PAPER *Special Section on Discrete Mathematics and Its Applications*

# The 3D-Packing by Meta Data Structure and Packing Heuristics

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**SUMMARY** The three dimensional (3D) packing problem is to arrange given rectangular boxes in a rectangular box of the minimum volume without overlapping each other. As an approach, this paper introduces the system of three sequences of the box labels, the sequence-triple, to encode the topology of the 3D-packing. The topology is the system of relative relations in pairs of boxes such as right-of, above, front-of, etc. It will be proved that the sequence-triple represents the topology of the tractable 3D-packings which is a 3D-packing such that there is an order of the boxes along which all the boxes are extracted one by one in a certain fixed direction without disturbing other remaining boxes. The idea is extended to the system of five ordered sequences, the sequence-quintuple. A decoding rule is given by which any 3D-packing is represented. These coding systems are applied to design heuristic algorithms by simulated annealing which search the codes for better 3D-packings. Experimental results were very convincing its usefulness as automated packing algorithms.

***key words:*** *3D-packing, placement, sequence-pair, sequence-triple, simulated annealing*

## 1. Introduction

The 3D-packing is to arrange a given set of 3-dimensional rectangular objects (parallelepipeds, or simply, boxes) without overlapping each other in a 3-dimensional rectangular box, the package. Our concern is to develop an efficient way for 3D-packing with the objective of minimizing the volume of the package. Any efficient automated 3D-packing algorithm is expected very useful in many applications, for example in packing loads in as small box. However, this problem is NP-hard since its 2D version was proved NP-hard [2]. There have been no contributions except [3]–[5], which are theoretical results to give upper-bounds of approximation of their proposed algorithms.

Our approach is different from [3]–[5] but a faithful extension of [1], [2]. First, we try to establish a coding system of the topology of the boxes in a 3D-packing. Here the topology is a set of relative relations assigned to pairs of boxes in such a way that one box shall be either "left-of," "above," or "front-of" the other. Then, letting the topology as the constraint, the 1D compaction is applied along each of three orthogonal axes, the x-, y-, and z-axis.

This framework was established in [1], [2], [6], [7] where the prime concern is to fix the coding system of the 2D-packing such that a code is easily decoded to the unique topology and so, to a unique packing and that there is a code which leads to an optimal solution, i.e. a minimum area packing. The coding by the sequence-pair was successful in this sense. It is applied to implement a stochastic search algorithm. Experiments showed a remarkable performance which is believed to have achieved a breakthrough practically. Even a set of thousand boxes (rectangles) was packed by a flat computation of several hours with a quality enough for any practical use. Our motivation is very simple. Let the sequence-pair be a sequence-triple, or sequence-$k$-tuple if necessary for 3D-packing.

We start with the coding by the sequence-triple. It will be reasoned however, together with counter examples, that it codes part of the limited class of 3D-packings such that there is an ordering of boxes along which all the boxes are able to be unloaded one after another without moving other remained boxes.

For the minimum volume 3D-packing, based on the observation of the feature of the sequence-triple, we develop a coding by the system of five sequences, the sequence-quintuple. It is proved that there is a sequence-quintuple which is decoded to a topology corresponding to an optimal 3D-packing.

A simulated annealing algorithm is implemented. It generates sequence-triples or sequence-quintuples one by one and evaluates each by the volume of the corresponding package. Experiments showed that one hundred of boxes were very satisfiably packed.

## 2. Preliminaries

For convenience to refer to the position of a box, assume that our 3D space is in a coordinate system $(x, y, z)$ as shown in Fig. 1, where the x-, y-, and z- axis indicate right-left, rear-front, and above-below directions, respectively. The topology of a 3D packing is a system of relative relations between pairs of boxes in such a way as: box $a$ is said to be left-of box $b$ when any point of $a$ is left-of any point of $b$, i.e. the x-coordinate of the former is not larger than that of $b$. Relations

Manuscript received July 28, 1999.
Manuscript revised October 5, 1999.

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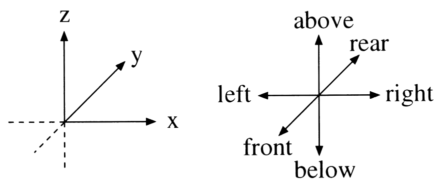


**Fig. 1** 3D space with orthogonal coordinate system.

"right-of," "above," "below," "front-of," and "rear-of" are analogously defined.

