Proofs you probably weren't taught

Friday, October 21, 2011

Why is the error function minimized in logistic regression convex?

We want to prove that the error/objective function of logistic regression :

$$J(\theta) = \sum_{i=1}^{m} y^i \left[-\log\left(h_\theta\left(x^i\right)\right)\right] + \left(1-y^i\right)\left[-\log\left(1-h_\theta\left(x^i\right)\right)\right] \quad (1)$$

$$where \;\; h_\theta(x) = \frac{1}{1+e^{-\theta^T x}}$$

is convex.

Proof:

Before beginning the proof, i would first like to make you review/recollect a few definitions/rules/facts/results related to convex functions:

- **Definition of a convex function:** A function f(x) is said to be convex if the following inequality holds true:

  $$f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y) \quad \forall x, y \in Domain(f) \; and \; \alpha \in [0,1]$$

- **First-order condition of convexity:** A function f(x) which is differentiable is convex if the following inequality condition holds true:

  $$f(y) \;\ge\; f(x) + \nabla_x^T f(x)(y-x)\,; \quad \forall x, y \in Domain(f)$$

  Intuitively, this condition says that the tangent/first-order-taylor-series approximation of f(x) is globally an under-estimator.

- **Second-order condition of convexity:** A function f(x) which is twice-differentiable is convex if and only if its hessian matrix (matrix of second-order partial derivatives) is positive semi-definite, i.e.

  $$\forall z : \;\; z^T \, \nabla_x^2 f(x) \; z \ge 0 \;\; where \;\; \nabla_x^2 f(x) \;\; is \; the \; hessian$$

- **Sum/Linear-combination of two or more convex functions is also convex:** Let f(x) and g(x) be two convex functions. Then any linear combination of these two functions

  $$(\lambda_1 f + \lambda_2 g)(x) = \lambda_1 f(x) + \lambda_2 g(x)$$

  is also a convex function (this can be easily proved using the definition of the convex function).

Now notice that if we can prove that the two functions

$$-\log\left(h_\theta\left(x^i\right)\right) \;\; and \;\; -\log\left(1-h_\theta\left(x^i\right)\right)$$

are convex, then our objective function

$$J(\theta) = \sum_{i=1}^{m} y^i \left[-\log\left(h_\theta\left(x^i\right)\right)\right] + \left(1-y^i\right)\left[-\log\left(1-h_\theta\left(x^i\right)\right)\right]\Big)$$

must also be convex since any linear combination of two or more convex functions is also convex.

Let us now try to prove that

$$-\log\left(h_\theta(x)\right) = -\log\left(\frac{1}{1+e^{-\theta^T x}}\right) = \log\left(1+e^{-\theta^T x}\right)$$

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Deepak Roy Chittajallu

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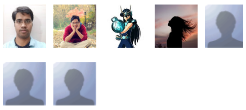

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is a convex function of theta. In order to do this, we will use the second-order condition of convexity described above. Let us first compute the hessian matrix:

$$
\begin{aligned}
&grad: \\
\nabla_\theta \left[-\log\left(h_\theta(x)\right)\right] &= \nabla_\theta \left[\log\left(1+e^{-\theta^T x}\right)\right] \\
&= \left(\frac{-e^{-\theta^T x}}{1+e^{-\theta^T x}}\right) x \\
&= \left(\frac{1}{1+e^{-\theta^T x}} - 1\right) x \\
&= \left(h_\theta(x) - 1\right) x
\end{aligned}
$$

$$
\begin{aligned}
&hessian: \\
\nabla_\theta^2 \left[-\log\left(h_\theta(x)\right)\right] &= \nabla_\theta \left(\nabla_\theta \left[-\log\left(h_\theta(x)\right)\right]\right) \\
&= \nabla_\theta \left(\left(h_\theta(x) - 1\right) x\right) \\
&= h_\theta(x)\left(1 - h_\theta(x)\right) x x^T
\end{aligned}
$$

Now below is the proof that this hessian matrix is positive semi-definite:

$$
\begin{aligned}
\forall z: \quad z^T \nabla_x^2 \left(-\log\left(h_\theta(x)\right)\right) &= z^T \left[h_\theta(x)\left(1 - h_\theta(x)\right) x x^T\right] z \\
&= h_\theta(x)\left(1 - h_\theta(x)\right)\left(x^T z\right)^2 \geq 0 \qquad (2)
\end{aligned}
$$

