IMPERIAL

Project Notes

Background & Progress Report

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Chapter 1

Sum Soundness

1.1 Current State

There exists a source of unsoundness in the current definition of the sum state model, related to the representation of empty states and the flipping between both sides of the sum, as there may exist multiple different observably empty states, leading to composition rules being unsound if not accounted for.

To demonstrate this, we may consider the Freeable state. It's full and compositional states are defined as:

$$\underline{\text{Freeable}}(X) \stackrel{\text{def}}{=} \varnothing \mid \mathit{Val}\ X$$
$$\text{Freeable}(X) \stackrel{\text{def}}{=} \bot \mid \varnothing \mid \mathit{Val}\ X$$

With composition:

$$f \cdot \bot = f$$
$$\bot \cdot f = f$$
$$\mathit{Val}\ f_1 \cdot \mathit{Val}\ f_2 = \mathit{Val}\ (f_1 \cdot f_2)$$

This forms a valid partially commutative monoid (PCM). We may note that this definition makes Freeable(X) a type of Sum state, with X the left hand side and the unique state model "Freed" on the right hand side.

Additionally, Freeable(X) defines a predicate, `Freed`, and a single action, `free`, that, when the state is *fully owned* (part of the complete states $\underline{X}$), sets the state to $\varnothing$, effectively freeing that bit of memory. Similarly to other state models, predicates and actions that arent `Freed` and `free` respectively are passed to the contained state model. We define the implementation of `consume`, `free` and that of the auxiliary `is_fully_owned` function as:

```
let consume σ, δ, v⃗ᵢ =
  match δ, σ with
  | Freed, Val σ → lfail (MismatchedState, σ)
  | Freed, ∅ → ok (() ⊥)
  | δ, Val σ →
    let* (v', σ') = X.consume σ δ v⃗ᵢ in
    (v', Val σ')
  | δ, Val σ → lfail (MismatchedState, σ)
```

```
  | _, ⊥ → miss (MissingResource, σ)

let free σ =
  match σ with
  | Val σ when is_fully_owned σ → ok ((), ∅)
  | ∅ → error (DoubleFree, σ)
  | _ → miss (MissingResource, σ)
```

$$\texttt{is\_fully\_owned}\ \sigma \stackrel{\text{def}}{=} \sigma \in \underline{\Sigma}$$

We may now look at an example of using this state model that proves to be unsound. Assume the state model Freeable(Exc), in an initial state $Val\ a$. We want to execute the specification of a simple function that simply executes `free`. It thus has a precondition $\langle PointsTo \rangle(; a)$ and a postcondition $\langle Freed \rangle(;)$. To execute it we thus first consume the precondition, and the produce then postcondition.

State	Operation applied
$Val\ a$	Initial State
$Val\ \bot$	Consume precondition $\langle PointsTo \rangle(; a)$
undefined!	Produce postcondition $\langle Freed \rangle(;)$

This is, of course, unsound, as the state at the end should indeed be $\varnothing$. This is because $Val\ \bot$, which is observably empty, is different from the Freeable empty state, $\bot$, rendering the composition $Val\ \bot \cdot \varnothing$ undefined.

This error can also be shown to exist in the implementation itself, by writing a specification with `False` as the postcondition.

```
spec free_cell()
  [[ <points_to>(;#anything) ]]
  [[ (ret == null) * <freed>(;) ]]
  normal
proc free_cell() {
  n := [free]();
  ret := null;
  return
};

spec test_unsoundness()
  [[ <points_to>(;#anything) ]]
  [[ False ]]
  normal
proc test_unsoundness() {
  n := "free_cell"(x);
  ret := null;
  return
};
```

This GIL code is then succesfuly verified, with the following output.

```
Parsing and compiling...
Preprocessing...
Obtaining specs to verify...
Obtaining lemmas to verify...
Obtained 2 symbolic tests in total
Running symbolic tests: 0.002729
Verifying one spec of procedure free_cell... s Success
Verifying one spec of procedure test_unsoundness... Success
```

```
All specs succeeded: 0.004729
```

1.2 Proof of Unsoundness

The sum state model transformer, $\mathbb{S}_1 \oplus \mathbb{S}_2$ with $\mathbb{S}_1 = (\Sigma_1, 0_1, \cdot)$ and $\mathbb{S}_2 = (\Sigma_2, 0_2, \cdot)$ two valid state models, is currently unsound. In particular, any action that allows flipping the sum from one side to the other is unsound, as it doesn't satisfy frame substraction. This is, for instance, the case with the `free` action of the Freeable state model, which is just a type of sum. It is $\mathbb{S}_1 \oplus \mathbb{S}_2 \stackrel{\text{def}}{=} \texttt{|}\ \bot\ \texttt{|}\ \texttt{S1}\ \Sigma_1\ \texttt{|}\ \texttt{S2}\ \Sigma_2$, and composition is defined as:

$$\sigma \cdot \bot = \sigma$$
$$\bot \cdot \sigma = \sigma$$
$$(\texttt{S1}\ \sigma) \cdot (\texttt{S1}\ \sigma') = \texttt{S1}\ (\sigma \cdot \sigma')$$
$$(\texttt{S2}\ \sigma) \cdot (\texttt{S2}\ \sigma') = \texttt{S2}\ (\sigma \cdot \sigma')$$
$$\text{undefined otherwise}$$

We also remind the frame substraction property, defined as:

$$\begin{aligned} &p \vdash (\sigma \cdot \sigma_f, e) \Downarrow_\theta o : (\sigma', v) \implies \\ &\quad (\exists o', v', \sigma''.\ p \vdash (\sigma, e) \Downarrow_\theta o' : (\sigma'', v') \wedge \\ &\qquad (o' \neq \texttt{Miss} \implies \sigma' = \sigma'' \cdot \sigma_f \wedge o = o' \wedge v = v')) \end{aligned}$$

Proof.

Proposition: Sum actions that swap sides are not frame preserving
Assuming

(H1) $\mathbb{S}_1$ is a well formed PCM, $\mathbb{S}_1 \stackrel{\text{def}}{=} (\Sigma_1, 0_1, \cdot)$

(H2) $\mathbb{S}_2$ is a well formed PCM, $\mathbb{S}_2 \stackrel{\text{def}}{=} (\Sigma_2, 0_2, \cdot)$

(H3) There is an action `swap` that may flip the side of a sum from one side to the other, ie.

$$\exists \sigma_1, \sigma_2', v.\ \texttt{swap}(\texttt{S1}\ \sigma_1) \rightsquigarrow (\texttt{Ok}, \texttt{S2}\ \sigma_2', v)$$

We want to prove frame substraction does not hold with action `swap`.

(H4) From (H3) we have $\exists \sigma_1, \sigma_2', v.\ \texttt{swap}(\texttt{S1}\ \sigma_1) \rightsquigarrow (\texttt{Ok}, \texttt{S2}\ \sigma_2', v)$

(H5) From the definition of composition, $\texttt{S1}\ \sigma_1 \cdot \texttt{S1}\ 0_1 = \texttt{S1}\ \sigma_1$

(H6) From (H5) and (H4), $\texttt{swap}((\texttt{S1}\ \sigma_1 \cdot \texttt{S1}\ 0_1)) \rightsquigarrow (\texttt{Ok}, \texttt{S2}\ \sigma_2', v)$

To satisfy frame substraction, as the outcome in (H4) is `Ok`, we would need $\texttt{S2}\ \sigma_2 = \texttt{S2}\ \sigma_2 \cdot \texttt{S1}\ 0_1$, which is not the case, as per the definition of composition, $\texttt{S2}\ \sigma_2 \cdot \texttt{S1}\ 0_1$ is undefined. Frame substraction thus does not hold for `swap`. □

Note the same can be proven analogously for actions that flip from the right-hand side to the left-hand side.

1.3 Sound Sum

To define a sound version of sum that supports flipping the side, we thus need to remove the 0 element of both sides from the allowed states, resulting in $\mathbb{S}_1 \oplus \mathbb{S}_2 \stackrel{\text{def}}{=} \texttt{|}\ \bot\ \texttt{|}\ \texttt{S1}\ (\Sigma_1 \backslash \{0_1\})\ \texttt{|}\ \texttt{S2}\ (\Sigma_2 \backslash \{0_2\})$. This ensure the states $\texttt{S1}\ 0_1$ and $\texttt{S2}\ 0_2$ aren't allowed, avoiding the problem seen in frame substraction.

Additionally, to allow for a frame-preserving `swap` action, a stronger requirement than a complete state must be met: the state must be *exclusively owned* – no other non-empty state can be composed with it. This thus requires for a `is_exclusively_owned` function to be provided, defined as:

$$\texttt{is\_exclusively\_owned}\ \sigma \stackrel{\text{def}}{=} \forall \sigma'.\ \sigma' \neq 0 \implies \neg(\sigma \# \sigma')$$

Proof. **Proposition:**

If a valid frame-preserving sum actions swaps sides, then that sum cannot permit empty elements on either sides and the swapped states must be exclusively owned
Assuming

(H1) $\mathbb{S}_1$ is a well formed PCM, $\mathbb{S}_1 \stackrel{\text{def}}{=} (\Sigma_1, 0_1, \cdot)$

(H2) $\mathbb{S}_1$ is cancellative, $\sigma_A \cdot \sigma_B = \sigma_A \cdot \sigma_C \implies \sigma_B = \sigma_C$

(H3) $\mathbb{S}_2$ is a well formed PCM, $\mathbb{S}_2 \stackrel{\text{def}}{=} (\Sigma_2, 0_2, \cdot)$

(H4) $\mathbb{S}_2$ is cancellative, $\sigma_A \cdot \sigma_B = \sigma_A \cdot \sigma_C \implies \sigma_B = \sigma_C$

(H5) A sum state model is defined as $\mathbb{S}_1 \oplus \mathbb{S}_2 \stackrel{\text{def}}{=} \bot \mid \varnothing \mid \texttt{S1}\ \Sigma_1' \mid \texttt{S2}\ \Sigma_2'$ with $\Sigma_1' \subseteq \Sigma_1$ and $\Sigma_2' \subseteq \Sigma_2$.

(H6) Composition of the sum state model is defined as:

$$\begin{gathered}
\sigma \cdot \bot = \sigma \\
\bot \cdot \sigma = \sigma \\
(\texttt{S1}\ \sigma) \cdot (\texttt{S1}\ \sigma') = \texttt{S1}\ (\sigma \cdot \sigma') \\
(\texttt{S2}\ \sigma) \cdot (\texttt{S2}\ \sigma') = \texttt{S2}\ (\sigma \cdot \sigma') \\
\text{undefined otherwise}
\end{gathered}$$

(H7) There is an action `swap` that may flip the side of a sum from the left to then right-hand side.

$$\exists \sigma_1, \sigma_2', v.\ \texttt{swap}(\texttt{S1}\ \sigma_1) \rightsquigarrow (\texttt{Ok}, \texttt{S2}\ \sigma_2', v)$$

This `swap` action is both valid and frame preserving, thus satisfying the following:

(H8) Frame substraction:

$$\begin{aligned}&\mathsf{swap}(\sigma \cdot \sigma_f) \rightsquigarrow (o, \sigma', v) \implies \\ &\quad (\exists o', v', \sigma''.\ \mathsf{swap}(\sigma) \rightsquigarrow (o', \sigma'', v') \wedge \\ &\qquad (o' \neq \mathtt{Miss} \implies \sigma' = \sigma'' \cdot \sigma_f \wedge o = o' \wedge v = v'))\end{aligned}$$

(H9) Frame addition:

$$\begin{aligned}&\mathsf{swap}(\sigma) \rightsquigarrow (o, \sigma', v) \wedge \sigma' \# \sigma_f \wedge o \neq \mathtt{Miss} \\ &\quad \implies \mathsf{swap}(\sigma \cdot \sigma_f) \rightsquigarrow (o, \sigma' \cdot \sigma_f, v)\end{aligned}$$

(H10) `Miss` executions may be completed:

$$\begin{aligned}&\mathsf{swap}(\sigma) \rightsquigarrow (\mathtt{Miss}, \sigma', v) \\ &\quad \implies (\exists \sigma_f.\ \forall o, \sigma'', v'.\ \mathsf{swap}(\sigma \cdot \sigma_f) \rightsquigarrow (o, \sigma'', v') \implies o \neq \mathtt{Miss})\end{aligned}$$

We aim to prove:

(G1) The zero elements of the left side of the sum is not permitted: $0_1 \notin \Sigma_1'$

(G2) The zero elements of the right side of the sum is not permitted: $0_2 \notin \Sigma_2'$

(G3) If `swap` flips the side of the sum, any `swap` of "smaller" states will result in a `Miss`:

$$\begin{aligned}&\mathsf{swap}(\mathtt{S1}\ \sigma_1) \rightsquigarrow (\mathtt{Ok}, \mathtt{S2}\ \sigma_2', v) \wedge \mathtt{S1}\ \sigma_1 = \mathtt{S1}\ \sigma_A \cdot \mathtt{S1}\ \sigma_B \wedge \mathsf{swap}(\mathtt{S1}\ \sigma_A) \rightsquigarrow (o, \sigma'', v') \\ &\quad \implies o = \mathtt{Miss}\end{aligned}$$

(G4) If `swap` flips the side of the sum, the new state must be exclusively owned:

$$\mathsf{swap}(\mathtt{S1}\ \sigma_1) \rightsquigarrow (\mathtt{Ok}, \mathtt{S2}\ \sigma_2', v) \implies \mathtt{is\_exclusively\_owned}\ \sigma_2'$$

We first prove (G1).

(H11) Assume $0_1 \in \Sigma_1'$.

(H12) From (H6), (H11) and (H7), $\mathsf{swap}((\mathtt{S1}\ \sigma_1 \cdot \mathtt{S1}\ 0_1)) \rightsquigarrow (\mathtt{Ok}, \mathtt{S2}\ \sigma_2', v)$

From (H8) and (H12),

(H13) $\exists p', v', \sigma''.\ \mathsf{swap}(\mathtt{S1}\ \sigma_1) \rightsquigarrow (o', \sigma'', v)$

(H14) $o' \neq \mathtt{Miss} \implies \mathtt{S2}\ \sigma_2' = \sigma'' \cdot \mathtt{S1}\ 0_1 \wedge \mathtt{Ok} = o' \wedge v = v'$

(H15) From (H7) and (H13), $o' = \mathtt{Ok} \wedge \sigma'' = \mathtt{S2}\ \sigma_2' \wedge v' = v$

However, from (H6), $\mathtt{S2}\ \sigma_2' \cdot \mathtt{S1}\ 0_1$ is undefined, thus despite $o' = \mathtt{Ok}$ (H15), $\mathtt{S2}\ \sigma_2' \neq \mathtt{S2}\ \sigma_2' \cdot \mathtt{S1}\ 0_1$ – our hypothesis (H11) is thus wrong, giving us our goal $0_1 \notin \Sigma_1'$ (G1).
We now prove (G2).

(H16) Assume $0_2 \in \Sigma_2'$.

(H17) Per (H16) and (H6), $\mathtt{S2}\ \sigma_2' \# \mathtt{S2}\ 0_2$, and $\mathtt{S2}\ \sigma_2' \cdot \mathtt{S2}\ 0_2 = \mathtt{S2}\ \sigma_2'$

Thus per (H7), (H17) and (H9), we would have $\mathtt{swap}((\mathtt{S1}\ \sigma_1 \cdot \mathtt{S2}\ 0_2)) \rightsquigarrow (\mathtt{Ok}, \mathtt{S2}\ \sigma_2' \cdot \mathtt{S2}\ 0_2, v)$. This is however not the case, as per (H6), $\mathtt{S1}\ \sigma_1 \cdot \mathtt{S2}\ 0_2$ is undefined. Our hypothesis (H16) is thus wrong, giving us our goal $0_2 \notin \Sigma_2'$ (G2).
We may now prove our third goal (G3).

(H18) Let $\mathtt{swap}(\mathtt{S1}\ \sigma_1) \rightsquigarrow (\mathtt{Ok}, \mathtt{S2}\ \sigma_2', v) \wedge \mathtt{S1}\ \sigma_1 = \mathtt{S1}\ \sigma_A \cdot \mathtt{S1}\ \sigma_b \wedge \mathtt{swap}(\mathtt{S1}\ \sigma_A) \rightsquigarrow (o, \sigma'', v')$.

(H19) Assume $o \neq \mathtt{Miss}$.

(H20) From (H8), (H18) and (H19), we have $\mathtt{S2}\ \sigma_2' = \sigma'' \cdot \mathtt{S1}\ \sigma_B$, and $o = \mathtt{Ok}$. From (H6) this is however not possible, as no composition from $\mathtt{S1}$ yields $\mathtt{S2}$. Our supposition (H19) is thus false, thus $o = \mathtt{Miss}$.

We may finally prove our fourth and last goal (G4).

(H21) Let $\mathtt{swap}(\mathtt{S1}\ \sigma_1) \rightsquigarrow (\mathtt{Ok}, \mathtt{S2}\ \sigma_2', v)$.

(H22) Assume $\neg\mathtt{is\_exclusively\_owned}\ \sigma_2'$.

(H23) From (H22), $\exists\sigma'.\ \sigma' \neq 0 \wedge \sigma_2 \# \sigma'$, thus $\sigma_2 \cdot \sigma'$ is defined and from (H6) must be expressed as $\sigma' = \mathtt{S2}\ \sigma_2''$.

(H24) From (H9), we must have $\mathtt{swap}((\mathtt{S1}\ \sigma_1 \cdot \mathtt{S2}\ \sigma_2'')) \rightsquigarrow (\mathtt{Ok}, \mathtt{S2}\ \sigma_2' \cdot \mathtt{S2}\ \sigma_2'', v)$. This is however not possible, as per (H6), $\mathtt{S1}\ \sigma_1 \cdot \mathtt{S2}\ \sigma_2''$ is undefined. Thus our hypothesis (H22) is false, giving us our goal $\mathtt{is\_exclusively\_owned}\ \sigma_2'$ (G4).

The same can of course be analogously proven for a `swap`that goes from the right-hand side to the left.

□

Because Freeable(X) is a type of sum, with X on the left-hand side and a single-state "Freed" state on the right, we may now present an implementation of `consume` and the `free` action of Freeable, that satisfy the above requirements relating to its validity and soundness, and that can be proven to be sound. We assume the input state model X provides an `is_exclusively_owned` function.

```
let consume σ, δ, v⃗ᵢ =
  match δ, σ with
  | Freed, Val σ → lfail (MismatchedState, σ)
  | Freed, ∅ → ok (() ⊥)
  | _, ⊥ → miss (MissingResource, σ)
  | δ, Val σ →
    let* (v', σ') = X.consume σ δ v⃗ᵢ in
    if σ' = X.0 then ⊥ else (v', Val σ')
  | δ, Val σ → lfail (MismatchedState, σ)
  | δ, ⊥ →
    let* (v', σ') = X.consume X.0 δ v⃗ᵢ in
    if σ' = X.0 then ⊥ else (v', Val σ')

let free σ =
  match σ with
  | Val σ when is_exclusively_owned σ → ok ((), ∅)
  | ∅ → error (DoubleFree, σ)
  | _ → miss (MissingResource, σ)
```

The main differences with the previous unsound version are the use of `is_exclusively_owned` over `is_fully_owned`, which better describes the requirement for no non-0 frame to exist, and the handling of `consume`, which verifies if the new state is *Val* 0, and if so sets it to $\bot$. We may note also that consuming a predicate δ while in $\bot$ doesn't automatically result in a `Miss`anymore, as it may be the case that X satisfies δ – this would be the case, for instance, if it were to expose an "emp" predicate, as is seen in linear separation logic.

1.4 Sound Emptiness

A fix to this is to define a "global" emptiness, that replaces the different "local" empty states each state model may define. Consuming a predicate, or executing an actiyeson, may result in a new state *or* a global $\bot$. This forces state model transformers to handle such empty states, and ensures a state becoming empty deep within a now-observably-empty construction will naturally unwrap into a shallow empty. We also lift the composition operation defined by state models to handle $\bot$: $a \cdot \bot = \bot \cdot a = a$.

A side-effect of this is that a non-$\bot$ state is never considered observably-empty, as otherwise it would be $\bot$ – this allows us to remove the `is_empty` function that was used for some optimisations, as it is sufficient to compare a given state with $\bot$.

An advantage of this approach is that lifting a full state model to a complete state model doesn't require anything (aside from handling $\bot$ in `produce` and `consume`), as the empty compositional state is already added!

We now redefine some of the constructs used in the engine according to this new definition.

$$\Sigma^? \stackrel{\text{def}}{=} \Sigma \uplus \bot$$

$$\texttt{consume} : \Sigma \to \Delta \to \text{Val } \texttt{list} \to (\mathcal{O}_l^+ \times \text{Val } \texttt{list} \times \Sigma^?)$$

$$\texttt{produce} : \Sigma^? \to \Delta \to \text{Val } \texttt{list} \to \text{Val } \texttt{list} \to \Sigma \texttt{ set}$$

$$\texttt{eval\_action} : \mathcal{A} \to \Sigma^? \to \text{Val } \texttt{list} \to (\mathcal{O}_l^+ \times \text{Val } \texttt{list} \times \Sigma^?)$$

A consideration with this is that `consume` only works on non-$\bot$ elements, as no predicate is satisfied by the empty state (not sure about this). For produce however, a predicate can be produced from the absence of state.

We may now redefine some of the state models with this new idea.

1.4.1 State Sum

A state sum $\mathbb{S}_1 \oplus \mathbb{S}_2$ is now defined as `type` Σ `=` `S1 of` $\mathbb{S}_1.\Sigma$ `|` `S2 of` $\mathbb{S}_2.\Sigma$. We define composition and `produce` as follows:

$$(\texttt{S1}\ \ \sigma) \cdot (\texttt{S1}\ \ \sigma') = \texttt{S1}\ \ (\sigma \cdot \sigma')$$
$$(\texttt{S2}\ \ \sigma) \cdot (\texttt{S2}\ \ \sigma') = \texttt{S2}\ \ (\sigma \cdot \sigma')$$
$$\text{undefined otherwise}$$

```
produce σ? δ v⃗ᵢ v⃗ₒ =
  match σ?, δ with
  | S1 σ₁, P1 δ₁ →
```

```
  let* δ1' = produce σ1 δ1 v⃗i v⃗o in
  S1 δ1'
| ⊥, P1 δ1 →
  let* δ1' = produce 𝕊1.0 δ1 v⃗i v⃗o in
  S1 δ1'
| S2 σ2, P2 δ2 →
  let* δ2' = produce σ2 δ2 v⃗i v⃗o in
  S2 δ2'
| ⊥, P2 δ2 →
  let* δ2' = produce 𝕊2.0 δ2 v⃗i v⃗o in
  S2 δ2'
| _, _ → vanish
```

Chapter 2

Resource Algebras for a CSE

2.1 Current State

2.1.1 CSE

CSE and Sacha's thesis do the traditional choice of using Partial Commutative Monoids (PCMs) to model state. They are defined as the tuple $(M, (\cdot) : M \times M \rightharpoonup M, 0)$. They are further equipped with a set of actions $\mathcal{A}$, an `execute_action` function, a set of core predicates Δ and a pair of `consume` and `produce` functions.

These additions are necessary for the engine to be parametric on the state model, as it provides an interface for interaction with the state.

The usage of PCMs comes with issues: the requirement of a single 0 for each state model means that state models such as the sum state model $\mathbb{S}_1 + \mathbb{S}_2$ come with unwieldy requirements to prove soundness – this comes into play for the `Freeable` state model, that could use a sum (like what is done in [1]) but can't because of this.

2.1.2 Iris

Iris [2] departs from this tradition and introduces Resource Algebras (RAs) to model state, defined as a tuple $(M, \overline{\mathcal{V}} : M \to \mathbb{B}, |-| : M \to M^?, (\cdot) : M \times M \to M)$, being respectively the state elements, a validity function, a partial core function and a composition function.

This makes Iris states more powerful, in that they have more flexibility in what they can express; for instance sum state models can be easily and soundly expressed, which isn't possible with PCMs due to the requirement of a single 0 element.

Furthermore, Iris RAs comes with plenty proofs and properties making them easy to use and adapt, whereas PCMs can prove unwieldy even for simpler state models (eg. with the Freeable state model transformer).

A similarity however is that the global RA in Iris must be unital, meaning it must have a single ϵ element, very much as it is the case with the 0 in PCMs. Any RA can be trivially extended to have a unit, which is what Iris defines as the option resource algebra [3].

2.2 Partial RAs

A property of Iris RAs is that composition is *total* – to take into account invalid composition, states are usually extended with a $\lightning$ state, such that $\neg\overline{\mathcal{V}}(\lightning)$ (while for states $\sigma \neq \lightning$, $\overline{\mathcal{V}}(\sigma)$

A *resource algebra* (RA) is a triple $(M, |-| : M \to M^?, (\cdot) : M \times M \rightharpoonup M)$

$$\forall a, b, c.\, (a \cdot b) \cdot c = a \cdot (b \cdot c) \qquad \text{(RA-Assoc)}$$
$$\forall a, b.\, a \cdot b = b \cdot a \qquad \text{(RA-Comm)}$$
$$\forall a.\, |a| \in M \Rightarrow |a| \cdot a = a \qquad \text{(RA-Core-ID)}$$
$$\forall a.\, |a| \in M \Rightarrow ||a|| = |a| \qquad \text{(RA-Core-Idem)}$$
$$\forall a, b.\, |a| \in M \land a \preccurlyeq b \Rightarrow |b| \in M \land |a| \preccurlyeq |b| \qquad \text{(RA-Core-Mono)}$$

where $M^? \stackrel{\text{def}}{=} M \uplus \{\bot\}$, with $a \cdot \bot \stackrel{\text{def}}{=} \bot \cdot a \stackrel{\text{def}}{=} a$

$a \preccurlyeq b \stackrel{\text{def}}{=} \exists c.\, b = a \cdot c$

$a \# b \stackrel{\text{def}}{=} a \cdot b$ is defined

A *unital* resource algebra is a resource algebra M with an element $\epsilon \in M$ such that:

$$\forall a \in M. \epsilon \cdot a = a \qquad |\epsilon| = \epsilon$$

Figure 2.1: Definition of Resource Algebras

holds). While this is needed in the Iris framework for higher-order ghost state and step-index, this doesn't come into play when only manipulating RAs. As such, because this is quite unwieldy, we can remove it by adding partiality instead, such that invalid ($\lightning$) states simply don't exist and the need for a $\overline{\mathcal{V}}$ function vanishes. This is also inline with the core function $(-)$ being partial.

It is worth noting that *partial* RAs are equivalent to regular RAs, *so long as* $\overline{\mathcal{V}}$ *always holds for valid states*[1]. Indeed, compositions that yield $\lightning$ can be made undefined, and the validity function removed, to gain partiality, and inversely to go back to the Iris definition.

An interesting property of this is that because validity is replaced by the fact composition is defined, the validity of a composition is equivalent to the fact two states are disjoint: $\overline{\mathcal{V}}(a \cdot b) \iff a \# b$.

We now define the properties of RAs taking this change into account – see Figure 2.1. From now, the term RA will be used to refer to these partial RAs.

2.3 CSE State Models + RAs

We now propose to redefine the notion of state models. To follow the spirit of CSE, that comes with a core engine, a compositional engine and a bi-abduction engine all built onto each other, we go through each layer, presenting what is for that part of the engine to function.

2.3.1 Core Engine

The core engine enables whole-program symbolic execution. For this state models must firstly define the set of states the execution will happen on; this is done via a partial resource algebra: a tuple $(M, |-| : M \to M^?, (\cdot) : M \times M \rightharpoonup M)$. They are further

[1] This, to our knowledge, is the case for all of the simpler RAs defined in Iris: Ex, Ag_0, sum, product, etc.

equipped with a set of actions $\mathcal{A}$, an `execute_action` function and a `sat` relation.

$$\texttt{execute\_action} : \hat{\Sigma}^? \to \mathcal{A} \to \text{Val } \texttt{list} \to \mathcal{P}(\underline{\mathcal{O}_e} \times \hat{\Sigma}^? \times \text{Val } \texttt{list} \times \Pi)$$
$$\texttt{sat} : \Theta \to \text{Store} \to \hat{\Sigma} \to \mathcal{P}(\Sigma)$$

The arguments of `execute_action` are, in order: the *optional* state the action is executed on, the action, and the received arguments. It returns a set of branches, with an outcome, the new state, the returned values, and the path condition of that branch. It is pretty-printed as $\alpha(\hat{\sigma}, \vec{\hat{v}_i}) \rightsquigarrow (o, \hat{\sigma}', \vec{\hat{v}_o}, \pi)$.

Here, the outcome is an outcome in the set of *full execution outcomes* $\underline{\mathcal{O}_e} = \{\texttt{Ok}, \texttt{Err}\}$. In the next subsection, this set will be extended to account for misses and logical failures, but these do not exist with full semantics.

The main difference here is that the state may be $\bot$, if the action is executed on empty state. This ensures non-unital RAs are not ruled out as invalid – indeed, many useful RAs are not unital and sometimes don't have a unit at all, as is the case for instance for Ex, the exclusively owned cell. One could decide to internally make all RAs of state models unital, and have the state model provide an `empty` function that returns said unit (this is what happens in Gillian). However this introduces unsoundness to certain state model constructions (in particular the sum), as this means the state cannot be *exclusively owned* – the empty state could always be composed with it.

Whole-program symbolic execution is, by definition, non-compositional – it thus operates on *full state*, a notion introduced in [4]. As such, the only valid outcomes here are `Ok` and `Err`.

Because we operate in symbolic memory, an additional piece of information is the *path condition*, the set of constraints accumulated throughout execution. A path condition $\pi \in \Pi$ is a *list of symbolic values*, that evaluates to a boolean. We decide to define it as a list rather than a single conjunction of boolean symbolic values, as this allows us to easily check if a path condition is an extension (or a strengthening) of another, with $\pi' \supseteq \pi$. We further define the predicate `SAT`, that is true if, given the substitution θ, store s and a path condition π, the conjunction of the elements of π resolves to `true` once evaluated:

$$\texttt{SAT}_{\theta,s}(\pi) \stackrel{\text{def}}{=} \left[\!\!\left[\bigwedge_i^{|\pi|} \pi(i) \right]\!\!\right]_{\theta,s} = \texttt{true}$$

We note that **cse2** also has an 'SV' argument in action execution, that contains all existing symbolic variables, and that must be used when creating a new symbolic variable to ensure it is fresh. While necessary for proofs within the engine with Rocq, we omit it here. It is only used for allocation and it can instead be kept implicit, by assuming we can always generate a fresh symbolic variable.

Finally, the user must define the `sat` relation, relating concrete and symbolic states. It is pretty printed as $\theta, s, \sigma \models \hat{\sigma}$, meaning that given a substitution θ and a store s, the concrete state σ can be matched by a symbolic state $\hat{\sigma}$. We define this relation for non-$\bot$ states; we can then lift it to the option state model $\Sigma^?$, by simply adding that $\forall \theta, s.\, \theta, s, \bot \models \bot$. This relieves users from needing to take $\bot$ into account when defining $\models$ and from proving the fairly trivial axiom Empty Memory.

2.3.2 Compositional Engine

The compositional engine, built on top of the core engine, allows for verification of function specifications, and handles calling functions by their specification. As such, the state model must be extended with a set of core predicates Δ and a pair of `consume` and `produce` functions (equivalent, respectively, to a resource assert and assume). Finally, to link core predicates to states, it provides a $\texttt{sat}_\Delta$ relation.

$$
\begin{aligned}
\text{for } M = \{&\texttt{OX}, \texttt{UX}\} \\
&\texttt{consume} : M \to \hat{\Sigma}^? \to \Delta \to \text{Val } \texttt{list} \to \mathcal{P}(\mathcal{O}_l \times \hat{\Sigma}^? \times \text{Val } \texttt{list} \times \Pi) \\
&\texttt{produce} : \hat{\Sigma}^? \to \Delta \to \text{Val } \texttt{list} \to \text{Val } \texttt{list} \to \mathcal{P}(\hat{\Sigma}^? \times \Pi) \\
&\texttt{sat}_\Delta : \Sigma^? \to \Delta \to \text{Val } \texttt{list} \to \mathcal{P}(\text{Val } \texttt{list})
\end{aligned}
$$

Similarly to `execute_action`, the input state can be $\bot$. While intuitively one may assume that the input state of `consume` and the output state of `produce` may never be $\bot$, this would limit what core predicates can do. In particular, this means an *emp* predicate couldn't be defined, since it's production on an empty state results in an empty state.

The arguments of `consume` are, in order: the mode of execution to distinguish between under-approximate and over-approximate reasoning, the state, the core predicate being consumed, the ins of the predicate. It outputs a *logical outcome*, the state with the matching predicate removed (which may result in an empty state $\bot$), the outs of the predicate and the associated path condition. It is pretty-printed as $\texttt{consume}(m, \hat{\sigma}, \delta, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}_f, \vec{\hat{v}}_o, \pi)$, and when the consumption is valid in both OX and UX the mode is omitted.

For `produce`, the arguments are the state, the core predicate being produced, the ins and the outs of the predicate, resulting in a set of new states and their associated path condition. As an example, producing $x \mapsto 0$ in a state $[1 \mapsto 2]$ results in a new state $[1 \mapsto 2, x \mapsto 0]$ with the path condition $x \neq 1$. If the produced predicate is incompatible with the state (eg. by producing $1 \mapsto y$ in a state containing $1 \mapsto x$), the producer *vanishes*. Inversely, if the assertion can be interpreted in several ways, the producer may branch. It is pretty-printed as $\texttt{produce}(\hat{\sigma}, \delta, \vec{\hat{v}}_i, \vec{\hat{v}}_o) \rightsquigarrow (\hat{\sigma}', \pi)$.

The $\texttt{sat}_\Delta$ relation relates a possibly empty *concrete* state, core predicate and in-values to a set of out-values. It is pretty-printed as $\sigma \models_\Delta \langle\delta\rangle(\vec{v}_i; \vec{v}_o)$ for $\vec{v}_o \in \texttt{sat}_\Delta\ \sigma\ \delta\ \vec{v}_i$. For instance in the linear heap state model, we have $[1 \mapsto 2] \models_\Delta \langle\texttt{points\_to}\rangle(1; 2)$. We also lift this relation to the symbolic realm:

$$
\theta, s, \sigma \models_\Delta \langle\delta\rangle(\vec{\hat{v}}_i; \vec{\hat{v}}_o) \stackrel{\text{def}}{=} [\![\vec{\hat{v}}_i]\!]_{\theta,s} = \vec{v}_i \wedge [\![\vec{\hat{v}}_o]\!]_{\theta,s} = \vec{v}_o \wedge \sigma \models_\Delta \langle\delta\rangle(\vec{v}_i; \vec{v}_o)
$$

Here we define logical outcomes $\mathcal{O}_l = \{\texttt{Ok}, \texttt{LFail}, \texttt{Miss}\}$. These are outcomes that happen during reasoning; in particular, `LFail` equates to a logical failure due to an incompatibility between the consumed predicate and the state. For instance, consuming $1 \mapsto 1$ when in state $1 \mapsto 2$ would yield a `LFail`, while consuming it in state $5 \mapsto 3$ would yield a `Miss`, as a state $1 \mapsto x$ could be composed with it to yield a non-miss outcome.

We must also modify the signature of `execute_action`, to include `Miss` outcomes, via the set of execution outcomes $\mathcal{O}_e = \{\texttt{Ok}, \texttt{Err}, \texttt{Miss}\}$.

An addition to what CSE previously defined is thus the split of what was the `Abort` outcome into `LFail` and `Miss`, this improves the quality of error messages and allows fixing consumption errors due to missing state – this will be described in the next subsection.

A last change compared to CSE is that we drop the path condition parameter to con-

sume and produce – the function instead directly returns the path condition required for the resulting branches, and the engine can filter these. For instance, in UX all consumption branches that result in `LFail` can be dropped, as dropping branches is allowed in UX. This has the advantage of simplifying the axioms, as the path condition is strengthened by definition; the function itself has no way of weakening it. *Mention that this also makes checkpointing with the SAT engine trivial – just set a checkpoint before* `consume/produce`*, add whatever that returns. No need to separate the old from new, avoids duplicating/deduplicating, etc.*

2.3.3 Bi-abduction Engine

To support bi-abduction in the style of Infer:Pulse [5], `Miss` outcomes must be fixed. These outcomes may happen during consumption or during action execution. For this, the state model must provide a `fix` function, that given the details of a miss error (these details being of type Val and returned with the outcome) returns a list of sets of assertions that must be produced to fix the missing error.

$$\texttt{fix} : \text{Val} \to \mathcal{P}(\text{Asrt})\ \texttt{list}$$

Note here we return a *list* of different fixes, which themselves are a set of assertions – this is because, for a given missing error, multiple fixes may be possible which causes branching. For instance, in the typical linear heap, accessing a cell that is not in the state fragment at address a results in a miss that has two fixes: either the cell exists and points to some existentially quantified variable (the fix is thus $\exists x.\, a \mapsto x$), or the cell exists and has been freed ($a \mapsto \varnothing$).