Before getting into coding of 3D-packings, we start with a briefing of the theory of 2D-packing by the sequence-pair since our 3D-packing goes along with it.

Suppose $n$ boxes (rectangles) are given, each with height and width. A sequence-pair, or seq-pair, is an ordered pair of sequences of all box labels as $SP = (\Gamma_+, \Gamma_-)$. $\Gamma_*$ ($*$ : $+$ or $-$) is regarded as a mapping of box labels to a linear order. If a box $b$ is the $k$-th element of $\Gamma_*$, we use the notation $\Gamma_*(k) = b$ or $\Gamma_*^{-1}(b) = k$. The seq-pair is assumed as a code of the topology of a packing with the decoding rule as follows.

**Decode rule: Seq-Pair to 2D Topology**

Given a seq-pair $SP = (\Gamma_+, \Gamma_-)$, any pair of boxes $a$ and $b$ is assigned the topology by:

**RL-topology:**
$\Gamma_+^{-1}(a) < \Gamma_+^{-1}(b)$ and $\Gamma_-^{-1}(a) < \Gamma_-^{-1}(b)$
$\rightarrow$ $a$ is left-of $b$ ($b$ is right of $a$)

**AB-topology:**
$\Gamma_+^{-1}(a) < \Gamma_+^{-1}(b)$ and $\Gamma_-^{-1}(a) > \Gamma_-^{-1}(b)$
$\rightarrow$ $a$ is above $b$ ($b$ is below $a$)

Construct a pair of vertex weighted directed graphs $G_{RL}$ and $G_{AB}$ as follows. The vertex set consists of $s$, $t$ and other $n$ vertices. $s$ and $t$ are called the source and sink, respectively. The other vertices correspond to boxes. They are referred to by box-labels as long as no confusion arises. If $SP$ is decoded as box $a$ is left-of box $b$, assign a directed edge $(a, b)$ in $G_{RL}$. While, if $SP$ says that $a$ is above $b$, add a directed edge $(a, b)$ to $G_{AB}$. Finally, edges $(s, b)$ and $(b, t)$ are added for every vertex $b$ in common in both graphs. The edge has no weight while each vertex $b$ has the weight of width in $G_{RL}$ and height in $G_{AB}$ of box $b$.

It is clear that these graphs are acyclic. Hence a longest path from $s$ to every vertex can be found in polynomial time in each graph. Let the length to $b$ be $\ell_H(b)$ in $G_{RL}$, and $\ell_V(b)$ in $G_{AB}$. Then, a packing is realized by locating $b$ at $(x, y)$ where $x = \ell_{RL}(b), y = \ell_{AB}(b)$. See Fig. 2 and 3.

Finding a longest path uses $O(number\ of\ edges)$, which is $O(n^2)$ in our graphs since for every pair of vertices an edge exists in exactly one of these graphs. This is the dominant term in total computation time.

It is clear that the resultant packing attains the minimum width and minimum height in the package of all the packings satisfying the topology decoded from $SP$.

