Séminaire Lotharingien de Combinatoire **XX** (2023)
Article #YY, 11 pp.

Proceedings of the 35th Conference on Formal Power Series and Algebraic Combinatorics (Davis)

The isomorphism problem for cominuscule Schubert varieties

Edward Richmond*[1], Mihail Ţarigradschi†[2], and Weihong Xu‡[3]

[1]*Department of Mathematics, Oklahoma State University, Stillwater, OK*
[2]*Department of Mathematics, Rutgers University, Piscataway, NJ*
[3]*Department of Mathematics, Virginia Tech, Blacksburg, VA*

Abstract. Cominuscule flag varieties generalize Grassmannians to other Lie types. Schubert varieties in cominuscule varieties are indexed by posets of roots labeled long/short. These labeled posets generalize Young diagrams. We prove that Schubert varieties in potentially different cominuscule flag varieties are isomorphic as varieties if and only if their corresponding labeled posets are isomorphic, generalizing the classification of Grassmannian Schubert varieties using Young diagrams by the last two authors.

Keywords: Schubert variety, flag variety, cominuscule

1 Introduction

Cominuscule (flag) varieties correspond to algebraic varieties that admit the structure of a compact Hermitian symmetric space and have been studied extensively due their shared properties with Grassmannians [1, 2, 13, 20, 3, 18, 4, 8, 6, 17, 5]. These varieties come in four infinite families and two exceptional types and are determined by a pair $(\mathcal{D}, \gamma)$ of a Dynkin diagram $\mathcal{D}$ of a reductive Lie group and a cominuscule simple root γ. See Table 1 for a classification of cominuscule varieties. Let X denote the cominuscule variety corresponding to $(\mathcal{D}, \gamma)$ and R denote the root system of the Dynkin diagram $\mathcal{D}$. Set $\mathcal{P}_X := \{\alpha \in R : \alpha \geq \gamma\}$ with the partial order $\alpha \leq \beta$ if $\beta - \alpha$ is a non-negative sum of simple roots, and give $\mathcal{P}_X$ a labeling of long/short roots. By [7, Theorem 2.4], Schubert varieties in X are indexed by lower order ideals in $\mathcal{P}_X$, generalizing the fact that Schubert varieties in a Grassmannian are indexed by Young diagrams.

Our main result Theorem 1 is a combinatorial criterion for distinguishing isomorphism classes of Schubert varieties coming from cominuscule varieties.

*edward.richmond@okstate.edu
†mt994@math.rutgers.edu
‡weihong@vt.edu

Theorem 1. *Let $X_\lambda \subseteq X$ and $Y_\mu \subseteq Y$ be cominuscule Schubert varieties indexed by lower order ideals $\lambda \subseteq \mathcal{P}_X$ and $\mu \subseteq \mathcal{P}_Y$, respectively. Then X_λ and Y_μ are algebraically isomorphic if and only if λ and μ are isomorphic as posets in a way that respects the labelling of long/short roots.*

For illustrative examples of Theorem 1, please see Section 2.

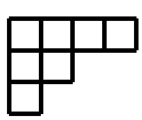
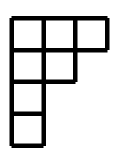
Since Grassmannians are cominuscule varieties, Theorem 1 extends the work of Țarigradschi and Xu in [19], where they prove two Grassmannian Schubert varieties are isomorphic if and only if their Young diagrams are the same or the transpose of each other. Other related works include Richmond and Slofstra's characterization of the isomorphism classes of Schubert varieties coming from complete flag varieties in [16] using Cartan equivalence. However, they also note that Cartan equivalence is neither necessary nor sufficient to distinguish Schubert varieties in partial flag varieties. A class of smooth Schubert varieties in partial flag varieties are classified by Develin, Martin, and Reiner in [9]. Yet many Schubert varieties are singular, with the first example being the Schubert divisor in the Grassmannian $\mathrm{Gr}(2,4)$.

