Adding and Optimising
Resource Algebras for a Parametric CSE

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Abstract

Compositional symbolic execution is a technique allowing for the scalable verification of code, using separation logic. Gillian is a compositional symbolic execution engine that has the unique property of being parametric on the state model, allowing verification of any language for which a state model can be defined.

Abstract separation logic is a branch of separation logic that allows reasoning about an abstract state. Iris is a state of the art theory in the field that introduces the notion of resource algebras, enabling the construction of sound complex state from simple elements.

Previous work introduces state model constructions in Gillian; though these transformers were only described on paper. Furthermore, they were formalised using partial commutative monoids, limiting their applicability and causing soundness issues. Following from such works, we show how resource algebras can be adapted into an abstract verified CSE engine, greatly facilitating the construction of sound state models. We then define a wide range of different state model transformers that can be used to assemble state models that carry the desired soundness properties by construction. We also prove at the theory level that some optimisations of these transformers are sound.

We implement the defined state model transformers, creating a library of easy to use modules and functors that can be combined to model real world languages in Gillian, while being verifiably sound. We instantiate Gillian to JavaScript, C, and WISL using them, greatly reducing developer effort.

The instantiations built with our library are evaluated and compared to previous instantiations of Gillian, showing a clear performance improvement for JavaScript and C. This allows us to conclude that our library enables a developer to instantiate Gillian to real world languages more easily, with less code, and with exceptional performance – all the while guaranteeing soundness.

Acknowledgements

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Chapter 1

Introduction

1.1 Motivation

As software becomes increasingly central to critical infrastructure, the need to formally prove its correctness increases. Symbolic execution tools have been developed to allow for the verification of programs automatically, though with poor scalability [4].

A framework that enables scalable program verification is *separation logic* [53, 59], by allowing one to reason about fragments of the state. Properties are shown to hold for a part of memory, and from this any disjoint memory can be composed with it while guaranteeing the properties still hold.

Compositional symbolic execution (CSE) uses separation logic to allow for the verification of functions independently from the rest of the code, improving drastically scalability [1] and allowing for its adoption in Continuous Integration pipelines [40].

A variety of CSE tools exist to verify different code written in different languages, such as C [14, 32, 40], Java [14, 21, 32, 40], JavaScript [29] or intermediate languages [11, 50, 60]. For this project we focus on Gillian [27, 46]; it has the unique property of being *parametric on the state model*, allowing it to be used to verify any language, if its state model is provided by the developer. A state model is a partial commutative monoid (PCM), endowed with actions, core predicates, and functions operating on it to modify it. It supports over-approximate and under-approximate reasoning, using separation logic and incorrectness separation logic, respectively. Gillian has succesfuly been instantiated to JavaScript, C and Rust [3, 60], enabling code verification, true bug finding, and whole program symbolic testing. These instantiations are monolithic, implemented as one tightly coupled piece of code, and make use of specific optimisations to improve performance, which are never justified in the theory.

From Gillian stemmed research on an abstract CSE engine [44, 45]; it defines at the theory level what properties must be satisfied by a state model for the engine to be sound, and provides a mechanised proof of the soundness of the engine.

While separation logic reasons about a specific memory model, the most typical example of which being a linear heap, *abstract* separation logic is a branch of separation logic that allows one to reason about properties of the memory independently of its actual representation [8, 16]. An outstanding logic in this field is Iris [35], a certified logic that is implemented in Rocq (formerly Coq) [18] and which proved itself to be capable of enabling reasoning for a wide range of applications. In particular, Iris introduces the *resource algebra* (RA), an algebraic structure that allows *building up more complex RAs from simpler elements* while carrying the soundness properties of the underlying elements. These logics have however been limited to manual pen and paper proofs, or mechanisations with Rocq, both requiring extensive technical and theoretical knowledge.

Gillian is particularly well suited for this concept of constructing complex state from simpler elements: because it is parametric on the state model, it can already technically support any construction provided

to it. In [2], the idea of constructing state models from simpler elements was thus ported to Gillian, however using PCMs, the traditional choice of structure for abstract separation logic [8, 16, 22, 36]. Only definitions were given however, with no usable implementation alongside it. Furthermore, the use of PCMs limited what could be done with these constructions, and introduced some soundness issues. More specifically, PCMs enforce a unit, which can become problematic when considering frame preserving actions and some particular constructions.

We are interested in bridging the gap between Iris and Gillian with regards to state model constructions, by replacing the PCMs used with a structure more akin to Iris's resource algebras. This should simplify the proof of soundness for state model transformers, while bringing the theory of an abstract CSE engine closer to the theory developed in Iris. In particular we shall interest ourselves on what changes need to be applied to this abstract CSE engine to function with RAs.

We shall then revisit the state model and state model transformer definitions that were first defined in [2] and that were ported from Iris [35], bringing them back to their original RA representation while adapting the additional functions that make them state models to use RAs. Furthermore, we are interested in seeing if we can bring the optimisations that already exist in monolithic Gillian instantiations to our state model transformers *within the theory*, proving their soundness, which has never been attempted before in this context.

While having sound constructions is essential, Gillian is before all a tool that should be used. We must thus see if the state model definitions we establish in the theory and prove the soundness of can be transferred to Gillian with sufficiently good performance, while permitting the construction of state models that can be used to verify real world languages. We are also interested in seeing how we may reproduce the state models for C and JavaScript using state model transformers, allowing for sound constructions without the burden of proving this soundness.

1.2   Contributions

The main contribution of this project is the adoption of resource algebras for the construction of state models that are compatible with CSE engines.

We define at the theory level seven different state models and state model transformers, which are all proven to be sound. These can then be combined seamlessly to build up complex state models that are sound by construction.

We also define in the theory three optimisations of the partial map transformer that originally only existed in the implementation of Gillian, proving the soundness of two of them.

We then implement these state models along with additional ones in OCaml, creating a state model library compatible with the CSE engine Gillian. Using this library, we instantiate Gillian to allow for the verification of code written in WISL, JavaScript and C. These instantiations have full parity with the pre-existing instantiations of Gillian, providing the same verification results with a fraction of the code needed for the instantiation.

Finally, we measure in great detail the performance of the instantiations and optimisations described, comparing them to the performance of the monolithic instantiations; we show that our constructions are, for C and JavaScript, more performant, while requiring significantly less code and proof effort.

1.3   Outline

In chapter 2 we introduce the necessary theoretical background needed for the project; we go over separation logic and its variations, with incorrectness separation logic and abstract separation logic. In particular we retrace the history of abstract separation logic and its evolution, leading to the creation

of Iris. We then describe Gillian, a compositional symbolic execution engine parametric on the memory model. We list existing tools that enable symbolic execution and their differences. Finally, we briefly cover additional requirements relevant to this research area.

In chapter 3 we motivate the transition from partial commutative monoids to resource algebras. We then go through the changes brought to our abstract CSE engine to accommodate the change, and define three different state models and four different state model transformers. We also cover optimisations at the theoretical level for the partial map transformer.

In chapter 4 we describe the implementation of these state models as a library, and how our theoretical constructs interface with the pre-existing definitions of Gillian. We also cover how a developer may use this library to instantiate Gillian with the desired state model.

In chapter 5 we evaluate the work done so far, discussing first the usability of the theory and of the library. We then compare the performance of our instantiations with the reference Gillian instantiations. We also briefly observe the performance improvements offered by our partial map optimisations.

In chapter 6 we conclude this report, retracing the work done, and outlining the key achievements, the relevant limitations, and give different directions to build upon what is defined here, for future work.

Chapter 2

Background

In this chapter we introduce some of the core ideas used throughout this document, while mentioning pre-existing work in the field. In §2.1 we introduce Separation Logic, its variations, and the idea of separation algebras; in §2.2 we introduce Gillian, a CSE engine; in §2.3 we mention other similar verification tools; finally, in §2.4 we evoke the legal, social, ethical and professional impacts of this work.

2.1 Separation Logic

As the need to formally verify programs grew, methods needed to be built to provide a framework to do so. Hoare Logic [31] allows proving properties of programs, by describing them axiomatically. Every statement can be expressed as having a precondition – the state before execution – and a postcondition – the state after *successful* execution – expressed as $\{P\} \, \mathsf{S} \, \{Q\}$. For assignment for instance, one could have $\{\mathsf{x} = x\} \, \mathsf{x} \ := \ \mathsf{0} \, \{\mathsf{x} = 0\}$.

While already helpful to reason about programs, an issue remains: shared mutable state. How does one describe that there is a list in memory of unknown length, that may be mutated at multiple places in the program, while ensuring properties hold at a specific point in time? Being able to describe the state and constraints upheld by global state is difficult, and past solutions scaled poorly.

Separation Logic (SL) [53, 59] is an extension of Hoare logic that permits this in a clear and scalable way. Its main addition is the separating conjunction $*$: $P * Q$ means not only that the heap satisfies P and Q, but that it can be split into two *disjoint* parts, such that one satisfies P and the other Q. This allows us to reason compositionally about the state, by not only stating what properties are upheld, but how they may be split for further proofs. For instance, given a *list* predicate, when calling a function that mutates the list one can simply substitute the part of the state corresponding to that list with the postcondition of the function, with the guarantee that the rest of the state is untouched, by using the *frame rule*:

$$\frac{\textsc{Frame}}{\{P\} \, \mathsf{S} \, \{Q\}}{\{P * F\} \, \mathsf{S} \, \{Q * F\}}$$

Because we know that S only uses (either by reading or modifying) P, any disjoint frame F can be added to the state without altering the execution of S. This is very powerful: one can prove properties of smaller parts of code (like a function, or a loop), and these properties will be able to be carried to a different context that may have a more complex state. For instance, this can be used in a Continuous Integration (CI) setting, by only analysing functions of which code is modified, while reusing past analyses of unchanged code, allowing for incremental analysis.

SL also comes equipped with the *emp* predicate, representing an empty state (i.e. $P * emp = P$), and the separating implication $-\!*$. $P_0 -\!* P_1$ states that if the current state is extended with a disjoint part satisfying P_0, then P_1 holds in the extended heap [59].

Further *spatial* assertions can then be defined; for instance the "points to" predicate, $a \mapsto x$, stating that the address a stores values x. It then follows that $a \mapsto x * a \mapsto x$ does not hold, since we cannot have two disjoint states both containing the address a. Similarly, $a \mapsto x * b \mapsto y$ only holds if $a \neq b$.

2.1.1 Incorrectness Separation Logic

Separation Logic is, by definition, *over-approximate* (OX): $\{P\} \, \mathsf{S} \, \{Q\}$ means that given a precondition P, we are guaranteed to reach a state that satisfies Q. In other words, Q may encompass *more* than only the states reachable from P. This can become an issue when used for real-world code, where errors that cannot occur are detected and flag the code as incorrect, hindering the use of a verification tool (for instance in a CI setting).

To solve this problem, a recent innovation in the field is Incorrectness Separation Logic (ISL) [58], derived from Incorrectness Logic [54]. It is the *under-approximate* (UX) equivalent to SL, instead ensuring that the detected errors actually exist, at the cost of possibly missing some executions of the code.

Where SL uses triplets of the form $\{$precondition$\} \, \mathsf{S} \, \{$postcondition$\}$, incorrectness separation logic uses $[$presumption$] \, \mathsf{S} \, [$result$]$, with the result being an under-approximation of the actual result of the code. The reasoning is thus flipped, where we start from a stronger assertion at the end of the function, and then step back and broaden our assumptions, until reaching the initial presumption. This means that all paths explored this way are guaranteed to exist, but may not encompass all possible paths. This enables us to do *true bug-finding.*

Another way of comparing SL to ISL is with consequence: in SL, the precondition implies the postcondition, whereas with ISL the result implies the presumption. Furthermore, ISL triplets are extended with an *outcome* ϵ, resulting in $[P] \, \mathsf{S} \, [\epsilon : Q]$, where ϵ is the outcome of the function, for instance Ok, or an error.

While SL is over-approximate and ISL is under-approximate, we may call *exact* (EX) specifications that are both OX-sound and UX-sound [47]. Such triplets are then written $\langle\!\langle P \rangle\!\rangle \, \mathsf{S} \, \langle\!\langle o : Q \rangle\!\rangle$, with o the outcome, and P and Q the pre and postcondition respectively.

2.1.2 Separation Algebras

While the above examples of separation logic use a simple *abstract heap* mapping locations to values, we often need more complex state representations to verify real-world programming languages. For instance, the model used by the C language (in particular CompCert C [41], a verified C compiler) uses the notion of memory blocks and offsets: memory is more than just an assortment of different atomic cells. Furthermore, the basic "points to" predicate, while useful, has limited applications; for contexts such as concurrency, one needs to have a more precise level of sharing that goes beyond the *exclusive ownership* of "points to".

An example of such extension is fractional permissions [9, 10]: the "points to" predicate is extended with a *permission*, a fraction q in the $(0; 1]$ range, written $a \underset{q}{\mapsto} x$. A permission of 1 gives read and write permission, while anything lower only gives read permission. This allows one to split permissions, for instance $a \underset{1}{\mapsto} x$ is equivalent to $a \underset{0.5}{\longmapsto} x * a \underset{0.5}{\longmapsto} x$, a program can thus concurrently execute two routines that read the same part of the state, while remaining provably sound – permissions can then be summed back as the routines exit, regaining full permissions over the cell.

Further changes and improvements to separation logic have been made, usually with the aim of adapting a particular language feature or mechanism that could not be expressed otherwise. "The Next 700 Separation Logics" [55] argues that indeed too many subtly different separation logics are being created

to accommodate specific challenges, which in turn require new soundness proofs that are incompatible with eachother [17]. To tackle this, research has emerged around the idea of providing one sound metatheory, that would provide the tools necessary to building more complex abstractions *within* that logic, rather than in parallel to it.

A first step in this quest towards a single core logic is the definition of an abstraction over the state. The main concept introduced for this is that of *separation algebras* [16, 22], which allow for the *construction* of complex separation algebras from simpler elements. Because soundness is proven for these smaller elements, constructions made using them also carry this soundness, and alleviate users of complicated proofs.

Different authors used different definitions and axioms to define separation algebras. In [16] they are initially defined as a *cancellative* partial commutative monoid (PCM) $(\Sigma, \cdot, 0)$. This definition is however too strong: for instance, the cancellativity property implies $\sigma \cdot \sigma' = \sigma \Rightarrow \sigma' = 0$. While this is true for some cases such as the linear heap where $[a \mapsto x] \cdot \sigma' = [a \mapsto x]$ implies σ' is the empty heap, this would invalidate useful constructions such as that of an *agreement*, where knowledge can be duplicated. For an agreement separation algebra, we can have $\sigma \neq 0$ and $\sigma \cdot \sigma = \sigma$, which dissatisfies cancellativity.

This approach is also taken in [22], where separation algebras are defined with a functional ternary relation $J(x, y, z)$ written $x \oplus y = z$ – the choice of using a relation rather than a partial function being due to the authors wanting to construct their models in Rocq (formerly Coq) [18], which only supports computable total functions. While the distinction is mostly syntactical and both relational and functional approaches are equivalent as long at the relation is functional, the former makes proof-work less practical when working with partial functions [57], whereas the functional approach allows one to reason equationally [35]. This paper also introduces the notion of *multi-unit separation algebras*, separation algebras that do not have a single unit 0, but rather enforce that every state has a unit, that can be different from other states' units. Formally, this means that instead of having $\exists u. \forall x. x \oplus u = x$, we have $\forall x. \exists u_x. x \oplus u_x = x$ – this allows one to properly define the disjoint union (or sum) of separation algebras, with both sides of the sum having a distinct unit. We may note that this is a first distinction from PCMs, as a partial commutative monoid cannot have two distinct units; according to the above definition, separation algebras are rather a type of partial commutative semigroup.

In [57], further improvements are done to the axiomatisation of separation algebras. Most notably, the property of cancellativity is removed, as it is too strong and unpractical for some cases, as shown. Furthermore, the idea of unit, or *core*, is made explicit, with the definition of the total function $\hat{-}$, the core of a resource, which is its *duplicable part*. A state can have non-duplicable information, like what value is stored in what cells, as well as duplicable information; for instance an invariant about some data structure, or the value of an immutable variable.

Finally, Iris [34–36, 38] is a state of the art "higher-order concurrent separation logic". It places itself as a solution to the problem mentioned in [55], and aims to provide a sound base logic that can be reused and extended to fit all needs. It defines *resource algebras* (RAs), objects that are similar to separation algebras. Because Iris supports "higher-order ghost state", more sophisticated mechanisms are needed to model state, and as such the composition function $\cdot$ needs to be total. To rule out invalid compositions, a validity function $\overline{\mathcal{V}}$ is instead used, which returns whether a given state is valid. The core function, written $|-|$, is made *partial*, rather than total – this, they argue, makes constructing state models from smaller components easier. This was confirmed when developing state models for Gillian, where having the core be a total function made some state models more complicated to make sound, as will be shown later. This is the algebraic structure we will use when defining our state models.

2.2 Program Verification with Gillian

While having a logic to prove the correctness of programs is a great step towards verifying codebases, doing it manually is tedious and time-consuming. Tools have been developed to automate this, such as *Gillian* [27, 46, 60]. It is a *compositional symbolic execution engine*, with the added property of being *parametric on the state model*. It allows reasoning about *correctness and incorrectness*. We may now go over exactly what this means.

2.2.1 Symbolic Execution

Traditionally, software is tested by calling the code with a predetermined or randomly generated input (for instance with fuzzing), and verifying that the output is as expected. These approaches, where the code is executed directly, are in the realm of *concrete* execution, as the code is run with concrete (i.e. real, existing) values. While this method is straightforward to execute, it comes with flaws: it is limited by the imagination of the person responsible for writing the tests when written manually, or by the probability of a given input to reach a specific part of the codebase when generated. With fuzzing, methods exist to improve the odds of finding new paths [5], but it all amounts to luck nevertheless.

A solution to this is symbolic execution: rather than running the code with concrete values, *symbolic* values are used [4]. These values are abstract, and are restricted as an interpreter steps through the code and conditional statements are encountered. Once a branch of the code terminates, a constraint or SMT (Satisfiability Modulo Theories) solver can be used against the accumulated constraints to obtain a possible concrete value, called an *interpretation*. This method appeared in the 70s, notably with the Select [12] and EFFIGY [37] systems.

For a simple program that checks if a given number is positive or negative and even or odd (see Figure 2.1), symbolic execution would thus branch thrice, once at each condition.[1] This would then result in 5 different branches, each with different constraints on the program variable x: $x = 0$, $x\%2 = 0 \land x < 0$, $x\%2 = 0 \land x > 0$, $x\%2 \neq 0 \land x < 0$ and $x\%2 \neq 0 \land x > 0$.[2]

```
1   if (x == 0) {
2     print("zero");
3     return;
4   }
5   if (x % 2 == 0) {
6     print("even");
7   } else {
8     print("odd");
9   }
10  if (x < 0) {
11    print("negative");
12  } else {
13    print("positive");
14  }
```

(a) Simple branching program
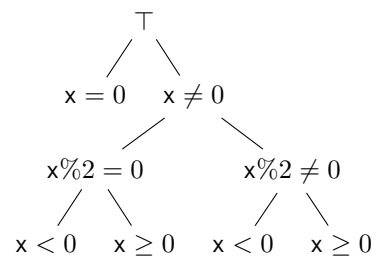
(b) Paths obtained from symbolically executing the code, with the added constraint at each node

Figure 2.1: Symbolic execution of a simple program

2.2.2 Compositional Symbolic Execution

An issue that arises from symbolic execution is that is scales quite poorly because of path explosion [1, 4], as each branching can potentially multiply by two the total number of branches. Furthermore, it lacks compositionality, as the code must be tested in its entirety, from `main` to `return`.

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[1]For simplicity, we omit the case where x is not an integer, which would lead to an error.

[2]The four last constraints also contain $x \neq 0$, which is included in $x > 0$ and $x < 0$

A solution to this is *compositional* symbolic execution (CSE), in which the code is tested by fragments, function by function. This allows for gradual adoption, as only parts of the code base can be tested. This minimises the effect of path explosion, since it limits the amount of branching that may occur.

A great tool to enable this is separation logic: one can specify the precondition and postcondition of a function, and the symbolic execution can then step through the code, by starting from a state satisfying the precondition and ensuring that once the branch terminates the postcondition is satisfied. Because of the frame rule, function specifications can then be re-used when calling functions, by symbolically executing the function specification on the corresponding state fragment. If the called function does not have a specification that can be used, it can instead be inlined, executing the code within the function, as would standard symbolic execution.

This approach was pioneered in Smallfoot [6], and was followed by later tools, such as JaVerT [29, 30], Infer [14], VeriFast [32], SpaceInvaders [62] and others – their individual contributions will be discussed in more detail in the next section.

2.2.3 Parametricity

An innovation of Gillian compared to other existing tools is the fact is is parametric on the memory model (also called *state model*).[3] Typical CSE engines usually come with a built-in notion of memory, ranging from simple heaps with memory chunks [32] to more complex state models tailored to a language, like in JaVerT [29, 30] to represent JavaScript objects.

While a fixed state model is useful for specific languages or feature sets, this can become a limitation when porting the engine to other languages. For instance, the Viper infrastructure is only capable of verifying statically typed Python code [24], despite its powerful generic state model. The goal behind Gillian's parametricity is to allow it to be used for virtually any language, as long as the language's state model can be encoded into a separation logic equivalent; improvements done to the engine can then be carried to all of its instantiations for free. This enables it to verify programs written in JavaScript, C, or Rust [3, 60], despite their significant differences in terms of state representation and level of abstraction.

To define a memory model to use in Gillian, a developer must implement it in OCaml [42]. A memory model is a partial commutative monoid, comprised of a set of states, which can be modified via *actions* and be represented in specifications via *core predicates*.

Actions serve as the interface for programs to interact with the state, for instance by storing or loading values into the state, or allocating a new block of memory and then freeing it. The language Gillian targets, GIL (Gillian's Intermediate Language), provides the base semantics of an imperative goto-language (variables, errors, functions...) which do not affect the state – actions are the only way of manipulating it.

Core predicates are predicates which have in and out parameters, and represent fragments of the state. They are the encoding of the state into separation logic, and can be used within function specifications. For example, the separation logic assertion $a \mapsto x$ could be represented as a core predicate: $\langle \texttt{points\_to} \rangle (a; x)$.

2.2.4 Correctness, Incorrectness

Another key difference between Gillian and other CSE engines is that it is capable of reasoning about programs both with correctness, in SL, and incorrectness, in ISL, thus enabling OX program verification and UX true bug-finding respectively. It also supports whole-program symbolic testing, as a non-compositional engine would.

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[3]We use "state model" rather than "memory model", as in fact it allows modelling *more* than just the memory of the program.

This allows one to instantiate Gillian to their preferred target language, and automatically benefit from all three modes of symbolic execution. This is in part possible thanks to the exact [47] semantics of GIL, enabling automated proofs to be done both with forwards and backwards implications, representing OX and UX respectively.

Thanks to this and the aforementioned parametricity of the memory model, Gillian is a CSE engine that allows scalable verification of software in a unified setting, without requiring one to adapt to two different engines, which may be prohibitively time-consuming.

2.3 Existing Tools

A wide range of engines exist to verify the correctness of code, most targeted towards a specific language. This allows them to implement behaviour that is appropriate for that particular language. For instance, CBMC [39] is a bounded model checker, that allows for the verification of C programs, via whole program symbolic testing – that is, it is not compositional, and instead symbolically executes a given codebase, and verifies that assertions hold on a generated formula. While clearly useful, as shown by its success, this solution does not scale particularly well; larger applications need to be split modularly to be verified, and manual stubbing is required to support testing parts of the code separately.

Slightly separate from symbolic execution, KLEE [13] allows for the *concolic* execution of LLVM code – thus supporting C, C++, Rust, and any other language that can be compiled to LLVM. Concolic execution is a hybrid of concrete and symbolic execution, allowing the use of symbolic values along a concrete execution path. Similarly to CBMC, it does whole program testing, and as such does not scale with larger codebases. MACKE [52] is an adaptation of KLEE enabling compositional testing, proving that concolic execution can be extended to be more scalable without losing accuracy in reported errors.

As codebases grow, there comes a need for compositional tools that can verify smaller parts of the code, enabling better scaling and integration in CI pipelines, which allow rapidly giving feedback to developers. Separation logic enables this, by allowing one to reason about a specific function and ignoring the rest of the program's state.

While Smallfoot [6] was the first tool to enable compositional symbolic execution with SL, jStar [21] was the first to allow automated verification of a real-world language, Java. It is in particular tailored towards handling design patterns that are commonly found in object-oriented programming and that may be hard to reason about with verification. It uses an intermediate representation of Java called Jimple, taken from the Java optimisation framework Soot. Similarly, VeriFast [32] is a CSE tool that is targeted at Java and a subset of C, while SpaceInvader [62] allows for the verification of C code in device drivers.

The Verified Software Toolchain [48] also allows verifying correctness of C code using SL; work has been done recently to bring some ideas from Iris such as ghost state into the toolchain. This makes it the only example known to the author of engine using Iris.

Infer [14] is a tool developed by Meta that uses separation logic and bi-abduction to find bugs in C, C++, Java and Objective-C programs, by attempting to prove correctness and extracting bugs from failures in the proof. It is the tool that pioneered *bi-abduction* [15], a technique enabling execution to proceed despite resources missing, via the construction of an anti-frame. Its approach, while initially stemming from separation logic, was used for finding bugs rather than proving correctness; this led to the creation of Incorrectness Logic [54], and subsequently ISL [58], to provide a sound theory justifying this approach. Following this, Pulse [40] followed and fully exploited the advances made in ISL to show that this new theory served as more than a justification to past tools, and to prove the existing of *true* bugs: bugs that must exist in a concrete execution.

Also using separation logic, JaVerT [29, 30] is a CSE engine aimed at verifying JavaScript, with support for whole program testing, verification, and bi-abduction in the same fashion as Infer – it followed from Cosette [28], which already supported whole program testing.

While the above examples are targeted at specific target languages, some tools have also been designed with modularity in mind. To this end, they instead support a general intermediate language – one that is not designed specifically for a language, like Jimple for Java – that a target language must be compiled to. This allows for greater flexibility in regards to what languages can be analysed by the engine, at the cost of requiring a compiler to the intermediate language.

For instance Viper [50] is an infrastructure capable of program verification with a permission-based separation logic, akin to what is presented in [9]. Its intermediate language is a rich object-oriented imperative typed language, that operates on a built-in heap. A range of Viper frontends exist, allowing for the verification of Scala, Java, OpenCL or typed Python.

One can still take genericity further, by also abstracting the state model the engine operates on. A tool that does this is coreStar [11], the CSE engine backing jStar with all the Java-related components removed and replaced. It does not target Java, but instead a generic intermediate language. Furthermore, instead of having predetermined spatial assertions defined, it requires user-defined predicates to be provided, which are then used throughout the proof. They are defined in a language specified by coreStar, and are thus however quite limited in expressivity. Still, this is to our knowledge the first example of a tool that allows some flexibility in the underlying logic theory.

This is also what Gillian [2, 27, 46, 60] does, a CSE engine parametric on the state model. It does not support any one language, but instead can target any language that provides a state model implementation in OCaml and a compiler to GIL, its intermediary language. Currently Gillian has been instantiated to C (via CompCert-C), JavaScript, and Rust [3]. It is a generalisation of JaVerT, which only targeted JavaScript. Following Gillian, work has also been done on formalising a theoretical CSE engine, providing axioms an implementation must follow [44, 45] as well as an implementation in Rocq of the engine to prove its soundness.

In Table 2.1 we show a side-by-side comparison of the aforementioned tools.

Table 2.1: Comparison of verification tools

	Target Language	Compositional	Parametric State Model	Mode
CBMC	C	✗	✗	WPST
KLEE	LLVM	✗	✗	Concolic
MACKE	LLVM	✓	✗	Concolic
jStar*	Java	✓	✗	OX
VeriFast	C, Java	✓	✗	OX
coreStar*	IR (coreStarIL)	✓	✓	OX
Infer	C, C++, Java, Objective-C	✓	✗	OX
Cosette*	JavaScript	✓	✗	WPST
JaVerT*	JavaScript	✓	✗	WPST, OX
Viper	IR (Viper)	✓	✗	OX
Pulse	C, C++, Java, Objective-C	✓	✗	UX
Gillian	IR (GIL)	✓	✓	WPST, OX, UX

WPST = Whole Program Symbolic Testing.

*Unmaintained

2.4 Legal, Social, Ethical and Professional Requirements

There are no particular legal, social or ethical considerations associated with compositional symbolic execution. To the contrary, improving code verification and facilitating its use and adoption may result in positive outcomes in a wide range of fields where software is critical. This includes, among others, medical, nuclear and spatial fields, where program failures can cause life loss, catastrophic environmental degradation, or significant economic loss. For instance, the infamous Therac-25 radiation therapy machine

caused the death of four patients and the injury of two others between 1985 and 1987 due to an error when handling shared mutable state [43]. This could have been detected and avoided if proper verification of the program had been done.

Verification tools have the potential to cause harm, if used on open source software by malevolent users that want to find possible exploits.[4] Exploits such as Heartbleed [19] or more recently the `xz` backdoor [20] happened on open-source code, and an attacker could very well automate verification of these libraries privately in an effort to find new exploits. It is thus important to facilitate and democratise verification tools to counteract these actors.

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[4]Though, to our knowledge, no exploit was found by attackers this way.

Chapter 3

Theory

In this chapter we will cover the theoretical foundations for resource algebras in a CSE setting. We will first cover in §3.1 the current state of the art regarding state models, then in §3.2 what modifications are needed for a CSE to handle RAs. This is followed by a list of simple state models in §3.3, that can be used to construct more complex state models via transformers, as shown in §3.4. Finally, in §3.5 we look at the theory justifying some optimisations of the PMAP state model transformer.

3.1 State of Affairs

3.1.1 State Models with PCMs for CSE

Abstract separation logic historically has mostly used partial commutative monoids (PCMs) [8, 16, 22, 36]. These are defined as a tuple $(\Sigma, (\cdot) \colon \Sigma \times \Sigma \rightharpoonup \Sigma, 0)$, corresponding to the carrier set, composition operator, and a unique identity (or unit) element, which must satisfy the properties highlighted in Figure 3.1. PCMs are useful for representing state in SL, as they "mathematically represent the essential notions of state ownership and ownership transfer" [26], while being simple and abstract enough that they can represent a variety of concrete states (the traditional case being that of heaps).

When bringing this concept to a Compositional Symbolic Execution (CSE) engine, the PCM must be endowed with a set of actions $\mathcal{A}$, of core predicates Δ, and functions that allow modifying it; in particular, an `execute_action` function that reflects program commands that modify the state, a `produce` function to add an assertion to the state, a `consume` function to remove an assertion from the state, and a `fix` function to provide fixes for missing errors in bi-abduction. In [2, 44, 45], the notion of state model is thus introduced, to represent a PCM equipped with these attributes, which can be seen in Figure 3.2. A state model is denoted $\mathbb{S}$, and does not include the variable store or the path condition; these are handled separately by the engine.

All of these together enable a CSE engine to be *parametric* on the state model; as long as the provided functions satisfy a set of properties, the engine using the state model can be proven to be

$$\forall a, b, c.\, (a \cdot b) \cdot c = a \cdot (b \cdot c) \qquad\qquad \text{(PCM-Assoc)}$$
$$\forall a, b.\, a \cdot b = b \cdot a \qquad\qquad \text{(PCM-Comm)}$$
$$\forall a.\, a \cdot 0 = a \qquad\qquad \text{(PCM-Identity)}$$

Equality means either both sides are defined and equal, or both are undefined.

Figure 3.1: Properties of Partial Commutative Monoids.

Figure 3.2: High-level description of the elements of a
state model $\mathbb{S}$

sound, either in over-approximate (OX) or under-approximate (UX) mode, in turn allowing for program verification or true bug finding.

Core predicates (sometimes simply called predicates) are used to make the assertion language of the engine parametric; they are written $\delta \in \Delta$. To enable this parametricity, the assertions of the language (see Figure 3.3) are extended with a core predicate assertion, written $\langle\delta\rangle(\vec{v}_i; \vec{v}_o)$, where δ is the core predicate, $\vec{v}_i$ are the *in-values* (or "ins") of the predicate, and $\vec{v}_o$ are the *out-values* (or "outs"). This distinction is used for automation purposes, to create matching plans: they tell the engine that given the values $\vec{v}_i$, the state can return the corresponding $\vec{v}_o$ associated with it; this is akin to the "parameter modes" described in [51]. For instance, the traditional "points to" assertion $a \mapsto v$, where a is an address and v a value, can be written as $\langle\texttt{points\_to}\rangle(a; v)$: given the address a, a heap can return the value store at the location v. Core predicate also imply a satisfiability relation, denoted $\sigma \models_\Delta \langle\delta\rangle(\vec{v}_i; \vec{v}_o)$ to signify a state satisfies a given predicate – this is akin to [16, 63], where predicates over states are a set of states. Predicate satisfiability happens at the concrete level, and is later lifted to the symbolic realm by evaluating the ins and outs using a substitution θ and variable store s. Another traditional example is the *emp* assertion, that is satisfied by the empty memory; it can be defined as a core predicate: $\langle\texttt{emp}\rangle(;)$.

Remark (Assertion language). *We follow the approach outlines in [2], by only defining a minimal assertion language, consisting of the separating conjunction, the existential quantifier, core predicates and logical values. First order formulae are part of LVal's grammar, and contain the usual operators: disjunction, implication, etc. This introduces a clear divide between logical assertions, expressed via logical values, and spatial assertions, expressed via core predicate assertions.*

Remark (Semantics of emp). *There are two accepted and clashing semantics of the emp assertion; either it is equivalent to* `true` *and any state satisfies emp, or it represents strictly the empty memory and nothing else. The former holds in intuitionistic separation logics, and is more appropriate for garbage collected languages, while the latter holds in classical separation logic and is better suited for* `alloc`/`free` *languages [17].*

Here we make the arbitrary choice of defining it as only being satisfied by the empty memory. Because the engine is parametric on the core predicates, a user is free to chose the semantics appropriate to their use case – this is a strength of the parametric approach.

$$\hat{v} \in \text{LVal}$$
$$\hat{x} \in \text{LVar} \subset \text{LVal}$$
$$\delta \in \Delta$$
$$P, Q, ... \in \text{Asrt} \triangleq P * Q \mid \hat{v} \mid \exists \hat{x}.\, P \mid \langle \delta \rangle (\vec{v}_i; \vec{v}_o)$$
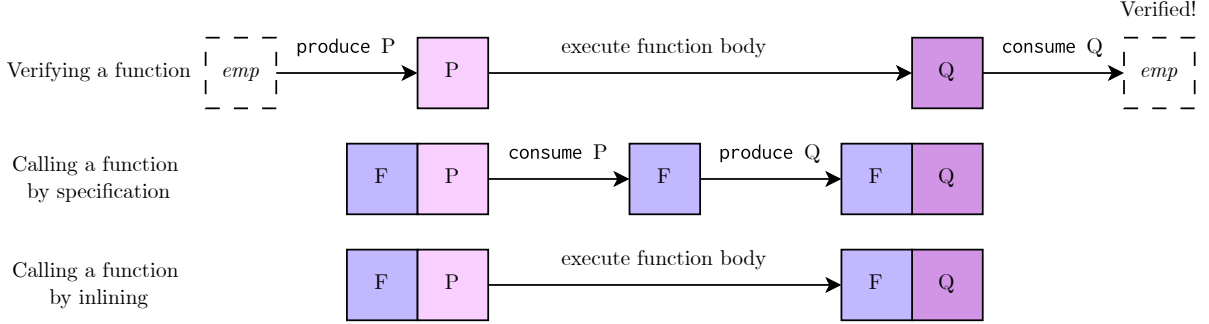
Figure 3.3: Assertion language

Figure 3.4: Function verification and function calling,
for a function with precondition P and postcondition Q.

Given the predicate satisfiability $\models_\Delta$, producing a core predicate assertion $\langle \delta \rangle (\vec{v}_i; \vec{v}_o)$ onto a state σ is equivalent to composing with it a state σ_δ such that $\sigma_\delta \models_\Delta \langle \delta \rangle (\vec{v}_i; \vec{v}_o)$; in other words, we would get $\mathsf{produce}(\sigma, \delta, \vec{v}_i, \vec{v}_o) \rightsquigarrow (\sigma \cdot \sigma_\delta)$. Consuming is then extracting the state satisfying the core predicate with the given ins from the state: $\mathsf{consume}(\sigma \cdot \sigma_\delta, \delta, \vec{v}_i) \rightsquigarrow (\mathsf{Ok}, \sigma, \vec{v}_o)$.[1]

Actions are used to represent program actions that interact with the state. Typical actions include `load`, `store`, `alloc` or `free`. These actions are called with a list of arguments, return a list of results, and may modify the state, as long as the modification are sound with respect to a set of properties, presented later. A key property of these state actions is that they are *local*: they only operate on a fragment of state, and do not necessarily need it in its entirety to succeed [16].

We may now look at an example of function, to understand how the state model is used. Consider the function `set_value` which has the following specification:

$$\langle\!\langle \mathsf{a} = \hat{a} * \mathsf{x} = \hat{x} * \langle \mathsf{points\_to} \rangle (\hat{a}; \hat{v}) \rangle\!\rangle \ \mathsf{set\_value(a, x)} \ \langle\!\langle \mathsf{ret} = \hat{a} * \langle \mathsf{points\_to} \rangle (\hat{a}; \hat{x}) \rangle\!\rangle$$

If the engine aims to verify the function, it will start with an empty state and *produce* all assertions of the pre-condition (the order being determined by the matching plan). Here, this means adding the assertions $\mathsf{a} = \hat{a}$ and $\mathsf{x} = \hat{x}$ to the path condition, and producing the core predicate $\langle \mathsf{points\_to} \rangle (\hat{a}; \hat{v})$ into the state. It will then execute the function's body until it terminates. Finally, to ensure the state at the end of the function matches the postcondition, the assertions are consumed: $\langle \mathsf{points\_to} \rangle (\hat{a}; \hat{x})$ is consumed, effectively removing the binding from the state, and we verify the returned value is $\hat{a}$.

For the case of function calls using function specifications, the inverse is done: when calling a function its precondition is consumed, and its postcondition is produced, effectively *framing off* the corresponding state, and then framing back on the post condition. For function calls by inlining, the specification is not used; instead, the code of the function is directly executed on the state. See Figure 3.4 for an illustration.

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[1] `consume` and `produce` act on symbolic states; here we treat them as concrete to explain how they work more simply, as this allows us to use composition.

The Linear Heap State Model

To exemplify how a state model works, we consider the traditional linear heap. We use a definition similar to what is shown in [58]: the two spatial assertions are the cell $a \mapsto x$, and the freed resource $a \not\mapsto$, meaning the cell at a was freed. We also need four memory-related commands: `load`, `store`, `alloc` and `free`. Our concrete state is a mapping from integers to values or to a special $\varnothing$ value representing a freed state.[2] The symbolic state is then the same, but lifting addresses and values to the symbolic realm. As such, our state model $\mathbb{S}_{\text{LinHeap}}$ thus defines:

- The concrete PCM Σ, of which the carrier set is $\mathbb{N} \rightharpoonup \text{Val} \uplus \{\varnothing\}$ – composition is the disjoint union, and the 0 is the empty map.
- The symbolic states $\hat{\Sigma} = \text{LVal} \rightharpoonup \text{LVal} \uplus \{\varnothing\}$.
- The actions $\mathcal{A} = \{\text{load}, \text{store}, \text{alloc}, \text{free}\}$.
- The core predicates $\Delta = \{\text{points\_to}, \text{freed}\}$; we can then write $\langle\text{points\_to}\rangle(a; x)$ and $\langle\text{freed}\rangle(a;)$ for $a \mapsto x$ and $a \not\mapsto$ respectively.

The definition of `execute_action` is straightforward, modifying the state accordingly. `consume` of `points_to` and `freed` suceeds and removes the binding appropriately when the mapped-to value is in LVal or is $\varnothing$ respectively. `produce` does the opposite, adding the binding to the address and vanishing (resulting in no branches) if the binding already exists, since two values cannot be at the same address.

Finally we define the satisfiability relations: $\models$ is defined with the evaluation of the addresses and values of the symbolic state. We then define $\models_\Delta$ as $[a \mapsto x] \models_\Delta \langle\text{points\_to}\rangle(a; x)$ and $[a \mapsto \varnothing] \models_\Delta \langle\text{freed}\rangle(a;)$.

3.1.2 Where it goes wrong

While PCMs are an obvious and straightforward choice for representing state ownership, they come with certain drawbacks. Firstly, PCMs can be overly restrictive; in particular, the presence of a unit 0 is restrictive, as it makes construction of sums impossible: for that, one needs to allow for multiple units in the object. In [22], it is shown that where a PCM fulfils $\exists u. \forall \sigma. \sigma \cdot u = \sigma$, one can allow for more units by simply swapping the existentials: $\forall \sigma. \exists u_\sigma. \sigma \cdot u_\sigma = \sigma$. Finally, Iris [35] shows that one *does not need a unit at all*; in fact, not having a unit makes RA constructions easier to do soundly.

In [2], the concept of RA constructions from [35] is brought into a CSE setting, using PCMs. The original work however contained some errors in constructions, that created unsoundness in the verification tool. These errors stemmed, from the most part, to the use of PCMs, and the existence of the unit 0. Indeed, this made some constructions, such as sum, freeable or the partial map unsound. We will now look at examples of these.

Unsoundness in Freeable

The first example is with $\text{Freeable}_{\text{PCM}}(\mathbb{S})$. This state model *transformer* is given a state model $\mathbb{S}$, and extends it with the capacity to free state, via the `free` action. It also adds a `freed` predicate, to represent freed memory.

Now consider the simple state model $\text{Freeable}_{\text{PCM}}(\text{Ex}_{\text{PCM}}(\text{Val}))$. This represents exactly the state of a singular cell in our previous linear heap example. Here $\text{Ex}_{\text{PCM}}(X)$ is the exclusive state model, that represents exclusive ownership of a value of type X.[3] Going back to our linear heap, the cell is also exclusively owned; one cannot have two different values at the same address, we either have it all, or have none of it. Ownership is indicated via the `ex` core predicate: $\langle\text{ex}\rangle(; x)$ means the state currently has the value x – this state model will be further defined later, and is only used here for the example.

For now, we define $\text{Ex}_{\text{PCM}}(X)$ as:

$$\text{Ex}_{\text{PCM}}(X) \triangleq \text{ex}(x\colon X) \mid 0$$

The notation $\text{ex}(x\colon X)$ is equivalent to $\{\text{ex}(x) : \forall x \in X\}$. We thus define $\text{Ex}_{\text{PCM}}(X)$ as the set of elements of X *and* the 0 element. We also define composition such that $0 \cdot \sigma = \sigma \cdot 0 = \sigma$ and $\text{ex}(x) \cdot \text{ex}(y)$ is always undefined: since the value is owned exclusively, we own it as a whole and nothing can be added to it. This satisfies the axioms of PCMs. We now consider a naive definition of $\text{Freeable}_{\text{PCM}}(\mathbb{S})$'s PCM, where Σ is the carrier set of $\mathbb{S}$'s PCM:

$$\text{Freeable}_{\text{PCM}}(\mathbb{S}) \triangleq \text{sub}(\sigma\colon \Sigma) \mid \varnothing \mid 0$$

We then again define composition as $0 \cdot \sigma = \sigma \cdot 0 = \sigma$, and $\text{sub}(\sigma) \cdot \text{sub}(\sigma') = \text{sub}(\sigma \cdot \sigma')$, which forms a valid PCM.[4] $\text{Freeable}_{\text{PCM}}(\mathbb{S})$ exposes a core predicate freed, as well as the core predicates of $\mathbb{S}$; similarly, consume and produce are lifted. The core predicate $\langle\text{freed}\rangle(;)$ is equivalent to the negative cell assertion $\not\mapsto$, modulo the index. While we omit the full details of the state model, we note that producing freed only succeeds when in state 0 (since the whole state is freed, it cannot also be something else), producing something other that freed calls the produce function of the underlying state model $\mathbb{S}$, and stores the resulting state σ as $\text{sub}(\sigma)$. Similarly, consuming a predicate of $\mathbb{S}$ while in state $\text{sub}(\sigma)$ calls consume on that predicate with σ, and then wraps the result σ' back into $\text{sub}(\sigma')$.