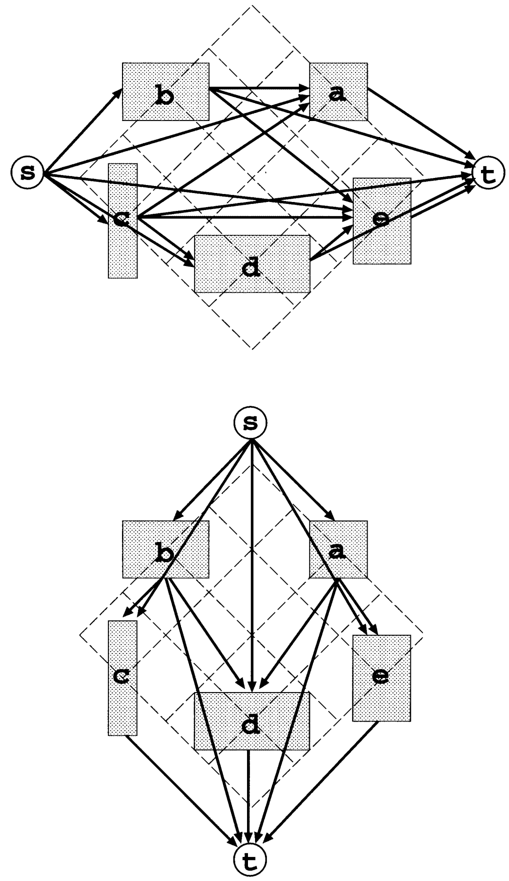


**Fig. 2** $G_{RL}$ and $G_{AB}$ for $(\Gamma_+, \Gamma_-) = (bcade,\ cdbea)$.

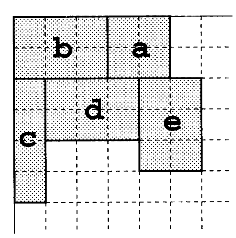


**Fig. 3** 2D-packing by $(\Gamma_+, \Gamma_-) = (bcade,\ cdbea)$.

It seems that the packing thus obtained is special since it has a feature: There exists a box which is simultaneously left-of and above every other box. The box is called the corner box. It is the box $\Gamma_+(1)$, because it is assigned no relation with others than "left-of" or "above." Hence there is no box in "left-of and above" direction of the concerned box.

However, this feature does not restrict generality at all since we see that any 2D-packing contains at least one corner box in diagonal directions. Actually, the following property was proved.

**(*)** There is a seq-pair such that the packing obtained from it by the above procedure is optimal.

This fact encourages us to search the seq-pairs with evaluation by area. Though the total number of seq-pairs is finite, it is $(n!)^2$, too large to exhaust. In [1], [2], they implemented a simulated annealing in searching. The experiments showed a performance that is admitted to have made a breakthrough as an automated packing algorithm.

## 3. The Sequence-Triple

The seq-pair is extended to the system of ordered three sequences of box labels. We call this system the sequence-triple, or seq-triple, and denote one as $ST = (\Gamma_1, \Gamma_2, \Gamma_3)$. For example, if the boxes are $a$, $b$, and $c$, a seq-triple is shown in Fig. 4.

The seq-triple is considered as a code of the topology of the 3D-packing. The decode rule is given following the same principle as in the seq-pair, which has not been explicitly described though.

**Pairwise-Uniqueness:** Every pair of boxes is assigned with a unique topology.

**Order-Dependency:** Only the order of box labels is significant in the sequence.

**Symmetricity:** The topology shall be reversely decoded if the label ordering is reversed. For example, if a pair of boxes $a$ and $b$ is assigned as "$a$ is left-of $b$," it is also decoded as "$b$ is right-of $a$."

By Pairwise-Uniqueness, the rule is completely described if it is given how an arbitrary pair of boxes $(a, b)$ is decoded. Furthermore, by Symmetricity, for each axis, one direction of "right-of," "rear-of," and "below" is enough to be mentioned in decoding.

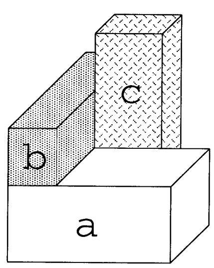


**Fig. 4** A 3D-packing with the topology decoded from seq-triple $(bac, acb, abc)$.

**Decode rule: Seq-Triple to 3D Topology**

Given a sequence-triple $(\Gamma_1, \Gamma_2, \Gamma_3)$, any pair of boxes $a$ and $b$ is assigned the topology by:

**RL-topology:**
$\Gamma_1^{-1}(a) < \Gamma_1^{-1}(b)$ *and* $\Gamma_2^{-1}(a) > \Gamma_2^{-1}(b)$ *and* $\Gamma_3^{-1}(a) < \Gamma_3^{-1}(b) \ \rightarrow\ b$ is right-of $a$
$\Gamma_1^{-1}(a) > \Gamma_1^{-1}(b)$ *and* $\Gamma_2^{-1}(a) > \Gamma_2^{-1}(b)$ *and* $\Gamma_3^{-1}(a) < \Gamma_3^{-1}(b) \ \rightarrow\ b$ is right-of $a$

**FR-topology:**
$\Gamma_1^{-1}(a) < \Gamma_1^{-1}(b)$ *and* $\Gamma_2^{-1}(a) < \Gamma_2^{-1}(b)$ *and* $\Gamma_3^{-1}(a) < \Gamma_3^{-1}(b) \ \rightarrow\ b$ is rear-of $a$

**AB-topology:**
$\Gamma_1^{-1}(a) < \Gamma_1^{-1}(b)$ *and* $\Gamma_2^{-1}(a) > \Gamma_2^{-1}(b)$ *and* $\Gamma_3^{-1}(a) > \Gamma_3^{-1}(b) \ \rightarrow\ b$ is below $a$

See a packing in Fig. 4 which satisfies the topology decoded.