We discuss preliminaries in Section 3, and then in Section 4, we sketch the proof of Theorem 1 and illustrate with examples. In Section 4.1, we construct the labeled poset of roots for a Schubert variety from geometric invariants depending only on its isomorphism class; in Section 4.2, we prove that Schubert varieties with the same labeled poset are isomorphic by embedding each Schubert variety in a "minimal" flag variety uniquely determined by the labeled poset.

2 Examples

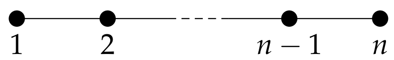
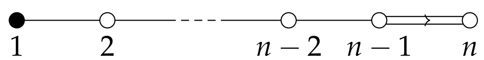
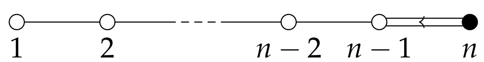
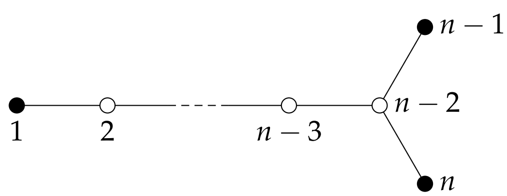
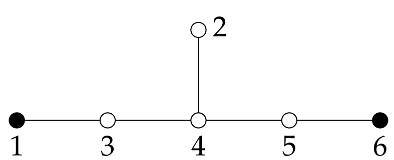
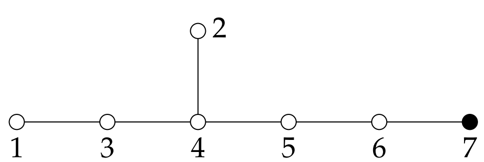
For the following examples, recall that cominuscule Schubert varieties are indexed by lower order ideals in $\mathcal{P}_X$. Examples of $\mathcal{P}_X$ are illustrated in Table 2, where each element in $\mathcal{P}_X$ is drawn as a box. The partial order on boxes is given by $\alpha \leq \beta$ if and only if α is weakly north-west of β. In particular, lower order ideals are given by subsets of boxes that are closed under moving to the north and west.

Example 2. As illustrated below, transposing a Young diagram does not change the poset structure. Therefore, two Grassmannian Schubert varieties are isomorphic if their indexing Young diagrams are the transpose of each other. Geometrically, this is related to the isomorphism $Gr(m, m+k) \cong Gr(k, m+k)$.

 $\cong$ 

Example 3. Using Table 2, it is not hard to see that if a Grassmannian Schubert variety is isomorphic to a non-type A cominuscule Schubert variety, then they are both isomorphic