This approach is different from how Gillian handles it. There the function `fix` returns *pure* assertions (type information, pure formulae) and arbitrary values of type `fix_t`, which can then be used with the $\texttt{apply\_fix} : \Sigma \to \texttt{fix\_t} \to \Sigma$ method of the monadic state. This means fixes can be arbitrary modifications to the state that don't necessarily equate to new assertions to add to the anti-frame.

This is a source of unsoundness, as the engine may interpret these modifications as fixes despite them not reliably modifying the state. This can be seen in [4], where not finding the binding in a PMap(X) returns a `MissingBinding` error. While being labelled as a miss, this error can actually not be fixed; PMap simply *lifts* predicates with an additional in parameter for the index. An implementation of that version of PMap(X) could attempt to fix this state by add a binding to $X.0$ (PMaps were originally made for PCMs, which always have a 0 element), which would then eventually lead to another error once the action gets called on the empty state. On top of being under-performing (as several fixes would need to be generated for one action), this requires PMap(X) to allow empty states in the codomain, which means a PMap is never exclusively owned (as a state with a singleton map to $X.0$ can always be composed with it), which limits its usability; aside from not being modelable using RAs, since $\bot$ is not an element of X's carrier set. Finally, if the underlying state model doesn't provide any additional fixes, then the fix for `MissingBinding` cannot be added to the UX specification of a function: there is no assertion generatable from within PMap to represent this modification. As such, having `fix` returns assertions without modifying any state directly ensures fixes are always soundly handled.

To finish this, we may note the solution to the above bug is to proceed executing the action on the underlying state model, giving it an empty state – it will then raise the

appropriate `Miss`, which can be fixed, as it is aware of what core predicates are needed to create the required state. For instance, for PMap(Exc) a `load` action on a missing binding would be executed against $\bot$, which would return a `MissingValue` error. The PMap could then wrap the error with information about the index at which the error occurred, $\texttt{SubError}(i, \texttt{MissingValue})$. When getting the fix, PMap can then call Exc.`fix`, which returns $\exists x.\, \langle points\_to \rangle(; x)$, and lift the fix by adding the index as an in-argument, resulting in the final fix $\exists x.\, \langle points\_to \rangle(i; x)$, which is a valid assertion and can be added to the UX specification for this execution.

2.3.4 Axioms

We may now go over the axioms that must be respected by the above defined functions for the soundness of the engine. Note we will thus focus on the axioms related to the state models in particular, and not the general semantics of the engine.

It is worth noting that Gillian supports both over-approximate (OX) and under-approximate (UX) reasoning – for which *frame subtraction* or *frame addition* must hold, respectively.

For all of the axioms we assume we have a symbolic state model $\mathbb{S}$, made of the RA $\hat{\Sigma} \ni \hat{\sigma}$. We consider the initial state $\hat{\sigma}$ well-formed.

Symbolicness Axioms *a nicer name would be good*

$$\theta, s, \sigma \models \bot_{\hat{\Sigma}^?} \iff \sigma = \bot_{\Sigma^?} \qquad \text{(Empty Memory)}$$

The above axiom is obtained for free, by lifting the $\models$ relation to the option RA. While in the axiom we distinguish between the two $\bot$ elements such that $\bot_{\hat{\Sigma}^?} \in \hat{\Sigma}^?$ and $\bot_{\Sigma^?} \in \Sigma^?$, we otherwise will omit the distinction

$$\begin{array}{l} \theta, s, \sigma \models \hat{\sigma} \land \alpha(\sigma, \vec{v}_i) = (o, \sigma', \vec{v}_o) \land [\![\vec{\hat{v}}_i]\!]_{\theta,s} = \vec{v}_i \land \\ (\forall o', \hat{\sigma}', \vec{\hat{v}}'_o, \pi'.\, \hat{\alpha}(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (o', \hat{\sigma}', \vec{\hat{v}}'_o, \pi') \Rightarrow o' \in \{\texttt{Ok}, \texttt{Err}\}) \implies \exists \hat{\sigma}', \vec{\hat{v}}_o, \pi, \theta'. \\ \quad \hat{\alpha}(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{\hat{v}}_o, \pi) \land \theta', s, \sigma' \models \hat{\sigma}' \land \texttt{SAT}_{\theta',s}(\pi) \land [\![\vec{\hat{v}}_o]\!]_{\theta',s} = \vec{v}_o \end{array}$$
(Memory Model OX Soundness)

$$\begin{array}{l} \hat{\alpha}(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{\hat{v}}_o, \pi) \land \texttt{SAT}_{\theta',s}(\pi) \land \theta, s, \sigma' \models \hat{\sigma}' \land \\ [\![\vec{\hat{v}}_o]\!]_{\hat{s},\pi} \rightsquigarrow (\vec{v}_o, \pi') \land [\![\vec{\hat{v}}_i]\!]_{\hat{s},\pi'} \rightsquigarrow (\vec{v}_i, \pi'') \implies \\ \quad \exists \sigma.\, \theta, s, \sigma \models \hat{\sigma} \land \alpha(\sigma, \vec{v}_i) = (o, \sigma', \vec{v}_o) \end{array} \qquad \text{(Memory Model UX Soundness)}$$

Compositionality Axioms

$$\begin{array}{l} \sigma \# \sigma_f \land \alpha(\sigma \cdot \sigma_f, \vec{\hat{v}}_i) = (o, \sigma', \vec{v}_o) \implies \\ \quad \exists \sigma'', o', \vec{v}'_o.\, \alpha(\sigma, \vec{v}_i) = (o', \sigma'', \vec{v}'_o) \land \\ \quad\quad (o' \neq \texttt{Miss} \implies o' = o \land \vec{v}'_o = \vec{v}_o \land \sigma' = \sigma'' \cdot \sigma_f) \end{array} \qquad \text{(Frame subtraction)}$$

$$\begin{array}{l} \alpha(\sigma, \vec{v}_i) = (o, \sigma', \vec{v}_o) \land o \neq \texttt{Miss} \land \sigma' \# \sigma_f \implies \\ \quad \sigma \# \sigma_f \land \alpha(\sigma \cdot \sigma_f, \vec{v}_i) = (o, \sigma' \cdot \sigma_f, \vec{v}_o) \end{array} \qquad \text{(Frame Addition)}$$

Here, we may note that the frame-preserving update $a \rightsquigarrow b$ from Iris is a form of frame subtraction: it guarantees $\forall c.\, \overline{\mathcal{V}}(a \cdot c) \Rightarrow \overline{\mathcal{V}}(b \cdot c)$, with c a frame that can be added to the state (σ_f in the axiom). This becomes evident when noticing that disjointness of partial RAs equates to validity in Iris RAs, giving us $\forall c.\, a \# c \Rightarrow b \# c$. In fact, the Iris frame-preserving update implies frame subtraction modulo action outcomes. This makes sense,

as Iris is used for OX reasoning, and frame subtraction is the property needed for OX soundness. *Maybe move this elsewhere.*

$$\mathsf{consume}(\hat{\sigma}, \delta, \vec{\hat{v}}_i) \rightsquigarrow (\mathsf{Ok}, \hat{\sigma}_f, \vec{\hat{v}}_o, \pi) \implies \forall \theta, s, \sigma.$$
$$\theta, s, \sigma \models \hat{\sigma} \land \mathsf{SAT}_{\theta,s}(\pi) \implies \exists \sigma_\delta, \sigma_f.$$
$$\sigma_\delta \# \sigma_f \land \sigma = \sigma_\delta \cdot \sigma_f \land \theta, s, \sigma_\delta \models_\Delta \langle\delta\rangle(\vec{\hat{v}}_i; \vec{\hat{v}}_o) \land \theta, s, \sigma_f \models \hat{\sigma}_f$$
(Consume UX Soundness)

$$\mathsf{consume}(\hat{\sigma}, \delta, \vec{\hat{v}}_i) \rightsquigarrow (\mathsf{Ok}, \hat{\sigma}_f, \vec{\hat{v}}_o, \pi) \implies \forall \theta, s, \sigma_f, \sigma_\delta.$$
$$\theta, s, \sigma_f \models \hat{\sigma}_f \land \theta, s, \sigma_\delta \models_\Delta \langle\delta\rangle(\vec{\hat{v}}_i; \vec{\hat{v}}_o) \land \sigma_f \# \sigma_\delta \implies$$
$$\exists \sigma.\, \sigma = \sigma_\delta \cdot \sigma_f \land \theta, s, \sigma \models \hat{\sigma} \land \mathsf{SAT}_{\theta,s}(\pi)$$
(Consume Complete)

$$(\forall o, \hat{\sigma}_f, \vec{\hat{v}}_o, \pi.\, \mathsf{consume}(\mathsf{OX}, \hat{\sigma}, \delta, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}_f, \vec{\hat{v}}_o, \pi) \Rightarrow o_c = \mathsf{Ok}) \implies$$
$$\exists \hat{\sigma}'_f, \vec{\hat{v}}'_o, \pi'.\, \mathsf{consume}(\mathsf{OX}, \hat{\sigma}, \delta, \vec{v}_i, \pi) \rightsquigarrow (\mathsf{Ok}, \hat{\sigma}'_f, \vec{\hat{v}}'_o, \pi')$$
(Consume OX: No Path Drops)

$$\mathsf{produce}(\hat{\sigma}_f, \delta, \vec{\hat{v}}_i, \vec{\hat{v}}_o) \rightsquigarrow (\hat{\sigma}, \pi) \implies$$
$$\forall \theta, s, \sigma.\, \mathsf{SAT}_{\theta,s}(\pi) \land \theta, s, \sigma \models \hat{\sigma} \implies \exists \sigma_\delta, \sigma_f.$$
$$\sigma_\delta \# \sigma_f \land \sigma = \sigma_\delta \cdot \sigma_f \land \theta, s, \sigma_\delta \models_\Delta \langle\delta\rangle(\vec{\hat{v}}_i; \vec{\hat{v}}_o) \land \theta, s, \sigma_f \models \hat{\sigma}_f$$
(Produce: Soundness)

$$\theta, s, \sigma_f \models \hat{\sigma}_f \land \theta, s, \sigma_\delta \models_\Delta \langle\delta\rangle(\vec{\hat{v}}_i; \vec{v}_o) \land \sigma_f \# \sigma_\delta \implies$$
$$\exists \hat{\sigma}.\, \mathsf{produce}(\hat{\sigma}_f, \delta, \vec{\hat{v}}_i, \vec{\hat{v}}_o) \rightsquigarrow (\hat{\sigma}, \pi) \land \mathsf{SAT}_{\theta,s}(\pi) \land \theta, s, (\sigma_f \cdot \sigma_\delta) \models \hat{\sigma}$$
(Produce: Completeness)

2.4 RAs and State Models

We now define the state transformers defined in [4], taking advantage of RAs. We will first define the "leaf" state models: the state models that are don't take any state model as input, Ex, Ag and Frac. We will then look at 3 simple transformer state models, Sum, Product and PProduct. Finally, we will discuss more complex state models, with Freeable, PMap and List.

2.4.1 Exclusive

The exclusive state model Ex(Val) is a simple state model, that represent exclusively owned cells: the cell can can only be owned once, and cannot be composed with any other cell. It is parametric on the values it stores – for the traditional symbolic execution cell, this would be `SVal`. When clear from context, the type of values is omitted. It's RA is defined as:

$$\mathrm{Ex}(X) \stackrel{\text{def}}{=} \mathrm{ex}(x : X)$$
$$|\mathrm{ex}(x)| \stackrel{\text{def}}{=} \bot$$
$$\mathrm{ex}(x_1) \cdot \mathrm{ex}(x_2) \text{ is always undefined}$$

The first notation, $\mathrm{Ex}(X)$ is the state model instantiation from the set X to the Ex resource algebra. The notation $\mathrm{ex}(x : X)$ stands for $\{\mathrm{ex}(x) : \forall x \in X\}$ – here '$\mathrm{ex}(x)$' refers to the particular element of the $\mathrm{Ex}(X)$ RA where the value is $x \in X$.

Note that the above definition is identical[2] to that in Iris [2], showing that little to no modification is needed to adapt RAs to state models.

It defines two actions, $\mathcal{A} = \{\mathtt{load}, \mathtt{store}\}$ and a predicate $\Delta = \{\mathtt{ex}\}$ We now define the functions for the state model: `execute_action`, `produce`, `consume` and `fix`.

ExLoadOk
$\mathtt{load}(\mathrm{ex}(x), []) \rightsquigarrow (\mathtt{Ok}, \mathrm{ex}(x), [x], [])$

ExLoadMiss
$\mathtt{load}(\bot, []) \rightsquigarrow (\mathtt{Miss}, \bot, [], [])$

ExStoreOk
$\mathtt{store}(\mathrm{ex}(x), [x']) \rightsquigarrow (\mathtt{Ok}, \mathrm{ex}(x'), [], [])$

ExStoreMiss
$\mathtt{store}(\bot, [x']) \rightsquigarrow (\mathtt{Miss}, \bot, [], [])$

ExConsOk
$\mathtt{consume}(\mathrm{ex}(x), \mathtt{ex}, []) \rightsquigarrow (\mathtt{Ok}, \bot, [x], [])$

ExConsMiss
$\mathtt{consume}(\bot, \mathtt{ex}, []) \rightsquigarrow (\mathtt{Miss}, \bot, [], [])$

ExProd
$\mathtt{produce}(\bot, \mathtt{ex}, [], [x]) \rightsquigarrow (\mathrm{ex}(x), [])$

ExFix
$\mathtt{fix}\ [] = [\{\exists x.\, \langle \mathtt{ex} \rangle(; x)\}]$

2.4.2 Agreement

The agreement state model Ag(Val) is the state model to represent an agreement algebra (sometimes referred to as *knowledge* in the literature [6]): information that can be duplicated.

$$\mathrm{Ag}(X) \stackrel{\text{def}}{=} \mathrm{ag}(x : X)$$
$$|\mathrm{ag}(x)| \stackrel{\text{def}}{=} \mathrm{ag}(x)$$
$$\mathrm{ag}(x) \cdot \mathrm{ag}(x') \stackrel{\text{def}}{=} \begin{cases} \mathrm{ag}(x) & \text{if } x = x' \\ \text{undefined} & \text{otherwise} \end{cases}$$

Again, this definition is identical to the one of Ag_0 in Iris (the non-step-indexed version of agreement).

Because knowledge is duplicable, it cannot be modified: indeed, one would need to modify all instances of the knowledge to ensure frame preservation still holds. Its actions are thus $\mathcal{A} = \{\mathtt{load}\}$, and it has one predicate, $\Delta = \{\mathtt{ag}\}$.

AgLoadOk
$\mathtt{load}(\mathrm{ag}(x), []) \rightsquigarrow (\mathtt{Ok}, \mathrm{ag}(x), [x], [])$

AgLoadMiss
$\mathtt{load}(\bot, []) \rightsquigarrow (\mathtt{Miss}, \bot, [], [])$

AgConsOk
$\mathtt{consume}(\mathrm{ag}(x), \mathtt{ag}, []) \rightsquigarrow (\mathtt{Ok}, \mathrm{ag}(x), [x], [])$

AgConsMiss
$\mathtt{consume}(\bot, \mathtt{ag}, []) \rightsquigarrow (\mathtt{Miss}, \bot, [][])$

AgProdBot
$\mathtt{produce}(\bot, \mathtt{ag}, [], [x]) \rightsquigarrow (\mathrm{ag}(x), [])$

AgProdEq
$\mathtt{produce}(\mathrm{ag}(x), \mathtt{ag}, [], [x']) \rightsquigarrow (\mathrm{ag}(x), [x = x'])$

AgFix
$\mathtt{fix}\ [] = [\{\exists x.\, \langle \mathtt{ag} \rangle(; x)\}]$

2.4.3 Fractional

The fractional state model Frac(Val) is used to handle *fractional permissions* [7], [8]. This allows a cell to be partly owned and its information shared, for instance in multithreading.

[2] Modulo partiality of composition and the validity function, as explained previously.

This is done by pairing every value with a fraction $0 < q \leq 1$, and ensuring the value can only be modified if we own the entire value (ie. $q = 1$).

$$\textsc{Frac}(X) \stackrel{\text{def}}{=} \text{frac}(x : X, q : (0; 1])$$

$$|\text{frac}(x, q)| \stackrel{\text{def}}{=} \bot$$

$$\text{frac}(x, q) \cdot \text{frac}(x', q') \stackrel{\text{def}}{=} \begin{cases} \text{frac}(x, q + q') & \text{if } x = x' \land q + q' \leq 1 \\ \text{undefined} & \text{otherwise} \end{cases}$$

We may note this definition is different from that in Iris, that chooses to define the fractional state model as $\textsc{Frac} \times \textsc{Ag}_0(\text{Val})$, where $\textsc{Frac}$ is the RA for strictly positive rationals. This work in their case, because they can define actions for any state model easily, so they can define `load` for the product having knowledge of the underlying state models. However, because the state models presented here are aimed at being reused in a variety of contexts while minimising the need for defining new actions and predicates, using their approach would hurt usability, as the `load` action would need to be redefined for this specific instantiation of the product. Furthermore, this construction would yield two predicates: `ag` and `frac`, making its use unpractical.

We instead define the actions $\mathcal{A} = \{\texttt{load}, \texttt{store}\}$ and the core predicate $\Delta = \{\texttt{frac}\}$ as follows:

FracLoadOk
$$\texttt{load}(\text{frac}(x, q), []) \rightsquigarrow (\texttt{Ok}, \text{frac}(x, q), [x], [], [])$$

FracLoadMiss
$$\texttt{load}(\bot, []) \rightsquigarrow (\texttt{Miss}, \bot, [1], [])$$

FracStoreOk
$$\texttt{store}(\text{frac}(x, q), [x']) \rightsquigarrow (\texttt{Ok}, \text{frac}(x', q), [], [q = 1])$$

FracStorePerm
$$\texttt{store}(\text{frac}(x, q), [x']) \rightsquigarrow (\texttt{Miss}, \text{frac}(x, q), [1 - q], [q < 1])$$

FracStoreMiss
$$\texttt{store}(\bot, [x']) \rightsquigarrow (\texttt{Miss}, \bot, [1], [])$$

FracConsAll
$$\texttt{consume}(\text{frac}(x, q), \texttt{frac}, [q']) \rightsquigarrow (\texttt{Ok}, \bot, [x], [q = q'])$$

FracConsSome
$$\texttt{consume}(\text{frac}(x, q), \texttt{frac}, [q']) \rightsquigarrow (\texttt{Ok}, \text{frac}(x, q - q'), [x], [0 < q' < q])$$

FracConsMiss
$$\texttt{consume}(\text{frac}(x, q), \texttt{frac}, [q']) \rightsquigarrow (\texttt{Miss}, \text{frac}(x, q), [q' - q], [q < q' \leq 1])$$

FracConsFail
$$\texttt{consume}(\text{frac}(x, q), \texttt{frac}, [q']) \rightsquigarrow (\texttt{LFail}, \text{frac}(x, q), [], [q' \leq 0 \lor 1 < q'])$$

FracProdBot
$$\texttt{produce}(\bot, \texttt{frac}, [q], [x]) \rightsquigarrow (\text{frac}(x, q), [0 < q \leq 1])$$

FracProdEq
$$\texttt{produce}(\text{frac}(x, q), \texttt{frac}, [q'], [x']) \rightsquigarrow (\text{frac}(x, q + q'), [x = x' \land 0 < q' \land q + q' \leq 1])$$

FracFix
$$\texttt{fix}\ [q] = [\{\exists x.\, \langle \texttt{frac} \rangle (q; x)\}]$$

Here we note that the fraction part of the state is an in-parameter, whereas the value is an out-parameter. This allows one to explicitly specify the required fraction of the state that is consumed.

2.4.4 Sum

The sum of two state models, denoted $\mathbb{S}_1 + \mathbb{S}_2$, represents all states that are in either one of the two states. Sums are one of the reasons for which the 0 of PCMs was removed, in favour of the core, as it allows both sides of the sum to have a different unit (if any). We re-use the definition of sum from Iris.

$$\textsc{Sum}(X, Y) \stackrel{\text{def}}{=} X + Y \stackrel{\text{def}}{=} l(x : X) \mid r(y : Y)$$

$$l(x) \cdot l(x') \stackrel{\text{def}}{=} l(x \cdot x')$$

$$r(y) \cdot r(y') \stackrel{\text{def}}{=} r(y \cdot y')$$

$$|l(x)| \stackrel{\text{def}}{=} \begin{cases} \bot & \text{if } |x| = \bot \\ l(|x|) & \text{otherwise} \end{cases}$$

$$|r(y)| \stackrel{\text{def}}{=} \begin{cases} \bot & \text{if } |y| = \bot \\ r(|y|) & \text{otherwise} \end{cases}$$

Similarly for the actions and predicates, we simply re-use the underlying function. The actions are defined as $\mathcal{A} = \{\alpha_l : \alpha \in \mathbb{S}_1.\mathcal{A}\} \uplus \{\alpha_r : \alpha \in \mathbb{S}_2.\mathcal{A}\}$, and the core predicates $\Delta = \{\delta_l : \delta \in \mathbb{S}_1.\Delta\} \uplus \{\delta_r : \delta \in \mathbb{S}_2.\Delta\}$.

$$\text{Given } \mathit{wrap}_l(x) = \begin{cases} \bot & \text{if } x = \bot \\ l(x) & \text{otherwise} \end{cases} \text{ and } \mathit{unwrap}_l(x_l) = \begin{cases} \bot & \text{if } x = \bot \\ x_l & \text{if } x = l(x_l) \\ \text{undefined} & \text{otherwise} \end{cases}$$

SumLAction
$$\frac{x = \mathit{unwrap}_l(x_l) \qquad \alpha(x, \vec{v}_i) \rightsquigarrow (o, x', \vec{v}_o, \pi) \qquad x'_l = \mathit{wrap}_l(x') \qquad o \neq \texttt{Miss}}{\alpha_l(x_l, \vec{v}_i) \rightsquigarrow (o, x'_l, \vec{v}_o, \pi)}$$

SumLActionMiss
$$\frac{x = \mathit{unwrap}_l(x_l) \qquad \alpha(x, \vec{v}_i) \rightsquigarrow (o, x', \vec{v}_o, \pi) \qquad x'_l = \mathit{wrap}_l(x') \qquad o = \texttt{Miss}}{\alpha_l(x_l, \vec{v}_i) \rightsquigarrow (o, x'_l, \texttt{`l'::}\vec{v}_o, \pi)}$$

SumLActionIncompat
$$\alpha_l(r(y), \vec{v}_i) \rightsquigarrow (\texttt{Err}, r(y), [], [])$$

SumLCons
$$\frac{x = \mathit{unwrap}_l(x_l) \qquad \texttt{consume}(x, \delta_l, \vec{v}_i) \rightsquigarrow (o, x', \vec{v}_o, \pi) \qquad x'_l = \mathit{wrap}_l(x') \qquad o \neq \texttt{Miss}}{\texttt{consume}(x_l, \delta_l, \vec{v}_i) \rightsquigarrow (o, x'_l, \vec{v}_o, \pi)}$$

SumLConsMiss
$$\frac{x = \mathit{unwrap}_l(x_l) \qquad \texttt{consume}(x, \delta_l, \vec{v}_i) \rightsquigarrow (o, x', \vec{v}_o, \pi) \qquad x'_l = \mathit{wrap}_l(x') \qquad o = \texttt{Miss}}{\texttt{consume}(x_l, \delta_l, \vec{v}_i) \rightsquigarrow (o, x'_l, \texttt{`l'::}\vec{v}_o, \pi)}$$

SumLConsIncompat
$$\texttt{consume}(r(y), \delta_l, \vec{v}_i) \rightsquigarrow (\texttt{LFail}, r(y), [], [])$$

SumLProd
$$\frac{x = \mathit{unwrap}_l(x_l) \qquad \texttt{produce}(x, \delta_l, \vec{v}_i, \vec{v}_o) \rightsquigarrow (x', \pi) \qquad x'_l = \mathit{wrap}_l(x')}{\texttt{produce}(x_l, \delta_l, \vec{v}_i, \vec{v}_o) \rightsquigarrow (x'_l, \pi)}$$

SumLFix
$$\frac{\mathbb{S}_1.\texttt{fix}\ \vec{v}_i = a}{\texttt{fix `l'::}\vec{v}_i = a}$$

We only describe the rules for the left side of the sum – the equivalent rules for the right hand side are defined analogously.

For fixes to be retrieved from the correct side of the sum, `Miss` outcomes must be extended with an indicator of what side the information comes from, so that the correct `fix` function is then called. Some extra care also needs to be taken to handle $\bot$ states separately, since it is not an element of the underlying state models and as such $l(\bot)$ or $r(\bot)$ are not valid – the auxiliary $wrap_l$ and $unwrap_l$ functions help do this without multiplying by four the number of rules.

The sum is one of the main reasons for the switch from PCMs to RAs, as being able to handle $\bot$ separately is an advantage: it avoids situations where the underlying state may be *observably empty*, but the state of the sum is $l(\mathbb{S}_1.0)$ (if we use PCMs). This causes unsoundness, as for instance actions and predicates belonging to the right side of the sum would then yield `Err` and `LFail` respectively, despite the fact that if the state of the sum was simply $\bot$ they'd succeed.

We may give an example to illustrate: let there be the state model $\mathrm{Ex}(\{1\}) + \mathrm{Ex}(\{2\})$. A valid function specification in this state model is $\{\!\{\langle \mathtt{ex}_l \rangle(;1)\}\!\}\ \mathtt{swap()}\ \{\!\{\langle \mathtt{ex}_r \rangle(;2)\}\!\}$[3], where the `swap` function switches the state from the left hand side to the right hand side. Now if this was constructed using PCMs, the engine would first consume the core predicate for the precondition. The consumption for $\mathrm{Ex}(\{1\})$ is $\mathtt{consume}(\mathrm{ex}(1), \mathtt{ex}_l, []) \rightsquigarrow (\mathtt{Ok}, 0_l, [1], [])$. The consumption for the sum would thus be $\mathtt{consume}(l(\mathrm{ex}(1)), \mathtt{ex}_l, []) \rightsquigarrow (\mathtt{Ok}, l(0_l), [1], [])$. Note, here, that the sum state model has *no way of knowing if the state became empty*, and must thus keep it as $l(0_l)$. The engine would then produce the postcondition, which is $\langle \mathtt{ex}_r \rangle(;2)$ – this would however result in the branch vanishing, as $\mathtt{ex}_r$ is not a predicate that can be produced into some $l(x)$. The function call would thus vanish, which is unsound in OX (and would result in no branches in UX, which is sound but useless). *I have a formal proof of this unsoundness, I'll add it here eventually, or put it in the appendix, TBD.*

Of course one may decide state models must expose an `is_empty` function, and use that instead – however this would needlessly complexify state models and would reinvent the wheel; multi-core resource algebras were specifically created to solve this problem [9], and the partial core of Iris was *also* created for this reason among others [2].

2.4.5 Product

The product $\mathbb{S}_1 \times \mathbb{S}_2$ of two state models is the cartesian product of both sets of states. Its RA is defined by lifting all elements pointwise:

$$\begin{aligned}
\textsc{Product}(X, Y) &\stackrel{\text{def}}{=} X \times Y \\
(x, y) \cdot (x', y') &\stackrel{\text{def}}{=} (x \cdot x', y \cdot y') \\
|(x, y)| &\stackrel{\text{def}}{=} \begin{cases} (|x|, |y|) & \text{if } |x| \neq \bot \land |y| \neq \bot \\ \bot & \text{otherwise} \end{cases}
\end{aligned}$$

An interesting property of the product is that if one side of the product has no core, then so does the entire product; this also means that both sides of the product must be defined, or neither are defined. This creates a challenge: if an empty ($\bot$) product produces a core predicate for one of its sides, what happens to the other side of the product? Indeed, while one side becomes, defined, $x \times \bot$ is not a valid state, since $\bot \notin \mathbb{S}_2.\hat{\Sigma}$.

[3]The signature of this `swap` function is analogous to that of the `free` action, for the Freeable state model that will later be described.

It seems here that the consume-produce interface of our engine, which allows creating state bit by bit, can thus be unadapted for certain RAs. Instead, we define an alternative product RA that is more suited to this engine.

2.4.6 Partial Product

The *partial product* state model, denoted $A \bowtie B$, is an alternative to the Iris product RA, that carries some of its useful properties while being adapted to a produce-consume interface. This product supports having only one side be empty – this is different from the usual product RA, that must have both sides have a value.

$$\textsc{PProduct}(X, Y) \stackrel{\text{def}}{=} X \bowtie Y \stackrel{\text{def}}{=} X^? \times Y^? \backslash \{(\bot, \bot)\}$$
$$(x, y) \cdot (x', y') \stackrel{\text{def}}{=} (x \cdot x', y \cdot y')$$
$$|(x, y)| \stackrel{\text{def}}{=} \begin{cases} \bot & \text{if } |x| = \bot \wedge |y| = \bot \\ (|x|, |y|) & \text{otherwise} \end{cases}$$

The advantage of this definition is twofold. Firstly, it allows expressing fragments of products, where one side may not have a defined core. For instance, given the product state $\mathrm{ex}(a) \times \mathrm{ex}(b)$, one can't express this as the composition of some $\mathrm{ex}(A) \times \bot$ and $\bot \times \mathrm{ex}(b)$, which becomes needed for some state transformers (notably, PMap and List). This is in turn possible with the partial product . The second advantage of the partial product and one of the main motivations behind it is that it *carries on the exclusivity of its components.* Given $\mathrm{exclusive}(a) \stackrel{\text{def}}{=} \forall c.\, \neg(a \# c)$, the following rule holds:

$$\frac{\mathrm{exclusive}(a) \qquad \mathrm{exclusive}(b)}{\mathrm{exclusive}(lr(a, b))}\ \textsc{PartProductEx}$$

This is needed to allow frame-preserving transitions from one side of a sum to another. We may note also that the state model transformer itself could be generalised to handle an arbitrary number of state models, as $(A \bowtie B) \bowtie C \equiv A \bowtie (B \bowtie C)$ – for the brevity of this presentation, we will only consider the partial product of two state models.

Similarly to the sum, its actions are $\mathcal{A} = \{\alpha_l : \alpha \in \mathbb{S}_1.\mathcal{A}\} \uplus \{\alpha_r : \alpha \in \mathbb{S}_2.\mathcal{A}\}$, and core predicates $\Delta = \{\delta_l : \delta \in \mathbb{S}_1.\Delta\} \uplus \{\delta_r : \delta \in \mathbb{S}_2.\Delta\}$.

$$\text{Given } \mathit{wrap}(x, y) = \begin{cases} \bot & \text{if } x = \bot \wedge y = \bot \\ (x, y) & \text{otherwise} \end{cases} \text{ and } \mathit{unwrap}(s) = \begin{cases} (\bot, \bot) & \text{if } s = \bot \\ (x, y) & \text{otherwise} \end{cases}$$

$$\frac{(x, y) = \mathit{unwrap}(s) \qquad \alpha(x, \vec{\hat{v}}_i) \rightsquigarrow (o, x', \vec{\hat{v}}_o, \pi) \qquad s' = \mathit{wrap}(x', y) \qquad o \neq \texttt{Miss}}{\alpha_l(s, \vec{\hat{v}}_i) \rightsquigarrow (o, s', \vec{\hat{v}}_o, \pi)}\ \textsc{PProductLAction}$$

$$\frac{(x, y) = \mathit{unwrap}(s) \qquad \alpha(x, \vec{\hat{v}}_i) \rightsquigarrow (o, x', \vec{\hat{v}}_o, \pi) \qquad s' = \mathit{wrap}(x', y) \qquad o = \texttt{Miss}}{\alpha_l(s, \vec{\hat{v}}_i) \rightsquigarrow (o, s', \texttt{`l'} :: \vec{\hat{v}}_o, \pi)}\ \textsc{PProductLActionMiss}$$

PProductLCons

$$\frac{(x, y) = \mathit{unwrap}(s) \qquad \mathtt{consume}(x, \delta_l, \vec{\hat{v}_i}) \rightsquigarrow (o, x', \vec{\hat{v}_o}, \pi) \qquad s' = \mathit{wrap}(x', y) \qquad o \neq \mathtt{Miss}}{\mathtt{consume}(s, \delta_l, \vec{\hat{v}_i}) \rightsquigarrow (o, s', \vec{\hat{v}_o}, \pi)}$$

PProductLConsMiss

$$\frac{(x, y) = \mathit{unwrap}(s) \qquad \mathtt{consume}(x, \delta_l, \vec{\hat{v}_i}) \rightsquigarrow (o, x', \vec{\hat{v}_o}, \pi) \qquad s' = \mathit{wrap}(x', y) \qquad o = \mathtt{Miss}}{\mathtt{consume}(s, \delta_l, \vec{\hat{v}_i}) \rightsquigarrow (o, s', \text{`l'}::\vec{\hat{v}_o}, \pi)}$$

PProductLProd

$$\frac{(x, y) = \mathit{unwrap}(s) \qquad \mathtt{produce}(x, \delta_l, \vec{\hat{v}_i}, \vec{\hat{v}_o}) \rightsquigarrow (x', \pi) \qquad s' = \mathit{wrap}(x', y)}{\mathtt{produce}(s, \delta_l, \vec{\hat{v}_i}, \vec{\hat{v}_o}) \rightsquigarrow (s', \pi)}$$

PProductLFix

$$\frac{\mathbb{S}_1.\mathtt{fix}\ \vec{\hat{v}_i} = a}{\mathtt{fix}\ \text{`l'}::\vec{\hat{v}_i} = a}$$

2.4.7 Freeable

The Freeable($\mathbb{S}$) state model transformer allows extending a state model with a `free` action, that allows freeing a part of memory. The freed memory can then not be used, and attempting to access it raises a `UserAfterFree` error. This is similar to the OneShot RA of Iris, with the key difference that the `freed` predicate used to mark a resource as freed is *not duplicable* – whereas Iris defines OneShot(X) $\stackrel{\text{def}}{=}$ Frac + Ag(X), this definition is not UX-sound. *Maybe a small proof of this? Or is it evident?* This is not a problem for Iris, that is only concerned with OX soundness, but justifies this modification for this engine.

To ensure only full states are freed, the underlying state model must provide a function $\mathtt{is\_exclusively\_owned} : \hat{\Sigma} \to \mathtt{LVal}$, which equates to the property 'exclusive' presented before. Note here this returns a *symbolic value*, that can be appended to the path condition (it must thus be a boolean). This allows symbolic ownership, for instance having the fraction of Frac be a symbolic number.

While not needed for the core and compositional engines, the bi-abductive engine requires that misses may be fixed; Freeable($\mathbb{S}$) however cannot access directly the core predicates of $\mathbb{S}$ to provide fixes when freeing an empty state. As such, the state model must also provide a $\mathtt{fix\_owned} : \Sigma^? \to \mathcal{P}(\text{Asrt})$ `list` function that returns possible fixes to make the given state exclusively owned. This also implies that full states of $\mathbb{S}$ are exclusively owned – this is not the case, for instance, for Ag. As such, constructions such as Freeable(Ag) are not sound.

Its RA is defined as a construction: Freeable($\mathbb{S}$) $\stackrel{\text{def}}{=}$ $\mathbb{S}$ + Ex({freed}), which allows all associated rules to be kept. For clarity, we rename the core predicate $\mathtt{ex}_r$ (defined by Ex({freed})) as `freed`. We also extend its actions with the `free` action.

Because Freeable is constructed via other state model transformers, we only need to describe the rules for `free` – the rest of the construction is already sound. This is an example of how simpler state models can be extended while alleviating the user from the burden of proving the soundness of the base construction.