The description by Table 1 is more clear. Note that only the 1st, 3rd, and 4th rows are significant. The 5th, 7th and 8th are derived from the 1st, 3rd, and 4th rows, respectively. Note that the 2nd and 6th rows are defined superfluous rules. This is reasoned as follows: There are $2^3 = 8$ different combinations for the pair of boxes while 6 varieties in topology because it is described along the axes. To design a reasonable decode system, one way is to neglect them. It would result in the existence of infeasible seq-triples. The other choice is to assign some superfluous but non conflicting rule. It would result in a non-uniqueness encoding. Our decode rule follows the latter.

It must be noted that the decoded topology satisfies the transitivity: "$a$ is $R$ $b$" and "$b$ is $R$ $c$" implies "$a$ is $R$ $c$" where $R$ denotes one of "right-of," "left-of," "above," "below," "front-of," and "rear-of." It is not difficult to prove this fact.

Given a seq-triple, the realization of the 3D-packing is as follows: We decode it to the system of RL-, FR-, and AB-topology. Then, construct three constraint graphs $G_{RL}$, $G_{FR}$, and $G_{AB}$, all analogously as in 2D-packing. Then, the longest path length to each vertex locates the corresponding box, i.e. the coordinate $(x, y, z)$ of the left-front-bottom corner.

The issue we have to note is that the resultant 3D-packing attains the minimum simultaneously in each of width, height, and depth of all the realizations under

**Table 1** Decode rule of the seq-triple sorted w.r.t. the 1st sequence.

| $\Gamma_1$ | $\Gamma_2$ | $\Gamma_3$ | | |
|---|---|---|---|---|
| $(\cdots a \cdots b \cdots$ | $\cdots a \cdots b \cdots$ | $\cdots a \cdots b \cdots)$ | $\rightarrow$ | $b$ is rear-of $a$ |
| $(\cdots a \cdots b \cdots$ | $\cdots a \cdots b \cdots$ | $\cdots b \cdots a \cdots)$ | $\rightarrow$ | $b$ is left-of $a$ |
| $(\cdots a \cdots b \cdots$ | $\cdots b \cdots a \cdots$ | $\cdots a \cdots b \cdots)$ | $\rightarrow$ | $b$ is right-of $a$ |
| $(\cdots a \cdots b \cdots$ | $\cdots b \cdots a \cdots$ | $\cdots b \cdots a \cdots)$ | $\rightarrow$ | $b$ is below $a$ |
| $(\cdots b \cdots a \cdots$ | $\cdots b \cdots a \cdots$ | $\cdots b \cdots a \cdots)$ | $\rightarrow$ | $b$ is front-of $a$ |
| $(\cdots b \cdots a \cdots$ | $\cdots b \cdots a \cdots$ | $\cdots a \cdots b \cdots)$ | $\rightarrow$ | $b$ is right-of $a$ |
| $(\cdots b \cdots a \cdots$ | $\cdots a \cdots b \cdots$ | $\cdots b \cdots a \cdots)$ | $\rightarrow$ | $b$ is left-of $a$ |
| $(\cdots b \cdots a \cdots$ | $\cdots a \cdots b \cdots$ | $\cdots a \cdots b \cdots)$ | $\rightarrow$ | $b$ is above $a$ |

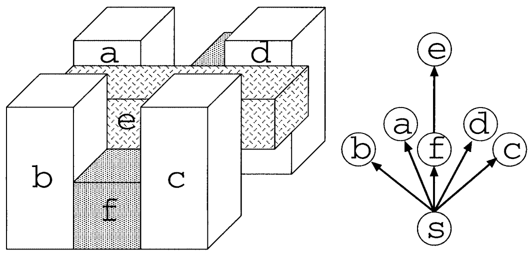


**Fig. 5** A 3D-packing whose topology is not covered by the seq-triple, while it is by seq-quintuple ($ebafcd$, $abfdce$, $fcbeda$, $bceadf$, $abcdfe$).