Going back to our state model $\text{Freeable}_{\text{PCM}}(\text{Ex}_{\text{PCM}}(\text{Val}))$, it thus exposes two predicates, ex and freed. Its set of states is $\{\text{sub}(x) : x \in \text{Val} \uplus \{0\}\} \uplus \{\varnothing, 0\}$. To show the unsoundness of our definition, consider calling the dispose function:

$$\langle\!\langle \langle\text{ex}\rangle(;\hat{x}) \rangle\!\rangle \; \texttt{dispose()} \; \langle\!\langle \text{ret} = 0 * \langle\text{freed}\rangle(;) \rangle\!\rangle$$

Assume we are in the state $\text{sub}(\text{ex}(1))$. To call dispose by its specification, we first need to consume its precondition: we consume $\langle\text{ex}\rangle(;\hat{x})$ from the state. To do this, $\text{Freeable}_{\text{PCM}}$ calls the consume function of $\text{Ex}_{\text{PCM}}(\text{Val})$, with the state $\text{ex}(1)$, which returns 0. $\text{Freeable}_{\text{PCM}}$ then wraps this state back, resulting in $\text{sub}(0)$. We then produce the post-condition of dispose, thus producing $\langle\text{freed}\rangle(;)$ into $\text{sub}(0)$. This however is not valid, according to the rules outlined earlier! Indeed the state cannot both be freed and something else. Of course, $\text{sub}(0)$ represents an empty state, but that means that the consume and produce rules of $\text{Freeable}_{\text{PCM}}$ must take this into account – a simple implementation could miss it, resulting in such errors.

This difficulty is in fact caused by the existence of both 0 and $\text{sub}(0)$ as two distinct states, despite them being the same semantically: empty. To fix this, one must either make $\text{sub}(0)$ invalid, defining it as $\text{sub}(x\colon \Sigma \setminus \{0\})$, or instead remove the definition of 0 and instead use $\text{sub}(0)$ as the unit of $\text{Freeable}_{\text{PCM}}$. Either cases require one to be quite careful around the unit, as forgetting about it may lead to unwanted errors.

Unsoundness in Sum

The sum state model $\text{Sum}(\mathbb{S}_1, \mathbb{S}_2)$, also written $\mathbb{S}_1 + \mathbb{S}_2$, represents a state where either one of the two sides is owned, but not both. It is useful when one wants to represent states that can transition from one set of states to another – for instance, $\text{Freeable}(\mathbb{S})$ can also be defined as $\mathbb{S} + \text{Ex}(\{\varnothing\})$.

While one of the primary purposes of the sum state model transformer is allowing for transitions between one side of the sum to the other, it is in fact non-trivial to define in a way that is both OX-sound and UX-sound, *particularly when using PCMs*. Consider a transition from a state of $\mathbb{S}_1$ to a state of $\mathbb{S}_2$:

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[4]All unspecified compositions are considered undefined.

- If one wants the transition to be OX-sound, the right hand side of the sum must forbid the 0 element of $\mathbb{S}_1$, and for any successfuly swapped state σ_1, any attempt to swap from a smaller σ_1' such that $\sigma_1' \preccurlyeq \sigma_1$ must result in a Miss.
- If one wants the transition to be UX-sound, the left hand side of the sum must forbid the 0 element of $\mathbb{S}_2$, and the resulting state must be *exclusively owned*. Being exclusively owned means the state has no valid frame; this property is taken from [35], and can be defined for PCMs as:

$$\texttt{is\_exclusively\_owned}_{\text{PCM}} \; \sigma \triangleq \forall \sigma'. \sigma' \neq 0 \implies \neg(\sigma \# \sigma')$$

If these properties do not hold, then action is not frame-preserving. The proof that the above properties must hold can be found in appendix C. Again we note how the presence of a 0 adds requirements around these constructions, cluttering otherwise elegant definitions.

Unsoundness in PMap

Finally, we consider how PCMs can cause unsoundness in the partial map state model transformer, $\text{PMAP}(I, \mathbb{S})$. It maps indices from a domain $I \subseteq \text{Val}$ to a codomain state from $\mathbb{S}$, by *lifting* all predicates of $\mathbb{S}$ with an additional in-value corresponding to the location $i \in I$ to which the predicate corresponds. All action executions also receive an additional argument with the location of the state the action is executed on.

The linear heap we presented earlier is equivalent to $\text{PMAP}_{\text{PCM}}(\mathbb{N}, \text{FREEABLE}_{\text{PCM}}(\text{EX}_{\text{PCM}}(\text{Val})))$; the core predicate $\langle\texttt{points\_to}\rangle(a; x)$ (better known as $a \mapsto x$) is then $\langle\texttt{ex}\rangle(a; x)$. Similarly, $\langle\texttt{freed}\rangle(a;)$ (i.e. $a \not\mapsto$) becomes $\langle\texttt{freed}\rangle(a;)$. In both cases, PMAP uses the in-value a as the address of the substate. We define the PCM of PMAP_{PCM} as:

$$\text{PMAP}_{\text{PCM}}(I, \mathbb{S}) \triangleq (I \xrightarrow{fin} \Sigma) \times \mathcal{P}(I)^?$$

We omit the precise details of how exactly it works – it will be extensively discussed in §3.4.

The right hand side of the product, $\mathcal{P}(I)^?$, represents the domain set of the partial map, in other words the set of addresses that are known to exist – any address outside of the set *cannot* exist. The $-^?$ signifies that we extend $\mathcal{P}(I)$ with the element $\bot$, in which case the domain set is not known. We use it to allow distinguishing Miss errors from out of bounds errors, when accessing a binding the state does not own: if the domain set d is not $\bot$ and the index is not in d, then the action must be an out of bounds error. From this follows an invariant: given a state (h, d) where h is the heap and d the domain set, $d \neq \bot \Rightarrow \text{dom}(h) \subseteq d$ must always hold; in other words, we cannot own any state outside of the domain set. The domain set is *exclusively owned*: if two PMAP states are disjoint, they cannot both have a defined domain set; this allows it to be *modified*, for instance during allocation. The core predicate $\langle\texttt{domainset}\rangle(; d)$ is satisfied by the state with exactly the domain set d. When producing the domainset core predicate onto a state, the produce function of PMAP_{PCM} must thus ensure the invariant constraint is upheld, vanishing otherwise.

$\text{PMAP}_{\text{PCM}}(I, \mathbb{S})$ lifts all core predicates and actions with an index, and as such all consume and produce calls must also have the index in their ins. The behaviour of a naive implementation would then be similar to what was explained for $\text{FREEABLE}_{\text{PCM}}$: calling consume or produce when the binding is not present executes it against the 0 element of the underlying PCM, and otherwise calls it on the state at the location and stores the result back at that same location, overriding the previous binding.

For this example we consider the SL linear heap state model, $\text{PMAP}_{\text{PCM}}(\mathbb{N}, \text{EX}_{\text{PCM}}(\text{Val}))$.[5] For simplicity, we re-use the traditional notation for cells, writing $a \mapsto x$ for $\langle\texttt{ex}\rangle(a; x)$.

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[5] We use the SL linear heap, that does not have the freed assertion $a \not\mapsto$, for simplicity.

Now, consider the following function specification that moves a value:

$$\langle\!\langle 1 \mapsto 1 * \langle\mathsf{domainset}\rangle(; \{1\})\rangle\!\rangle \, \mathtt{move()} \, \langle\!\langle \mathtt{ret} = 0 * 2 \mapsto 1 * \langle\mathsf{domainset}\rangle(; \{2\})\rangle\!\rangle$$

This specification is valid, as it is frame preserving. More precisely, because the domain set of the precondition is $\{1\}$ and we own the cell at address 1, the only state compatible with the precondition is the empty state $([], \bot)$; the same also applies for the postcondition. As such, one should be able to soundly call this function.

Let the initial state be $([1 \mapsto \mathrm{ex}(1)], \{1\})$: a heap with one cell, at address 1 – this matches the precondition of $\mathtt{move}$. If we want to call $\mathtt{move}$ by specification, the engine must first consume the entire precondition, assertion by assertion (effectively framing it off). It first consumes $1 \mapsto 1$, or $\langle\mathsf{ex}\rangle(1; 1)$; PMAP in turn gets the state at address 1, $\mathrm{ex}(1)$, and consumes $\langle\mathsf{ex}\rangle(; 1)$ from it, unlifting the index from the assertion. The state at 1 then becomes the unit 0 of $\mathrm{Ex}_{\mathrm{PCM}}(\mathrm{Val})$, and the outer state becomes $(\{1 \mapsto 0\}, \{1\})$. The $\mathtt{domainset}$ predicate is then successfully consumed too, leaving $(\{1 \mapsto 0\}, \bot)$, since the domain set must be consumed as a whole.

Next, the post-condition is produced onto the state. First, the engine produces $2 \mapsto 1$, or $\langle\mathsf{ex}\rangle(2; 1)$, which adds a binding to $\mathrm{PMAP}_{\mathrm{PCM}}$ and creates the cell: our bigger state is now $([1 \mapsto 0, 2 \mapsto \mathrm{ex}(1)], \bot)$. Finally, $\langle\mathsf{domainset}\rangle(; \{2\})$ is produced onto the state, however this *does not succeed*. Indeed, we have $\mathrm{dom}([1 \mapsto 0, 2 \mapsto \mathrm{ex}(1)]) = \{1, 2\}$, and of course $\{1, 2\} \not\subseteq \{2\}$: the domain of the heap is not a subset of the produced domain set $\{2\}$. This is because again we forgot to check if the state at location 1 became 0 when consuming $\langle\mathsf{ex}\rangle(1; 1)$. We would thus need to check for this, and potentially redefine the PCM of $\mathrm{PMAP}_{\mathrm{PCM}}$ to forbid mappings to 0. Another solution is to reformulate the invariant, such that indices pointing to 0 are ignored, since they are empty anyways. In both cases, we must be careful about how we choose to handle the unit.

These examples hopefully showed that PCMs enforce a unit that is unwieldy, and can easily cause unsound behaviour when mishandled. Even when taken into account, it leads to unpleasant definitions, as every state model must define a 0, and every state model transformer must then either exclude it in its construction, or explicitly take it into account. What if we could simply eliminate the 0 entirely, avoiding the need to handle it?

3.1.3 RAs for Iris

Iris [34–36, 38] departs from the tradition of PCMs, and introduces resource algebras (RAs) to model state, defined as a tuple $(\Sigma, \overline{\mathcal{V}}\colon \Sigma \to \mathrm{iProp}, |-|\colon \Sigma \to \Sigma^?, (\cdot)\colon \Sigma \times \Sigma \to \Sigma)$,[6] containing the carrier set, a validity function, a partial core function and a *total* composition operator. This definition makes use of the option RA $\Sigma^? \triangleq \Sigma \uplus \{\bot\}$ [7], that extends the set of Σ with an "empty" element $\bot$, such that $\sigma \cdot \bot = \bot \cdot \sigma = \sigma$. Similarly to PCMs, a set of axioms must be satisfied, as seen in Figure 3.5.

The *core function* $|-|\colon \Sigma \to \Sigma^?$ associates to an element σ of the RA its *duplicable core*. The crucial difference here, compared to PCMs, is that aside from allowing multiple units, it allows having *no unit*, in which case $|\sigma| = \bot$. As we will see later, having no unit is a requirement for a state to be exclusively owned (i.e. it has no frame). Indeed, if the state σ has a core $|\sigma| \neq \bot$, then that core always constitutes a valid frame.

The *validity function* $\overline{\mathcal{V}}\colon \Sigma \to \mathrm{iProp}$ is used to rule out invalid compositions, rather than relying on the partiality of the composition operator. It returns an Iris property, iProp, which allows for step indexing and more sophisticated logic, that is out of the scope of this project. Most RAs are then extended with

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[6] The original papers on Iris use the symbol M rather than Σ; we keep Σ for uniformity with the rest of the presentation.

An *Iris resource algebra* is a tuple $(\Sigma, \overline{\mathcal{V}} \colon \Sigma \to \mathrm{iProp}, |-| \colon \Sigma \to \Sigma^?, (\cdot) \colon \Sigma \times \Sigma \to \Sigma)$

$$\forall a, b, c.\, (a \cdot b) \cdot c = a \cdot (b \cdot c) \qquad\qquad \text{(Iris-RA-Assoc)}$$
$$\forall a, b.\, a \cdot b = b \cdot a \qquad\qquad \text{(Iris-RA-Comm)}$$
$$\forall a.\, |a| \in \Sigma \Rightarrow |a| \cdot a = a \qquad\qquad \text{(Iris-RA-Core-ID)}$$
$$\forall a.\, |a| \in \Sigma \Rightarrow ||a|| = |a| \qquad\qquad \text{(Iris-RA-Core-Idem)}$$
$$\forall a, b.\, |a| \in \Sigma \wedge a \preccurlyeq b \Rightarrow |b| \in \Sigma \wedge |a| \preccurlyeq |b| \qquad\qquad \text{(Iris-RA-Core-Mono)}$$
$$\forall a, b.\, \overline{\mathcal{V}}(a \cdot b) \Rightarrow \overline{\mathcal{V}}(a) \qquad\qquad \text{(Iris-RA-Valid-Op)}$$

$$\text{where} \qquad \Sigma^? \triangleq \Sigma \uplus \{\bot\}, \text{ with } \sigma \cdot \bot \triangleq \bot \cdot \sigma \triangleq \sigma$$
$$a \preccurlyeq b \triangleq \exists c.\, b = a \cdot c$$
$$a \rightsquigarrow B \triangleq \forall c^? \in \Sigma^?.\, \overline{\mathcal{V}}(a \cdot c^?) \Rightarrow \exists b \in B.\, \overline{\mathcal{V}}(b \cdot c^?)$$
$$a \rightsquigarrow b \triangleq a \rightsquigarrow \{b\}$$

A *unital* resource algebra is an RA Σ with an element $\epsilon \in \Sigma$ such that:

$$\overline{\mathcal{V}}(\epsilon) \qquad\qquad \forall \sigma \in \Sigma.\, \epsilon \cdot \sigma = \sigma \qquad\qquad |\epsilon| = \epsilon$$

Figure 3.5: Definition of resource algebras in Iris

an invalid element $\frac{1}{2}$, and validity is often defined as $\overline{\mathcal{V}}(\sigma) \triangleq \sigma \neq \frac{1}{2}$. For instance for the $\mathrm{Ex}_{\mathrm{IRIS}}$ RA:

$$\mathrm{Ex}_{\mathrm{IRIS}}(X) \triangleq \mathrm{ex}(x \colon X) \mid \frac{1}{2}$$
$$\overline{\mathcal{V}}(\sigma) \triangleq \sigma \neq \frac{1}{2}$$
$$|\mathrm{ex}(x)| \triangleq \bot$$
$$\sigma \cdot \sigma' \triangleq \frac{1}{2}$$

The validity and core functions thus make Iris states more powerful, as they have more flexibility in what can be expressed; while a regular PCM can be trivially converted to an RA, the opposite is not true. An example of construction that can easily be more easily constructed is the sum state model. The requirement for a transition from one side of the sum to the other can be more elegantly expressed, without having to worry about the existence of the unit 0 [35]. Given $\mathrm{exclusive}_{\mathrm{IRIS}}(\sigma) \triangleq \forall \sigma'.\, \neg\overline{\mathcal{V}}(\sigma \cdot \sigma')$, we have:

$$\frac{\text{ExclusiveUpdate}}{\mathrm{exclusive}_{\mathrm{IRIS}}(\sigma) \qquad \overline{\mathcal{V}}(\sigma')}{\sigma \rightsquigarrow \sigma'}$$

The Iris update $\sigma \rightsquigarrow \sigma'$ is a form of frame preserving update: if we update σ into σ', we know that any frame that was compatible with σ remains compatible with σ'. For the above rule, because σ is owned exclusively, we can replace it with any state, as $\sigma \rightsquigarrow \sigma'$ is vacuously true. This rule is not the only way for one to have a frame preserving update, of course; but it is a rule that can be applied for any RA Σ, and is thus crucial. We will often implicitly use a form of it when wanting to prove the soundness of actions.

Furthermore, Iris is a battle-tested and certified logic,[7] that comes with plenty of proofs and properties making them easy to use and adapt, whereas PCMs can prove unwieldy even for simpler state models, as shown previously.

We note an interesting similarity between Iris and the PCM approach, which is that "the global RA must be unital, which means it should have a unit element ϵ" [35]. This is similar to what one would

---

[7]See https://iris-project.org/#publications, where more than a hundred publications are cited as using Iris, allowing for the verification of, among others, C, Go, OCaml, Rust, Scala or WASM.

A *partial resource algebra* (RA) is a triple $(\Sigma, |-| \colon \Sigma \to \Sigma^?, (\cdot) \colon \Sigma \times \Sigma \rightharpoonup \Sigma)$

$$\forall a, b, c.\, (a \cdot b) \cdot c = a \cdot (b \cdot c) \tag{RA-Assoc}$$
$$\forall a, b.\, a \cdot b = b \cdot a \tag{RA-Comm}$$
$$\forall a.\, |a| \in \Sigma \Rightarrow |a| \cdot a = a \tag{RA-Core-ID}$$
$$\forall a.\, |a| \in \Sigma \Rightarrow ||a|| = |a| \tag{RA-Core-Idem}$$
$$\forall a, b.\, |a| \in \Sigma \wedge a \preccurlyeq b \Rightarrow |b| \in \Sigma \wedge |a| \preccurlyeq |b| \tag{RA-Core-Mono}$$

$$\text{where} \quad \Sigma^? \triangleq \Sigma \uplus \{\bot\}, \text{ with } \sigma \cdot \bot \triangleq \bot \cdot \sigma \triangleq \sigma$$
$$a \preccurlyeq b \triangleq \exists c.\, b = a \cdot c$$
$$a \,\#\, b \triangleq a \cdot b \text{ is defined}$$

A *unital* resource algebra is a resource algebra Σ with an element $\epsilon \in \Sigma$ such that:

$$\forall \sigma \in \Sigma.\, \epsilon \cdot \sigma = \sigma \qquad\qquad\qquad |\epsilon| = \epsilon$$

Figure 3.6: Definition of (partial) resource algebras

find in a PCM, where we have the 0 element. Any RA can be trivially extended to have a unit, via the option RA. This in particular means that while the building blocks of both approaches are different (PCMs or RAs), the end result that is used is similar. This is good news: it means that adapting an RA construction to a CSE engine that uses PCMs should be fairly straightforward, as one can rely on the added unit as a de facto 0.

3.2 State Models with RAs

We have seen how PCMs are unpractical for a CSE engine, as they easily create incorrect behaviour that can be quite tricky to notice. We will now see how we may adapt a CSE to instead use RAs. We first define the notion of partial RAs, followed by the description of the changes done to the engine's layers to support partial RAs. Finally, we list the new properties that must be satisfied by the functions of the state model.

3.2.1 Partial RAs

A property of Iris RAs is that composition is *total* – to take into account invalid composition, states are usually extended with a $\notin$ state, such that $\overline{\mathcal{V}}(\sigma) \Leftrightarrow \sigma \neq \notin$. This is needed in the Iris framework because they have a mechanisation in Rocq, where all functions must be total. In our case however we do not have this constraint; as such, because having to define and later verify the validity of states can be quite unwieldy, we remove validity by instead making use of partiality, as was originally the case in PCMs. This translation is done by removing the invalid state $\notin$ when present, and instead defining compositions that lead to it as undefined. From this, we can get rid of the validity function, and define composition as being partial: $(\cdot) \colon \Sigma \times \Sigma \rightharpoonup \Sigma$. This is also in line with the core function $(-)$ being partial.

Remark (Partiality and the option RA). *One may note that we use the term "partial" to refer to two different signatures; the core $|-| \colon \Sigma \to \Sigma^?$ is called partial but is in fact total, with undefined mappings being equal to $\bot$, while composition $(\cdot) \colon \Sigma \times \Sigma \rightharpoonup \Sigma$ is a "proper" partial function.*

While one could adapt the core to be $\Sigma \rightharpoonup \Sigma$, this would prove unpractical, in cases where one still wants to use $|\sigma|$ even when it is $\bot$. On the other hand, defining composition as $\Sigma \times \Sigma \to \Sigma^?$ is simply incorrect: let $a \cdot b$ undefined, it is untrue that $(a \cdot b) \cdot c = c$, while defining an undefined composition to equal $\bot$ would render this statement true.

$$\text{Ex}_{\text{Iris}}(X) \triangleq \text{ex}(x \colon X) \mid \lightning$$
$$\overline{\mathcal{V}}(\sigma) \triangleq \sigma \neq \lightning$$
$$|\text{ex}(x)| \triangleq \bot$$
$$\sigma \cdot \sigma' \triangleq \lightning$$

$$\text{Ex}(X) \triangleq \text{ex}(x \colon X)$$
$$|\text{ex}(x)| \triangleq \bot$$
$$\sigma \cdot \sigma' \text{ is always undefined}$$

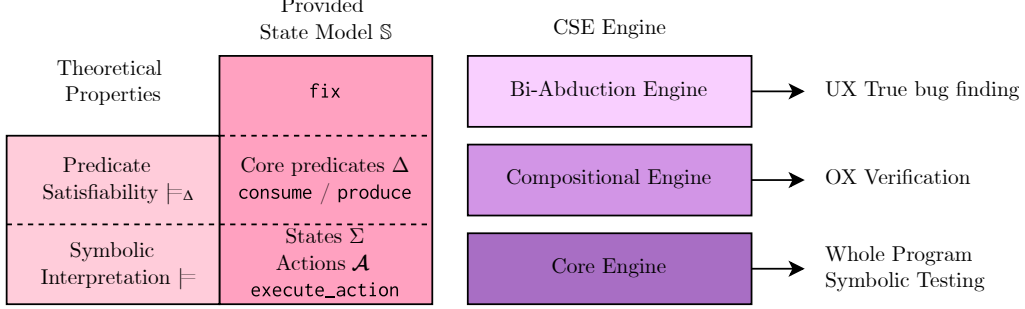Figure 3.7: Comparison of the definition of Ex with Iris and partial RAs

Figure 3.8: Layers of the CSE engine

The partiality of the core and the composition are opposite: whereas the core when undefined yields a $\bot$, the composition would yield a $\lightning$ in Iris, which is akin to $\top$. In fact this is exactly how faulting is defined in [16].

Partial RAs, defined as a triple $(\Sigma, |-| \colon \Sigma \to \Sigma^?, (\cdot) \colon \Sigma \times \Sigma \rightharpoonup \Sigma)$, are equivalent to regular RAs, as long as $\neg\overline{\mathcal{V}}(a) \Rightarrow \neg\overline{\mathcal{V}}(a \cdot b)$ holds; luckily for us, this is equivalent to the axiom (Iris-RA-Valid-Op), that states $\overline{\mathcal{V}}(a \cdot b) \Rightarrow \overline{\mathcal{V}}(a)$. Compositions that yield $\lightning$ (or any state σ such that $\neg\overline{\mathcal{V}}(\sigma)$) can be made undefined, and the validity function removed, to gain partiality, and inversely to go back to the Iris definition.

An interesting property of this change is that because validity is replaced by the fact composition is defined, the validity of a composition $\overline{\mathcal{V}}(a \cdot b)$ in a RA is exactly equivalent to the fact two elements are disjoint in a partial RA $a \,\#\, b$. We now define the properties of partial RAs taking this change into account – see Figure 3.6. From now, the term RA will be used to refer to these partial RAs.

We may compare the definition of Ex_{Iris}, presented earlier, with the equivalent definition using partial RAs, Ex. The main difference is the removal of $\overline{\mathcal{V}}$ and the $\lightning$ element, making the definition simpler (see Figure 3.7).

3.2.2 Core Engine

We now consider what changes need to be brought to our CSE to handle RAs. To do this, we use the properties and definitions stated in [44]. We will also specify additional changes that exist in Gillian and that were formalised in [2], but that have not been added to the aforementioned formalisation.

This CSE engine is split into three layers, each adding functionality to the layer below, allowing for a clear separation of concerns. These layers are, from bottom to top: the core engine, the compositional engine, and the bi-abduction engine (see Figure 3.8).

The core engine enables whole program symbolic testing. For this, state models must firstly define the set of states the execution will happen on. While *concrete* compositional state is modelled as an RA, *symbolic* compositional state is simply a set of elements that does not come with composition or cores. This is simply because defining composition on symbolic states is not necessary; soundness of

symbolic states can be verified with symbolic interpretation ($\models$), and modification of symbolic state can be done via `consume` and `produce`.

Once the set of symbolic states $\hat{\Sigma} \ni \hat{\sigma}$ is defined, it must be further equipped with a set of actions $\mathcal{A}$ and an `execute_action` function that allows the program to modify the state. We also include the definition of the $\models$ relation, called `sat`, noting that this is a purely theoretical construct that is not implemented in the actual engine.

$$\texttt{execute\_action}\colon \Sigma^? \to \mathcal{A} \to \mathrm{Val}\ \texttt{list} \to \mathcal{P}(\mathcal{O}_e \times \Sigma^? \times \mathrm{Val}\ \texttt{list})$$

$$\texttt{execute\_action}\colon \hat{\Sigma}^? \to \mathcal{A} \to \mathrm{LVal}\ \texttt{list} \to \mathcal{P}(\mathcal{O}_e \times \hat{\Sigma}^? \times \mathrm{LVal}\ \texttt{list} \times \Pi)$$

$$\texttt{sat}\colon \Theta \to \mathrm{Store} \to \hat{\Sigma} \to \mathcal{P}(\Sigma)$$

We define two different versions of `execute_action`: one operating on concrete states, and one operating on symbolic states. The concrete version of the function is used in the theory, to prove the soundness of its symbolic counterpart, while the symbolic version is what is actually used by the engine to execute the analysed program.

In the concrete case, the arguments of `execute_action` are, in order: the *optional* state the action is executed on, the action, and the received arguments. It returns a set of branches, with the outcome, the new state and the returned values. Note how concrete semantics are non-deterministic; this is needed for instance for allocation, which will be discussed in §3.4. It is pretty-printed as $\alpha(\sigma, \vec{v}_i) \rightsquigarrow (o, \sigma', \vec{v}_o)$. We also must accept a $\perp$ state as an input, to allow for example a "pure" state model, for which the set of states is the empty set [2].

For its symbolic counterpart, the arguments are lifted to the symbolic realm. It also returns a set of branches, along with a path condition (PC) π representing the condition for the branch to exist – if the PC is not satisfiable, the branch must be cut. It is pretty-printed as $\alpha(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{\hat{v}}_o, \pi)$.

Here, the outcome is in the set of *execution outcomes* $\mathcal{O}_e = \{\texttt{Ok}, \texttt{Err}, \texttt{Miss}\}$. Where `Ok` represents successful execution and `Err` program errors, `Miss` outcomes come from the compositional nature of the state. An action that leads to a `Miss` does not represent a bug in the code, but rather that additional state is needed to determine whether execution is successful or not.

The main difference with [44] is that the state may be $\perp$, if the action is executed on empty state; this is made explicit from using $\hat{\Sigma}^?$ (rather than $\hat{\Sigma}$). This ensures non-unital RAs are not ruled out as invalid – indeed, many useful RAs are not unital and sometimes do not have a unit at all, as is the case for instance for Ex.

We also modify the definition of a path condition. Because it only contains first order formulae, [44, 45] makes the choice of defining it as a logical value that must evaluate to a boolean, and which is extended via conjunction. Internally, the engine however still needs to split the conjunction into a list of terms to manipulate it. As such, to make the theory closer to what is effectively done within the engine, we make the choice of defining the path condition as a *list of logical values*: $\pi \in \Pi = \mathrm{LVal}\ \texttt{list}$. Extension is then defined as concatenating new logical values to the path condition, and strengthening of the path condition can be trivially checked with $\pi' \supseteq \pi$. Substitutions $\theta \in \Theta$ and variable stores (or simply stores) $s \in \mathrm{Store}$ are variable environments, for logical and program variables respectively. We define the predicate `SAT`, that is true if, given the substitution θ, store s and a path condition π, the conjunction of the elements of π resolves to `true` once evaluated:

$$\mathsf{SAT}_{\theta, s}(\pi) \triangleq \left[\!\!\left[\bigwedge_i^{|\pi|} \pi(i) \right]\!\!\right]_{\theta, s} = \texttt{true}$$

Remark (Formality of fresh variables). *The original definition of the symbolic `execute_action` presented in [44] also has an $SV \in \mathcal{P}(LVar)$ argument, which is the set containing all existing symbolic variables.*

This set is used when an action needs to create a fresh symbolic variable $\hat{v}$, to ensure the variable is fresh by checking $\hat{v} \notin SV$. While necessary for proofs within the engine in Rocq, we omit it here, as it is only relevant for one action (alloc), and pen and paper proofs do not require the same rigidity as a Rocq proof. Including it would be straightforward.

Finally, the user must define the sat relation, relating concrete and symbolic states. It is pretty printed as $\theta, s, \sigma \models \hat{\sigma}$ for $\sigma \in \mathsf{sat}\ \theta\ s\ \hat{\sigma}$, meaning that given a substitution θ and a store s, the concrete state σ can be matched by a symbolic state $\hat{\sigma}$. The user must only define this relation for non-$\perp$ states; we can then lift it to the option RA, by simply adding that $\forall \theta, s.\ \mathsf{sat}\ \theta\ s\ \perp = \{\perp\}$. This relieves users from needing to take $\perp$ into account when defining $\models$ and from proving Empty Memory.

3.2.3 Compositional Engine

The compositional engine, built on top of the core engine, allows for verification of function specifications, and handles calls by specification. The state model must be extended with a set of core predicates Δ and a pair of consume and produce functions (equivalent, respectively, to a resource assert and assume). Finally, to link core predicates to states, a core predicate satisfiability relation $\models_\Delta$ (called sat_Δ) must be defined in the theory, to prove soundness of consume and produce.

$$
\begin{aligned}
&\text{for } M = \{\mathsf{OX}, \mathsf{UX}\}\\
&\qquad \mathsf{consume}\colon M \to \hat{\Sigma}^? \to \Delta \to \mathrm{LVal\ list} \to \mathcal{P}(\mathcal{O}_l \times \hat{\Sigma}^? \times \mathrm{LVal\ list} \times \Pi)\\
&\qquad \mathsf{produce}\colon \hat{\Sigma}^? \to \Delta \to \mathrm{LVal\ list} \to \mathrm{LVal\ list} \to \mathcal{P}(\hat{\Sigma}^? \times \Pi)\\
&\qquad\quad \mathsf{sat}_\Delta\colon \Sigma^? \to \Delta \to \mathrm{Val\ list} \to \mathcal{P}(\mathrm{Val\ list})
\end{aligned}
$$

Similarly to execute_action, the input state can be $\perp$. While intuitively one may assume that the input state of consume and the output state of produce may never be $\perp$ (as there must be a non-empty state to consume an assertion from or produce an assertion into), this would limit what core predicates can do. In particular, this means an *emp* predicate could not be defined, since its production on an empty state results in an empty state.

The arguments of consume are, in order: the mode of execution, to distinguish between under-approximate and over-approximate reasoning, the state, the core predicate being consumed, the ins of the predicate. It outputs a *logical outcome*, the state with the matching predicate removed (which may result in an empty state $\perp$), the outs of the predicate and the associated path condition. It is pretty-printed as $\mathsf{consume}(m, \hat{\sigma}, \delta, \vec{v}_i) \rightsquigarrow (o, \hat{\sigma}_f, \vec{v}_o, \pi)$, and when the consumption is valid for both OX and UX the mode is omitted.

For produce, the arguments are the state, the core predicate being produced, the ins and the outs of the predicate, resulting in a set of new states and their associated path condition. As an example, producing $\hat{x} \mapsto 0$ in a state $[1 \mapsto 2]$ results in a new state $[1 \mapsto 2, \hat{x} \mapsto 0]$ with the path condition $[\hat{x} \neq 1]$. If the produced predicate is incompatible with the state (e.g. producing $1 \mapsto y$ in a state containing $1 \mapsto x$), the producer *vanishes*. Inversely, if the assertion can be interpreted in several ways, the producer may branch. It is pretty-printed as $\mathsf{produce}(\hat{\sigma}, \delta, \vec{v}_i, \vec{v}_o) \rightsquigarrow (\hat{\sigma}', \pi)$.

The sat_Δ relation relates a possibly empty *concrete* state, core predicate and in-values to a set of out-values. It is pretty-printed as $\sigma \models_\Delta \langle \delta \rangle (\vec{v}_i; \vec{v}_o)$ for $\vec{v}_o \in \mathsf{sat}_\Delta\ \sigma\ \delta\ \vec{v}_i$. For instance in the linear heap state model, we have $[1 \mapsto 2] \models_\Delta \langle \mathsf{points\_to} \rangle (1; 2)$. Denoting expression evaluation as $[\![\hat{e}]\!]_{\theta, s} = e$, we lift sat_Δ to the symbolic realm:

$$
\theta, s, \sigma \models_\Delta \langle \delta \rangle (\vec{\hat{v}}_i; \vec{\hat{v}}_o) \triangleq [\![\vec{\hat{v}}_i]\!]_{\theta, s} = \vec{v}_i \wedge [\![\vec{\hat{v}}_o]\!]_{\theta, s} = \vec{v}_o \wedge \sigma \models_\Delta \langle \delta \rangle (\vec{v}_i; \vec{v}_o)
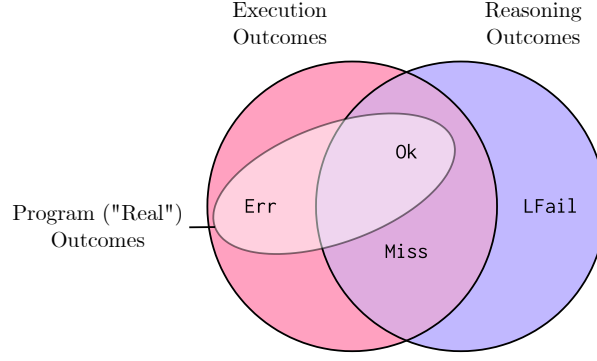$$

Figure 3.9: Overlap of outcomes

Here we define logical outcomes $\mathcal{O}_l = \{\text{Ok}, \text{LFail}, \text{Miss}\}$. These are outcomes that happen during reasoning; in particular, LFail equates to a logical failure due to an *incompatibility* between the consumed predicate and the state. For instance, consuming $1 \mapsto 1$ from a state $[1 \mapsto 2]$ would yield a LFail, while consuming it from state $[2 \mapsto 1]$ would yield a Miss, as the state $[1 \mapsto 1]$ could be composed with it to yield a non-miss outcome. Err and LFail are distinct because they represent to different types of error: Err is a language error that can occur at runtime (e.g. a division by zero, a null pointer dereference) while LFail corresponds to a reasoning failure, that happens in the logic and does not exist when running the code.

An addition to what [44] previously defined, taking inspiration from [2], is thus the split of what was the Abort outcome into LFail and Miss, which improves the quality of error messages and allows *fixing* consumptions that yield a Miss – this will be described in the next subsection. This means function calls by specification with missing resources can be done in bi-abduction.

Remark (Miss in consume). *It is argued in [45] that* consume *should not report* Miss*, as the missing resource might not actually be used by the function, which would then be UX unsound; we believe this preoccupation is not linked to* consume *but to function call semantics. If including* Miss *here is problematic, it should be up to the engine to correctly abort when doing function calls by specification in UX mode, rather than to* consume *to unnecessarily limit its capabilities.*

We thus have $\mathcal{O}_e$ for execution outcomes, and $\mathcal{O}_l$ for reasoning outcomes – one is returned by action execution, and one for predicate consumption (see Figure 3.9). They overlap on one particular feature of the engine, which is function calls by specification: when calling a function, any outcome can happen: Ok if the function call succeeds, Err if the function call faults (as defined by its specification), Miss if additional resources are needed to satisfy the precondition of the function specification, and LFail if the state is incompatible with the function specification.

A last change done to what [44] defines is that the path condition input of consume and produce is removed, disallowing both functions from inspecting the path condition – the functions instead directly return the path condition required for the resulting branches. This brings several advantages: firstly, it simplifies the properties for consume and produce, as there is no need to require that they strengthen the PC. Instead, the returned PC is always concatenated with the current PC, ensuring it can only be strengthened. Secondly, it simplifies the definition of consume, as whether the branch is cut or not (to satisfy OX and UX soundness) is also delegated to the engine. For instance, in UX all consumption branches that result in LFail can be dropped, as dropping branches is UX-sound – in OX however, this is not sound. A typical pattern in consume implementations is thus to evaluate the modified PC, and if an LFail is reached drop the branch depending on the mode. By omitting the PC from the arguments, this responsibility is thus delegated to the engine. Finally, this makes a particular optimisation of the

engine simpler to implement: incremental SMT solving.[8] This is a mechanism that allows storing an intermediary state in the SMT solver (a "checkpoint"), adding to the state, and later backtracking to the checkpointed state to evaluate another branch if needed. By ensuring `consume` and `produce` only return what to append to the PC, the checkpoint can easily be added before the call, and then each branch's PC added and evaluated. Allowing `consume` and `produce` to modify the PC directly would mean the PC must be parsed and filtered, to tell apart old from new terms – modified terms would make this even harder.

3.2.4 Bi-Abduction Engine

To support bi-abduction in the style of Infer:Pulse [40], `Miss` outcomes must be fixed. This enables true UX bug finding in under-specified or unspecified functions. For this, the state model must provide a `fix` function, that given the details of a miss error (these details being of type LVal `list` and returned with the outcome) returns a *list of sets of assertions* that must be produced to fix the missing error.

$$\text{fix}: \text{LVal list} \rightarrow \mathcal{P}(\text{Asrt}) \text{ list}$$

A *list* of different fixes is returned, which themselves are a set of assertions – this is because, for a given missing error, multiple fixes may be possible which causes branching. For instance, trying to load a cell that is not found in the state can lead to two possible fixes: either the cell is present, and the fix is $\{\langle \texttt{points\_to} \rangle (\hat{a}; \hat{x})\}$, either the cell is not present any more because it has been freed (which would then lead to a use after free error), giving the fix $\{\langle \texttt{freed} \rangle (\hat{a};)\}$.

Furthermore, we define fixes as assertions rather than lists of core predicates. This is because a fix often contains *more than just spatial information*. The simplest example is the fix for a missing cell at address $\hat{a}$ in the linear heap: we must ensure the fix can be soundly applied to an existing state, without any variable clashes. For this, we must use an existential, giving us the fix $\exists \hat{x}. \langle \texttt{points\_to} \rangle (\hat{a}; \hat{x})$. More generally, it is sometimes needed that the fix contains logical information, for instance specifying constraints on a variable.

3.2.5 Properties

We may now go over the properties that must be respected by the above defined functions for the soundness of the engine. Note we only mention the properties relating to state models – properties for the engine's overall soundness or the core semantics are out of the scope of this report.

For all of the properties we assume we have a symbolic state model $\mathbb{S}$, made of the concrete states RA $\Sigma \ni \sigma$, symbolic states set $\hat{\Sigma} \ni \hat{\sigma}$, actions $\mathcal{A}$ and core predicates Δ.

We split the properties into two parts: those relating to the symbolic nature of state and ensuring soundness with respect to a concrete state, and those relating to the parametricity of the state.

Symbolicness Properties

$$\theta, s, \sigma \models \bot \iff \sigma = \bot \qquad \text{(Empty Memory)}$$

The above is obtained for free, by lifting the $\models$ relation to the option RA.

$$\begin{aligned} &\alpha(\sigma, \vec{v}_i) \rightsquigarrow (o, \sigma', \vec{v}_o) \wedge \theta, s, \sigma \models \hat{\sigma} \wedge [\![\vec{\hat{v}}_i]\!]_{\theta,s} = \vec{v}_i \implies \exists \hat{\sigma}', \vec{\hat{v}}_o, \pi, \theta'. \\ &\quad \alpha(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{\hat{v}}_o, \pi) \wedge \mathsf{SAT}_{\theta',s}(\pi) \wedge \theta', s, \sigma' \models \hat{\sigma}' \wedge [\![\vec{\hat{v}}_o]\!]_{\theta',s} = \vec{v}_o \end{aligned} \qquad \text{(Memory Model OX Soundness)}$$

$$\alpha(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{\hat{v}}_o, \pi) \land \mathsf{SAT}_{\theta',s}(\pi) \land \theta, s, \sigma' \models \hat{\sigma}' \land$$
$$[\![\vec{\hat{v}}_o]\!]_{\hat{s},\pi} \rightsquigarrow (\vec{v}_o, \pi') \land [\![\vec{\hat{v}}_i]\!]_{\hat{s},\pi'} \rightsquigarrow (\vec{v}_i, \pi'') \implies \qquad\qquad \text{(Memory Model UX Soundness)}$$
$$\exists \sigma. \, \alpha(\sigma, \vec{v}_i) \rightsquigarrow (o, \sigma', \vec{v}_o) \land \theta, s, \sigma \models \hat{\sigma}$$

The above two properties are at the core of OX and UX soundness, respectively. The former ensures that if a concrete execution exists, then it must also exist in the symbolic semantics; it however does not ensure that more executions that those that actually exist may happen. The latter ensures that if a symbolic execution exists, then it must also exist in the concrete world; this time, no guarantees are made regarding the existence of such symbolic transition: for instance, symbolic semantics that never result in any branches are UX-sound (though they are of limited use).

Because incorrectness separation logic does execution backwards, we also prove UX soundness of the memory model backwards: if the resulting state has a concrete model, then so must the state before. In OX, soundness is instead forwards: if the starting state is sound, so must be the resulting state.