FreeableActionFree

$$\mathtt{free}(l(x), []) \rightsquigarrow (\mathtt{Ok}, r(\mathrm{ex}(\mathrm{freed})), [], [\mathbb{S}.\mathtt{is\_exclusively\_owned}\ x])$$

FreeableActionFreeErr

$$\mathtt{free}(l(x), []) \rightsquigarrow (\mathtt{Miss}, l(x), \mathbb{S}.\mathtt{fix\_owned}\ x, [\neg\mathbb{S}.\mathtt{is\_exclusively\_owned}\ x])$$

FreeableActionFreeMiss

$$\texttt{free}(\bot, []) \rightsquigarrow (\texttt{Miss}, \bot, \mathbb{S}.\texttt{fix\_owned}\ \bot, [])$$

FreeableActionDoubleFree

$$\texttt{free}(r(\text{ex}(\text{freed})), []) \rightsquigarrow (\texttt{Err}, r(\text{ex}(\text{freed})), [], [])$$

We may note that use-after-free errors are already handled by the sum construction, thanks to the SumLActionIncompat rule.

2.4.8 PMap

The partial map transformer, PMap$(I, \mathbb{S})$, allows modelling partial finite maps. It receives a domain I, and the codomain state model $\mathbb{S}$.

$$\text{PMap}(I, \mathbb{S}) \stackrel{\text{def}}{=} I \xrightarrow{fin} \mathbb{S}.\Sigma \times \mathcal{P}(I)^?$$

$$(h, d) \cdot (h', d') \stackrel{\text{def}}{=} (h'', d'')$$

$$\text{where } h'' \stackrel{\text{def}}{=} \lambda i. \begin{cases} h(i) \cdot h'(i) & \text{if } i \in \text{dom}(h) \cap \text{dom}(h') \\ h(i) & \text{if } i \in \text{dom}(h) \setminus \text{dom}(h') \\ h'(i) & \text{if } i \in \text{dom}(h') \setminus \text{dom}(h) \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$\text{and } d'' \stackrel{\text{def}}{=} \begin{cases} d & \text{if } d' = \bot \\ d' & \text{if } d = \bot \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$\text{and } d'' = \bot \vee \text{dom}(h'') \subseteq d''$$

$$|(h, d)| \stackrel{\text{def}}{=} \begin{cases} \bot & \text{if } \text{dom}(h') = \emptyset \\ (h', \bot) & \text{otherwise} \end{cases}$$

$$\text{where } h' \stackrel{\text{def}}{=} \lambda i. \begin{cases} |h(i)| & \text{if } i \in \text{dom}(h) \wedge |h(i)| \neq \bot \\ \text{undefined} & \text{otherwise} \end{cases}$$

The state is thus the bindings from index to substate, as well as a possibly unset *domain set*. This domain set allows distinguishing, when an index is not found in the map, if the outcome is an `Err` or `LFail` (because the binding cannot exist), or if the state at that index is actually $\bot$ (in which case the `execute_action`, `consume` or `produce` call is done with $\bot$ as an input). More than improving automation, this is a requirement, for compatibility with the full state model (**??**) and to preserve Frame subtraction and Frame Addition.

PMap has a well-formedness condition, to ensure the domain of the bindings is a subset of the domain set – for instance, the state $(\{2 \mapsto x\}, \{1\})$ is not valid, as $\{2\} \not\subseteq \{1\}$. If the domain set is missing, then there is no restrictions on the bindings.

PMap$(I, \mathbb{S})$ defines one action, `alloc`, and one predicate, `domainset`. Furthermore, it lifts all predicates of the wrapped state model, by adding the index of the cell as an in-parameter. Furthermore, to allow allocation, $\mathbb{S}$ must provide an `instantiate` : Val `list` $\to \Sigma$ function, to instantiate a new state from a list of arguments. This is needed because PMap has no awareness of how $\mathbb{S}$ works, or or what it should be initialised to. *Maybe specify that CSE/Sacha don't mention this: PMap allocation requires an interface for allocation.*

For the rules below, we define $h[i \leftarrow s]$ as setting the binding at index i of h to s, and $h[i \not\leftarrow]$ as removing the binding at i from h.

We first define a helper methods `get` and `set`, that allows modifying a symbolic map with branching. After a look up, it returns the value at the location (which may be $\bot$ if it's not found), and the path condition corresponding to the branch. This allows simplifying shared rules for `execute_action`, `produce` and `consume` for PMap.

$$\texttt{get} : ((I \xrightarrow{fin} X) \times \mathcal{P}(I)^?) \to I \to \mathcal{P}(I \times X \times \Pi)$$
$$\texttt{set} : ((I \xrightarrow{fin} X) \times \mathcal{P}(I)^?) \to I \to X \to (I \xrightarrow{fin} X \times \mathcal{P}(I)^?)$$

We pretty-print `get` and `set` as $\texttt{get}(s, i) \rightsquigarrow (i', x, \pi)$ and $\texttt{set}(s, i, x) = s'$.

$$\text{Given } \mathit{wrap}(h, d) \stackrel{\text{def}}{=} \begin{cases} \bot & \text{if } \mathrm{dom}(h) = \emptyset \wedge d = \bot \\ (h, d) & \text{otherwise} \end{cases}$$
$$\mathit{unwrap}(s) \stackrel{\text{def}}{=} \begin{cases} ([], \bot) & \text{if } s = \bot \\ (h, d) & \text{if } s = (h, d) \end{cases}$$

PMapGetMatch
$$\frac{(h, d) = \mathit{unwrap}(s) \qquad i' \in \mathrm{dom}(h) \qquad s_{i'} = h(i')}{\texttt{get}(s, i) \rightsquigarrow (i', s_{i'}, [i = i'])}$$

PMapGetAdd
$$\frac{(h, d) = \mathit{unwrap}(s) \qquad i \notin \mathrm{dom}(h) \qquad d \neq \bot}{\texttt{get}(s, i) \rightsquigarrow (i, \bot, [i \notin \mathrm{dom}(h) \wedge i \in d])}$$

PMapGetBotDomain
$$\frac{(h, d) = \mathit{unwrap}(s) \qquad i \notin \mathrm{dom}(h) \qquad d = \bot}{\texttt{get}(s, i) \rightsquigarrow (i, \bot, [i \notin \mathrm{dom}(h)])}$$

PMapSetSome
$$\frac{(h, d) = \mathit{unwrap}(s) \qquad s_i \neq \bot \qquad h' = h[i \leftarrow s_i] \qquad s' = \mathit{wrap}(h', d)}{\texttt{set}(s, i, s_i) = s'}$$

PMapSetNone
$$\frac{(h, d) = \mathit{unwrap}(s) \qquad s_i = \bot \qquad h' = h[i \not\leftarrow] \qquad s' = \mathit{wrap}(h', d)}{\texttt{set}(s, i, s_i) = s'}$$

An interesting thing to outline here is that we may branch in three different ways when retrieving a binding:

- PMapGetMatch: the index is equal to some index in the map, and we execute the action for that location – note here the resulting path condition has $i = i'$, such that branches where the index isn't equal will be cut. This allows taking into account symbolic values.
- PMapGetAdd: the index is not already in the map, but is part of the domain set; this means the state is $\bot$.
- PMapGetBotDomain: the index is not already in the map, and the domain set is not owned; the given index may thus be valid, and we again return $\bot$.

There is no branching when setting the binding, as the index is already known.

We now define the rules for PMap(I,$\mathbb{S}$):

$$\text{Given } \mathit{lift\_if\_miss}(o, i, \vec{v}_i) \stackrel{\text{def}}{=} \begin{cases} i :: \vec{v}_i & \text{if } o = \mathtt{Miss} \\ \vec{v}_i & \text{otherwise} \end{cases}$$

PMapAction
$$\frac{\mathtt{get}(s, i) \rightsquigarrow (i', s_{i'}, \pi) \qquad \alpha(s_{i'}, \vec{v}_i) \rightsquigarrow (o, s'_{i'}, \vec{v}_o, \pi') \qquad \mathtt{set}(s, i', s'_{i'}) = s'}{\alpha(s, i :: \vec{v}_i) \rightsquigarrow (o, s', i' :: \vec{v}_o, \pi :: \pi')}$$

PMapActionOutOfBounds
$$\frac{d \neq \bot}{\alpha((h, d), i :: \vec{v}_i) \rightsquigarrow (\mathtt{Err}, (h, d), [], [i \notin d])}$$

PMapAlloc
$$\frac{d \neq \bot \qquad i = \mathtt{fresh} \qquad s_i = \mathtt{instantiate}(\vec{v}_i) \qquad h' = h[i \leftarrow s_i] \qquad d' = d \uplus i}{\mathtt{alloc}((h, d), \vec{v}_i) \rightsquigarrow (\mathtt{Ok}, (h', d'), [i], [i = i])}$$

PMapAllocMiss
$$\frac{(h, d) = \mathit{unwrap}(s) \qquad d = \bot}{\mathtt{alloc}(s, \vec{v}_i) \rightsquigarrow (\mathtt{Miss}, s, [\text{`}\mathtt{domainset}\text{'}], [])}$$

PMapCons
$$\frac{\mathtt{get}(s, i) \rightsquigarrow (i', s_{i'}, \pi) \qquad \mathtt{consume}(s_{i'}, \delta, \vec{v}_i) \rightsquigarrow (o, s'_{i'}, \vec{v}_o, \pi') \qquad \mathtt{set}(s, i', s'_{i'}) = s' \qquad \vec{v}'_o = \mathit{lift\_if\_miss}(o, i', \vec{v}_o)}{\mathtt{consume}(s, \delta, i :: \vec{v}_i) \rightsquigarrow (o, s', \vec{v}'_o, \pi :: \pi')}$$

PMapConsIncompat
$$\frac{d \neq \bot}{\mathtt{consume}((h, d), \delta, i :: \vec{v}_i) \rightsquigarrow (\mathtt{LFail}, (h, d), [], [i \notin d])}$$

PMapConsDomainSet
$$\frac{d \neq \bot}{\mathtt{consume}((h, d), \mathtt{domainset}, []) \rightsquigarrow (\mathtt{Ok}, (h, \bot), [d], [])}$$

PMapConsDomainSetMiss
$$\frac{(h, d) = \mathit{unwrap}(s) \qquad d = \bot}{\mathtt{consume}(s, \mathtt{domainset}, []) \rightsquigarrow (\mathtt{Miss}, s, [\text{`}\mathtt{domainset}\text{'}], [])}$$

PMapProd
$$\frac{\mathtt{get}(s, i) \rightsquigarrow (i', s_{i'}, \pi) \qquad \mathtt{produce}(s_{i'}, \delta, \vec{v}_i, \vec{v}_o) \rightsquigarrow (s'_{i'}, \pi') \qquad \mathtt{set}(s, i', s'_{i'}) = s'}{\mathtt{produce}(s, \delta, i :: \vec{v}_i, \vec{v}_o) \rightsquigarrow (s', \pi :: \pi')}$$

PMapProdDomainSet
$$\frac{(h, \bot) = \mathit{unwrap}(s)}{\mathtt{produce}(s, \mathtt{domainset}, [], [d]) \rightsquigarrow ((h, d), [\mathrm{dom}(h) \subseteq d])}$$

PMapFix
$$\frac{\mathbb{S}.\mathtt{fix}\ \vec{v}_i = a \qquad a' = \mathit{lift}(a, i)}{\mathtt{fix}\ i :: \vec{v}_i = a'}$$

PMapFixDomainSet
$$\mathtt{fix}\ [\text{`}\mathtt{domainset}\text{'}] = \exists d.\, \langle \mathtt{domainset} \rangle (; d)$$

This simple construction is thus enough to recreate the standard "points to" predicate of standard separation logic [10], [11]: with PMap($\mathbb{N}$, Ex(Val)), one can formulate the core predicates $\langle \mathtt{ex} \rangle (i; x)$, which is equivalent to $i \mapsto x$.

Note here we must unwrap and re-wrap the state model after every action, to properly handle the case where there are no bindings and no domain set as $\bot$. For the `fix` function we must also lift all core predicates, by recursing through the assertions and adding the index to the in-values of all assertions of type $\langle\delta\rangle(\vec{\hat{v}_i};\vec{\hat{v}_o})$.

We note here that action execution and predicate consumption and production may branch. For instance, if the state is $(\{1 \mapsto y\}, \{1,2\})$ and a `load` action is executed with an unconstrained symbolic index $\hat{x}$, then three branches are created: one where $\hat{x} = 1$ and the action is executed on the cell y, one where $\hat{x} = 2$ and the action is executed on $\bot$, and finally one where $\hat{x} \notin \{1,2\}$, and an `Err` is raised for out of bounds access. A fourth branch also gets created, with $\hat{x} \notin \{1,2\} \land \hat{x} \in \{1,2\}$ – it however gets cut because of course this path condition is false. This outlines a key difficulty of PMap, that will be discussed more in depth later: implementation-wise, it is a complex and expensive transformer that has the potential to create many branches – this makes it an important target for optimisation.

Using RAs rather than PCMs makes well-formedness easier to uphold, as a binding to $\bot$ is not valid, since $\bot \notin \mathbb{S}.\Sigma$. To exemplify why this is helpful, consider the state model $\text{PMap}(\mathbb{N}, \text{Ex}(\{1\}))$ and the function `move`, that relocates a memory cell:

$$\{\!\{\langle \texttt{ex}\rangle(1;1) \star \langle \texttt{domainset}\rangle(;\{1\})\}\!\}\ \texttt{move()}\ \{\!\{\langle \texttt{ex}\rangle(2;1) \star \langle \texttt{domainset}\rangle(;\{2\})\}\!\}$$

Let the initial state be $(\{1 \mapsto \text{ex}(1)\}, \{1\})$: a heap with one cell, at address 1 – this matches exactly the precondition of `move`.

If we are using PCMs, Ex(1) is defined as ex(1) | 0, with 0 the unit of the PCM. Note here that executing actions, consuming and producing doesn't return an element of $\Sigma^?$, but of Σ directly, since the 0 is already in its carrier set; one has no reason to extend it with $\bot$. The engine first consumes the `ex` predicate, at address 1 – this means the pointed-to state becomes 0 (or empty), and the bigger (or outer) state becomes $(\{1 \mapsto 0\}, \{1\})$. The `domainset` predicate is then successfully consumed too, leaving $(\{1 \mapsto 0\}, \bot)$. Now, the post-condition is produced onto the state, as calling the function modified the state of our program. First, the engine thus produces $\langle \texttt{ex}\rangle(2;1)$, which adds a binding to PMap and creates the cell: our bigger state is now $(\{1 \mapsto 0, 2 \mapsto \text{ex}(1)\}, \bot)$. Finally, $\langle \texttt{domainset}\rangle(;\{2\})$ is produced onto the state, however this *doesn't succeed*. Indeed, the bindings of the heap are $\{1,2\}$, which are not a subset of the produced domain set $\{2\}$. This is because the engine cannot tell apart empty (0) states from non-empty states. Of course one could handle this by checking for 0, or by explicitly excluding units from the codomain of PMap. This however makes rules and definitions more complex and prone to error, as it is easy to forget to check for units.

If this examples is now reproduced with RAs, the consumption of $\langle \texttt{ex}\rangle(1;1)$ makes the cell return $\bot$, which cannot be added into the heap – the PMap must thus removes the binding, and the postcondition can be produced properly, as the bindings of the heap is only $\{2\}$. RAs are a solution to the above problem, as they facilitate the sound construction of state models and state model transformers, by allowing unitless state models.

2.4.9 Dynamic PMap

The *dynamic* partial map state transformer, $\text{DynPMap}(I, \mathbb{S})$ is similar to the regular (or *static*) PMap transformer, but allows modelling "dynamic" maps that can be modified without allocation. This is used, for instance, to model the JavaScript memory model, where one can set a field of an object directly if it doesn't exist, and where reading a field

that doesn't exist does not raise an error but simply returns a default `undefined` value. In other words, the domain set is used not to distinguish `Miss` from `Err` but `Miss` from `Ok`.

This transformer has the same RA as PMap, as well as the same predicates. It however only lifts the actions of the underlying state model $\mathbb{S}$, without adding an `alloc` action (since allocation does not exist; the field is just created on access). However, since this requires instantiating the underlying state model, it must still implement an `instantiate` : Val `list` $\to \Sigma$ function. Here we keep Val `list` for compatibility with PMap, but in fact the arguments are always the empty list.

We now present the rules for DynPMap$(I, \mathbb{S})$, highlighting the differences with PMap$(I, \mathbb{S})$. We reuse the definition of `get` and `set`.

$$\text{Given } \mathit{lift\_if\_miss}(o, i, \vec{v}_i) \stackrel{\text{def}}{=} \begin{cases} i :: \vec{v}_i & \text{if } o = \texttt{Miss} \\ \vec{v}_i & \text{otherwise} \end{cases}$$

DynPMapAction

$$\frac{\texttt{get}(s, i) \rightsquigarrow (i', s_{i'}, \pi) \qquad \alpha(s_{i'}, \vec{v}_i) \rightsquigarrow (o, s'_{i'}, \vec{v}_o, \pi') \qquad \texttt{set}(s, i', s'_{i'}) = s'}{\alpha(s, i :: \vec{v}_i) \rightsquigarrow (o, s', i' :: \vec{v}_o, \pi :: \pi')}$$

DynPMapActionOutOfBounds

$$\frac{d \neq \bot \qquad s_i = \texttt{instantiate}\ [] \qquad \alpha(s_i, \vec{v}_i) \rightsquigarrow (o, s'_i, \vec{v}_o, \pi) \qquad h' = h[i \leftarrow s'_i] \qquad d' = d \uplus \{i\}}{\alpha((h, d), i :: \vec{v}_i) \rightsquigarrow (o, (h', d'), \vec{v}_o, [i \notin d] :: \pi)}$$

DynPMapCons

$$\frac{\texttt{get}(s, i) \rightsquigarrow (i', s_{i'}, \pi) \qquad \texttt{consume}(s_{i'}, \delta, \vec{v}_i) \rightsquigarrow (o, s'_{i'}, \vec{v}_o, \pi') \qquad \texttt{set}(s, i', s'_{i'}) = s' \qquad \vec{v}'_o = \mathit{lift\_if\_miss}(o, i', \vec{v}_o)}{\texttt{consume}(s, \delta, i :: \vec{v}_i) \rightsquigarrow (o, s', \vec{v}'_o, \pi :: \pi')}$$

DynPMapConsOutOfBounds

$$\frac{d \neq \bot \qquad s_i = \texttt{instantiate}\ [] \qquad \texttt{consume}(s_i, \delta, \vec{v}_i) \rightsquigarrow (o, s'_i, \vec{v}_o, \pi) \qquad h' = h[i \leftarrow s'_i] \qquad d' = d \uplus \{i\}}{\texttt{consume}((h, d), \delta, i :: \vec{v}_i) \rightsquigarrow (o, (h', d'), \vec{v}_o, [i \notin d] :: \pi)}$$

DynPMapConsDomainSet

$$\frac{d \neq \bot}{\texttt{consume}((h, d), \texttt{domainset}, []) \rightsquigarrow (\texttt{Ok}, (h, \bot), [d], [])}$$

DynPMapConsDomainSetMiss

$$\frac{(h, d) = \mathit{unwrap}(s) \qquad d = \bot}{\texttt{consume}(s, \texttt{domainset}, []) \rightsquigarrow (\texttt{Miss}, s, [\texttt{`domainset'}], [])}$$

DynPMapProd

$$\frac{\texttt{get}(s, i) \rightsquigarrow (i', s_{i'}, \pi) \qquad \texttt{produce}(s_{i'}, \delta, \vec{v}_i, \vec{v}_o) \rightsquigarrow (s'_{i'}, \pi') \qquad \texttt{set}(s, i', s'_{i'}) = s'}{\texttt{produce}(s, \delta, i :: \vec{v}_i, \vec{v}_o) \rightsquigarrow (s', \pi :: \pi')}$$

DynPMapProdDomainSet

$$\frac{(h, \bot) = \mathit{unwrap}(s)}{\texttt{produce}(s, \texttt{domainset}, [], [d]) \rightsquigarrow ((h, d), [\mathrm{dom}(h) \subseteq d])}$$

PMapFix

$$\frac{\mathbb{S}.\texttt{fix}\ \vec{v}_i = a \qquad a' = \mathit{lift}(a, i)}{\texttt{fix}\ i :: \vec{v}_i = a'}$$

DynPMapFixDomainSet

$$\texttt{fix}\ [\texttt{`domainset'}] = \exists d.\, \langle \texttt{domainset} \rangle (; d)$$

We see here the difference in behaviour is minimal, as only the `consume` and `execute_action` functions are affected, and only in the out of bounds case. In fact, the implementation of PMap for Gillian receives a `mode` flag that allows one to instantiate the transformer to static or dynamic mode, avoiding code repetition.

2.4.10 List

The List state model transformer is similar to PMap, but instead of receiving an domain I as a parameter it only operates on positive integers. It allows representing a list of cells, up to a bounded size – it can be used, for instance, to represent a block of memory, with the index serving as the offset of the base address of the block. Similarly to PMap, it represents state as a finite partial map from integers to state, as well as a *bound*: a strictly positive integer specifying the size of the list, and allowing one to distinguish out of bounds from $\bot$ elements.

$$\textsc{List}(\mathbb{S}) \stackrel{\text{def}}{=} \mathbb{N} \xrightarrow{fin} \mathbb{S}.\Sigma \times \mathbb{N}^?$$

$$(b, n) \cdot (b', n') \stackrel{\text{def}}{=} (b'', n'')$$

$$\text{where } b'' \stackrel{\text{def}}{=} \lambda i. \begin{cases} b(i) \cdot b'(i) & \text{if } i \in \text{dom}(b) \cap \text{dom}(b') \\ b(i) & \text{if } i \in \text{dom}(b) \setminus \text{dom}(b') \\ b'(i) & \text{if } i \in \text{dom}(b') \setminus \text{dom}(b) \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$\text{and } n'' \stackrel{\text{def}}{=} \begin{cases} n & \text{if } n' = \bot \\ n' & \text{if } n = \bot \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$\text{and } \forall i \in \text{dom}(b''). \, 0 \leq i \wedge (n'' = \bot \vee i < n'')$$

$$|(b, n)| \stackrel{\text{def}}{=} \begin{cases} \bot & \text{if } \text{dom}(b') = \emptyset \\ (b', \bot) & \text{otherwise} \end{cases}$$

$$\text{where } b' \stackrel{\text{def}}{=} \lambda i. \begin{cases} |b(i)| & \text{if } i \in \text{dom}(b) \wedge |b(i)| \neq \bot \\ \text{undefined} & \text{otherwise} \end{cases}$$

Just like in PMap, all core predicates are lifted, and a `bound` core predicate is added for the bound. It does not include an `alloc` action – instead, all cells are created when the list is instantiated (or taken from the specification of the executed function).

For brevity, its rules will not be outlined – they are analogous to those of PMap, with the main difference being that checks for the membership of an index in the domain set are replaced with checks for the index in the range $[0, n)$ with the bound n.

2.4.11 General Map

In [4], the PMap and List transformers are presented as two different transformers, however they can both be merged into a single transformer quite succinctly, to prove the modularity of transformers.

We thus introduce the *general map* transformer, $\textsc{GMap}(I, \mathbb{S}, \mathbb{S}_D)$. It is built from a domain set I, a codomain state model $\mathbb{S}$ and a *discriminator* state model $\mathbb{S}_D$. This last state model serves as a way to tell apart invalid accesses from $\bot$ elements – modelling it as a state model allows us to make it more flexible on what predicates and state are used to represent it.

The discriminator state model $\mathbb{S}_D$ must also provide a $\texttt{is\_within} : \mathbb{S}_D.\Sigma \to I \to \texttt{LVal}$ function. `is_within` returns a symbolic boolean, that evaluates to true if and only if a state fragment containing a singleton partial map with the given index could be validly composed into the state while upholding the desired GMap's invariant. A consequence of this is that when the state is a*full* state, `is_within` is only used for out of bounds accesses (as otherwise the key is already in the map, since we're dealing with full states), and as such always returns false. If we write $\operatorname{dom}\sigma$ the domain of a GMap's map, given a set of states Σ with the subset of full states $\underline{\Sigma} \subseteq \Sigma$, we have that $i \notin \operatorname{dom}\sigma \land \sigma \in \underline{\Sigma} \Rightarrow \texttt{SAT}_{\theta,s}(\neg\texttt{is\_within}\ \sigma\ i)$.

To replicate the usual PMap, one would have $\mathbb{S}_D = \textsc{Ex}(\mathcal{P}(I))$, with `is_within` true if the value is in the set: $\texttt{is\_within}_{\textsc{PMap}}\ \sigma_D\ i = i \in \sigma_D$. For List, one has $\mathbb{S}_D = \textsc{Ex}(\mathbb{N})$ with $\texttt{is\_within}_{\textsc{List}}\ \sigma_D\ i = 0 \leq i < \sigma_D$. Note here that state transformer may automatically assume the cell can exist if the key is not found in the map and the discriminator state is $\bot$ in the state tuple. Note for both of the above examples the `load` and `store` actions of Ex should be removed, as modifying the domain set or bound directly is not sound – the discriminator is used for the compositional state to distinguish outcomes, and *does not exist in full states*, so actions to modify the discriminator directly cannot exist in the full state either. Allowing them to exist in the compositional state model would break compatibility.

$$
\begin{array}{l}
\textsc{GMap}(I, \mathbb{S}, \mathbb{S}_D) \stackrel{\text{def}}{=} I \xrightarrow{fin} \mathbb{S}.\Sigma \times \mathbb{S}_D.\Sigma^? \\
\quad (h, d) \cdot (h', d') \stackrel{\text{def}}{=} (h'', d'') \\
\qquad \text{where } h'' \stackrel{\text{def}}{=} \lambda i. \begin{cases} h(i) \cdot h'(i) & \text{if } i \in \operatorname{dom}(h) \cap \operatorname{dom}(h') \\ h(i) & \text{if } i \in \operatorname{dom}(h) \setminus \operatorname{dom}(h') \\ h'(i) & \text{if } i \in \operatorname{dom}(h') \setminus \operatorname{dom}(h) \\ \text{undefined} & \text{otherwise} \end{cases} \\
\qquad \text{and } d'' \stackrel{\text{def}}{=} d \cdot d' \\
\qquad\quad \text{and } d'' = \bot \lor (\forall i \in \operatorname{dom}(h''). \texttt{is\_within}\ d''\ i) \\
\qquad |(h, d)| \stackrel{\text{def}}{=} \begin{cases} \bot & \text{if } \operatorname{dom}(h') = \emptyset \land d' = \bot \\ (h', d') & \text{otherwise} \end{cases} \\
\qquad\quad \text{where } h' \stackrel{\text{def}}{=} \lambda i. \begin{cases} |h(i)| & \text{if } i \in \operatorname{dom}(h) \land |h(i)| \neq \bot \\ \text{undefined} & \text{otherwise} \end{cases} \\
\qquad\quad \text{and } d' \stackrel{\text{def}}{=} |d|
\end{array}
$$

We now define the rules for $\textsc{GMap}(I, \mathbb{S}, \mathbb{S}_D)$. Its actions are only that of $\mathbb{S}$ (not that of the discriminator state model), as explained previously; it does however inherit predicates from both. As such, $\mathcal{A} = \mathbb{S}.\mathcal{A}$, and $\Delta = \mathbb{S}.\Delta \uplus \{\delta_D : \delta \in \mathbb{S}_D.\Delta\}$.

We also redefine the `get` and `set` helper functions, with `get` branching again. This requires lifting `get` to do checks using `is_within` (the rest of the rule is unaffected).

We first define a helper methods `get` and `set`, that allows modifying a symbolic map

with branching. After a look up, it returns the value at the location (which may be $\bot$ if it's not found), and the path condition corresponding to the branch. This allows simplifying shared rules for `execute_action`, `produce` and `consume` for PMap.

$$\texttt{get} : ((I \xrightarrow{fin} X) \times \mathbb{S}_D.\Sigma^?) \to I \to \mathcal{P}(I \times X \times \Pi)$$
$$\texttt{set} : ((I \xrightarrow{fin} X) \times \mathbb{S}_D.\Sigma^?) \to I \to X \to (I \xrightarrow{fin} X \times \mathbb{S}_D.\Sigma^?)$$

$$\text{Given } \mathit{wrap}(h,d) \stackrel{\text{def}}{=} \begin{cases} \bot & \text{if } \mathrm{dom}(h) = \emptyset \wedge d = \bot \\ (h,d) & \text{otherwise} \end{cases}$$

$$\mathit{unwrap}(s) \stackrel{\text{def}}{=} \begin{cases} ([], \bot) & \text{if } s = \bot \\ (h,d) & \text{if } s = (h,d) \end{cases}$$

$$\mathit{lift\_if\_miss}(o, i, \vec{\hat{v}}_i) \stackrel{\text{def}}{=} \begin{cases} i :: \vec{\hat{v}}_i & \text{if } o = \texttt{Miss} \\ \vec{\hat{v}}_i & \text{otherwise} \end{cases}$$

GMapGetMatch
$$\frac{(h,d) = \mathit{unwrap}(s) \qquad i' \in \mathrm{dom}(h) \qquad s_{i'} = h(i')}{\texttt{get}(s,i) \rightsquigarrow (i', s_{i'}, [i = i'])}$$

GMapGetAdd
$$\frac{(h,d) = \mathit{unwrap}(s) \qquad i \notin \mathrm{dom}(h) \qquad d \neq \bot}{\texttt{get}(s,i) \rightsquigarrow (i, \bot, [i \notin \mathrm{dom}(h) \wedge \texttt{is\_within}\ d\ i])}$$

GMapGetBotDomain
$$\frac{(h,d) = \mathit{unwrap}(s) \qquad i \notin \mathrm{dom}(h) \qquad d = \bot}{\texttt{get}(s,i) \rightsquigarrow (i, \bot, [i \notin \mathrm{dom}(h)])}$$

GMapSetSome
$$\frac{(h,d) = \mathit{unwrap}(s) \qquad s_i \neq \bot \qquad h' = h[i \leftarrow s_i] \qquad s' = \mathit{wrap}(h', d)}{\texttt{set}(s,i,s_i) = s'}$$

GMapSetNone
$$\frac{(h,d) = \mathit{unwrap}(s) \qquad s_i = \bot \qquad h' = h[i \not\leftarrow] \qquad s' = \mathit{wrap}(h', d)}{\texttt{set}(s,i,s_i) = s'}$$

GMapAction
$$\frac{\texttt{get}(s,i) \rightsquigarrow (i', s_{i'}, \pi) \qquad \alpha(s_{i'}, \vec{\hat{v}}_i) \rightsquigarrow (o, s'_{i'}, \vec{\hat{v}}_o, \pi') \qquad \texttt{set}(s, i', s'_{i'}) = s'}{\alpha(s, i :: \vec{\hat{v}}_i) \rightsquigarrow (o, s', i' :: \vec{\hat{v}}_o, \pi :: \pi')}$$

GMapActionOutOfBounds
$$\frac{d \neq \bot}{\alpha((h,d), \vec{\hat{v}}_i) \rightsquigarrow (\texttt{Err}, s, [], [\neg\texttt{is\_within}\ d\ i])}$$

GMapCons
$$\frac{\texttt{get}(s,i) \rightsquigarrow (i', s_{i'}, \pi) \qquad \texttt{consume}(s_{i'}, \delta, \vec{\hat{v}}_i) \rightsquigarrow (o, s'_{i'}, \vec{\hat{v}}_o, \pi') \qquad \vec{\hat{v}}'_o = \mathit{lift\_if\_miss}(o, i', \vec{\hat{v}}_o) \qquad \texttt{set}(s, i', s'_{i'}) = s'}{\texttt{consume}(s, \delta, i :: \vec{\hat{v}}_i) \rightsquigarrow (o, s', \vec{\hat{v}}'_o, \pi :: \pi')}$$

GMapConsIncompat

$$\frac{d \neq \bot}{\texttt{consume}((h,d),\delta,i :: \vec{v}_i) \rightsquigarrow (\texttt{LFail},(h,d),[],[\neg \texttt{is\_within}\ d\ i])}$$

GMapConsDiscr

$$\frac{(h,d) = \mathit{unwrap}(s) \qquad \texttt{consume}(d,\delta_d,\vec{v}_i) \rightsquigarrow (o,d',\vec{v}_o,\pi) \qquad o \neq \texttt{Miss} \qquad s' = \mathit{wrap}(h,d')}{\texttt{consume}(s,\delta_D,\vec{v}_i) \rightsquigarrow (\texttt{Ok},s',\vec{v}_o,\pi)}$$

GMapConsDiscrMiss

$$\frac{(h,d) = \mathit{unwrap}(s) \qquad \texttt{consume}(d,\delta_d,\vec{v}_i) \rightsquigarrow (o,d',\vec{v}_o,\pi) \qquad o = \texttt{Miss} \qquad s' = \mathit{wrap}(h,d')}{\texttt{consume}(s,\delta_D,\vec{v}_i) \rightsquigarrow (\texttt{Ok},s',\texttt{'D'} :: \vec{v}_o,\pi)}$$

GMapProd

$$\frac{\texttt{get}(s,i) \rightsquigarrow (i',s_{i'},\pi) \qquad \texttt{produce}(s_{i'},\delta,\vec{v}_i,\vec{v}_o) \rightsquigarrow (s'_{i'},\pi') \qquad \texttt{set}(s,i',s'_{i'}) = s'}{\texttt{produce}(s,\delta,i :: \vec{v}_i,\vec{v}_o) \rightsquigarrow (s',\pi :: \pi')}$$

GMapProdDiscr

$$\frac{(h,d) = \mathit{unwrap}(s) \qquad \texttt{produce}(d,\delta_D,\vec{v}_i,\vec{v}_o) \rightsquigarrow (d',\pi)}{\texttt{produce}(s,\delta_D,\vec{v}_i,\vec{v}_o) \rightsquigarrow (s',[\forall i \in \mathrm{dom}(h).\ \texttt{is\_within}\ d\ i] :: \pi)}$$

GMapFix

$$\frac{\mathbb{S}.\texttt{fix}\ \vec{v}_i = a \qquad a' = \mathit{lift}(a,i)}{\texttt{fix}\ i :: \vec{v}_i = a'}$$

GMapFixDiscr

$$\frac{\mathbb{S}_D.\texttt{fix}\ \vec{v}_i = a}{\texttt{fix}\ \texttt{'D'} :: \vec{v}_i = a}$$

These rules are very similar to those of PMap, or the rules List would have, only with the discriminator check instead, as well as more generic `consume`, `produce` and `fix` rules for the discriminator. We also do not provide an `alloc` action with GMap, as the discriminator may need to be modified with a new index; it is up to the user to define any additional actions over it.

An example instantiation of GMap is with the empty state model Emp, a state model that provides no actions or predicates, and that can only be $\bot$. `is_within` may then always return `true`. This allows emulating the traditional separation logic linear heap, where one can always do allocation (this is not the case in PMap, where one must own the domain set to allocate). We note that this instantiation does not uphold **??**: any out of bounds access will result in a `Miss`, rather that properly separating misses from errors.

2.4.12 Miscellaneous Transformers

I'm not sure if this is worth mentionning at all – if I have an 'implementation' section in my report I'll put it there.

Along with the above defined transformers, a range of additional simpler transformers are provided, to make customising constructions easier.

ActionAdd$(\mathbb{S}, A)$ is a transformer that equips the given state model with additional actions defined by A, without needing to define any predicates, overriding functions, or re-implementing repetitive functions.

Filter$(\mathbb{S}, \mathcal{A}, \Delta)$ is a transformer that allows filtering the actions and predicates exposed by a state model, for instance to limit the functionality of a state model if one of its provided actions shouldn't exist in the semantics of the language. This is used, for instance, to mask the `get_all_props` action that is defined in the implementation of PMap as required by JavaScript, but that shouldn't exist in C.

MAPPER($\mathbb{S}, F_{\mathcal{A}}, F_{\Delta}$) allows renaming predicates or actions. This is extremely useful when implementing state models using transformers that need to match a pre-existing interface.

INJECTOR($\mathbb{S}, I$) adds hooks into `consume`, `produce` and `execute_action`, allowing the developer to apply transformations to the inputs or outputs of these functions. This is also convenient when implementing a state model that needs to satisfy a pre-existing interface. For instance, this makes re-ordering the arguments of certain actions trivial, or allows treating an out-value as an in-value that makes the branch vanish if it doesn't correspond (this is the case for the `domainset` predicate in JSIL).