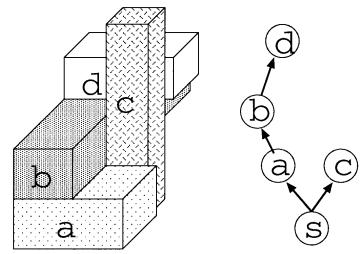


**Fig. 6** Another example whose topology corresponds to no seq-triple, while to seq-quintuple ($adbc$, $bcda$, $acdb$, $bacd$, $acbd$).

the constraint by the seq-triple. This is trivial by the property of the longest path in each constraint graph. A 3D-packing satisfying this property is said minimal. Of course the minimality has nothing to do with the minimum volume solution. Therefore, the most we are concerned now is if the seq-triple covers all the packings, or at least one that attains the minimum volume. The answer is not affirmative yet to answer Yes or No. In the following, we analyze the situation from two different viewpoints. This is the most exciting part of this paper.

Consider a 3D-packing shown in Fig. 5. The transitivity holds in the form that ($d$ is rear-of $e$) and ($e$ is rear-of $b$) lead to ($d$ is rear-of $b$). Simultaneously, ($d$ is right-of $f$) and ($f$ is right-$b$) leads ($d$ is right-of $b$). Then, we have double topology in pair $(b, d)$, a contradiction to Pairwise Uniqueness. Hence this is a packing with topology to which no seq-triple is decoded. In fact, if we were to admit the topology ($d$ is right-of $f$) and ($f$ is right-of $b$), the decode rule would not simultaneously allow ($d$ is rear-of $e$) and ($e$ is rear of $b$) . Therefore, if it were to hold the assumption that this topology (and its symmetric) is the only solution of the minimum volume, this example would be a counter-example to the claim that the seq-triple covers an optimal solution. Of course, the assumption does not hold in this small instance: another arrangement provides the same volume solution. This fact suggests that, as long as we follow the principle, the seq-triple has a limited capacity to convey enough information of the 3D-packing. It must be extended to a larger system which is the subject of the next section.

However we find a significant merit in the seq-triple. A 3D-packing is said *tractable* if there is a linear order of all the boxes that the box is extracted one by one in above-front without moving the remaining boxes.

**Lemma 1:** Any 3D-packing by the topology of a seq-triple is tractable.

Let a seq-triple be $(\Gamma_1, \Gamma_2, \Gamma_3)$. Then, all the boxes are taken off one by one along the order $\Gamma_1$, in above-front direction, without moving remaining boxes. The class of tractable 3D-packings are preferable in applications, for example, in determining an acceptable loading order in a container.

See an example packing shown in Fig. 6. The packing has no box whose upper-front is open to outside. Then we can conclude that this is another example whose topology is not represented by any seq-triple.

A natural question is "if the seq-triple covers all the tractable packings". The answer is negative. We have a counter example shown in Fig. 5, which can be tractable in above-front direction.

## 4. The Sequence-Quintuple

We reasoned that three sequences are not enough to convey the information of all 3D-packings. We introduce here the system of five sequences denoted as $SQ = (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5)$, which we call the sequence-quintuple, or simply, seq-quintuple.

Given $n$ boxes, the $SQ$ defines a 3D-packing uniquely by the following algorithm.

**Algorithm: From Seq-Quintuple to 3D-packing**

**Step 1.** By a seq-pair $(\Gamma_1, \Gamma_2)$, construct the right-left constraint graph $G_{RL}$ to represent the RL-topology. It follows the same definition as for the seq-pair but restricted only to the right-left relation that ($\Gamma_1^{-1}(a) < \Gamma_1^{-1}(b)$ and $\Gamma_2^{-1}(a) < \Gamma_2^{-1}(b) \rightarrow$ ($a$ is left-of $b$)). Vertices are $s$, $t$ and $n$ vertices labeled with box labels with weight of the width of the corresponding boxes. Edges are $(s, b)$ and $(b, t)$ for every $b$, and edge ($a$, $b$) only if ($a$ is left-of $b$).
Also construct the front-rear constraint graph $G_{FR}$ from $(\Gamma_3, \Gamma_4)$ by the rule of the topology that ($\Gamma_3^{-1}(a) < \Gamma_3^{-1}(b)$ and $\Gamma_4^{-1}(a) < \Gamma_4^{-1}(b) \rightarrow$ ($a$ is front-of $b$)). The vertices with weight and edges are defined analogously.