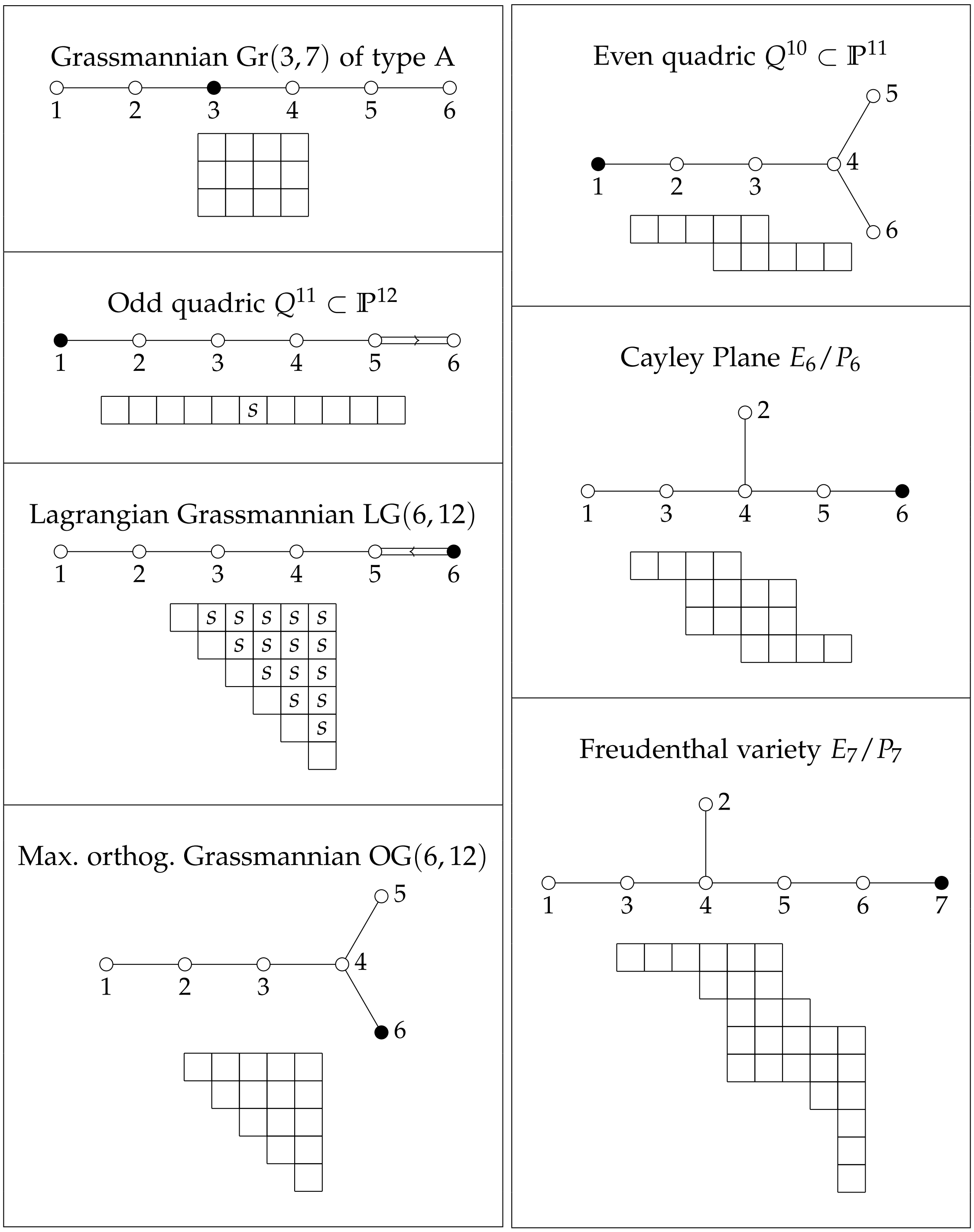
Table 1: Cominuscule varieties. Cominuscule roots are denoted by the filled-in circles. This Table is a modification of [7, Table 1]

	Type A_n: $A_n/P_m = \mathrm{Gr}(m, n+1)$ Grassmannian of type A.
	Type B_n ($n \geq 2$): $B_n/P_1 = Q^{2n-1}$ Odd quadric.
	Type C_n ($n \geq 3$): $C_n/P_n = \mathrm{LG}(n, 2n)$ Lagrangian Grassmannian.
	Type D_n ($n \geq 4$): $D_n/P_1 = Q^{2n-2}$ Even quadric. $D_n/P_{n-1} \cong D_n/P_n = \mathrm{OG}(n, 2n)$ Max. orthogonal Grassmannian.
	Type E_6: $E_6/P_1 \cong E_6/P_6$ Cayley plane.
	Type E_7: E_7/P_7 Freudenthal variety.

to a projective space. Indeed, in order to fit inside a $\mathcal{P}_X$ of another type, the lower order ideal is forced to be a chain.

As a special case, we also see that any Schubert curve in any cominuscule variety is isomorphic to $\mathbb{P}^1$. In fact, any Schubert curve in any flag variety is isomorphic to $\mathbb{P}^1$, which follows from the more general statements that Schubert varieties are rational normal projective varieties and $\mathbb{P}^1$ is the only rational normal projective curve.

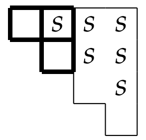
Table 2: $\mathcal{P}_X$ for cominuscule varieties. In each case the partial order is given by $\alpha \le \beta$ if and only if α is weakly north-west of β, and boxes decorated with an "s" correspond to short roots. This table is a modification of [5, Table 1].