2.5 Optimising maps

As mentioned above, GMAP is an ideal target for optimisation: it is a common transformer (used once in the C and WISL memory models, and twice in the JS memory model) that has a high performance cost, due to branching: for instance, executing an action in a map with n cells can lead to $n + 2$ branches. While most of these branches eventually get dropped, as the path condition doesn't hold, there is a cost to checking the satisfiability of the path condition. The *ideal* GMAP transformer only returns feasible branches, minimising SAT checks.

Another aspect of optimisation is the need for simplifying *substitution*. While not described here, substituting a GMAP's state requires recursing through all key-value pairs, and applying the substitution to both the key and the value, before rebuilding a binding and possibly composing values that end with the same key. For instance, given a construction GMAP($\mathbb{N}$, EX($\{1\}$) $\bowtie$ EX($\{2\}$)) (we ignore the discriminator for brevity) applying the substitution $\theta[\hat{x} \to 0]$ to the mappings $[\hat{x} \mapsto (1, \bot), 0 \mapsto (\bot, 2)]$ would yield $[0 \mapsto (1, 2)]$. Evaluation of Gillian with different state model constructions has shown that, especially for states with large partial maps (as is the case in JavaScript), substitution can take up to *50%* of the execution time directly within the state model (here, we ignore the difference in total execution time of the engine).

We will here explore three different optimisations of PMAP that have been ported to Gillian, by justifying their theoretical soundness. While they have been adapted for PMAP, they could also be adapted for GMAP with little additional effort – we choose not to, to keep the presentation simpler.

2.5.1 Syntactic Checking

A technique introduced in [4] that we may reuse is syntactic equality checking for a matching key before attempting to branch on symbolic equality. This means that if the map contains the exact key already, we do not need to do any branching. This method also benefits from *hash consing*, where a hash is associated to every expression's AST, avoiding the need for recursing through possibly deep trees. *We note hash consing for expressions is currently not implemented in Gillian.*

We present here the rules for this technique. We highlight the new rule and condition, noting the last two rules are unchanged.

$$\frac{(h, d) = \mathit{unwrap}(s) \qquad i \in \mathrm{dom}(h) \qquad s_i = h(i)}{\mathtt{get}(s, i) \rightsquigarrow (i, s_i, [])} \text{ SYNTACTICPMAPGETMATCH}$$

$$\textsc{SyntacticPMapGetBranch}\quad \frac{(h,d) = \mathit{unwrap}(s) \qquad i \notin \mathrm{dom}(h) \qquad i' \in \mathrm{dom}(h) \qquad s_{i'} = h(i')}{\mathtt{get}(s,i) \rightsquigarrow (i', s_{i'}, [i = i'])}$$

$$\textsc{SyntacticPMapGetAdd}\quad \frac{(h,d) = \mathit{unwrap}(s) \qquad i \notin \mathrm{dom}(h) \qquad d \neq \bot}{\mathtt{get}(s,i) \rightsquigarrow (i, \bot, [i \notin \mathrm{dom}(h) \wedge i \in d])}$$

$$\textsc{SyntacticPMapGetBotDomain}\quad \frac{(h,d) = \mathit{unwrap}(s) \qquad i \notin \mathrm{dom}(h) \qquad d = \bot}{\mathtt{get}(s,i) \rightsquigarrow (i, \bot, [i \notin \mathrm{dom}(h)])}$$

This optimisation is also the reason why actions on PMap always return the accessed index; it allows further accesses to the same element to benefit from this syntactic equality check, and thus avoids repetitive SAT checks.

2.5.2 Split PMap

The idea behind this first optimisation is to split the bindings of the map between *concrete* and *symbolic*, effectively storing two maps at once. This has a few advantages: firstly, substitution must only be done on the symbolic part of the map, as only symbolic variables can be substituted. The second effect this has is that when doing lookups of a key, if the key is not concrete we may avoid checking for syntactic equality in the concrete part of the map, since the binding cannot be there. Note this doesn't hold for the opposite: the key may be concrete, but if the associated value is symbolic then the binding will be in the symbolic map – a concrete key does thus require doing a lookup on both maps.

Of course, to distinguish concrete from non-concrete cells, it requires the underlying state model to provide an $\mathtt{is\_concrete}_\Sigma : \Sigma \to \mathbb{B}$ function.

We first define the resource algebra of this state model, where h_c denotes the concrete part of the map and h_s the symbolic (or better said, non-concrete) part:

$$\textsc{PMap}_{\textsc{Split}}(I, \mathbb{S}) \stackrel{\text{def}}{=} I \xrightarrow{\mathit{fin}} \mathbb{S}.\Sigma \times I \xrightarrow{\mathit{fin}} \mathbb{S}.\Sigma \times \mathcal{P}(I)^?$$

We note that composition requires checking for the presence of the index in both maps; if the first state has a binding in its concrete heap while the second has a binding in its symbolic heap, both need to be composed and then verify again if the result of the composition is concrete or symbolic.

Its predicates and actions are the same as for PMap: $\mathcal{A} = \mathbb{S}.\mathcal{A}$ and $\Delta = \mathbb{S}.\Delta \uplus \{\mathtt{domainset}\}$. We assume the engine also provides an `is_concrete` function that given an expression returns true if it is concrete; this can be implemented by recursing through the expression's AST.

We again only define the rules for the `get` and `set` helper functions – everything else behaves the same (showing the advantage of having these two functions abstracted away).

$$\text{Given } \mathit{wrap}(h_c, h_s, d) \stackrel{\text{def}}{=} \begin{cases} \bot & \text{if } \mathrm{dom}(h_c) = \emptyset \wedge \mathrm{dom}(h_s) = \emptyset \wedge d = \emptyset \\ (h_c, h_s, d) & \text{otherwise} \end{cases}$$

$$\mathit{unwrap}(s) \stackrel{\text{def}}{=} \begin{cases} ([], [], \emptyset) & \text{if } s = \bot \\ (h_c, h_s, d) & \text{if } s = (h_c, h_s, d) \end{cases}$$

SplitPMapGetMatchCon

$$\frac{(h_c, h_s, d) = \mathit{unwrap}(s) \qquad \texttt{is\_concrete}\ i \qquad i \in \mathrm{dom}(h_c) \qquad s_i = h_c(i)}{\texttt{get}(s, i) \rightsquigarrow (i, s_i, [])}$$

SplitPMapGetMatchSym

$$\frac{(h_c, h_s, d) = \mathit{unwrap}(s) \qquad i \in \mathrm{dom}(h_s) \qquad s_i = h_s(i)}{\texttt{get}(s, i) \rightsquigarrow (i, s_i, [])}$$

SplitPMapGetBranch

$$\frac{\begin{array}{c}(h_c, h_s, d) = \mathit{unwrap}(s) \\ h_{all} = h_c \cup h_s \qquad i \notin \mathrm{dom}(h_{all}) \qquad i' \in \mathrm{dom}(h_{all}) \qquad s_{i'} = h_{all}(i')\end{array}}{\texttt{get}(s, i) \rightsquigarrow (i', s_{i'}, [i = i'])}$$

SplitPMapGetAdd

$$\frac{(h_c, h_s, d) = \mathit{unwrap}(s) \qquad h_{all} = h_c \cup h_s \qquad i \notin \mathrm{dom}(h_{all}) \qquad d \neq \bot}{\texttt{get}(s, i) \rightsquigarrow (i, \bot, [i \notin \mathrm{dom}(h_{all}) \wedge i \in d])}$$

SplitPMapGetBotDomain

$$\frac{(h_c, h_s, d) = \mathit{unwrap}(s) \qquad h_{all} = h_c \cup h_s \qquad i \notin \mathrm{dom}(h_{all}) \qquad d = \bot}{\texttt{get}(s, i) \rightsquigarrow (i, \bot, [i \notin \mathrm{dom}(h_{all})])}$$

SplitPMapSetSomeCon

$$\frac{\begin{array}{c}(h_c, h_s, d) = \mathit{unwrap}(s) \qquad s_i \neq \bot \qquad \texttt{is\_concrete}_{\Sigma}\ s_i \\ \texttt{is\_concrete}\ i \qquad h'_c = h_c[i \leftarrow s_i] \qquad h'_s = h_s[i \not\leftarrow] \qquad s' = \mathit{wrap}(h'_c, h'_s, d)\end{array}}{\texttt{set}(s, i, s_i) = s'}$$

SplitPMapSetSomeSym

$$\frac{\begin{array}{c}(h_c, h_s, d) = \mathit{unwrap}(s) \qquad s_i \neq \bot \qquad \neg(\texttt{is\_concrete}_{\Sigma}\ s_i \vee \texttt{is\_concrete}\ i) \\ h'_c = h_c[i \not\leftarrow] \qquad h'_s = h_s[i \leftarrow s_i] \qquad s' = \mathit{wrap}(h'_c, h'_s, d)\end{array}}{\texttt{set}(s, i, s_i) = s'}$$

SplitPMapSetNone

$$\frac{\begin{array}{c}(h_c, h_s, d) = \mathit{unwrap}(s) \\ s_i = \bot \qquad h'_c = h_c[i \not\leftarrow] \qquad h'_s = h_s[i \not\leftarrow] \qquad s' = \mathit{wrap}(h'_c, h'_s, d)\end{array}}{\texttt{set}(s, i, s_i) = s'}$$

We note here that this optimisation comes with a cost: if syntactic matches are rare, then the two maps need to be merged for the branching check. Furthermore, modifying the map also requires removing the binding from the other map, as the state might have been there and changed (for instance, if the wrapped state was symbolic, but was modified to be concrete, then it needs to be removed from the symbolic map). Verifying if a given state fragment is concrete or not also comes with a cost, as it requires traversing the entire

state, which may be expensive – though this could be partly solved by caching whether it's concrete or not, and invalidating or modifying the cache when the state is modified. This caching has not been implemented, as in practice the cost of verifying concreteness is less than the gain of the optimisation.

In evaluation we will discuss the impact this optimisation has. *I need to measure the proportion of size between concrete/symbolic, to see in what executions it's more useful and in which it's not. Similarly, need to profile the time for a lookup depending on the size of the joined map!*

2.5.3 Abstract location PMap

This second optimisation is new to Gillian, and uses *abstract locations* (ALocs). We expand the definition of symbolic values, `SVal`, with abstract locations denotes $\text{aloc}(a)$, with a the name of the location. These are a form of *semantic hash consing*: they can be used to distinguish different locations from their name, when not doing "matching". *Matching* is a mode that is enabled when consuming or producing, and that is disabled when executing actions and doing substitutions. When matching is disabled, two given abstract locations that are syntactically different (have different names) are semantically different (are distinct). When matching is enabled, two syntactically different abstract locations may be equal, depending on the current path condition, which may lead to branching (in which cases the two states at the clashing locations are composed). This is a powerful optimisation: inside the body of a function, all lookups are branchless, as two syntactically different ALocs are considered different (matching is disabled), however when producing the pre-condition or consuming the post-condition then locations may clash and we thus merge together states at clashing locations.

A limitation of this optimisation is that only abstract locations may be used as the domain type; it is thus a specialisation of PMap, where $\text{PMap}_{\text{Split}}$ was a more general optimisation.

Let's first define the RA for this optimisation, $\text{PMap}_{\text{ALoc}}$.

$$\text{PMap}_{\text{ALoc}}(\mathbb{S}) \stackrel{\text{def}}{=} \text{Str} \xrightarrow{fin} \mathbb{S}.\Sigma \times \mathcal{P}(\text{Str})^?$$

We make use of the helper function $\texttt{to\_aloc} : \Pi \to \text{Val} \to \text{Str}^?$, which returns the name of the abstract location matching a given value, if it exists. This function receives the path condition, as for instance given a symbolic variable $\hat{x}$ with the path condition $\hat{x} = \text{aloc}(\texttt{loc\_x})$, the path condition is required to know what abstract location is associated to $\hat{x}$. The engine presented thus far however never receives the path condition, as to ensure it can only be strengthened – this is not the case in Gillian, where this optimisation was first implemented. We thus assume that `to_aloc` is capable of reading the current path condition without getting it as an input, to avoid modifying the previously defined signatures, noting it is trivial to add the path condition as a parameter of `execute_action`, `consume` and `produce`. We thus instead use the signature $\texttt{to\_aloc} : \text{Val} \to \text{Str}^?$. We also use the auxiliary function $\texttt{fresh\_aloc} : \texttt{unit} \to \text{Str}$ to generate fresh abstract location names.

We now present the rules for this state model; in particular, we again only need to concern ourselves with the `get` and `set` internal methods; actions and predicate consumption and production remain the same. We also extend `get` to receive a mode $M = \{\texttt{MATCH}, \texttt{NO\_MATCH}\}$; it is `MATCH` in `produce` and `consume`, and `NO_MATCH` otherwise.

ALocPMapGetMatch

$$\frac{(h, d) = \mathit{unwrap}(s) \qquad a = \texttt{to\_aloc}\ i \qquad a \neq \bot \qquad a \in \mathrm{dom}(h) \qquad s_a = h(a)}{\texttt{get}(s, i, m) \rightsquigarrow (\mathrm{aloc}(a), s_a, [])}$$

ALocPMapGetMatchBot

$$\frac{(h, d) = \mathit{unwrap}(s) \qquad a = \texttt{to\_aloc}\ i \qquad a \neq \bot \qquad a \notin \mathrm{dom}(h) \qquad d \neq \bot}{\texttt{get}(s, i, m) \rightsquigarrow (\mathrm{aloc}(a), \bot, [\mathrm{aloc}(a) \in d])}$$

ALocPMapGetNewLoc

$$\frac{(h, d) = \mathit{unwrap}(s) \qquad a = \texttt{to\_aloc}\ i \qquad a = \bot \qquad d = \bot \qquad a' = \texttt{fresh\_aloc ()}}{\texttt{get}(s, i, m) \rightsquigarrow (\mathrm{aloc}(a'), \bot, [i = \mathrm{aloc}(a')])}$$

ALocPMapMatching

$$\frac{(h, d) = \mathit{unwrap}(s) \qquad a = \texttt{to\_aloc}\ i \qquad a \neq \bot \qquad a \notin \mathrm{dom}(h) \qquad a' \in \mathrm{dom}(h) \qquad s_{a'} = h(a')}{\texttt{get}(s, i, \texttt{MATCH}) \rightsquigarrow (\mathrm{aloc}(a'), s_{a'}, [\mathrm{aloc}(a) = \mathrm{aloc}(a')])}$$

ALocPMapMatchingBot

$$\frac{(h, d) = \mathit{unwrap}(s) \qquad a = \texttt{to\_aloc}\ i \qquad a = \bot \qquad a' \in \mathrm{dom}(h) \qquad s_{a'} = h(a')}{\texttt{get}(s, i, \texttt{MATCH}) \rightsquigarrow (\mathrm{aloc}(a'), s_{a'}, [i = \mathrm{aloc}(a')])}$$

ALocPMapSetSome

$$\frac{(h, d) = \mathit{unwrap}(s) \qquad a = \texttt{to\_aloc}\ i \qquad a \neq \bot \qquad s_i \neq \bot \qquad h' = h[a \leftarrow s_i] \qquad s' = \mathit{wrap}(h', d)}{\texttt{set}(s, i, s_i) = s'}$$

ALocPMapSetNone

$$\frac{(h, d) = \mathit{unwrap}(s) \qquad a = \texttt{to\_aloc}\ i \qquad a \neq \bot \qquad s_i = \bot \qquad h' = h[a \not\leftarrow] \qquad s' = \mathit{wrap}(h', d)}{\texttt{set}(s, i, s_i) = s'}$$

The main advantage of $\mathrm{PMap}_{\mathrm{ALoc}}$ is made evident from the rules: looking up an index does not branch when not matching; it either is found directly in the map, or can be added. This is sound, because abstract locations cannot be calculated or generated freely by the program: they can only be created when producing or consuming predicates (in which case we enable matching and the performance is similar to that of the unoptimised PMap), or when allocating (at which point we know it is a new ALoc different to all others, since it is a fresh ALoc, by definition). It is also noteworthy that we index on strings, rather than expressions; this makes other implementation-specific optimisations easy to apply, for instance using a prefix map or a hash map.

There is a (minor) price to pay for this: it requires the `to_aloc` function, that needs to simplify an expression and check the path condition to see if it equates a particular abstract location – this is non-trivial, in particular for more complex expressions when used with large path conditions.

Chapter 3

Soundness of allocation in PMap

3.1 Current State

We initially defined PMap as:

$$\textsc{PMap}(I, \mathbb{S}) \stackrel{\text{def}}{=} I \xrightarrow{fin} \mathbb{S}.\Sigma \times \mathcal{P}(I)^?$$

$$(h, d) \cdot (h', d') \stackrel{\text{def}}{=} (h'', d'')$$

$$\text{where } h'' \stackrel{\text{def}}{=} \lambda i. \begin{cases} h(i) \cdot h'(i) & \text{if } i \in \text{dom}(h) \cap \text{dom}(h') \\ h(i) & \text{if } i \in \text{dom}(h) \setminus \text{dom}(h') \\ h'(i) & \text{if } i \in \text{dom}(h') \setminus \text{dom}(h) \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$\text{and } d'' \stackrel{\text{def}}{=} \begin{cases} d & \text{if } d' = \bot \\ d' & \text{if } d = \bot \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$\text{and } d'' = \bot \vee \text{dom}(h'') \subseteq d''$$

$$|(h, d)| \stackrel{\text{def}}{=} \begin{cases} \bot & \text{if } \text{dom}(h') = \emptyset \\ (h', \bot) & \text{otherwise} \end{cases}$$

$$\text{where } h' \stackrel{\text{def}}{=} \lambda i. \begin{cases} |h(i)| & \text{if } i \in \text{dom}(h) \wedge |h(i)| \neq \bot \\ \text{undefined} & \text{otherwise} \end{cases}$$

The `alloc` action could then be executed; if the domain set was present, it was simply extended with the new index:

PMapAlloc

$$\frac{d \neq \bot \qquad i \notin SV \qquad i \notin \mathrm{dom}(h) \qquad i \notin d \qquad s_i = \texttt{instantiate}(\vec{v}_i) \qquad h' = h[i \leftarrow s_i] \qquad d' = d \uplus \{i\}}{\texttt{alloc}(SV, (h, d), \vec{v}_i) \rightsquigarrow (\texttt{Ok}, (h', d'), [i], [i = i])}$$

PMapAllocBot

$$\frac{(h, d) = \mathit{unwrap}(s) \qquad d = \bot \qquad i \notin SV \qquad i \notin \mathrm{dom}(h) \qquad s_i = \texttt{instantiate}(\vec{v}_i) \qquad h' = h[i \leftarrow s_i]}{\texttt{alloc}(SV, s, \vec{v}_i) \rightsquigarrow (\texttt{Ok}, (h', \bot), [i], [i = i])}$$

This however breaks Frame subtraction:

Let $\sigma = ([0 \mapsto a], \bot), \quad \sigma_f = ([], \{0\})$

$\sigma \cdot \sigma_f = ([0 \mapsto a], \bot) \cdot ([], \{0\}) = ([0 \mapsto a], \{0\})$

We have that $\texttt{alloc}(\sigma \cdot \sigma_f, []) \rightsquigarrow (o, ([0 \mapsto a, 1 \mapsto b], \{0, 1\}), [])$ with $o = \texttt{Ok}$

and $\texttt{alloc}((\sigma, \bot), []) \rightsquigarrow (o', ([0 \mapsto a, 1 \mapsto b], \bot), [])$ with $o' = \texttt{Ok}$

$o' \neq \texttt{Miss}$ but $([0 \mapsto a, 1 \mapsto b], \{0, 1\}) \neq ([0 \mapsto a, 1 \mapsto b], \bot) \cdot ([], \{1\})$ since that's undefined, because

$\mathrm{dom}([0 \mapsto a, 1 \mapsto b]) = \{0, 1\} \not\subseteq \{1\}$

To solve this, `alloc` on a missing domain set should result in a `Miss`, and only succeed when owning the domain set:

PMapAllocMiss

$$\frac{(h, d) = \mathit{unwrap}(s) \qquad d = \bot}{\texttt{alloc}(SV, s, \vec{v}_i) \rightsquigarrow (\texttt{Miss}, s, [\texttt{'domainset'}], [])}$$

3.2 Concurrency

This however does not work in a concurrency setting, as two threads cannot simultaneously own and modify the domain set. This means that the partial semantics of `alloc` shown above are not compatible with full semantics in which this is permitted (as is the case, for instance, in the traditional model of separation logic).

A solution to this is to make the domain set a set of all indices *known* to exist – indices not in it don't necessarily mean an out of bounds, but may mean the index was created in

another thread that hasn't "rejoined" into the current thread. Its RA is defined as:

$$\begin{aligned}
&\textsc{PMap}_{\textsc{Fixed}}(I, \mathbb{S}) \stackrel{\text{def}}{=} I \xrightarrow{\mathit{fin}} \mathbb{S}.\Sigma \times \mathcal{P}(I) \\
&\quad (h, d) \cdot (h', d') \stackrel{\text{def}}{=} (h'', d \cup d') \\
&\qquad \text{where } h'' \stackrel{\text{def}}{=} \lambda i. \begin{cases} h(i) \cdot h'(i) & \text{if } i \in \text{dom}(h) \cap \text{dom}(h') \\ h(i) & \text{if } i \in \text{dom}(h) \backslash \text{dom}(h') \\ h'(i) & \text{if } i \in \text{dom}(h') \backslash \text{dom}(h) \\ \text{undefined} & \text{otherwise} \end{cases} \\
&\qquad\quad \text{and } \text{dom}(h'') \subseteq (d \cup d') \\
&\quad |(h, d)| \stackrel{\text{def}}{=} (h', d) \\
&\qquad \text{where } h' \stackrel{\text{def}}{=} \lambda i. \begin{cases} |h(i)| & \text{if } i \in \text{dom}(h) \wedge |h(i)| \neq \bot \\ \text{undefined} & \text{otherwise} \end{cases}
\end{aligned}$$

This means however that $\textsc{PMap}_{\textsc{Fixed}}$ cannot be exclusively owned, as a partial map with a subset of its domain can always be composed with it. Its rule for allocation is the same as before, except there is no miss case:

PMapFixedAlloc

$$\frac{i \notin \text{dom}(h) \qquad i \notin d \qquad s_i = \texttt{instantiate}(\vec{v}_i) \qquad h' = h[i \leftarrow s_i] \qquad d' = d \cup \{i\}}{\texttt{alloc}(SV, (h, d), \vec{v}_i) \rightsquigarrow (\texttt{Ok}, (h', d'), [i], [i = i])} \; i \notin SV$$

Rules are otherwise similar, with consuming the domainset not modifying it, and producing it producing the union with the current domain set.

While allowing concurrent allocation, this state model does not fulfill **??** in its current state. Indeed, it cannot reliably distinguish out of bounds from missing outcomes on accesses - as the index not being in the domain set may mean it still exists in another thread.

Chapter 4

Proofs of soundness

4.1 Exclusive

4.1.1 Resource Algebra

$$\mathrm{Ex}(X) \stackrel{\text{def}}{=} \mathrm{ex}(x : X)$$
$$|\mathrm{ex}(x)| \stackrel{\text{def}}{=} \bot$$
$$\mathrm{ex}(x_1) \cdot \mathrm{ex}(x_2) \text{ is always undefined}$$

We define the actions of the state model as $\mathcal{A} = \{\mathsf{load}, \mathsf{store}\}$, and the predicates $\Delta = \{\mathsf{ex}\}$. We define predicate satisfiability and symbolic interpretation as:

$$\textsc{ExPredSat}\ \frac{\sigma = \mathrm{ex}(x)}{\sigma \models_\Delta \langle \mathsf{ex} \rangle([];[x])} \qquad \textsc{ExSymInterpretation}\ \frac{[\![\hat{x}]\!]_{\theta,s} = x}{\theta, s, \mathrm{ex}(x) \models \mathrm{ex}(\hat{x})}$$

4.1.2 Compositional Concrete Rules

$$\textsc{CExLoadOk}\ \mathsf{load}(\mathrm{ex}(x), []) = (\mathsf{Ok}, \mathrm{ex}(x), [x]) \qquad \textsc{CExLoadMiss}\ \mathsf{load}(\bot, []) \rightsquigarrow (\mathsf{Miss}, \bot, [])$$

$$\textsc{CExStoreOk}\ \mathsf{store}(\mathrm{ex}(x), [x']) \rightsquigarrow (\mathsf{Ok}, \mathrm{ex}(x'), []) \qquad \textsc{CExStoreMiss}\ \mathsf{store}(\bot, [x']) \rightsquigarrow (\mathsf{Miss}, \bot, [])$$

4.1.3 Compositional Symbolic Rules

$$\textsc{ExLoadOk}\ \mathsf{load}(\mathrm{ex}(\hat{x}), []) \rightsquigarrow (\mathsf{Ok}, \mathrm{ex}(\hat{x}), [\hat{x}], []) \qquad \textsc{ExLoadMiss}\ \mathsf{load}(\bot, []) \rightsquigarrow (\mathsf{Miss}, \bot, [], [])$$

$$\textsc{ExStoreOk}\ \mathsf{store}(\mathrm{ex}(\hat{x}), [\hat{x}']) \rightsquigarrow (\mathsf{Ok}, \mathrm{ex}(\hat{x}'), [], []) \qquad \textsc{ExStoreMiss}\ \mathsf{store}(\bot, [\hat{x}']) \rightsquigarrow (\mathsf{Miss}, \bot, [], [])$$

$$\textsc{ExConsOk}\ \mathsf{consume}(\mathrm{ex}(\hat{x}), \mathsf{ex}, []) \rightsquigarrow (\mathsf{Ok}, \bot, [\hat{x}], []) \qquad \textsc{ExConsMiss}\ \mathsf{consume}(\bot, \mathsf{ex}, []) \rightsquigarrow (\mathsf{Miss}, \bot, [], [])$$

ExProd

$\texttt{produce}(\bot, \texttt{ex}, [], [\hat{x}]) \rightsquigarrow (\text{ex}(\hat{x}), [])$

ExFix

$\texttt{fix}\ [] = [\{\exists \hat{x}.\, \langle \texttt{ex} \rangle (; \hat{x})\}]$

4.1.4 Soundness Proofs

Proof.

Proposition: OX Soudness
Assume

(H1) $\theta, s, \sigma \models \hat{\sigma} \wedge \alpha(\sigma, \vec{v}_i) = (o, \sigma', \vec{v}_o) \wedge [\![\vec{\hat{v}}_i]\!]_{\theta,s} = \vec{v}_i$

(H2) $\forall o', \hat{\sigma}', \vec{\hat{v}}'_o, \pi'.\, \alpha(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (o', \hat{\sigma}', \vec{\hat{v}}'_o, \pi') \Rightarrow o' \in \{\texttt{Ok}, \texttt{Err}\}$

To prove

(G1) $\exists \hat{\sigma}', \vec{\hat{v}}_o, \pi, \theta'.\, \hat{\alpha}(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{\hat{v}}_o, \pi) \wedge \theta', s, \sigma' \models \hat{\sigma}' \wedge \texttt{SAT}_{\theta',s}(\pi) \wedge [\![\vec{\hat{v}}_o]\!]_{\theta',s} = \vec{v}_o$

The proof is analogous for both actions, so we only consider the case where $\alpha = \texttt{load}$. Given (H1) and the definition of $\models$, there are two further cases: $\sigma = \text{ex}(x)$ and $\hat{\sigma} = \text{ex}(\hat{x})$, or $\sigma = \bot$ and $\hat{\sigma} = \bot$. Again, both cases are analogous, so we only consider $\sigma = \text{ex}(x)$, $\hat{\sigma} = \text{ex}(\hat{x})$.

(H3) From our hypothesis, $\sigma = \text{ex}(x)$ and $\hat{\sigma} = \text{ex}(\hat{x})$

(H4) From the definition of concrete actions CExLoadOk, we get $\vec{v}_i = []$, $o = \texttt{Ok}$, $\sigma' = \sigma$, $\vec{v}_o = [x]$.

(H5) From the definition of $\models$ we also have $[\![\hat{x}]\!]_{\theta,s} = x$

(H6) We can pick $\vec{\hat{v}}_o$, π and θ' such that $\vec{\hat{v}}_o = [\hat{x}]$, $\pi = []$ and $\theta' = \theta$.

(H7) From (H6), we get $[\![\vec{\hat{v}}_o]\!]_{\theta,s} = \vec{v}_o$, as well as $\texttt{SAT}_{\theta,s}(\pi)$

(H8) From ExLoadOk and (H6), we have $\texttt{load}(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (\texttt{Ok}, \hat{\sigma}', \vec{\hat{v}}_o, \pi)$ with $\hat{\sigma}' = \hat{\sigma}$.

(H9) Finally, from (H4) and (H8), we have $\sigma' = \sigma$ and $\hat{\sigma}' = \hat{\sigma}$, thus from (H1) it follows that $\theta, s, \sigma' \models \hat{\sigma}'$.

Combining (H7), (H8) and (H9), we get our goal (G1).

Proposition: UX Soudness
Assume

(H1) $\hat{\alpha}(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{\hat{v}}_o, \pi) \wedge \texttt{SAT}_{\theta,s}(\pi) \wedge \theta, s, \sigma' \models \hat{\sigma}' \wedge$
$[\![\vec{\hat{v}}_o]\!]_{\hat{s},\pi} \rightsquigarrow (\vec{v}_o, \pi') \wedge [\![\vec{\hat{v}}_i]\!]_{\hat{s},\pi'} \rightsquigarrow (\vec{v}_i, \pi'')$

To prove

(G1) $\exists \sigma.\, \theta, s, \sigma \models \hat{\sigma} \wedge \alpha(\sigma, \vec{v}_i) = (o, \sigma', \vec{v}_o)$

We again only consider the case where $\alpha = \texttt{load}$ and $\sigma' = \text{ex}(x)$, $\hat{\sigma}' = \text{ex}(\hat{x})$ – the other three cases are analogous.

(H2) From ExLoadOk, we get $\hat{\sigma} = \hat{\sigma}' = \text{ex}(\hat{x})$, $\vec{\hat{v}}_i = []$, $o = \texttt{Ok}$, $\vec{\hat{v}}_o = [\hat{x}]$, $\pi = []$, with $s = [x \mapsto \hat{x}]$

(H3) From (H2) and (H1) we have $\vec{v}_i = []$ and $\vec{v}_o = [x]$

(H4) We can pick $\sigma = \sigma' = \mathrm{ex}(x)$, which from (H1) and (H2) give us $\theta, s, \sigma \models \hat{\sigma}$

(H5) From CExLoadOk, (H3) and (H4), we have $\mathtt{load}(\sigma, \vec{v}_i) = (\mathtt{Ok}, \sigma', \vec{v}_o)$.

Combinding (H4) and (H5) gives our goal (G1).

Proposition: Frame subtraction is satisfied
Assume

(H1) $\sigma \# \sigma_f \wedge \alpha(\sigma \cdot \sigma_f, \vec{v}_i) = (o, \sigma', \vec{v}_o)$

To prove

(G1) $\exists \sigma'', o', \vec{v}'_o.\, \alpha(\sigma, \vec{v}_i) = (o', \sigma'', \vec{v}'_o) \wedge$
$(o' \neq \mathtt{Miss} \implies o' = o \wedge \vec{v}'_o = \vec{v}_o \wedge \sigma' = \sigma'' \cdot \sigma_f)$

(H2) `load` and `store` are always defined for respectively 0 and 1 arguments, so from (H1) we know $\exists \sigma'', o', \vec{v}'_o.\, \alpha(\sigma, \vec{v}_i) = (o', \sigma'', \vec{v}'_o)$.

(H3) Assume $o' \neq \mathtt{Miss}$

(H4) If $\sigma = \bot$, the rules say $o' = \mathtt{Miss}$, contradicting (H3). Thus $\sigma = \mathrm{ex}(x)$.

(H5) From (H1), we know $\sigma \cdot \sigma_f$ is defined, so it must be that $\sigma_f = \bot$ as $\mathrm{ex}(x) \cdot \mathrm{ex}(y)$ is undefined.

From (H5) and composition rules we know $\sigma \cdot \sigma_f = \mathrm{ex}(x) \cdot \bot = \mathrm{ex}(x) = \sigma$. This gives our goal (G1).

Proposition: Frame addition is satisfied
Assume

(H1) $\alpha(\sigma, \vec{v}_i) = (o, \sigma', \vec{v}_o)$

(H2) $\sigma_f \# \sigma'$

(H3) $o \neq \mathtt{Miss}$

To prove

(G1) $\sigma \# \sigma_f \wedge \alpha(\sigma \cdot \sigma_f, \vec{v}_i) = (o, \sigma' \cdot \sigma_f, \vec{v}_o)$

(H4) From (H3) and the action rules, we know $\sigma' = \mathrm{ex}(x)$.

(H5) From composition, (H4) and (H2), we get $\sigma_f = \bot$.

From (H5) we get $\sigma' \cdot \sigma_f = \sigma' \cdot \bot = \sigma'$, and from (H1) our goal (G1) follows.

Proposition: Consume soundness
Assume

(H1) $\mathtt{consume}(\hat{\sigma}, \delta, \vec{\hat{v}}_i) \rightsquigarrow (\mathtt{Ok}, \hat{\sigma}_f, \vec{\hat{v}}_o, \pi)$

(H2) $\theta, s, \sigma \models \hat{\sigma} \wedge \mathtt{SAT}_{\theta,s}(\pi)$

To prove

(G1) $\exists \sigma_\delta, \sigma_f.\, \sigma_\delta \# \sigma_f \wedge \sigma = \sigma_\delta \cdot \sigma_f \wedge \theta, s, \sigma_\delta \models_\Delta \langle \delta \rangle (\vec{\hat{v}}_i; \vec{\hat{v}}_o) \wedge \theta, s, \sigma_f \models \hat{\sigma}_f$

(H3) There is only one `consume` rules yielding `Ok`, giving us $\delta = \mathsf{ex}$, $\hat{\sigma} = \mathrm{ex}(\hat{x})$, $\hat{\sigma}_f = \bot$, $\vec{\hat{v}}_i = []$, $\vec{\hat{v}}_o = [\hat{x}]$, $\pi = []$

(H4) From the definition of $\models$, we must have $\sigma = \mathrm{ex}(x)$ and $\sigma_f = \bot$ such that $\theta, s, \sigma \models \hat{\sigma}$ and $\theta, s, \sigma_f \models \hat{\sigma}_f$.

(H5) From (H4), we have $[\![\hat{x}]\!]_{\theta,s} = x$, such that $[\![\vec{\hat{v}}_i]\!]_{\theta,s} = []$ and $[\![\vec{\hat{v}}_o]\!]_{\theta,s} = [x]$.

(H6) From the definition of composition, we must have $\sigma_\delta = \mathrm{ex}(x)$ such that $\sigma = \sigma_\delta \cdot \sigma_f$, which also implies $\sigma_\delta \# \sigma_f$

(H7) From (H6), (H5) and the definition of $\models_\Delta$, we have $\theta, s, \sigma_\delta \models_\Delta \langle\delta\rangle(\vec{\hat{v}}_i; \vec{\hat{v}}_o)$.

Combining (H4), (H6) and (H7) gives our goal (G1).

Proposition: Consume completeness
Assume

(H1) $\mathsf{consume}(\hat{\sigma}, \delta, \vec{\hat{v}}_i) \rightsquigarrow (\mathsf{Ok}, \hat{\sigma}_f, \vec{\hat{v}}_o, \pi)$

(H2) $\theta, s, \sigma_f \models \hat{\sigma}_f \wedge \theta, s, \sigma_\delta \models_\Delta \langle\delta\rangle(\vec{\hat{v}}_i; \vec{\hat{v}}_o) \wedge \sigma_f \# \sigma_\delta$

To prove

(G1) $\exists \sigma.\, \sigma = \sigma_\delta \cdot \sigma_f \wedge \theta, s, \sigma \models \hat{\sigma} \wedge \mathsf{SAT}_{\theta,s}(\pi)$

(H3) There is only one `consume` rules yielding `Ok`, giving us $\delta = \mathsf{ex}$, $\hat{\sigma} = \mathrm{ex}(\hat{x})$, $\hat{\sigma}_f = \bot$, $\vec{\hat{v}}_i = []$, $\vec{\hat{v}}_o = [\hat{x}]$, $\pi = []$

(H4) From (H3) and the definition of $\models$, if $\theta, s, \sigma_f \models \hat{\sigma}_f$ we have $\sigma_f = \bot$.