Let us now try to prove that

$$
\begin{aligned}
-\log\left(1 - h_\theta(x)\right) &= -\log\left(1 - \frac{1}{1+e^{-\theta^T x}}\right) \\
&= -\log\left(\frac{e^{-\theta^T x}}{1+e^{-\theta^T x}}\right) \\
&= \theta^T x + \log\left(1 + e^{-\theta^T x}\right)
\end{aligned}
$$

is a convex function of theta. In order to do this, we will again use the second-order condition of convexity described above. Let us first compute its hessian matrix:

$$
\begin{aligned}
&grad: \\
\nabla_\theta \left[-\log\left(1 - h_\theta(x)\right)\right] &= \nabla_\theta \left[\theta^T x + \log\left(1 + e^{-\theta^T x}\right)\right] \\
&= x + \nabla_\theta \left[\log\left(1 + e^{-\theta^T x}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&hessian: \\
\nabla_\theta^2 \left[-\log\left(1 - h_\theta(x)\right)\right] &= \nabla_\theta \left(\nabla_\theta \left[-\log\left(1 - h_\theta(x)\right)\right]\right) \\
&= \nabla_\theta \left(x + \nabla_\theta \left[\log\left(1 + e^{-\theta^T x}\right)\right]\right) \\
&= \nabla_\theta^2 \left[-\log\left(h_\theta(x)\right)\right] \\
&\quad (we\ have\ proved\ in\ Eq.\ (2)\ above \\
&\quad that\ this\ is\ positive\ semi-definite)
\end{aligned}
$$

Above, we have proved that both

$$-\log\left(h_\theta\left(x^i\right)\right) \quad and \quad -\log\left(1 - h_\theta\left(x^i\right)\right)$$

are convex functions. And, the error/objective function of logistic regression

$$J(\theta) = \sum_{i=1}^{m} y^i \left[-\log\left(h_\theta\left(x^i\right)\right)\right] + \left(1 - y^i\right)\left[-\log\left(1 - h_\theta\left(x^i\right)\right)\right])$$

is essentially a linear-combination of several such convex functions. Now, since a linear combination of two or more convex functions is convex, we conclude that the objective function of logistic regression is convex.

Hence proved ...

Following the same line of approach/argument it can be easily proven that the objective function of logistic regression is convex even if regularization is used.

Posted by Deepak Roy Chittajallu at 9:38 PM

Labels: Logistic Regression, Machine Learning

12 comments:

Unknown November 2, 2015 at 3:42 PM

Just to correct a little mistake: only a positive linear combination of convex functions is guaranteed to be convex again. However, in the logistic regression case y_i are positive, so it works indeed.

Reply

Replies

for. October 17, 2020 at 8:54 AM

The development of artificial intelligence (AI) has propelled more programming architects, information scientists, and different experts to investigate the plausibility of a vocation in machine learning. Notwithstanding, a few newcomers will in general spotlight a lot on hypothesis and insufficient on commonsense application. machine learning projects for final year In case you will succeed, you have to begin building machine learning projects in the near future.

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Unknown February 13, 2016 at 10:44 AM

This comment has been removed by the author.

Reply

Unknown February 13, 2016 at 11:09 AM

I think for equation(2) you lost a z in the left part.

Reply

Unknown August 14, 2017 at 11:42 PM

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Reply

Unknown March 9, 2018 at 9:26 AM

Awesome post, thank you.

Agree with above that (2) missing a z on L.H.S.

Reply

Effesian March 12, 2018 at 5:22 AM

I think in Eq.(2) you should write grad_{\theta} and not w.r.t. x.
This is of course just a typo.
Nice post btw.!

Reply

Unknown September 24, 2018 at 4:30 AM

I think things like this are really interesting. I absolutely love to find unique places like this. It really looks super creepy though!! Best Machine Learning institute in Chennai | Best Machine learning training | best machine learning training certification

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Robotic Process Automation Tutorial November 4, 2018 at 9:49 PM

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Diya shree July 27, 2019 at 2:46 AM

I feel very grateful that I read this. It is very helpful and very informative and I really learned a lot from it.

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Reply

Javier T August 13, 2020 at 3:10 PM

Thank you a lot! It has been very useful to me :)

Reply

tuan March 23, 2021 at 8:48 PM

who know paper proves that this loss function is convex

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