Compositionality Properties

$$\sigma \# \sigma_f \land \alpha(\sigma \cdot \sigma_f, \vec{\hat{v}}_i) \rightsquigarrow (o, \sigma', \vec{v}_o) \implies$$
$$\exists \sigma'', o', \vec{v}_o'. \, \alpha(\sigma, \vec{v}_i) \rightsquigarrow (o', \sigma'', \vec{v}_o') \land \qquad\qquad \text{(Frame subtraction)}$$
$$(o' \neq \mathtt{Miss} \implies o' = o \land \vec{v}_o' = \vec{v}_o \land \sigma' = \sigma'' \cdot \sigma_f)$$

$$\alpha(\sigma, \vec{v}_i) \rightsquigarrow (o, \sigma', \vec{v}_o) \land o \neq \mathtt{Miss} \land \sigma' \# \sigma_f \implies \qquad\qquad \text{(Frame Addition)}$$
$$\sigma \# \sigma_f \land \alpha(\sigma \cdot \sigma_f, \vec{v}_i) \rightsquigarrow (o, \sigma' \cdot \sigma_f, \vec{v}_o)$$

The above two properties are what enable the traditional frame rule of separation logic, but bringing it to operational semantics. Frame subtraction in particular is almost identical to the "Frame property" defined in [63]: if executing an action on a state from which a frame was removed yields a non-$\mathtt{Miss}$, then that action did not need that frame, and the frame can be added to the result. For frame addiction, which is needed in UX, we go backwards again: if a frame can be added to the state after executing an action and the action does not cause a $\mathtt{Miss}$, then it can also be added to the state before without interfering with the action execution.

$$\mathsf{consume}(\hat{\sigma}, \delta, \vec{\hat{v}}_i) \rightsquigarrow (\mathtt{Ok}, \hat{\sigma}_f, \vec{\hat{v}}_o, \pi) \implies \forall \theta, s, \sigma_f, \sigma_\delta.$$
$$\theta, s, \sigma_f \models \hat{\sigma}_f \land \theta, s, \sigma_\delta \models_\Delta \langle \delta \rangle (\vec{\hat{v}}_i; \vec{\hat{v}}_o) \land \sigma_f \# \sigma_\delta \implies \qquad \text{(Consume OX Soundness)}$$
$$\exists \sigma. \, \sigma = \sigma_\delta \cdot \sigma_f \land \theta, s, \sigma \models \hat{\sigma} \land \mathsf{SAT}_{\theta,s}(\pi)$$

$$(\forall o, \hat{\sigma}_f, \vec{\hat{v}}_o, \pi. \, \mathsf{consume}(\mathtt{OX}, \hat{\sigma}, \delta, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}_f, \vec{\hat{v}}_o, \pi) \Rightarrow o_c = \mathtt{Ok}) \implies \qquad \text{(Consume OX: No Path Drops)}$$
$$\exists \hat{\sigma}_f', \vec{\hat{v}}_o', \pi'. \, \mathsf{consume}(\mathtt{OX}, \hat{\sigma}, \delta, \vec{v}_i, \pi) \rightsquigarrow (\mathtt{Ok}, \hat{\sigma}_f', \vec{\hat{v}}_o', \pi')$$

$$\mathsf{consume}(\hat{\sigma}, \delta, \vec{\hat{v}}_i) \rightsquigarrow (\mathtt{Ok}, \hat{\sigma}_f, \vec{\hat{v}}_o, \pi) \implies \forall \theta, s, \sigma.$$
$$\theta, s, \sigma \models \hat{\sigma} \land \mathsf{SAT}_{\theta,s}(\pi) \implies \exists \sigma_\delta, \sigma_f. \qquad\qquad \text{(Consume UX Soundness)}$$
$$\sigma_\delta \# \sigma_f \land \sigma = \sigma_\delta \cdot \sigma_f \land \theta, s, \sigma_\delta \models_\Delta \langle \delta \rangle (\vec{v}_i; \vec{\hat{v}}_o) \land \theta, s, \sigma_f \models \hat{\sigma}_f$$

The above three properties show the soundness of $\mathsf{consume}$ for OX and UX operations. They are similar to Memory Model OX Soundness and Memory Model UX Soundness respectively. We also have an additional requirement for OX execution, stating that if there is no erroneous execution of $\mathsf{consume}$ then at least one successful execution exists.

$$\text{E{\scriptsize X}S{\scriptsize TORE}O{\scriptsize K}}$$
$$\mathsf{store}(\mathsf{ex}(\hat{x}), [\hat{x}']) \rightsquigarrow (\mathsf{Ok}, \mathsf{ex}(\hat{x}'), [], [])$$

$$\text{E{\scriptsize X}S{\scriptsize TORE}M{\scriptsize ISS}}$$
$$\mathsf{store}(\bot, [\hat{x}']) \rightsquigarrow (\mathsf{Miss}, \bot, [], [])$$

$$\text{E{\scriptsize X}C{\scriptsize ONS}O{\scriptsize K}}$$
$$\mathsf{consume}(\mathsf{ex}(\hat{x}), \mathsf{ex}, []) \rightsquigarrow (\mathsf{Ok}, \bot, [\hat{x}], [])$$

$$\text{E{\scriptsize X}P{\scriptsize ROD}}$$
$$\mathsf{produce}(\bot, \mathsf{ex}, [], [\hat{x}]) \rightsquigarrow (\mathsf{ex}(\hat{x}), [])$$

$$\text{E{\scriptsize X}F{\scriptsize IX}}$$
$$\mathsf{fix}\,[] = [\{\exists \hat{x}.\, \langle \mathsf{ex} \rangle (; \hat{x})\}]$$

Figure 3.10: Some rules for Ex

$$\theta, s, \sigma_f \models \hat{\sigma}_f \wedge \theta, s, \sigma_\delta \models_\Delta \langle \delta \rangle (\vec{v}_i; \vec{v}_o) \wedge \sigma_f \,\#\, \sigma_\delta \implies$$
$$\exists \hat{\sigma}.\, \mathsf{produce}(\hat{\sigma}_f, \delta, \vec{v}_i, \vec{v}_o) \rightsquigarrow (\hat{\sigma}, \pi) \wedge \mathsf{SAT}_{\theta,s}(\pi) \wedge \theta, s, (\sigma_f \cdot \sigma_\delta) \models \hat{\sigma}$$

(Produce: OX Soundness)

$$\mathsf{produce}(\hat{\sigma}_f, \delta, \vec{v}_i, \vec{v}_o) \rightsquigarrow (\hat{\sigma}, \pi) \implies$$
$$\forall \theta, s, \sigma.\, \mathsf{SAT}_{\theta,s}(\pi) \wedge \theta, s, \sigma \models \hat{\sigma} \implies \exists \sigma_\delta, \sigma_f.$$
$$\sigma_\delta \,\#\, \sigma_f \wedge \sigma = \sigma_\delta \cdot \sigma_f \wedge \theta, s, \sigma_\delta \models_\Delta \langle \delta \rangle (\vec{v}_i; \vec{v}_o) \wedge \theta, s, \sigma_f \models \hat{\sigma}_f$$

(Produce: UX Soundness)

Similarly, these two properties show soundness of produce for OX and UX operation. For the former, we show that if two disjoint states exist in the concrete world and one of the two satisfies a core predicate, then producing that predicate onto the other yields their composition. For the latter, we state that if produce yields a branch that is satisfiable then there must exist a model of the result.

There are no properties that fix must satisfy – as it simply results in assertions, it cannot break soundness. Either it produces no fixes or only incompatible fixes and we vanish, which is UX sound, or it finds fixes that work, in which case we already know the execution exists in the concrete world and we remain UX sound.

3.3 State Models

Now that the CSE engine is defined and the properties of state models are defined, we focus on state models that can be later used to build up complex state. We first look at the base case, Ex, followed by the state model of knowledge or agreement, A{\scriptsize G}. Finally, we look at an in-between, F{\scriptsize RAC}, that allows exclusive ownership with a form of sharing.

3.3.1 Exclusive

The exclusive state model $\mathrm{Ex}(X)$ presented earlier is the simplest form of state model. It asserts ownership of an element of the set X, provides one core predicate ex, and two actions to modify the state, load and store. The ex predicate has no ins, and one out: the value stored.

$$\mathrm{Ex}(X) \triangleq \mathsf{ex}(x \colon X)$$
$$|\mathsf{ex}(x)| \triangleq \bot$$
$$\mathsf{ex}(x_1) \cdot \mathsf{ex}(x_2) \text{ is always undefined}$$

The first notation, $\mathrm{Ex}(X)$ is the state model instantiation from the set X to the Ex resource algebra. The notation $\mathsf{ex}(x \colon X)$ stands for $\{\mathsf{ex}(x) : \forall x \in X\}$ – here "$\mathsf{ex}(x)$" refers to the particular element of the $\mathrm{Ex}(X)$ RA where the value is $x \in X$.

CExStoreOk
$$\mathsf{store}(\mathrm{ex}(x), [x']) \rightsquigarrow (\mathsf{Ok}, \mathrm{ex}(x'), [])$$

ExStoreOk
$$\mathsf{store}(\mathrm{ex}(\hat{x}), [\hat{x}']) \rightsquigarrow (\mathsf{Ok}, \mathrm{ex}(\hat{x}'), [], [])$$

CExStoreMiss
$$\mathsf{store}(\bot, [x']) \rightsquigarrow (\mathsf{Miss}, \bot, [])$$

ExStoreMiss
$$\mathsf{store}(\bot, [\hat{x}']) \rightsquigarrow (\mathsf{Miss}, \bot, [], [])$$

Figure 3.11: Side by side comparison of concrete and symbolic action rules for Ex – concrete rules are prefixed with C.

AgLoadOk
$$\mathsf{load}(\mathrm{ag}(\hat{x}), []) \rightsquigarrow (\mathsf{Ok}, \mathrm{ag}(\hat{x}), [\hat{x}], [])$$

AgConsOk
$$\mathsf{consume}(\mathrm{ag}(\hat{x}), \mathsf{ag}, []) \rightsquigarrow (\mathsf{Ok}, \mathrm{ag}(\hat{x}), [\hat{x}], [])$$

AgProdBot
$$\mathsf{produce}(\bot, \mathsf{ag}, [], [\hat{x}]) \rightsquigarrow (\mathrm{ag}(\hat{x}), [])$$

AgProdEq
$$\mathsf{produce}(\mathrm{ag}(\hat{x}), \mathsf{ag}, [], [\hat{x}']) \rightsquigarrow (\mathrm{ag}(\hat{x}), [\hat{x} = \hat{x}'])$$

Figure 3.12: Some rules for Ag

See Figure 3.10 for an extract of the rules for Ex – the full definition can be found in section A.1. We also show side by side the rules for the concrete and symbolic actions in Figure 3.11: in particular we note that they are strikingly similar, the latter simply being lifted from the former.

The only difference for Ex between RAs and PCMs is the absence of a 0, which is instead $\bot$; in fact $\mathrm{Ex}(X)^?$ is strictly equivalent to $\mathrm{Ex}_{\mathrm{PCM}}(X)$.

To provide at least one full of example of state model, we also include the definition of the symbolic interpretation and core predicate satisfiability relations – both are trivial, simply checking for equality:

ExSymInterpretation
$$\frac{[\![\hat{x}]\!]_{\theta,s} = x}{\theta, s, \mathrm{ex}(x) \models \mathrm{ex}(\hat{x})}$$

ExPredSat
$$\frac{\sigma = \mathrm{ex}(x)}{\sigma \models_\Delta \langle \mathsf{ex} \rangle([]; [x])}$$

3.3.2 Agreement

The agreement state model $\mathrm{Ag}(X)$ is the counterpart to the exclusively owned state model – it allows sharing, and is duplicable. It defines one predicate, ag, and a single action, load. Again, ag has no ins and one out, the value. Mutating state is not sound with a duplicable resource, as that does not satisfy Frame subtraction or Frame Addition: a frame that is compatible with the original value is not compatible with the modified value, as they are not equal any more.

$$\mathrm{Ag}(X) \triangleq \mathrm{ag}(x\colon X)$$

$$|\mathrm{ag}(x)| \triangleq \mathrm{ag}(x)$$

$$\mathrm{ag}(x) \cdot \mathrm{ag}(x') \triangleq \begin{cases} \mathrm{ag}(x) & \text{if } x = x' \\ \text{undefined} & \text{otherwise} \end{cases}$$

See Figure 3.12 for an extract of its rules. Its duplicable nature can be seen in the rules AgConsOk and AgProdEq: consuming the predicate does not turn the state into $\bot$ but leaves it untouched, while producing into an already present state succeeds as long as the predicate's out and the state are equal, as seen in the path condition: $[\hat{x} = \hat{x}']$. An observation that can be made from this is that it is always sound for consumption of a state to result in *at least* the input's core; as this knowledge is duplicable, it can always be kept anyway.

FRACCONSALL
$\mathsf{consume}(\mathrm{frac}(\hat{x},\hat{q}),\mathsf{frac},[\hat{q}']) \rightsquigarrow (\mathsf{Ok},\bot,[\hat{x}],[\hat{q}=\hat{q}'])$

FRACCONSSOME
$\mathsf{consume}(\mathrm{frac}(\hat{x},\hat{q}),\mathsf{frac},[\hat{q}']) \rightsquigarrow (\mathsf{Ok},\mathrm{frac}(\hat{x},\hat{q}-\hat{q}'),[\hat{x}],[0<\hat{q}'<\hat{q}])$

FRACCONSMISS
$\mathsf{consume}(\mathrm{frac}(\hat{x},\hat{q}),\mathsf{frac},[\hat{q}']) \rightsquigarrow (\mathsf{Miss},\mathrm{frac}(\hat{x},\hat{q}),[\hat{q}'-\hat{q}],[\hat{q}<\hat{q}'\leq 1])$

FRACCONSFAIL
$\mathsf{consume}(\mathrm{frac}(\hat{x},\hat{q}),\mathsf{frac},[\hat{q}']) \rightsquigarrow (\mathsf{LFail},\mathrm{frac}(\hat{x},\hat{q}),[],[\hat{q}'\leq 0 \vee 1<\hat{q}'])$

Figure 3.13: consume rules for FRAC

Unlike EX, AG is *not cancellative*: in other words, $a \cdot b = a \cdot c \nRightarrow b = c$, since from the definition of composition we have $\mathrm{ag}(x) \cdot \bot = \mathrm{ag}(x) \cdot \mathrm{ag}(x)$ and $\bot \neq \mathrm{ag}(x)$. Cancellativity is a property that was initially thought to be required for separation algebras [16, 22], but that has since been dropped [35, 57].

The agreement state model is thus useful to model immutable, shareable state – for instance, it is used in the JavaScript instantiation (discussed in §4.2) to represent the address of an object's metadata, as this information is immutable. This was first brought to a CSE engine in JaVerT [29, 30], though when that was developed the notion of an "agreement state model" did not exist and as such it was not identified as one.

3.3.3 Fraction

The fractional state model $\mathrm{FRAC}(X)$ allows for exclusive ownership while allowing sharing, via fractional permissions. This idea originates from [9, 10], where the "points to" assertion is equipped with a fraction q: a state satisfying $a \underset{1}{\mapsto} x$ has full ownership (read and write) of x, and can be split into $a \underset{q}{\mapsto} x * a \underset{1-q}{\longmapsto} x$, creating two states with read-only permissions. The FRAC memory model is a generalisation of this at the value level. It defines one predicate, frac, that has one in, the fraction q, and one out, the value x. Just like EX, it provides a load and a store action – the only difference being that a fraction of 1 is required to modify the value. Because the maximum fraction is 1, we know that when it is 1 no frame can have ownership of the value, and it can thus safely be modified.

$$\mathrm{FRAC}(X) \triangleq \mathrm{frac}(x\colon X, q\colon (0;1])$$

$$|\mathrm{frac}(x,q)| \triangleq \bot$$

$$\mathrm{frac}(x,q) \cdot \mathrm{frac}(x',q') \triangleq \begin{cases} \mathrm{frac}(x,q+q') & \text{if } x=x' \wedge q+q' \leq 1 \\ \text{undefined} & \text{otherwise} \end{cases}$$

We put the fraction q in the in-values of the frac core predicate to allow consume to subtract exactly the permission required from the state – if both the permission and the value were out-values, then consume would always evaluate to $\bot$, cancelling the main advantage of this state model. The rules for consume can be seen in Figure 3.13. We note here that in the symbolic state model, the fraction part is also lifted to the symbolic realm, allowing for a symbolic permission.

Because the outcome of actions depends on the amount of permission owned, FRAC is also a good example to show how one may adapt a concrete state model into a symbolic state model, by moving the relevant concrete conditions into the path condition. The rules for the concrete and symbolic store are shown in Figure 3.14. Note how, between CFRACSTOREPERM and FRACSTOREPERM, the condition $q < 1$

CFRACSTOREOK
$$\frac{q = 1}{\texttt{store}(\mathrm{frac}(x, q), [x']) \rightsquigarrow (\texttt{Ok}, \mathrm{frac}(x', q), [])}$$

CFRACSTOREPERM
$$\frac{q < 1}{\texttt{store}(\mathrm{frac}(x, q), [x']) \rightsquigarrow (\texttt{Miss}, \mathrm{frac}(x, q), [1 - q])}$$

FRACSTOREOK
$$\texttt{store}(\mathrm{frac}(\hat{x}, \hat{q}), [\hat{x}']) \rightsquigarrow (\texttt{Ok}, \mathrm{frac}(\hat{x}', \hat{q}), [], [\hat{q} = 1])$$

FRACSTOREPERM
$$\texttt{store}(\mathrm{frac}(\hat{x}, \hat{q}), [\hat{x}']) \rightsquigarrow (\texttt{Miss}, \mathrm{frac}(\hat{x}, \hat{q}), [1 - \hat{q}], [\hat{q} < 1])$$

Figure 3.14: `store` rules for FRAC

is replaced by $\hat{q} < 1$, and is returned in the path condition, rather than being checked for in the rule itself, since the value of $\hat{q}$ depends on the current PC.

While this state model is conceived taking into account fractional permissions, other separation algebras exist to tackle the same problem. For instance, [9] also proposes a counting permissions system: $a \underset{0}{\mapsto} x$ is a writing permission, and it can issue reading permissions denoted $a \rightharpoonup x$, such that $a \underset{q}{\mapsto} x \iff a \underset{q+1}{\longmapsto} x * a \rightharpoonup x$. Though we do not define it in full, this could also be implemented as a state model, providing two core predicates: $\langle \texttt{write} \rangle(q; x)$ and $\langle \texttt{read} \rangle(; x)$, with every consumption of `read` from a writeable state increasing q, and inversely every production of `read` decreasing it, and checking $q = 0$ when using `store`. This shows state models seem to be the right abstraction, as they support a variety of resource algebras.

3.4 State Model Transformers

While the above state models can prove useful, they are limited in use, only allowing the storage of one value. One would likely want to model state that is more complex than this, storing a range of values in a variety of ways. Iris introduces the idea of RA constructions [35] – this idea has then been adapted to Gillian in [2], with PCMs. Here we revisit these constructions, porting them back to RAs. We call them *state model transformers*,[9] or state transformers.

3.4.1 Sum

The sum state model $\textsc{Sum}(\mathbb{S}_1, \mathbb{S}_2)$, also written $\mathbb{S}_1 + \mathbb{S}_2$, allows representing states that can be in either the states of $\mathbb{S}_1$ or $\mathbb{S}_2$. This state model transformer does not introduce any core predicates or actions, and instead simply lifts those of the two underlying state models. Let $\mathcal{A}_1$ and Δ_1 the actions and core predicates of $\mathbb{S}_1$, and $\mathcal{A}_2$ and Δ_2 those of $\mathbb{S}_2$, the actions of the sum are defined as $\mathcal{A} = \{\texttt{L}\alpha : \alpha \in \mathcal{A}_1\} \uplus \{\texttt{R}\alpha : \alpha \in \mathcal{A}_2\}$, and the core predicates as $\Delta = \{\texttt{L}\delta : \delta \in \Delta_1\} \uplus \{\texttt{R}\delta : \delta \in \Delta_2\}$. Actions and predicates are simply tagged with what side of the sum they originate from.

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[9]Not to be confused with "state monad transformers", which allow adding state to a monad.

$$\text{Given } \mathit{wrap}_l(x) = \begin{cases} \bot & \text{if } x = \bot \\ l(x) & \text{otherwise} \end{cases} \text{ and } \mathit{unwrap}_l(x_l) = \begin{cases} \bot & \text{if } x = \bot \\ x & \text{if } x_l = l(x) \\ \text{undefined} & \text{otherwise} \end{cases}$$

SUMLACTION
$$\frac{\hat{\sigma} = \mathit{unwrap}_l(\hat{\sigma}_l) \qquad \alpha(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{\hat{v}}_o, \pi) \qquad \hat{\sigma}'_l = \mathit{wrap}_l(\hat{\sigma}') \qquad o \neq \texttt{Miss}}{\mathsf{L}\alpha(\hat{\sigma}_l, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}'_l, \vec{\hat{v}}_o, \pi)}$$

SUMLACTIONMISS
$$\frac{\hat{\sigma} = \mathit{unwrap}_l(\hat{\sigma}_l) \qquad \alpha(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (\texttt{Miss}, \hat{\sigma}', \vec{\hat{v}}_o, \pi) \qquad \hat{\sigma}'_l = \mathit{wrap}_l(\hat{\sigma}')}{\mathsf{L}\alpha(\hat{\sigma}_l, \vec{\hat{v}}_i) \rightsquigarrow (\texttt{Miss}, \hat{\sigma}'_l, \text{`l'} :: \vec{\hat{v}}_o, \pi)}$$

SUMLFIX
$$\frac{\mathbb{S}_1.\texttt{fix } \vec{\hat{v}}_i = a}{\texttt{fix `l'} :: \vec{\hat{v}}_i = a}$$

Figure 3.15: Rules for handling $\texttt{Miss}$ in SUM

Let Σ_1 and Σ_2 be the carrier sets of the RAs of $\mathbb{S}_1$ and $\mathbb{S}_2$ respectively; the RA of $\text{SUM}(\mathbb{S}_1, \mathbb{S}_2)$ is:

$$\text{SUM}(\mathbb{S}_1, \mathbb{S}_2) \triangleq \mathbb{S}_1 + \mathbb{S}_2 \triangleq l(\sigma \colon \Sigma_1) \mid r(\sigma \colon \Sigma_2)$$
$$l(\sigma) \cdot l(\sigma') \triangleq l(\sigma \cdot \sigma')$$
$$r(\sigma) \cdot r(\sigma') \triangleq r(\sigma \cdot \sigma')$$
$$|l(\sigma)| \triangleq \begin{cases} \bot & \text{if } |\sigma| = \bot \\ l(|\sigma|) & \text{otherwise} \end{cases}$$
$$|r(\sigma)| \triangleq \begin{cases} \bot & \text{if } |\sigma| = \bot \\ r(|\sigma|) & \text{otherwise} \end{cases}$$

In §3.1.2 we showed how SUM_{PCM} must disallow the unit on each side to be sound. In RAs however, this is not an issue any more, as the underlying state models can simply not define a unit. We may also redefine $\texttt{is\_exclusively\_owned}$ to take into account that the unit is not present in the carrier set, giving a much simpler definition:

$$\texttt{is\_exclusively\_owned } \sigma \triangleq \nexists \sigma' \in \Sigma. \, \sigma \, \# \, \sigma'$$

The rules of the SUM state model are otherwise straightforward, lifting $\texttt{execute\_action}$, $\texttt{consume}$ and $\texttt{produce}$ to the sum from the underlying state models. The only minor difference is when a $\texttt{Miss}$ occurs – because fixes are only generated from a LVal $\texttt{list}$, SUM has no way of knowing which side raised the $\texttt{Miss}$, and can thus not call the $\texttt{fix}$ function of the correct side. To avoid this, when the outcome of $\texttt{execute\_action}$ or $\texttt{consume}$ is a $\texttt{Miss}$ we concatenate an identifier ('l' or 'r') to the values to allow discriminating between the two. An extract of these rules is shown in Figure 3.15.

This example is interesting in that it also shows the constraint imposed by the fact $\bot \notin \hat{\Sigma}$: it is often needed to define auxiliary *unwrap* and *wrap* functions, that get the state out a value if it is not $\bot$, and returns $\bot$ otherwise. In other words, $\mathit{unwrap} \colon \hat{\Sigma}_1^? \rightharpoonup \hat{\Sigma}_2^?$ and $\mathit{wrap} \colon \hat{\Sigma}_2^? \rightarrow \hat{\Sigma}_1^?$.[10] While these are straightforward to define and use, they can clutter rules a bit, in favour of not needing to handle $\bot$ and non-$\bot$ cases separately.

Compared to SUM_{PCM}, we now need two less properties to guarantee a sound transition from one side of the sum to the other: indeed, we do not need to exclude the unit of either sides from the carrier set, since $\bot \notin \Sigma_1$ and $\bot \notin \Sigma_2$ already, by definition.

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[10]Note how *unwrap* is a partial function; for instance in the case of a sum, attempting to unwrap a state from the left side into the right side is undefined.

3.4.2 Product

The product state model $\textsc{Product}(\mathbb{S}_1, \mathbb{S}_2)$, also written $\mathbb{S}_1 \times \mathbb{S}_2$, allows representing pairs of states, where each side belongs to a specific state model. For instance, one could chose to represent the $\textsc{Frac}(X)$ state model as $\textsc{Ex}(X) \times \textsc{Frac}_\mathbb{Q}$, where $\textsc{Frac}_\mathbb{Q}$ is an exclusively owned rational in $(0; 1]$, and with composition defined as addition. This is, in fact, the way Iris defines $\textsc{Frac}$ [33] – however, this approach makes using the state model less practical, as one would need to define a separate predicate for each side.

Iris also defines the product RA, by simply lifting pointwise the operations on each side of the product. We call this initial definition $\textsc{Product}_0$, defining it as:

$$\textsc{Product}_0(\mathbb{S}_1, \mathbb{S}_2) \triangleq \Sigma_1 \times \Sigma_2$$
$$(\sigma_l, \sigma_r) \cdot (\sigma_l', \sigma_r') \triangleq (\sigma_l \cdot \sigma_l', \sigma_r \cdot \sigma_r')$$
$$|(\sigma_l, \sigma_r)| \triangleq \begin{cases} (|\sigma_l|, |\sigma_r|) & \text{if } |\sigma_l| \neq \bot \wedge |\sigma_r| \neq \bot \\ \bot & \text{otherwise} \end{cases}$$

This definition, while straightforward, has a problem: while its sets of states is $\Sigma_1 \times \Sigma_2$, we still have that $\bot \notin \Sigma$. If an empty state ($\bot$) produces a core predicate for one of its sides, what should happen to the other side? While one side becomes defined, such that $\sigma_l \in \Sigma_1$, $(\sigma_l, \bot)$ is not a valid state, since $\bot \notin \Sigma_2$. This is a problem, since in our CSE engine all states are initialised as $\bot$ and can only be built upon via actions or predicate production, one predicate at a time. It seems here that the consume-produce interface of our engine, which allows creating state bit by bit, can be unadapted for certain RAs. Of course one can define additional core predicates for a specific product instantiation that combines predicates of underlying state models, however this goes against the core idea of state model transformers to minimise the amount of effort needed to construct new working state models. We also note that this problem is not encountered in $\textsc{Product}_{\text{PCM}}$, since its unit could be defined as $(0_1, 0_2)$ and work straightforwardly.

We define an alternative $\textsc{Product}$ RA that is more suited to this engine:

$$\textsc{Product}(\mathbb{S}_1, \mathbb{S}_2) \triangleq \mathbb{S}_1 \times \mathbb{S}_2 \triangleq (\Sigma_1^? \times \Sigma_2^?) \setminus \{(\bot, \bot)\}$$
$$(\sigma_l, \sigma_r) \cdot (\sigma_l', \sigma_r') \triangleq (\sigma_l \cdot \sigma_l', \sigma_r \cdot \sigma_r')$$
$$|(\sigma_l, \sigma_r)| \triangleq \begin{cases} \bot & \text{if } |\sigma_l| = \bot \wedge |\sigma_r| = \bot \\ (|\sigma_l|, |\sigma_r|) & \text{otherwise} \end{cases}$$

We take advantage of the option state model $-^?$ and use that instead, thus allowing either sides of the product to be $\bot$. Crucially however, we exclude the $(\bot, \bot)$ state and ensure in the *wrap* helper that it never occurs (see Figure 3.16). This is important, because it means *exclusivity* is carried: if for (σ_l, σ_r) we have `is_exclusively_owned` $\sigma_l \wedge$ `is_exclusively_owned` σ_r, then it also follows that `is_exclusively_owned` (σ_l, σ_r). If we had not excluded $(\bot, \bot)$, then it would always be the case that $(\sigma_l, \sigma_r) \cdot (\bot, \bot)$ holds, meaning that the state could never be exclusively owned. This is unpractical, as it would mean that disposing of a product is not frame preserving.

3.4.3 Freeable

The $\textsc{Freeable}(\mathbb{S})$ state model transformer allows extending a state model with a `free` action, that allows freeing a part of memory. The freed memory can then not be accessed, and attempting to use it raises a use-after-free error. The notion of marking freed memory as freed, rather than simply discarding it, comes from Incorrectness Separation Logic [58], as indeed simply disposing of the memory would break Frame Addition.

Given $wrap(x, y) = \begin{cases} \bot & \text{if } x = \bot \wedge y = \bot \\ (x, y) & \text{otherwise} \end{cases}$ and $unwrap(s) = \begin{cases} (\bot, \bot) & \text{if } s = \bot \\ (x, y) & \text{otherwise} \end{cases}$

PRODUCTLACTION

$$\frac{(\hat{\sigma}_l, \hat{\sigma}_r) = unwrap(\hat{\sigma}) \qquad \alpha(\hat{\sigma}_l, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}'_l, \vec{\hat{v}}_o, \pi) \qquad \hat{\sigma}' = wrap(\hat{\sigma}'_l, \hat{\sigma}_r) \qquad o \neq \texttt{Miss}}{\mathsf{L}\alpha(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{\hat{v}}_o, \pi)}$$

Figure 3.16: Example of action rules in PRODUCT

FREEABLEACTIONFREE
$\texttt{free}(l(\hat{\sigma}), []) \rightsquigarrow (\texttt{Ok}, r(\text{ex}(\varnothing)), [], [\texttt{is\_exclusively\_owned } \hat{\sigma}])$

FREEABLEACTIONFREEERR
$\texttt{free}(l(\hat{\sigma}), []) \rightsquigarrow (\texttt{Miss}, l(\hat{\sigma}), \texttt{fix\_owned } \hat{\sigma}, [\neg\texttt{is\_exclusively\_owned } \hat{\sigma}])$

FREEABLEACTIONFREEMISS
$\texttt{free}(\bot, []) \rightsquigarrow (\texttt{Miss}, \bot, \texttt{fix\_owned } \bot, [])$

FREEABLEACTIONDOUBLEFREE
$\texttt{free}(r(\text{ex}(\varnothing)), []) \rightsquigarrow (\texttt{Err}, r(\text{ex}(\varnothing)), [], [])$

Figure 3.17: Symbolic action rule of FREEABLE

FREEABLE is similar to the ONESHOT RA of Iris, with the key difference being that the $\texttt{freed}$ predicate used to mark a resource as freed is *not duplicable*. Iris defines $\text{ONESHOT}(X) \triangleq \text{Ex}(\{1\}) + \text{AG}(X)$ [35], which is not UX-sound; in order for the switch of the side of a sum to be UX-sound, the resulting state (here, $\varnothing$) must be exclusively owned, which AG never is. This is not a problem for Iris, that is only concerned with OX soundness, but because our goal is to make our state models both OX and UX sound for additional flexibility, we need to make this change.

To ensure frame preservation, the underlying state model must provide an additional function, equivalent to the aforementioned property "exclusive": $\texttt{is\_exclusively\_owned}: \Sigma \rightarrow \mathbb{B}$. Because actions in the symbolic realm need to be sound with respect to the concrete realm, we then lift its definition to $\texttt{is\_exclusively\_owned}: \hat{\Sigma} \rightarrow \text{LVal}$, making it return a symbolic value that can be appended to the path condition.

While not needed for the core and compositional engines, the bi-abductive engine requires that misses may be fixed; FREEABLE($\mathbb{S}$) however cannot access directly the core predicates of $\mathbb{S}$ to provide fixes when freeing an empty state. As such, the state model must also provide a $\texttt{fix\_owned}: \hat{\Sigma}^? \rightarrow \text{LVal } \texttt{list}$ function that returns possible fixes to make the given state exclusively owned. This also implies that for any state σ of $\mathbb{S}$, if it is not exclusively owned then there must exist a bigger state σ' such that $\sigma \preccurlyeq \sigma'$ and $\texttt{is\_exclusively\_owned } \sigma'$. This is not the case, for instance, for AG, meaning a construction such as FREEABLE(AG(X)) is not sound.

Like Iris, we define the FREEABLE($\mathbb{S}$) state model by construction: FREEABLE($\mathbb{S}$) $\triangleq \mathbb{S} + \text{Ex}(\{\varnothing\})$, which allows all associated rules to be kept. For clarity, we rename the core predicate R ex (defined by $\text{Ex}(\{\varnothing\})$) as $\texttt{freed}$. We also extend its actions with the $\texttt{free}$ action.

Because FREEABLE is constructed via other state model transformers, we only need to describe the rules for $\texttt{free}$, which are shown in Figure 3.17 – the rest of the construction is already defined. This shows how simpler state models can be extended while alleviating the user from the burden of proving the soundness of the base construction.

Conveniently, use-after-free errors are already handled by the sum construction, thanks to the rule SUMLACTIONINCOMPAT which forbids actions from one side of the sum to be executed on the other side.

3.4.4 Partial Map

The partial map state model transformer $\text{PMAP}(I, \mathbb{S})$ is one of the most important state model transformers, as it allows having multiple state instances at different addresses and accessing them. It is constructed from a domain, $I \subseteq \text{Val}$, and a state model for the codomain. For instance, the traditional SL linear heap can be modelled as $\text{PMAP}(\mathbb{N}, \text{Ex}(\text{Val}))$. The C, JavaScript and Rust memory models can also all be modelled using a PMAP [2], despite them being radically different languages.

It defines one action, `alloc`, and one core predicate, `domainset`. It also *lifts* all actions and predicates of the underlying model, adding to them an index argument or in-value. For instance, while for $\text{Ex}(X)$ one has the core predicate $\langle \text{ex} \rangle(; x)$, for $\text{PMAP}(\text{Ex}(X))$ one would instead have predicates of the form $\langle \text{ex} \rangle(i; x)$, which corresponds to the "points to" assertion $i \mapsto x$. We define its RA as:

$$\text{PMAP}(I, \mathbb{S}) \triangleq (I \xrightarrow{fin} \mathbb{S}.\Sigma) \times \mathcal{P}(I)^?$$

$$(h, d) \cdot (h', d') \triangleq (h'', d'')$$

$$\text{where } h'' \triangleq \lambda i. \begin{cases} h(i) \cdot h'(i) & \text{if } i \in \text{dom}(h) \cap \text{dom}(h') \\ h(i) & \text{if } i \in \text{dom}(h) \setminus \text{dom}(h') \\ h'(i) & \text{if } i \in \text{dom}(h') \setminus \text{dom}(h) \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$\text{and } d'' \triangleq \begin{cases} d & \text{if } d' = \bot \\ d' & \text{if } d = \bot \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$\text{and } d'' = \bot \lor \text{dom}(h'') \subseteq d''$$

$$|(h, d)| \triangleq \begin{cases} \bot & \text{if } \text{dom}(h') = \emptyset \\ (h', \bot) & \text{otherwise} \end{cases}$$

$$\text{where } h' \triangleq \lambda i. \begin{cases} |h(i)| & \text{if } i \in \text{dom}(h) \land |h(i)| \neq \bot \\ \text{undefined} & \text{otherwise} \end{cases}$$

An element of PMAP is of the form (h, d), where h are the mappings from locations to states, and d is the *domain set*, the set of indices that are known to exist. It is defined such that $\text{dom}(h) \subseteq d$ always holds. The domain set can also be $\bot$, in which case there is no knowledge on what addresses exist. The aim of the domain set is to allow separating invalid accesses from misses: if $d \neq \bot$, then any access to $i \notin d$ leads to an `Err` outcome, however if $i \in d$ and the binding does not exist in the heap, then a `Miss` is raised.

As a consequence of this, $\text{PMAP}(I, \mathbb{S})$ has an additional requirement for $\mathbb{S}$: any action execution on $\bot$ must lead to a `Miss`.

$$\alpha(\bot, \vec{v}_i) \rightsquigarrow (o, \sigma', \vec{v}_o, \pi) \implies o = \texttt{Miss}$$

This is needed for frame preservation, otherwise one could run into a case where the action of a cell in the heap succeeds on an empty heap, but an empty domain set would be compatible with the state, despite its composition with the state leading to a different outcome (an `Err`).

This constraint also comes from the fact that PMAP cannot automatically yield a `Miss` when executing actions on missing cells, and must instead execute the action on $\bot$. Indeed, because $\text{PMAP}(I, \mathbb{S})$ can accept any underlying state model, it does not "know" what values are needed to fix the missing cell. Instead, it needs $\mathbb{S}$ to raise a `Miss` that it can then lift with an index.

PMAPGETMATCH
$$\frac{(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad \hat{i}' \in \mathrm{dom}(\hat{h}) \qquad \hat{\sigma}_i = \hat{h}(\hat{i}')}{\mathsf{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, [\hat{i} = \hat{i}'])}$$

PMAPGETADD
$$\frac{(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad \hat{i} \notin \mathrm{dom}(\hat{h}) \qquad \hat{d} \neq \bot}{\mathsf{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \bot, [\hat{i} \notin \mathrm{dom}(\hat{h}) \wedge \hat{i} \in \hat{d}])}$$

PMAPGETBOTDOMAIN
$$\frac{(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad \hat{i} \notin \mathrm{dom}(\hat{h}) \qquad \hat{d} = \bot}{\mathsf{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \bot, [\hat{i} \notin \mathrm{dom}(\hat{h})])}$$

Figure 3.18: Rules for `get` for PMAP

PMAPALLOC
$$\frac{\hat{d} \neq \bot \qquad \hat{i} = \mathsf{fresh}\ I \qquad \hat{\sigma}_i = \mathsf{instantiate}(\vec{\hat{v}}_i) \qquad \hat{h}' = \hat{h}[\hat{i} \leftarrow \hat{\sigma}_i] \qquad \hat{d}' = \hat{d} \uplus \{\hat{i}\}}{\mathsf{alloc}((\hat{h}, \hat{d}), \vec{\hat{v}}_i) \rightsquigarrow (\mathsf{Ok}, (\hat{h}', \hat{d}'), [\hat{i}], [\hat{i} = \hat{i}])}$$

PMAPALLOCMISS
$$\frac{(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \qquad \hat{d} = \bot}{\mathsf{alloc}(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (\mathsf{Miss}, \hat{\sigma}, [\text{`domainset'}], [])}$$

Figure 3.19: Rules for `alloc` for PMAP

As mentioned before, PMAP lifts actions and core predicates, adding to them an index. Getting the state associated to an index is not simple, and will in fact be the target of several optimisations in §3.5. For now we consider the simplest form of symbolic matching: either the index is present in the heap, or it is not, in which case if the domain set is not $\bot$, we also know that it must be in the domain set. This creates three rules one must consider. For instance, given a state $([\hat{a} \mapsto \hat{x}, \hat{b} \mapsto \hat{y}], \bot)$, when an action is executed with index $\hat{c}$, three branches are created: either $\hat{a} = \hat{c}$, or $\hat{b} = \hat{c}$, or $\hat{c}$ is not part of the current heap, giving us $\hat{c} \notin \{\hat{a}, \hat{b}\}$.

Because most `execute_action`, `consume` and `produce` rules of PMAP require retrieving states at a specific index, we introduce an abstraction over this: the `get` and `set` functions. The former allows getting the state at an index *with branching*: $\mathsf{get} \colon \mathrm{PMAP}(I, \mathbb{S})^? \to I \to \mathcal{P}(I \times \Sigma^? \times \Pi)$. It returns the index to modify exactly the returned state, the state itself, and a path condition corresponding to the branch. The `set` function has signature $\mathsf{set} \colon \mathrm{PMAP}(I, \mathbb{S})^? \to I \to \Sigma^? \to \mathrm{PMAP}(I, \mathbb{S})^?$. Unlike for `get`, we do not allow branching when setting: because branching already happened when getting, and `get` returns the index to use to modify the state, we can use this new index and be guaranteed that it is sound. For instance for the above example, the three executions are $\mathsf{get}(\hat{\sigma}, \hat{c}) \rightsquigarrow (\hat{a}, \hat{\sigma}_a, [\hat{c} = \hat{a}])$, $\mathsf{get}(\hat{\sigma}, \hat{c}) \rightsquigarrow (\hat{b}, \hat{\sigma}_b, [\hat{c} = \hat{b}])$ and $\mathsf{get}(\hat{\sigma}, \hat{c}) \rightsquigarrow (\hat{c}, \bot, [\hat{c} \notin \{\hat{a}, \hat{b}\}])$. We present the rules for `get` in Figure 3.18.

Another unseemingly complex aspect of PMAP is allocation, for which the rules are presented in Figure 3.19. To allow instantiating, the wrapped state model must provide an `instantiate` function defined as $\mathsf{instantiate} \colon \mathrm{Val} \to \Sigma$, such that given the arguments passed to `alloc`, it returns a newly instantiated state. An initial definition of PMAP had `alloc` always succeed; for instance, a $\bot$ state could allocate, resulting in $([\hat{i} \mapsto \sigma_i], \bot)$, with σ_i the instantiated state. This however does not satisfy Frame subtraction: the frame $(\emptyset, \{\})$, made of an empty heap and the empty domain set, is compatible with the state before the allocation, but is not compatible with the state after, since the domain of the heap $\{\hat{i}\}$ is not a subset of $\{\}$. We must thus enforce ownership of the domain set when allocating. This is odd: in allocation-based languages like C, one can always allocate a new cell, without needing to "own" anything, as the allocator is global. With this definition of `alloc` however, one cannot allocate without it, which can be problematic: for instance in a multi-threaded setting, two threads cannot own the domain set, since it is exclusively owned (in order to allow modification).

This is a weakness of this PMAP definition, as we require a more restrictive behaviour than what is usually admitted with allocation. This also has the unfortunate side effect that whenever a function needs to allocate, it must specify a `domainset` core predicate in its pre and postcondition, which can become quite verbose and unpractical.[11] Attempts have been made to preserve the domain set while allowing for more flexible allocation semantics, for instance by having the domain set be a duplicable resource that allows modification and defining its composition as set union. However these relaxations often had as a consequence that out of bounds accesses could not reliably be detected any more, as indeed more heap cells with a larger domain set could always be composed to yield a successful outcome. In fact, *allocation seems to be a topic that is in essence hard to tackle with separation logic*; [59, 63] already mentioned some of these difficulties regarding freshness of the new index. Even when using sophisticated logics like Iris, works on verifying CompCert-C code does not differentiate misses from out of bounds: "An `rmap` does not distinguish between an unallocated block and a block on which it holds no ownership" [49]. This difficulty stems from the fact that allocators are inherently non-deterministic (a program cannot "guess" what address is allocated) and global (two threads allocating are guaranteed to yield different addresses).

A solution to the above problem is to simply get rid of the domain set, and define PMAP only as the mappings from indices to states. This works, however one loses the ability to detect out of bounds accesses entirely, as additional cells can always be composed with the state to yield a non-`Miss` outcome. For our purposes we keep the definition using a domain set, as it provides better automation and is adapted to our needs.

Using RAs here again means the problem we faced in §3.1 does not occur, because $\perp \notin \Sigma$, the binding $i \mapsto \perp$ is not valid. This is handled in the definition of the `set` function, where if the set state is $\perp$ the binding is removed. Using RAs forces us to think about this case, as otherwise the definition is not valid.

3.4.5 Partial Map Variations

The behaviour of PMAP is not always adapted for the state one wants to model. Here we describe in a shorter form some state models that function similarly to PMAP.

Dynamic Partial Map

The *dynamic* partial map, DYNPMAP($I, \mathbb{S}$), allows dynamically allocating missing cells when accessed. It is used, for instance, when modelling objects in JavaScript. In JavaScript, accessing a property that has not been set simply yields a default value, `undefined`, and any property of an object can be set directly without needing to allocate it first. This can be easily achieved by modifying PMAP, such that any out of bounds accesses have the effect of instantiating the state at that location, storing it, and executing the action against it. In other words, instead of using the domain set to distinguish between `Err` and `Miss`, we use it to distinguish between `Ok` and `Miss` [2].

We also need to remove the `alloc` action, since allocation has no reason to exist; the cell can be modified directly instead.

List

The list state model, LIST($\mathbb{S}$), allows storing a list of states up to a bound. Instead of having a domain set, we define it as having a bound n such that all indices are in $[0; n($. It is useful when wanting to represent continuous blocks of memory of known size. We use it for the state model of WISL, which uses a simple block-offset memory model: each memory access requires an address of the block, and the offset within the block to read from.

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[11] In fact, the instantiations of Gillian ignore the domain set altogether; this comes with a slight automation cost, as out of bounds accesses can never be detected.

