## Elementary row operations:

**Operation 1:** switch 2 rows

E.g.

$$\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix} \rightsquigarrow \begin{bmatrix}5&6\\3&4\\1&2\end{bmatrix} = \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}$$

**Operation 2:** scale one row by a real/complex number

E.g.

$$\begin{bmatrix}1+i&1-i\\3&4\\5&6\end{bmatrix} \times (2+2i) \rightsquigarrow \begin{bmatrix}(1+i)(2+2i)&(1-i)(2+2i)\\3&4\\5&6\end{bmatrix} = \begin{bmatrix}4i&4\\3&4\\5&6\end{bmatrix}$$

$$= \begin{bmatrix}2+2i&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1+i&1-i\\3&4\\5&6\end{bmatrix}$$

**Operation 3:** add a $\mathbb{R}/\mathbb{C}$-multiple of one row to another

E.g.

$$\begin{bmatrix}1+i&2\\3&4\\4&4\end{bmatrix} + (-2+2i)\cdot\text{row }1 \rightsquigarrow \begin{bmatrix}1+i&2\\3&4\\4+(-2+2i)(1+i)&4+(-2+2i)2\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0\\0&1&0\\-2+2i&0&1\end{bmatrix}\begin{bmatrix}1+i&2\\3&4\\4&4\end{bmatrix} = \begin{bmatrix}1+i&2\\3&4\\0&4i\end{bmatrix}$$

## Elementary column operations:

**Operation 1:** switch 2 columns

E.g.

$$\begin{bmatrix}1&3&5\\2&4&6\end{bmatrix} \rightsquigarrow \begin{bmatrix}5&3&1\\6&4&2\end{bmatrix} = \begin{bmatrix}1&3&5\\2&4&6\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$

Operation 2: scale one column by a real/complex number

E.g.
$$\begin{bmatrix} 1+i & 3 & 5 \\ 1-i & 4 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} (1+i)(2+2i) & 3 & 5 \\ (1-i)(2+2i) & 4 & 6 \end{bmatrix} = \begin{bmatrix} 4i & 3 & 5 \\ 4 & 4 & 6 \end{bmatrix}$$

$\times(2+2i)$

$$\begin{bmatrix} 1+i & 3 & 5 \\ 1-i & 4 & 6 \end{bmatrix} \begin{bmatrix} 2+2i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operation 3: add a $\mathbb{R}/\mathbb{C}$-multiple of one column to another

E.g.
$$\begin{bmatrix} 1+i & 3 & 4 \\ 2 & 4 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1+i & 3 & 4+(-2+2i)(1+i) \\ 2 & 4 & 4+(-2+2i)2 \end{bmatrix} = \begin{bmatrix} 1+i & 3 & 0 \\ 2 & 4 & 4i \end{bmatrix}$$

$+(-2+2i)\cdot$ Column 1

$$\begin{bmatrix} 1+i & 3 & 4 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2+2i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Def. The matrices by which multiplications give elementary row/column operations are called elementary matrices

E.g. Let $T: \mathbb{R}^5 \to \mathbb{R}^3$ be given by the matrix

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 2 & -4 & 2 & 0 & 8 \\ 1 & -2 & 3 & -3 & 16 \end{bmatrix}$$

under standard bases. Compute $K(T), R(T)$.

$K(T) =$ the set of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ in $\mathbb{R}^5$ such that

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 2 & -4 & 2 & 0 & 8 \\ 1 & -2 & 3 & -3 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

, i.e., $K(T)$ is given by the set of real solutions to

$$\begin{cases} 1x_1 + (-2)x_2 + 0x_3 + 2x_4 + (-3)x_5 = 0 \\ 2x_1 + (-4)x_2 + 2x_3 + 0x_4 + 8x_5 = 0 \\ 1x_1 + (-2)x_2 + 3x_3 + (-3)x_4 + 16x_5 = 0 \end{cases}$$

The 3 elementary row operations on the matrix

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 2 & -4 & 2 & 0 & 8 \\ 1 & -2 & 3 & -3 & 16 \end{bmatrix}$$

correspond to switching 2 equations, scaling one equation by a real number, and adding a real-multiple of one equation to another. Therefore, they do not change the set of solutions.

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 2 & -4 & 2 & 0 & 8 \\ 1 & -2 & 3 & -3 & 16 \end{bmatrix}\begin{array}{l} \\ +(-2)\cdot \text{row } 1 \\ +(-1)\cdot \text{row } 1 \end{array}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 0 & 0 & 2 & -4 & 14 \\ 0 & 0 & 3 & -5 & 19 \end{bmatrix}\begin{array}{l} \\ \times \frac{1}{2} \\ \\ \end{array}$$

this process is called Gaussian Elimination

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 0 & 0 & 1 & -2 & 7 \\ 0 & 0 & 3 & -5 & 19 \end{bmatrix}\begin{array}{l} \\ \\ +(-3)\cdot \text{row } 2 \end{array}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 0 & 0 & 1 & -2 & 7 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}\begin{array}{l} +(-2)\cdot \text{row } 3 \\ +2\cdot \text{row } 3 \\ \\ \end{array}$$

$$\Downarrow$$

$$\begin{array}{c} \begin{bmatrix} 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \\ \begin{array}{ccccc} & x_2 & & & x_5 \end{array} \end{array} \leftarrow \text{reduced row echelon form of the matrix}$$

$x_2$ $x_5$ are free variables

$$1x_4 + (-2)x_5 = 0 \Rightarrow x_4 = 2x_5$$

$$1x_3 \quad + 3x_5 = 0 \Rightarrow x_3 = -3x_5$$

$$1x_1+(-2)x_2 \qquad +1x_5=0 \Rightarrow x_1=2x_2-x_5$$

$$K(T)=\left\{\begin{bmatrix}2x_2-x_5\\x_2\\-3x_5\\2x_5\\x_5\end{bmatrix},\ x_2,x_5\in\mathbb{R}\right\}$$

✳ Note that R(T) is exactly the span of the column vectors of the matrix.

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**Definitions** A matrix is said to be in **row echelon form** if it satisfies the following three conditions:

1. Each nonzero row lies above every zero row.
2. The leading entry of a nonzero row lies in a column to the right of the column containing the leading entry of any preceding row.
3. If a column contains the leading entry of some row, then all entries of that column below the leading entry are 0.[5]

If a matrix also satisfies the following two additional conditions, we say that it is in **reduced row echelon form**.[6]

4. If a column contains the leading entry of some row, then all the other entries of that column are 0.
5. The leading entry of each nonzero row is 1.

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✳ It can be shown that the reduced row echelon form of a matrix is unique.