(H5) From (H3) and the definition of $\models_\Delta$, if $\theta, s, \sigma_\delta \models_\Delta \langle\delta\rangle(\vec{\hat{v}}_i; \vec{\hat{v}}_o)$, we have $\sigma_\delta = \mathrm{ex}(x)$, with $\vec{v}_i = []$ and $\vec{v}_o = [x]$ such that $[\![\vec{\hat{v}}_i]\!]_{\theta,s} = \vec{v}_i$ and $[\![\vec{\hat{v}}_o]\!]_{\theta,s} = \vec{v}_o$.

(H6) From (H4), (H5) and the rule for composition, we have $\sigma_f \# \sigma_\delta$.

(H7) We can pick $\sigma = \sigma_\delta = \mathrm{ex}(x)$.

(H8) From the definition of composition, (H4) and (H5), we have $\sigma = \sigma_\delta \cdot \sigma_f$.

(H9) Given (H3) and (H7), we have $\theta, s, \sigma \models \hat{\sigma}$.

(H10) From (H3), $\pi = []$ thus $\mathsf{SAT}_{\theta,s}(\pi)$. Together with (H8) and (H9), this gives our goal (G1).

Proposition: Consume: OX sound
Assume

(H1) $\forall o, \hat{\sigma}_f, \vec{\hat{v}}_o, \pi.\, \mathsf{consume}(\mathsf{OX}, \hat{\sigma}, \delta, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}_f, \vec{\hat{v}}_o, \pi) \Rightarrow o_c = \mathsf{Ok}$

To prove

(G1) $\exists \hat{\sigma}'_f, \vec{\hat{v}}'_o, \pi'.\, \mathsf{consume}(\mathsf{OX}, \hat{\sigma}, \delta, \vec{v}_i, \pi) \rightsquigarrow (\mathsf{Ok}, \hat{\sigma}'_f, \vec{\hat{v}}'_o, \pi')$

From the `consume` rules, we never vanish, so if all consumptions are `Ok` we must have $\hat{\sigma} = \mathrm{ex}(\hat{x})$, $\delta = \mathsf{ex}$, $\vec{\hat{v}}_i = []$, $\hat{\sigma}'_f = \bot$, $\vec{\hat{v}}'_o = [\hat{x}]$, $\pi' = []$. Our goal (G1) follows.

Proposition: Produce soundness
Assume

(H1) $\mathsf{produce}(\hat{\sigma}_f, \delta, \vec{\hat{v}}_i, \vec{\hat{v}}_o) \rightsquigarrow (\hat{\sigma}, \pi)$

(H2) $\mathsf{SAT}_{\theta,s}(\pi) \wedge \theta, s, \sigma \models \hat{\sigma}$

To prove

(G1) $\exists \sigma_\delta, \sigma_f.\, \sigma_\delta \# \sigma_f \wedge \sigma = \sigma_\delta \cdot \sigma_f \wedge \theta, s, \sigma_\delta \models_\Delta \langle\delta\rangle(\vec{\hat{v}}_i; \vec{\hat{v}}_o) \wedge \theta, s, \sigma_f \models \hat{\sigma}_f$

(H3) From the produce rule, we have $\hat{\sigma}_f = \bot$, $\delta = \mathsf{ex}$, $\vec{\hat{v}}_i = []$, $\vec{\hat{v}}_o = [\hat{x}]$, $\hat{\sigma} = \mathrm{ex}(\hat{x})$ and $\pi = []$.

(H4) From (H2) and the definition of $\models$ we have $\sigma = \mathrm{ex}(x)$ and $[\![\hat{x}]\!]_{\theta,s} = x$

(H5) Given (H4), we have $\sigma_\delta = \mathrm{ex}(x)$ and $\sigma_f = \bot$, such that $\sigma_\delta \# \sigma_f$ and $\sigma = \sigma_\delta \cdot \sigma_f$.

(H6) From (H4), we have $[\![\vec{\hat{v}}_i]\!]_{\theta,s} = []$ and $[\![\vec{\hat{v}}_o]\!]_{\theta,s} = [x]$, thus from (H5) and the definition of $\models_\Delta$ we have $\theta, s, \sigma_\delta \models_\Delta \langle\delta\rangle(\vec{\hat{v}}_i; \vec{\hat{v}}_o)$.

(H7) From (H3) and (H5) we have $\sigma_f = \bot$ and $\hat{\sigma}_f = \bot$, thus from the definition of $\models$, we have $\theta, s, \sigma_f \models \hat{\sigma}_f$. This with (H6) gives our goal (G1).

Proposition: Produce completeness
Assume

(H1) $\theta, s, \sigma_f \models \hat{\sigma}_f \wedge \theta, s, \sigma_\delta \models_\Delta \langle\delta\rangle(\vec{\hat{v}}_i; \vec{v}_o) \wedge \sigma_f \# \sigma_\delta$

To prove

(G1) $\exists \hat{\sigma}.\, \mathsf{produce}(\hat{\sigma}_f, \delta, \vec{\hat{v}}_i, \vec{\hat{v}}_o) \rightsquigarrow (\hat{\sigma}, \pi) \wedge \mathsf{SAT}_{\theta,s}(\pi) \wedge \theta, s, (\sigma_f \cdot \sigma_\delta) \models \hat{\sigma}$

(H2) From (H9), $\sigma_\delta \models_\Delta \langle\delta\rangle(\vec{\hat{v}}_i; \vec{\hat{v}}_o)$, thus from the definition of $\models_\Delta$ we have $\sigma_\delta = \mathrm{ex}(x)$, $\delta = \mathsf{ex}$, $\vec{\hat{v}}_i = []$, $\vec{\hat{v}}_o = [\hat{x}]$ with $[\![\hat{x}]\!]_{\theta,s} = x$.

(H3) From (H9), $\sigma_f \# \sigma_\delta$, thus from (H2) and the rules of composition, we have $\sigma_f = \bot$.

(H4) From the rules of $\models$ and (H3), we have $\hat{\sigma}_f = \bot$.

(H5) We can pick $\hat{\sigma} = \mathrm{ex}(\hat{x})$.

(H6) From (H4), (H2), (H5) and the rules of produce, we have $\mathsf{produce}(\sigma_f, \delta, \vec{\hat{v}}_i, \vec{\hat{v}}_o) \rightsquigarrow (\hat{\sigma}, \pi)$, with $\pi = []$.

(H7) From (H6), $\mathsf{SAT}_{\theta,s}(\pi)$.

(H8) From (H3) and (H2), $\sigma_f = \bot$ and $\sigma_\delta = \mathrm{ex}(x)$, thus from the rules of composition, $\sigma_f \cdot \sigma_\delta = \mathrm{ex}(x)$.

(H9) From (H2), $[\![\hat{x}]\!]_{\theta,s} = x$, thus from (H5) and (H8) we have $\theta, s, (\sigma_f \cdot \sigma_\delta) \models \hat{\sigma}$, which combined with (H6) and (H7) gives our goal (G1).

□

4.2 Partial Map

4.2.1 Resource Algebra

$$\textsc{PMap}(I, \mathbb{S}) \stackrel{\text{def}}{=} I \xrightarrow{fin} \mathbb{S}.\Sigma \times \mathcal{P}(I)^?$$

$$(h, d) \cdot (h', d') \stackrel{\text{def}}{=} (h'', d'')$$

$$\text{where } h'' \stackrel{\text{def}}{=} \lambda i. \begin{cases} h(i) \cdot h'(i) & \text{if } i \in \text{dom}(h) \cap \text{dom}(h') \\ h(i) & \text{if } i \in \text{dom}(h) \setminus \text{dom}(h') \\ h'(i) & \text{if } i \in \text{dom}(h') \setminus \text{dom}(h) \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$\text{and } d'' \stackrel{\text{def}}{=} \begin{cases} d & \text{if } d' = \bot \\ d' & \text{if } d = \bot \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$\text{and } d'' = \bot \vee \text{dom}(h'') \subseteq d''$$

$$|(h, d)| \stackrel{\text{def}}{=} \begin{cases} \bot & \text{if } \text{dom}(h') = \emptyset \\ (h', \bot) & \text{otherwise} \end{cases}$$

$$\text{where } h' \stackrel{\text{def}}{=} \lambda i. \begin{cases} |h(i)| & \text{if } i \in \text{dom}(h) \wedge |h(i)| \neq \bot \\ \text{undefined} & \text{otherwise} \end{cases}$$

For a wrapped state model $\mathbb{S} = \{\Sigma_\mathbb{S}, \mathcal{A}_\mathbb{S}, \Delta_\mathbb{S}\}$, we define the actions of $\textsc{PMap}(I, \mathbb{S})$ as $\mathcal{A} = \mathcal{A}_\mathbb{S} \uplus \{\texttt{alloc}\}$, and the predicates $\Delta = \Delta_\mathbb{S} \uplus \{\texttt{domainset}\}$.

We define predicate satisfiability as:

$$\textsc{PMapPredSat}\ \frac{\sigma_i \models_\Delta \langle\delta\rangle(\vec{v}_i; \vec{v}_o)}{([i \mapsto \sigma_i], \bot) \models_\Delta \langle\delta\rangle(i :: \vec{v}_i; \vec{v}_o)} \qquad \textsc{PMapPredSatBot}\ \frac{\bot \models_\Delta \langle\delta\rangle(\vec{v}_i; \vec{v}_o)}{\bot \models_\Delta \langle\delta\rangle(i :: \vec{v}_i; \vec{v}_o)} \qquad \textsc{PMapPredDomainSet}\ (\emptyset, d) \models_\Delta \langle\texttt{domainset}\rangle(; d)$$

We define symbolic interpretation as:

$$\textsc{PMapSymInterpretation}\ \frac{\forall \hat{i} \in \text{dom}(\hat{h}).\ [\![\hat{i}]\!]_{\theta,s} = i \wedge i \in \text{dom}(h) \wedge \theta, s, h(i) \models \hat{h}(\hat{i}) \qquad [\![\text{dom}(\hat{h})]\!]_{\theta,s} = \text{dom}(h) \qquad [\![\hat{d}]\!]_{\theta,s} = d}{\theta, s, (h, d) \models (\hat{h}, \hat{d})}$$

4.2.2 Compositional Concrete Rules

We define the helper functions `get` and `set` as:

$$\texttt{get} : \textsc{PMap}(I, \mathbb{S}) \to I \to \mathbb{S}.\Sigma$$

$$\texttt{set} : \textsc{PMap}(I, \mathbb{S}) \to I \to \mathbb{S}.\Sigma \to \textsc{PMap}(I, \mathbb{S})$$

We pretty-print `get` and `set` as $\texttt{get}(\sigma, i) = \sigma_i$ and $\texttt{set}(\sigma, i, \sigma_i) = \sigma'$.

$$\text{Given } \mathit{wrap}(h, d) \stackrel{\text{def}}{=} \begin{cases} \bot & \text{if } \mathrm{dom}(h) = \emptyset \wedge d = \bot \\ (h, d) & \text{otherwise} \end{cases}$$

$$\mathit{unwrap}(\sigma) \stackrel{\text{def}}{=} \begin{cases} ([], \bot) & \text{if } \sigma = \bot \\ (h, d) & \text{if } \sigma = (h, d) \end{cases}$$

$$\frac{(h, d) = \mathit{unwrap}(\sigma) \qquad i \in \mathrm{dom}(h) \qquad \sigma_i = h(i)}{\texttt{get}(\sigma, i) = \sigma_i} \text{ CPMapGetMatch}$$

$$\frac{(h, d) = \mathit{unwrap}(\sigma) \qquad i \notin \mathrm{dom}(h) \qquad d \neq \bot \qquad i \in d}{\texttt{get}(\sigma, i) = \bot} \text{ CPMapGetAdd}$$

$$\frac{(h, d) = \mathit{unwrap}(\sigma) \qquad i \notin \mathrm{dom}(h) \qquad d = \bot}{\texttt{get}(\sigma, i) = \bot} \text{ CPMapGetBotDomain}$$

$$\frac{(h, d) = \mathit{unwrap}(\sigma) \qquad \sigma_i \neq \bot \qquad h' = h[i \leftarrow \sigma_i] \qquad \sigma' = \mathit{wrap}(h', d)}{\texttt{set}(\sigma, i, \sigma_i) = \sigma'} \text{ CPMapSetSome}$$

$$\frac{(h, d) = \mathit{unwrap}(\sigma) \qquad \sigma_i = \bot \qquad h' = h[i \not\leftarrow] \qquad \sigma' = \mathit{wrap}(h', d)}{\texttt{set}(\sigma, i, \sigma_i) = \sigma'} \text{ CPMapSetNone}$$

The action rules are then:

$$\frac{\texttt{get}(\sigma, i) = \sigma_i \qquad \alpha(\sigma_i, \vec{v}_i) = (o, \sigma_i', \vec{v}_o) \qquad \texttt{set}(\sigma, i, \sigma_i') = \sigma'}{\alpha(\sigma, i :: \vec{v}_i) = (o, \sigma', i :: \vec{v}_o)} \text{ CPMapAction}$$

$$\frac{d \neq \bot \qquad i \notin d}{\alpha((h, d), i :: \vec{v}_i) = (\texttt{Err}, (h, d), [])} \text{ CPMapActionOutOfBounds}$$

$$\frac{d \neq \bot \qquad i = \texttt{fresh} \qquad \sigma_i = \texttt{instantiate}(\vec{v}_i) \qquad h' = h[i \leftarrow \sigma_i] \qquad d' = d \uplus \{i\}}{\texttt{alloc}((h, d), \vec{v}_i) = (\texttt{Ok}, (h', d'), [i])} \text{ CPMapAlloc}$$

$$\frac{(h, d) = \mathit{unwrap}(\sigma) \qquad d = \bot}{\texttt{alloc}(\sigma, \vec{v}_i) = (\texttt{Miss}, \sigma, [\texttt{`domainset'}])} \text{ CPMapAllocMiss}$$

4.2.3 Compositional Symbolic Rules

We now re-define `get` and `set`, lifting them to the symbolic realm (note `set` is unchanged, as it does not perform any sort of matching). `get` now returns a state and an index, that may be different from the input index – it corresponds to the actual index of the state in the map. For example, for a map $[1 \mapsto \hat{x}]$, calling `get` with index $\hat{y}$ may return index 1, along with the state $\hat{x}$ and the path condition $[\hat{y} = 1]$.

$$\texttt{get} : \textsc{PMap}(I, \mathbb{S}) \to I \to \mathcal{P}(I \times \mathbb{S}.\hat{\Sigma} \times \Pi)$$
$$\texttt{set} : \textsc{PMap}(I, \mathbb{S}) \to I \to \mathbb{S}.\hat{\Sigma} \to \textsc{PMap}(I, \mathbb{S})$$

We pretty-print `get` and `set` as $\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi)$ and $\texttt{set}(\hat{\sigma}, \hat{i}, \hat{\sigma}_i) = \hat{\sigma}'$.

$$\text{Given } \mathit{wrap}(\hat{h}, \hat{d}) \stackrel{\text{def}}{=} \begin{cases} \bot & \text{if } \mathrm{dom}(\hat{h}) = \emptyset \wedge \hat{d} = \bot \\ (\hat{h}, \hat{d}) & \text{otherwise} \end{cases}$$
$$\mathit{unwrap}(\hat{\sigma}) \stackrel{\text{def}}{=} \begin{cases} ([], \bot) & \text{if } \hat{\sigma} = \bot \\ (\hat{h}, \hat{d}) & \text{if } \hat{\sigma} = (\hat{h}, \hat{d}) \end{cases}$$

PMapGetMatch
$$\frac{(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad \hat{i}' \in \mathrm{dom}(\hat{h}) \qquad \hat{\sigma}_i = \hat{h}(\hat{i}')}{\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, [\hat{i} = \hat{i}'])}$$

PMapGetAdd
$$\frac{(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad \hat{i} \notin \mathrm{dom}(\hat{h}) \qquad \hat{d} \neq \bot}{\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \bot, [\hat{i} \notin \mathrm{dom}(\hat{h}) \wedge \hat{i} \in \hat{d}])}$$

PMapGetBotDomain
$$\frac{(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad \hat{i} \notin \mathrm{dom}(\hat{h}) \qquad \hat{d} = \bot}{\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \bot, [\hat{i} \notin \mathrm{dom}(\hat{h})])}$$

PMapSetSome
$$\frac{(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad \hat{\sigma}_i \neq \bot \qquad \hat{h}' = \hat{h}[\hat{i} \leftarrow \hat{\sigma}_i] \qquad \hat{\sigma}' = \mathit{wrap}(\hat{h}', \hat{d})}{\texttt{set}(\hat{\sigma}, \hat{i}, \hat{\sigma}_i) = \hat{\sigma}'}$$

PMapSetNone
$$\frac{(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad \hat{\sigma}_i = \bot \qquad \hat{h}' = \hat{h}[\hat{i} \not\leftarrow] \qquad \hat{\sigma}' = \mathit{wrap}(\hat{h}', \hat{d})}{\texttt{set}(\hat{\sigma}, \hat{i}, \hat{\sigma}_i) = \hat{\sigma}'}$$

$$\text{Given } \mathit{lift\_if\_miss}(o, \hat{i}, \vec{\hat{v}}_i) \stackrel{\text{def}}{=} \begin{cases} \hat{i} :: \vec{\hat{v}}_i & \text{if } o = \texttt{Miss} \\ \vec{\hat{v}}_i & \text{otherwise} \end{cases}$$

PMapAction
$$\frac{\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \qquad \alpha(\hat{\sigma}_i, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}_i', \vec{\hat{v}}_o, \pi') \qquad \texttt{set}(\hat{\sigma}, \hat{i}', \hat{\sigma}_i') = \hat{\sigma}'}{\alpha(\hat{\sigma}, \hat{i} :: \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}', \hat{i}' :: \vec{\hat{v}}_o, \pi :: \pi')}$$

PMapActionOutOfBounds
$$\frac{\hat{d} \neq \bot}{\alpha((\hat{h}, \hat{d}), \hat{i} :: \vec{\hat{v}}_i) \rightsquigarrow (\texttt{Err}, (\hat{h}, \hat{d}), [], [\hat{i} \notin \hat{d}])}$$

PMapAlloc
$$\frac{\hat{d} \neq \bot \qquad \hat{i} = \texttt{fresh} \qquad \hat{\sigma}_i = \texttt{instantiate}(\vec{\hat{v}}_i) \qquad \hat{h}' = \hat{h}[\hat{i} \leftarrow \hat{\sigma}_i] \qquad \hat{d}' = \hat{d} \uplus \{\hat{i}\}}{\texttt{alloc}((\hat{h}, \hat{d}), \vec{\hat{v}}_i) \rightsquigarrow (\texttt{Ok}, (\hat{h}', \hat{d}'), [\hat{i}], [\hat{i} = \hat{i}])}$$

PMapAllocMiss
$$\frac{(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad \hat{d} = \bot}{\texttt{alloc}(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (\texttt{Miss}, \hat{\sigma}, [\texttt{`domainset'}], [])}$$

PMapCons

$$\frac{\mathsf{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \quad \mathsf{consume}(\hat{\sigma}_i, \delta, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}_i', \vec{\hat{v}}_o, \pi') \quad \mathsf{set}(\hat{\sigma}, \hat{i}', \hat{\sigma}_i') = \hat{\sigma}' \quad \vec{\hat{v}}_o' = \mathit{lift\_if\_miss}(o, \hat{i}', \vec{\hat{v}}_o)}{\mathsf{consume}(\hat{\sigma}, \delta, \hat{i} :: \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{\hat{v}}_o', \pi :: \pi')}$$

PMapConsIncompat

$$\frac{\hat{d} \neq \bot}{\mathsf{consume}((\hat{h}, \hat{d}), \delta, \hat{i} :: \vec{\hat{v}}_i) \rightsquigarrow (\mathsf{LFail}, (\hat{h}, \hat{d}), [], [\hat{i} \notin \hat{d}])}$$

PMapConsDomainSet

$$\frac{\hat{d} \neq \bot}{\mathsf{consume}((\hat{h}, \hat{d}), \mathsf{domainset}, []) \rightsquigarrow (\mathsf{Ok}, (\hat{h}, \bot), [\hat{d}], [])}$$

PMapConsDomainSetMiss

$$\frac{(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \quad \hat{d} = \bot}{\mathsf{consume}(\hat{\sigma}, \mathsf{domainset}, []) \rightsquigarrow (\mathsf{Miss}, \hat{\sigma}, [\texttt{`domainset'}], [])}$$

PMapProd

$$\frac{\mathsf{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \quad \mathsf{produce}(\hat{\sigma}_i, \delta, \vec{\hat{v}}_i, \vec{\hat{v}}_o) \rightsquigarrow (\hat{\sigma}_i', \pi') \quad \mathsf{set}(\hat{\sigma}, \hat{i}', \hat{\sigma}_i') = \hat{\sigma}'}{\mathsf{produce}(\hat{\sigma}, \delta, \hat{i} :: \vec{\hat{v}}_i, \vec{\hat{v}}_o) \rightsquigarrow (\hat{\sigma}', \pi :: \pi')}$$

PMapProdDomainSet

$$\frac{(\hat{h}, \bot) = \mathit{unwrap}(\hat{\sigma})}{\mathsf{produce}(\hat{\sigma}, \mathsf{domainset}, [], [\hat{d}]) \rightsquigarrow ((\hat{h}, \hat{d}), [\mathrm{dom}(\hat{h}) \subseteq \hat{d}])}$$

PMapFix

$$\frac{\mathbb{S}.\mathsf{fix}\ \vec{\hat{v}}_i = a \quad a' = \mathit{lift}(a, \hat{i})}{\mathsf{fix}\ \hat{i} :: \vec{\hat{v}}_i = a'}$$

PMapFixDomainSet

$$\mathsf{fix}\ [\texttt{`domainset'}] = \exists \hat{d}.\, \langle \mathsf{domainset} \rangle (; \hat{d})$$

Note in the above we define $\mathit{lift}(a, \hat{i})$ as a function that traverses an assertion a and lifts all core predicate assertions by adding the value $\hat{i}$ at the start of its in-values, such that $\mathit{lift}(\langle \delta \rangle (\vec{\hat{v}}_i; \vec{\hat{v}}_o), \hat{i}) = \langle \delta \rangle (\hat{i} :: \vec{\hat{v}}_i; \vec{\hat{v}}_o)$.

4.2.4 Soundness Proofs

To facilitate the soundness proof for PMap, and due to its extensive use of the `get` and `set` helper methods to modify the elements in the heap, we first define and prove axioms about these auxiliary functions.

We introduce a pair of axioms for `get`, similar to OX and UX soundness of actions, to ensure that the symbolic function retrieves the right state if it exists in the concrete counterpart of the state.

$$\begin{aligned} &\theta, s, \sigma \models \hat{\sigma} \wedge \mathsf{get}(\sigma, i) = \sigma_i \wedge [\![\hat{i}]\!]_{\theta,s} = i \implies \exists \hat{\sigma}_i, \hat{i}', \pi. \\ &\quad \mathsf{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \wedge \theta, s, \sigma_i \models \hat{\sigma}_i \wedge [\![\hat{i}']\!]_{\theta,s} = i \wedge \mathsf{SAT}_{\theta,s}(\pi) \end{aligned} \qquad \text{(Get OX Soundness)}$$

$$\begin{aligned} &\mathsf{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \wedge [\![\hat{i}]\!]_{\theta,s} = [\![\hat{i}']\!]_{\theta,s} = i \wedge \mathsf{SAT}_{\theta,s}(\pi) \implies \\ &\quad \forall \sigma.\, \theta, s, \sigma \models \hat{\sigma} \implies \exists \sigma_i.\, \mathsf{get}(\sigma, i) = \sigma_i \wedge \theta, s, \sigma_i \models \hat{\sigma}_i \end{aligned} \qquad \text{(Get UX Soundness)}$$

Proof.

Proposition: Get OX Soundness
Assume

(H1) $\theta, s, \sigma \models \hat{\sigma} \wedge \mathtt{get}(\sigma, i) = \sigma_i \wedge [\![\hat{i}]\!]_{\theta,s} = i$

To prove

(G1) $\exists \hat{\sigma}_i, \hat{i}', \pi.\, \mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \wedge \theta, s, \sigma_i \models \hat{\sigma}_i \wedge [\![\hat{i}']\!]_{\theta,s} = i \wedge \mathtt{SAT}_{\theta,s}(\pi)$

We proceed by proving the property holds for all rules of the concrete `get`, resulting in three cases.

Case CPMapGetMatch**:**

(H2) Assume $(h, d) = \mathit{unwrap}(\sigma) \wedge i \in \mathrm{dom}(h) \wedge \sigma_i = h(i)$.

(H3) It follows from (H1) and the definition of $\models$ that $\hat{\sigma} = (\hat{h}, \hat{d})$ such that $\exists \hat{i}'.\, \hat{i}' \in \mathrm{dom}(\hat{h})$, $[\![\hat{i}']\!]_{\theta,s} = i$ and $\theta, s, \sigma_i \models \hat{h}(\hat{i}')$.

(H4) We can then apply PMapGetMatch, giving us $\mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{h}(\hat{i}'), [\hat{i}' = \hat{i}])$.

(H5) From (H3) and (H1) it follows that $\mathtt{SAT}_{\theta,s}([\hat{i}' = \hat{i}])$, which completes our goal (G1).

Case CPMapGetAdd**:**

(H6) Assume $(h, d) = \mathit{unwrap}(\sigma) \wedge i \notin \mathrm{dom}(h) \wedge d \neq \bot \wedge i \in d$.

(H7) It follows from (H1) and the definition of $\models$ that $\hat{\sigma} = (\hat{h}, \hat{d})$ such that $\hat{i} \in \hat{d}$ and $\hat{i} \notin \mathrm{dom}(\hat{h})$.

(H8) We can then apply PMapGetAdd, giving us $\mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \bot, [\hat{i} \notin \mathrm{dom}(\hat{h}) \wedge \hat{i} \notin \hat{d}])$.

(H9) From (H8) and (H7) it follows that $\mathtt{SAT}_{\theta,s}([\hat{i} \notin \mathrm{dom}(\hat{h}) \wedge \hat{i} \notin \hat{d}])$, which completes our goal (G1).

Case CPMapGetBotDomain**:**

(H10) Assume $(h, d) = \mathit{unwrap}(\sigma) \wedge i \notin \mathrm{dom}(h) \wedge d = \bot$.

(H11) From (H10), (H1) and CPMapGetBotDomain, we have $\sigma_i = \bot$.

(H12) It follows from (H1) and the definition of $\models$ that $\hat{\sigma} = (\hat{h}, \bot)$ such that $\hat{i} \notin \mathrm{dom}(\hat{h})$.

(H13) We can then apply PMapGetBotDomain, giving us $\mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \bot, [\hat{i} \notin \mathrm{dom}(\hat{h})])$.

(H14) From (H13) and (H12) it follows that $\mathtt{SAT}_{\theta,s}([\hat{i} \notin \mathrm{dom}(\hat{h})])$, which completes our goal (G1).

Proposition: Get UX Soundness
Assume

(H1) $\mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \wedge [\![\hat{i}]\!]_{\theta,s} = [\![\hat{i}']\!]_{\theta,s} = i \wedge \mathtt{SAT}_{\theta,s}(\pi)$

(H2) $\theta, s, \sigma \models \hat{\sigma}$

To prove

(G1) $\exists \sigma_i.\, \mathtt{get}(\sigma, i) = \sigma_i \wedge \theta, s, \sigma_i \models \hat{\sigma}_i$

We proceed by proving the property holds for all rules of the symbolic `get`, resulting in three cases.

Case PMapGetMatch**:**

(H3) Assume $(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \wedge \hat{i}' \in \mathrm{dom}(\hat{h}) \wedge \hat{\sigma}_i = \hat{h}(\hat{i}')$

(H4) From (H3), (H1) and PMapGetMatch, we have $\pi = [\hat{i} = \hat{i}']$

(H5) From (H3), we have $\hat{\sigma} \neq \bot$, thus from (H2) we have $\sigma = (h, d)$

(H6) From (H2), (H3) and the definition of $\models$, we have $i \in h$ such that $\sigma_i = h(i)$ and $\theta, s, \sigma_i \models \hat{\sigma}_i$

(H7) From (H6) we can apply CPMapGetMatch, thus $\mathtt{get}(\sigma, i) = \sigma_i$. This completes our goal (G1).

Case PMapGetAdd**:**

(H8) Assume $(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \wedge \hat{i} \notin \mathrm{dom}(\hat{h}) \wedge \hat{d} \neq \bot$

(H9) From (H8), (H1) and PMapGetAdd, we have $\hat{i}' = \hat{i}$, $\hat{\sigma}_i = \bot$ and $\pi = [\hat{i} \notin \mathrm{dom}(\hat{h}) \wedge \hat{i} \in \hat{d}]$

(H10) From (H8) we have $\hat{d} \neq \bot$, thus $\hat{\sigma} = (\hat{h}, \hat{d})$ and from (H2) $\sigma = (h, d)$ such that $[\![\hat{d}]\!]_{\theta,s} = d$.

(H11) From (H1) $\mathtt{SAT}_{\theta,s}(\pi)$, thus from (H9) and (H2), $i \notin \mathrm{dom}(h) \wedge i \in d$

(H12) From (H11) and (H10) we can apply CPMapGetAdd, thus $\mathtt{get}(\sigma, i) = \bot$.

(H13) From Empty Memory, $\theta, s, \bot \models \bot$, which along with (H12) completes our goal (G1).

Case PMapGetBotDomain**:**

(H14) Assume $(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \wedge \hat{i} \notin \mathrm{dom}(\hat{h}) \wedge \hat{d} = \bot$

(H15) From (H14), (H1) and PMapGetBotDomain, we have $\hat{i}' = \hat{i}$, $\hat{\sigma}_i = \bot$ and $\pi = [\hat{i} \notin \mathrm{dom}(\hat{h})]$

(H16) From (H14) we have $\hat{d} = \bot$, so given $(h, d) = \mathit{unwrap}(\sigma)$ and (H2), we have $d = \bot$.

(H17) From (H1) $\mathtt{SAT}_{\theta,s}(\pi)$, thus from (H15) and (H2), $i \notin \mathrm{dom}(h)$.

(H18) From (H17) and (H16) we can apply CPMapGetBotDomain, thus $\mathtt{get}(\sigma, i) = \bot$.

(H19) From Empty Memory, $\theta, s, \bot \models \bot$, which along with (H18) completes our goal (G1).

□

We can now proceed with the standard proofs.

Proof.

Proposition: OX Soudness
Assume

(H1) $\theta, s, \sigma \models \hat{\sigma} \wedge \alpha(\sigma, \vec{v}_i) = (o, \sigma', \vec{v}_o) \wedge [\![\vec{\hat{v}}_i]\!]_{\theta,s} = \vec{v}_i$

(H2) $\forall o', \hat{\sigma}', \vec{\hat{v}}'_o, \pi'. \alpha(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (o', \hat{\sigma}', \vec{\hat{v}}'_o, \pi') \Rightarrow o' \in \{\texttt{Ok}, \texttt{Err}\}$

(H3) The actions on $\mathbb{S}$ are OX sound.

To prove

(G1) $\exists \hat{\sigma}', \vec{\hat{v}}_o, \pi, \theta'. \hat{\alpha}(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{\hat{v}}_o, \pi) \wedge \theta', s, \sigma' \models \hat{\sigma}' \wedge \texttt{SAT}_{\theta',s}(\pi) \wedge [\![\vec{\hat{v}}_o]\!]_{\theta',s} = \vec{v}_o$

There are two action cases to consider: $\alpha \in \mathcal{A}_{\mathbb{S}}$ and $\alpha = \texttt{alloc}$.

Case $\alpha \in \mathcal{A}_{\mathbb{S}}$**:**

(H4) If the action gets executed succesfully (it is not an out of bounds error), from CPMapAction we have $\texttt{get}(\sigma, i) = \sigma_i$, $\alpha(\sigma_i, \vec{v}'_i) = (o, \sigma'_i, \vec{v}'_o)$ and $\texttt{set}(\sigma, i, \sigma'_i) = \sigma'$, with $\vec{v}_i = i :: \vec{v}'_i$ and $\vec{v}_o = i :: \vec{v}'_o$

(H5) From (H1) and (H4), we know we have $\vec{\hat{v}}_i = \hat{i} :: \vec{\hat{v}}'_i$ such that $[\![\hat{i}]\!]_{\theta,s} = i$ and $[\![\vec{\hat{v}}'_i]\!]_{\theta,s} = \vec{\hat{v}}_i$.

(H6) From (H1), (H4) and (H5), we can apply Get OX Soundness, giving $\exists \hat{\sigma}_i, \hat{i}', \pi. \texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \wedge \theta, s, \sigma_i \models \hat{\sigma}_i \wedge [\![\hat{i}']\!]_{\theta,s} = i \wedge \texttt{SAT}_{\theta,s}(\pi)$

(H7) From (H4), (H6), (H2), (H5) and (H3), we can apply Memory Model OX Soundness, giving us $\exists \vec{\hat{v}}'_o, \hat{\sigma}'_i, \pi', \theta'. \alpha(\hat{\sigma}_i, \vec{\hat{v}}'_i) \rightsquigarrow (o, \hat{\sigma}'_i, \vec{\hat{v}}'_o, \pi') \wedge \theta', s, \sigma'_i \models \hat{\sigma}'_i \wedge \texttt{SAT}_{\theta',s}(\pi') \wedge [\![\vec{\hat{v}}'_o]\!]_{\theta',s} = \vec{v}'_o$.

(H8) From (H1) and (H7), we have $\theta, s, \sigma \models \hat{\sigma}$ and $\theta', s, \sigma'_i \models \hat{\sigma}'_i$. From the definition of set, it follows that only modifying the state at $\hat{i}'$ preserves symbolic interpretation, thus given $\texttt{set}(\hat{\sigma}, \hat{i}', \hat{\sigma}'_i) = \hat{\sigma}'$, we have $\theta', s, \sigma' \models \hat{\sigma}'$.

(H9) By applying PMapAction and from (H7), (H6), (H8) our goal (G1) follows.

(H10) If this is an out of bounds access, we have $\sigma = \sigma' = (h, d)$ with $d \neq \bot$ and $\vec{v}_i = i :: \vec{v}'_i$, with $i \notin d$, as well as $\vec{\hat{v}}_i = \hat{i} :: \vec{\hat{v}}'_i$

(H11) From (H1) and the definition of $\models$, we have $\hat{\sigma} = (\hat{h}, \hat{d})$ such that $\pi = [\hat{i} \notin \hat{d}]$ and $\texttt{SAT}_{\theta,s}(\pi)$.

(H12) From the rules of action execution and (H11), we get $\alpha(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (\texttt{Err}, \hat{\sigma}, [], \pi)$. The rest of our goal (G1) follows.

Case $\alpha = \texttt{alloc}$**:**

Both cases (when $d \neq \bot$ and $d = \bot$) are trivially lifted from the concrete to the symbolic realm, with $\pi = [i = i]$ and $\pi = []$ respectively. It follows that $\texttt{SAT}_{\theta,s}(\pi)$ in both cases, giving our goal (G1).

For the successful allocation case, we note there is an additional requirement for soundness of state instantiation to be able to prove the resulting states $\hat{\sigma}'$ and σ' are compatible.

$$[\![\vec{\hat{v}}_i]\!]_{\theta,s} = \vec{v}_i \implies (\forall \sigma, \hat{\sigma}. \sigma = \texttt{instantiate}\ \vec{v}_i \wedge \hat{\sigma} = \texttt{instantiate}\ \vec{\hat{v}}_i \implies \theta, s, \sigma \models \hat{\sigma})$$

Proposition: UX Soundness

Assume

(H1) $\hat{\alpha}(\hat{\sigma}, \vec{\hat{v}_i}) \rightsquigarrow (o, \hat{\sigma}', \vec{\hat{v}_o}, \pi) \wedge \mathsf{SAT}_{\theta,s}(\pi) \wedge \theta, s, \sigma' \models \hat{\sigma}' \wedge$
$[\![\vec{\hat{v}_o}]\!]_{\hat{s},\pi} \rightsquigarrow (\vec{v}_o, \pi') \wedge [\![\vec{\hat{v}_i}]\!]_{\hat{s},\pi'} \rightsquigarrow (\vec{v}_i, \pi'')$

(H2) The actions on $\mathbb{S}$ are UX sound.