Because the bound can also be $\bot$, it serves the exact same purpose as the domain set in PMAP: distinguishing out of bounds from misses.

General Map

Due to the strong similarity between PMAP and LIST, it is tempting to define a more general sort of map state model, that allows replicating both behaviours. The general map GMAP allows this; constructed as $\mathrm{GMAP}(I, \mathbb{S}, \mathbb{S}_D)$, it receives the same domain I and codomain $\mathbb{S}$ as before, but it also receives a *discriminator* state model $\mathbb{S}_D$, which represents the state which is capable of discriminating out of bounds accesses from missing accesses.

The discriminator must come equipped with an `is_within`: $\Sigma_D \to I \to \mathbb{B}$ function that given an index returns whether or not the state is within bounds. Here, the input discriminator state is never $\bot$: if the discriminator is not known then a `Miss` must be issued, since composing a state with a non-$\bot$ discriminator can change the outcome. `is_within` must also be lifted to the symbolic realm, resulting in `is_within`: $\hat{\Sigma}_D \to I \to \mathcal{P}(\mathrm{LVal})$.

Defining PMAP and LIST with a GMAP would require using the discriminator state models $\mathrm{Ex}(\mathcal{P}(I))$ and $\mathrm{Ex}(\mathbb{N})$ respectively, with the following `is_within` definitions:

$$\texttt{is\_within}_{\mathrm{PMAP}} \ d \ i \triangleq \{i \in d\}$$
$$\texttt{is\_within}_{\mathrm{LIST}} \ n \ i \triangleq \{0 \leq i \wedge i < n\}$$

For the above two examples it is also needed to exclude the `store` actions defined by the discriminator (via EX), as modifying the domain set of a partial map or the bound of a list is not frame preserving. In general, it is unsound to modify the discriminator directly without also modifying the map (like `alloc` does): "increasing" it breaks Frame Addition (as states compatible with the new state are not compatible with the old state), while "decreasing" it breaks Frame subtraction (for the inverse reason). Here increasing and decreasing refer to modifying the discriminator such that more or less indices are considered *within* it.

We have thus defined a wide range of different state models and state model transformers, that can be easily combined to build up a state model. Most notably, while in our examples we only considered variations of the linear heap, there is no reason why we cannot define more complex state models the same way. For instance, $\mathrm{FREEABLE}(\mathrm{PMAP}(\mathbb{N}, \mathrm{FREEABLE}(\mathrm{Ex}(\mathrm{Val}) \times \mathrm{Ex}(\mathrm{Str})))) \times (\mathrm{AG}(\mathrm{Val}) + \mathrm{FRAC}(\mathbb{N}))$ is a perfectly valid and sound construction, and we do not need to verify any of it despite its complexity; if each application of a state model transformer is valid, then the result is sound.

3.5 Optimising Partial Maps

Map-like state models are both widely used and of varied applications, as seen before. They are also quite complex, as they can lead to many branches, even when only a small number of the branches is actually feasible. Consider the PMAP state $([\hat{a}_1 \mapsto x_1, ..., \hat{a}_n \mapsto x_n], \bot)$. Executing a lookup on it will yield $n + 1$ branches: one for each possible match, and one for the case where the index is different from all others. Such a state also needs to carry the fact that all addresses are distinct; for n addresses, this is $\frac{n(n+1)}{2}$ inequalities. While at the theory level this is not an issue, as we only worry about the soundness of the actions when the path condition is satisfiable, this can have a significant performance impact on the implementation.

Optimisations exist to reduce the number of branches generated, and while they have been implemented successfully in Gillian, there currently exists no theoretical justification to them proving their

$$\text{SYNTACTIC PMAP GET MATCH}$$
$$\frac{(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \qquad \hat{i} \in \text{dom}(\hat{h}) \qquad \hat{\sigma}_i = \hat{h}(\hat{i})}{\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \hat{\sigma}_i, [])}$$

$$\text{SYNTACTIC PMAP GET BRANCH}$$
$$\frac{(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \qquad \hat{i} \notin \text{dom}(\hat{h}) \qquad \hat{i}' \in \text{dom}(\hat{h}) \qquad \hat{\sigma}_i = \hat{h}(\hat{i}')}{\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, [\hat{i} = \hat{i}'])}$$

Figure 3.20: Modified rules for syntactic matching

soundness. Here, we *theoretically* define how to optimise PMAP-like state models, how to prove the soundness of these optimisations, and explore three different optimisations that already exist in Gillian.

3.5.1 Syntactic Checking

The simplest optimisation done to lookups is *syntactic* checking: before attempting to branch on every index on the map, we check if the symbolic index is already present in the map, in which case no branching is needed.

This only requires modifying the matching rule for the $\texttt{get}$ function, adding $\hat{i} \notin \text{dom}(\hat{h})$ to the precondition, and adding a rule that directly gets the relevant state; both rules can be seen in Figure 3.20.

This optimisation is what justifies returning an index for every action in PMAP. As an example, take the state model $\text{PMAP}(\text{Ex}(\text{Val}))$ in the state $([\hat{a}_1 \mapsto x_a, ..., \hat{a}_n \mapsto x_n], \bot)$. When executing $\texttt{load}(\hat{b})$ with a fresh symbolic variable $\hat{b}$, we branch $n + 1$ times, and return the value stored there *along with the used index*, for instance returning $[\hat{a}_3, \hat{x}_3]$. The program is then free to use $\hat{a}_3$ instead of $\hat{b}$ for further accesses to that cell, and thanks to syntactic checking no branching will be needed. Without syntactic checking, every access invariably causes branching, even if the index is already present.

An observation is that thanks to our $\texttt{get}$ and $\texttt{set}$ abstractions over accessing states at a given index, we can restrict ourselves to only modify these two definitions and carry over the rest of the rules of PMAP. Because these optimisations are purely due to the symbolicness of the heap indices, the concrete state model is untouched and we can prove soundness with respect to the concrete PMAP. Furthermore, we define symbolic $\texttt{get}$ *properties and lemmas*, which further ease proving soundness of the optimisations: when proving soundness of PMAP we only use them, meaning that any optimisation that satisfies them is also automatically sound.

$$\theta, s, \sigma \models \hat{\sigma} \wedge \texttt{get}(\sigma, i) = \sigma_i \wedge [\![\hat{i}]\!]_{\theta,s} = i \implies \exists \hat{\sigma}_i, \hat{i}', \pi.$$
$$\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \wedge \theta, s, \sigma_i \models \hat{\sigma}_i \wedge [\![\hat{i}']\!]_{\theta,s} = i \wedge \mathsf{SAT}_{\theta,s}(\pi) \qquad \text{(Get OX Soundness)}$$

$$\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \wedge [\![\hat{i}]\!]_{\theta,s} = [\![\hat{i}']\!]_{\theta,s} = i \wedge \mathsf{SAT}_{\theta,s}(\pi) \implies$$
$$\forall \sigma. \theta, s, \sigma \models \hat{\sigma} \implies \exists \sigma_i. \texttt{get}(\sigma, i) = \sigma_i \wedge \theta, s, \sigma_i \models \hat{\sigma}_i \qquad \text{(Get UX Soundness)}$$

They are quite simple: for the OX case, any $\texttt{get}$ call that exists in the concrete world must exist in the symbolic world. For the UX soundness, if a symbolic $\texttt{get}$ returns a satisfiable branch then there is also a matching state σ_i in the concrete world.

We also define two additional lemmas that we do not prove but that must hold for all $\texttt{get}$ and $\texttt{set}$ pairs:

$$\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i) \wedge \texttt{set}(\hat{\sigma}, \hat{i}', \hat{\sigma}_i') = \hat{\sigma}' \implies$$
$$(\exists \sigma, \sigma_i'. \theta, s, \sigma \models \hat{\sigma} \wedge \theta, s, \sigma_i' \models \hat{\sigma}_i') \implies \qquad \text{(Get-Set Forwards Soundness)}$$
$$\exists \sigma'. \theta, s, \sigma' \models \hat{\sigma}'$$

$$\text{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i) \wedge \text{set}(\hat{\sigma}, \hat{i}', \hat{\sigma}_i') = \hat{\sigma}' \implies$$
$$(\exists \sigma', \sigma_i. \theta, s, \sigma' \models \hat{\sigma}' \wedge \theta, s, \sigma_i \models \hat{\sigma}_i) \implies \qquad \text{(Get-Set Backwards Soundness)}$$
$$\exists \sigma. \theta, s, \sigma \models \hat{\sigma}$$

These are quite straightforward: the former states that if a state exists in the concrete world and is modified with a substate that also exists in the concrete world, then the result must exist too. The latter states the reverse: if a state after set exists in the concrete world, and we know that the substate that used to be at the modified location also exists in the concrete world, then the original state also must exist.

These two lemmas simply follow from the fact that $\hat{i}'$ represents the location to directly modify the returned state, and that set only modifies the state at the given index, the rest being untouched. They are used when proving OX and UX soundness respectively, as they each use either forwards or backwards reasoning.

3.5.2 Split PMap

The *split* partial map, $\text{PMAP}_{\text{Split}}$, goes further: it splits the heap into two, one side being the *concrete* part, and the other the *symbolic*. A concrete value is a value that contains no symbolic variables, such that its evaluation is unaffected by substitutions: $\exists e. \forall \theta, s. [\![\hat{e}]\!]_{\theta,s} = e$. We introduce the predicate is_concrete that is true if a value is concrete. The symbolicness of a binding in the heap is determined from the value (the substate) *and the key*; a concrete key pointing to a symbolic substate, or a symbolic key pointing to a concrete substate, both go in the symbolic part of the heap. Similarly to values, a state is concrete if its interpretation is unaffected by substitutions: $\exists \sigma. \forall \theta, s. \theta, s, \sigma \models \hat{\sigma}$. $\text{PMAP}_{\text{Split}}(I, \mathbb{S})$ thus requires the state model to implement an $\text{is\_concrete}_{\mathbb{S}} \colon \hat{\Sigma} \to \mathbb{B}$ function, which determines whether the entire state is concrete. For instance for the $\text{Ex}(X)$ state model, we have $\text{is\_concrete}_{\text{Ex}} \text{ex}(\hat{x}) \triangleq \text{is\_concrete } \hat{x}$.

This has two advantages; the first minor advantage is that when looking for syntactic matches in the heaps, if the address is symbolic then we only must check in the symbolic part of the heap, as symbolic keys are not permitted in the concrete part.[12] The second major advantage of this technique is related to an implementation detail of some CSE engines: *substitution.*

Substitution is the process of replacing logical values in a state with a new value.[13] For instance when learning $\hat{a}_1 = \hat{a}_2$, the engine may decide to substitute all occurrences of $\hat{a}_2$ with $\hat{a}_1$ and remove the equality from the PC if it is sound to do so (if $\hat{a}_2$ never occurs again). Because this process applies to the entire state, it can lead to unfortunate consequences, such as needing to merge two state fragments that live at the same location. Reusing the above example, if the state contained $[\hat{a}_1 \mapsto \hat{\sigma}_1, \hat{a}_2 \mapsto \hat{\sigma}_2]$, then the substituted state would be $[\hat{a}_1 \mapsto \hat{\sigma}_1 \cdot \hat{\sigma}_2]$. This is problematic, as composition is not defined for symbolic state; symbolic states are an arbitrary set, unlike concrete states which form an RA. Current implementations usually mimic the composition of the concrete counterpart, branching when relevant; this is however not justified in the theory presented in this project, as it is caused by an implementation quirk rather than a theoretical necessity. In fact, what causes this is also a "defect" from the implementation, as it would mean the engine did not assert that two different locations are in fact different. This phenomenon exists in Gillian under the name substitutions [46], and in Viper under the name consolidation [25].

Despite the fact it is only needed due to implementation details, we argue that optimisations are also something that is only needed because of an implementation, which justifies attempting to prove soundness of techniques that are caused by theoretically unsound solutions.

The second advantage of $\text{PMAP}_{\text{Split}}$ relating to substitutions is that these can be entirely skipped for the concrete part of the heap, since we already know that it does not contain any symbolic variables.

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[12] This does not hold for concrete keys, as if the concrete key maps to a symbolic state the binding goes in the symbolic part of the heap.

[13] Here we define substitution in states; this is different to substitution in expressions – also known as α-conversion – which cannot be avoided. State substitution usually uses expression substitution.

SPLITPMAPSETSOMECON

$$\frac{\texttt{is\_concrete}_\Sigma\ \hat{\sigma}_i \qquad \texttt{is\_concrete}\ \hat{i} \qquad (\hat{h}_c, \hat{h}_s, \hat{d}) = unwrap(\hat{\sigma}) \qquad \hat{\sigma}_i \neq \bot \qquad \hat{h}'_c = \hat{h}_c[\hat{i} \leftarrow \hat{\sigma}_i] \qquad \hat{h}'_s = \hat{h}_s[\hat{i} \nleftarrow] \qquad \hat{\sigma}' = wrap(\hat{h}'_c, \hat{h}'_s, \hat{d})}{\texttt{set}(\hat{\sigma}, \hat{i}, \hat{\sigma}_i) = \hat{\sigma}'}$$

SPLITPMAPSETSOMESYM

$$\frac{\neg(\texttt{is\_concrete}_\Sigma\ \hat{\sigma}_i \vee \texttt{is\_concrete}\ \hat{i}) \qquad (\hat{h}_c, \hat{h}_s, \hat{d}) = unwrap(\hat{\sigma}) \qquad \hat{s}_i \neq \bot \qquad \hat{h}'_c = \hat{h}_c[\hat{i} \nleftarrow] \qquad \hat{h}'_s = \hat{h}_s[\hat{i} \leftarrow \hat{\sigma}_i] \qquad \hat{\sigma}' = wrap(\hat{h}'_c, \hat{h}'_s, \hat{d})}{\texttt{set}(\hat{\sigma}, \hat{i}, \hat{\sigma}_i) = \hat{\sigma}'}$$

SPLITPMAPSETNONE

$$\frac{(\hat{h}_c, \hat{h}_s, \hat{d}) = unwrap(\hat{\sigma}) \qquad \hat{\sigma}_i = \bot \qquad \hat{h}'_c = \hat{h}_c[\hat{i} \nleftarrow] \qquad \hat{h}'_s = \hat{h}_s[\hat{i} \nleftarrow] \qquad \hat{\sigma}' = wrap(\hat{h}'_c, \hat{h}'_s, \hat{d})}{\texttt{set}(\hat{\sigma}, \hat{i}, \hat{\sigma}_i) = \hat{\sigma}'}$$

Figure 3.21: Rules for set in $\text{PMAP}_{\text{Split}}$

Substitutions are computationally expensive, as they require traversing the entire state and attempting substitutions on all values; for large states, this can take a significant amount of time.

In Figure 3.21 we present the rules relating to set for $\text{PMAP}_{\text{Split}}$. In particular, when this requires modifying the binding for one side of the heap, we must delete it from the other, as it can be that the state was concrete and became symbolic, or inversely.

We also note that this optimisation, while at times performant, also comes with a cost: checking if a state is concrete also requires traversing the tree (although it does not require modifying it), which can be expensive for larger trees. This could be further optimised, for instance by caching whether larger states are concrete, and invalidating the cache on modification. However, we have not found this cost to be significant enough to warrant this modification. This improvement is thus highly dependent on what code is verified; if there is a majority of symbolic values, then the speed gain is minimal, and it could even be cancelled by the cost of needing to check for concreteness of substates.

3.5.3 Abstract Location PMap

The last, and by far most aggressive optimisation we present is the *abstract location* optimisation. Rather than straightforwardly stating how the optimisation works, we will first take a detour, explaining what abstract locations are.

Abstract Locations?

First, we define locations. A location $\text{loc}(x) \in \text{Loc} \subseteq \text{Val}$ is an uninterpreted value, representing a location in memory. It only has a name x, such that two locations with different names are always different; $\text{Loc} = \{\text{loc}(s) : s \in \text{Str}\}$.

Abstract locations (or ALocs) are, as their name implies, locations lifted to the symbolic realm. An abstract location $\text{aloc}(x) \in \text{ALoc} \subseteq \text{LVal}$ also is characterised by its name, and evaluates to a location: $\forall a.\ \exists b.\ [\![\text{aloc}(a)]\!]_{\theta,s} = \text{loc}(b)$. Unlike for locations, two syntactically distinct ALocs can still be equal once concretised. Whether they are equal is decided by the engine according to an additional flag of the path condition: the *matching mode*. We denote it $m \in \{\texttt{MATCH}, \texttt{NO\_MATCH}\}$. When matching is enabled, abstract locations behave similarly to symbolic variables: their equality is decided by the path condition, such that for an empty path condition and $a \neq b$, both $\text{aloc}(a) = \text{aloc}(b)$ and $\text{aloc}(a) \neq \text{aloc}(b)$ are satisfiable. When matching is disabled however, equality of ALocs is only decided by their name: $a = b \Leftrightarrow \text{aloc}(a) = \text{aloc}(b)$. Matching is always enabled for consume, and always disabled for produce and execute_action.

To be able to use abstract locations, the engine must be able to get the ALoc associated to a value; to this end, we introduce the function $\mathtt{to\_aloc}\colon \mathrm{LVal} \to \mathrm{Str}^?$. Given a logical value, it *attempts* to find a location or abstract location that it is equal to, and returns its name, if it finds it – otherwise, it returns $\bot$. It attempts to guess the ALoc by traversing the value's AST, and by looking in the PC for an equality. The crucial point of this is that this is a best effort function, that can fail to find an ALoc the value is associated to. When $\mathtt{to\_aloc}\ e = \bot$, the engine is then free to use $\mathtt{fresh\_aloc}$ to generate a fresh new abstract location, and add $e = \mathrm{aloc}(a)$ to the PC, where a is the name of the fresh ALoc. This mechanism is a form of *semantic hash consing*, where the engine looks for any value semantically equal to the expression, and looks for an ALoc associated to it. This approach is sound, as the generated location is fresh, such that even if the value was already bound to a location, it can always be found later that the new location is equal to the old one.

Abstract Location Optimisation

We now present the optimisation of PMAP using abstract locations: $\mathrm{PMAP}_{\mathrm{ALoc}}(\mathbb{S})$. Firstly, and unlike the syntactic optimisation or $\mathrm{PMAP}_{\mathrm{Split}}$, this optimisation is only applicable to $\mathrm{PMAP}(\mathrm{Loc}, \mathbb{S})$, since abstract locations can only evaluate to locations. The optimisation consists in retrieving the abstract location associated with the given index, and then looking for syntactic matches only. The strength (and as will be shown later, weakness) of this optimisation is that *there is no branching when matching is disabled*, as indeed the SMT solver is not needed to match the given abstract location with the locations in the heap: syntactic equality is sufficient. Its set of symbolic elements is very similar to PMAP, but with strings as keys (the name of the abstract location):

$$\mathrm{PMAP}_{\mathrm{ALoc}}(\mathbb{S}) \triangleq \mathrm{Str} \xrightarrow{fin} \mathbb{S}.\Sigma \times \mathcal{P}(\mathrm{LVal})^?$$

To take advantage of matching, the definition of $\mathtt{get}$ is modified, adding an argument for the matching mode, $m \in \{\mathtt{MATCH}, \mathtt{NO\_MATCH}\}$. Two rules, in their matching and non-matching form, are shown in Figure 3.22. The only difference between the two is the inclusion in the path condition of $\hat{i} \notin \{\mathrm{aloc}(a'') : a'' \in \mathrm{dom}(\hat{h})\}$, such that the branch is only valid if we are *sure* that the address is not already in the heap. This cannot be included when matching is disabled, because it is too strong a condition: when not matching we already do not attempt to match the abstract location to any of the other indices, so if it results that we skipped a correct branch then the action would vanish (result in no branches), which is not OX sound.

This optimisation is particularly powerful in the case where a value that is not known to be equal to any already present location is accessed: for a heap of size n, PMAP would branch $n + 1$ times, whereas $\mathrm{PMAP}_{\mathrm{ALoc}}$ does not branch (if not matching), from the rule ALOCPMAPNOMATCHNEW. However, this is also too strong an optimisation. Consider the state model $\mathrm{PMAP}(\mathrm{Loc}, \mathrm{PMAP}(\mathrm{Str}, \mathrm{Ex}(\mathrm{Val})))$: the state maps locations to objects, which are string-value mappings.[14] It exposes predicates of the form $\langle\mathtt{ex}\rangle(a, n; x)$, where a is the address in the heap, n is the name of the attribute, and x its value. Now consider the following precondition:

$$\langle\mathtt{ex}\rangle(\hat{a}_1, \text{`val'}; \hat{x}) * \langle\mathtt{ex}\rangle(\hat{a}_2, \text{`len'}; \hat{n})$$

When producing this precondition with PMAP, we get two branches: one where $\hat{a}_1 = \hat{a}_2$ and both properties live in the same object, and one where $\hat{a}_1 \neq \hat{a}_2$. However with $\mathrm{PMAP}_{\mathrm{ALoc}}$ this only results in one branch: each property is in a different object and *the path condition only contains* $[\hat{a}_1 = \mathrm{aloc}(loc1), \hat{a}_2 = \mathrm{aloc}(loc2)]$. In the current definition of symbolic interpretation for $\mathrm{PMAP}_{\mathrm{ALoc}}$ (shown in Figure 3.23), this is not OX sound. Substitutions play a role here, because if at a later point in execution

---

[14]This is very similar to the JavaScript state model.

$$\text{with } \hat{i} \in^? \hat{d} \triangleq \begin{cases} \texttt{true} & \text{if } \hat{d} = \bot \\ \hat{i} \in \hat{d} & \text{otherwise} \end{cases}$$

ALocPMapGetNoMatchNotFound
$$\frac{(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \qquad a = \texttt{to\_aloc } \hat{i} \qquad a \neq \bot \qquad a \notin \text{dom}(\hat{h})}{\texttt{get}(\hat{\sigma}, \hat{i}, \texttt{NO\_MATCH}) \rightsquigarrow (\text{aloc}(a), \bot, [\hat{i} \in^? \hat{d}])}$$

ALocPMapGetMatchNotFound
$$\frac{(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \qquad a = \texttt{to\_aloc } \hat{i} \qquad a \neq \bot \qquad a \notin \text{dom}(\hat{h})}{\texttt{get}(\hat{\sigma}, \hat{i}, \texttt{MATCH}) \rightsquigarrow (\text{aloc}(a), \bot, [\hat{i} \notin \{\text{aloc}(a') : a' \in \text{dom}(\hat{h})\} \wedge \hat{i} \in^? \hat{d}])}$$

ALocPMapGetNoMatchNew
$$\frac{(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \qquad a = \texttt{to\_aloc } \hat{i} \qquad a = \bot \qquad a' = \texttt{fresh\_aloc ()}}{\texttt{get}(\hat{\sigma}, \hat{i}, \texttt{NO\_MATCH}) \rightsquigarrow (\text{aloc}(a'), \bot, [\hat{i} = \text{aloc}(a') \wedge \hat{i} \in^? \hat{d}])}$$

ALocPMapGetMatchNew
$$\frac{(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \qquad a = \texttt{to\_aloc } \hat{i} \qquad a = \bot \qquad a' = \texttt{fresh\_aloc ()}}{\texttt{get}(\hat{\sigma}, \hat{i}, \texttt{MATCH}) \rightsquigarrow (\text{aloc}(a'), \bot, [\hat{i} = \text{aloc}(a') \wedge \hat{i} \notin \{\text{aloc}(a'') : a'' \in \text{dom}(\hat{h})\} \wedge \hat{i} \in^? \hat{d}])}$$

Figure 3.22: Extract of rules for $\texttt{get}$ in $\text{PMAP}_{\text{ALoc}}$

ALocPMapSymInterpretation
$$\frac{\forall a \in \text{dom}(\hat{h}). \, [\![\text{aloc}(a)]\!]_{\theta,s} = i \wedge i \in \text{dom}(h) \wedge \theta, s, h(i) \models \hat{h}(a)}{[\![\{\text{aloc}(a) : a \in \text{dom}(\hat{h})\}]\!]_{\theta,s} = \text{dom}(h) \qquad [\![\hat{d}]\!]_{\theta,s} = d}$$
$$\frac{}{\theta, s, (h, d) \models (\hat{h}, \hat{d})}$$

Figure 3.23: Symbolic state satisfiability for $\text{PMAP}_{\text{ALoc}}$

the engine finds $\hat{a}_1 = \hat{a}_2$ then it may do a substitution and merge the two states, however this is not guaranteed. A solution to this would be to redefine satisfiability of $\text{PMAP}_{\text{ALoc}}$, to take into account the merging of states. Informally, this would mean that a symbolic state models a concrete state if for all concrete addresses, the *composition of all symbolic substates at matching symbolic addresses* is modelled by the concrete substate. For example, the symbolic state $[\hat{a}_1 \mapsto [\text{`val'} \mapsto \hat{x}], \hat{a}_2 \mapsto [\text{`len'} \mapsto \hat{n}]]$ would then satisfy a concrete state $[a \mapsto [\text{`val'} \mapsto x, \text{`len'} \mapsto n]]$[15] for an empty path condition, since we could compose the two substates together (assuming again that composition is defined for symbolic states, which is not the case in the current formalism).

In fact, this unsoundness is also visible when attempting to prove Get OX Soundness, and another similar issue occurs in the proof for Get UX Soundness. In both cases, the issue only occurs when the matching mode is $\texttt{NO\_MATCH}$, confirming this is the source of the problems.

In Defense of Abstract Locations

As shown above, there is currently an unsoundness in optimisations using abstract locations, due to the *matching mode*, and in particular when it is disabled. This occurs because the engine eagerly assumes difference, when there is no reason to. We argue here that this behaviour, while currently unsound, can be made sound, and has a practical justification.

Consider a simple linear heap, $\text{PMAP}(\text{Loc}, \text{Ex}(\text{Val}))$, with the core predicate $\langle \texttt{ex} \rangle(\hat{i}; \hat{x})$, that we denote $\hat{i} \mapsto \hat{x}$. In a heap $\hat{i}_0 \mapsto \hat{x}_0 * ... * \hat{i}_n \mapsto \hat{x}_n$, one's intuition is of course that all locations are different, and indeed both $\text{PMAP}(\text{Loc}, \text{Ex}(\text{Val}))$ and $\text{PMAP}_{\text{ALoc}}(\text{Ex}(\text{Val}))$ will create the same state when producing

---

[15]We take some liberties when representing the states, for the sake of readability, by omitting the domain sets.
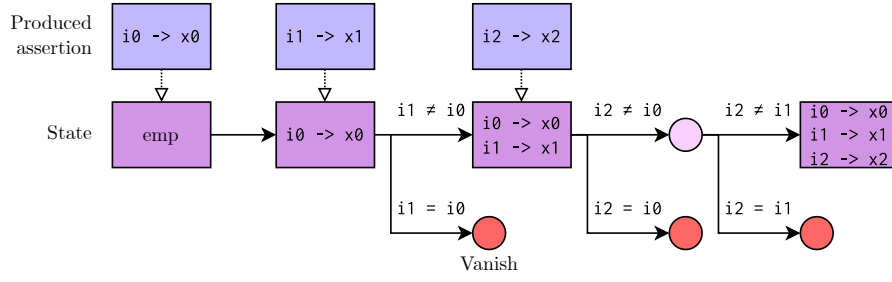
Figure 3.24: Branching when producing cells

these assertions. In the optimised case, because each index will be new when producing the associated core predicate, $\text{PMAP}_{\text{ALoc}}$ will always generate a new abstract location to associate to it, and consider that it is different from all other locations. This is done *implicitly* – that is, the path condition is not extended to take this into account. The base version, on the other side, will need to branch for each new assertion. Because these assertions do not already explicitly specify that all addresses are distinct $(\hat{i}_0 \neq \hat{i}_1 \wedge \hat{i}_0 \neq \hat{i}_2 \wedge ...)$, the branch $\hat{i}_0 = \hat{i}_1$ is satisfiable; however of course producing $\hat{i}_1 \mapsto \hat{x}_1$ into a state $[\hat{i}_0 \mapsto \hat{x}_0]$ when $\hat{i}_0 = \hat{i}_1$ will vanish. This then happens for each address combination; for n addresses, this results in $\frac{n(n+1)}{2}$ inequalities. This makes the PC grow quadratically, slowing down queries sent to the SMT solver, as well as branching and vanishing right after just as many times (see Figure 3.24).

One could of course specify, along with the assertions, that all addresses are distinct – this is however quite effortful, and detracts from the already complex task of formulating function specifications.

Furthermore, one cannot design a general map state model that makes this assumption – while for a simple case like $\text{PMAP}(\text{Loc}, \text{Ex}(\text{Val}))$ it is always true that different cells must have different addresses (since the cell is exclusively owned), this is not true for more complex state models. For instance take $\text{PMAP}(\text{Loc}, \text{PMAP}(\text{Str}, \text{Val}))$, where the map contains objects, which are string-value pairs. When describing two objects with non-overlapping keys, if the addresses of the two objects are not explicitly said to be different then both the branches where they are distinct and where they are equal should be considered. While these two branches should work the same, it results that disabling this abstract location optimisation when producing (i.e. enabling matching on produce) results in several functions not verifying with Gillian, in C and JavaScript. It thus seems that this lack of completeness is at least in part caused by developer intuition, where one "knows" when writing the specification that the locations should be distinct, without formalising it, and instead relying on this unsound behaviour. This problem seems partly caused by recursive user predicates, where addresses are "discovered" as they are unfolded; these locations are, before the unfolding, not named, and as such cannot be stated to be distinct from the others.

This difference between intuition and actual application to a verification setting seems to be caused by "Barendregt Variable Convention" [61], where variables named differently are informally assumed to be different, which can become an issue for an engine that does not properly take it into account. A solution to this would be to better formalise the principles governing abstract locations, for instance using *Nominal Logic* [56, 61]. It provides a sound framework for reasoning about syntactic equivalence of terms within recursive structures, such as recursive user defined predicates.

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Chapter 4

Implementation

We now present the implementation of the state models and state model transformers presented. In §4.1, we present the library which exposes these state models, some relevant implementation details, and notable differences with the theory. In §4.2, we then present the instantiations of Gillian for three different target languages using this library.

4.1 State Model Library

All of the presented state models have been implemented in OCaml 5.2 [42] for Gillian [27, 46, 60], a CSE engine that is parametric on the state model. We first present the general interface for state models, and how it connects to the Gillian engine. We then look more in depth into the implementation of two state models: Ex for its canonical simplicity, and PMAP for its complexity. Finally, we list some additional "utility" state models that were omitted in the previous chapter but that are frequently used when instantiating state models.

4.1.1 State Model Interface

State models are built as standalone *modules*, while state model transformers are *functors*, that accept one or more state model modules, and return a new state model module. Gillian already provides an interface for state models, `MonadicSMemory`, whose name originates from its extensive use of the `Delayed` monad: a *symbolic execution monad* that allows branching and SMT queries in a sugared syntax that is easy to read and use, as seen in Figure 4.1. The signature of the monad, in OCaml syntax, is `'a Delayed.t = Pc.t -> ('a * Pc.t) list`: given a PC, we return a list of branches, containing the new value and an updated PC. We make extensive use of *let bindings*:[1] `let* v = d in e` is the `bind` operator, passing the value `v` contained in the delayed element `d` to the expression `e`, which must result in another delayed value. `let+` is the `map` operator, which modifies the value without branching. For the code shown, we branch when validating the index, and then branch again when calling `S.produce`; we then update the entry with `ss'`, without branching.

The interface provided by Gillian is not satisfactory: it is quite complex, requiring the implementation of 26 different functions whose purpose is at times unclear.[2] We thus create a simpler interface, `MyMonadicSMemory`, reducing the number of required functions and type definitions, and simplifying the definition of some functions (removing arguments when unneeded). Gillian also passes actions and core predicates as strings; while this is manageable for monolithic state models, this is unpractical. We instead require state models to define `action` and `pred` types, along with their `to_string` and `from_string`

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[1] See https://ocaml.org/manual/5.2/bindingops.html

[2] This is due to the fact Gillian has been in development for several years, and as it grew some unused functions were never cleaned up.

```
1   let produce pred s args =
2     match (pred, args) with
3     | SubPred pred, idx :: args →
4       let* ss = validate_index s idx in
5       let+ ss' = S.produce pred ss args in
6       update_entry s idx ss'
```

Figure 4.1: Example of usage of the `Delayed` monad

parsers, ensuring decoupling between parsing and execution. See Table 4.1 for a side-by-side comparison of the interfaces. Because Gillian still requires the state model to satisfy `MonadicSMemory`, we provide a `Make` functor converting from our interface to Gillian's.

Most of the removals are possible because some default function can be used for all state models, under the assumption they behave a particular way; for instance, we remove the `copy` function, because we assume all state models only work on immutable code. Most of the additions are due to the modular nature of our state models, or to enforce a stronger typing of actions and predicates.

A significant difference with the theory presented is that the `execute_action`, `consume` and `produce` functions do not take as input states of type `t option` (to imitate $\hat{\Sigma}^?$), but instead `t`. While both options are possible, using the `option` monad proved to at times be somewhat unweildy (for the same reason it is unpractical to use *wrap* and *unwrap* in the rules) – we instead rely on the `is_empty` function when necessary. We also require a `compose` function, despite it not being justified in the theory, because of substitutions.

One of the most significant improvements of our interface is the re-work of fixes in the engine, to behave precisely like the definition of `fix` in the theory. Gillian uses an intermediary `fix_t` type, that is obtained from an error with `get_fixes`, and must then be applied with `apply_fix: t -> fix_ t -> (t, err_t) result Delayed.t`. Aside from being an unpractical system to use, generating a fix and then applying it separately, it *introduces unsoundness into the bi-abductive engine.* Indeed, applying a fix modifies the state directly; there is no requirement for this fix to be describable in terms of assertions. This means that applying the same fix on the state and the anti-frame may give different results, that do not result in the appropriate change for the function's generated pre-condition.

For instance in [2], an example implementation of PMAP is given, where executing an action on an empty cell creates a `Miss`, called `MissingBinding`. A naive implementation of PMAP for Gillian could resolve this fix by simply adding a binding that points to the empty substate in the heap at the corresponding index. This however is not sound, as it *cannot be expressed as an assertion*: it manipulates the representation of state, but no assertion exists that creates an empty binding, since PMAP does not define core predicates for bindings – it only lifts the core predicates of the wrapped state model. This would therefore result in a fix that does not change the anti-frame or the specification of the function, which is invalid.

We fix this by enforcing `fix_t` to be of type `MyAsrt.t`: strongly typed assertions. The implementation of `apply_fix` then produces all the fixes into the state model. This change thus not only makes state models less prone to error, but also simplifies their code.

4.1.2 Exclusive

We now present the implementation of the Ex state model, step by step (skipping the uninteresting details such as parsing or auxiliary functions). Firstly, we must define the types for the state model: the state type, the errors (which include misses), the actions and core predicates. This is trivial:

```
1   type t = Expr.t option
2   type err_t = MissingState
3   type action = Load | Store
4   type pred = Ex
```

Table 4.1: Difference between Gillian's and our state model interface

Gillian interface	Our interface	Outcome
apply_fix		Removed
clean_up		Removed
clear		Removed
copy		Removed
get_failing_constraint		Removed
get_init_data		Removed
get_print_info		Removed
is_overlapping_asrt		Removed
mem_constraints		Removed
pp_by_need		Removed
pp_c_fix		Removed
split_further		Removed
sure_is_nonempty		Removed
alocs	alocs	
assertions	assertions	Typed the core predicates
can_fix	can_fix	
consume	consume	Typed the core predicate
execute_action	execute_action	Typed the action
get_fixes	get_fixes	Simplified, use assertions
get_recovery_tactic	get_recovery_tactic	Simplified
init	empty	Renamed
lvars	lvars	
pp	pp	
pp_err	pp_err	
produce	produce	Typed core predicate
substitution_in_place	substitution_in_place	
	action_from_str	Added
	action_to_str	Added
	assertions_others	Added
	compose	Added
	instantiate	Added
	is_concrete	Added
	is_empty	Added
	is_exclusively_owned	Added
	pred_from_str	Added
	pred_to_str	Added

Here we define states as being an optional expression, where the `None` case stands for ⊥. There is also one singular error, for misses. We then must define the function of the state model: `execute_action`, `consume` and `produce`, following the rules outlined in appendix A.1:

```
1   let execute_action action s args =
2     match (action, s, args) with
3     | _, None, _ → DR.error MissingState
4     | Load, Some v, [] → DR.ok (Some v, [ v ])
5     | Store, Some _, [ v' ] → DR.ok (Some v', [])
6     | _ → failwith "Invalid Ex execute_action"
7
8   let consume core_pred s ins =
9     match (core_pred, s, ins) with
10    | Ex, Some v, [] → DR.ok (None, [ v ])
11    | Ex, None, _ → DR.error MissingState
12    | _ → failwith "Invalid Ex consume"
13
14  let produce core_pred s insouts =
15    match (core_pred, s, insouts) with
16    | Ex, None, [ v ] → Delayed.return (Some v)
17    | Ex, Some _, _ → Delayed.vanish ()
18    | _ → failwith "Invalid Ex produce"
```

Again, this is straightforward; a simple pattern matching suffices. `DR.ok` stands for `Delayed.return (Ok -)`, returning the input in a `Result` monad, wrapped in a `Delayed` monad. The outcome is thus `Ok` – for `Err`, `LFail` and `Miss` outcomes, we use `DR.error`; the state model then provides a `can_fix` function that given an error returns whether it is fixable, signifying it is a `Miss`. Experiments were done to create an `Outcome` monad, with variants `Ok | Miss | Err | LFail`, as a replacement for the `Result` monad with `Ok | Err`. While this allowed getting rid of the `can_fix` function, it was unpractical to use, as it was purely "aesthetic"; the engine still needed to parse everything back to a `Result`. Furthermore, this made lifting errors quite tedious: instead of simply defining a variant `SubError of S.err_t` for the error type and lifting the errors of the underlying state model through it, one had to handle all three error cases, creating a lot of boilerplate code for little to no added value.

Finally here we mention some of the additional functions that the state model must implement:

```
1   let empty () = None
2   let is_empty = Option.is_none
3   let is_exclusively_owned = Option.is_some
4   let instantiate = function
5     | [] → (Some (Expr.Lit Undefined), [])
6     | [ v ] → (Some v, [])
7     | _ → failwith "Invalid Ex instantiation"
8   let assertions = function
9     | None → []
10    | Some v → [ (Ex, [], [ v ]) ]
11  let can_fix MissingState = true
12  let get_fixes MissingState =
13    [ [ MyAsrt.CorePred (Ex, [], [ LVar (Generators.fresh_svar ()) ]) ] ]
```

These functions are, in order: the constructor for the empty (⊥) state, a function to tell if the state is ⊥, the `is_exclusively_owned` presented before, the `instantiate` function required by PMAP, a function to return all the core predicates associated with a state, the function to tell apart `Err` and `LFail` from `Miss`, and the `fix` function presented in chapter 3. For `fix`, we note that specifying the existential for the fix $\exists \hat{x}. \langle ex \rangle(; \hat{x})$ is not needed, as all unbound logical variables in Gillian are existentially quantified by default.

This is – skipping the irrelevant functions – all of the code for a simple state model in Gillian, using our interface `MyMonadicSMemory`. The entire code is 77 lines long, with the biggest function taking 9 lines. This shows that for simple state models, our interface allows for just as simple implementations.

4.1.3 Partial Map

We now present some details of the $\text{PMAP}(I, \mathbb{S})$ implementation. It is a functor, receiving a `PMapIndex` module and a `MyMonadicSMemory` module, corresponding to the I and $\mathbb{S}$ arguments of the constructor. Both PMAP and DYNPMAP are implemented as one state model; what enables the "dynamic" behaviour is the value `mode: Static | Dynamic` that is contained in the input index. The only difference in behaviour this causes is enabling the `alloc` action in static mode, and catching any `NotAllocated` errors and replacing them with an implicit allocation in dynamic mode. Aside from this, `PMapIndex` has an `is_valid_index`, which returns the "actual" index associated with an input index. This is where the abstract location optimisations plays a role: for the `LocationIndex` module, it is:

```
1  module LocationIndex : PMapIndex = struct
2    let mode = Static
3
4    let make_fresh () =
5      let loc = ALoc.alloc () in
6      Expr.loc_from_loc_name loc
7
8    let is_valid_index = function
9      | (Expr.Lit (Loc _) | Expr.ALoc _) as l → Delayed.return (Some l)
10     | e when not (Expr.is_concrete e) → (
11         let* loc = Delayed.resolve_loc e in
12         match loc with
13         | Some l → Delayed.return (Some (Expr.loc_from_loc_name l))
14         | None →
15             let loc' = make_fresh () in
16             Delayed.return ~learned:[ (loc' #== e) ] (Some loc'))
17     | _ → Delayed.return None
18 end
```

We see here how `Delayed.resolve_loc`, named `to_aloc` in §3.5, may not find a result, in which case we generate a new fresh abstract location to assign to the expression, by extending the PC: "`~learned: [ (loc' #== e) ]`" appends the expression `loc'` = e to it, with `#==` signifying the symbolic equality.

This is only one part of the implementation for what was called `get` in the theory. The rest of it is two functions:[3]

```
1  let validate_index (h, d) idx =
2    let* idx' = I.is_valid_index idx in
3    match idx' with
4    | None → DR.error (InvalidIndexValue idx)
5    | Some idx' → (
6        let* match_val = ExpMap.sym_find_opt idx' h in
7        match (match_val, d) with
8        | Some (idx'', v), _ → DR.ok (idx'', v)
9        | None, None → DR.ok (idx', S.empty ())
10       | None, Some d →
11           if%sat Formula.SetMem (idx', d) then DR.ok (idx', S.empty ())
12           else DR.error (NotAllocated idx'))
```

This is the function in PMAP responsible for retrieving the substate associated to an index; it calls `ExpMap.sym_find_opt`, which symbolically tries to match an index against all entries of the map, and if that finds no result it either returns an empty substate $\bot$ with `S.empty ()`, or raises an error if a bound exists and $\hat{i}' \notin \hat{d}$. Here `if%sat` is sugared syntax to allow branching according to a symbolic formula, using `Delayed`. If both the formula and its negation can be true then the value it returns evaluates to two branches, extended with the formula or its negation in the PC.

```
1  module ExpMap = struct
2    include Map.Make (Expr)
3
4    let sym_find_opt k m =
5      match find_opt k m with
6      | Some v → Delayed.return (Some (k, v))
7      | None →
8          let rec find_match = function
```

---
[3]We omit some optimisations for clarity.

```
9              | [] → Delayed.return None
10             | (k', v) :: tl →
11                 if%sat k' #== k then Delayed.return (Some (k', v))
12                 else find_match tl
13         in
14         find_match (bindings m)
15   end
```

This last function is the symbolic matching function, which checks if $\mathrm{SAT}_{\theta,s}(\hat{i} = \hat{i}')$ for every index $\hat{i}'$ in the map, and creates a branch for every possible match. It also does *syntactic matching*, as seen at the start of the function: it first attempts to find the matching key directly, using `Map.find_opt`, and only if that fails does it proceed to match with every key.

The `Delayed` monad and the associated syntactic sugar, initially described in [2], allow us to write branching-heavy code with ease, abstracting the details of the PC away from us. We finally take a brief look at the implementation of `produce`, to show how a state model transformer accesses the wrapped state model via the interface we defined:

```
1    let produce pred s args =
2      match (pred, args) with
3      | SubPred pred, idx :: args →
4          let*? idx, ss = validate_index s idx in
5          let+ ss' = S.produce pred ss args in
6          update_entry s idx idx ss'
7      | DomainSet, [ d' ] → (
8          match s with
9          | _, Some _ → Delayed.vanish ()
10         | h, None → Delayed.return (h, Some d'))
11     | _ → failwith "Invalid PMap produce"
```

Here again, and even for a transformer as complex as $\mathrm{PMAP}(I, \mathbb{S})$, the code is short and minimal. For the case of a core predicate from $\mathbb{S}$ (`SubPred pred`), we get the substate associated with the index (`let*?` gets the value associated with a `Result` and vanishes if it is an error), we produce the predicate onto the substate, and then update the entry. For the domain set, we simply add it to the state, vanishing if a domain set is already present as it is exclusively owned.