**Step 2.** Find the length of the longest path in $G_{RL}$ from the source to each vertex, and let it be the x-coordinate of the corresponding box. Also determine the y-coordinate by $G_{FR}$.
Now every box is located its xy-coordinate. Two boxes are said *xy-overlapping* if they share the same xy-coordinate, i.e. they overlap in the projected xy-plane.

**Step 3.** Construct the above-below constraint graph

$G_{AB}$ as follows. The vertices and edges incident to $s$ and $t$ are defined the same way as other constraint graphs. For each pair of boxes $a$ and $b$, an edge from $a$ to $b$ is added if and only if 1) $a$ and $b$ are xy-overlapping and 2) $\Gamma_5^{-1}(a) < \Gamma_5^{-1}(b)$.

**Step 4.** Determine the z-coordinate by the longest path in $G_{AB}$. As will be proved in the following, the seq-quintuple revives the generality. The packing shown in Figs. 5 and 6, which are out of the scope by the seq-triple, are obtained from seq-quintuples.

Further important notices follows. The principle behind the 2D-packing by the seq-pair and the 3D-packing by the seq-triple is in the fashion that topology and physical dimension are separately decoded. In other words, the strategy is to decode the topology and use it as the constraint in 1D compaction. This gives a beautiful framework because the constraint is represented by DAGs whose structures are invariant (universal) for box sizes. Most of greedy and dynamical algorithms work consulting with the topology and dimension of the boxes simultaneously. However, the 3D-packing by the seq-quintuple under our decode rule lost this elegancy. It generates three constraint graphs, one of which completely depends on the dimensions. However we can prove the following fact.

**Theorem 1:** For any input of boxes and any seq-quintuple,

1. algorithm "From Seq-Quintuple to 3D-packing" provides a unique 3D-packing, and
2. there is a seq-quintuple that leads an optimal 3D-packing.

**Proof:**

1. Any two boxes $a$ and $b$ to which a topology "$a$ is left-of $b$" are forced to separate in right-left direction so as not to overlap each other because an edge from $a$ to $b$ is assigned in $G_{RL}$ and the weight of $a$ is its width. Similarly, any two boxes will not overlap in y-direction.

If two boxes $a$ and $b$ are xy-overlapping, there is an edge from one to the other according to $\Gamma_5$. Hence they will not overlap in z-direction. As conclusion, no overlap exists in the output arrangement of the boxes. The uniqueness is clear because the longest path outputs the unique value.

2. We first define the 1D compaction of our style. We assume without loss of generality that every box is in the positive domain, i.e. every coordinate of $(x, y, z)$ of the left-front-bottom corner of the box is nonnegative. The 1D compaction is to move one box continuously along one axis, or more precisely, to move one box so that strictly one coordinate continuously decreases as long as it hits no other box.

Consider a 3D-packing $\mathcal{P}$ which is optimal. Without loss of generality, we can assume that any box is impossible to move by any 1D compaction. This means that with respect to each direction, say the direction along the x-axis, any box $b$ can indicate one box or the yz-plane that touches $b$ and stops $b$ to move leftward. Such a box or the xy-plane is called the *left-stop* of $b$. The *front-stop*, *below-stop* are defined analogously. We are going to define a seq-quintuple that leads $\mathcal{P}$ using these informations.

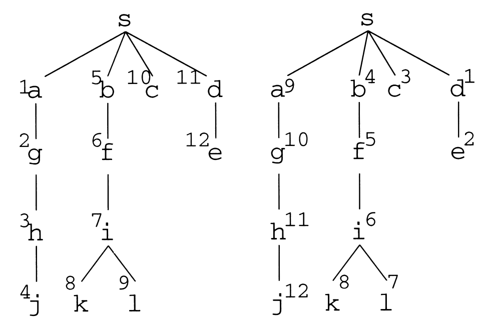


**Fig. 7** Two orderings of box labels: left-preorder and right-preorder.

Construct a graph $T_x$ consisting of the vertex $s$ and other vertices. Vertex $s$ corresponds to the yz-plane and the others one-to-one to the boxes. They are referred to by the corresponding box-labels. An edge $(s, b)$ exists if and only if the yz-plane is the left-stop of box $b$. An edge $(a, b)$ exists if and only if box $a$ is the left-stop of box $b$. The vertices are weighted by width of the corresponding boxes. Analogously, $T_y$ and $T_z$ are defined from $\mathcal{P}$. Trivially, these graphs are rooted trees with root $s$.