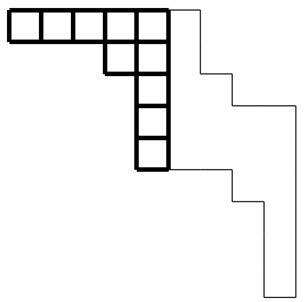
Example 4. The Schubert divisor in Q^3 is not isomorphic to $\mathbb{P}^2$, because the labeling of their posets do not match:

We can also see it geometrically, as the Schubert divisor in Q^3 is singular.

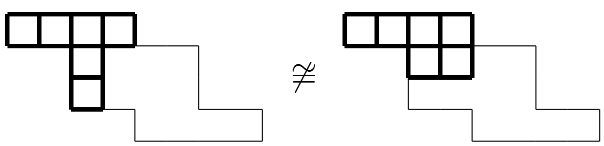
Example 5. The quadric Q^3 embeds in $\mathrm{LG}(n, 2n)$ ($n \geq 3$) as a Schubert variety, as illustrated by



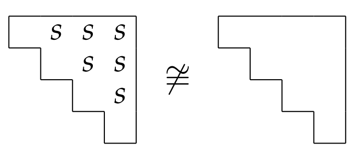
Example 6. The quadric Q^{10} embeds in E_7/P_7 as a Schubert variety, as illustrated by



Example 7. There are two non-isomorphic 6-dimensional Schubert varieties in E_6/P_6, given by the two order ideals illustrated below.



Example 8. While $\mathcal{P}_{\mathrm{LG}(n,2n)}$ and $\mathcal{P}_{\mathrm{OG}(n+1,2n+2)}$ are isomorphic as posets, this isomorphism does not preserve the labeling of long/short roots (see illustration below). As a result, $\mathrm{LG}(n, 2n)$ and $\mathrm{OG}(n + 1, 2n + 2)$ do not contain isomorphic Schubert varieties of dimension greater than one.



3 Preliminaries

Let G be a complex reductive linear algebraic group. We fix subgroups $T \subset B \subset G$, where T is a maximal torus and B is a Borel subgroup. With this setup, $T \subset G$ determines a root system R of G, with corresponding Weyl group $W := N(T)/T$ and B determines a set of simple roots $\Delta \subseteq R$. The set of roots decomposes into positive and negative roots: $R = R^+ \sqcup R^-$, with R^+ being non-negative sums of simple roots. The Weyl group W is generated by the set of simple reflections $S := \{s_\alpha : \alpha \in \Delta\}$.

To each subset $I \subseteq S$ one can associate a Weyl subgroup $W_I := \langle s : s \in I \rangle \subseteq W$, a parabolic subgroup $P_I = BW_IB \subseteq G$ and the corresponding (partial) flag variety $X = G/P_I$. Schubert varieties in X are indexed by W^I, the set of minimal length poset representatives of W/W_I. Explicitly, for $w \in W^I$, the Schubert variety

$$X_w := \overline{BwP_I/P_I}$$

has dimension the Coxeter length of w, denoted $\ell(w)$.

From now on, X is a cominuscule variety. In other words, $I = S \setminus \{s_\gamma\}$, where γ is a cominuscule simple root, i.e. γ appears with coefficient 1 in the highest root of R. Cominuscule roots are illustrated by filled-in circles in Table 1 and Table 2.

In [14], Proctor proves that W^I is a distributive lattice under the induced Bruhat partial order from W. Birkhoff's representation theorem implies there is a bijection between W^I and the set of lower order ideals in $\mathcal{P}_X$. In particular, the join-irreducible elements of W^I are identified with principal lower order ideals of $\mathcal{P}_X$ and hence with $\mathcal{P}_X$ itself. See Figure 1 for an illustration when $X = \mathrm{Gr}(2,4)$. Explicitly, to each $w \in W^I$ we associate its inversion set

$$R(w) := \{\alpha \in R^+ : w.\alpha < 0\},$$

viewed as a sub-poset of $\mathcal{P}_X$. It is well known that $\ell(w) = |R(w)|$. Moreover, the following proposition was proved in [7, Theorem 2.4 and Corollary 2.6]:

Proposition 9 (Buch–Samuel)**.** *For any $w \in W^I$, the inversion set $R(w)$ is a lower order ideal in $\mathcal{P}_X$. Moreover:*

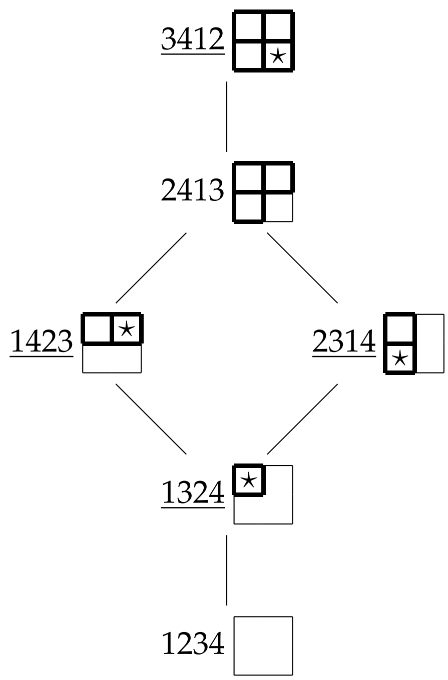
1. *The map $w \mapsto R(w)$ is a bijection between W^I and the set of lower order ideals in $\mathcal{P}_X$.*
2. *For any $u \in W^I$, we have $u \leq w$ in Bruhat order if and only if $R(u) \subseteq R(w)$.*
3. *If $\alpha \in R(w)$ and $R(w) \setminus \{\alpha\}$ is a lower order ideal, then $ws_\alpha \in W^I$ and $R(ws_\alpha) = R(w) \setminus \{\alpha\}$, where $s_\alpha \in W$ is the reflection corresponding to α.*

4 Proof sketch

4.1 Forward direction: the isomorphism class of X_w determines the labeled poset $R(w)$

We use the Picard group $\mathrm{Pic}(X_w)$ and the Chow group $A_*(X_w)$, which depend only on the isomorphism class of X_w. Recall that the k-th Chow group $A_k(Y)$ of a scheme Y is the

Figure 1: The Bruhat poset W^I when $X = \mathrm{Gr}(2,4)$. Permutations in W^I are denoted using one-line notation, and next to each is the corresponding lower order ideal in $\mathcal{P}_X$. Join-irreducible elements of W^I are the ones underlined, and the generator of the corresponding principal lower order ideal in $\mathcal{P}_X$ is decorated with a $\star$.



free abelian group on the k-dimensional subvarieties of Y modulo rational equivalence. When Y is a normal variety, the Picard group $\mathrm{Pic}(Y)$ can be identified with the subgroup of $A_{\dim(Y)-1}(Y)$ generated by classes of locally principal divisors.

Our idea is to construct the labeled poset $R(w)$ from the map

$$\mathrm{Pic}(X_w) \times A_*(X_w) \to A_*(X_w)$$

given by the intersection product.

Definition 10. Let Y be a scheme. The effective cone in the Chow group $A_*(Y)$ is the semigroup in $A_*(Y)$ generated by the classes of closed subvarieties of Y.

Let $X = G/P_I$ be a cominuscule variety. There is a unique Schubert variety of codimension 1 in X, called the Schubert divisor, and its class D is the unique effective generator of the Picard group $\mathrm{Pic}(X) \subseteq A_*(X)$. Fix $w \in W^I$, and let $i_w : X_w \hookrightarrow X$ be the closed embedding. For $u \leq w$ in W^I, we will write $[X_u]$ for the class of X_u in $A_*(X_w)$ and $A_*(X)$, in an abuse of notation.

Lemma 11 below is well known.

Lemma 11. *For $w \neq id$ in W^I, the map $i_w^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X_w)$ is an isomorphism.*

Lemma 12 below is a special case of [10, Corollary of Thereom 1].