The entire PMAP code is 343 lines of code – a sizeable amount, but still rather short given its complexity.

4.1.4   Helper State Models

We now define some additional state model transformers that are often used when one wants to instantiate a state model with the aforementioned library. These transformers focus on applying small, customisable modifications to state models, allowing for an even greater re-use. In particular these utility state model transformers are useful when instantiating our state model with the goal to mimic a pre-existing state model.

$\mathrm{ACTIONADD}(\mathbb{S}, A)$ is a transformer that equips the given state model with additional actions defined by A, without needing to define any predicates, overriding functions, or re-implementing repetitive functions.

$\mathrm{FILTER}(\mathbb{S}, \mathcal{A}, \Delta)$ is a transformer that allows filtering the actions and predicates exposed by a state model, for instance to limit the functionality of a state model if one of its provided actions should not exist in the semantics of the language.

$\mathrm{MAPPER}(\mathbb{S}, F_{\mathcal{A}}, F_{\Delta})$ allows renaming predicates or actions.

$\mathrm{INJECTOR}(\mathbb{S}, I)$ adds hooks into `consume`, `produce` and `execute_action`, allowing the developer to apply transformations to the inputs or outputs of these functions. For instance, this makes re-ordering the arguments of actions trivial.

4.2 Instantiations

We now look at how this library of state models can be used to instantiate Gillian. In particular we focus on instantiating Gillian to target languages (TLs) that have already been instantiated for it: WISL, JavaScript and C.

To instantiate a language to Gillian, the developer must provide its state model, and a compiler from the TL to GIL. Because this project focuses on constructing state models, it is thus natural that we picked languages for which the compiler already exists.

We note that all forms of PMAP(Loc, $\mathbb{S}$) implicitly make use of the abstract location optimisation described in §3.5. This is because all of Gillian's instantiations also make use of it, and the goal of our instantiations is to have *parity* with the reference implementation, rather than to have additional completeness. Indeed, while removing the unsoundness caused by ALocs is straightforward in our library, it causes a majority of tests to fail, making any comparison with Gillian impossible.

4.2.1 WISL

WISL (WhIle language for Separation Logic) is an imperative toy language used to teach students the principles of software verification. It uses a simple block-offset memory:

$$\text{PMAP}(\text{Loc}, \text{FREEABLE}(\text{LIST}(\text{Ex}(\text{Val}))))$$

Unsurprisingly for a simple state model like this, its instantiation is also trivial when brought to code:

```
1   module ExclusiveNull = struct
2     include Exclusive
3
4     let instantiate = function
5       | [] → (Some (Expr.Lit Null), [])
6       | [ v ] → (Some v, [])
7       | _ → failwith "ExclusiveNull: instantiate: too many arguments"
8   end
9
10  module BaseMemory = PMap (LocationIndex) (Freeable (MList (ExclusiveNull)))
11
12  module WISLSubst : NameMap = struct
13    let action_substitutions =
14      [
15        ("alloc", "alloc");
16        ("dispose", "free");
17        ("setcell", "store");
18        ("getcell", "load");
19      ]
20
21    let pred_substitutions =
22      [ ("cell", "points_to"); ("freed", "freed"); ("bound", "length") ]
23  end
24
25  module WISLMonadicSMemory = Mapper (WISLSubst) (BaseMemory)
```

This is <u>all</u> the code needed for the instantiation of WISL to match exactly Gillian's original monolithic instantiation.

A consequence of our modular approach to state models is functor-heavy code, with many intermediary modules to allow for further constructions. This can be a bit unwieldy if not organised properly, which explains why here we take some intermediary steps, instead of writing down one large and confusing functor application.[4]

We now describe, module by module, how the instantiation of WISL is done. We first override the Ex state model, to modify its default value on instantiation, as WISL requires instantiation to null

---

[4]It is often not possible to write the whole state model as a single functor application anyway, due to constraints in OCaml about naming modules – see https://github.com/ocaml/ocaml/issues/6917#issuecomment-626586129.

and the base implementation of Ex initialises it to `undefined` instead. We then create the intermediary module `BaseMemory`, equivalent to $\mathrm{PMap}(\mathrm{Loc}, \mathrm{Freeable}(\mathrm{List}(\mathrm{Ex})))$,[5] which corresponds to the bulk of the state model. We finish by applying the substitutions needed to match the target instantiation, via the `Mapper` module, and voilà! Thanks to our library, what used to be about 800 lines of code is now 25 lines. We point out that our instantiation is also easier to verify the soundness of, as all modules are independent and do not require understanding how the other state model transformers work.

4.2.2 JavaScript

Gillian originated from JaVerT [29, 30], which eventually became Gillian-JS, a JavaScript instantiation for Gillian, which targets ECMAScript 5.1 [23]. Its state model is:

$$\mathrm{PMap}(\mathrm{Loc}, \mathrm{DynPMap}(\mathrm{Str}, \mathrm{Ex}(\mathrm{Val})) \times \mathrm{Ag}(\mathrm{Loc}))$$

Here $\mathrm{DynPMap}(\mathrm{Str}, \mathrm{Ex}(\mathrm{Val}))$ represents objects, and $\mathrm{Ag}(\mathrm{Loc})$ is the location of the object's metadata, which itself is also an object. An object's metadata is immutable, and thus shareable, explaining the choice of an agreement.

Its implementation with our library is more involved than for WISL; indeed, Gillian-JS does not quite line up with the actions and predicates we defined for our state models. These differences mostly stem from different design, meaning most of the work required to fix them is juggling values to fit the right shape.

For instance in Gillian-JS the `domainset` predicate has one in-value, the domain set, and no out. In our definitions however, the same predicate has no in, and one out, the domain set. To this end, we override `consume` for `domainset`, and vanish when the given in does not match the out.

```
1  module MoveInToOut (S : States.MyMonadicSMemory.S) :
2    States.MyMonadicSMemory.S with type t = S.t = struct
3    include S
4
5    let consume pred s ins =
6      match (pred_to_str pred, ins) with
7      | "domainset", [ out ] → (
8          let** s', outs = S.consume pred s [] in
9          match outs with
10         | [ out' ] →
11             if%sat out #== out' then Delayed_result.ok (s', [])
12             else Delayed.vanish ()
13         | _ → Delayed_result.ok (s', outs))
14     | _ → consume pred s ins
15   end
```

Another difference is that for the $\mathrm{DynPMap}$, when accessing a field that did not exist, Gillian-JS instantiates it to `Nono`, a special value to represent values that are not actually there. When using the `get_domainset` action or consuming the `domainset` predicate, all keys mapping to a `Nono` value must thus be removed:

```
1  module PatchDomainsetObject (S : sig
2    include MyMonadicSMemory
3    val is_not_nono : t → Expr.t → bool
4  end) =
5    Injector
6      (struct
7        include DummyInject (S)
8
9        let post_execute_action a (s, args, rets) =
10         match (a, rets) with
11         | "get_domainset", [ Expr.EList dom ] →
12             let dom = List.filter (S.is_not_nono s) dom in
13             Delayed.return (s, args, [ Expr.EList dom ])
14         | _ → Delayed.return (s, args, rets)
15
16        let post_consume p (s, outs) =
```

---

[5]The List state model transformer is called `MList` to avoid shadowing OCaml's `List` module.

```
17        match (p, outs) with
18        | "domainset", [ Expr.ESet dom ] →
19            let dom = List.filter (S.is_not_nono s) dom in
20            Delayed.return (s, [ Expr.ESet dom ])
21        | _ → Delayed.return (s, outs)
22      end)
23      (S)
```

This is done via an injection, that receives a state model *enhanced with the* `is_not_nono` *function*; we do this instead of hardcoding the lookup to ease the process of instantiating the JavaScript stack with different optimisations of PMAP that may not have the same state type. An example implementation is then:

```
1  module ObjectBase = struct
2    include PMap (StringIndex) (Exclusive)
3
4    let is_not_nono (h, _) idx =
5      match ExpMap.find_opt idx h with
6      | Some (Some (Expr.Lit Nono)) → false
7      | _ → true
8  end
```

The entire instantiation to JavaScript is 253 lines of code, including all versions of the state model with different optimisations enabled. Most of these lines are used on functor and module definitions, to build up the state one modification at a time. This type of code is hard to read and understand, but this complexity is not caused by the target state model as much as the need for matching the Gillian-JS interface; if the compiler from JavaScript to GIL was reworked to be more aligned with our state model definitions, the instantiation of JavaScript would likely be much simpler and succinct.

4.2.3   C

Gillian-C [60] targets a subset of C, via the CompCert-C verified compiler [41]. Due to the intricacies of the C memory model, its construction uses custom components, not part of the shared library. Its state model is:

$$\text{PMAP}(\text{LOC}, \text{FREEABLE}(\text{BLOCKTREE})) \times \text{CGENV}$$

The first custom state model used is the BLOCKTREE component, which represents chunks of memory as a binary tree, where each node has a range, a value, and zero or two children. Discussing the behaviour of this state model is out of the scope of this report, and is described in more detail in [2].

The second custom state model is CGENV, the C global environment. Gillian-C handles function pointers and global variables by constructing an "initial data" when parsing the input code, and then passing this data to the state model on instantiation. While this behaviour was hard-coded in Gillian-C, we generalise it, considering simply that the state is a product of the memory and the global environment.

For the implementation of BLOCKTREE and CGENV, the code from Gillian-C was reused and adapted where necessary to our interface. A particular improvement that was done to it is that in Gillian-C the notion of FREEABLE and of BLOCKTREE were coupled, meaning any action executed on the BLOCKTREE had to first check if the memory had been freed before proceeding. Decoupling this behaviour was straightforward, and allowed for simpler code in BLOCKTREE.

The bulk of the new code needed for the C instantiation lies in implementing a new action, that was previously hardcoded in the Gillian-C state model: `move`. This action allows copying some memory from a source into a target location, merging the BLOCKTREEs when appropriate. It is quite inelegant to implement, as it cannot rely on the abstraction provided by state models, since it needs to "cross layers" between the state models: it operates on two different indices of the PMAP, but applies a merge operation that only exists for BLOCKTREE. It seems that for such *crossing actions*, our modular state model approach is not sufficient; one instead needs to implement these actions for a specific instantiation, making the proof work much more intricate.

Omitting this addition done via ACTIONADD, the rest of the state model is straightforward to instantiate:

```
1   module BaseMemory = PMap (LocationIndex) (Freeable (BlockTree))
2
3   module Wrap (S : C_PMapType) =
4     Product
5       (struct
6         let id1 = "mem_"
7         let id2 = "genv_"
8       end)
9       (ExtendMemory (S))
10      (CGEnv)
11
12  module CMonadicSMemory = Wrap (BaseMemory)
```

Here ExtendMemory is the functor responsible for adding the move action to the state model. We make use of the Wrap functor to avoid code duplication when building the state model with different optimisations of PMAP. The module defined on lines 5-8 specifies the suffixes used to distinguish actions and core predicates from the left and right side of the product.

The instantiation of C with our library is a good example of mixing modular, general use state models with specialised state models; one can still make use of this library while implementing a perhaps more performant or more complex state model for some of the construction.

Chapter 5

Evaluation

In this chapter we evaluate the work produced in this document. In §5.1 we evaluate the usability of the theory developed; in §5.2 we assess the ease of use of the developed library; in §5.3 we compare the performance of the developed instantiations with their monolithic equivalent in Gillian; finally in §5.4 we evaluate the performance improvement of the various PMAP optimisations.

5.1 Theory Usability

The theory we have presented contrasts with previous work for CSE engines in its use of resource algebras rather than partial commutative monoids or separation algebras. This allows us to easily prove the soundness of the presented constructions.

Still, some aspects of the theory can be at times somewhat unpractical to work with. Representing the empty memory $\perp$ as an element outside of an RA's carrier avoids common mistakes, but comes with the cost of requiring more ceremony when writing the rules for a state model transformers: in particular, we frequently need to define *wrap* and *unwrap* auxiliary functions to allow seamlessly handling the case where the state is $\perp$, which can clutter the rules. We note however that this additional care would also be needed with the PCM approach, though it would not be forced by the framework. The trade-off here is therefore between a more restrictive notation that avoids some errors, or a more flexible notation that can be error-prone.

The RA constructions presented are relatively primitive; in particular we only focus on simpler forms of ghost state, without exploring more advanced concepts that may make sense in a logic but not in an engine. The presented state models are sufficient to represent JavaScript, but fall short when attempting to model C, requiring specialised state models. For low-level languages where precision is required, it seems like one still needs to create a tailored state model, though transformers can still help alleviate the effort required. Other examples of this come to mind: for Gillian-Rust, the state model still needs to define the RUSTHEAP, PCY and OBS state models [2, 3].[1] Though some of these are partly derived from other state models (RUSTHEAP uses PMAP), significant effort is needed to represent concepts such as prophecies or observations. It seems unavoidable for one to need to define some additional state models when targeting real-world languages. The fact these custom state models still use the generic constructions we defined show their versatility, and that they are still capable of simplfying part of the construction of a complex state model.

---

[1] These state models have not yet been implemented in our library, though they exist as a monolithic construction as part of Gillian-Rust.

5.2 Library Usability

The developed library exposes a total of 14 state models, including utility state models and optimised versions. These are all easy to use and combine when little additional functionality is required, for instance with WISL. Fine grained modifications of the constructions are also possible when needed, though sometimes non-trivial to implement. The effort needed to make an instantiation is much lesser than what would be required when creating the state model from scratch, as most behaviour is already available.

The key improvement our approach has over defining entire state models from scratch is *verifiability*. Because all state models are relatively small (less than 400 lines) and isolated, ensuring a specific construction is sound is *much easier* than verifying an entire state model is sound. Modifications to constructions are also relatively easy to verify, as they usually only impact a specific action or predicate, meaning one only needs to re-verify the soundness of the modified element. This is a major improvement on monolithic approaches, as verifying tightly coupled code spanning thousands of lines is no easy task.

5.3 Comparison with Gillian Instantiations

We now measure the performance of the instantiations created using our library against the performance of the original monolithic Gillian state models.

The evaluation is focused on testing the performance of state models across all modes supported by Gillian: whole program symbolic testing (WPST), OX verification and UX bi-abduction. The files tested are either small programs used to test the engine, or larger verification targets extracted from real world code. Furthermore, when possible, the tests are also run with different optimisations of PMAP: $\text{PMAP}_{\text{ALoc}}$ and $\text{PMAP}_{\text{Split}}$.

All test logs were verified to ensure full parity between all versions: the instantiations built using state model transformers yield the same results as the original instantiations. All passing tests pass, and all failing tests fail for the same reasons. All tests were run on a 2020 MacBook Pro, with an M1 processor and 8GB of memory.

This evaluation is split among the three different stacks originally supported by Gillian: WISL, JavaScript and C. A more in-depth presentation of the instantiations is done in §4.2. We note that all Gillian instantiations make use of the abstract location optimisation internally, which as shown in §3.5 has some unsoundness issues. Therefore, to ensure our instantiations have the same results as the reference implementation, we evaluate our transformer constructions using the ALoc optimisation where relevant.

5.3.1 WISL

The WISL state model is $\text{PMAP}(\text{LOC}, \text{FREEABLE}(\text{LIST}(\text{EX}(\text{VAL}))))$. The Gillian-WISL instantiation uses the abstract location optimisation, so our instantiations also do; using PMAP without the optimisation would yield different test outcomes, due to its unsoundness. To measure the performance impact of different optimisations, we build the following transformer stacks:

$$\text{PMAP}_{\text{ALoc}}(\text{FREEABLE}(\text{LIST}(\text{EX}(\text{VAL})))) \tag{TR}$$

$$\text{PMAP}_{\text{ALoc}}^{\text{Str}}(\text{FREEABLE}(\text{LIST}(\text{EX}(\text{VAL})))) \tag{TR-ALoc}$$

$$\text{PMAP}_{\text{ALocSplit}}(\text{FREEABLE}(\text{LIST}(\text{EX}(\text{VAL})))) \tag{TR-Split}$$

Here $\text{PMAP}_{\text{ALocSplit}}$ represents a PMAP with both the ALoc and the split optimisations.

The benchmark is split into three parts: whole program symbolic testing (WPST), verification and bi-abduction, which are made up of respectively 1, 6 and 2 files. We note that the benchmark is small because it only has small code samples used for teaching purposes, rather than any real world code (since
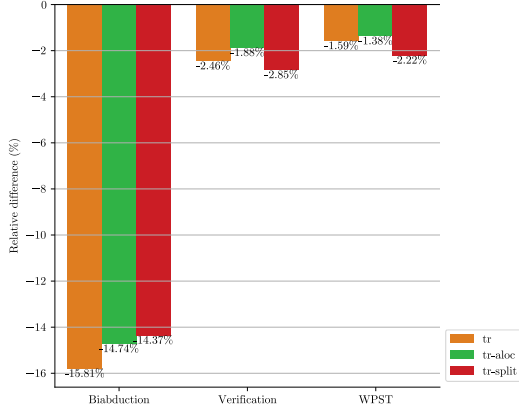
Figure 5.1: Average of the relative difference in execution time for different WISL test suites, per transformer stack
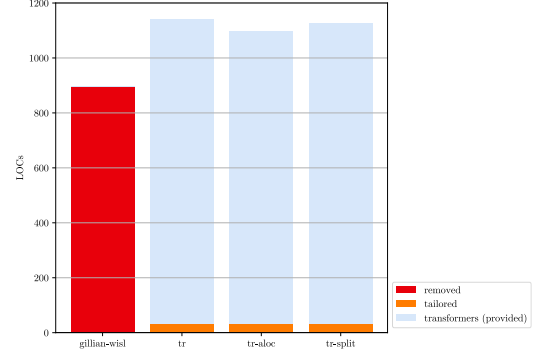

Figure 5.2: Number of lines of code per WISL instantiation

WISL is used for teaching). For this reason also we will not delve into too much detail for this comparison. The tests were run 50 times each, and the results are shown in Figure 5.1.

The performance here seems to be worsened by the use of our library. This makes sense: WISL has a small toy-like state model, allowing for relatively simple code that can work straightforwardly. Gillian-WISL also uses an older and lower level API for Gillian, allowing it for instance to manipulate the PC without using `Delayed`. While this approach is more error-prone, it works well for small state models, and explains the better performance observed here. In comparison, using state model transformers is probably overkill and the added indirection worsens performance compared to a simple implementation. The difference is however negligible: no more than 3% performance loss in verification and WPST.

While not a perfect measure of complexity, lines of code (LOCs) are used to compare the amount of code needed to instantiate each state model to get equivalent results. We only measure the lines of code in implementation files (`.ml`), ignoring interface files (`.mli`) and whitespace. What is gained from using transformers is a shorter implementation, as shown in Figure 5.2: the amount of work required from the developer has been reduced to almost none, as most functionality comes with the provided library. There are more LOCs needed as a whole, but the developer of the state model does not need to worry about the size of the library.

5.3.2 JavaScript

The JavaScript state model is $\text{PMap}(\text{Loc}, \text{DynPMap}(\text{Str}, \text{Ex}(\text{Val})) \times \text{Ag}(\text{Loc}))$. Gillian-JS comes with some optimisations built-in, to improve performance. Firstly, it splits the first level PMap entries between concrete and symbolic *values* (not keys!), avoiding substitutions in the concrete part. Secondly, it uses the abstract location mechanism described in §3.5. Finally, it uses the OCaml `Hashtbl` module, a mutable data structure, rather than the immutable `Map` module; this avoids creating copies of the map on modification. All of these optimisations are important, as the state in JavaScript code tends to be significantly large, with the first PMap regularly reaching 200 to 600 entries when running real-world code: the highest recorded map size reaches 1179 entries for the `set3.gil` file.
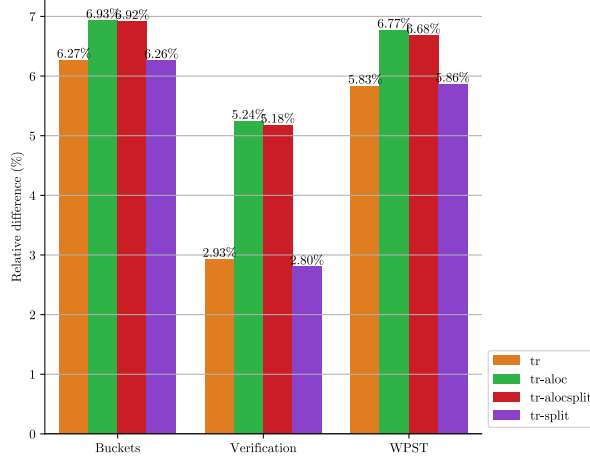
Figure 5.3: Average of the relative difference in execution time for different JavaScript test suites, per transformer stack

To test all combinations of optimisations available via our library, we get the following four transformer stacks:

$$\text{PMAP}_{\text{ALoc}}(\text{DynPMap}(\text{Str}, \text{Ex}(\text{Val})) \times \text{AG}(\text{Loc})) \tag{TR}$$

$$\text{PMAP}^{\text{Str}}_{\text{ALoc}}(\text{DynPMap}(\text{Str}, \text{Ex}(\text{Val})) \times \text{AG}(\text{Loc})) \tag{TR-ALoc}$$

$$\text{PMAP}_{\text{ALoc}}(\text{DynPMap}_{\text{Split}}(\text{Str}, \text{Ex}(\text{Val})) \times \text{AG}(\text{Loc})) \tag{TR-Split}$$

$$\text{PMAP}^{\text{Str}}_{\text{ALoc}}(\text{DynPMap}_{\text{Split}}(\text{Str}, \text{Ex}(\text{Val})) \times \text{AG}(\text{Loc})) \tag{TR-ALocSplit}$$

The last version, TR-ALocSplit, is the closest in terms of applied optimisations to what Gillian-JS does. The biggest difference is the use of `Hashtbl`, as all transformers use immutable data structures.

Gillian-JS is capable of analysing real world code: the Buckets-JS[2] library, which provides several data structures such as linked lists, dictionaries or sets has been verified by it, using whole program symbolic testing. Our instantiations having parity with the reference instantations, we are also capable of verifying Buckets-JS, with no additional modifications.

This benchmark is split into three parts: WPST, verification, and Buckets-JS (in WPST mode), which are made up of respectively 21, 6, and 78 files. All tests were then run 30 times, for each state transformer stack and for Gillian-JS (labelled "base"). The results can be seen in Figure 5.3. The first insight this give us is that transformers seem to, on the whole, outperform the monolithic Gillian-JS, with an improvement ranging from 2.8 to 6.9%. We also see that the ALoc optimisation that uses strings as keys is the best improvement; the Split optimisation seems to have no effect, hinting that maybe the cost of splitting the map and checking what is concrete is too high to compensate the time gain on actual substitutions. We also note that the improvement given by the ALoc optimisation is greater when doing verification than with WPST; this is likely caused because verification uses more non-concrete locations, which implies less syntactic matches when doing lookups.

The initial conclusion we can reach from this is that transformer stacks are more performant than the monolithic alternative. One could, of course, imagine that a hyper-optimised monolith could be made with optimisations specific to the state model, which would then out-perform the more generic transformers. However, in practice, the size of such monoliths makes these optimisations hard to apply, as code becomes complex quickly and makes changes harder. In contrast, transformers are very simple to optimise, as they maintain a more generic structure.

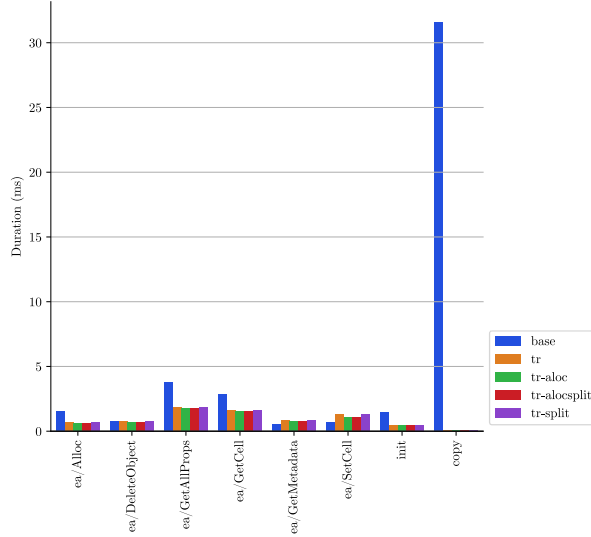[2]See https://www.npmjs.com/package/buckets-js.

Figure 5.4: Average time spent per 1000 function calls
in the Buckets test suite

Another hypothesis this experiment seems to confirm is that the improvement given by optimisations is highly dependent on the context in which the engine is used: ALoc has a much lesser impact when doing WPST than when doing verification. Transformers thus allow users to tailor the optimisations they use in their stack according to what code is verified and how, by empirically measuring the performance of different alternatives (which are trivial to construct).

We may also take a closer look at how this time is spent within the engine. This is done by measuring the time before and after the entry point of each exposed method of the memory model and summing their difference. By looking at the average time per function call in the Buckets test suite (see Figure 5.4), we see that most memory actions (prefixed with "ea/", shorthand for `execute_action`) are faster than in the base instantiation. Furthermore, copying is orders of magnitude faster with transformers, since no work needs to be done thanks to the use of immutable data structures.

Looking at the average total duration for an execution of the test suite (see Figure 5.5) however, we note that the time taken by copying the state is minimal, as seen by the size of the "other" category. Instead, most time is spent during the `load` (`GetCell`) action, which is called 11400 times (more than twice as many times as `store`). The most notable improvement from the transformer construction is the reduction of the time spent getting a cell, reducing total time spent by 43% (from 28.55 to 16.18ms). Because `load` dominates the amount of action calls (see Figure 5.6), this is sufficient to make up for the performance loss in `store` and "GetMetadata" (`load` on the right side of the product).

If, instead of focusing on WPST with the Buckets code we focus on verification, the picture painted is significantly different (see Figure 5.7). Indeed, while in WPST the memory model is only used for memory actions (as only the core engine is used), verification exercises the memory model quite differently, notably with `produce` and `consume`.

Most notably, this shows us that one of the leading differences in total time between different transformer instantiations is substitutions: TR and TR-Split spend more than twice as much time during substitutions than TR-ALoc and TR-ALocSplit. This is because with the ALoc optimisation, substitutions for the keys (abstract location names) can be filtered: if there is no substitution for abstract locations, we can simply map the values of the heap without worrying about key conflicts. Without this, substitutions need to be applied to both the key and the value, and then all resulting keys need to be compared to compose clashing entries – this is expensive.
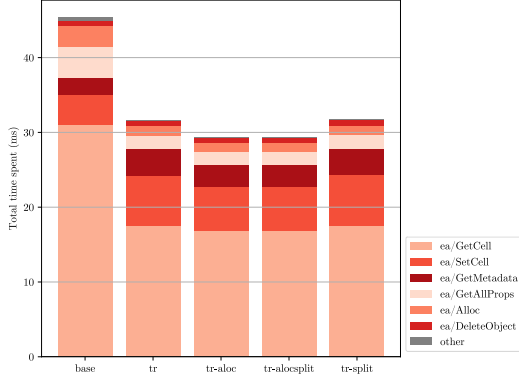
Figure 5.5: Average total time spent per function in an execution of the Buckets test suite
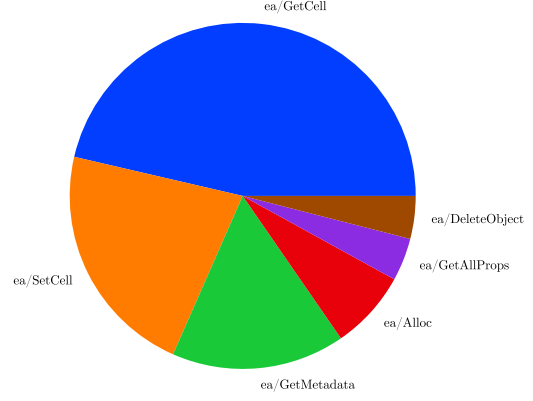

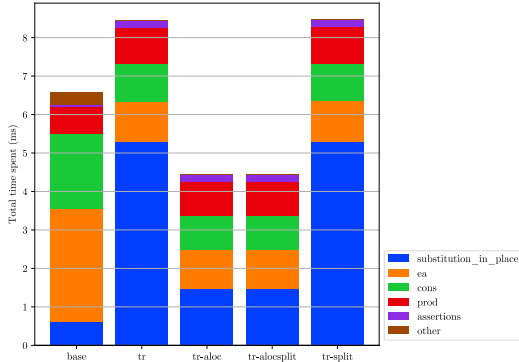Figure 5.6: Share of function calls for one execution of the Buckets test suite

Figure 5.7: Share of grouped function calls for for one execution of the verification test suite
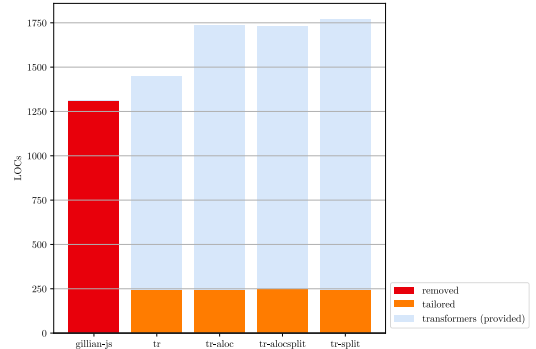

Figure 5.8: Number of lines of code per JS instantiation

While this has not been measured, it is possible that the reason why Gillian-JS spends so little time in substitutions is because it uses mutable structures, allowing modifications to be done in-place, rather than creating new copies.

Substitutions aside, we note how significantly less time is spent in `execute_action` and `consume` calls compared to the base state model; about 65% and 48% respectively. Again, this seems to indicate that simpler transformers tend to be more efficient than large monoliths. We have not been able to pinpoint specific reasons for this. An hypothesis is that Gillian-JS overcomplicates its actions; for instance, the implementation of `GetCell` alone is over 100 lines long and is surprisingly intricate.

Finally, we may compare development effort between the two state models. The results are shown in Figure 5.8. Here, we note that the amount of tailored code needed for each instantiation (in yellow) is minimal compared to Gillian-JS: only 246 LOCs. Most of the LOCs are in the state model library, which is shared and can be reused for different state models; a user of the engine does not need to worry about them.

5.3.3 C

Gillian-C allows verifying a subset of the C language, via the CompCert-C verified compiler. Its state model can be defined as $\text{PMAP}(\text{LOC}, \text{FREEABLE}(\text{BLOCKTREE})) \times \text{CGENV}$. To verify that the improvements from the optimised PMAP versions are carried between instantiations, we also provide several C
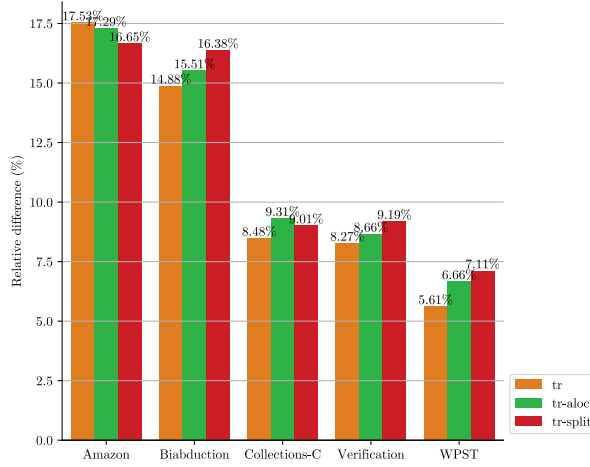
Figure 5.9: Average of the relative difference in execution time for different C test suites, per transformer stack

instantiations. Again we note Gillian-C already makes use of the ALoc optimisation, so in an effort to preserve parity we do not compare with instantiations that do not use it.

$$\text{PMap}_{\text{ALoc}}(\text{Freeable}(\text{BlockTree})) \times \text{CGEnv} \qquad \text{(TR)}$$

$$\text{PMap}^{\text{Str}}_{\text{ALoc}}(\text{Freeable}(\text{BlockTree})) \times \text{CGEnv} \qquad \text{(TR-ALoc)}$$

$$\text{PMap}_{\text{ALocSplit}}(\text{Freeable}(\text{BlockTree})) \times \text{CGEnv} \qquad \text{(TR-Split)}$$

Like Gillian-JS, Gillian-C has been used to verify code that is used in real-world libraries. It verified the Collections-C library[3], that provides data structures, using whole program symbolic testing, as well as the AWS encryption SDK[4], an encryption library, using OX verification. Our instantiation is also capable of verifying these two libraries, providing the exact same results as the original Gillian instantiation.

This benchmark is split into five parts: WPST, verification, bi-abduction, Collections-C (in WPST mode) and AWS (in verification mode), each made up of respectively 8, 6, 7, 159, and 10 files. All tests were run 50 times, except the AWS test suite that was executed 10 times[5].

The general results can be seen in Figure 5.9. Here again, we note a significant performance improvement compared to the monolithic Gillian-C, in particular for the AWS and bi-abduction suites. As for the optimisations, we get inconsistent results: they seem to slightly improve performance for all modes but for AWS.

We may now look into the detailed rundown of the time spent; we will focus on Collections-C WPST and the AWS encryption SDK verification, as these correspond to real-world code usage.

For Collections-C, when looking at the time spent in each function for each instantiation (see Figure 5.10), the main takeaway is that the two most common actions, `mem_store` and `mem_load`, are both faster than the original version; about 20% and 25% respectively. This is a crucial improvement, considering more than 75% of memory actions in Collections-C is a load or a store.

The story is however wildly different for AWS code, which uses verification rather than WPST. Here, memory actions only represent a fraction of the total time spent, which is instead dominated by `consume`, as seen in Figure 5.11. Here we note slight improvements in time for most categories, again seeming to indicate that the more generic approach is generally more performant, though the exact reason as to why is still unclear.

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[3]See https://github.com/srdja/Collections-C.

[4]See https://github.com/aws/aws-encryption-sdk-c.

[5]This is because verifying the AWS code takes *significantly* longer than the rest, due to its size.
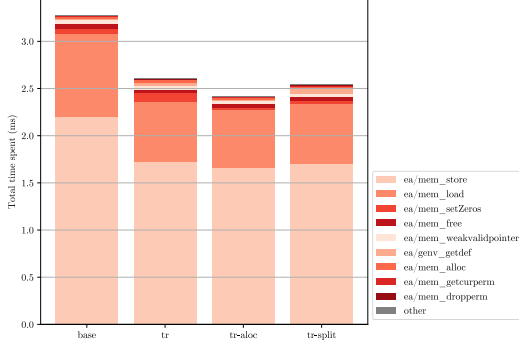
Figure 5.10: Average total time spent per function in an execution of the Collections-C test suite
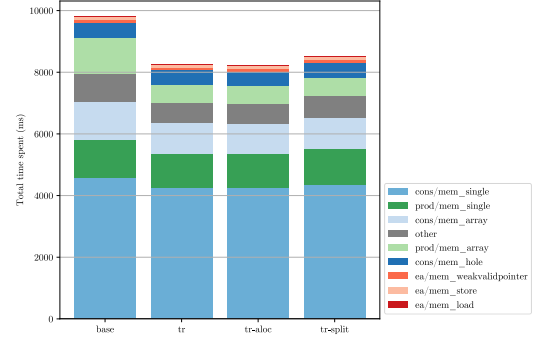

Figure 5.11: Average total time spent per function in an execution of the AWS header encryption SDK test suite
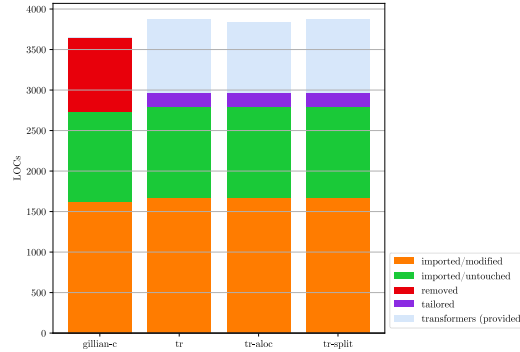

Figure 5.12: Number of lines of code per C instantiation

The charts for TR, TR-ALoc and TR-Split are extremely similar, for both Collections-C and AWS code; we can thus hardly make a hypothesis as to which optimisation is better suited. This is in part due to the fact these optimisations excel in larger maps, as was the case for JavaScript. In C, maps are instead rather small; for Collections-C, the map size tends to fluctuate between 15 and 35 elements, with the maximum recorder reaching 52 elements. For AWS, its size is between 10 and 20 elements, maxing at 20. This in particular explains why the optimisations are at times detrimental for the AWS test suite: because the maps are always small, the cost of both optimisations regularly outweighs their improvement.

We now compare the complexity of instantiations, using LOCs, delving into more details on the cause of each line count metric; see Figure 5.12. In particular, we see that the majority of the state model is the same; the modified part representing BLOCKTREE and CGENV, which are lightly modified to fit into the transformers setup, while the untouched part represents the internal modules used by the C instantiations (for instance, to encode permissions, or memory chunks). Finally, the removed part of the code is trivially replaced with constructions, using PMAP and FREEABLE. Some tailored code is also required, to ensure the construction matches the interface of Gillian-C. This again shows that even for more complex state models that require a significant amount of custom code, using our state model library reduces the amount of code required exclusively for that instantiation, while providing performance gains for free.

5.4   Partial Map Performance

Finally, and to confirm our intuition, we do a brief evaluation of the time taken to symbolically `get` elements from the different PMAP optimisations. This is done by measuring the time at the start and end of the `validate_index` function of the various PMAP implementations during the execution of the
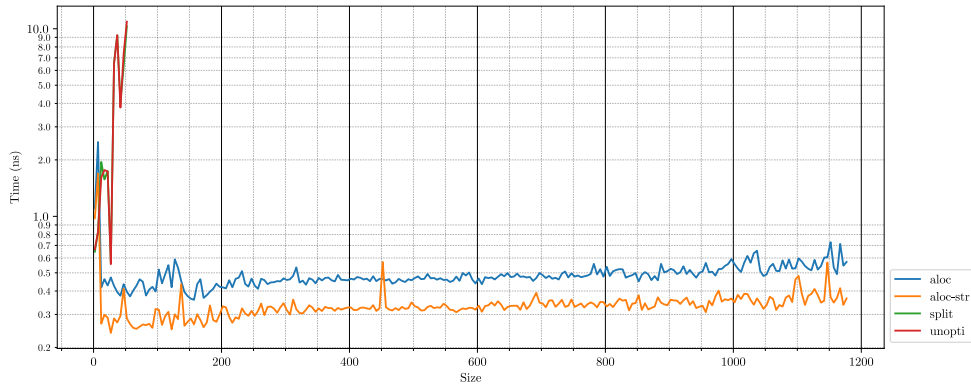
Figure 5.13: Average time taken to access states in PMap, depending on the size of the heap and the optimisations used.

Buckets-JS tests.[6] The results are shown in Figure 5.13. `unopti` is the base PMap implementation with no optimisation, `split` is $\text{PMap}_{\text{Split}}$, `aloc` is $\text{PMap}_{\text{ALoc}}$, and `aloc-str` is $\text{PMap}_{\text{ALoc}}^{\text{Str}}$ – the ALoc optimisation with strings as keys.

We first notice the significant difference between the PMap with and without the ALoc optimisation: while the data only exists for map sizes up to 50 elements, the operation takes up to 10ns without ALocs, while taking on average 0.5ns with them. The use of the split optimisation does not seem to have an impact in this regard.

We can also compare performance when using expressions as keys (`aloc`) against using strings (`aloc-str`); here the impact seems to be constant, with an approximate improvement of 30% when using strings. In both cases, the time taken for a lookup seems to logarithmically scale with the size, which makes sense given `Map`, the used data structure, is a balanced binary tree.

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[6]We choose to do it over this sample because it represents real-world code, and because JavaScript makes use of more combinations of optimisations, since the PMap for objects does not have a Loc domain.

Chapter 6

Conclusion

6.1 Achievements

Throughout this project we adapted and developed a theory to construct state models, as well as developed a range of different state models to allow for the construction of complex state from easy to understand and modify building blocks. To this end, we firstly highlighted the difficulty with using partial commutative monoids when considering state model constructions. We then introduced the concept of *partial resource algebras*, an adaptation of the resource algebras presented in Iris [35]. We adapted the notion of state models introduced in the context of an abstract parametric CSE engine [44] to use these partial resource algebras, and re-defined the properties these state models must satisfy. We then introduced a range of different state models and state model transformers, taking inspiration from work done in [2, 35]; *these state models are proven to be OX and UX sound*. These state models and state model transformers can be combined to build more complex state models, allowing real-world memory models to be recreated with ease, while being sound by construction. For instance, the JavaScript state model which is capable of verifying real world code is sound, and no additional proof is needed to show it, as it only uses the state models we defined. We have also demonstrated that two optimisations of the partial map transformer are sound, and shown exactly why a third optimisation is not.

We then brought these state model constructions to Gillian, a CSE engine. We have adapted and simplified the state model interface it provides to suit our needs, and implemented the presented state models as an OCaml library that exposes a total of *14 different modules and functors* usable for state model constructions. The implemented state models are OX and UX sound, allowing them to be used in Gillian to perform whole program symbolic testing, verification and true bug finding of real world code. These modules are all short and independent, allowing them to be modified, verified and optimised easily.

Furthermore, we used our library to create instantiations of Gillian for JavaScript, C and WISL, which have the same capabilities as the pre-existing monolithic instantiations, but with stronger soundness guarantees, thanks to the ease of verification of the library. This has shown that the state models we defined are capable of representing real languages in their entirety, like JavaScript, but can also be used in combination with more specialised state models while still providing better performance and a simpler instantiation, with C as an example.

We have performed an extensive evaluation of the performance of our instantiations, analysing in detail what is improved compared to the reference Gillian instantiations, and what areas still need further refinement. The evaluation was done for WISL and C with whole program symbolic testing (WPST), verification and true bug finding; for JavaScript with WPST and verification. In the case of JavaScript and C, we saw a net improvement in performance in all execution modes.

Finally, we also evaluated the performance offered by the optimisations of the partial map we defined theoretically, showing mixed results.

6.2 Limitations and Future Work

The presented state models are quite simple in their construction; in particular more ambitious resource algebras such as what Iris calls "higher-order ghost state" [35] have not been replicated. Overall a lot of the logic used is still embedded in Gillian, rather than defined externally via state models and core predicates. More research would be needed to see how more deeply Iris-enabled reasoning can be embedded in Gillian's logic.

User-defined predicates, the bi-abduction anti-frame and function calls by specification can all be defined via state model transformers, rather than as part of a CSE engine [2]. Formalising and implementing them using our definitions could prove useful, as these functionalities could then be removed from Gillian's core engine and instead defined externally, reducing its complexity. Doing this would also facilitate the adoption of these features in new parametric CSE engines, as they could be added on top of the engine later in development, rather than being deeply embedded within it. This work paves the way to a new CSE engine that puts more emphasis on state models and modularity in general.

The partial map state model transformer, PMAP, only supports allocation in a non-concurrent setting. While not a limitation for our use case, since Gillian does not support concurrency, it is a flaw of the theory; more advanced state models could be developed to support parallel allocation while still providing guarantees of OX and UX soundness and allow distinguishing out of bounds from missing errors. This could be done, for instance, via an authoritative resource algebra [35].