Consider $T_x$. For any box $b$, the path length from $s$ to $b$ is the x-coordinate of the left face of $b$ in $\mathcal{P}$. Find two kinds of linear orders of the box labels along the left-preorder and right-preorder, the definition of which will be clearly understood by an example shown in Fig. 7. Then, let them be $\Gamma_1$ and $\Gamma_2$, respectively. It is easily checked that a box $a$ is on the path to $b$ if and only if $\Gamma_1(a) < \Gamma_1(b)$ and $\Gamma_2(a) < \Gamma_2(b)$. Therefore, the longest path algorithm applied to $G_{RL}$ calculates the path length from $a$ to $b$ which is exactly the separation between them in $\mathcal{P}$.

Similarly, $(\Gamma_3, \Gamma_4)$ is found from $\mathcal{P}$ which gives the same arrangement of $\mathcal{P}$ in y-direction.

Finally we define $\Gamma_5$. It can be any order as long as it is the inverse of a topological order tree $T_z$. A topological order of a rooted tree where directions are from the source is defined as follows. "Take a leaf and delete it. Continue until only the source remains".

See an example in Fig. 5 where $T_z$ is shown. So $\Gamma_5$ is the inverse of $(abcdef)$, $(bcadef)$, $\cdots$, or $(efdcba)$, that is, all the sequences as long as $e$ comes after $f$. Also see Fig. 6 where $\Gamma_5$ can be any if ordering $a-b-d$ is not violated. Since an edge between $a$ and $b$ in $G_{AB}$ exists if and only if they xy-overlap and $\Gamma_5^{-1}(a) < \Gamma_5^{-1}(b)$, their

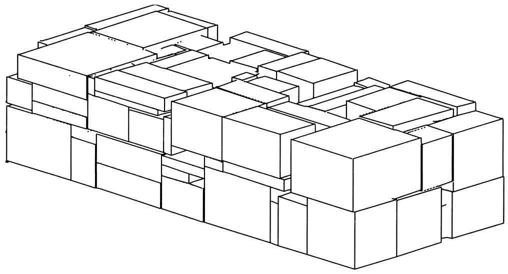

**Fig. 8** 3D-packing of 100 boxes by the seq-triple based search.

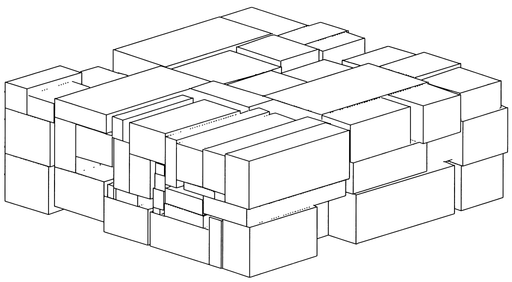

**Fig. 9** 3D-packing of 100 boxes by the seq-quintuple based search.

z-coordinates are determined as they are in $\mathcal{P}$. $\square$

## 5. Experiments and Analysis

We implemented two 3D-packing heuristic algorithms, one is based on the seq-triple the other on the seq-quintuple. They are controlled by the simulated annealing with objective function "volume of the package." The move is chosen every time randomly from

- exchange of two random labels in randomly chosen one of $\Gamma_i$ $(i = 1, 2, ..)$
- exchange of two random labels in every $\Gamma_i$.
- 90 degree rotation of one random box on a random choice of xy-, yz-, xz-plane.
- (only in case of seq-quintuple based algorithm) 90 degree rotation of the whole packing. (This has a meaning since three orthogonal axes are not symmetric. )

The input is a set of randomly generated boxes. The resultant 3D-packings of 100 boxes by the seq-triple and the seq-quintuple are shown in Fig. 8 and Fig. 9, respectively. These computation times are 3.5 hours and 8.5 hours, and volume ratios (*volume of package / sum of module volumes*) are 1.152 and 1.243, respectively.