Lemma 12 (Fulton–MacPherson–Sottile–Sturmfels)**.** *The Schubert classes $[X_u]$ such that $u \leq w$ are exactly the minimal elements in the extremal rays of the effective cone in $A_*(X_w)$.*

Lemma 11 allows us to identify the unique effective generator $i_w^* D$ of $\mathrm{Pic}(X_w)$, and Lemma 12 allows us to identify Schubert classes in $A_*(X_w)$, or equivalently, their indexing set $[e, w] := \{u \in W^I : u \leq w\}$. We shall see that the Bruhat order on $[e, w]$ can be recovered from the intersection products $i_w^* D \cdot [X_u]$, $u \in [e, w]$.

Let $j : X_u \hookrightarrow X_w$ be an inclusion of Schubert varieties, where $u \in [e, w]$. By the definition of intersection product,

$$i_w^* D \cdot [X_u] = [j^* i_w^* D] = [(i_w \circ j)^* D] = D \cdot [X_u],$$

which says that the intersection products $i_w^* D \cdot [X_u]$ on X_w can be computed on X. By [12, Example 19.1.11], $A_*(X)$ can be identified with the homology group $H_*(X)$, and by [12, Proposition 19.1.2], the intersection product $D \cdot [X_u]$ can be identified with a cap product, which corresponds to the cup product of the cohomology classes corresponding to D and $[X_u]$ via the Poincaré duality isomorphism for X. This cup product is given by the Chevalley formula [11, Lemma 8.1]. By translating backwards, we restate the Chevalley formula for cominuscule varieties as Lemma 13 below.

Lemma 13 (Fulton–Woodward)**.** *For $u \in [e, w]$, we have*

$$i_w^* D \cdot [X_u] = \sum \frac{(\gamma, \gamma)}{(\alpha, \alpha)} [X_{us_\alpha}],$$

the sum over all positive roots α such that $us_\alpha \in W^I$ and $\ell(us_\alpha) = \ell(u) - 1$. Here $(\cdot, \cdot)$ denotes the usual inner product.

Note that by Lemma 13, the class $[X_v]$ appears with a nonzero coefficient in $i_w^* D \cdot [X_u]$ if and only if $v < u$ and $\ell(v) = \ell(u) - 1$. Therefore, the Bruhat order on $[e, w]$ is determined by the intersection products $i_w^* D \cdot [X_u]$, $u \in [e, w]$.

By our discussion in Section 3, the poset $R(w)$ can then be identified as the join-irreducible elements in the Bruhat interval $[e, w]$. For each $\alpha \in R(w)$, let λ_α denote the principal lower order ideal generated by α, and $z_{\lambda_\alpha} \in [e, w]$ the element corresponding to λ_α in Proposition 9. Since z_{λ_α} is join-irreducible, the Chevalley formula implies

$$i_w^* D \cdot [X_{z_{\lambda_\alpha}}] = \frac{(\gamma, \gamma)}{(\alpha, \alpha)} [X_{z_{\lambda_\alpha} s_\alpha}]$$

with no other terms in the sum. Now the length of α can be recovered from the coefficient. Specifically, if $(\gamma, \gamma) = (\alpha, \alpha)$, then α is long; otherwise, α is short.

Example 14. By Lemma 13, the following equalities hold for $X = \mathrm{LG}(3,6)$:

$$D \cdot \left[X_{\begin{ytableau} {} & s & s \\ \none & {} \end{ytableau}}\right] = 2\left[X_{\begin{ytableau} {} & s \\ \none & {} \end{ytableau}}\right] + \left[X_{\begin{ytableau} {} & s & s \end{ytableau}}\right],$$

$$D \cdot \left[X_{\begin{ytableau} {} & s \\ \none & {} \end{ytableau}}\right] = \left[X_{\begin{ytableau} {} & s \end{ytableau}}\right],$$

$$D \cdot \left[X_{\begin{ytableau} {} & s \end{ytableau}}\right] = 2\left[X_{\begin{ytableau} {} \end{ytableau}}\right],$$

where we have identified W^I with the set of lower order ideals in $\mathcal{P}_X$. Note that a coefficient 2 occurs whenever the removed box (root) is short.

4.2 Converse direction: the labeled poset $R(w)$ determines the isomorphism class of X_w

Definition 15. For $w \in W$, the support of w is defined as

$$S(w) := \{s \in S : s \leq w\}.$$

Equivalently, $S(w)$ is the set of simple reflections appearing in any reduced decomposition of w.