Instantiating Gillian using the developed state model library is still a convoluted process at times, requiring functor-heavy code, which is hard to write and understand. The developed interface for state models is also not ideal, requiring a significant amount of functions to be defined; ideally only the core `execute_action`, `consume`, `produce` and `fix` functions should be needed, the rest being optional. The goal is here to again simplify implementation as much as possible, allowing it to be as close to the theory as possible. An idea that comes to mind is the use of OCaml's `ppxlib` library[1] to do meta-programming and simplify some tedious requirements of the state model interface.

While we have obtained full parity with the Gillian instantiations via our transformers, future work could attempt to further improve performance, with more aggressive optimisations or attempting to tie in the state models with lower level code. Going further than what has been done with Gillian before, instantiating it to new languages, should also be considered.

Finally, we showed the system of abstract locations Gillian uses extensively is currently unsound; finding a sound justification to its use, for instance taking inspiration from nominal logic, would be crucial to reinforce Gillian's claims of soundness.

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[1] See https://github.com/ocaml-ppx/ppxlib.

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Appendix A

State Model Definitions

We list here the different state models developed. In particular, we list their components in the following order: the definition of the Resource Algebra, along with its core $|-|$ and composition $(\cdot)$ rules, the list of actions $\mathcal{A}$ and core predicates Δ it has, the rules for symbolic interpretation $\models$, the rules for predicate satisfiability $\models_\Delta$, the rules for `execute_action` on concrete states, the rules for `execute_action` on symbolic state, the rules for `consume`, the rules for `produce`, and finally the rules for `fix`.

A.1 Exclusive

A.1.1 Resource Algebra

$$\mathrm{Ex}(X) \triangleq \mathrm{ex}(x\colon X)$$
$$|\mathrm{ex}(x)| \triangleq \bot$$
$$\mathrm{ex}(x_1) \cdot \mathrm{ex}(x_2) \text{ is always undefined}$$

It defines two actions, $\mathcal{A} = \{\texttt{load}, \texttt{store}\}$ and a predicate $\Delta = \{\mathrm{ex}\}$.

A.1.2 Satisfiability

$$\frac{\text{ExPredSat}}{\sigma = \mathrm{ex}(x)}{\sigma \models_\Delta \langle \mathrm{ex}\rangle([]; [x])} \qquad \frac{\text{ExSymInterpretation}}{[\![\hat{x}]\!]_{\theta,s} = x}{\theta, s, \mathrm{ex}(x) \models \mathrm{ex}(\hat{x})}$$

A.1.3 Compositional Concrete Rules

CExLoadOk
$$\texttt{load}(\mathrm{ex}(x), []) \rightsquigarrow (\texttt{Ok}, \mathrm{ex}(x), [x])$$

CExLoadMiss
$$\texttt{load}(\bot, []) \rightsquigarrow (\texttt{Miss}, \bot, [])$$

CExStoreOk
$$\texttt{store}(\mathrm{ex}(x), [x']) \rightsquigarrow (\texttt{Ok}, \mathrm{ex}(x'), [])$$

CExStoreMiss
$$\texttt{store}(\bot, [x']) \rightsquigarrow (\texttt{Miss}, \bot, [])$$

A.1.4 Compositional Symbolic Rules

ExLoadOk
$$\mathtt{load}(\mathrm{ex}(\hat{x}), []) \rightsquigarrow (\mathtt{Ok}, \mathrm{ex}(\hat{x}), [\hat{x}], [])$$

ExLoadMiss
$$\mathtt{load}(\bot, []) \rightsquigarrow (\mathtt{Miss}, \bot, [], [])$$

ExStoreOk
$$\mathtt{store}(\mathrm{ex}(\hat{x}), [\hat{x}']) \rightsquigarrow (\mathtt{Ok}, \mathrm{ex}(\hat{x}'), [], [])$$

ExStoreMiss
$$\mathtt{store}(\bot, [\hat{x}']) \rightsquigarrow (\mathtt{Miss}, \bot, [], [])$$

ExConsOk
$$\mathtt{consume}(\mathrm{ex}(\hat{x}), \mathsf{ex}, []) \rightsquigarrow (\mathtt{Ok}, \bot, [\hat{x}], [])$$

ExConsMiss
$$\mathtt{consume}(\bot, \mathsf{ex}, []) \rightsquigarrow (\mathtt{Miss}, \bot, [], [])$$

ExProd
$$\mathtt{produce}(\bot, \mathsf{ex}, [], [\hat{x}]) \rightsquigarrow (\mathrm{ex}(\hat{x}), [])$$

ExFix
$$\mathtt{fix} \; [] = [\{\exists \hat{x}. \langle \mathsf{ex} \rangle (; \hat{x})\}]$$

A.2 Agreement

A.2.1 Resource Algebra

$$\mathrm{Ag}(X) \triangleq \mathrm{ag}(x \colon X)$$

$$|\mathrm{ag}(x)| \triangleq \mathrm{ag}(x)$$

$$\mathrm{ag}(x) \cdot \mathrm{ag}(x') \triangleq \begin{cases} \mathrm{ag}(x) & \text{if } x = x' \\ \text{undefined} & \text{otherwise} \end{cases}$$

It defines $\mathcal{A} = \{\mathtt{load}\}$, and one predicate, $\Delta = \{\mathsf{ag}\}$.

A.2.2 Satisfiability

AgPredSat
$$\dfrac{\sigma = \mathrm{ag}(x)}{\sigma \models_\Delta \langle \mathsf{ag} \rangle ([]; [x])}$$

AgSymInterpretation
$$\dfrac{[\![\hat{x}]\!]_{\theta,s} = x}{\theta, s, \mathrm{ag}(x) \models \mathrm{ag}(\hat{x})}$$

A.2.3 Compositional Concrete Rules

CAgLoadOk
$$\mathtt{load}(\mathrm{ag}(x), []) \rightsquigarrow (\mathtt{Ok}, \mathrm{ag}(x), [x])$$

CAgLoadMiss
$$\mathtt{load}(\bot, []) \rightsquigarrow (\mathtt{Miss}, \bot, [])$$

A.2.4 Compositional Symbolic Rules

AgLoadOk
$$\mathtt{load}(\mathrm{ag}(\hat{x}), []) \rightsquigarrow (\mathtt{Ok}, \mathrm{ag}(\hat{x}), [\hat{x}], [])$$

AgLoadMiss
$$\mathtt{load}(\bot, []) \rightsquigarrow (\mathtt{Miss}, \bot, [], [])$$

AgConsOk
$$\mathtt{consume}(\mathrm{ag}(\hat{x}), \mathsf{ag}, []) \rightsquigarrow (\mathtt{Ok}, \mathrm{ag}(\hat{x}), [\hat{x}], [])$$

AgConsMiss
$$\mathtt{consume}(\bot, \mathsf{ag}, []) \rightsquigarrow (\mathtt{Miss}, \bot, [][])$$

AgProdBot
$$\mathtt{produce}(\bot, \mathsf{ag}, [], [\hat{x}]) \rightsquigarrow (\mathrm{ag}(\hat{x}), [])$$

AgProdEq
$$\mathtt{produce}(\mathrm{ag}(\hat{x}), \mathsf{ag}, [], [\hat{x}']) \rightsquigarrow (\mathrm{ag}(\hat{x}), [\hat{x} = \hat{x}'])$$

AgFix
$$\mathtt{fix} \; [] = [\{\exists \hat{x}. \langle \mathsf{ag} \rangle (; \hat{x})\}]$$

A.3 Fractional

A.3.1 Resource Algebra

$$\text{FRAC}(X) \triangleq \text{frac}(x \colon X, q \colon (0;1])$$

$$|\text{frac}(x,q)| \triangleq \bot$$

$$\text{frac}(x,q) \cdot \text{frac}(x',q') \triangleq \begin{cases} \text{frac}(x, q+q') & \text{if } x = x' \wedge q + q' \leq 1 \\ \text{undefined} & \text{otherwise} \end{cases}$$

We define the actions $\mathcal{A} = \{\texttt{load}, \texttt{store}\}$ and the core predicate $\Delta = \{\texttt{frac}\}$.

A.3.2 Satisfiability

$$\frac{\sigma = \text{frac}(x, q)}{\sigma \models_\Delta \langle \texttt{frac} \rangle ([q]; [x])} \quad \text{FRACPREDSAT}$$

$$\frac{[\![\hat{x}]\!]_{\theta,s} = x \qquad [\![\hat{q}]\!]_{\theta,s} = q}{\theta, s, \text{frac}(x,q) \models \text{frac}(\hat{x}, \hat{q})} \quad \text{FRACSYMINTERPRETATION}$$

A.3.3 Compositional Concrete Rules

CFRACLOADOK
$$\texttt{load}(\text{frac}(x,q), []) \rightsquigarrow (\texttt{Ok}, \text{frac}(x,q), [x], [])$$

CFRACLOADMISS
$$\texttt{load}(\bot, []) \rightsquigarrow (\texttt{Miss}, \bot, [1])$$

CFRACSTOREOK
$$\frac{q = 1}{\texttt{store}(\text{frac}(x,q), [x']) \rightsquigarrow (\texttt{Ok}, \text{frac}(x',q), [])}$$

CFRACSTOREPERM
$$\frac{q < 1}{\texttt{store}(\text{frac}(x,q), [x']) \rightsquigarrow (\texttt{Miss}, \text{frac}(x,q), [1 - q])}$$

CFRACSTOREMISS
$$\texttt{store}(\bot, [x']) \rightsquigarrow (\texttt{Miss}, \bot, [1])$$

A.3.4 Compositional Symbolic Rules

FRACLOADOK
$$\texttt{load}(\text{frac}(\hat{x}, \hat{q}), []) \rightsquigarrow (\texttt{Ok}, \text{frac}(\hat{x}, \hat{q}), [\hat{x}], [])$$

FRACLOADMISS
$$\texttt{load}(\bot, []) \rightsquigarrow (\texttt{Miss}, \bot, [1], [])$$

FRACSTOREOK
$$\texttt{store}(\text{frac}(\hat{x}, \hat{q}), [\hat{x}']) \rightsquigarrow (\texttt{Ok}, \text{frac}(\hat{x}', \hat{q}), [], [\hat{q} = 1])$$

FRACSTOREPERM
$$\texttt{store}(\text{frac}(\hat{x}, \hat{q}), [\hat{x}']) \rightsquigarrow (\texttt{Miss}, \text{frac}(\hat{x}, \hat{q}), [1 - \hat{q}], [\hat{q} < 1])$$

FRACSTOREMISS
$$\texttt{store}(\bot, [\hat{x}']) \rightsquigarrow (\texttt{Miss}, \bot, [1], [])$$

FRACCONSALL
$$\texttt{consume}(\text{frac}(\hat{x}, \hat{q}), \texttt{frac}, [\hat{q}']) \rightsquigarrow (\texttt{Ok}, \bot, [\hat{x}], [\hat{q} = \hat{q}'])$$

FRACCONSSOME
$$\texttt{consume}(\text{frac}(\hat{x}, \hat{q}), \texttt{frac}, [\hat{q}']) \rightsquigarrow (\texttt{Ok}, \text{frac}(\hat{x}, \hat{q} - \hat{q}'), [\hat{x}], [0 < \hat{q}' < \hat{q}])$$

FRACCONSMISS
$$\texttt{consume}(\text{frac}(\hat{x}, \hat{q}), \texttt{frac}, [\hat{q}']) \rightsquigarrow (\texttt{Miss}, \text{frac}(\hat{x}, \hat{q}), [\hat{q}' - \hat{q}], [\hat{q} < \hat{q}' \leq 1])$$

FRACCONSFAIL
$$\texttt{consume}(\text{frac}(\hat{x}, \hat{q}), \texttt{frac}, [\hat{q}']) \rightsquigarrow (\texttt{LFail}, \text{frac}(\hat{x}, \hat{q}), [], [\hat{q}' \leq 0 \vee 1 < \hat{q}'])$$

FRACPRODBOT

$\mathsf{produce}(\bot, \mathsf{frac}, [\hat{q}], [\hat{x}]) \rightsquigarrow (\mathsf{frac}(\hat{x}, \hat{q}), [0 < \hat{q} \leq 1])$

FRACPRODEQ

$\mathsf{produce}(\mathsf{frac}(\hat{x}, \hat{q}), \mathsf{frac}, [\hat{q}'], [\hat{x}']) \rightsquigarrow (\mathsf{frac}(\hat{x}, \hat{q} + \hat{q}'), [\hat{x} = \hat{x}' \wedge 0 < \hat{q}' \wedge \hat{q} + \hat{q}' \leq 1])$

FRACFIX

$\mathsf{fix}\ [\hat{q}] = [\{\exists \hat{x}.\ \langle \mathsf{frac} \rangle(\hat{q}; \hat{x})\}]$

A.4   Sum

A.4.1   Resource Algebra

Let Σ_1 and Σ_2 be the carrier sets of the RAs of $\mathbb{S}_1$ and $\mathbb{S}_2$ respectively.

$$\text{SUM}(\mathbb{S}_1, \mathbb{S}_2) \triangleq \mathbb{S}_1 + \mathbb{S}_2 \triangleq l(\sigma\colon \Sigma_1) \mid r(\sigma\colon \Sigma_2)$$
$$l(\sigma) \cdot l(\sigma') \triangleq l(\sigma \cdot \sigma')$$
$$r(\sigma) \cdot r(\sigma') \triangleq r(\sigma \cdot \sigma')$$
$$|l(\sigma)| \triangleq \begin{cases} \bot & \text{if } |\sigma| = \bot \\ l(|\sigma|) & \text{otherwise} \end{cases}$$
$$|r(\sigma)| \triangleq \begin{cases} \bot & \text{if } |\sigma| = \bot \\ r(|\sigma|) & \text{otherwise} \end{cases}$$

Given the sum $\mathbb{S}_1 + \mathbb{S}_2$ such that the actions and predicates of $\mathbb{S}_1$ and $\mathbb{S}_2$ are $\mathcal{A}_1$ and Δ_1, and $\mathcal{A}_2$ and Δ_2 respectively, the actions of the sum are defined as $\mathcal{A} = \{\mathsf{L}\alpha : \alpha \in \mathcal{A}_1\} \uplus \{\mathsf{R}\alpha : \alpha \in \mathcal{A}_2\}$, and the core predicates as $\Delta = \{\mathsf{L}\delta : \delta \in \Delta_1\} \uplus \{\mathsf{R}\delta : \delta \in \Delta_2\}$.

A.4.2   Satisfiability

SUMPREDSATL
$$\frac{\sigma_l = l(\sigma) \qquad \sigma \models_\Delta \langle \delta \rangle(\vec{v}_i; \vec{v}_o)}{\sigma_l \models_\Delta \langle \mathsf{L}\delta \rangle(\vec{v}_i; \vec{v}_o)}$$

SUMPREDSATR
$$\frac{\sigma_r = r(\sigma) \qquad \sigma \models_\Delta \langle \delta \rangle(\vec{v}_i; \vec{v}_o)}{\sigma_r \models_\Delta \langle \mathsf{R}\delta \rangle(\vec{v}_i; \vec{v}_o)}$$

SUMPREDSATLBOT
$$\frac{\bot \models_\Delta \langle \delta \rangle(\vec{v}_i; \vec{v}_o)}{\bot \models_\Delta \langle \mathsf{L}\delta \rangle(\vec{v}_i; \vec{v}_o)}$$

SUMPREDSATRBOT
$$\frac{\bot \models_\Delta \langle \delta \rangle(\vec{v}_i; \vec{v}_o)}{\bot \models_\Delta \langle \mathsf{R}\delta \rangle(\vec{v}_i; \vec{v}_o)}$$

SUMSYMINTERPRETATIONL
$$\frac{\sigma_l = l(\sigma) \qquad \hat{\sigma}_l = l(\hat{\sigma}) \qquad \theta, s, \sigma \models \hat{\sigma}}{\theta, s, \sigma_l \models \hat{\sigma}_l}$$

SUMSYMINTERPRETATIONR
$$\frac{\sigma_r = r(\sigma) \qquad \hat{\sigma}_r = r(\hat{\sigma}) \qquad \theta, s, \sigma \models \hat{\sigma}}{\theta, s, \sigma_r \models \hat{\sigma}_r}$$

A.4.3 Compositional Concrete Rules

Given $wrap_l(x) = \begin{cases} \bot & \text{if } x = \bot \\ l(x) & \text{otherwise} \end{cases}$ $\qquad$ $wrap_r(x) = \begin{cases} \bot & \text{if } x = \bot \\ r(x) & \text{otherwise} \end{cases}$

$unwrap_l(x_l) = \begin{cases} \bot & \text{if } x = \bot \\ x & \text{if } x_l = l(x) \\ \text{undefined} & \text{otherwise} \end{cases}$ $\qquad$ $unwrap_r(x_r) = \begin{cases} \bot & \text{if } x = \bot \\ x & \text{if } x_r = r(x) \\ \text{undefined} & \text{otherwise} \end{cases}$

CSumLAction
$$\frac{\sigma = unwrap_l(\sigma_l) \qquad \alpha(\sigma, \vec{v}_i) \rightsquigarrow (o, \sigma', \vec{v}_o) \qquad \sigma_l' = wrap_l(\sigma') \qquad o \neq \text{Miss}}{\mathsf{L}\alpha(\sigma_l, \vec{v}_i) \rightsquigarrow (o, \sigma_l', \vec{v}_o)}$$

CSumLActionMiss
$$\frac{\sigma = unwrap_l(\sigma_l) \qquad \alpha(\sigma, \vec{v}_i) \rightsquigarrow (\text{Miss}, \sigma', \vec{v}_o) \qquad \sigma_l' = wrap_l(\sigma')}{\mathsf{L}\alpha(\sigma_l, \vec{v}_i) \rightsquigarrow (\text{Miss}, \sigma_l', \text{`l'} :: \vec{v}_o)}$$

CSumRAction
$$\frac{\sigma = unwrap_r(\sigma_r) \qquad \alpha(\sigma, \vec{v}_i) \rightsquigarrow (o, \sigma', \vec{v}_o) \qquad \sigma_r' = wrap_r(\sigma') \qquad o \neq \text{Miss}}{\mathsf{R}\alpha(\sigma_r, \vec{v}_i) \rightsquigarrow (o, \sigma_r', \vec{v}_o)}$$

CSumRActionMiss
$$\frac{\sigma = unwrap_r(\sigma_r) \qquad \alpha(\sigma, \vec{v}_i) \rightsquigarrow (\text{Miss}, \sigma', \vec{v}_o) \qquad \sigma_r' = wrap_r(\sigma')}{\mathsf{R}\alpha(\sigma_r, \vec{v}_i) \rightsquigarrow (\text{Miss}, \sigma_r', \text{`r'} :: \vec{v}_o)}$$

CSumLActionIncompat
$$\mathsf{L}\alpha(r(\sigma), \vec{v}_i) \rightsquigarrow (\text{Err}, r(\sigma), [])$$

CSumRActionIncompat
$$\mathsf{R}\alpha(l(\sigma), \vec{v}_i) \rightsquigarrow (\text{Err}, l(\sigma), [])$$

A.4.4 Compositional Symbolic Rules

SumLAction
$$\frac{\hat{\sigma} = unwrap_l(\hat{\sigma}_l) \qquad \alpha(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{\hat{v}}_o, \pi) \qquad \hat{\sigma}_l' = wrap_l(\hat{\sigma}') \qquad o \neq \text{Miss}}{\mathsf{L}\alpha(\hat{\sigma}_l, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}_l', \vec{\hat{v}}_o, \pi)}$$

SumLActionMiss
$$\frac{\hat{\sigma} = unwrap_l(\hat{\sigma}_l) \qquad \alpha(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (\text{Miss}, \hat{\sigma}', \vec{\hat{v}}_o, \pi) \qquad \hat{\sigma}_l' = wrap_l(\hat{\sigma}')}{\mathsf{L}\alpha(\hat{\sigma}_l, \vec{\hat{v}}_i) \rightsquigarrow (\text{Miss}, \hat{\sigma}_l', \text{`l'} :: \vec{\hat{v}}_o, \pi)}$$

SumRAction
$$\frac{\hat{\sigma} = unwrap_r(\hat{\sigma}_r) \qquad \alpha(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{\hat{v}}_o, \pi) \qquad \hat{\sigma}_r' = wrap_r(\hat{\sigma}') \qquad o \neq \text{Miss}}{\mathsf{R}\alpha(\hat{\sigma}_r, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}_r', \vec{\hat{v}}_o, \pi)}$$

SumRActionMiss
$$\frac{\hat{\sigma} = unwrap_r(\hat{\sigma}_r) \qquad \alpha(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (\text{Miss}, \hat{\sigma}', \vec{\hat{v}}_o, \pi) \qquad \hat{\sigma}_r' = wrap_r(\hat{\sigma}')}{\mathsf{R}\alpha(\hat{\sigma}_r, \vec{\hat{v}}_i) \rightsquigarrow (\text{Miss}, \hat{\sigma}_r', \text{`r'} :: \vec{\hat{v}}_o, \pi)}$$

SumLActionIncompat
$$\mathsf{L}\alpha(r(\hat{\sigma}), \vec{\hat{v}}_i) \rightsquigarrow (\text{Err}, r(\hat{\sigma}), [], [])$$

SumRActionIncompat
$$\mathsf{R}\alpha(l(\hat{\sigma}), \vec{\hat{v}}_i) \rightsquigarrow (\text{Err}, l(\hat{\sigma}), [], [])$$

SumLCons
$$\frac{\hat{\sigma} = unwrap_l(\hat{\sigma}_l) \qquad \mathsf{consume}(\hat{\sigma}, \delta, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{\hat{v}}_o, \pi) \qquad \hat{\sigma}_l' = wrap_l(\hat{\sigma}') \qquad o \neq \text{Miss}}{\mathsf{consume}(\hat{\sigma}_l, \mathsf{L}\delta, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}_l', \vec{\hat{v}}_o, \pi)}$$

SUMLCONSMISS
$$\frac{\hat{\sigma} = unwrap_l(\hat{\sigma}_l) \qquad \mathsf{consume}(\hat{\sigma}, \delta, \vec{\hat{v}}_i) \rightsquigarrow (\mathtt{Miss}, \hat{\sigma}', \vec{\hat{v}}_o, \pi) \qquad \hat{\sigma}'_l = wrap_l(\hat{\sigma}')}{\mathsf{consume}(\hat{\sigma}_l, \mathsf{L}\delta, \vec{\hat{v}}_i) \rightsquigarrow (\mathtt{Miss}, \hat{\sigma}'_l, \text{`l'} :: \vec{\hat{v}}_o, \pi)}$$

SUMRCONS
$$\frac{\hat{\sigma} = unwrap_r(\hat{\sigma}_r) \qquad \mathsf{consume}(\hat{\sigma}, \delta, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{\hat{v}}_o, \pi) \qquad \hat{\sigma}'_r = wrap_r(\hat{\sigma}') \qquad o \neq \mathtt{Miss}}{\mathsf{consume}(\hat{\sigma}_r, \mathsf{R}\delta, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}'_r, \vec{\hat{v}}_o, \pi)}$$

SUMRCONSMISS
$$\frac{\hat{\sigma} = unwrap_r(\hat{\sigma}_r) \qquad \mathsf{consume}(\hat{\sigma}, \delta, \vec{\hat{v}}_i) \rightsquigarrow (\mathtt{Miss}, \hat{\sigma}', \vec{\hat{v}}_o, \pi) \qquad \hat{\sigma}'_r = wrap_r(\hat{\sigma}')}{\mathsf{consume}(\hat{\sigma}_r, \mathsf{R}\delta, \vec{\hat{v}}_i) \rightsquigarrow (\mathtt{Miss}, \hat{\sigma}'_r, \text{`r'} :: \vec{\hat{v}}_o, \pi)}$$

SUMLCONSINCOMPAT
$$\mathsf{consume}(r(\hat{\sigma}), \mathsf{L}\delta, \vec{\hat{v}}_i) \rightsquigarrow (\mathtt{LFail}, r(\hat{\sigma}), [], [])$$

SUMRCONSINCOMPAT
$$\mathsf{consume}(l(\hat{\sigma}), \mathsf{R}\delta, \vec{\hat{v}}_i) \rightsquigarrow (\mathtt{LFail}, l(\hat{\sigma}), [], [])$$

SUMLPROD
$$\frac{\hat{\sigma} = unwrap_l(\hat{\sigma}_l) \qquad \mathsf{produce}(\hat{\sigma}, \delta, \vec{\hat{v}}_i, \vec{\hat{v}}_o) \rightsquigarrow (\hat{\sigma}', \pi) \qquad \hat{\sigma}'_l = wrap_l(\hat{\sigma}')}{\mathsf{produce}(\hat{\sigma}_l, \mathsf{L}\delta, \vec{\hat{v}}_i, \vec{\hat{v}}_o) \rightsquigarrow (\hat{\sigma}'_l, \pi)}$$

SUMRPROD
$$\frac{\hat{\sigma} = unwrap_r(\hat{\sigma}_r) \qquad \mathsf{produce}(\hat{\sigma}, \delta, \vec{\hat{v}}_i, \vec{\hat{v}}_o) \rightsquigarrow (\hat{\sigma}', \pi) \qquad \hat{\sigma}'_r = wrap_r(\hat{\sigma}')}{\mathsf{produce}(\hat{\sigma}_r, \mathsf{R}\delta, \vec{\hat{v}}_i, \vec{\hat{v}}_o) \rightsquigarrow (\hat{\sigma}'_r, \pi)}$$

SUMLFIX
$$\frac{\mathbb{S}_1.\mathtt{fix} \ \vec{\hat{v}}_i = a}{\mathtt{fix} \ \text{`l'} :: \vec{\hat{v}}_i = a}$$

SUMRFIX
$$\frac{\mathbb{S}_2.\mathtt{fix} \ \vec{\hat{v}}_i = a}{\mathtt{fix} \ \text{`r'} :: \vec{\hat{v}}_i = a}$$

A.5   Product

A.5.1   Resource Algebra

Let Σ_1 and Σ_2 be the carrier sets of the RAs of $\mathbb{S}_1$ and $\mathbb{S}_2$ respectively.

$$\mathrm{PRODUCT}(\mathbb{S}_1, \mathbb{S}_2) \triangleq \mathbb{S}_1 \times \mathbb{S}_2 \triangleq \Sigma_1^? \times \Sigma_2^? \setminus \{(\bot, \bot)\}$$

$$(\sigma_l, \sigma_r) \cdot (\sigma'_l, \sigma'_r) \triangleq (\sigma_l \cdot \sigma'_l, \sigma_r \cdot \sigma'_r)$$

$$|(\sigma_l, \sigma_r)| \triangleq \begin{cases} \bot & \text{if } |\sigma_l| = \bot \wedge |\sigma_r| = \bot \\ (|\sigma_l|, |\sigma_r|) & \text{otherwise} \end{cases}$$

Given the product $\mathbb{S}_1 \times \mathbb{S}_2$ such that the actions and predicates of $\mathbb{S}_1$ and $\mathbb{S}_2$ are $\mathcal{A}_1$ and Δ_1, and $\mathcal{A}_2$ and Δ_2 respectively, the actions of the product are defined as $\mathcal{A} = \{\mathsf{L}\alpha : \alpha \in \mathcal{A}_1\} \uplus \{\mathsf{R}\alpha : \alpha \in \mathcal{A}_2\}$, and the core predicates as $\Delta = \{\mathsf{L}\delta : \delta \in \Delta_1\} \uplus \{\mathsf{R}\delta : \delta \in \Delta_2\}$.

A.5.2 Satisfiability

$$
\frac{\text{PRODUCTPREDSATL}}{\sigma_l \models_\Delta \langle\delta\rangle(\vec{v}_i; \vec{v}_o)}{(\sigma_l, \sigma_r) \models_\Delta \langle\mathsf{L}\delta\rangle(\vec{v}_i; \vec{v}_o)}
\qquad
\frac{\text{PRODUCTPREDSATR}}{\sigma_r \models_\Delta \langle\delta\rangle(\vec{v}_i; \vec{v}_o)}{(\sigma_l, \sigma_r) \models_\Delta \langle\mathsf{R}\delta\rangle(\vec{v}_i; \vec{v}_o)}
$$

$$
\frac{\text{PRODUCTPREDSATLBOT}}{\bot \models_\Delta \langle\delta\rangle(\vec{v}_i; \vec{v}_o)}{\bot \models_\Delta \langle\mathsf{L}\delta\rangle(\vec{v}_i; \vec{v}_o)}
\qquad
\frac{\text{PRODUCTPREDSATRBOT}}{\bot \models_\Delta \langle\delta\rangle(\vec{v}_i; \vec{v}_o)}{\bot \models_\Delta \langle\mathsf{R}\delta\rangle(\vec{v}_i; \vec{v}_o)}
$$

$$
\frac{\text{PRODUCTSYMINTERPRETATION}}{\theta, s, \sigma_l \models \hat{\sigma}_l \qquad \theta, s, \sigma_r \models \hat{\sigma}_r}{\theta, s, (\sigma_l, \sigma_r) \models (\hat{\sigma}_l, \hat{\sigma}_r)}
$$

A.5.3 Compositional Concrete Rules

Given $wrap(x, y) = \begin{cases} \bot & \text{if } x = \bot \wedge y = \bot \\ (x, y) & \text{otherwise} \end{cases}$ and $unwrap(s) = \begin{cases} (\bot, \bot) & \text{if } s = \bot \\ (x, y) & \text{otherwise} \end{cases}$

$$
\frac{\text{CPRODUCTLACTION}}{(\sigma_l, \sigma_r) = unwrap(\sigma) \qquad \alpha(\sigma_l, \vec{v}_i) \rightsquigarrow (o, \sigma_l', \vec{v}_o) \qquad \sigma' = wrap(\sigma_l', \sigma_r) \qquad o \neq \texttt{Miss}}{\mathsf{L}\alpha(\sigma, \vec{v}_i) \rightsquigarrow (o, \sigma', \vec{v}_o)}
$$

$$
\frac{\text{CPRODUCTLACTIONMISS}}{(\sigma_l, \sigma_r) = unwrap(\sigma) \qquad \alpha(\sigma_l, \vec{v}_i) \rightsquigarrow (\texttt{Miss}, \sigma_l', \vec{v}_o) \qquad \sigma' = wrap(\sigma_l', \sigma_r)}{\mathsf{L}\alpha(\sigma, \vec{v}_i) \rightsquigarrow (\texttt{Miss}, \sigma', \text{`l'} :: \vec{v}_o)}
$$

$$
\frac{\text{CPRODUCTRACTION}}{(\sigma_l, \sigma_r) = unwrap(\sigma) \qquad \alpha(\sigma_r, \vec{v}_i) \rightsquigarrow (o, \sigma_r', \vec{v}_o) \qquad \sigma' = wrap(\sigma_l, \sigma_r') \qquad o \neq \texttt{Miss}}{\mathsf{R}\alpha(\sigma, \vec{v}_i) \rightsquigarrow (o, \sigma', \vec{v}_o)}
$$

$$
\frac{\text{CPRODUCTRACTIONMISS}}{(\sigma_l, \sigma_r) = unwrap(\sigma) \qquad \alpha(\sigma_r, \vec{v}_i) \rightsquigarrow (\texttt{Miss}, \sigma_r', \vec{v}_o) \qquad \sigma' = wrap(\sigma_l, \sigma_r')}{\mathsf{R}\alpha(\sigma, \vec{v}_i) \rightsquigarrow (\texttt{Miss}, \sigma', \text{`r'} :: \vec{v}_o)}
$$

A.5.4 Compositional Symbolic Rules

$$
\frac{\text{PRODUCTLACTION}}{(\hat{\sigma}_l, \hat{\sigma}_r) = unwrap(\hat{\sigma}) \qquad \alpha(\hat{\sigma}_l, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}_l', \vec{\hat{v}}_o, \pi) \qquad \hat{\sigma}' = wrap(\hat{\sigma}_l', \hat{\sigma}_r) \qquad o \neq \texttt{Miss}}{\mathsf{L}\alpha(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{\hat{v}}_o, \pi)}
$$

$$
\frac{\text{PRODUCTLACTIONMISS}}{(\hat{\sigma}_l, \hat{\sigma}_r) = unwrap(\hat{\sigma}) \qquad \alpha(\hat{\sigma}_l, \vec{\hat{v}}_i) \rightsquigarrow (\texttt{Miss}, \hat{\sigma}_l', \vec{\hat{v}}_o, \pi) \qquad \hat{\sigma}' = wrap(\hat{\sigma}_l', \hat{\sigma}_r)}{\mathsf{L}\alpha(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (\texttt{Miss}, \hat{\sigma}', \text{`l'} :: \vec{\hat{v}}_o, \pi)}
$$

$$
\frac{\text{PRODUCTRACTION}}{(\hat{\sigma}_l, \hat{\sigma}_r) = unwrap(\hat{\sigma}) \qquad \alpha(\hat{\sigma}_r, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}_r', \vec{\hat{v}}_o, \pi) \qquad \hat{\sigma}' = wrap(\hat{\sigma}_l, \hat{\sigma}_r') \qquad o \neq \texttt{Miss}}{\mathsf{R}\alpha(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{\hat{v}}_o, \pi)}
$$

$$\text{PRODUCTRACTIONMISS}$$
$$\frac{(\hat{\sigma}_l, \hat{\sigma}_r) = unwrap(\hat{\sigma}) \qquad \alpha(\hat{\sigma}_r, \vec{v}_i) \rightsquigarrow (\texttt{Miss}, \hat{\sigma}'_r, \vec{v}_o, \pi) \qquad \hat{\sigma}' = wrap(\hat{\sigma}_l, \hat{\sigma}'_r)}{\texttt{R}\alpha(\hat{\sigma}, \vec{v}_i) \rightsquigarrow (\texttt{Miss}, \hat{\sigma}', \text{`r'} :: \vec{v}_o, \pi)}$$

$$\text{PRODUCTLCONS}$$
$$\frac{(\hat{\sigma}_l, \hat{\sigma}_r) = unwrap(\hat{\sigma}) \qquad \mathsf{consume}(\hat{\sigma}_l, \delta, \vec{v}_i) \rightsquigarrow (o, \hat{\sigma}'_l, \vec{v}_o, \pi) \qquad \hat{\sigma}' = wrap(\hat{\sigma}'_l, \hat{\sigma}_r) \qquad o \neq \texttt{Miss}}{\mathsf{consume}(\hat{\sigma}, \mathsf{L}\delta, \vec{v}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{v}_o, \pi)}$$

$$\text{PRODUCTLCONSMISS}$$
$$\frac{(\hat{\sigma}_l, \hat{\sigma}_r) = unwrap(\hat{\sigma}) \qquad \mathsf{consume}(\hat{\sigma}_l, \delta, \vec{v}_i) \rightsquigarrow (\texttt{Miss}, \hat{\sigma}'_l, \vec{v}_o, \pi) \qquad \hat{\sigma}' = wrap(\hat{\sigma}'_l, \hat{\sigma}_r)}{\mathsf{consume}(\hat{\sigma}, \mathsf{L}\delta, \vec{v}_i) \rightsquigarrow (\texttt{Miss}, \hat{\sigma}', \text{`l'} :: \vec{v}_o, \pi)}$$

$$\text{PRODUCTRCONS}$$
$$\frac{(\hat{\sigma}_l, \hat{\sigma}_r) = unwrap(\hat{\sigma}) \qquad \mathsf{consume}(\hat{\sigma}_r, \delta, \vec{v}_i) \rightsquigarrow (o, \hat{\sigma}'_r, \vec{v}_o, \pi) \qquad \hat{\sigma}' = wrap(\hat{\sigma}_l, \hat{\sigma}'_r) \qquad o \neq \texttt{Miss}}{\mathsf{consume}(\hat{\sigma}, \mathsf{R}\delta, \vec{v}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{v}_o, \pi)}$$

$$\text{PRODUCTRCONSMISS}$$
$$\frac{(\hat{\sigma}_l, \hat{\sigma}_r) = unwrap(\hat{\sigma}) \qquad \mathsf{consume}(\hat{\sigma}_r, \delta, \vec{v}_i) \rightsquigarrow (\texttt{Miss}, \hat{\sigma}'_r, \vec{v}_o, \pi) \qquad \hat{\sigma}' = wrap(\hat{\sigma}_l, \hat{\sigma}'_r)}{\mathsf{consume}(\hat{\sigma}, \mathsf{R}\delta, \vec{v}_i) \rightsquigarrow (\texttt{Miss}, \hat{\sigma}', \text{`r'} :: \vec{v}_o, \pi)}$$

$$\text{PRODUCTLPROD}$$
$$\frac{(\hat{\sigma}_l, \hat{\sigma}_r) = unwrap(\hat{\sigma}) \qquad \mathsf{produce}(\hat{\sigma}_l, \delta, \vec{v}_i, \vec{v}_o) \rightsquigarrow (\hat{\sigma}'_l, \pi) \qquad \hat{\sigma}' = wrap(\hat{\sigma}'_l, \hat{\sigma}_r)}{\mathsf{produce}(\hat{\sigma}, \mathsf{L}\delta, \vec{v}_i, \vec{v}_o) \rightsquigarrow (\hat{\sigma}', \pi)}$$

$$\text{PRODUCTRPROD}$$
$$\frac{(\hat{\sigma}_l, \hat{\sigma}_r) = unwrap(\hat{\sigma}) \qquad \mathsf{produce}(\hat{\sigma}_r, \delta, \vec{v}_i, \vec{v}_o) \rightsquigarrow (\hat{\sigma}'_r, \pi) \qquad \hat{\sigma}' = wrap(\hat{\sigma}_l, \hat{\sigma}'_r)}{\mathsf{produce}(\hat{\sigma}, \mathsf{R}\delta, \vec{v}_i, \vec{v}_o) \rightsquigarrow (\hat{\sigma}', \pi)}$$

$$\text{PRODUCTLFIX}$$
$$\frac{\mathbb{S}_1.\mathtt{fix}\ \vec{v}_i = a}{\mathtt{fix}\ \text{`l'} :: \vec{v}_i = a}$$

$$\text{PRODUCTRFIX}$$
$$\frac{\mathbb{S}_1.\mathtt{fix}\ \vec{v}_i = a}{\mathtt{fix}\ \text{`r'} :: \vec{v}_i = a}$$

A.6 Freeable

A.6.1 Resource Algebra

The RA for FREEABLE($\mathbb{S}$) is defined as a construction: FREEABLE($\mathbb{S}$) $\triangleq \mathbb{S} + \mathrm{Ex}(\{\varnothing\})$, which allows all associated rules to be kept. For clarity, we rename the core predicate R ex (defined by $\mathrm{Ex}(\{\varnothing\})$) as freed. We also extend its actions with the free action.

A.6.2 Satisfiability

The symbolic interpretation and core predicate satisfiability are carried by construction via the sum.

A.6.3 Compositional Concrete Rules

We only define the additional action free, the rest being carried by construction via the sum. The wrapped concrete state model $\mathbb{S}$ must provide an is_exclusively_owned: $\Sigma \to \mathbb{B}$ function such that:

$$\texttt{is\_exclusively\_owned}\ \sigma \iff \nexists \sigma'. \sigma \,\#\, \sigma'$$

It also must provide a fix_owned: $\Sigma \to \mathsf{Val}$ list giving values to fix the Miss.

$$\frac{\texttt{is\_exclusively\_owned } \sigma}{\texttt{free}(l(\sigma), []) \rightsquigarrow (\texttt{Ok}, r(\texttt{ex}(\varnothing)), [])}$$

$$\frac{\neg\texttt{is\_exclusively\_owned } \sigma}{\texttt{free}(l(\sigma), []) \rightsquigarrow (\texttt{Miss}, l(\sigma), \texttt{fix\_owned } x)}$$

CFREEABLEACTIONFREEMISS

$$\texttt{free}(\bot, []) \rightsquigarrow (\texttt{Miss}, \bot, \texttt{fix\_owned } \bot)$$

CFREEABLEACTIONDOUBLEFREE

$$\texttt{free}(r(\texttt{ex}(\varnothing)), []) \rightsquigarrow (\texttt{Err}, r(\texttt{ex}(\varnothing)), [])$$

A.6.4 Compositional Symbolic Rules

Similarly, we carry the symbolic rules for execute_action, consume, produce and fix by construction via the sum. The wrapped symbolic state model $\mathbb{S}$ must provide an is_exclusively_owned: $\hat{\Sigma} \to$ LVal function such that:

$$\texttt{is\_exclusively\_owned } \hat{\sigma} = \texttt{true} \iff (\forall \theta, s, \sigma. \theta, s, \sigma \models \hat{\sigma} \implies \nexists.\sigma'. \sigma \# \sigma')$$

It also must provide a fix_owned: $\hat{\Sigma}^? \to$ LVal list giving values to fix the Miss.

FREEABLEACTIONFREE

$$\texttt{free}(l(\hat{\sigma}), []) \rightsquigarrow (\texttt{Ok}, r(\texttt{ex}(\varnothing)), [], [\texttt{is\_exclusively\_owned } \hat{\sigma}])$$

FREEABLEACTIONFREEERR

$$\texttt{free}(l(\hat{\sigma}), []) \rightsquigarrow (\texttt{Miss}, l(\hat{\sigma}), \texttt{fix\_owned } \hat{\sigma}, [\neg\texttt{is\_exclusively\_owned } \hat{\sigma}])$$

FREEABLEACTIONFREEMISS

$$\texttt{free}(\bot, []) \rightsquigarrow (\texttt{Miss}, \bot, \texttt{fix\_owned } \bot, [])$$

FREEABLEACTIONDOUBLEFREE

$$\texttt{free}(r(\texttt{ex}(\varnothing)), []) \rightsquigarrow (\texttt{Err}, r(\texttt{ex}(\varnothing)), [], [])$$

A.7 Partial Map

A.7.1 Resource Algebra

$$\text{PMAP}(I, \mathbb{S}) \triangleq (I \xrightarrow{fin} \mathbb{S}.\Sigma) \times \mathcal{P}(I)^?$$

$$(h, d) \cdot (h', d') \triangleq (h'', d'')$$

$$\text{where } h'' \triangleq \lambda i. \begin{cases} h(i) \cdot h'(i) & \text{if } i \in \text{dom}(h) \cap \text{dom}(h') \\ h(i) & \text{if } i \in \text{dom}(h) \setminus \text{dom}(h') \\ h'(i) & \text{if } i \in \text{dom}(h') \setminus \text{dom}(h) \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$\text{and } d'' \triangleq \begin{cases} d & \text{if } d' = \bot \\ d' & \text{if } d = \bot \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$\text{and } d'' = \bot \vee \text{dom}(h'') \subseteq d''$$

$$|(h, d)| \triangleq \begin{cases} \bot & \text{if } \text{dom}(h') = \emptyset \\ (h', \bot) & \text{otherwise} \end{cases}$$

$$\text{where } h' \triangleq \lambda i. \begin{cases} |h(i)| & \text{if } i \in \text{dom}(h) \wedge |h(i)| \neq \bot \\ \text{undefined} & \text{otherwise} \end{cases}$$

For a wrapped state model $\mathbb{S}$ such that its actions and predicates are $\mathcal{A}_{\mathbb{S}}$ and $\Delta_{\mathbb{S}}$, we define the actions of $\mathrm{PMAP}(I, \mathbb{S})$ as $\mathcal{A} = \mathcal{A}_{\mathbb{S}} \uplus \{\texttt{alloc}\}$, and the predicates $\Delta = \Delta_{\mathbb{S}} \uplus \{\texttt{domainset}\}$.