To prove

(G1) $\exists \sigma.\, \theta, s, \sigma \models \hat{\sigma} \wedge \alpha(\sigma, \vec{v}_i) = (o, \sigma', \vec{v}_o)$

Case $\alpha \in \mathcal{A}_{\mathbb{S}}$**:**

(H3) If the action gets executed succesfully (it is not an out of bounds error), from PMapAction we have $\mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi')$, $\alpha(\hat{\sigma}_i, \vec{\hat{v}}_i') \rightsquigarrow (o, \hat{\sigma}_i', \vec{\hat{v}}_o', \pi'')$ and $\mathtt{set}(\hat{\sigma}, \hat{i}', \hat{\sigma}_i') = \hat{\sigma}'$, with $\vec{\hat{v}}_i = \hat{i} :: \vec{\hat{v}}_i'$, $\vec{\hat{v}}_o = \hat{i}' :: \vec{\hat{v}}_o'$ and $\pi = \pi' :: \pi''$

(H4) From (H2), (H3) and (H1), we get $\exists \sigma_i.\, \theta, s, \sigma_i \models \hat{\sigma}_i \wedge \alpha(\sigma_i, \vec{v}_i') = (p, \sigma', \vec{v}_o')$

(H5) From the definition of $\models$, it follows that if $\theta, s, \sigma' \models \hat{\sigma}'$ where $\mathtt{set}(\hat{\sigma}, \hat{i}', \hat{\sigma}_i') = \hat{\sigma}'$, then if $\theta, s, \sigma_i \models \hat{\sigma}_i$ there exists an σ and i such that $\mathtt{set}(\sigma, i, \sigma_i') = \sigma'$, giving us $\theta, s, \sigma \models \hat{\sigma}$

(H6) From (H3), (H1) and (H5) we can use Get UX Soundness, giving us $\exists \sigma_i.\, \mathtt{get}(\sigma, i) = \sigma_i \wedge$
$\theta, s, \sigma_i \models \hat{\sigma}_i$

(H7) From (H6), (H4) and (H5) we can apply CPMapAction, giving us $\alpha(\sigma, \vec{v}_i) = (o, \sigma', \vec{v}_o)$, which together with (H5) gives our goal (G1).

(H8) If the action fails as it is out of bounds, from PMapActionOutOfBounds we have $\vec{\hat{v}}_i = \hat{i} :: \vec{\hat{v}}_i'$ and $\pi = [\hat{i} \notin \hat{d}]$.

(H9) From (H1) and the definition of $\models$ we thus have $\sigma = (h, d)$, with $\vec{v}_i = i :: \vec{v}_i'$, $d \neq \bot$ and $i \notin d$

(H10) From (H9), we can apply CPMapActionOutOfBounds, which along with the fact the state is unmodified gives our goal (G1).

Case $\alpha = \mathtt{alloc}$**:**
Again, both cases (when the domains set d is owned and when it is $\bot$) are directly lifted from the concrete cases; it follows that if $\theta, s, \sigma' \models \hat{\sigma}'$ then $\theta, s, \sigma \models \hat{\sigma}$ where σ is the state without the added binding.

Proposition: Frame subtraction is satisfied
Assume

(H1) $\sigma \# \sigma_f \wedge \alpha(\sigma \cdot \sigma_f, \vec{v}_i) = (o, \sigma', \vec{v}_o)$

(H2) The actions on $\mathbb{S}$ satisfy frame subtraction.

(H3) $\forall \alpha, \vec{v}_i, \vec{v}_o.\, \alpha(\bot, \vec{v}_i) = (\mathtt{Miss}, \sigma', \vec{v}_o)$

To prove

(G1) $\exists \sigma'', o', \vec{v}_o'.\, \alpha(\sigma, \vec{v}_i) = (o', \sigma'', \vec{v}_o') \wedge$
$(o' \neq \mathtt{Miss} \implies o' = o \wedge \vec{v}_o' = \vec{v}_o \wedge \sigma' = \sigma'' \cdot \sigma_f)$

Case $\alpha \in \mathcal{A}_{\mathbb{S}}$**:**

(H4) If the action causes an out of bounds error, from CPMapActionOutOfBounds we have $\sigma \cdot \sigma_f = (h, d)$ and $\vec{v}_i = i :: \vec{v}_i'$ such that $i \notin d$.

(H5) From composition, the domain set d is either part of σ or σ_f. If it is part of σ such that $\sigma = (h_a, d)$ and $\sigma_f = (h_b, \bot)$, then we still have $i \notin d$, resulting in the same error and outcome, giving our goal (G1).

(H6) If the domain is in σ_f such that $\sigma = (h_a, \bot)$ and $\sigma_f = (h_b, d)$, then the action is executed on the underlying state model. Because we have $\text{dom}(h) \subseteq d$, from (H4) we have $i \notin \text{dom}(h_a)$, so from the rules of `get` and CPMapAction the action is executed on $\bot$. From (H3), we have $o' = \texttt{Miss}$, satisfying (G1).

(H7) If the action is not out of bounds, we have $\alpha(\sigma_i, \vec{v}_i') = (o, \sigma_i', \vec{v}_o')$ with $\vec{v}_i = i :: \vec{v}_i'$ and $\vec{v}_o = i :: \vec{v}_o'$.

(H8) If σ_f only includes the domain set and indices different than i, then the action is executed on the same σ_i and gives the same outcomes – it can then be composed with the result, giving our goal (G1).

(H9) Let σ_f such that $\sigma_f = (h_b, d_b)$ and $i \in \text{dom}(h_b)$. We denote $\sigma_{b,i} = h_b(i)$. For $\sigma = (h_a, d_a)$, we also have $\sigma_{a,i} = \bot$ if $i \notin \text{dom}(h_a)$, and $\sigma_{a,i} = h_a(i)$ otherwise, such that $\sigma_{a,i} \# \sigma_{b,i}$.

(H10) With (H7) and (H9), from (H2), we can apply Frame subtraction to the action on the state at index i, which allows re-applying CPMapAction and completes our goal (G1).

Case $\alpha = \texttt{alloc}$**:**

(H11) `alloc` is always defined, so we know $\exists \sigma'', o', \vec{v}_o. \alpha(\sigma, \vec{v}_i) = (o', \sigma'', \vec{v}_o')$.

(H12) Assume $o' \neq \texttt{Miss}$.

(H13) From (H12) and the rules of `alloc`, $\sigma = (h, d)$ with $d \neq \bot$.

(H14) From (H13) and (H1), if $\sigma \# \sigma_f$, then $\sigma_f = (h_f, \bot)$.

(H15) From the definition of `alloc`, the heap is not modified to the exception of the added entry, so from the definition of composition we have $\sigma' = \sigma'' \cdot \sigma_f$. The returns value $\vec{v}_o$ are the same (a fresh value), and the outcome is `Ok` in both cases, giving our goal (G1).

Proposition: Frame addition is satisfied
Assume

(H1) $\alpha(\sigma, \vec{v}_i) = (o, \sigma', \vec{v}_o)$

(H2) $\sigma_f \# \sigma'$

(H3) $o \neq \texttt{Miss}$

(H4) The actions on $\mathbb{S}$ satisfy frame addition.

(H5) $\forall \alpha, \vec{v}_i, \vec{v}_o.\, \alpha(\bot, \vec{v}_i) = (\texttt{Miss}, \sigma', \vec{v}_o)$

To prove

(G1) $\sigma \# \sigma_f \land \alpha(\sigma \cdot \sigma_f, \vec{v}_i) = (o, \sigma' \cdot \sigma_f, \vec{v}_o)$

Case $\alpha \in \mathcal{A}_{\mathbb{S}}$**:**

(H6) From the rules of concrete actions we have $\vec{v}_i = i :: \vec{v}_i'$ and $\vec{v}_o = i :: \vec{v}_o'$.

(H7) If the execution of the action is an out of bounds error, by CPMapActionOutOfBounds we have $\sigma = (h, d)$ such that $d \neq \bot$ and $i \notin d$.

(H8) From composition rules, we know $\sigma_f = (h_f, \bot)$, meaning that we still get $\alpha(\sigma \cdot \sigma_f, \vec{v}_i) = (\texttt{Err}, \sigma' \cdot \sigma_f, [])$ as we still have $i \notin d$. This satisfies our goal (G1).

(H9) We now look at the case where we do not have an out of bounds error. Assume we have $\sigma = (h, d)$ such that $i \notin \mathrm{dom}(h)$. From the rules for `get`, we know we executed the action on $\bot$. From (H5), we thus have $o = \texttt{Miss}$, which is a contradiction with (H3). Thus we know $i \in \mathrm{dom}(h)$.

(H10) From (H9) and the rules for `get`, we know we have $\sigma_i = h(i)$ and $\alpha(\sigma_i, \vec{v}_i') = (o, \sigma_i', \vec{v}_o')$.

(H11) If we have $\sigma_f = (h_f, d_f)$ such that $i \notin \mathrm{dom}(h_f)$, then it follows from (H2) that $\sigma \# \sigma_f$ as the action does not modify any other cell. The action will also have the same outcome, completing our goal (G1).

(H12) If we have $\sigma_f = (h_f, d_f)$ such that $i \in \mathrm{dom}(h_f)$, it follows from (H2) that $\sigma_i' \# h_f(i)$. We also know the outcome is not `Miss` from the rules of action execution and (H3).

(H13) From (H12), (H10) and (H4) we can apply Frame Addition and get $\sigma_i \# h_f(i)$ and $\alpha(\sigma_i \cdot h_f(i), \vec{v}_i') = (o, \sigma_i' \cdot h_f(i), \vec{v}_o')$.

(H14) From (H13) and CPMapAction, we get $\sigma \# \sigma_f$ and $\alpha(\sigma \cdot \sigma_f, \vec{v}_i) = (o, \sigma' \cdot \sigma_f, \vec{v}_o)$, giving our goal (G1).

Case $\alpha = \texttt{alloc}$**:**

(H15) From (H1), (H3) and the rules for `alloc`, we have $\sigma' = (h', d')$ such that $d' \neq \bot$.

(H16) From (H15), (H2) and the rules of composition it follows that $\sigma_f = (h_f, \bot)$.

(H17) Because `alloc` does not modify any state aside from adding a cell, and because σ_f has no domain set, it follows from (H2) that $\sigma \# \sigma_f$ and that the action has the saame outcome, thus $\alpha(\sigma \cdot \sigma_f, \vec{v}_i) = (o, \sigma' \cdot \sigma_f, \vec{v}_o)$, giving our goal (G1).

Proposition: Consume soundness
Assume

(H1) $\texttt{consume}(\hat{\sigma}, \delta, \vec{\hat{v}}_i) \rightsquigarrow (\texttt{Ok}, \hat{\sigma}_f, \vec{\hat{v}}_o, \pi)$

(H2) $\theta, s, \sigma \models \hat{\sigma} \land \texttt{SAT}_{\theta,s}(\pi)$

(H3) `consume` on $\mathbb{S}$ is sound.

To prove

(G1) $\exists \sigma_\delta, \sigma_f.\, \sigma_\delta \# \sigma_f \wedge \sigma = \sigma_\delta \cdot \sigma_f \wedge \theta, s, \sigma_\delta \models_\Delta \langle \delta \rangle(\vec{\hat{v}}_i; \vec{\hat{v}}_o) \wedge \theta, s, \sigma_f \models \hat{\sigma}_f$

Case $\delta \in \Delta_{\mathbb{S}}$**:**

(H4) From (H1) and PMapCons, given $\vec{\hat{v}}_i = \hat{i} :: \vec{\hat{v}}'_i$ and $\pi = \pi' :: \pi''$ we have $\mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi')$, $\mathtt{consume}(\hat{\sigma}_i, \delta, \vec{\hat{v}}'_i) \rightsquigarrow (o, \hat{\sigma}'_i, \vec{\hat{v}}_o, \pi'')$ and $\mathtt{set}(\hat{\sigma}, \hat{i}', \hat{\sigma}'_i) = \hat{\sigma}_f$.

(H5) From (H4) and (H2), we have $\vec{\hat{v}}_i = \hat{i} :: \vec{\hat{v}}'_i \wedge [\![\hat{i}]\!]_{\theta,s} = i$

(H6) From (H2) and Get UX Soundness, we know $\exists \sigma_i.\, \mathtt{get}(\sigma, i) = \sigma_i \wedge \theta, s, \sigma_i \models \hat{\sigma}_i$.

(H7) From (H2) and (H4), we have $\mathtt{SAT}_{\theta,s}(\pi'')$.

(H8) From (H6), (H7) and (H3) we can apply Consume UX Soundness, giving us $\exists \sigma_{\delta,i}, \sigma_{f,i}.\, \sigma_{\delta,i} \# \sigma'_i \wedge \sigma_i = \sigma_{\delta,i} \cdot \sigma'_i \wedge \theta, s, \sigma_{\delta,i} \models_\Delta \langle \delta \rangle(\vec{\hat{v}}'_i; \vec{\hat{v}}_o) \wedge \theta, s, \sigma_{f,i} \models \hat{\sigma}_{f,i}$.

(H9) We can then pick $\sigma_\delta = ([i \mapsto \sigma_{\delta,i}], \bot)$, and from (H8) and the definition of composition we know $\sigma_\delta \# \sigma$ and (H4) and the definition of $\mathtt{set}$, $\sigma_f = \sigma \cdot \sigma_\delta$.

(H10) From the definition of $\models_\Delta$ and (H8) we then have $\theta, s, \sigma_\delta \models \langle \delta \rangle(\vec{v}_i; \vec{v}_o)$, which with (H9) gives our goal (G1).

Case $\delta = \mathtt{domainset}$**:**

(H11) From (H1) and the rules for $\mathtt{consume}$, we have $\hat{\sigma} = (\hat{h}, \hat{d})$ and $\hat{\sigma}_f = (\hat{h}, \bot)$ such that $\hat{d} \neq \bot$, as well as $\vec{\hat{v}}_i = []$ and $\vec{\hat{v}}_o = [\hat{d}]$

(H12) From the definition of $\models$ and (H11) we have $\sigma = (h, d)$ and $\sigma_f = (h, \bot)$ such that $d \neq \bot$ and $[\![\hat{d}]\!]_{\theta,s} = d$.

(H13) We can pick $\sigma_\delta = ([], d)$ – it follows that $\sigma_f \# \sigma_\delta$ and from the composition rules, $\sigma = \sigma_f \cdot \sigma_\delta$.

(H14) From the definition of $\models_\Delta$ and (H12) it follows that $\theta, s, \sigma_\delta \models_\Delta \langle \mathtt{domainset} \rangle(\vec{\hat{v}}_i; \vec{\hat{v}}_o)$. Along with (H13), this gives our goal (G1).

Proposition: Consume completeness
Assume

(H1) $\mathtt{consume}(\hat{\sigma}, \delta, \vec{\hat{v}}_i) \rightsquigarrow (\mathtt{Ok}, \hat{\sigma}_f, \vec{\hat{v}}_o, \pi)$

(H2) $\theta, s, \sigma_f \models \hat{\sigma}_f \wedge \theta, s, \sigma_\delta \models_\Delta \langle \delta \rangle(\vec{\hat{v}}_i; \vec{\hat{v}}_o) \wedge \sigma_f \# \sigma_\delta$

(H3) $\mathtt{consume}$ on $\mathbb{S}$ is complete.

To prove

(G1) $\exists \sigma.\, \sigma = \sigma_\delta \cdot \sigma_f \wedge \theta, s, \sigma \models \hat{\sigma} \wedge \mathtt{SAT}_{\theta,s}(\pi)$

Case $\delta \in \Delta_{\mathbb{S}}$**:**

(H4) From (H1) and PMapCons, given $\vec{\hat{v}}_i = \hat{i} :: \vec{\hat{v}}'_i$ and $\pi = \pi' :: \pi''$ we have $\mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi')$, $\mathtt{consume}(\hat{\sigma}_i, \delta, \vec{\hat{v}}'_i) \rightsquigarrow (o, \hat{\sigma}'_i, \vec{\hat{v}}_o, \pi'')$ and $\mathtt{set}(\hat{\sigma}, \hat{i}', \hat{\sigma}'_i) = \hat{\sigma}_f$.

(H5) There are two cases here, $\sigma_\delta = \bot$ and $\sigma_\delta \neq \bot$. If $\sigma_\delta = \bot$, we trivially get $\sigma = \bot \cdot \sigma_f$ and the rest of our goal (G1).

(H6) Assume now $\sigma_\delta \neq \bot$. From the definition of $\models_\Delta$ and (H2), we have $\exists i, \sigma_{\delta,i}.\ [\![\hat{i}]\!]_{\theta,s} = i \wedge \sigma_\delta = ([i \mapsto \sigma_{\delta,i}], \bot) \wedge \sigma_{\delta,i} \models_\Delta \langle\delta\rangle(\vec{\hat{v}}'_i; \vec{\hat{v}}_o)$.

(H7) From (H2), (H4) and the definitions of $\models$ and composition we have $\exists \sigma'_i.\ \theta, s, \sigma'_i \models \hat{\sigma}'_i \wedge \sigma'_i \# \sigma_{\delta,i}$.

(H8) From (H7), (H6) and (H3), we can apply Consume Complete, giving us $\exists \sigma_i.\ \sigma_i = \sigma_{\delta,i} \cdot \sigma'_i \wedge \theta, s, \sigma_i \models \hat{\sigma}_i \wedge \mathsf{SAT}_{\theta,s}(\pi'')$.

(H9) From the definition of $\models$, because `consume` doesn't affect any other cell, given $\theta, s, \sigma_i \models \hat{\sigma}_i$ and $\theta, s, \sigma_f \models \hat{\sigma}_f$ and $\mathsf{set}(\hat{\sigma}, \hat{i}', \hat{\sigma}'_i) = \hat{\sigma}_f$, we obtain $\theta, s, \sigma \models \hat{\sigma}$.

(H10) From (H6), we have σ_δ of the form $\sigma_\delta = (h_\delta, \bot)$ where $i \in \mathrm{dom}(h_\delta)$. From composition rules, it follows that if $\sigma = \sigma_\delta \cdot \sigma_f$ we have $\sigma = (h, d)$ where $i \in \mathrm{dom}(h)$ too.

(H11) From (H10) and (H9), it follows that we can apply CPMapGetMatch such that $\mathsf{get}(\sigma, i) = \sigma_i$. We can then use Get OX Soundness, giving us, from (H4) that $\mathsf{SAT}_{\theta,s}(\pi')$.

(H12) It follows from (H11) and (H8) that $\mathsf{SAT}_{\theta,s}(\pi)$. This along with (H9) gives our goal (G1).

Case $\delta = \mathsf{domainset}$**:**

(H13) From (H1) and the rules for `consume`, we have $\hat{\sigma} = (\hat{h}, \hat{d})$ and $\hat{\sigma}_f = (\hat{h}, \bot)$ such that $\hat{d} \neq \bot$, as well as $\vec{\hat{v}}_i = []$, $\vec{\hat{v}}_o = [\hat{d}]$ and $\pi = []$

(H14) From the definition of $\models_\Delta$ and (H2) we have $\sigma_\delta = (\emptyset, d)$ such that $[\![\hat{d}]\!]_{\theta,s} = d$.

(H15) It follows from (H13), (H2) and the definition of $\models$ that σ_f is of the form $\sigma_f = (h, \bot)$.

(H16) From (H14), (H15) and the definition of $\models$ it follows that for $\sigma = \sigma_\delta \cdot \sigma_f$, we have $\theta, s, \sigma \models \hat{\sigma}$. This with the fact $\mathsf{SAT}_{\theta,s}([])$ gives our goal (G1).

Proposition: Consume: OX sound
Assume

(H1) $\forall o, \hat{\sigma}_f, \vec{\hat{v}}_o, \pi.\ \mathsf{consume}(\mathsf{OX}, \hat{\sigma}, \delta, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}_f, \vec{\hat{v}}_o, \pi) \Rightarrow o_c = \mathsf{Ok}$

(H2) `consume` on $\mathbb{S}$ are OX sound.

To prove

(G1) $\exists \hat{\sigma}'_f, \vec{\hat{v}}'_o, \pi'.\ \mathsf{consume}(\mathsf{OX}, \hat{\sigma}, \delta, \vec{v}_i, \pi) \rightsquigarrow (\mathsf{Ok}, \hat{\sigma}'_f, \vec{\hat{v}}'_o, \pi')$

Case $\delta \in \Delta_{\mathbb{S}}$**:**
From PMapProd we know `consume` for $\delta \in \Delta_{\mathbb{S}}$ only returns `Ok` if the underlying's `consume` returns `Ok`. It follows that if that is the case, then by (H2) there exists an execution of `consume` that succeeds for $\mathbb{S}$, that is lifted by PMap, giving us our goal (G1).

Case $\delta = \texttt{domainset}$**:**

From the `consume` rules, we never vanish, so if all consumptions are `Ok` we must have $\hat{\sigma} = (\hat{h}, \hat{d})$ with $\hat{d} \neq \bot$, $\vec{\hat{v}}_i = []$, $\hat{\sigma}'_f = (\hat{h}, \bot)$, $\vec{\hat{v}}'_o = [\hat{d}]$, $\pi' = []$, which gives our goal (G1).

Proposition: Produce soundness
Assume

(H1) $\texttt{produce}(\hat{\sigma}_f, \delta, \vec{\hat{v}}_i, \vec{\hat{v}}_o) \rightsquigarrow (\hat{\sigma}, \pi)$

(H2) $\texttt{SAT}_{\theta,s}(\pi) \wedge \theta, s, \sigma \models \hat{\sigma}$

(H3) `produce` on $\mathbb{S}$ is sound.

To prove

(G1) $\exists \sigma_\delta, \sigma_f.\, \sigma_\delta \# \sigma_f \wedge \sigma = \sigma_\delta \cdot \sigma_f \wedge \theta, s, \sigma_\delta \models_\Delta \langle\delta\rangle(\vec{\hat{v}}_i; \vec{\hat{v}}_o) \wedge \theta, s, \sigma_f \models \hat{\sigma}_f$

Case $\delta \in \Delta_{\mathbb{S}}$**:**

(H4) Given (H1), from PMapProd we have $\texttt{get}(\hat{\sigma}_f, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_{f,i}, \pi') \wedge \texttt{produce}(\hat{\sigma}_{f,i}, \delta, \vec{\hat{v}}'_i, \vec{\hat{v}}_o) \rightsquigarrow (\hat{\sigma}_i, \pi'') \wedge \texttt{set}(\hat{\sigma}_f, \hat{i}', \hat{\sigma}_i) = \hat{\sigma}$, given $\vec{\hat{v}}_i = \hat{i} :: \vec{\hat{v}}'_i$ and $\pi = \pi' :: \pi''$.

(H5) From (H2) and $\pi = \pi' :: \pi''$, it follows that $\texttt{SAT}_{\theta,s}(\pi')$ and $\texttt{SAT}_{\theta,s}(\pi'')$.

(H6) From the definition of `set`, (H4) and (H2), we have $\exists \sigma_i.\, \theta, s, \sigma_i \models \hat{\sigma}_i$.

(H7) From (H4), (H6), (H5) and (H3), we can apply Produce: Soundness, giving us $\exists \sigma_{\delta,i}, \sigma_{f,i}.\, \sigma_{\delta,i} \# \sigma_{f,i} \wedge \sigma_i = \sigma_{\delta,i} \cdot \sigma_{f,i} \wedge \theta, s, \sigma_{\delta,i} \models_\Delta \langle\delta\rangle(\vec{\hat{v}}'_i; \vec{\hat{v}}_o) \wedge \theta, s, \sigma_{f,i} \models \hat{\sigma}_{f,i}$.

(H8) Given $[\![\hat{i}']\!]_{\theta,s} = i$, we can then have $\sigma_\delta = ([i \mapsto \sigma_{\delta,i}], \bot)$, or $\sigma_\delta = \bot$ if $\sigma_{\delta,i} = \bot$. By the definition of $\models_\Delta$, $\theta, s, \sigma_\delta \models_\Delta \langle\delta\rangle(\vec{\hat{v}}_i; \vec{\hat{v}}_o)$ in both cases.

(H9) Given `set` only modifies the state at a specific location leaving the rest untouched, from (H2) and (H7) we have $\theta, s, \sigma \models \hat{\sigma}$ and $\theta, s, \sigma_{f,i} \models \hat{\sigma}_{f,i}$, thus it follows that $\exists \sigma_f.\, \theta, s, \sigma_f \models \sigma_f$.

(H10) From (H7) and by the rules of composition, we know that if the two states $\sigma_{\delta,i}$ and $\sigma_{f,i}$ at index i are disjoint then the entire state σ_f and the singleton σ_δ are disjoint. Thus $\sigma_\delta \# \sigma_f \wedge \sigma = \sigma_\delta \cdot \sigma_f$, giving our goal (G1).

Case $\delta = \texttt{domainset}$**:**

(H11) Given (H1), from PMapProdDomainSet we have $\hat{\sigma}_f = (\hat{h}, \bot)$, $\hat{\sigma} = (\hat{h}, \hat{d})$ and $\pi = [\mathrm{dom}(\hat{h}) \subseteq \hat{d}]$, with $\vec{\hat{v}}_i = []$ and $\vec{\hat{v}}_o = [\hat{d}]$.

(H12) We can pick σ_δ such that $\sigma_\delta = (\emptyset, d)$ with $[\![\hat{d}]\!]_{\theta,s} = d$, which from the definition of $\models_\Delta$ gives $\theta, s, \sigma_\delta \models_\Delta \langle\texttt{domainset}\rangle(\vec{\hat{v}}_i; \vec{\hat{v}}_o)$.

(H13) We can pick σ_f such that $\sigma_f = (h, \bot)$ and $[\![\hat{h}]\!]_{\theta,s} = h$. From the definition of $\models$, we have $\theta, s, \sigma_f \models \hat{\sigma}_f$.

(H14) From (H12), (H13) and composition rules we have $\sigma_f \# \sigma_\delta$ and $\sigma = \sigma_f \cdot \sigma_\delta$. This completes our goal (G1).

Proposition: Produce completeness

Assume

(H1) $\theta, s, \sigma_f \models \hat{\sigma}_f \wedge \theta, s, \sigma_\delta \models_\Delta \langle\delta\rangle(\vec{\hat{v}}_i; \vec{\hat{v}}_o) \wedge \sigma_f \# \sigma_\delta$

(H2) `produce` on $\mathbb{S}$ is complete.

To prove

(G1) $\exists \hat{\sigma}.\, \mathsf{produce}(\hat{\sigma}_f, \delta, \vec{\hat{v}}_i, \vec{\hat{v}}_o) \rightsquigarrow (\hat{\sigma}, \pi) \wedge \mathsf{SAT}_{\theta,s}(\pi) \wedge \theta, s, (\sigma_f \cdot \sigma_\delta) \models \hat{\sigma}$

Case $\delta \in \Delta_{\mathbb{S}}$**:**

(H3) From (H1) and the definition of $\models_\Delta$, we have $\exists \hat{i}, i, \sigma_{\delta,i}.\, \vec{\hat{v}}_i = \hat{i} :: \vec{\hat{v}}'_i \wedge [\![\hat{i}]\!]_{\theta,s} = i \wedge \theta, s, \sigma_{\delta,i} \models_\Delta \langle\delta\rangle(\vec{\hat{v}}'_i; \vec{\hat{v}}_o) \wedge \sigma_\delta = ([i \mapsto \sigma_{\delta,i}], \bot)$.

(H4) Given the fact $\sigma_f \# \sigma_\delta$ and $\sigma_\delta \models_\Delta \langle\delta\rangle(\hat{i} :: \vec{\hat{v}}'_i; \vec{\hat{v}}_o)$, it means the index i is compatible with σ_f, so the state located at i is obtainable via `get`. We thus have $\exists \sigma_{f,i}.\, \mathsf{get}(\sigma_f, i) = \sigma_{f,i}$.

(H5) From (H4), (H3) and (H1), we can use Get OX Soundness, giving us $\exists \hat{\sigma}_{f,i}, \hat{i}', \pi'.\, \mathsf{get}(\hat{\sigma}_f, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_{f,i}, \pi) \wedge \theta, s, \sigma_{f,i} \models \hat{\sigma}_{f,i} \wedge [\![\hat{i}']\!]_{\theta,s} = i \wedge \mathsf{SAT}_{\theta,s}(\pi')$.

(H6) From (H1), because $\sigma_{f,i}$ and $\sigma_{\delta,i}$ are at the same index and $\sigma_f \# \sigma_\delta$, from the rules for composition we have $\sigma_{f,i} \# \sigma_{\delta,i}$.

(H7) From (H3), (H5) and (H6) we can apply Produce: Completeness for $\mathbb{S}$, giving us $\exists \hat{\sigma}_i.\, \mathsf{produce}(\hat{\sigma}_{f,i}, \delta, \vec{\hat{v}}'_i, \vec{\hat{v}}_o) \rightsquigarrow (\hat{\sigma}_i, \pi'') \wedge \mathsf{SAT}_{\theta,s}(\pi'') \wedge \theta, s, (\sigma_{f,i} \cdot \sigma_{\delta,i}) \models \hat{\sigma}_i$.

(H8) We may define $\hat{\sigma}$ such that $\mathsf{set}(\hat{\sigma}_f, \hat{i}', \hat{\sigma}_i) = \hat{\sigma}$.

(H9) We have $\theta, s, \sigma_f \models \hat{\sigma}_f$ and $\theta, s, (\sigma_{f,i} \cdot \sigma_{\delta,i}) \models \hat{\sigma}_i$ from (H1) and (H7). Furthermore, we know $\sigma_\delta = ([i \mapsto \sigma_{\delta,i}], \bot)$ from (H3). From (H8) and the rules for `set` we know no state is lost and only the state at i is extended by $\sigma_{\delta,i}$, it thus follows that $\theta, s, (\sigma_f \cdot \sigma_\delta) \models \hat{\sigma}$.

(H10) From and (H5), (H7) and this we can apply PMapProd, giving us $\mathsf{produce}(\hat{\sigma}_f, \delta, \vec{\hat{v}}_i, \vec{\hat{v}}_o) \rightsquigarrow (\hat{\sigma}, \pi' :: \pi'')$.

(H11) From (H5) and (H7), we have $\mathsf{SAT}_{\theta,s}(\pi' :: \pi'')$. This, along with (H9) and (H10), gives our goal (G1).

Case $\delta = \mathsf{domainset}$**:**

(H12) From (H1) and the definition of $\models_\Delta$ we have $\sigma_\delta = (\emptyset, d)$ such that $\vec{\hat{v}}_i = []$, $\vec{\hat{v}}_o = [\hat{d}]$ and $[\![\hat{d}]\!]_{\theta,s} = d$.

(H13) From (H12) and (H1), by the definition of composition, we must have $\hat{\sigma}_f = (\hat{h}, \bot)$, $\sigma_f = (h, \bot)$ and $\mathrm{dom}(h) \subseteq d$.

(H14) From (H12) and (H13) we may apply PMapProdDomainSet, resulting in $\mathsf{produce}(\hat{\sigma}_f, \mathsf{domainset}, \vec{\hat{v}}_i, \vec{\hat{v}}_o) \rightsquigarrow ((\hat{h}, \hat{d}), [\mathrm{dom}(\hat{h}) \subseteq \hat{d}])$.

(H15) From (H1) and (H13) we have $\mathsf{SAT}_{\theta,s}([\mathrm{dom}(\hat{h}) \subseteq \hat{d}])$.

(H16) From (H12), (H13) and composition rules we have we have $\sigma_f \cdot \sigma_\delta = (h, d)$. It follows from (H14) that we have $\hat{\sigma} = (\hat{h}, \hat{d})$ such that $\theta, s, (\sigma_f \cdot \sigma_\delta) \models \hat{\sigma}$. This along with (H14) and (H15) gives our goal (G1).

□

4.3 Syntactic Partial Map

For the partial map with syntactic checks, we re-use the RA of PMap, as well as the concrete and symbolic action rules, and the `produce` and `consume` rules, as we only modify the behaviour of the `get` helper function. From this, we can thus re-use the soundness proofs for all of the axioms; the only part that needs to be proved again are the axioms for symbolic `get`, with regards to the concrete version that is taken from PMap.

4.3.1 `get` rules

$$\frac{(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad \hat{i} \in \mathrm{dom}(\hat{h}) \qquad \hat{\sigma}_i = \hat{h}(\hat{i})}{\mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \hat{\sigma}_i, [])} \text{ SyntacticPMapGetMatch}$$

$$\frac{(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad \hat{i} \notin \mathrm{dom}(\hat{h}) \qquad \hat{i}' \in \mathrm{dom}(\hat{h}) \qquad \hat{\sigma}_i = \hat{h}(\hat{i}')}{\mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, [\hat{i} = \hat{i}'])} \text{ SyntacticPMapGetBranch}$$

$$\frac{(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad \hat{i} \notin \mathrm{dom}(\hat{h}) \qquad \hat{d} \neq \bot}{\mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \bot, [\hat{i} \notin \mathrm{dom}(\hat{h}) \wedge \hat{i} \in \hat{d}])} \text{ SyntacticPMapGetAdd}$$

$$\frac{(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad \hat{i} \notin \mathrm{dom}(\hat{h}) \qquad \hat{d} = \bot}{\mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \bot, [\hat{i} \notin \mathrm{dom}(\hat{h})])} \text{ SyntacticPMapGetBotDomain}$$

4.3.2 Soundness Proofs

Proof.

Proposition: Get OX Soundness
Assume

(H1) $\theta, s, \sigma \models \hat{\sigma} \wedge \mathtt{get}(\sigma, i) = \sigma_i \wedge [\![\hat{i}]\!]_{\theta,s} = i$

To prove

(G1) $\exists \hat{\sigma}_i, \hat{i}', \pi.\, \mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \wedge \theta, s, \sigma_i \models \hat{\sigma}_i \wedge [\![\hat{i}']\!]_{\theta,s} = i \wedge \mathtt{SAT}_{\theta,s}(\pi)$

We proceed by proving the property holds for all rules of the concrete `get`, resulting in three cases.

Case CPMapGetMatch**:**

(H2) Assume $(h, d) = \mathit{unwrap}(\sigma) \wedge i \in \mathrm{dom}(h) \wedge \sigma_i = h(i)$.

(H3) It follows from (H1) and the definition of $\models$ that $\hat{\sigma} = (\hat{h}, \hat{d})$.

(H4) Either $\hat{i} \in \hat{h}$ or not. If $\hat{i} \in \hat{h}$, we apply SyntacticPMapGetMatch, giving us $\mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \hat{h}(\hat{i}), [])$. It also follows from the definition of $\models$ that $\theta, s, \sigma_i \models \hat{h}(\hat{i})$, completing our goal (G1).

(H5) Otherwise $\hat{i} \notin \hat{h}$. From the definition of $\models$ we know $\exists \hat{i}'. [\![\hat{i}']\!]_{\theta,s} = i \wedge \hat{i}' \in \hat{h} \wedge \theta, s, \sigma_i \models \hat{h}(\hat{i}')$. We can apply SyntacticPMapGetBranch, giving us $\mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, [\hat{i} = \hat{i}'])$. It follows that $\mathtt{SAT}_{\theta,s}([\hat{i} = \hat{i}'])$, which completes our goal (G1).

Case CPMapGetAdd**:** The rule SyntacticPMapGetAdd is exactly the same as PMapGetAdd – we omit the proof.

Case CPMapGetBotDomain**:** The rule SyntacticPMapGetBotDomain is exactly the same as PMapGetBotDomain – we omit the proof.

Proposition: Get UX Soundness
Assume

(H1) $\mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \wedge [\![\hat{i}]\!]_{\theta,s} = [\![\hat{i}']\!]_{\theta,s} = i \wedge \mathtt{SAT}_{\theta,s}(\pi)$

(H2) $\theta, s, \sigma \models \hat{\sigma}$.

To prove

(G1) $\exists \sigma_i. \mathtt{get}(\sigma, i) = \sigma_i \wedge \theta, s, \sigma_i \models \hat{\sigma}_i$

We proceed by proving the property holds for all rules of the symbolic get, resulting in four cases.

Case SyntacticPMapGetMatch**:**

(H3) Assume $(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \wedge \hat{i} \in \mathrm{dom}(\hat{h}) \wedge \hat{\sigma}_i = \hat{h}(\hat{i})$.

(H4) From (H3), (H1) and SyntacticPMapGetMatch we have $\pi = []$ and $\hat{i}' = \hat{i}$.