Let $X = G/P_I$ and $w \in W^I$. Let G' be the reductive subgroup of $P_{S(w)}$ with Weyl group $W' := W_{S(w)}$ and $P' := G' \cap P_I$ be the reductive subgroup of G' corresponding to $I' := I \cap S(w)$. Set $X' := G'/P'$. Note that X' is a cominuscule variety and that $w \in {W'}^{I'}$. A key step in our proof is [15, Lemma 4.8], restated as Lemma 16 below.

Lemma 16 (Richmond–Slofstra)**.** *The inclusion $X' \hookrightarrow X$ induces an isomorphism $X'_w \to X_w$.*

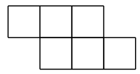
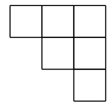
Every reduced decomposition of w, and in particular, $S(w)$, can be read out from the poset $R(w)$ [5, Section 4]. Then Lemma 16 allows us to place X_w in a "minimal" cominuscule variety X'. If X has Dynkin diagram $\mathcal{D}_X$ with vertex set Δ, then the Dynkin diagram $\mathcal{D}_X^w$ of X' is the full subgraph of $\mathcal{D}_X$ with vertex set $\{\alpha \in \Delta : s_\alpha \in S(w)\}$.

We finish the proof with Proposition 17, which implies that the "minimal" cominuscule flag varieties for Schubert varieties with isomorphic labeled posets are isomorphic.

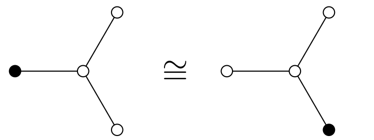
Proposition 17. *Let $R(u) \subseteq \mathcal{P}_X$ and $R(v) \subseteq \mathcal{P}_Y$ be inversion sets corresponding to cominuscule Schubert varieties X_u and Y_v. Every labeled poset isomorphism between $R(u)$ and $R(v)$ induces a graph isomorphism between $\mathcal{D}_X^u$ and $\mathcal{D}_Y^v$ that identifies reduced decompositions of u and v.*

We illustrate the above process with Example 18 and Example 19 below.

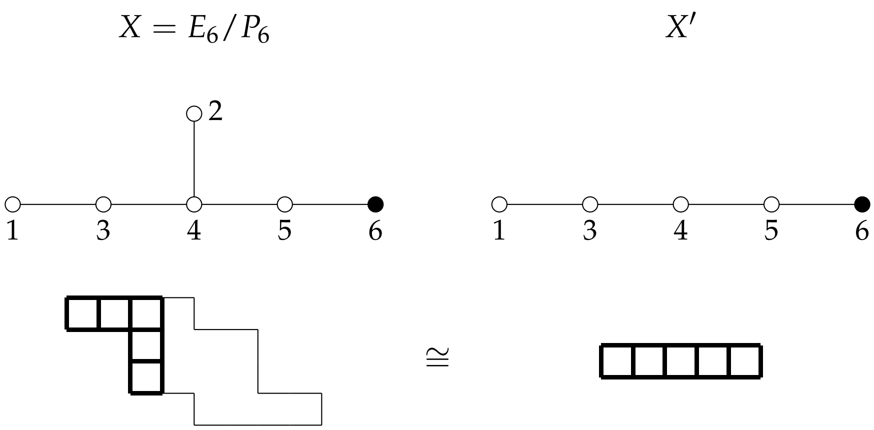
Example 18.

$\mathcal{P}_{Q^6} =$  $\cong$  $= \mathcal{P}_{OG(4,8)}$,

therefore, $Q^6 \cong \mathrm{OG}(4,8)$. This isomorphism comes from a symmetry of the D_4 Dynkin diagram:



Example 19. Let $X = E_6/P_6$ and w be the Weyl group element corresponding to the lower order ideal depicted on the left below. Then $S(w) = \{s_1, s_3, s_4, s_5, s_6\}$, where s_i is the simple reflection corresponding to the simple root labeled by i. Therefore, the pair $(\mathcal{D}_X^w, \gamma)$ is as depicted on the right, isomorphic to that of $\mathbb{P}^5$, showing $X_w \cong X' \cong \mathbb{P}^5$.