A.7.2 Satisfiability

PMapPredSat
$$\frac{\sigma_i \models_\Delta \langle \delta \rangle (\vec{v}_i; \vec{v}_o)}{([i \mapsto \sigma_i], \bot) \models_\Delta \langle \delta \rangle (i :: \vec{v}_i; \vec{v}_o)}$$

PMapPredSatBot
$$\frac{\bot \models_\Delta \langle \delta \rangle (\vec{v}_i; \vec{v}_o)}{\bot \models_\Delta \langle \delta \rangle (i :: \vec{v}_i; \vec{v}_o)}$$

PMapPredDomainSet
$$(\emptyset, d) \models_\Delta \langle \texttt{domainset} \rangle (; d)$$

PMapSymInterpretation
$$\frac{\forall \hat{i} \in \mathrm{dom}(\hat{h}). \ [\![\hat{i}]\!]_{\theta,s} = i \wedge i \in \mathrm{dom}(h) \wedge \theta, s, h(i) \models \hat{h}(\hat{i}) \qquad [\![\mathrm{dom}(\hat{h})]\!]_{\theta,s} = \mathrm{dom}(h) \qquad [\![\hat{d}]\!]_{\theta,s} = d}{\theta, s, (h, d) \models (\hat{h}, \hat{d})}$$

A.7.3 Compositional Concrete Rules

We define the helper functions `get` and `set` as:

$$\texttt{get} \colon \mathrm{PMAP}(I, \mathbb{S}) \to I \to \mathbb{S}.\Sigma$$
$$\texttt{set} \colon \mathrm{PMAP}(I, \mathbb{S}) \to I \to \mathbb{S}.\Sigma \to \mathrm{PMAP}(I, \mathbb{S})$$

We pretty-print `get` and `set` as $\texttt{get}(\sigma, i) = \sigma_i$ and $\texttt{set}(\sigma, i, \sigma_i) = \sigma'$.

$$\text{Given } wrap(h, d) \triangleq \begin{cases} \bot & \text{if } \mathrm{dom}(h) = \emptyset \wedge d = \bot \\ (h, d) & \text{otherwise} \end{cases}$$

$$unwrap(\sigma) \triangleq \begin{cases} ([], \bot) & \text{if } \sigma = \bot \\ (h, d) & \text{if } \sigma = (h, d) \end{cases}$$

CPMapGetMatch
$$\frac{(h, d) = unwrap(\sigma) \qquad i \in \mathrm{dom}(h) \qquad \sigma_i = h(i)}{\texttt{get}(\sigma, i) = \sigma_i}$$

CPMapGetAdd
$$\frac{(h, d) = unwrap(\sigma) \qquad i \notin \mathrm{dom}(h) \qquad d \neq \bot \qquad i \in d}{\texttt{get}(\sigma, i) = \bot}$$

CPMapGetBotDomain
$$\frac{(h, d) = unwrap(\sigma) \qquad i \notin \mathrm{dom}(h) \qquad d = \bot}{\texttt{get}(\sigma, i) = \bot}$$

CPMapSetSome
$$\frac{(h, d) = unwrap(\sigma) \qquad \sigma_i \neq \bot \qquad h' = h[i \leftarrow \sigma_i] \qquad \sigma' = wrap(h', d)}{\texttt{set}(\sigma, i, \sigma_i) = \sigma'}$$

CPMapSetNone
$$\frac{(h, d) = unwrap(\sigma) \qquad \sigma_i = \bot \qquad h' = h[i \not\leftarrow] \qquad \sigma' = wrap(h', d)}{\texttt{set}(\sigma, i, \sigma_i) = \sigma'}$$

The action rules are then:

$$\frac{\mathsf{get}(\sigma,i)=\sigma_i \qquad \alpha(\sigma_i,\vec{v}_i)\rightsquigarrow(o,\sigma'_i,\vec{v}_o) \qquad \mathsf{set}(\sigma,i,\sigma'_i)=\sigma'}{\alpha(\sigma,i::\vec{v}_i)\rightsquigarrow(o,\sigma',i::\vec{v}_o)}$$

CPMAPACTIONOUTOFBOUNDS

$$\frac{d\neq\bot \qquad i\notin d}{\alpha((h,d),i::\vec{v}_i)\rightsquigarrow(\mathsf{Err},(h,d),[])}$$

CPMAPALLOC

$$\frac{d\neq\bot \qquad i=\mathsf{fresh} \qquad \sigma_i=\mathsf{instantiate}(\vec{v}_i) \qquad h'=h[i\leftarrow\sigma_i] \qquad d'=d\uplus\{i\}}{\mathsf{alloc}((h,d),\vec{v}_i)\rightsquigarrow(\mathsf{Ok},(h',d'),[i])}$$

CPMAPALLOCMISS

$$\frac{(h,d)=unwrap(\sigma) \qquad d=\bot}{\mathsf{alloc}(\sigma,\vec{v}_i)\rightsquigarrow(\mathsf{Miss},\sigma,['domainset'])}$$

A.7.4 Compositional Symbolic Rules

We now redefine `get` and `set`, lifting them to the symbolic realm (note `set` is unchanged, as it does not perform any sort of matching). `get` now returns a state and an index, that may be different from the input index – it corresponds to the actual index of the state in the map.

$$\mathsf{get}\colon \mathrm{PMAP}(I,\mathbb{S})\to I\to\mathcal{P}(I\times\mathbb{S}.\hat{\Sigma}\times\Pi)$$
$$\mathsf{set}\colon \mathrm{PMAP}(I,\mathbb{S})\to I\to\mathbb{S}.\hat{\Sigma}\to\mathrm{PMAP}(I,\mathbb{S})$$

We pretty-print `get` and `set` as $\mathsf{get}(\hat{\sigma},\hat{i})\rightsquigarrow(\hat{i}',\hat{\sigma}_i,\pi)$ and $\mathsf{set}(\hat{\sigma},\hat{i},\hat{\sigma}_i)=\hat{\sigma}'$.

$$\text{Given } wrap(\hat{h},\hat{d})\triangleq\begin{cases}\bot & \text{if } \mathrm{dom}(\hat{h})=\emptyset\wedge\hat{d}=\bot\\(\hat{h},\hat{d}) & \text{otherwise}\end{cases}$$

$$unwrap(\hat{\sigma})\triangleq\begin{cases}([],\bot) & \text{if } \hat{\sigma}=\bot\\(\hat{h},\hat{d}) & \text{if } \hat{\sigma}=(\hat{h},\hat{d})\end{cases}$$

PMAPGETMATCH

$$\frac{(\hat{h},\hat{d})=unwrap(\hat{\sigma}) \qquad \hat{i}'\in\mathrm{dom}(\hat{h}) \qquad \hat{\sigma}_i=\hat{h}(\hat{i}')}{\mathsf{get}(\hat{\sigma},\hat{i})\rightsquigarrow(\hat{i}',\hat{\sigma}_i,[\hat{i}=\hat{i}'])}$$

PMAPGETADD

$$\frac{(\hat{h},\hat{d})=unwrap(\hat{\sigma}) \qquad \hat{i}\notin\mathrm{dom}(\hat{h}) \qquad \hat{d}\neq\bot}{\mathsf{get}(\hat{\sigma},\hat{i})\rightsquigarrow(\hat{i},\bot,[\hat{i}\notin\mathrm{dom}(\hat{h})\wedge\hat{i}\in\hat{d}])}$$

PMAPGETBOTDOMAIN

$$\frac{(\hat{h},\hat{d})=unwrap(\hat{\sigma}) \qquad \hat{i}\notin\mathrm{dom}(\hat{h}) \qquad \hat{d}=\bot}{\mathsf{get}(\hat{\sigma},\hat{i})\rightsquigarrow(\hat{i},\bot,[\hat{i}\notin\mathrm{dom}(\hat{h})])}$$

PMAPSETSOME

$$\frac{(\hat{h},\hat{d})=unwrap(\hat{\sigma}) \qquad \hat{\sigma}_i\neq\bot \qquad \hat{h}'=\hat{h}[\hat{i}\leftarrow\hat{\sigma}_i] \qquad \hat{\sigma}'=wrap(\hat{h}',\hat{d})}{\mathsf{set}(\hat{\sigma},\hat{i},\hat{\sigma}_i)=\hat{\sigma}'}$$

PMAPSETNONE

$$\frac{(\hat{h},\hat{d})=unwrap(\hat{\sigma}) \qquad \hat{\sigma}_i=\bot \qquad \hat{h}'=\hat{h}[\hat{i}\nleftarrow] \qquad \hat{\sigma}'=wrap(\hat{h}',\hat{d})}{\mathsf{set}(\hat{\sigma},\hat{i},\hat{\sigma}_i)=\hat{\sigma}'}$$

And now for the relevant rules:

$$\text{Given } lift\_if\_miss(o,\hat{i},\vec{v}_i)\triangleq\begin{cases}\hat{i}::\vec{v}_i & \text{if } o=\mathsf{Miss}\\\vec{v}_i & \text{otherwise}\end{cases}$$

$$\frac{\mathsf{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \qquad \alpha(\hat{\sigma}_i, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}'_i, \vec{\hat{v}}_o, \pi') \qquad \mathsf{set}(\hat{\sigma}, \hat{i}', \hat{\sigma}'_i) = \hat{\sigma}'}{\alpha(\hat{\sigma}, \hat{i} :: \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}', \hat{i}' :: \vec{\hat{v}}_o, \pi :: \pi')}$$

$$\frac{\text{PMapActionOutOfBounds}}{\hat{d} \neq \bot}{\alpha((\hat{h}, \hat{d}), \hat{i} :: \vec{\hat{v}}_i) \rightsquigarrow (\mathsf{Err}, (\hat{h}, \hat{d}), [], [\hat{i} \notin \hat{d}])}$$

$$\frac{\text{PMapAlloc}}{\hat{d} \neq \bot \qquad \hat{i} = \mathsf{fresh} \qquad \hat{\sigma}_i = \mathsf{instantiate}(\vec{\hat{v}}_i) \qquad \hat{h}' = \hat{h}[\hat{i} \leftarrow \hat{\sigma}_i] \qquad \hat{d}' = \hat{d} \uplus \{\hat{i}\}}{\mathsf{alloc}((\hat{h}, \hat{d}), \vec{\hat{v}}_i) \rightsquigarrow (\mathsf{Ok}, (\hat{h}', \hat{d}'), [\hat{i}], [\hat{i} = \hat{i}])}$$

$$\frac{\text{PMapAllocMiss}}{(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \qquad \hat{d} = \bot}{\mathsf{alloc}(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (\mathsf{Miss}, \hat{\sigma}, [\text{`domainset'}], [])}$$

$$\frac{\text{PMapCons}}{\mathsf{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi)}{\mathsf{consume}(\hat{\sigma}_i, \delta, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}'_i, \vec{\hat{v}}_o, \pi') \qquad \mathsf{set}(\hat{\sigma}, \hat{i}', \hat{\sigma}'_i) = \hat{\sigma}' \qquad \vec{\hat{v}}'_o = lift\_if\_miss(o, \hat{i}', \vec{\hat{v}}_o)}{\mathsf{consume}(\hat{\sigma}, \delta, \hat{i} :: \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{\hat{v}}'_o, \pi :: \pi')}$$

$$\frac{\text{PMapConsIncompat}}{\hat{d} \neq \bot}{\mathsf{consume}((\hat{h}, \hat{d}), \delta, \hat{i} :: \vec{\hat{v}}_i) \rightsquigarrow (\mathsf{LFail}, (\hat{h}, \hat{d}), [], [\hat{i} \notin \hat{d}])} \qquad \frac{\text{PMapConsDomainSet}}{\hat{d} \neq \bot}{\mathsf{consume}((\hat{h}, \hat{d}), \mathsf{domainset}, []) \rightsquigarrow (\mathsf{Ok}, (\hat{h}, \bot), [\hat{d}], [])}$$

$$\frac{\text{PMapConsDomainSetMiss}}{(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \qquad \hat{d} = \bot}{\mathsf{consume}(\hat{\sigma}, \mathsf{domainset}, []) \rightsquigarrow (\mathsf{Miss}, \hat{\sigma}, [\text{`domainset'}], [])}$$

$$\frac{\text{PMapProd}}{\mathsf{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \qquad \mathsf{produce}(\hat{\sigma}_i, \delta, \vec{\hat{v}}_i, \vec{\hat{v}}_o) \rightsquigarrow (\hat{\sigma}'_i, \pi') \qquad \mathsf{set}(\hat{\sigma}, \hat{i}', \hat{\sigma}'_i) = \hat{\sigma}'}{\mathsf{produce}(\hat{\sigma}, \delta, \hat{i} :: \vec{\hat{v}}_i, \vec{\hat{v}}_o) \rightsquigarrow (\hat{\sigma}', \pi :: \pi')}$$

$$\frac{\text{PMapProdDomainSet}}{(\hat{h}, \bot) = unwrap(\hat{\sigma})}{\mathsf{produce}(\hat{\sigma}, \mathsf{domainset}, [], [\hat{d}]) \rightsquigarrow ((\hat{h}, \hat{d}), [\mathrm{dom}(\hat{h}) \subseteq \hat{d}])} \qquad \frac{\text{PMapFix}}{\mathbb{S}.\mathsf{fix}\ \vec{\hat{v}}_i = a \qquad a' = lift(a, \hat{i})}{\mathsf{fix}\ \hat{i} :: \vec{\hat{v}}_i = a'}$$

$$\text{PMapFixDomainSet}$$
$$\mathsf{fix}\ [\text{`domainset'}] = \exists \hat{d}.\ \langle \mathsf{domainset} \rangle(; \hat{d})$$

Note in the above we define $lift(a, \hat{i})$ as a function that traverses an assertion a and lifts all core predicate assertions by adding the value $\hat{i}$ at the start of its in-values, such that $lift(\langle \delta \rangle(\vec{\hat{v}}_i; \vec{\hat{v}}_o), \hat{i}) = \langle \delta \rangle(\hat{i} :: \vec{\hat{v}}_i; \vec{\hat{v}}_o)$.

A.8 Syntactic PMap

For the variation of PMap that has syntactic checking, we only need to modify the rules for the symbolic version of `get`.

SYNTACTICPMAPGETMATCH
$$\frac{(\hat{h},\hat{d}) = unwrap(\hat{\sigma}) \qquad \hat{i} \in \mathrm{dom}(\hat{h}) \qquad \hat{\sigma}_i = \hat{h}(\hat{i})}{\mathtt{get}(\hat{\sigma},\hat{i}) \rightsquigarrow (\hat{i},\hat{\sigma}_i,[])}$$

SYNTACTICPMAPGETBRANCH
$$\frac{(\hat{h},\hat{d}) = unwrap(\hat{\sigma}) \qquad \hat{i} \notin \mathrm{dom}(\hat{h}) \qquad \hat{i}' \in \mathrm{dom}(\hat{h}) \qquad \hat{\sigma}_i = \hat{h}(\hat{i}')}{\mathtt{get}(\hat{\sigma},\hat{i}) \rightsquigarrow (\hat{i}',\hat{\sigma}_i,[\hat{i}=\hat{i}'])}$$

SYNTACTICPMAPGETADD
$$\frac{(\hat{h},\hat{d}) = unwrap(\hat{\sigma}) \qquad \hat{i} \notin \mathrm{dom}(\hat{h}) \qquad \hat{d} \neq \bot}{\mathtt{get}(\hat{\sigma},\hat{i}) \rightsquigarrow (\hat{i},\bot,[\hat{i} \notin \mathrm{dom}(\hat{h}) \wedge \hat{i} \in \hat{d}])}$$

SYNTACTICPMAPGETBOTDOMAIN
$$\frac{(\hat{h},\hat{d}) = unwrap(\hat{\sigma}) \qquad \hat{i} \notin \mathrm{dom}(\hat{h}) \qquad \hat{d} = \bot}{\mathtt{get}(\hat{\sigma},\hat{i}) \rightsquigarrow (\hat{i},\bot,[\hat{i} \notin \mathrm{dom}(\hat{h})])}$$

A.9 Split PMap

A.9.1 Symbolic States

Because we re-use the concrete RA from PMAP, we only need to define the set of symbolic states, which does not come equipped with composition or core.

$$\mathrm{PMAP}_{\mathrm{Split}}(I,\mathbb{S}) \triangleq I \xrightarrow{fin} \mathbb{S}.\Sigma \times I \xrightarrow{fin} \mathbb{S}.\Sigma \times \mathcal{P}(I)^?$$

A.9.2 Satisfiability

Predicate satisfiability is defined with regards to the concrete compositional states, so we do not need to redefine it. We however need to redefine symbolic interpretation, as follows. We denote $\hat{h}_c$ and $\hat{h}_s$ the concrete and symbolic parts of the heap respectively.

SPLITPMAPSYMINTERPRETATION
$$\frac{\begin{array}{c} \forall \hat{i} \in \mathrm{dom}(\hat{h}_c).\,(\nexists \hat{i}' \in \hat{h}_s.\,[\![\hat{i}']\!]_{\theta,s} = [\![\hat{i}]\!]_{\theta,s}) \wedge [\![\hat{i}]\!]_{\theta,s} = i \wedge i \in \mathrm{dom}(h) \wedge \theta,s,h(i) \models \hat{h}_c(\hat{i}) \\ \forall \hat{i} \in \mathrm{dom}(\hat{h}_s).\,(\nexists \hat{i}' \in \hat{h}_c.\,[\![\hat{i}']\!]_{\theta,s} = [\![\hat{i}]\!]_{\theta,s}) \wedge [\![\hat{i}]\!]_{\theta,s} = i \wedge i \in \mathrm{dom}(h) \wedge \theta,s,h(i) \models \hat{h}_s(\hat{i}) \\ [\![\mathrm{dom}(\hat{h}_c) \uplus \mathrm{dom}(\hat{h}_s)]\!]_{\theta,s} = \mathrm{dom}(h) \qquad [\![\hat{d}]\!]_{\theta,s} = d \end{array}}{\theta,s,(h,d) \models (\hat{h}_c,\hat{h}_s,\hat{d})}$$

A.9.3 `get` and `set` rules

Given $wrap(\hat{h}_c,\hat{h}_s,\hat{d}) \triangleq \begin{cases} \bot & \text{if } \mathrm{dom}(\hat{h}_c) = \emptyset \wedge \mathrm{dom}(\hat{h}_s) = \emptyset \wedge \hat{d} = \emptyset \\ (\hat{h}_c,\hat{h}_s,\hat{d}) & \text{otherwise} \end{cases}$

$unwrap(\hat{\sigma}) \triangleq \begin{cases} ([],[],\emptyset) & \text{if } \hat{\sigma} = \bot \\ (\hat{h}_c,\hat{h}_s,\hat{d}) & \text{if } \hat{\sigma} = (\hat{h}_c,\hat{h}_s,\hat{d}) \end{cases}$

SPLITPMAPGETMATCHCON
$$\frac{(\hat{h}_c,\hat{h}_s,\hat{d}) = unwrap(\hat{\sigma}) \qquad \mathtt{is\_concrete}\ \hat{i} \qquad \hat{i} \in \mathrm{dom}(\hat{h}_c) \qquad \hat{\sigma}_i = \hat{h}_c(\hat{i})}{\mathtt{get}(\hat{\sigma},\hat{i}) \rightsquigarrow (\hat{i},\hat{\sigma}_i,[])}$$

SPLITPMAPGETMATCHSYM
$$\frac{(\hat{h}_c,\hat{h}_s,\hat{d}) = unwrap(\hat{\sigma}) \qquad \hat{i} \in \mathrm{dom}(\hat{h}_s) \qquad \hat{\sigma}_i = \hat{h}_s(i)}{\mathtt{get}(\hat{\sigma},\hat{i}) \rightsquigarrow (\hat{i},\hat{\sigma}_i,[])}$$

$$(\hat{h}_c, \hat{h}_s, \hat{d}) = unwrap(\hat{\sigma}) \qquad \hat{h}_{all} = \hat{h}_c \cup \hat{h}_s \qquad \hat{i} \notin \text{dom}(\hat{h}_{all}) \qquad \hat{i}' \in \text{dom}(\hat{h}_{all}) \qquad \hat{\sigma}_i = \hat{h}_{all}(\hat{i}')$$
$$\overline{\text{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, [\hat{i} = \hat{i}'])}$$

SPLITPMAPGETADD
$$(\hat{h}_c, \hat{h}_s, \hat{d}) = unwrap(\hat{\sigma}) \qquad \hat{h}_{all} = \hat{h}_c \cup \hat{h}_s \qquad \hat{i} \notin \text{dom}(\hat{h}_{all}) \qquad \hat{d} \neq \bot$$
$$\overline{\text{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \bot, [\hat{i} \notin \text{dom}(\hat{h}_{all}) \wedge \hat{i} \in \hat{d}])}$$

SPLITPMAPGETBOTDOMAIN
$$(\hat{h}_c, \hat{h}_s, \hat{d}) = unwrap(\hat{\sigma}) \qquad \hat{h}_{all} = \hat{h}_c \cup \hat{h}_s \qquad \hat{i} \notin \text{dom}(\hat{h}_{all}) \qquad \hat{d} = \bot$$
$$\overline{\text{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \bot, [\hat{i} \notin \text{dom}(\hat{h}_{all})])}$$

SPLITPMAPSETSOMECON
$$(\hat{h}_c, \hat{h}_s, \hat{d}) = unwrap(\hat{\sigma}) \qquad \hat{\sigma}_i \neq \bot$$
$$\frac{\text{is\_concrete}_\Sigma \; \hat{\sigma}_i \qquad \text{is\_concrete} \; \hat{i} \qquad \hat{h}'_c = \hat{h}_c[\hat{i} \leftarrow \hat{\sigma}_i] \qquad \hat{h}'_s = \hat{h}_s[\hat{i} \not\leftarrow] \qquad \hat{\sigma}' = wrap(\hat{h}'_c, \hat{h}'_s, \hat{d})}{\text{set}(\hat{\sigma}, \hat{i}, \hat{\sigma}_i) = \hat{\sigma}'}$$

SPLITPMAPSETSOMESYM
$$(\hat{h}_c, \hat{h}_s, \hat{d}) = unwrap(\hat{\sigma}) \qquad \hat{s}_i \neq \bot$$
$$\frac{\neg(\text{is\_concrete}_\Sigma \; \hat{\sigma}_i \vee \text{is\_concrete} \; \hat{i}) \qquad \hat{h}'_c = \hat{h}_c[\hat{i} \not\leftarrow] \qquad \hat{h}'_s = \hat{h}_s[\hat{i} \leftarrow \hat{\sigma}_i] \qquad \hat{\sigma}' = wrap(\hat{h}'_c, \hat{h}'_s, \hat{d})}{\text{set}(\hat{\sigma}, \hat{i}, \hat{\sigma}_i) = \hat{\sigma}'}$$

SPLITPMAPSETNONE
$$\frac{(\hat{h}_c, \hat{h}_s, \hat{d}) = unwrap(\hat{\sigma}) \qquad \hat{\sigma}_i = \bot \qquad \hat{h}'_c = \hat{h}_c[\hat{i} \not\leftarrow] \qquad \hat{h}'_s = \hat{h}_s[\hat{i} \not\leftarrow] \qquad \hat{\sigma}' = wrap(\hat{h}'_c, \hat{h}'_s, \hat{d})}{\text{set}(\hat{\sigma}, \hat{i}, \hat{\sigma}_i) = \hat{\sigma}'}$$

A.10 Abstract Locations PMap

A.10.1 Symbolic States

$$\text{PMAP}_{\text{ALoc}}(\mathbb{S}) \triangleq \text{Str} \xrightarrow{fin} \mathbb{S}.\Sigma \times \mathcal{P}(\text{LVal})^?$$

A.10.2 Satisfiability

ALOCPMAPSYMINTERPRETATION
$$\forall a \in \text{dom}(\hat{h}). \; [\![\text{aloc}(a)]\!]_{\theta,s} = i \wedge i \in \text{dom}(h) \wedge \theta, s, h(i) \models \hat{h}(a)$$
$$\frac{[\![\{\text{aloc}(a) : a \in \text{dom}(\hat{h})\}]\!]_{\theta,s} = \text{dom}(h) \qquad [\![\hat{d}]\!]_{\theta,s} = d}{\theta, s, (h, d) \models (\hat{h}, \hat{d})}$$

A.10.3 `get` and `set` rules

Here we use the matching mode $m \in \{\texttt{MATCH}, \texttt{NO\_MATCH}\}$.

$$\text{Given } wrap(\hat{h}, \hat{d}) \triangleq \begin{cases} \bot & \text{if } \text{dom}(\hat{h}) = \emptyset \wedge \hat{d} = \bot \\ (\hat{h}, \hat{d}) & \text{otherwise} \end{cases}$$

$$unwrap(\hat{\sigma}) \triangleq \begin{cases} ([], \bot) & \text{if } \hat{\sigma} = \bot \\ (\hat{h}, \hat{d}) & \text{if } \hat{\sigma} = (\hat{h}, \hat{d}) \end{cases}$$

$$\hat{i} \in^? \hat{d} \triangleq \begin{cases} \mathsf{true} & \text{if } \hat{d} = \bot \\ \hat{i} \in \hat{d} & \text{otherwise} \end{cases}$$

ALOCPMAPGETMATCH
$$\frac{(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \qquad a = \mathsf{to\_aloc}\ \hat{i} \qquad a \neq \bot \qquad a \in \mathrm{dom}(\hat{h}) \qquad \hat{\sigma}_i = \hat{h}(a)}{\mathsf{get}(\hat{\sigma}, \hat{i}, m) \rightsquigarrow (\mathrm{aloc}(a), \hat{\sigma}_i, [])}$$

ALOCPMAPGETNOMATCHNOTFOUND
$$\frac{(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \qquad a = \mathsf{to\_aloc}\ \hat{i} \qquad a \neq \bot \qquad a \notin \mathrm{dom}(\hat{h})}{\mathsf{get}(\hat{\sigma}, \hat{i}, \mathsf{NO\_MATCH}) \rightsquigarrow (\mathrm{aloc}(a), \bot, [\hat{i} \in^? \hat{d}])}$$

ALOCPMAPGETNOMATCHNEW
$$\frac{(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \qquad a = \mathsf{to\_aloc}\ \hat{i} \qquad a = \bot \qquad a' = \mathsf{fresh\_aloc}\ ()}{\mathsf{get}(\hat{\sigma}, \hat{i}, \mathsf{NO\_MATCH}) \rightsquigarrow (\mathrm{aloc}(a'), \bot, [\hat{i} = \mathrm{aloc}(a') \wedge \hat{i} \in^? \hat{d}])}$$

ALOCPMAPGETMATCHNOTFOUND
$$\frac{(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \qquad a = \mathsf{to\_aloc}\ \hat{i} \qquad a \neq \bot \qquad a \notin \mathrm{dom}(\hat{h})}{\mathsf{get}(\hat{\sigma}, \hat{i}, \mathsf{MATCH}) \rightsquigarrow (\mathrm{aloc}(a), \bot, [\hat{i} \notin \{\mathrm{aloc}(a') : a' \in \mathrm{dom}(\hat{h})\} \wedge \hat{i} \in^? \hat{d}])}$$

ALOCPMAPGETMATCHNEW
$$\frac{(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \qquad a = \mathsf{to\_aloc}\ \hat{i} \qquad a = \bot \qquad a' = \mathsf{fresh\_aloc}\ ()}{\mathsf{get}(\hat{\sigma}, \hat{i}, \mathsf{MATCH}) \rightsquigarrow (\mathrm{aloc}(a'), \bot, [\hat{i} = \mathrm{aloc}(a') \wedge \hat{i} \notin \{\mathrm{aloc}(a'') : a'' \in \mathrm{dom}(\hat{h})\} \wedge \hat{i} \in^? \hat{d}])}$$

ALOCPMAPMATCHING
$$\frac{(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \qquad a = \mathsf{to\_aloc}\ \hat{i} \qquad a \neq \bot \qquad a \notin \mathrm{dom}(\hat{h}) \qquad a' \in \mathrm{dom}(\hat{h}) \qquad \hat{\sigma}_i = \hat{h}(a')}{\mathsf{get}(\hat{\sigma}, \hat{i}, \mathsf{MATCH}) \rightsquigarrow (\mathrm{aloc}(a'), \hat{\sigma}_i, [\mathrm{aloc}(a) = \mathrm{aloc}(a')])}$$

ALOCPMAPMATCHINGBOT
$$\frac{(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \qquad a = \mathsf{to\_aloc}\ \hat{i} \qquad a = \bot \qquad a' \in \mathrm{dom}(\hat{h}) \qquad \hat{\sigma}_i = \hat{h}(a')}{\mathsf{get}(\hat{\sigma}, \hat{i}, \mathsf{MATCH}) \rightsquigarrow (\mathrm{aloc}(a'), \hat{\sigma}_i, [\hat{i} = \mathrm{aloc}(a')])}$$

ALOCPMAPSETSOME
$$\frac{(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \qquad a = \mathsf{to\_aloc}\ \hat{i} \qquad a \neq \bot \qquad \hat{\sigma}_i \neq \bot \qquad \hat{h}' = \hat{h}[a \leftarrow \hat{\sigma}_i] \qquad \hat{\sigma}' = wrap(\hat{h}', \hat{d})}{\mathsf{set}(\hat{\sigma}, \hat{i}, \hat{\sigma}_i) = \hat{\sigma}'}$$

ALOCPMAPSETNONE
$$\frac{(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \qquad a = \mathsf{to\_aloc}\ \hat{i} \qquad a \neq \bot \qquad \hat{\sigma}_i = \bot \qquad \hat{h}' = \hat{h}[a \not\leftarrow] \qquad \hat{\sigma}' = wrap(\hat{h}', \hat{d})}{\mathsf{set}(\hat{\sigma}, \hat{i}, \hat{\sigma}_i) = \hat{\sigma}'}$$

Appendix B

Proofs of soundness

Here we present proofs of soundness for some of the showcased state models. In appendix B.1 we show the soundness of the Ex state model, the simplest state model; in appendix B.2 we prove it for PMAP, a more complex state model transformer; in appendix B.3, appendix B.4 and appendix B.5 we show the soundness results for three optimisations of PMAP: syntactic checks, PMAP$_{\text{Split}}$ and PMAP$_{\text{ALoc}}$ respectively.

B.1 Exclusive

Proof.

Proposition 1: OX soundness
Assume

(H1) $\theta, s, \sigma \models \hat{\sigma} \wedge \alpha(\sigma, \vec{v}_i) \rightsquigarrow (o, \sigma', \vec{v}_o) \wedge [\![\vec{\hat{v}_i}]\!]_{\theta,s} = \vec{v}_i$

(H2) $\forall o', \hat{\sigma}', \vec{\hat{v}}_o', \pi'. \alpha(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (o', \hat{\sigma}', \vec{\hat{v}}_o', \pi') \Rightarrow o' \in \{\mathsf{Ok}, \mathsf{Err}\}$

To prove

(G1) $\exists \hat{\sigma}', \vec{\hat{v}}_o, \pi, \theta'. \hat{\alpha}(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{\hat{v}}_o, \pi) \wedge \theta', s, \sigma' \models \hat{\sigma}' \wedge \mathsf{SAT}_{\theta',s}(\pi) \wedge [\![\vec{\hat{v}}_o]\!]_{\theta',s} = \vec{v}_o$

The proof is analogous for both actions, so we only consider the case where $\alpha = \mathsf{load}$. Given (H1) and the definition of $\models$, there are two further cases: $\sigma = \mathrm{ex}(x)$ and $\hat{\sigma} = \mathrm{ex}(\hat{x})$, or $\sigma = \bot$ and $\hat{\sigma} = \bot$. Again, both cases are analogous, so we only consider $\sigma = \mathrm{ex}(x)$, $\hat{\sigma} = \mathrm{ex}(\hat{x})$.

(H3) From our hypothesis, $\sigma = \mathrm{ex}(x)$ and $\hat{\sigma} = \mathrm{ex}(\hat{x})$

(H4) From the definition of concrete actions CExLoadOk, we get $\vec{v}_i = []$, $o = \mathsf{Ok}$, $\sigma' = \sigma$, $\vec{v}_o = [x]$.

(H5) From the definition of $\models$ we also have $[\![\hat{x}]\!]_{\theta,s} = x$

(H6) We can pick $\vec{\hat{v}}_o$, π and θ' such that $\vec{\hat{v}}_o = [\hat{x}]$, $\pi = []$ and $\theta' = \theta$.

(H7) From (H6), we get $[\![\vec{\hat{v}}_o]\!]_{\theta,s} = \vec{v}_o$, as well as $\mathsf{SAT}_{\theta,s}(\pi)$

(H8) From ExLoadOk and (H6), we have $\mathsf{load}(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (\mathsf{Ok}, \hat{\sigma}', \vec{\hat{v}}_o, \pi)$ with $\hat{\sigma}' = \hat{\sigma}$.

(H9) Finally, from (H4) and (H8), we have $\sigma' = \sigma$ and $\hat{\sigma}' = \hat{\sigma}$, thus from (H1) it follows that $\theta, s, \sigma' \models \hat{\sigma}'$.

Combining (H7), (H8) and (H9), we get our goal (G1).

Proposition 2: UX soundness
Assume

(H1) $\hat\alpha(\hat\sigma, \vec{v}_i) \rightsquigarrow (o, \hat\sigma', \vec{v}_o, \pi) \wedge \mathsf{SAT}_{\theta,s}(\pi) \wedge \theta, s, \sigma' \models \hat\sigma' \wedge$
$[\![\vec{v}_o]\!]_{\hat{s},\pi} \rightsquigarrow (\vec{v}_o, \pi') \wedge [\![\vec{v}_i]\!]_{\hat{s},\pi'} \rightsquigarrow (\vec{v}_i, \pi'')$

To prove

(G1) $\exists \sigma. \theta, s, \sigma \models \hat\sigma \wedge \alpha(\sigma, \vec{v}_i) \rightsquigarrow (o, \sigma', \vec{v}_o)$

We again only consider the case where $\alpha = \mathtt{load}$ and $\sigma' = \mathrm{ex}(x)$, $\hat\sigma' = \mathrm{ex}(\hat{x})$ – the other three cases are analogous.

(H2) From ExLoadOk, we get $\hat\sigma = \hat\sigma' = \mathrm{ex}(\hat{x})$, $\vec{v}_i = []$, $o = \mathtt{Ok}$, $\vec{v}_o = [\hat{x}]$, $\pi = []$, with $s = [x \mapsto \hat{x}]$

(H3) From (H2) and (H1) we have $\vec{v}_i = []$ and $\vec{v}_o = [x]$

(H4) We can pick $\sigma = \sigma' = \mathrm{ex}(x)$, which from (H1) and (H2) give us $\theta, s, \sigma \models \hat\sigma$

(H5) From CExLoadOk, (H3) and (H4), we have $\mathtt{load}(\sigma, \vec{v}_i) \rightsquigarrow (\mathtt{Ok}, \sigma', \vec{v}_o)$.

Combinding (H4) and (H5) gives our goal (G1).

Proposition 3: Frame subtraction is satisfied
Assume

(H1) $\sigma \# \sigma_f \wedge \alpha(\sigma \cdot \sigma_f, \vec{v}_i) \rightsquigarrow (o, \sigma', \vec{v}_o)$

To prove

(G1) $\exists \sigma'', o', \vec{v}_o'. \alpha(\sigma, \vec{v}_i) \rightsquigarrow (o', \sigma'', \vec{v}_o') \wedge$
$(o' \neq \mathtt{Miss} \implies o' = o \wedge \vec{v}_o' = \vec{v}_o \wedge \sigma' = \sigma'' \cdot \sigma_f)$

(H2) $\mathtt{load}$ and $\mathtt{store}$ are always defined for respectively 0 and 1 arguments, so from (H1) we know
$\exists \sigma'', o', \vec{v}_o'. \alpha(\sigma, \vec{v}_i) \rightsquigarrow (o', \sigma'', \vec{v}_o')$.

(H3) Assume $o' \neq \mathtt{Miss}$

(H4) If $\sigma = \bot$, the rules say $o' = \mathtt{Miss}$, contradicting (H3). Thus $\sigma = \mathrm{ex}(x)$.

(H5) From (H1), we know $\sigma \cdot \sigma_f$ is defined, so it must be that $\sigma_f = \bot$ as $\mathrm{ex}(x) \cdot \mathrm{ex}(y)$ is undefined.

From (H5) and composition rules we know $\sigma \cdot \sigma_f = \mathrm{ex}(x) \cdot \bot = \mathrm{ex}(x) = \sigma$. This gives our goal (G1).

Proposition 4: Frame addition is satisfied
Assume

(H1) $\alpha(\sigma, \vec{v}_i) \rightsquigarrow (o, \sigma', \vec{v}_o)$

(H2) $\sigma_f \# \sigma'$

(H3) $o \neq \mathtt{Miss}$

To prove

(G1) $\sigma \# \sigma_f \wedge \alpha(\sigma \cdot \sigma_f, \vec{v}_i) \rightsquigarrow (o, \sigma' \cdot \sigma_f, \vec{v}_o)$

(H4) From (H3) and the action rules, we know $\sigma' = \mathrm{ex}(x)$.

(H5) From composition, (H4) and (H2), we get $\sigma_f = \bot$.

From (H5) we get $\sigma' \cdot \sigma_f = \sigma' \cdot \bot = \sigma'$, and from (H1) our goal (G1) follows.

Proposition 5: Consume UX soundness
Assume

(H1) $\mathsf{consume}(\hat{\sigma}, \delta, \vec{v}_i) \rightsquigarrow (\mathsf{Ok}, \hat{\sigma}_f, \vec{v}_o, \pi)$

(H2) $\theta, s, \sigma \models \hat{\sigma} \wedge \mathsf{SAT}_{\theta,s}(\pi)$

To prove

(G1) $\exists \sigma_\delta, \sigma_f. \sigma_\delta \# \sigma_f \wedge \sigma = \sigma_\delta \cdot \sigma_f \wedge \theta, s, \sigma_\delta \models_\Delta \langle \delta \rangle (\vec{v}_i; \vec{v}_o) \wedge \theta, s, \sigma_f \models \hat{\sigma}_f$

(H3) There is only one $\mathsf{consume}$ rules yielding Ok, giving us $\delta = \mathsf{ex}$, $\hat{\sigma} = \mathsf{ex}(\hat{x})$, $\hat{\sigma}_f = \bot$, $\vec{v}_i = []$, $\vec{v}_o = [\hat{x}]$, $\pi = []$

(H4) From the definition of $\models$, we must have $\sigma = \mathsf{ex}(x)$ and $\sigma_f = \bot$ such that $\theta, s, \sigma \models \hat{\sigma}$ and $\theta, s, \sigma_f \models \hat{\sigma}_f$.

(H5) From (H4), we have $[\![\hat{x}]\!]_{\theta,s} = x$, such that $[\![\vec{v}_i]\!]_{\theta,s} = []$ and $[\![\vec{v}_o]\!]_{\theta,s} = [x]$.

(H6) From the definition of composition, we must have $\sigma_\delta = \mathsf{ex}(x)$ such that $\sigma = \sigma_\delta \cdot \sigma_f$, which also implies $\sigma_\delta \# \sigma_f$

(H7) From (H6), (H5) and the definition of $\models_\Delta$, we have $\theta, s, \sigma_\delta \models_\Delta \langle \delta \rangle (\vec{v}_i; \vec{v}_o)$.

Combining (H4), (H6) and (H7) gives our goal (G1).

Proposition 6: Consume OX soundness
Assume

(H1) $\mathsf{consume}(\hat{\sigma}, \delta, \vec{v}_i) \rightsquigarrow (\mathsf{Ok}, \hat{\sigma}_f, \vec{v}_o, \pi)$

(H2) $\theta, s, \sigma_f \models \hat{\sigma}_f \wedge \theta, s, \sigma_\delta \models_\Delta \langle \delta \rangle (\vec{v}_i; \vec{v}_o) \wedge \sigma_f \# \sigma_\delta$

To prove

(G1) $\exists \sigma. \sigma = \sigma_\delta \cdot \sigma_f \wedge \theta, s, \sigma \models \hat{\sigma} \wedge \mathsf{SAT}_{\theta,s}(\pi)$

(H3) There is only one $\mathsf{consume}$ rules yielding Ok, giving us $\delta = \mathsf{ex}$, $\hat{\sigma} = \mathsf{ex}(\hat{x})$, $\hat{\sigma}_f = \bot$, $\vec{v}_i = []$, $\vec{v}_o = [\hat{x}]$, $\pi = []$

(H4) From (H3) and the definition of $\models$, if $\theta, s, \sigma_f \models \hat{\sigma}_f$ we have $\sigma_f = \bot$.

(H5) From (H3) and the definition of $\models_\Delta$, if $\theta, s, \sigma_\delta \models_\Delta \langle \delta \rangle (\vec{v}_i; \vec{v}_o)$, we have $\sigma_\delta = \mathsf{ex}(x)$, with $\vec{v}_i = []$ and $\vec{v}_o = [x]$ such that $[\![\vec{v}_i]\!]_{\theta,s} = \vec{v}_i$ and $[\![\vec{v}_o]\!]_{\theta,s} = \vec{v}_o$.

(H6) From (H4), (H5) and the rule for composition, we have $\sigma_f \# \sigma_\delta$.

(H7) We can pick $\sigma = \sigma_\delta = \mathsf{ex}(x)$.

(H8) From the definition of composition, (H4) and (H5), we have $\sigma = \sigma_\delta \cdot \sigma_f$.

(H9) Given (H3) and (H7), we have $\theta, s, \sigma \models \hat{\sigma}$.

(H10) From (H3), $\pi = []$ thus $\mathsf{SAT}_{\theta,s}(\pi)$. Together with (H8) and (H9), this gives our goal (G1).

Proposition 7: Consume OX no path drops

Assume

(H1) $\forall o, \hat{\sigma}_f, \vec{v}_o, \pi.\, \text{consume}(\text{OX}, \hat{\sigma}, \delta, \vec{v}_i) \rightsquigarrow (o, \hat{\sigma}_f, \vec{v}_o, \pi) \Rightarrow o_c = \text{Ok}$

To prove

(G1) $\exists \hat{\sigma}'_f, \vec{v}'_o, \pi'.\, \text{consume}(\text{OX}, \hat{\sigma}, \delta, \vec{v}_i, \pi) \rightsquigarrow (\text{Ok}, \hat{\sigma}'_f, \vec{v}'_o, \pi')$

From the consume rules, we never vanish, so if all consumptions are Ok we must have $\hat{\sigma} = \text{ex}(\hat{x})$, $\delta = \text{ex}$, $\vec{v}_i = []$, $\hat{\sigma}'_f = \bot$, $\vec{v}'_o = [\hat{x}]$, $\pi' = []$. Our goal (G1) follows.

Proposition 8: Produce soundness

Assume

(H1) $\text{produce}(\hat{\sigma}_f, \delta, \vec{v}_i, \vec{v}_o) \rightsquigarrow (\hat{\sigma}, \pi)$

(H2) $\text{SAT}_{\theta,s}(\pi) \wedge \theta, s, \sigma \models \hat{\sigma}$

To prove

(G1) $\exists \sigma_\delta, \sigma_f.\, \sigma_\delta \,\#\, \sigma_f \wedge \sigma = \sigma_\delta \cdot \sigma_f \wedge \theta, s, \sigma_\delta \models_\Delta \langle \delta \rangle (\vec{v}_i; \vec{v}_o) \wedge \theta, s, \sigma_f \models \hat{\sigma}_f$

(H3) From the produce rule, we have $\hat{\sigma}_f = \bot$, $\delta = \text{ex}$, $\vec{v}_i = []$, $\vec{v}_o = [\hat{x}]$, $\hat{\sigma} = \text{ex}(\hat{x})$ and $\pi = []$.

(H4) From (H2) and the definition of $\models$ we have $\sigma = \text{ex}(x)$ and $\llbracket \hat{x} \rrbracket_{\theta,s} = x$

(H5) Given (H4), we have $\sigma_\delta = \text{ex}(x)$ and $\sigma_f = \bot$, such that $\sigma_\delta \,\#\, \sigma_f$ and $\sigma = \sigma_\delta \cdot \sigma_f$.