The comparison of the performance by the seq-triple and that by the seq-quintuple was an interesting experiment. If we were allowed to use any amount of computation resources, the latter would give the better, actually the best solution. However, within a limited time, the former has an opportunity that a number of redundant solutions may be cut and that an optimal solution may be included. (As noted, it is our conjecture that the seq-triple does not lose generality with respect to the volume.)

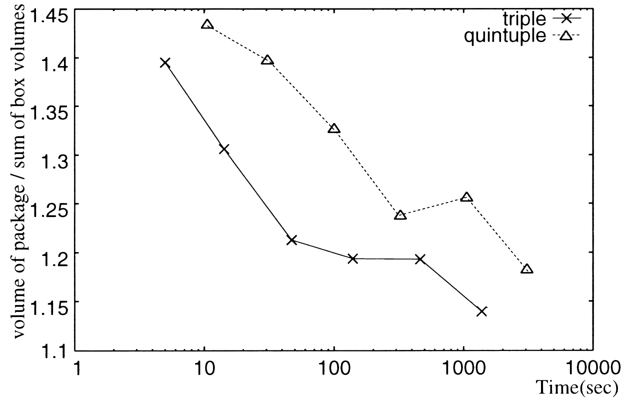


**Fig. 10** Comparison of quality improvement by the seq-triple and seq-quintuple in SA ($n = 30$).

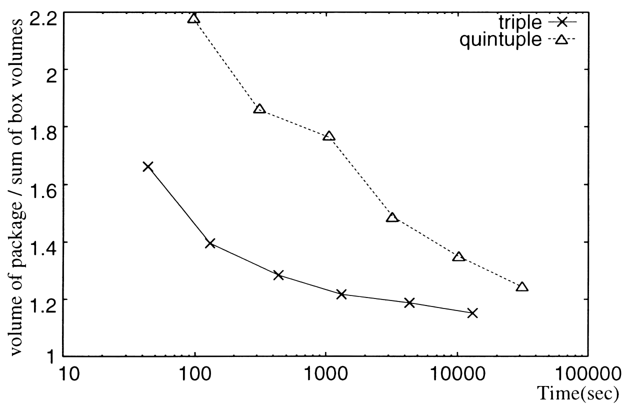


**Fig. 11** Comparison of quality improvement by the seq-triple and seq-quintuple in SA ($n = 100$).

The comparisons are shown in Fig. 10 and Fig. 11. At least in these situations (computation time is less than several ours) , our expectation is confirmed: the seq-triple based stochastic search finds the better solution in smaller time.

## 6. Concluding Remarks

It was shown that the sequence-triple is able to code the topology of the minimal 3D-packing, and that the 3D-packing of this class is tractable. It was also shown that the sequence-quintuple is able to code any minimal 3D-packing. They are enhanced to the heuristics by simulated annealing. They generate sequence-triples or sequence-quintuples one after another systematically and goes to the next according to the evaluation. It is believed that a breakthrough has been attained practically since they can now pack a hundred of boxes very nicely in practical time.

There may be future works to be considered: irregular box packing, coding by the sequence $(2k+1)$-tuple for $k$ dimensional packing, fragile box packing, conditional packing, etc. The extention of BSG [6], [7], which is another coding system of the 2D-packing, is also an interesting work.

This paper, beyond a faithful extension of the existing "sequence-pair" for the 2D-packing [1], [2], unveiled features particular to the 3D space. It is remarkable to have found the fact that a very naive sequence-triple represents a subset of the class of tractable 3D-packings, which we often meet in practical environment.

## Acknowledgement

Authors are grateful to Associate Prof. A. Takahashi of Tokyo Inst. of Tech. and Associate Prof. K. Hiraishi of JAIST for their helpful suggestions. This work is part of a project of CAD21 at Tokyo Inst. of Tech. and is supported in part by a Grant in Aid for Scientific Research of Ministry of Education, Science, and Culture of Japan (10750298), and a Research and Development Aid of Support Center for Advanced Telecommunications Technology Research.

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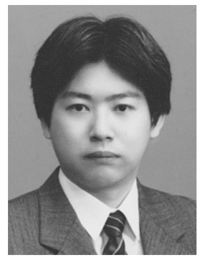

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