(H5) From (H3), we know $\hat{\sigma} \neq \bot$, thus σ is of the form $\sigma = (h, d)$.

(H6) From (H2), (H3) and the definition of $\models$, we have $i \in h$ such that $\sigma_i = h(i)$ and $\theta, s, \sigma_i \models \hat{\sigma}_i$.

(H7) From (H6) we can apply CPMapGetMatch, thus $\mathtt{get}(\sigma, i) = \sigma_i$. This completes our goal (G1).

Case SyntacticPMapGetBranch**:**

(H8) Assume $(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \wedge \hat{i}' \in \mathrm{dom}(\hat{h}) \wedge \hat{\sigma}_i = \hat{h}(\hat{i}')$

(H9) From (H8), (H1) and PMapGetMatch, we have $\pi = [\hat{i} = \hat{i}']$

(H10) From (H8), we have $\hat{\sigma} \neq \bot$, thus from (H2) we have $\sigma = (h, d)$

(H11) From (H2), (H8) and the definition of $\models$, we have $i \in h$ such that $\sigma_i = h(i)$ and $\theta, s, \sigma_i \models \hat{\sigma}_i$

(H12) From (H11) we can apply CPMapGetMatch, thus $\mathtt{get}(\sigma, i) = \sigma_i$. This completes our goal (G1).

Case SyntacticPMapGetAdd**:** The rule SyntacticPMapGetAdd is exactly the same as PMapGetAdd – we omit the proof.

Case SyntacticPMapGetBotDomain**:** The rule SyntacticPMapGetBotDomain is exactly the same as PMapGetBotDomain – we omit the proof. □

4.4 Split Partial Map

For the split partial map, we define a new RA, $\text{PMAP}_{\text{SPLIT}}$, which we only use for the symbolic representation; the concrete compositional RA is still that defined in PMAP. We re-use the definitions for action execution, `consume` and `produce`, as the only part of them that is modified is `get` and `set`. From this, we can thus re-use the soundness proofs for all of the axioms; the only part that needs to be proved again are the axioms for symbolic `get`, with regards to the concrete version that is taken from PMAP.

4.4.1 Resource Algebra

While we keep the default PMAP RA for the concrete compositional states, we need to define a new set of states for the symbolic compositional states, as defined below.

$$\text{PMAP}_{\text{SPLIT}}(I, \mathbb{S}) \stackrel{\text{def}}{=} I \xrightarrow{fin} \mathbb{S}.\Sigma \times I \xrightarrow{fin} \mathbb{S}.\Sigma \times \mathcal{P}(I)^?$$

Predicate satisfiability is defined with regards to the concrete compositional states, so we re-use it from PMAP. We however need to re-define symbolic interpretation, as follows. We denote $\hat{h}_c$ and $\hat{h}_s$ the concrete and symbolic parts of the heap respectively.

$$\textsc{SplitPMapSymInterpretation}\quad \frac{\begin{array}{c}\forall \hat{i} \in \text{dom}(\hat{h}_c).\, \hat{i} \notin \hat{h}_s \land [\![\hat{i}]\!]_{\theta,s} = i \land i \in \text{dom}(h) \land \theta, s, h(i) \models \hat{h}_c(\hat{i}) \\ \forall \hat{i} \in \text{dom}(\hat{h}_s).\, \hat{i} \notin \hat{h}_c \land [\![\hat{i}]\!]_{\theta,s} = i \land i \in \text{dom}(h) \land \theta, s, h(i) \models \hat{h}_s(\hat{i}) \\ [\![\text{dom}(\hat{h}_c) \uplus \text{dom}(\hat{h}_s)]\!]_{\theta,s} = \text{dom}(h) \qquad [\![\hat{d}]\!]_{\theta,s} = d\end{array}}{\theta, s, (h, d) \models (\hat{h}_c, \hat{h}_s, \hat{d})}$$

4.4.2 `get` and `set` rules

$$\text{Given } \mathit{wrap}(\hat{h}_c, \hat{h}_s, \hat{d}) \stackrel{\text{def}}{=} \begin{cases} \bot & \text{if } \text{dom}(\hat{h}_c) = \emptyset \land \text{dom}(\hat{h}_s) = \emptyset \land \hat{d} = \emptyset \\ (\hat{h}_c, \hat{h}_s, \hat{d}) & \text{otherwise} \end{cases}$$

$$\mathit{unwrap}(\hat{\sigma}) \stackrel{\text{def}}{=} \begin{cases} ([], [], \emptyset) & \text{if } \hat{\sigma} = \bot \\ (\hat{h}_c, \hat{h}_s, \hat{d}) & \text{if } \hat{\sigma} = (\hat{h}_c, \hat{h}_s, \hat{d}) \end{cases}$$

$$\textsc{SplitPMapGetMatchCon}\quad \frac{(\hat{h}_c, \hat{h}_s, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad \texttt{is\_concrete}\ \hat{i} \qquad \hat{i} \in \text{dom}(\hat{h}_c) \qquad \hat{\sigma}_i = \hat{h}_c(\hat{i})}{\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \hat{\sigma}_i, [])}$$

$$\textsc{SplitPMapGetMatchSym}\quad \frac{(\hat{h}_c, \hat{h}_s, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad \hat{i} \in \text{dom}(\hat{h}_s) \qquad \hat{\sigma}_i = \hat{h}_s(i)}{\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \hat{\sigma}_i, [])}$$

$$\textsc{SplitPMapGetBranch}\quad \frac{\begin{array}{c}(\hat{h}_c, \hat{h}_s, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \\ \hat{h}_{all} = \hat{h}_c \cup \hat{h}_s \qquad \hat{i} \notin \text{dom}(\hat{h}_{all}) \qquad \hat{i}' \in \text{dom}(\hat{h}_{all}) \qquad \hat{\sigma}_i = \hat{h}_{all}(\hat{i}')\end{array}}{\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, [\hat{i} = \hat{i}'])}$$

SplitPMapGetAdd

$$\frac{(\hat{h}_c, \hat{h}_s, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad \hat{h}_{all} = \hat{h}_c \cup \hat{h}_s \qquad \hat{i} \notin \mathrm{dom}(\hat{h}_{all}) \qquad \hat{d} \neq \bot}{\mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \bot, [\hat{i} \notin \mathrm{dom}(\hat{h}_{all}) \wedge \hat{i} \in \hat{d}])}$$

SplitPMapGetBotDomain

$$\frac{(\hat{h}_c, \hat{h}_s, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad \hat{h}_{all} = \hat{h}_c \cup \hat{h}_s \qquad \hat{i} \notin \mathrm{dom}(\hat{h}_{all}) \qquad \hat{d} = \bot}{\mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \bot, [\hat{i} \notin \mathrm{dom}(\hat{h}_{all})])}$$

SplitPMapSetSomeCon

$$\frac{\begin{array}{c}(\hat{h}_c, \hat{h}_s, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad \hat{\sigma}_i \neq \bot \qquad \mathtt{is\_concrete}_\Sigma\ \hat{\sigma}_i \\ \mathtt{is\_concrete}\ \hat{i} \qquad \hat{h}'_c = \hat{h}_c[\hat{i} \leftarrow \hat{\sigma}_i] \qquad \hat{h}'_s = \hat{h}_s[\hat{i} \not\leftarrow] \qquad \hat{\sigma}' = \mathit{wrap}(\hat{h}'_c, \hat{h}'_s, \hat{d})\end{array}}{\mathtt{set}(\hat{\sigma}, \hat{i}, \hat{\sigma}_i) = \hat{\sigma}'}$$

SplitPMapSetSomeSym

$$\frac{\begin{array}{c}(\hat{h}_c, \hat{h}_s, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad \hat{s}_i \neq \bot \qquad \neg(\mathtt{is\_concrete}_\Sigma\ \hat{\sigma}_i \vee \mathtt{is\_concrete}\ \hat{i}) \\ \hat{h}'_c = \hat{h}_c[\hat{i} \not\leftarrow] \qquad \hat{h}'_s = \hat{h}_s[\hat{i} \leftarrow \hat{\sigma}_i] \qquad \hat{\sigma}' = \mathit{wrap}(\hat{h}'_c, \hat{h}'_s, \hat{d})\end{array}}{\mathtt{set}(\hat{\sigma}, \hat{i}, \hat{\sigma}_i) = \hat{\sigma}'}$$

SplitPMapSetNone

$$\frac{\begin{array}{c}(\hat{h}_c, \hat{h}_s, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \\ \hat{\sigma}_i = \bot \qquad \hat{h}'_c = \hat{h}_c[\hat{i} \not\leftarrow] \qquad \hat{h}'_s = \hat{h}_s[\hat{i} \not\leftarrow] \qquad \hat{\sigma}' = \mathit{wrap}(\hat{h}'_c, \hat{h}'_s, \hat{d})\end{array}}{\mathtt{set}(\hat{\sigma}, \hat{i}, \hat{\sigma}_i) = \hat{\sigma}'}$$

4.4.3 Soundness Proofs

Proof.

Proposition: Get OX Soundness
Assume

(H1) $\theta, s, \sigma \models \hat{\sigma} \wedge \mathtt{get}(\sigma, i) = \sigma_i \wedge [\![\hat{i}]\!]_{\theta,s} = i$

To prove

(G1) $\exists \hat{\sigma}_i, \hat{i}', \pi.\, \mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \wedge \theta, s, \sigma_i \models \hat{\sigma}_i \wedge [\![\hat{i}']\!]_{\theta,s} = i \wedge \mathtt{SAT}_{\theta,s}(\pi)$

We proceed by proving the property holds for all rules of the concrete `get`, resulting in three cases.

Case CPMapGetMatch**:**

(H2) Assume $(h, d) = \mathit{unwrap}(\sigma) \wedge i \in \mathrm{dom}(h) \wedge \sigma_i = h(i)$.

(H3) It follows from (H1) and the definition of $\models$ that $\hat{\sigma} = (\hat{h}_c, \hat{h}_s, \hat{d})$. From here we have three cases: $\hat{i}$ is present directly in $\hat{h}_c$, or in $\hat{h}_s$, or neither.

(H4) From (H3), assume the binding is in the concrete part of the heap: $\hat{i} \in \hat{h}_c$. From the definition of `set`, we know that if an entry is in the concrete part it must be concrete; thus $\mathtt{is\_concrete}\ \hat{i}$. From this, we can apply SplitPMapGetMatchCon, giving us $\mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \hat{h}_c(\hat{i}), [])$. From the definition of $\models$, it also follows that $\theta, s, \sigma_i \models \hat{h}_c(\hat{i})$. This, along with the fact $\mathtt{SAT}_{\theta,s}([])$, gives our goal (G1).

(H5) Assume now from (H3) that the binding is in the symbolic part of the heap, $\hat{h}_s$. From this, we can apply SplitPMapGetMatchSym, giving us $\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \hat{h}_s(\hat{i}), [])$. From the definition of $\models$, it also follows that $\theta, s, \sigma_i \models \hat{h}_s(\hat{i})$. This again gives our goal (G1).

(H6) Finally, it may be that $\hat{i} \notin \hat{h}_c \wedge \hat{i} \notin \hat{h}_s$. From the definition of $\models$, there must however still exist a $\hat{i}'$ such that $[\![\hat{i}']\!]_{\theta,s} = i$ and given $\hat{h}_{all} = \hat{h}_c \cup \hat{h}_s$, $\theta, s, \sigma_i \models \hat{h}_{all}(\hat{i}')$. From this we may apply SplitPMapGetBranch, giving us $\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, [\hat{i} = \hat{i}'])$. Because we have $[\![\hat{i}]\!]_{\theta,s} = i$, it follows that $\texttt{SAT}_{\theta,s}([\hat{i} = \hat{i}'])$, completing our goal (G1).

Case CPMapGetAdd**:**

(H7) Assume $(h, d) = \mathit{unwrap}(\sigma) \wedge i \notin \mathrm{dom}(h) \wedge d \neq \bot \wedge i \in d$.

(H8) It follows from (H1) and the definition of $\models$ that $\hat{\sigma} = (\hat{h}_c, \hat{h}_s, \hat{d})$ such that $[\![\hat{i}]\!]_{\theta,s} = i$, $\hat{i} \in \hat{d}$ and $\hat{i} \notin \mathrm{dom}(\hat{h}_c \cup \hat{h}_s)$.

(H9) We can then apply SplitPMapGetAdd, giving us $\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \bot, [\hat{i} \notin \mathrm{dom}(\hat{h}_c \cup \hat{h}_s) \wedge \hat{i} \notin \hat{d}])$.

(H10) From (H8) it follows that $\texttt{SAT}_{\theta,s}([\hat{i} \notin \mathrm{dom}(\hat{h}) \wedge \hat{i} \notin \hat{d}])$, which completes our goal (G1).

Case CPMapGetBotDomain**:**

(H11) Assume $(h, d) = \mathit{unwrap}(\sigma) \wedge i \notin \mathrm{dom}(h) \wedge d = \bot$.

(H12) From (H11), (H1) and CPMapGetBotDomain, we have $\sigma_i = \bot$.

(H13) It follows from (H1) and the definition of $\models$ that $\hat{\sigma} = (\hat{h}_c, \hat{h}_s, \bot)$ such that $[\![\hat{i}]\!]_{\theta,s} = i$ and $\hat{i} \notin \mathrm{dom}(\hat{h}_c \cup \hat{h}_s)$.

(H14) We can then apply SplitPMapGetBotDomain, giving us $\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \bot, [\hat{i} \notin \mathrm{dom}(\hat{h}_c \cup \hat{h}_s)])$.

(H15) From (H13) it follows that $\texttt{SAT}_{\theta,s}([\hat{i} \notin \mathrm{dom}(\hat{h}_c \cup \hat{h}_s)])$, which completes our goal (G1).

Proposition: Get UX Soundness
Assume

(H1) $\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \wedge [\![\hat{i}]\!]_{\theta,s} = [\![\hat{i}']\!]_{\theta,s} = i \wedge \texttt{SAT}_{\theta,s}(\pi)$

(H2) $\sigma.\, \theta, s, \sigma \models \hat{\sigma}$

To prove

(G1) $\exists \sigma_i.\, \texttt{get}(\sigma, i) = \sigma_i \wedge \theta, s, \sigma_i \models \hat{\sigma}_i$

We proceed by proving the property holds for all rules of the symbolic `get`, resulting in five cases.

Case SplitPMapGetMatchCon**:**

(H3) Assume $(\hat{h}_c, \hat{h}_s, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \wedge \texttt{is\_concrete}\ \hat{i} \wedge \hat{i} \in \mathrm{dom}(\hat{h}_c) \wedge \hat{\sigma}_i = \hat{h}_c(\hat{i})$.

(H4) From (H3), (H1) and SplitPMapGetMatchCon we have $\pi = []$ and $\hat{i}' = \hat{i}$.

(H5) From (H3), we know $\hat{\sigma} \neq \bot$, thus σ is of the form $\sigma = (h, d)$.

(H6) From (H2), (H3) and the definition of $\models$, we have $i \in \mathrm{dom}(h)$ such that $\sigma_i = h(i)$ and $\theta, s, \sigma_i \models \hat{\sigma}_i$.

(H7) From (H6) we can apply CPMapGetMatch, thus $\mathtt{get}(\sigma, i) = \sigma_i$. This completes our goal (G1).

Case SplitPMapGetMatchSym**:** This is proved analogously to the above.

Case SplitPMapGetBranch**:**

(H8) Assume $(\hat{h}_c, \hat{h}_s, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \wedge \hat{h}_{all} = \hat{h}_c \cup \hat{h}_s \wedge \hat{i} \notin \mathrm{dom}(\hat{h}_{all}) \wedge \hat{i}' \in \mathrm{dom}(\hat{h}_{all}) \wedge \hat{\sigma}_i = \hat{h}_{all}(\hat{i}')$.

(H9) From (H8), (H1) and SplitPMapGetBranch we have $\pi = [\hat{i} = \hat{i}']$.

(H10) From (H8) we know $\hat{\sigma} \neq \bot$, thus σ is of the form $\sigma = (h, d)$.

(H11) From (H8), (H2) and the definition of $\models$, we have $i \in \mathrm{dom}(h)$ such that $\sigma_i = h(i)$ and $\theta, s, \sigma_i \models \hat{\sigma}_i$.

(H12) From (H11) we can apply CPMapGetMatch, thus $\mathtt{get}(\sigma, i) = \sigma_i$. This completes our goal (G1).

Case SplitPMapGetAdd**:**

(H13) Assume $(\hat{h}_c, \hat{h}_s, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \wedge \hat{h}_{all} = \hat{h}_c \cup \hat{h}_s \wedge \hat{i} \notin \mathrm{dom}(\hat{h}_{all}) \wedge \hat{d} \neq \bot$.

(H14) From (H13), (H1) and SplitPMapGetAdd we have $\pi = [\hat{i} \notin \mathrm{dom}(\hat{h}_{all}) \wedge \hat{i} \in \hat{d}]$, $\hat{i}' = \hat{i}$ and $\hat{\sigma}_i = \bot$.

(H15) From (H13) we know $\hat{\sigma} \neq \bot$, thus σ is of the form $\sigma = (h, d)$.

(H16) From (H1) and (H14), we know $\mathtt{SAT}_{\theta,s}(\pi)$, such that from (H2) we have $i \notin \mathrm{dom}(h) \wedge i \in d$.

(H17) From (H16) we can straightforwardly apply CPMapGetAdd, giving us $\mathtt{get}(\sigma, i) = \bot$ – together with (H14) and Empty Memory this completes our goal (G1).

Case SplitPMapGetBotDomain**:**

(H18) Assume $(\hat{h}_c, \hat{h}_s, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \wedge \hat{h}_{all} = \hat{h}_c \cup \hat{h}_s \wedge \hat{i} \notin \mathrm{dom}(\hat{h}_{all}) \wedge \hat{d} = \bot$.

(H19) From (H18), (H1) and SplitPMapGetBotDomain we have $\pi = [\hat{i} \notin \mathrm{dom}(\hat{h}_{all})]$ and $\hat{i} = \hat{i}$.

(H20) From (H18) we have $\hat{d} = \bot$, so given $(h, d) = \mathit{unwrap}(\sigma)$ and (H2), we have $d = \bot$.

(H21) From (H1) $\mathtt{SAT}_{\theta,s}(\pi)$, thus from (H19) and (H2), $i \notin \mathrm{dom}(h)$.

(H22) From (H21) and (H20) we can apply CPMapGetBotDomain, thus $\mathtt{get}(\sigma, i) = \bot$ – together with (H19) and Empty Memory this completes our goal (G1).

□

4.5 Abstract Location Partial Map

For the abstract location partial map, we define a new set of states, $\text{PMap}_{\text{ALoc}}$, which we only use for the symbolic representation; the concrete compositional RA is still that defined in PMap.

For the PMap with syntactic matching and $\text{PMap}_{\text{Split}}$ we considered compatiblity with any "regular" PMap that has the same domain I. Here however, because $\text{PMap}_{\text{ALoc}}$ enforces the domain be abstract locations (as strings), we must consider compatiblity with regards to $\text{PMap}(\text{Loc}, \mathbb{S})$ only.

Concrete locations (or just locations) are values of the form $\text{loc}(a)$, where a is the name of the location – it always holds that given two locations $\text{loc}(a)$ and $\text{loc}(b)$, $\text{loc}(a) = \text{loc}(b) \iff a = b$. They are uninterpreted values, and do not allow any operations; as such, they can be used as sorts of pointers, where they represent address in a map, to the difference that they do not allow pointer arithmetics.

Abstract locations are *symbolic* values of the form $\text{aloc}(a)$, where a is the name of the abstract location. Unlike with concrete locations, their name does not uniquely identify them: we may have a substitution θ such that for $\text{aloc}(a)$ and $\text{aloc}(b)$ with $a \neq b$, we have $[\![\text{aloc}(a)]\!]_{\theta,s} = [\![\text{aloc}(b)]\!]_{\theta,s}$.

We also introduce the function `to_aloc`, that returns the name of an abstract location associated with a symbolic value if it exists and is found, and $\bot$ otherwise. This is a best effort function, that may not find an abstract location that exists.

$$\texttt{to\_aloc}\ i = a \implies \exists a.\, [\![i]\!]_{\theta,s} = [\![\text{aloc}(a)]\!]_{\theta,s}$$

4.5.1 Resource Algebra

$$\text{PMap}_{\text{ALoc}}(\mathbb{S}) \stackrel{\text{def}}{=} \text{Str} \xrightarrow{fin} \mathbb{S}.\Sigma \times \mathcal{P}(\text{LVal})^?$$

ALocPMapSymInterpretation

$$\frac{\forall a \in \text{dom}(\hat{h}).\, [\![\text{aloc}(a)]\!]_{\theta,s} = i \wedge i \in \text{dom}(h) \wedge \theta, s, h(i) \models \hat{h}(a) \qquad [\![\{\text{aloc}(a) : a \in \text{dom}(\hat{h})\}]\!]_{\theta,s} = \text{dom}(h) \qquad [\![\hat{d}]\!]_{\theta,s} = d}{\theta, s, (h, d) \models (\hat{h}, \hat{d})}$$

4.5.2 `get` and `set` rules

We now present the rules for this state model; in particular, we again only need to concern ourselves with the `get` and `set` internal methods. We also extend `get` to receive a mode $M = \{\texttt{MATCH}, \texttt{NO\_MATCH}\}$; it is `MATCH` in `consume` and `produce`, and `NO_MATCH` during action execution. The rules for `execute_action`, `consume` and `produce` are omitted as they are analogous to those in PMap, to the exception that `get` receives a matching mode. To simplify the rules, we also add the notation $\in^?$, that checks for membership in a possibly $\bot$ set, in which case the result is `true`.

$$\text{Given } \mathit{wrap}(\hat{h},\hat{d}) \stackrel{\text{def}}{=} \begin{cases} \bot & \text{if } \mathrm{dom}(\hat{h}) = \emptyset \wedge \hat{d} = \bot \\ (\hat{h},\hat{d}) & \text{otherwise} \end{cases}$$

$$\mathit{unwrap}(\hat{\sigma}) \stackrel{\text{def}}{=} \begin{cases} ([],\bot) & \text{if } \hat{\sigma} = \bot \\ (\hat{h},\hat{d}) & \text{if } \hat{\sigma} = (\hat{h},\hat{d}) \end{cases}$$

$$a \in^? \hat{d} \stackrel{\text{def}}{=} \begin{cases} \texttt{true} & \text{if } \hat{d} = \bot \\ a \in \hat{d} & \text{otherwise} \end{cases}$$

ALocPMapGetMatch

$$\frac{(\hat{h},\hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad a = \texttt{to\_aloc}\ \hat{i} \qquad a \neq \bot \qquad a \in \mathrm{dom}(\hat{h}) \qquad \hat{\sigma}_i = \hat{h}(a)}{\texttt{get}(\hat{\sigma},\hat{i},m) \rightsquigarrow (\mathrm{aloc}(a),\hat{\sigma}_i,[])}$$

ALocPMapGetNoMatchNotFound

$$\frac{(\hat{h},\hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad a = \texttt{to\_aloc}\ \hat{i} \qquad a \neq \bot \qquad a \notin \mathrm{dom}(\hat{h})}{\texttt{get}(\hat{\sigma},\hat{i},\texttt{NO\_MATCH}) \rightsquigarrow (\mathrm{aloc}(a),\bot,[\hat{i} \in^? \hat{d}])}$$

ALocPMapGetNoMatchNew

$$\frac{(\hat{h},\hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad a = \texttt{to\_aloc}\ \hat{i} \qquad a = \bot \qquad a' = \texttt{fresh\_aloc ()}}{\texttt{get}(\hat{\sigma},\hat{i},\texttt{NO\_MATCH}) \rightsquigarrow (\mathrm{aloc}(a'),\bot,[\hat{i} = \mathrm{aloc}(a') \wedge \hat{i} \in^? \hat{d}])}$$

ALocPMapGetMatchNotFound

$$\frac{(\hat{h},\hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad a = \texttt{to\_aloc}\ \hat{i} \qquad a \neq \bot \qquad a \notin \mathrm{dom}(\hat{h})}{\texttt{get}(\hat{\sigma},\hat{i},\texttt{MATCH}) \rightsquigarrow (\mathrm{aloc}(a),\bot,[\hat{i} \notin \{\mathrm{aloc}(a') : a' \in \mathrm{dom}(\hat{h})\} \wedge \hat{i} \in^? \hat{d}])}$$

ALocPMapGetMatchNew

$$\frac{(\hat{h},\hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad a = \texttt{to\_aloc}\ \hat{i} \qquad a = \bot \qquad a' = \texttt{fresh\_aloc ()}}{\texttt{get}(\hat{\sigma},\hat{i},\texttt{MATCH}) \rightsquigarrow (\mathrm{aloc}(a'),\bot,[\hat{i} = \mathrm{aloc}(a') \wedge \hat{i} \notin \{\mathrm{aloc}(a'') : a'' \in \mathrm{dom}(\hat{h})\} \wedge \hat{i} \in^? \hat{d}])}$$

ALocPMapMatching

$$\frac{\begin{array}{c}(\hat{h},\hat{d}) = \mathit{unwrap}(\hat{\sigma}) \\ a = \texttt{to\_aloc}\ \hat{i} \qquad a \neq \bot \qquad a \notin \mathrm{dom}(\hat{h}) \qquad a' \in \mathrm{dom}(\hat{h}) \qquad \hat{\sigma}_i = \hat{h}(a')\end{array}}{\texttt{get}(\hat{\sigma},\hat{i},\texttt{MATCH}) \rightsquigarrow (\mathrm{aloc}(a'),\hat{\sigma}_i,[\mathrm{aloc}(a) = \mathrm{aloc}(a')])}$$

ALocPMapMatchingBot

$$\frac{(\hat{h},\hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad a = \texttt{to\_aloc}\ \hat{i} \qquad a = \bot \qquad a' \in \mathrm{dom}(\hat{h}) \qquad \hat{\sigma}_i = \hat{h}(a')}{\texttt{get}(\hat{\sigma},\hat{i},\texttt{MATCH}) \rightsquigarrow (\mathrm{aloc}(a'),\hat{\sigma}_i,[\hat{i} = \mathrm{aloc}(a')])}$$

ALocPMapSetSome

$$\frac{\begin{array}{c}(\hat{h},\hat{d}) = \mathit{unwrap}(\hat{\sigma}) \\ a = \texttt{to\_aloc}\ \hat{i} \qquad a \neq \bot \qquad \hat{\sigma}_i \neq \bot \qquad \hat{h}' = \hat{h}[a \leftarrow \hat{\sigma}_i] \qquad \hat{\sigma}' = \mathit{wrap}(\hat{h}',\hat{d})\end{array}}{\texttt{set}(\hat{\sigma},\hat{i},\hat{\sigma}_i) = \hat{\sigma}'}$$

ALocPMapSetNone

$$\frac{\begin{array}{c}(\hat{h},\hat{d}) = \mathit{unwrap}(\hat{\sigma}) \\ a = \texttt{to\_aloc}\ \hat{i} \qquad a \neq \bot \qquad \hat{\sigma}_i = \bot \qquad \hat{h}' = \hat{h}[a \not\leftarrow] \qquad \hat{\sigma}' = \mathit{wrap}(\hat{h}',\hat{d})\end{array}}{\texttt{set}(\hat{\sigma},\hat{i},\hat{\sigma}_i) = \hat{\sigma}'}$$

4.5.3 Soundness Proofs

Because we extend the signature of get with a matching mode, we proceed with the proofs for `get` and `set` by doing no assumption on the value of the mode m. Interestingly, we will see that *we cannot prove OX soundness* with $m = \texttt{NO\_MATCH}$, as it only holds for $m = \texttt{MATCH}$. This is central to the difference in behaviour between $\text{PMap}(\text{Loc}, \mathbb{S})$ and $\text{PMap}_{\text{ALoc}}(\mathbb{S})$.
Proof.

Proposition: Get OX Soundness
Assume

(H1) $\theta, s, \sigma \models \hat{\sigma} \land \texttt{get}(\sigma, i) = \sigma_i \land [\![\hat{i}]\!]_{\theta,s} = i$

To prove

(G1) $\exists \hat{\sigma}_i, \hat{i}', \pi.\, \texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \land \theta, s, \sigma_i \models \hat{\sigma}_i \land [\![\hat{i}']\!]_{\theta,s} = i \land \texttt{SAT}_{\theta,s}(\pi)$

We proceed by proving the property holds for all rules of the concrete `get`, resulting in three cases.

Case CPMapGetMatch**:**

(H2) Assume $(h, d) = \mathit{unwrap}(\sigma) \land i \in \text{dom}(h) \land \sigma_i = h(i)$.

(H3) It follows from (H1) and the definition of $\models$ that $\hat{\sigma} = (\hat{h}, \hat{d})$ such that $\exists \hat{i}'.\, \hat{i}' = \text{aloc}(a') \land a' \in \text{dom}(\hat{h})$, $[\![\hat{i}']\!]_{\theta,s} = i$ and $\theta, s, \sigma_i \models \hat{h}(a')$.

(H4) From here, multiple cases are possible: either $\texttt{to\_aloc}\ \hat{i} = a'$, or $\texttt{to\_aloc}\ \hat{i} = a$ such that $a \neq a' \land [\![\text{aloc}(a)]\!]_{\theta,s} = [\![\text{aloc}(a')]\!]_{\theta,s}$, or $\texttt{to\_aloc}\ \hat{i} = \bot$. Indeed, `to_aloc` is a best effort function, that may not find a match despite there being one; it can also be that $\hat{i}$ is a fresh symbolic variable, such that in the current path condition nothing binds it to $\text{aloc}(a)$.

(H5) In the first case, $\texttt{to\_aloc}\ \hat{i} = a'$ we can apply ALocPMapGetMatch, giving us our goal (G1).

(H6) In the second case, $\texttt{to\_aloc}\ \hat{i} = a \land a \neq a' \land [\![\text{aloc}(a)]\!]_{\theta,s} = [\![\text{aloc}(a')]\!]_{\theta,s}$, we may apply ALocPMapMatching, *but only when* $m = \texttt{MATCH}$. This gives our goal (G1). Otherwise, when $m = \texttt{NO\_MATCH}$, the only rule that suits is ALocPMapGetNoMatchNotFound, which gives us $\hat{\sigma}_i = \bot$, from which it follows that $\theta, s, \sigma_i \not\models \hat{\sigma}_i$.

(H7) In the last case, $\texttt{to\_aloc}\ \hat{i} = \bot$. Again, if $m = \texttt{MATCH}$, we can apply ALocPMapMatchingBot and get our goal (G1); otherwise, the only applicable rule is ALocPMapGetNoMatchNew, which invalidates our goal, as $\theta, s, \sigma_i \not\models \hat{\sigma}_i$.

Case CPMapGetAdd**:**

(H8) Assume $(h, d) = \mathit{unwrap}(\sigma) \land i \notin \text{dom}(h) \land d \neq \bot \land i \in d$.

(H9) From (H14), (H1) and CPMapGetAdd, we have $\sigma_i = \bot$.

(H10) It follows from (H1) and the definition of $\models$ that $\hat{\sigma} = (\hat{h}, \hat{d})$ such that $\hat{i} = \text{aloc}(a') \land \hat{i} \in \hat{d}$ and $a' \notin \text{dom}(\hat{h})$.

(H11) We again get three possible cases, depending on `to_aloc`: either $\texttt{to\_aloc}\ \hat{i} = a'$, or $\texttt{to\_aloc}\ \hat{i} = a$ such that $a \neq a' \wedge [\![\text{aloc}(a)]\!]_{\theta,s} = [\![\text{aloc}(a')]\!]_{\theta,s}$, or $\texttt{to\_aloc}\ \hat{i} = \bot$.

(H12) In the two first cases, from the definition of $\models$ and `to_aloc`, it still holds from (H10) that $a' \notin \hat{h}$ and $a \notin \hat{h}$, respectively. Depending on the mode m, we thus apply ALocPMapGetNoMatchNotFound or ALocPMapGetMatchNotFound, giving us $\hat{\sigma}_i = \bot$ and $\pi = [\hat{i} \in^? \hat{d}]$ or $\pi = [\hat{i} \notin \{\text{aloc}(a') : a' \in \text{dom}(\hat{h})\} \wedge \hat{i} \in^? \hat{d}]$, which from (H10) we know $\texttt{SAT}_{\theta,s}(\pi)$ in both cases, giving our goal (G1).

(H13) In the last case, we have $\texttt{to\_aloc}\ \hat{i} = \bot$, we thus apply ALocPMapGetNoMatchNew or ALocPMapGetMatchNew depending on m, giving us $a' = \texttt{fresh\_aloc ()}$, $\hat{\sigma}_i = \bot$, $\hat{i}' = \text{aloc}(a')$ and $\pi = [\hat{i} = \text{aloc}(a') \wedge \hat{i} \in^? \hat{d}]$ or $[\hat{i} = \text{aloc}(a') \wedge \hat{i} \notin \{\text{aloc}(a'') : a'' \in \text{dom}(\hat{h})\} \wedge \hat{i} \in^? \hat{d}]$. From (H9) and Empty Memory, we have $\theta, s, \sigma_i \models \hat{\sigma}_i$. Because a' is a fresh abstract location, it's equality to one term can always be satisfied; and from (H10) we know $\hat{i} \in^? \hat{d}$ holds – thus $\texttt{SAT}_{\theta,s}(\pi)$, completing our goal (G1).

Case CPMapGetBotDomain**:**

(H14) Assume $(h, d) = \mathit{unwrap}(\sigma) \wedge i \notin \text{dom}(h) \wedge d = \bot$.

(H15) From (H14), (H1) and CPMapGetBotDomain, we have $\sigma_i = \bot$.

(H16) It follows from (H1) and the definition of $\models$ that $\hat{\sigma} = (\hat{h}, \bot)$ such that $\hat{i} \notin \text{dom}(\hat{h})$.

(H17) The remainder of the proof is analogous to the above; three cases are possible depending on the result of $\texttt{to\_aloc}\ \hat{i}$ – in all three cases, we can apply one of the four [...]NotFound and [...]New rules, such that $\hat{\sigma}_i = \bot$, $[\![\hat{i}']\!]_{\theta,s} = i$ and $\texttt{SAT}_{\theta,s}(\pi)$, giving our goal (G1).

Proposition: Get UX Soundness
Assume

(H1) $\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \wedge [\![\hat{i}]\!]_{\theta,s} = [\![\hat{i}']\!]_{\theta,s} = i \wedge \texttt{SAT}_{\theta,s}(\pi)$

(H2) $\theta, s, \sigma \models \hat{\sigma}$

To prove

(G1) $\exists \sigma_i.\, \texttt{get}(\sigma, i) = \sigma_i \wedge \theta, s, \sigma_i \models \hat{\sigma}_i$

We proceed by proving the property holds for all rules of the symbolic `get`, resulting in five cases.

Case ALocPMapGetMatch**:**

(H3) Assume $(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \wedge a = \texttt{to\_aloc}\ \hat{i} \wedge a \neq \bot \wedge a \in \text{dom}(\hat{h}) \wedge \hat{\sigma}_i = \hat{h}(a)$

(H4) From (H3), (H1) and ALocPMapGetMatch, we have $\pi = []$

(H5) From (H3), we have $\hat{\sigma} \neq \bot$, thus from (H2) we have $\sigma = (h, d)$

(H6) From (H2), (H3) and the definition of $\models$, we have $i \in h$ such that $\sigma_i = h(i)$ and $\theta, s, \sigma_i \models \hat{\sigma}_i$

(H7) From (H6) we can apply CPMapGetMatch, thus $\texttt{get}(\sigma, i) = \sigma_i$. This completes our goal (G1).

Case ALocPMapGetNoMatchNotFound**:**

(H8) Assume $(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \wedge a = \texttt{to\_aloc}\ \hat{i} \wedge a \neq \bot \wedge a \notin \mathrm{dom}(\hat{h}) \wedge m = \texttt{NO\_MATCH}$

(H9) From (H21), (H1) and ALocPMapGetNoMatchNotFound, we have $\hat{\sigma}_i = \bot$ and $\pi = [\mathrm{aloc}(a) \in^? \hat{d}]$

(H10) Let $(h, d) = \mathit{unwrap}(\sigma)$.

(H11) From (H1) $\mathsf{SAT}_{\theta,s}(\pi)$, thus from (H22), either $d \neq \bot \wedge i \in d$, or $d = \bot$.

