N], low[N], root, rootChildren, cnt = 0;

void dfs(int u, int parent) {
    num[u] = low[u] = cnt++;
    vis[u] = true;
    for (int v : adj[u]) {
        if (!vis[u]) {
            dfs(v, u);
            if (u == root) rootChildren++;
            if (low[v] >= num[u] && u != root)
                // u is an articulation point
            if (low[v] > num[u])
                // (u,v) is a bridge
            low[u] = min(low[u], low[v]);
        } else if (v != parent)
            low[u] = min(low[u], num[v]);
    }
}

// loop in main():
if (!vis[u]) {
    root = u, rootChildren = 0;
    dfs(u, -1);
    if(rootChildren > 1)
        // u is an articulation point
}
```