Acknowledgements

We thank Anders Buch for helpful discussions.

References

[1] B. D. Boe. "Kazhdan-Lusztig polynomials for Hermitian symmetric spaces". *Trans. Amer. Math. Soc.* **309**.1 (1988), pp. 279–294. DOI.

[2] M. Brion and P. Polo. "Generic singularities of certain Schubert varieties". *Math. Z.* **231**.2 (1999), pp. 301–324. DOI.

[3] A. S. Buch, P.-E. Chaput, L. C. Mihalcea, and N. Perrin. "Finiteness of cominuscule quantum K-theory". *Annales scientifiques de l'École Normale Supérieure* **46**.3 (2013), pp. 477–494. Link.

[4] A. S. Buch, P.-E. Chaput, L. C. Mihalcea, and N. Perrin. "A Chevalley formula for the equivariant quantum K-theory of cominuscule varieties". *arXiv preprint arXiv:1604.07500* (2016).

[5] A. S. Buch, P.-E. Chaput, L. C. Mihalcea, and N. Perrin. "Positivity of minuscule quantum K-theory". *arXiv preprint arXiv:2205.08630* (2022).

[6] A. S. Buch, P.-E. Chaput, L. C. Mihalcea, and N. Perrin. "Projected Gromov-Witten varieties in cominuscule spaces". *Proceedings of the American Mathematical Society* **146** (2018), pp. 3647–3660. DOI.

[7] A. S. Buch and M. J. Samuel. "K-theory of minuscule varieties". *Journal für die reine und angewandte Mathematik (Crelles Journal)* **2016**.719 (2016), pp. 133–171.

[8] S. Chung. "Cominuscule flag varieties and their quantum K-theory: some results". *Dissertation* (2017).

[9] M. Develin, J. L. Martin, and V. Reiner. "Classification of Ding's Schubert Varieties: Finer Rook Equivalence". *Canadian Journal of Mathematics* **59**.1 (2007), 36–62. DOI.

[10] W. Fulton, R. MacPherson, F. Sottile, and B. Sturmfels. "Intersection theory on spherical varieties". *J. Algebraic Geom.* **4**.1 (1995), pp. 181–193.

[11] W Fulton and C Woodward. "On the quantum product of Schubert classes". *Journal of Algebraic Geometry* **13**.4 (2004), pp. 641–661.

[12] W. Fulton. *Intersection theory*. Second. Vol. 2. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 1998, pp. xiv+470. DOI.

[13] N. Perrin. "The Gorenstein locus of minuscule Schubert varieties". *Adv. Math.* **220**.2 (2009), pp. 505–522. DOI.

[14] R. A. Proctor. "Bruhat lattices, plane partition generating functions, and minuscule representations". *European J. Combin.* **5**.4 (1984), pp. 331–350. DOI.

[15] E. Richmond and W. Slofstra. "Billey-Postnikov decompositions and the fibre bundle structure of Schubert varieties". *Math. Ann.* **366**.1-2 (2016), pp. 31–55. DOI.

[16] E. Richmond and W. Slofstra. "The isomorphism problem for Schubert varieties". *arXiv preprint arXiv:2103.08114* (2021).

[17] E. Richmond, W. Slofstra, and A. Woo. "The Nash blow-up of a cominuscule Schubert variety". *J. Algebra* **559** (2020), pp. 580–600. DOI.

[18] C. Robles. "Singular loci of cominuscule Schubert varieties". *J. Pure Appl. Algebra* **218**.4 (2014), pp. 745–759. DOI.

[19] M. Tarigradschi and W. Xu. "The isomorphism problem for Grassmannian Schubert varieties". *arXiv preprint arXiv:2205.12814* (2022).

[20] H. Thomas and A. Yong. "A combinatorial rule for (co)minuscule Schubert calculus". *Adv. Math.* **222**.2 (2009), pp. 596–620. DOI.