(H12) From this, we get two cases: either $i \in \mathrm{dom}(h)$, or $i \notin \mathrm{dom}(h)$.

(H13) If $i \notin h$, we can apply either CPMapGetAdd or CPMapGetBotDomain depending on d, in both cases giving us $\texttt{get}(\sigma, i) = \bot$. Together with Empty Memory, this completes our goal.

(H14) However, if $i \in \mathrm{dom}(h)$, we can only apply CPMapGetMatch, which however gives us $\texttt{get}(\sigma, i) = \sigma_i$ where $\sigma_i \neq \bot$ – this however *is not compatible with* $\hat{\sigma}_i$.

Case ALocPMapGetNoMatchNew**:**

(H15) Assume $(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \wedge a = \texttt{to\_aloc}\ \hat{i} \wedge a = \bot \wedge a' = \texttt{fresh\_aloc ()} \wedge m = \texttt{NO\_MATCH}$

(H16) From (H26), (H1) and ALocPMapGetNoMatchNew, we have $\hat{i}' = \mathrm{aloc}(a')$, $\hat{\sigma}_i = \bot$ and $\pi = [\hat{i} = \mathrm{aloc}(a') \wedge \hat{i} \in^? \hat{d}]$

(H17) Let $(h, d) = \mathit{unwrap}(\sigma)$.

(H18) From here, we get two cases: either $i \in \mathrm{dom}(h)$, or $i \notin \mathrm{dom}(h)$ – as indeed it may be the case that $[\![\hat{i}]\!]_{\theta,s} = i$, but that due to the nature of $\texttt{to\_aloc}$, $a \notin \hat{h}$.

(H19) If $i \notin h$, we can apply either CPMapGetAdd or CPMapGetBotDomain depending on d, in either cases giving us $\texttt{get}(\sigma, i) = \bot$. Together with Empty Memory, this completes our goal.

(H20) However, if $i \in \mathrm{dom}(h)$, we can only apply CPMapGetMatch, which however gives us $\texttt{get}(\sigma, i) = \sigma_i$ where $\sigma_i \neq \bot$ – this however *is not compatible with* $\hat{\sigma}_i$.

Case ALocPMapGetMatchNotFound**:**

(H21) Assume $(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \wedge a = \texttt{to\_aloc}\ \hat{i} \wedge a \neq \bot \wedge a \notin \mathrm{dom}(\hat{h}) \wedge m = \texttt{MATCH}$

(H22) From (H21), (H1) and ALocPMapGetNoMatchNotFound, we have $\hat{\sigma}_i = \bot$ and $\pi = [\hat{i} \notin \{\mathrm{aloc}(a') : a' \in \mathrm{dom}(\hat{h})\} \wedge \hat{i} \in^? \hat{d}]$

(H23) Let $(h, d) = \mathit{unwrap}(\sigma)$.

(H24) From (H1) $\mathsf{SAT}_{\theta,s}(\pi)$, thus from (H22), either $d \neq \bot \wedge i \in d$, or $d = \bot$. We also have, crucially, that $i \notin \mathrm{dom}(h)$.

(H25) We can thus apply either CPMapGetAdd or CPMapGetBotDomain depending on d, in both cases giving us $\texttt{get}(\sigma, i) = \bot$. Together with Empty Memory, this completes our goal.

Case ALocPMapGetMatchNew**:**

(H26) Assume $(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \wedge a = \texttt{to\_aloc}\ \hat{i} \wedge a = \bot \wedge a' = \texttt{fresh\_aloc ()} \wedge m = \texttt{MATCH}$

(H27) Let $(h, d) = \mathit{unwrap}(\sigma)$.

(H28) From (H26), (H1) and ALocPMapGetNoMatchNew, we have $\hat{i}' = \mathrm{aloc}(a')$, $\hat{\sigma}_i = \bot$ and $\pi = [\hat{i} = \mathrm{aloc}(a') \wedge \hat{i} \notin \{\mathrm{aloc}(a'') : a'' \in \mathrm{dom}(\hat{h})\} \wedge \hat{i} \in^? \hat{d}]$

(H29) From (H1) $\texttt{SAT}_{\theta,s}(\pi)$, thus from (H28), either $d \neq \bot \wedge i \in d$, or $d = \bot$. We also have, crucially, that $[\![\mathrm{aloc}(a')]\!]_{\theta,s} = i$, and thus $i \notin \mathrm{dom}(h)$.

(H30) From (H29), we can apply either CPMapGetAdd or CPMapGetBotDomain depending on d, in either cases giving us $\texttt{get}(\sigma, i) = \bot$. Together with Empty Memory, this completes our goal.

Case ALocPMapMatching**:**

(H31) Assume $(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \wedge a = \texttt{to\_aloc}\ \hat{i} \wedge a \neq \bot \wedge a \notin \mathrm{dom}(\hat{h}) \wedge a' \in \mathrm{dom}(\hat{h}) \wedge \hat{\sigma}_i = \hat{h}(a')$

(H32) From (H31), (H1) and ALocPMapMatching, we have $\hat{i}' = \mathrm{aloc}(a')$, $\hat{\sigma}_i = \hat{h}(a')$, $\pi = [\mathrm{aloc}(a) = \mathrm{aloc}(a')]$ and $m = \texttt{MATCH}$.

(H33) From (H2), (H31) and the definition of $\models$, we have $i \in h$ such that $\sigma_i = h(i)$ and $\theta, s, \sigma_i \models \hat{\sigma}_i$

(H34) From (H33) we can apply CPMapGetMatch, thus $\texttt{get}(\sigma, i) = \sigma_i$. This completes our goal (G1).

Case ALocPMapMatchingBot**:**

(H35) Assume $(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \wedge a = \texttt{to\_aloc}\ \hat{i} \wedge a = \bot \wedge a' \in \mathrm{dom}(\hat{h}) \wedge \hat{\sigma}_i = \hat{h}(a')$

(H36) From (H35), (H1) and ALocPMapMatchingBot, we have $\hat{i}' = \mathrm{aloc}(a')$, $\hat{\sigma}_i = \hat{h}(a')$, $\pi = [\hat{i} = \mathrm{aloc}(a')]$ and $m = \texttt{MATCH}$.

(H37) From (H2), (H35) and the definition of $\models$, we have $i \in h$ such that $\sigma_i = h(i)$ and $\theta, s, \sigma_i \models \hat{\sigma}_i$

(H38) From (H37) we can apply CPMapGetMatch, thus $\texttt{get}(\sigma, i) = \sigma_i$. This completes our goal (G1).

□

Chapter 5

Evaluation

5.1 Performance compared to Gillian monoliths

We first measure the performance of the stacks created using state model transformers against the performance of the original Gillian state models, that are built as large monoliths.

The evaluation is focused on testing the performance of state models across all modes supported by Gillian: Whole Program Symbolic Testing (WPST), OX Verification and UX Bi-Abduction. The files tested are either small tests used to test the engine, or larger verification targets extracted from real world code. Furthermore, when possible, the tests are also run with different optimisations of PMap: $\text{PMap}_{\text{ALoc}}$ and $\text{PMap}_{\text{Split}}$.

All test logs were verified, to ensure full parity between all versions: the instantiations built using state model transformers yield the same results as the original instantiations. All passing tests pass, and all failing tests fail for the same reasons.

All tests were run on a 2020 MacBook Pro, with an M1 processor and 8GB of memory.

This evaluation will be split among the three different stacks originally supported by Gillian. As this evaluation concerns itself with performance only, we will not delve into the details of how these different languages work.

- WISL: a simple language, based on a linear heap, used to teach students. It supports WPST, verification and bi-abduction. Its state model can be constructed trivially, with the stack:

$$\text{PMap}(\text{Loc}, \text{Freeable}(\text{List}(\text{Ex}(\text{Val}))))$$

- JavaScript: taking inspiration from JaVerT [12], [13], Gillian comes with an instantiation for ES5 JavaScript. It supports WPST and verification – bi-abduction ceased working due to changes during past developments, but could be fixed. It also comes with a WPST test suite to verify the Buckets-JS library. Its state can be modelled as:

$$\text{PMap}(\text{Loc}, \text{DynPMap}(\text{Str}, \text{Ex}(\text{Val})) \bowtie \text{Ag}(\text{Loc}))$$

- C: the third and last instantiation of Gillian supports a subset of C, using the CompCert-C verified compiler as an intermediary compilation step. It supports WPST, verification and bi-abduction. It comes with a WPST test suite to verify the Collections-C library, as well as a verification test suite for the AWS-Encryption-

SDK library. Its state model is built as:

$$\textsc{PMap}(\text{Loc}, \textsc{Freeable}(\textsc{BlockTree})) \bowtie \text{CGEnv}$$

Add links / citations to Buckets-JS, Collections-C, CompCert-C, AWS-Encryption-SDK.

5.1.1 WISL

To do

5.1.2 JavaScript

Originally the JavaScript instantiation of Gillian, Gillian-JS, was ported from JaVerT [12], [13]. We can reconstruct it as $\textsc{PMap}(\text{Loc}, \textsc{DynPMap}(\text{Str}, \textsc{Ex}(\text{Val})) \bowtie \textsc{Ag}(\text{Loc}))$. Dissecting this construction, we first have $\textsc{DynPMap}(\text{Str}, \textsc{Ex}(\text{Val}))$, representing an object in JavaScript: a map from keys to values. On the other side of the partial product we have $\textsc{Ag}(\text{Loc})$, corresponding to the location of the metadata of the object, in the same map. This allows to save some memory, as multiple objects can share metadata.

It comes with a few optimisations, to improve performance. Firstly, it splits the first level PMap entries between concrete and symbolic *values*, avoiding substitutions in the concrete part. Secondly, it uses the abstract location mechanism mentioned previously, to index all entries by strings rather than expressions. Finally, it uses the OCaml `Hashtbl` module, a mutable data structure, rather than the immutable `Map` module – this could improve performance, by avoiding creating copies of the map on modification. All of these optimisations are important, as the state in JavaScript code tends to be significantly large, with the first PMap regularly reaching 200 to 600 entries when running real life code (Buckets-JS, in this case) – the highest recorded map size reaching 1179 entries for the `set3.gil` file.

Because the splitting between concrete and symbolic entries only occurs on the values of the first map, we can broadly replicate it by applying the split optimisation to the second map instead. To test all combinations of optimisations, we thus get the four following transformer stacks:

$$\textsc{PMap}(\text{Loc}, \textsc{DynPMap}(\text{Str}, \textsc{Ex}(\text{Val})) \bowtie \textsc{Ag}(\text{Loc})) \quad \text{(TR)}$$

$$\textsc{PMap}_{\textsc{ALoc}}(\textsc{DynPMap}(\text{Str}, \textsc{Ex}(\text{Val})) \bowtie \textsc{Ag}(\text{Loc})) \quad \text{(TR-ALoc)}$$

$$\textsc{PMap}(\text{Loc}, \textsc{DynPMap}_{\textsc{Split}}(\text{Str}, \textsc{Ex}(\text{Val})) \bowtie \textsc{Ag}(\text{Loc})) \quad \text{(TR-Split)}$$

$$\textsc{PMap}_{\textsc{ALoc}}(\textsc{DynPMap}_{\textsc{Split}}(\text{Str}, \textsc{Ex}(\text{Val})) \bowtie \textsc{Ag}(\text{Loc})) \quad \text{(TR-ALocSplit)}$$

The last version, TR-ALocSplit, is the closest in terms of applied optimisations to what Gillian-JS does. We note that the optimisation that uses `Hashtbl` is not applied to transformers, as they all use immutable data structures. While this has not been measured, it doesn't seem to have a significant performance impact. The opposite may be true, as while benchmarking we note that some time is spent copying the state in Gillian-JS; with transformers, a copy is instant.

This benchmark is split into three parts: WPST, verification, and Buckets (in WPST mode), each made up of respectively 21, 6 and 78 files. All tests were then run 30 times, for each state transformer stack and for Gillian-JS (labelled "base"). The results can be seen in Figure 5.1. The first insight this give us is that transformers seem to overall outperform the

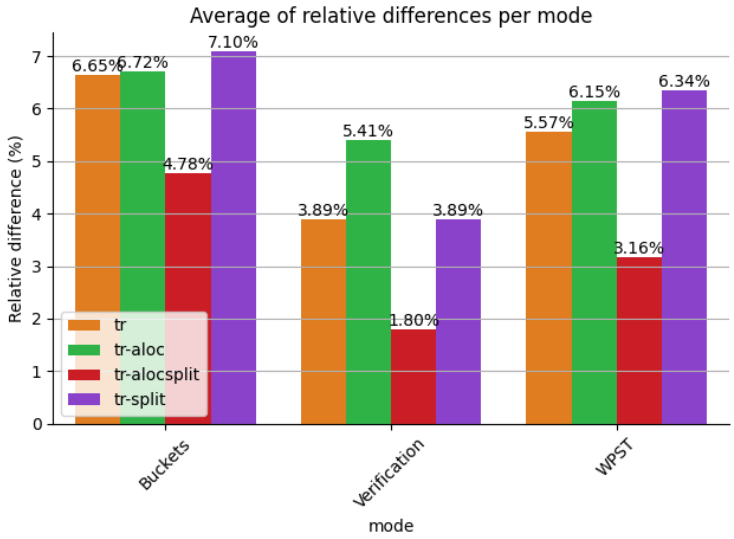

Figure 5.1: Average of the relative difference in execution time for different test suites, per transformer stack

monolithic Gillian-JS, with an improvement ranging from 1.8 to 7.1%. Another observation is that while both optimisations (ALoc and Split) seem to improve performance on the base instantiation, this improvement is dependent on what is the mode of execution, with ALoc being faster for verification, and Split being faster for WPST and Buckets (which is also WPST). This makes sense, as the split optimisation is only useful if there is a significant amount of concrete entries in the map, which there ought to be in whole program symbolic testing. We also note that the combination of the ALoc and Split optimisations seem to *decrease performance* compared to when used separately or not used at all. This would indicate that the cost of both optimisations (abstract location translating, and checking for concreteness, respectively) becomes too much compared to the benefits.

The initial conclusion we can reach from this is that transformer stacks are overall more performing than the monolithic alternative. One could of course imagine that a hyper-optimised monolith could be made with optimisations specific to the state model, which would then out-perform the more generic transformers. However in practice the size of such monoliths makes these optimisations hard to do, as code becomes complex quickly and makes changes harder – in contrast, transformers are very simple to optimise, as they maintain a more generic structure. They are also easier to prove to be sound, as proofs can be done on the smaller elements rather than on the full state model.

Another idea this experiment seems to confirm is that the improvement given by optimisations is highly dependent on the context in which the engine is used, as simply switching between OX and WPST means one optimisation is better than the other. Transformers thus allow users to tailor the optimisations they use in their stack according to what code is verified, and how, by empirically measuring the performance of different alternatives (which are trivial to construct).

We may also take a closer look at how this time is spent within the engine. This is done by measuring the time before and after the entry point of each exposed method of the memory model, and summing the time spent within it. By looking at the average time per function call in the Buckets test suite (see Figure 5.2), we note that most memory actions (prefixed with "ea/", shorthand for `execute_action`) are faster than in the base instantiation. Furthermore, and as mentioned previously, copying is orders of magnitude

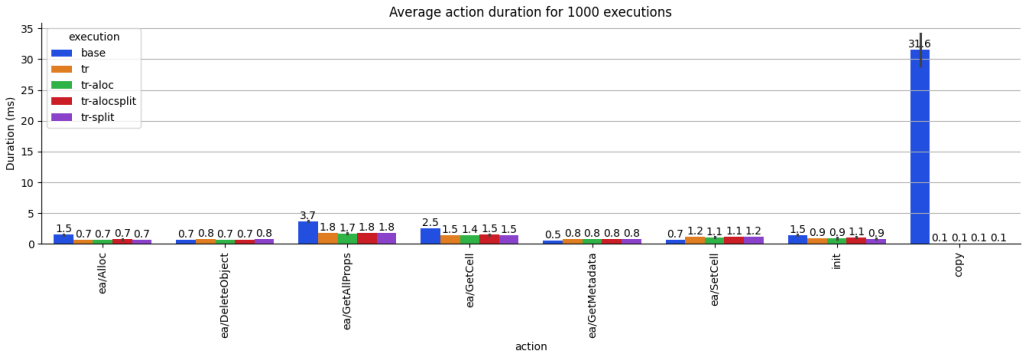

Figure 5.2: Average time spent per 1000 function calls in the Buckets test suite

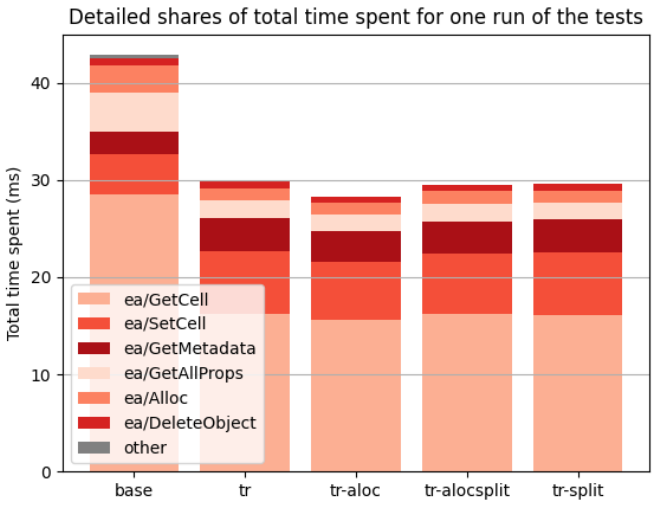

Figure 5.3: Average total time spent per function in an execution of the Buckets test suite

faster with transformers, since no work needs to be done.

Looking at the average total duration for an execution of the test suite however (see Figure 5.3), we note that the time taken by copying the state is minimal, as seen by the size of the "other" category. Instead, most time is spent during the `load` (`GetCell`) action, which is called 11400 times (more than twice as many times as `store`). The most notable improvement from the transformer construction is the reduction of the time spent getting a cell, reducing total time spent by 43% (from 28.55 to 16.18ms). Because `load` dominates the amount of action calls (see Figure 5.4), this is sufficient to make up for the performance loss in `store` and "`GetMetada`" (`load` on the right side of the product).

If, instead of focusing on WPST with the Buckets code we focus on verification, the painted picture is significantly different. Indeed, while in WPST the memory model is only used for memory actions (as only the core engine is used), verification exercises the memory model quite differently, notably with `produce` and `consume`.

Most notably, this shows us that one of the leading differences in total time between different transformer instantiations is substitutions: TR-ALoc and TR-ALocSplit spending more than half as much time during substitutions than TR and TR-Split. This can be explained by the fact that with the ALoc optimisation, substitutions for the keys (abstract locations) can be filtered, meaning that if there is no substitution concerning abstract locations, we can simply map the values of the map without concerning ourselves with key

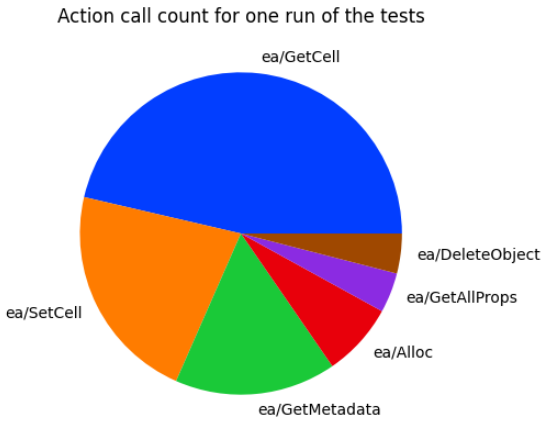

Figure 5.4: Share of function calls for one execution of the Buckets test suite

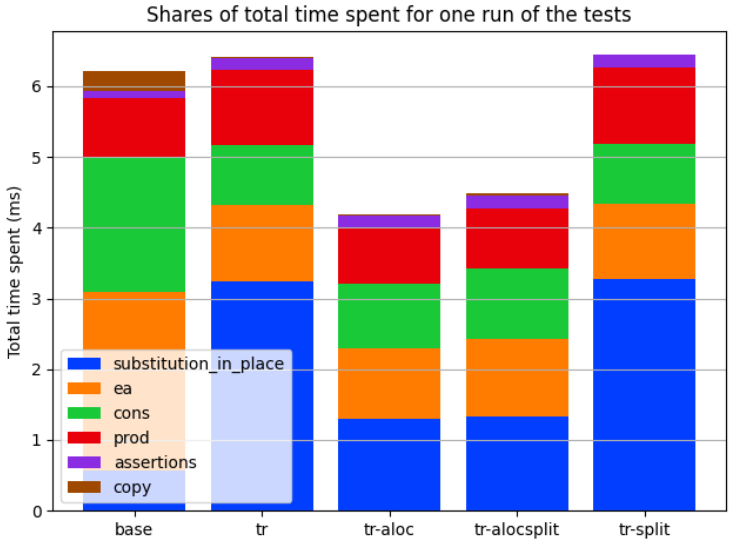

Figure 5.5: Share of grouped function calls for for one execution of the verification test suite

conflicts. Without this, substitutions need to be applied to both key and value, and then all resulting keys need to be compared (an expensive process) to compose clashing entries.

While this hasn't been measured, it is possible that the reason for the difference between time spent in substitutions between Gillian-JS and the transformer stacks is caused by the immutable data structures. With immutable data structures, copies of the full state are made for each substitution, which could be costly.

Substitutions aside, we note how significantly less time is spent in `execute_action` and `consume` calls compared to the base state model; about 58% and 55% respectively. Again, this seems to indicate that simpler transformers tend to be more efficient than large monoliths.

Finally, we may compare development effort between the two state models. While not a perfect measure of complexity, lines of code (LOCs) are used, to compare the amount of code needed to instantiate each state model to get equivalent results. We only measure the lines of code in implementation files (`.ml`), ignoring interface files (`.mli`). We also ignore comments and whitespace lines. The results are shown in Figure 5.6. Here, we note that the amount of code needed tailored for each instantiation, seen in yellow (this includes

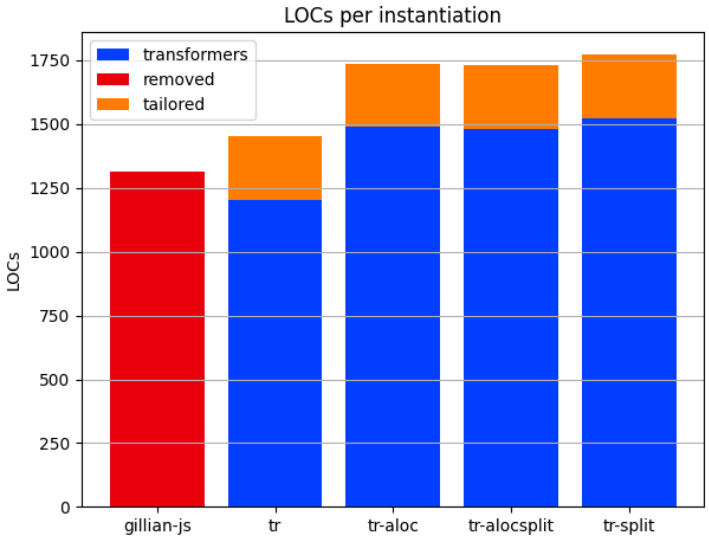

Figure 5.6: Number of lines of code per JS instantiation

constructing the stack, and applying necessary transformations, such as renaming actions and predicates, or adding a `delete` action) is minimal compared to Gillian-JS: only 246 LOCs. Most of the LOCs stem from the transformers code, which is shared and can be reused for different state models; a user of the engine would not need to worry about these. An engine developer is also likely to find it easier to work on the transformer code, as they are all independent from each other; unlike in the monolith where all elements have direct dependencies.

5.1.3 C

An instantiation of Gillian has been made for C, using the verified compiler CompCert-C [14]. It's memory model can be replicated as PMap(Loc, Freeable(BlockTree)) $\bowtie$ CGEnv. An important distinction from the JavaScript instantiation is that we can't replicate Gillian-C's behaviour only with the transformers introduced thus far. Indeed, C allows one to read parts of values (for instance, extracting two `int`s from a `long`), and Gillian-C also supports objects of symbolic size, which is not possible with List *why? Need to ask, I'm not sure*. Instead, we thus use the BlockTree state model, tailored for C, which supports the above uses. We will not go into details of how it works. Gillian-C also has a *global environment* system, which allows storing pointers to function definitions in an agreement-like map: we import this system in CGEnv.

As such, the Gillian instantiation for C using transformers uses a mix of generic transformers (PMap, Freeable) and of custom built state models (BlockTree). The BlockTree code is mostly imported from Gillian-C, with minor modifications to match the structure of state models used for transformers. In comparison, the instantiation for JavaScript only uses generic transformers. This shows the flexibility of our approach, as it allows both for constructions that only rely on state model transformers, as well as hybrid constructions using specialised elements along with the generic ones.

In an effort to verify that the improvements from the optimised PMap versions are

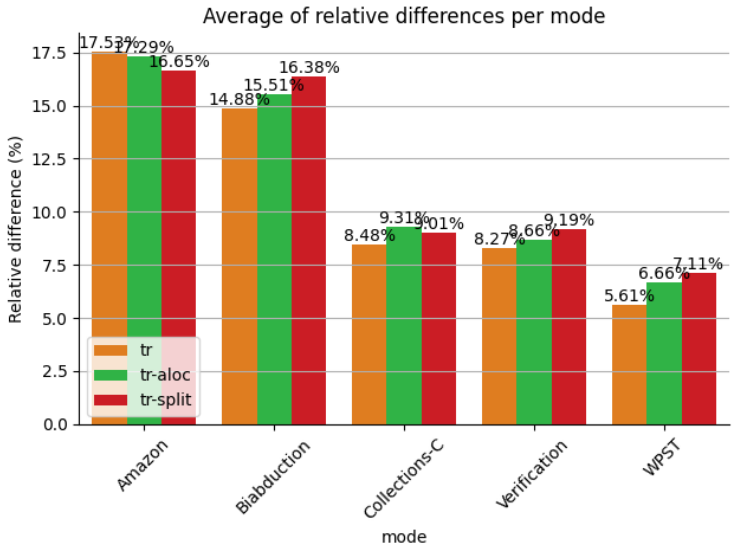

Figure 5.7: Average of the relative difference in execution time for different test suites, per transformer stack

carried between languages, we also provide several C instantiations:

$$\textsc{PMap}(\text{Loc}, \textsc{Freeable}(\textsc{BlockTree})) \bowtie \textsc{CGEnv} \qquad \text{(TR)}$$

$$\textsc{PMap}_{\textsc{ALoc}}(\textsc{Freeable}(\textsc{BlockTree})) \bowtie \textsc{CGEnv} \qquad \text{(TR-ALoc)}$$

$$\textsc{PMap}_{\textsc{Split}}(\text{Loc}, \textsc{Freeable}(\textsc{BlockTree})) \bowtie \textsc{CGEnv} \qquad \text{(TR-Split)}$$

Gillian-C supports WPST, verification and bi-abduction with UX reasoning. It uses an abstract location optimisation, similarly to what was described previously, and immutable data structures (unlike Gillian-JS's use of `Hashtbl`).

This benchmark is split into five parts: WPST, verification, bi-abduction, Collections-C (in WPST mode) and AWS code (in verification mode), each made up of respectively 8, 6, 7, 159 and 10 files. All tests were run 50 times, except the AWS test suite that was executed 10 times[1].

The general results can be seen in Figure 5.7. Here again, we note a significant performance improvement compared to the monoliths in Gillian-C, in particular for the AWS and bi-abduction suites. As for the optimisations, we get inconsistent results: they seem to improve performance for all modes but for AWS.

An interesting observation is that transformers seem to still yield a better performance even when mixed with more complex state models, where most of the implementation still resides in large complex elements.

We may now look into the detailed rundown of the time spent; we will focus on Collections-C WPST and the AWS encryption SDK verification, as these correspond to real world code (whereas the other test suites only use simple data structures, and are primarily useful for ensuring parity).

When looking at the time spent in each function for each instantiation (see Figure 5.8), the main takeaway is that two most common actions, `mem_store` and `mem_load`, are both faster than the original version; about 20% and 25% respectively. This is significant, considering more than 75% of memory actions in Collections-C is a load or a store.

[1]This is simply because verifying the AWS code takes *significantly* longer than the rest.

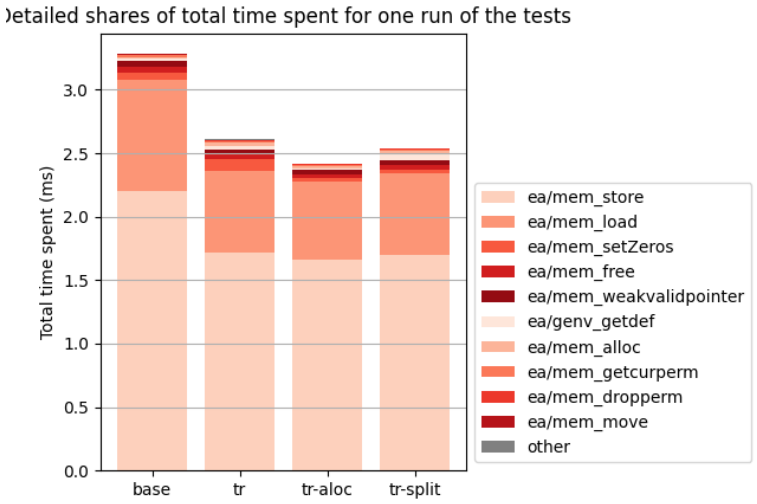

Figure 5.8: Average total time spent per function in an execution of the Collections-C test suite

This story is however wildly different for AWS code, which uses verification rather than WPST. Here, memory actions only represent a fraction of the total time spent, which is instead dominated by mainly `consume`, as well as `produce`, as seen in Figure 5.9. Here we note slight improvements in time for most categories, again seeming to indicate that the more generic approach is generally more performant.

Importantly, the charts for TR, TR-ALoc and TR-Split are extremely similar, for both Collections-C and AWS code; we can thurs hardly make a hypothesis as to which optimisation is better suited. This is in part due to the fact these optimisations excel in larger maps, as was the case for JavaScript. In C, maps are instead rather small; for Collections-C, the map size tends to fluctuate between 15 and 35 elements, with the maximum recorder reaching 52 elements. For AWS, its size is between 10 and 20 elements, maxing at 20. This in particular explains why the optimisations are detrimental for the AWS test suite: because the maps are always small, the cost of both optimisations regularly outweighs their improvement.

We now compare the complexity of instantiations, using LOCs. Here, we delve into more details on the reason for each line count. In particular, we see that the majority of the state model is the same; the modified part representing BlockTree and CGEnv, which are lightly modified to fit into the transformers setup, while the untouched part represents the internal modules used by the C instantiations (for instance, to encode permissions, or memory chunks). Finally, the removed part of the code is trivially replaced with minimal constructions, using PMap and Freeable. Some tailored code is also required, to ensure the construction matches the interface of Gillian-C. This again shows that even for more complex state models that require a significant amount of customised code, using transformers reduces the amount of code required exclusively for that instantiation – since some can be in a shared transformers library – while providing some performance gains for free.

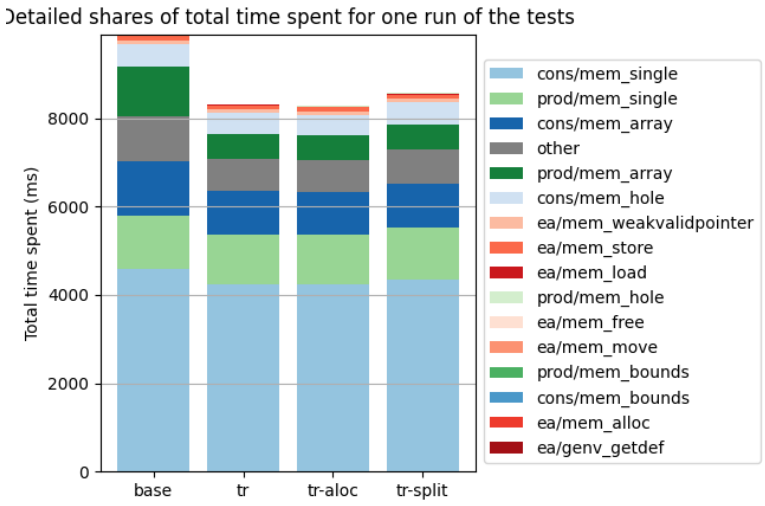

Figure 5.9: Average total time spent per function in an execution of the AWS header encryption SDK test suite

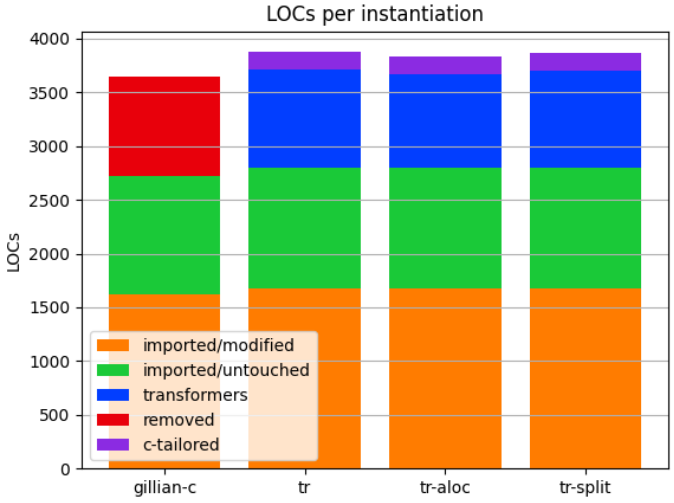

Figure 5.10: Number of lines of code per C instantiation

Chapter 6

// TODO:

Priority	Name
0	~~**Redefine state models**: Ex, Ag, Frac, PMap, DynPMap, List, GMap, Freeable, Sum, Product~~
0	~~**Define optimised state models**: ALocPMap, SplitPMap~~
0	**Proof of equivalence**: ALocPMap, SplitPMap
1	**Define stacks**: WISL, JS, C
1	**Write comparative evaluation**: WISL, ~~JS, C~~
1	**Write proofs**: ~~Ex~~, Ag, Frac, ~~PMap~~, List, GMap, Freeable, Sum, Product, DynPMap
2	**Results table**: make a large table for appendix, with for each tested file: mode, number of procedures, number of GIL commands, Gillian time to verify, instantiation time to verify
2	**Soudness of JS construction**
2	**Write absolute evaluation**: PMap perf according to size, split PMap split rate
3	**Define ea/consume/produce for** $\bot$: rules, proofs?
3	**Add `LFail` and `Miss` to consume**: rules, proofs?
4	**Redefine fixes as purely assertions**: signature, biabduction rules, proof of soundness

Small things to do/fix:

- ~~Mention that CSE2 has *SV*, but keep it implicit here.~~
- ~~Remove notion of compatibility / update: maybe Sacha has a fix? / update: not really~~
- Explain the difference between `LFail` and `Err`, maybe with that Venn diagram
- Give an example where `Miss` is relevant for `consume`.
- Explain what core predicates are – relevant: [15]
- Mention the `emp` "core predicate", if it has a relation with intuistic vs classic SL.
- Mention why we can't just lift `execute_action` (actions on $\bot$ don't only yield $\bot$).
- Figure out if JSIL is wrong with Ag for metadata (as it makes `delete` UX unsound) – replace with Frac without `store`? See [16]
- Explain/understand why we can't prove soundness of C construction like we can for JavaScript. Is there something about C that makes it incompatible with our nice proofs?

- Give example of why the sum is hard to define with PCMs
- Mention why although we can define the core as $M \xrightarrow{fin} M$, it's more pleasing to have $M \mapsto M^?$
- Find the tests in C that are consistently slower, emit hypothesis and/or find why and explain
- Mention the example of boolean formula state models, where $|a| \cdot |b| \preccurlyeq |a \cdot b|$
- For every state model transformer make it clear the different wrt PCMs
- Give an example of how ALocPMap works
- Explain somewhere that symbolic states don't need to have composition defined, and explain why (symbolic block tree composition may yield >1 results)
- Include quote from that Iris paper to explain why we need a domainset in PMap.
- Look into "Nominal Logic", as a better solution to ALocs, see [17]
- Mention the gap between theory and practice to jutify ALocs
- Maybe mention core predicates are atoms for constructing predicates etc.
- Maybe add diagram to show that when we call a function, we consume pre and produce post, but when verifying that function we produce pre and consume post.

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