(H6) From (H4), we have $\llbracket \vec{v}_i \rrbracket_{\theta,s} = []$ and $\llbracket \vec{v}_o \rrbracket_{\theta,s} = [x]$, thus from (H5) and the definition of $\models_\Delta$ we have $\theta, s, \sigma_\delta \models_\Delta \langle \delta \rangle (\vec{v}_i; \vec{v}_o)$.

(H7) From (H3) and (H5) we have $\sigma_f = \bot$ and $\hat{\sigma}_f = \bot$, thus from the definition of $\models$, we have $\theta, s, \sigma_f \models \hat{\sigma}_f$. This with (H6) gives our goal (G1).

Proposition 9: Produce completeness

Assume

(H1) $\theta, s, \sigma_f \models \hat{\sigma}_f \wedge \theta, s, \sigma_\delta \models_\Delta \langle \delta \rangle (\vec{v}_i; \vec{v}_o) \wedge \sigma_f \,\#\, \sigma_\delta$

To prove

(G1) $\exists \hat{\sigma}.\, \text{produce}(\hat{\sigma}_f, \delta, \vec{v}_i, \vec{v}_o) \rightsquigarrow (\hat{\sigma}, \pi) \wedge \text{SAT}_{\theta,s}(\pi) \wedge \theta, s, (\sigma_f \cdot \sigma_\delta) \models \hat{\sigma}$

(H2) From (H9), $\sigma_\delta \models_\Delta \langle \delta \rangle (\vec{v}_i; \vec{v}_o)$, thus from the definition of $\models_\Delta$ we have $\sigma_\delta = \text{ex}(x)$, $\delta = \text{ex}$, $\vec{v}_i = []$, $\vec{v}_o = [\hat{x}]$ with $\llbracket \hat{x} \rrbracket_{\theta,s} = x$.

(H3) From (H9), $\sigma_f \,\#\, \sigma_\delta$, thus from (H2) and the rules of composition, we have $\sigma_f = \bot$.

(H4) From the rules of $\models$ and (H3), we have $\hat{\sigma}_f = \bot$.

(H5) We can pick $\hat{\sigma} = \text{ex}(\hat{x})$.

(H6) From (H4), (H2), (H5) and the rules of produce, we have $\text{produce}(\sigma_f, \delta, \vec{v}_i, \vec{v}_o) \rightsquigarrow (\hat{\sigma}, \pi)$, with $\pi = []$.

(H7) From (H6), $\text{SAT}_{\theta,s}(\pi)$.

(H8) From (H3) and (H2), $\sigma_f = \bot$ and $\sigma_\delta = \text{ex}(x)$, thus from the rules of composition, $\sigma_f \cdot \sigma_\delta = \text{ex}(x)$.

(H9) From (H2), $\llbracket \hat{x} \rrbracket_{\theta,s} = x$, thus from (H5) and (H8) we have $\theta, s, (\sigma_f \cdot \sigma_\delta) \models \hat{\sigma}$, which combined with (H6) and (H7) gives our goal (G1).

$\square$

B.2 Partial Map

To facilitate the soundness proof for PMAP, and due to its extensive use of the `get` and `set` helper methods to modify the elements in the heap, we first define and prove properties about these auxiliary functions.

We introduce a pair of properties for `get`, similar to OX and UX soundness of actions, to ensure that the symbolic function retrieves the right state if it exists in the concrete counterpart of the state.

$$\theta, s, \sigma \models \hat{\sigma} \wedge \mathsf{get}(\sigma, i) = \sigma_i \wedge [\![\hat{i}]\!]_{\theta,s} = i \implies \exists \hat{\sigma}_i, \hat{i}', \pi. \\ \mathsf{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \wedge \theta, s, \sigma_i \models \hat{\sigma}_i \wedge [\![\hat{i}']\!]_{\theta,s} = i \wedge \mathsf{SAT}_{\theta,s}(\pi) \qquad \text{(Get OX Soundness)}$$

$$\mathsf{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \wedge [\![\hat{i}]\!]_{\theta,s} = [\![\hat{i}']\!]_{\theta,s} = i \wedge \mathsf{SAT}_{\theta,s}(\pi) \implies \\ \forall \sigma. \theta, s, \sigma \models \hat{\sigma} \implies \exists \sigma_i. \mathsf{get}(\sigma, i) = \sigma_i \wedge \theta, s, \sigma_i \models \hat{\sigma}_i \qquad \text{(Get UX Soundness)}$$

Proof.

Proposition 1: Get OX Soundness
Assume

(H1) $\theta, s, \sigma \models \hat{\sigma} \wedge \mathsf{get}(\sigma, i) = \sigma_i \wedge [\![\hat{i}]\!]_{\theta,s} = i$

To prove

(G1) $\exists \hat{\sigma}_i, \hat{i}', \pi. \mathsf{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \wedge \theta, s, \sigma_i \models \hat{\sigma}_i \wedge [\![\hat{i}']\!]_{\theta,s} = i \wedge \mathsf{SAT}_{\theta,s}(\pi)$

We proceed by proving the property holds for all rules of the concrete `get`, resulting in three cases.

Case CPMAPGETMATCH:

(H2) Assume $(h, d) = unwrap(\sigma) \wedge i \in \mathrm{dom}(h) \wedge \sigma_i = h(i)$.

(H3) It follows from (H1) and the definition of $\models$ that $\hat{\sigma} = (\hat{h}, \hat{d})$ such that $\exists \hat{i}'. \hat{i}' \in \mathrm{dom}(\hat{h})$, $[\![\hat{i}']\!]_{\theta,s} = i$ and $\theta, s, \sigma_i \models \hat{h}(\hat{i}')$.

(H4) We can then apply PMAPGETMATCH, giving us $\mathsf{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{h}(\hat{i}'), [\hat{i}' = \hat{i}])$.

(H5) From (H3) and (H1) it follows that $\mathsf{SAT}_{\theta,s}([\hat{i}' = \hat{i}])$, which completes our goal (G1).

Case CPMAPGETADD:

(H6) Assume $(h, d) = unwrap(\sigma) \wedge i \notin \mathrm{dom}(h) \wedge d \neq \bot \wedge i \in d$.

(H7) It follows from (H1) and the definition of $\models$ that $\hat{\sigma} = (\hat{h}, \hat{d})$ such that $\hat{i} \in \hat{d}$ and $\hat{i} \notin \mathrm{dom}(\hat{h})$.

(H8) We can then apply PMAPGETADD, giving us $\mathsf{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \bot, [\hat{i} \notin \mathrm{dom}(\hat{h}) \wedge \hat{i} \notin \hat{d}])$.

(H9) From (H8) and (H7) it follows that $\mathsf{SAT}_{\theta,s}([\hat{i} \notin \mathrm{dom}(\hat{h}) \wedge \hat{i} \notin \hat{d}])$, which completes our goal (G1).

Case CPMAPGETBOTDOMAIN:

(H10) Assume $(h, d) = unwrap(\sigma) \wedge i \notin \mathrm{dom}(h) \wedge d = \bot$.

(H11) From (H10), (H1) and CPMAPGETBOTDOMAIN, we have $\sigma_i = \bot$.

(H12) It follows from (H1) and the definition of $\models$ that $\hat{\sigma} = (\hat{h}, \bot)$ such that $\hat{i} \notin \mathrm{dom}(\hat{h})$.

(H13) We can then apply PMAPGETBOTDOMAIN, giving us $\mathsf{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \bot, [\hat{i} \notin \mathrm{dom}(\hat{h})])$.

(H14) From (H13) and (H12) it follows that $\mathsf{SAT}_{\theta,s}([\hat{i} \notin \mathrm{dom}(\hat{h})])$, which completes our goal (G1).

Proposition 2: Get UX Soundness

Assume

(H1) $\mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \wedge [\![\hat{i}]\!]_{\theta,s} = [\![\hat{i}']\!]_{\theta,s} = i \wedge \mathsf{SAT}_{\theta,s}(\pi)$

(H2) $\theta, s, \sigma \models \hat{\sigma}$

To prove

(G1) $\exists \sigma_i.\, \mathtt{get}(\sigma, i) = \sigma_i \wedge \theta, s, \sigma_i \models \hat{\sigma}_i$

We proceed by proving the property holds for all rules of the symbolic `get`, resulting in three cases.

Case PMapGetMatch**:**

(H3) Assume $(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \wedge \hat{i}' \in \mathrm{dom}(\hat{h}) \wedge \hat{\sigma}_i = \hat{h}(\hat{i}')$

(H4) From (H3), (H1) and PMapGetMatch, we have $\pi = [\hat{i} = \hat{i}']$

(H5) From (H3), we have $\hat{\sigma} \neq \bot$, thus from (H2) we have $\sigma = (h, d)$

(H6) From (H2), (H3) and the definition of $\models$, we have $i \in h$ such that $\sigma_i = h(i)$ and $\theta, s, \sigma_i \models \hat{\sigma}_i$

(H7) From (H6) we can apply CPMapGetMatch, thus $\mathtt{get}(\sigma, i) = \sigma_i$. This completes our goal (G1).

Case PMapGetAdd**:**

(H8) Assume $(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \wedge \hat{i} \notin \mathrm{dom}(\hat{h}) \wedge \hat{d} \neq \bot$

(H9) From (H8), (H1) and PMapGetAdd, we have $\hat{i}' = \hat{i}$, $\hat{\sigma}_i = \bot$ and $\pi = [\hat{i} \notin \mathrm{dom}(\hat{h}) \wedge \hat{i} \in \hat{d}]$

(H10) From (H8) we have $\hat{d} \neq \bot$, thus $\hat{\sigma} = (\hat{h}, \hat{d})$ and from (H2) $\sigma = (h, d)$ such that $[\![\hat{d}]\!]_{\theta,s} = d$.

(H11) From (H1) $\mathsf{SAT}_{\theta,s}(\pi)$, thus from (H9) and (H2), $i \notin \mathrm{dom}(h) \wedge i \in d$

(H12) From (H11) and (H10) we can apply CPMapGetAdd, thus $\mathtt{get}(\sigma, i) = \bot$.

(H13) From Empty Memory, $\theta, s, \bot \models \bot$, which along with (H12) completes our goal (G1).

Case PMapGetBotDomain**:**

(H14) Assume $(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \wedge \hat{i} \notin \mathrm{dom}(\hat{h}) \wedge \hat{d} = \bot$

(H15) From (H14), (H1) and PMapGetBotDomain, we have $\hat{i}' = \hat{i}$, $\hat{\sigma}_i = \bot$ and $\pi = [\hat{i} \notin \mathrm{dom}(\hat{h})]$

(H16) From (H14) we have $\hat{d} = \bot$, so given $(h, d) = unwrap(\sigma)$ and (H2), we have $d = \bot$.

(H17) From (H1) $\mathsf{SAT}_{\theta,s}(\pi)$, thus from (H15) and (H2), $i \notin \mathrm{dom}(h)$.

(H18) From (H17) and (H16) we can apply CPMapGetBotDomain, thus $\mathtt{get}(\sigma, i) = \bot$.

(H19) From Empty Memory, $\theta, s, \bot \models \bot$, which along with (H18) completes our goal (G1).

We can now proceed with the standard proofs.

Proof.

Proposition 1: OX soundness

Assume

(H1) $\theta, s, \sigma \models \hat{\sigma} \wedge \alpha(\sigma, \vec{v}_i) \rightsquigarrow (o, \sigma', \vec{v}_o) \wedge [\![\vec{\hat{v}}_i]\!]_{\theta,s} = \vec{v}_i$

(H2) $\forall o', \hat{\sigma}', \vec{\hat{v}}_o', \pi'. \, \alpha(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (o', \hat{\sigma}', \vec{\hat{v}}_o', \pi') \Rightarrow o' \in \{\mathsf{Ok}, \mathsf{Err}\}$

(H3) The actions on $\mathbb{S}$ are OX sound.

To prove

(G1) $\exists \hat{\sigma}', \vec{\hat{v}}_o, \pi, \theta'. \, \hat{\alpha}(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{\hat{v}}_o, \pi) \wedge \theta', s, \sigma' \models \hat{\sigma}' \wedge \mathsf{SAT}_{\theta',s}(\pi) \wedge [\![\vec{\hat{v}}_o]\!]_{\theta',s} = \vec{v}_o$

There are two action cases to consider: $\alpha \in \mathcal{A}_{\mathbb{S}}$ and $\alpha = \mathtt{alloc}$.

Case $\alpha \in \mathcal{A}_{\mathbb{S}}$**:**

(H4) If the action gets executed succesfully (it is not an out of bounds error), from CPMAPACTION we have $\mathtt{get}(\sigma, i) = \sigma_i$, $\alpha(\sigma_i, \vec{v}_i') \rightsquigarrow (o, \sigma_i', \vec{v}_o')$ and $\mathtt{set}(\sigma, i, \sigma_i') = \sigma'$, with $\vec{v}_i = i :: \vec{v}_i'$ and $\vec{v}_o = i :: \vec{v}_o'$

(H5) From (H1) and (H4), we know we have $\vec{\hat{v}}_i = \hat{i} :: \vec{\hat{v}}_i'$ such that $[\![\hat{i}]\!]_{\theta,s} = i$ and $[\![\vec{\hat{v}}_i']\!]_{\theta,s} = \vec{v}_i$.

(H6) From (H1), (H4) and (H5), we can apply Get OX Soundness, giving $\exists \hat{\sigma}_i, \hat{i}', \pi. \, \mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \wedge \theta, s, \sigma_i \models \hat{\sigma}_i \wedge [\![\hat{i}']\!]_{\theta,s} = i \wedge \mathsf{SAT}_{\theta,s}(\pi)$

(H7) From (H4), (H6), (H2), (H5) and (H3), we can apply Memory Model OX Soundness, giving us $\exists \vec{\hat{v}}_o', \hat{\sigma}_i', \pi', \theta'. \, \alpha(\hat{\sigma}_i, \vec{\hat{v}}_i') \rightsquigarrow (o, \hat{\sigma}_i', \vec{\hat{v}}_o', \pi') \wedge \theta', s, \sigma_i' \models \hat{\sigma}_i' \wedge \mathsf{SAT}_{\theta',s}(\pi') \wedge [\![\vec{\hat{v}}_o']\!]_{\theta',s} = \vec{v}_o'$.

(H8) From (H1) and (H7), we have $\theta, s, \sigma \models \hat{\sigma}$ and $\theta', s, \sigma_i' \models \hat{\sigma}_i'$. From the definition of $\mathtt{set}$, it follows that only modifying the state at $\hat{i}'$ preserves symbolic interpretation, thus given $\mathtt{set}(\hat{\sigma}, \hat{i}', \hat{\sigma}_i') = \hat{\sigma}'$, we have $\theta', s, \sigma' \models \hat{\sigma}'$.

(H9) By applying PMAPACTION and from (H7), (H6), (H8) our goal (G1) follows.

(H10) If this is an out of bounds access, we have $\sigma = \sigma' = (h, d)$ with $d \neq \bot$ and $\vec{v}_i = i :: \vec{v}_i'$, with $i \notin d$, as well as $\vec{\hat{v}}_i = \hat{i} :: \vec{\hat{v}}_i'$

(H11) From (H1) and the definition of $\models$, we have $\hat{\sigma} = (\hat{h}, \hat{d})$ such that $\pi = [\hat{i} \notin \hat{d}]$ and $\mathsf{SAT}_{\theta,s}(\pi)$.

(H12) From the rules of action execution and (H11), we get $\alpha(\hat{\sigma}, \vec{\hat{v}}_i) \rightsquigarrow (\mathsf{Err}, \hat{\sigma}, [], \pi)$. The rest of our goal (G1) follows.

Case $\alpha = \mathtt{alloc}$**:**

Both cases (when $d \neq \bot$ and $d = \bot$) are trivially lifted from the concrete to the symbolic realm, with $\pi = [i = i]$ and $\pi = []$ respectively. It follows that $\mathsf{SAT}_{\theta,s}(\pi)$ in both cases, giving our goal (G1).

For the successful allocation case, we note there is an additional requirement for soundness of state instantiation to be able to prove the resulting states $\hat{\sigma}'$ and σ' are compatible.

$$[\![\vec{\hat{v}}_i]\!]_{\theta,s} = \vec{v}_i \wedge \sigma = \mathtt{instantiate} \; \vec{v}_i \wedge \hat{\sigma} = \mathtt{instantiate} \; \vec{\hat{v}}_i \implies \theta, s, \sigma \models \hat{\sigma}$$

Proposition 2: UX soundness

Assume

(H1) $\hat{\alpha}(\hat{\sigma}, \vec{v}_i) \rightsquigarrow (o, \hat{\sigma}', \vec{v}_o, \pi) \wedge \mathsf{SAT}_{\theta,s}(\pi) \wedge \theta, s, \sigma' \models \hat{\sigma}' \wedge$
$[\![\vec{v}_o]\!]_{\hat{s},\pi} \rightsquigarrow (\vec{v}_o, \pi') \wedge [\![\vec{v}_i]\!]_{\hat{s},\pi'} \rightsquigarrow (\vec{v}_i, \pi'')$

(H2) The actions on $\mathbb{S}$ are UX sound.

To prove

(G1) $\exists \sigma. \, \theta, s, \sigma \models \hat{\sigma} \wedge \alpha(\sigma, \vec{v}_i) \rightsquigarrow (o, \sigma', \vec{v}_o)$

Case $\alpha \in \mathcal{A}_{\mathbb{S}}$**:**

(H3) If the action gets executed succesfully (it is not an out of bounds error), from PMAPACTION we have
$\mathsf{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi')$, $\alpha(\hat{\sigma}_i, \vec{v}_i') \rightsquigarrow (o, \hat{\sigma}_i', \vec{v}_o', \pi'')$ and $\mathsf{set}(\hat{\sigma}, \hat{i}', \hat{\sigma}_i') = \hat{\sigma}'$, with $\vec{v}_i = \hat{i} :: \vec{v}_i'$, $\vec{v}_o = \hat{i}' :: \vec{v}_o'$ and $\pi = \pi' :: \pi''$

(H4) From (H2), (H3) and (H1), we get $\exists \sigma_i. \, \theta, s, \sigma_i \models \hat{\sigma}_i \wedge \alpha(\sigma_i, \vec{v}_i') \rightsquigarrow (p, \sigma', \vec{v}_o')$

(H5) From the definition of $\models$, it follows that if $\theta, s, \sigma' \models \hat{\sigma}'$ where $\mathsf{set}(\hat{\sigma}, \hat{i}', \hat{\sigma}_i') = \hat{\sigma}'$, then if $\theta, s, \sigma_i \models \hat{\sigma}_i$ there exists an σ and i such that $\mathsf{set}(\sigma, i, \sigma_i') = \sigma'$, giving us $\theta, s, \sigma \models \hat{\sigma}$

(H6) From (H3), (H1) and (H5) we can use Get UX Soundness, giving us $\exists \sigma_i. \, \mathsf{get}(\sigma, i) = \sigma_i \wedge \theta, s, \sigma_i \models \hat{\sigma}_i$

(H7) From (H6), (H4) and (H5) we can apply CPMAPACTION, giving us $\alpha(\sigma, \vec{v}_i) \rightsquigarrow (o, \sigma', \vec{v}_o)$, which together with (H5) gives our goal (G1).

(H8) If the action fails as it is out of bounds, from PMAPACTIONOUTOFBOUNDS we have $\vec{v}_i = \hat{i} :: \vec{v}_i'$ and $\pi = [\hat{i} \notin \hat{d}]$.

(H9) From (H1) and the definition of $\models$ we thus have $\sigma = (h, d)$, with $\vec{v}_i = i :: \vec{v}_i'$, $d \neq \perp$ and $i \notin d$

(H10) From (H9), we can apply CPMAPACTIONOUTOFBOUNDS, which along with the fact the state is unmodified gives our goal (G1).

Case $\alpha = \mathtt{alloc}$**:**
Again, both cases (when the domains set d is owned and when it is $\perp$) are directly lifted from the concrete cases; it follows that if $\theta, s, \sigma' \models \hat{\sigma}'$ then $\theta, s, \sigma \models \hat{\sigma}$ where σ is the state without the added binding.

Proposition 3: Frame subtraction is satisfied
Assume

(H1) $\sigma \# \sigma_f \wedge \alpha(\sigma \cdot \sigma_f, \vec{v}_i) \rightsquigarrow (o, \sigma', \vec{v}_o)$

(H2) The actions on $\mathbb{S}$ satisfy frame subtraction.

(H3) $\forall \alpha, \vec{v}_i, \vec{v}_o. \, \alpha(\perp, \vec{v}_i) \rightsquigarrow (\mathtt{Miss}, \sigma', \vec{v}_o)$

To prove

(G1) $\exists \sigma'', o', \vec{v}_o'. \, \alpha(\sigma, \vec{v}_i) \rightsquigarrow (o', \sigma'', \vec{v}_o') \wedge$
$(o' \neq \mathtt{Miss} \implies o' = o \wedge \vec{v}_o' = \vec{v}_o \wedge \sigma' = \sigma'' \cdot \sigma_f)$

Case $\alpha \in \mathcal{A}_{\mathbb{S}}$**:**

(H4) If the action causes an out of bounds error, from CPMAPACTIONOUTOFBOUNDS we have $\sigma \cdot \sigma_f = (h, d)$ and $\vec{v}_i = i :: \vec{v}_i'$ such that $i \notin d$.

(H5) From composition, the domain set d is either part of σ or σ_f. If it is part of σ such that $\sigma = (h_a, d)$ and $\sigma_f = (h_b, \bot)$, then we still have $i \notin d$, resulting in the same error and outcome, giving our goal (G1).

(H6) If the domain is in σ_f such that $\sigma = (h_a, \bot)$ and $\sigma_f = (h_b, d)$, then the action is executed on the underlying state model. Because we have $\text{dom}(h) \subseteq d$, from (H4) we have $i \notin \text{dom}(h_a)$, so from the rules of `get` and CPMapAction the action is executed on $\bot$. From (H3), we have $o' = \texttt{Miss}$, satisfying (G1).

(H7) If the action is not out of bounds, we have $\alpha(\sigma_i, \vec{v}'_i) \rightsquigarrow (o, \sigma'_i, \vec{v}'_o)$ with $\vec{v}_i = i :: \vec{v}'_i$ and $\vec{v}_o = i :: \vec{v}'_o$.

(H8) If σ_f only includes the domain set and indices different than i, then the action is executed on the same σ_i and gives the same outcomes – it can then be composed with the result, giving our goal (G1).

(H9) Let σ_f such that $\sigma_f = (h_b, d_b)$ and $i \in \text{dom}(h_b)$. We denote $\sigma_{b,i} = h_b(i)$. For $\sigma = (h_a, d_a)$, we also have $\sigma_{a,i} = \bot$ if $i \notin \text{dom}(h_a)$, and $\sigma_{a,i} = h_a(i)$ otherwise, such that $\sigma_{a,i} \# \sigma_{b,i}$.

(H10) With (H7) and (H9), from (H2), we can apply Frame subtraction to the action on the state at index i, which allows re-applying CPMapAction and completes our goal (G1).

Case $\alpha = \texttt{alloc}$**:**

(H11) `alloc` is always defined, so we know $\exists \sigma'', o', \vec{v}_o.\, \alpha(\sigma, \vec{v}_i) \rightsquigarrow (o', \sigma'', \vec{v}'_o)$.

(H12) Assume $o' \neq \texttt{Miss}$.

(H13) From (H12) and the rules of `alloc`, $\sigma = (h, d)$ with $d \neq \bot$.

(H14) From (H13) and (H1), if $\sigma \# \sigma_f$, then $\sigma_f = (h_f, \bot)$.

(H15) From the definition of `alloc`, the heap is not modified to the exception of the added entry, so from the definition of composition we have $\sigma' = \sigma'' \cdot \sigma_f$. The returns value $\vec{v}_o$ are the same (a fresh value), and the outcome is `Ok` in both cases, giving our goal (G1).

Proposition 4: Frame addition is satisfied
Assume

(H1) $\alpha(\sigma, \vec{v}_i) \rightsquigarrow (o, \sigma', \vec{v}_o)$

(H2) $\sigma_f \# \sigma'$

(H3) $o \neq \texttt{Miss}$

(H4) The actions on $\mathbb{S}$ satisfy frame addition.

(H5) $\forall \alpha, \vec{v}_i, \vec{v}_o.\, \alpha(\bot, \vec{v}_i) \rightsquigarrow (\texttt{Miss}, \sigma', \vec{v}_o)$

To prove

(G1) $\sigma \# \sigma_f \wedge \alpha(\sigma \cdot \sigma_f, \vec{v}_i) \rightsquigarrow (o, \sigma' \cdot \sigma_f, \vec{v}_o)$

Case $\alpha \in \mathcal{A}_{\mathbb{S}}$**:**

(H6) From the rules of concrete actions we have $\vec{v}_i = i :: \vec{v}'_i$ and $\vec{v}_o = i :: \vec{v}'_o$.

(H7) If the execution of the action is an out of bounds error, by CPMAPACTIONOUTOFBOUNDS we have $\sigma = (h, d)$ such that $d \neq \perp$ and $i \notin d$.

(H8) From composition rules, we know $\sigma_f = (h_f, \perp)$, meaning that we still get $\alpha(\sigma \cdot \sigma_f, \vec{v}_i) \rightsquigarrow (\mathsf{Err}, \sigma' \cdot \sigma_f, [])$ as we still have $i \notin d$. This satisfies our goal (G1).

(H9) We now look at the case where we do not have an out of bounds error. Assume we have $\sigma = (h, d)$ such that $i \notin \mathrm{dom}(h)$. From the rules for $\mathtt{get}$, we know we executed the action on $\perp$. From (H5), we thus have $o = \mathtt{Miss}$, which is a contradiction with (H3). Thus we know $i \in \mathrm{dom}(h)$.

(H10) From (H9) and the rules for $\mathtt{get}$, we know we have $\sigma_i = h(i)$ and $\alpha(\sigma_i, \vec{v}'_i) \rightsquigarrow (o, \sigma'_i, \vec{v}'_o)$.

(H11) If we have $\sigma_f = (h_f, d_f)$ such that $i \notin \mathrm{dom}(h_f)$, then it follows from (H2) that $\sigma \# \sigma_f$ as the action does not modify any other cell. The action will also have the same outcome, completing our goal (G1).

(H12) If we have $\sigma_f = (h_f, d_f)$ such that $i \in \mathrm{dom}(h_f)$, it follows from (H2) that $\sigma'_i \# h_f(i)$. We also know the outcome is not $\mathtt{Miss}$ from the rules of action execution and (H3).

(H13) From (H12), (H10) and (H4) we can apply Frame Addition and get $\sigma_i \# h_f(i)$ and $\alpha(\sigma_i \cdot h_f(i), \vec{v}'_i) \rightsquigarrow (o, \sigma'_i \cdot h_f(i), \vec{v}'_o)$.

(H14) From (H13) and CPMAPACTION, we get $\sigma \# \sigma_f$ and $\alpha(\sigma \cdot \sigma_f, \vec{v}_i) \rightsquigarrow (o, \sigma' \cdot \sigma_f, \vec{v}_o)$, giving our goal (G1).

Case $\alpha = \mathtt{alloc}$**:**

(H15) From (H1), (H3) and the rules for $\mathtt{alloc}$, we have $\sigma' = (h', d')$ such that $d' \neq \perp$.

(H16) From (H15), (H2) and the rules of composition it follows that $\sigma_f = (h_f, \perp)$.

(H17) Because $\mathtt{alloc}$ does not modify any state aside from adding a cell, and because σ_f has no domain set, it follows from (H2) that $\sigma \# \sigma_f$ and that the action has the saame outcome, thus $\alpha(\sigma \cdot \sigma_f, \vec{v}_i) \rightsquigarrow (o, \sigma' \cdot \sigma_f, \vec{v}_o)$, giving our goal (G1).

Proposition 5: Consume UX soundness
Assume

(H1) $\mathtt{consume}(\hat{\sigma}, \delta, \vec{\hat{v}}_i) \rightsquigarrow (\mathsf{Ok}, \hat{\sigma}_f, \vec{\hat{v}}_o, \pi)$

(H2) $\theta, s, \sigma \models \hat{\sigma} \wedge \mathsf{SAT}_{\theta, s}(\pi)$

(H3) $\mathtt{consume}$ on $\mathbb{S}$ is UX sound.

To prove

(G1) $\exists \sigma_\delta, \sigma_f. \, \sigma_\delta \# \sigma_f \wedge \sigma = \sigma_\delta \cdot \sigma_f \wedge \theta, s, \sigma_\delta \models_\Delta \langle \delta \rangle (\vec{\hat{v}}_i; \vec{\hat{v}}_o) \wedge \theta, s, \sigma_f \models \hat{\sigma}_f$

Case $\delta \in \Delta_{\mathbb{S}}$**:**

(H4) From (H1) and PMAPCONS, given $\vec{\hat{v}}_i = \hat{i} :: \vec{\hat{v}}'_i$ and $\pi = \pi' :: \pi''$ we have $\mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi')$, $\mathtt{consume}(\hat{\sigma}_i, \delta, \vec{\hat{v}}'_i) \rightsquigarrow (o, \hat{\sigma}'_i, \vec{\hat{v}}_o, \pi'')$ and $\mathtt{set}(\hat{\sigma}, \hat{i}', \hat{\sigma}'_i) = \hat{\sigma}_f$.

(H5) From (H4) and (H2), we have $\vec{\hat{v}}_i = \hat{i} :: \vec{\hat{v}}'_i \wedge [\![\hat{i}]\!]_{\theta, s} = i$

(H6) From (H2) and Get UX Soundness, we know $\exists \sigma_i.\, \texttt{get}(\sigma, i) = \sigma_i \wedge \theta, s, \sigma_i \models \hat{\sigma}_i$.

(H7) From (H2) and (H4), we have $\mathsf{SAT}_{\theta,s}(\pi'')$.

(H8) From (H6), (H7) and (H3) we can apply Consume UX Soundness, giving us $\exists \sigma_{\delta,i}, \sigma_{f,i}.\, \sigma_{\delta,i} \,\#\, \sigma_i' \wedge \sigma_i = \sigma_{\delta,i} \cdot \sigma_i' \wedge \theta, s, \sigma_{\delta,i} \models_\Delta \langle \delta \rangle(\vec{v}_i'; \vec{v}_o) \wedge \theta, s, \sigma_{f,i} \models \hat{\sigma}_{f,i}$.

(H9) We can then pick $\sigma_\delta = ([i \mapsto \sigma_{\delta,i}], \bot)$, and from (H8) and the definition of composition we know $\sigma_\delta \,\#\, \sigma$ and (H4) and the definition of $\texttt{set}$, $\sigma_f = \sigma \cdot \sigma_\delta$.

(H10) From the definition of $\models_\Delta$ and (H8) we then have $\theta, s, \sigma_\delta \models \langle \delta \rangle(\vec{v}_i; \vec{v}_o)$, which with (H9) gives our goal (G1).

Case $\delta = \texttt{domainset}$:

(H11) From (H1) and the rules for $\texttt{consume}$, we have $\hat{\sigma} = (\hat{h}, \hat{d})$ and $\hat{\sigma}_f = (\hat{h}, \bot)$ such that $\hat{d} \neq \bot$, as well as $\vec{v}_i = []$ and $\vec{v}_o = [\hat{d}]$

(H12) From the definition of $\models$ and (H11) we have $\sigma = (h, d)$ and $\sigma_f = (h, \bot)$ such that $d \neq \bot$ and $[\![\hat{d}]\!]_{\theta,s} = d$.

(H13) We can pick $\sigma_\delta = ([], d)$ – it follows that $\sigma_f \,\#\, \sigma_\delta$ and from the composition rules, $\sigma = \sigma_f \cdot \sigma_\delta$.

(H14) From the definition of $\models_\Delta$ and (H12) it follows that $\theta, s, \sigma_\delta \models_\Delta \langle \texttt{domainset} \rangle(\vec{v}_i; \vec{v}_o)$. Along with (H13), this gives our goal (G1).

Proposition 6: Consume OX soundness
Assume

(H1) $\texttt{consume}(\hat{\sigma}, \delta, \vec{v}_i) \rightsquigarrow (\texttt{Ok}, \hat{\sigma}_f, \vec{v}_o, \pi)$

(H2) $\theta, s, \sigma_f \models \hat{\sigma}_f \wedge \theta, s, \sigma_\delta \models_\Delta \langle \delta \rangle(\vec{v}_i; \vec{v}_o) \wedge \sigma_f \,\#\, \sigma_\delta$

(H3) $\texttt{consume}$ on $\mathbb{S}$ is OX sound.

To prove

(G1) $\exists \sigma.\, \sigma = \sigma_\delta \cdot \sigma_f \wedge \theta, s, \sigma \models \hat{\sigma} \wedge \mathsf{SAT}_{\theta,s}(\pi)$

Case $\delta \in \Delta_\mathbb{S}$:

(H4) From (H1) and PMapCons, given $\vec{v}_i = \hat{i} :: \vec{v}_i'$ and $\pi = \pi' :: \pi''$ we have $\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi')$, $\texttt{consume}(\hat{\sigma}_i, \delta, \vec{v}_i') \rightsquigarrow (o, \hat{\sigma}_i', \vec{v}_o, \pi'')$ and $\texttt{set}(\hat{\sigma}, \hat{i}', \hat{\sigma}_i') = \hat{\sigma}_f$.

(H5) There are two cases here, $\sigma_\delta = \bot$ and $\sigma_\delta \neq \bot$. If $\sigma_\delta = \bot$, we trivially get $\sigma = \bot \cdot \sigma_f$ and the rest of our goal (G1).

(H6) Assume now $\sigma_\delta \neq \bot$. From the definition of $\models_\Delta$ and (H2), we have $\exists i, \sigma_{\delta,i}.\, [\![\hat{i}]\!]_{\theta,s} = i \wedge \sigma_\delta = ([i \mapsto \sigma_{\delta,i}], \bot) \wedge \sigma_{\delta,i} \models_\Delta \langle \delta \rangle(\vec{v}_i'; \vec{v}_o)$.

(H7) From (H2), (H4) and the definitions of $\models$ and composition we have $\exists \sigma_i'.\, \theta, s, \sigma_i' \models \hat{\sigma}_i' \wedge \sigma_i' \,\#\, \sigma_{\delta,i}$.

(H8) From (H7), (H6) and (H3), we can apply Consume OX Soundness, giving us $\exists \sigma_i.\, \sigma_i = \sigma_{\delta,i} \cdot \sigma_i' \wedge \theta, s, \sigma_i \models \hat{\sigma}_i \wedge \mathsf{SAT}_{\theta,s}(\pi'')$.

(H9) From the definition of $\models$, because $\mathsf{consume}$ does not affect any other cell, given $\theta, s, \sigma_i \models \hat{\sigma}_i$ and $\theta, s, \sigma_f \models \hat{\sigma}_f$ and $\mathsf{set}(\hat{\sigma}, \hat{i}', \hat{\sigma}'_i) = \hat{\sigma}_f$, we obtain $\theta, s, \sigma \models \hat{\sigma}$.

(H10) From (H6), we have σ_δ of the form $\sigma_\delta = (h_\delta, \bot)$ where $i \in \mathrm{dom}(h_\delta)$. From composition rules, it follows that if $\sigma = \sigma_\delta \cdot \sigma_f$ we have $\sigma = (h, d)$ where $i \in \mathrm{dom}(h)$ too.

(H11) From (H10) and (H9), it follows that we can apply CPMAPGETMATCH such that $\mathsf{get}(\sigma, i) = \sigma_i$. We can then use Get OX Soundness, giving us, from (H4) that $\mathsf{SAT}_{\theta,s}(\pi')$.

(H12) It follows from (H11) and (H8) that $\mathsf{SAT}_{\theta,s}(\pi)$. This along with (H9) gives our goal (G1).

Case $\delta = \mathsf{domainset}$:

(H13) From (H1) and the rules for $\mathsf{consume}$, we have $\hat{\sigma} = (\hat{h}, \hat{d})$ and $\hat{\sigma}_f = (\hat{h}, \bot)$ such that $\hat{d} \neq \bot$, as well as $\vec{\hat{v}}_i = [], \vec{\hat{v}}_o = [\hat{d}]$ and $\pi = []$

(H14) From the definition of $\models_\Delta$ and (H2) we have $\sigma_\delta = (\emptyset, d)$ such that $[\![\hat{d}]\!]_{\theta,s} = d$.

(H15) It follows from (H13), (H2) and the definition of $\models$ that σ_f is of the form $\sigma_f = (h, \bot)$.

(H16) From (H14), (H15) and the definition of $\models$ it follows that for $\sigma = \sigma_\delta \cdot \sigma_f$, we have $\theta, s, \sigma \models \hat{\sigma}$. This with the fact $\mathsf{SAT}_{\theta,s}([])$ gives our goal (G1).

Proposition 7: Consume OX no path drops
Assume

(H1) $\forall o, \hat{\sigma}_f, \vec{\hat{v}}_o, \pi.\, \mathsf{consume}(\mathsf{OX}, \hat{\sigma}, \delta, \vec{\hat{v}}_i) \rightsquigarrow (o, \hat{\sigma}_f, \vec{\hat{v}}_o, \pi) \Rightarrow o_c = \mathsf{Ok}$

(H2) $\mathsf{consume}$ on $\mathbb{S}$ does not drop paths in OX.

To prove

(G1) $\exists \hat{\sigma}'_f, \vec{\hat{v}}'_o, \pi'.\, \mathsf{consume}(\mathsf{OX}, \hat{\sigma}, \delta, \vec{v}_i, \pi) \rightsquigarrow (\mathsf{Ok}, \hat{\sigma}'_f, \vec{\hat{v}}'_o, \pi')$

Case $\delta \in \Delta_\mathbb{S}$:
From PMAPPROD we know $\mathsf{consume}$ for $\delta \in \Delta_\mathbb{S}$ only returns Ok if the underlying's $\mathsf{consume}$ returns Ok. It follows that if that is the case, then by (H2) there exists an execution of $\mathsf{consume}$ that succeeds for $\mathbb{S}$, that is lifted by PMAP, giving us our goal (G1).

Case $\delta = \mathsf{domainset}$:
From the $\mathsf{consume}$ rules, we never vanish, so if all consumptions are Ok we must have $\hat{\sigma} = (\hat{h}, \hat{d})$ with $\hat{d} \neq \bot, \vec{\hat{v}}_i = [], \hat{\sigma}'_f = (\hat{h}, \bot), \vec{\hat{v}}'_o = [\hat{d}], \pi' = []$, which gives our goal (G1).

Proposition 8: Produce UX soundness
Assume

(H1) $\mathsf{produce}(\hat{\sigma}_f, \delta, \vec{\hat{v}}_i, \vec{\hat{v}}_o) \rightsquigarrow (\hat{\sigma}, \pi)$

(H2) $\mathsf{SAT}_{\theta,s}(\pi) \wedge \theta, s, \sigma \models \hat{\sigma}$

(H3) $\mathsf{produce}$ on $\mathbb{S}$ is UX sound.

To prove

(G1) $\exists \sigma_\delta, \sigma_f.\, \sigma_\delta \,\#\, \sigma_f \wedge \sigma = \sigma_\delta \cdot \sigma_f \wedge \theta, s, \sigma_\delta \models_\Delta \langle \delta \rangle (\vec{\hat{v}}_i; \vec{\hat{v}}_o) \wedge \theta, s, \sigma_f \models \hat{\sigma}_f$

Case $\delta \in \Delta_{\mathbb{S}}$**:**

(H4) Given (H1), from PMAPPROD we have $\mathsf{get}(\hat{\sigma}_f, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_{f,i}, \pi') \wedge \mathsf{produce}(\hat{\sigma}_{f,i}, \delta, \vec{v}'_i, \vec{v}_o) \rightsquigarrow (\hat{\sigma}_i, \pi'') \wedge \mathsf{set}(\hat{\sigma}_f, \hat{i}', \hat{\sigma}_i) = \hat{\sigma}$, given $\vec{v}_i = \hat{i} :: \vec{v}'_i$ and $\pi = \pi' :: \pi''$.

(H5) From (H2) and $\pi = \pi' :: \pi''$, it follows that $\mathsf{SAT}_{\theta,s}(\pi')$ and $\mathsf{SAT}_{\theta,s}(\pi'')$.

(H6) From the definition of set, (H4) and (H2), we have $\exists \sigma_i. \theta, s, \sigma_i \models \hat{\sigma}_i$.

(H7) From (H4), (H6), (H5) and (H3), we can apply Produce: UX Soundness, giving us $\exists \sigma_{\delta,i}, \sigma_{f,i}. \sigma_{\delta,i} \# \sigma_{f,i} \wedge \sigma_i = \sigma_{\delta,i} \cdot \sigma_{f,i} \wedge \theta, s, \sigma_{\delta,i} \models_\Delta \langle\delta\rangle(\vec{v}'_i; \vec{v}_o) \wedge \theta, s, \sigma_{f,i} \models \hat{\sigma}_{f,i}$.

(H8) Given $[\![\hat{i}']\!]_{\theta,s} = i$, we can then have $\sigma_\delta = ([i \mapsto \sigma_{\delta,i}], \bot)$, or $\sigma_\delta = \bot$ if $\sigma_{\delta,i} = \bot$. By the definition of $\models_\Delta$, $\theta, s, \sigma_\delta \models_\Delta \langle\delta\rangle(\vec{v}_i; \vec{v}_o)$ in both cases.

(H9) Given set only modifies the state at a specific location leaving the rest untouched, from (H2) and (H7) we have $\theta, s, \sigma \models \hat{\sigma}$ and $\theta, s, \sigma_{f,i} \models \hat{\sigma}_{f,i}$, thus it follows that $\exists \sigma_f. \theta, s, \sigma_f \models \sigma_f$.

(H10) From (H7) and by the rules of composition, we know that if the two states $\sigma_{\delta,i}$ and $\sigma_{f,i}$ at index i are disjoint then the entire state σ_f and the singleton σ_δ are disjoint. Thus $\sigma_\delta \# \sigma_f \wedge \sigma = \sigma_\delta \cdot \sigma_f$, giving our goal (G1).

Case $\delta = \mathsf{domainset}$**:**

(H11) Given (H1), from PMAPPRODDOMAINSET we have $\hat{\sigma}_f = (\hat{h}, \bot)$, $\hat{\sigma} = (\hat{h}, \hat{d})$ and $\pi = [\mathsf{dom}(\hat{h}) \subseteq \hat{d}]$, with $\vec{v}_i = []$ and $\vec{v}_o = [\hat{d}]$.

(H12) We can pick σ_δ such that $\sigma_\delta = (\emptyset, d)$ with $[\![\hat{d}]\!]_{\theta,s} = d$, which from the definition of $\models_\Delta$ gives $\theta, s, \sigma_\delta \models_\Delta \langle\mathsf{domainset}\rangle(\vec{v}_i; \vec{v}_o)$.

(H13) We can pick σ_f such that $\sigma_f = (h, \bot)$ and $[\![\hat{h}]\!]_{\theta,s} = h$. From the definition of $\models$, we have $\theta, s, \sigma_f \models \hat{\sigma}_f$.

(H14) From (H12), (H13) and composition rules we have $\sigma_f \# \sigma_\delta$ and $\sigma = \sigma_f \cdot \sigma_\delta$. This completes our goal (G1).

Proposition 9: Produce OX soundness
Assume

(H1) $\theta, s, \sigma_f \models \hat{\sigma}_f \wedge \theta, s, \sigma_\delta \models_\Delta \langle\delta\rangle(\vec{v}_i; \vec{v}_o) \wedge \sigma_f \# \sigma_\delta$

(H2) $\mathsf{produce}$ on $\mathbb{S}$ is OX sound.

To prove

(G1) $\exists \hat{\sigma}. \mathsf{produce}(\hat{\sigma}_f, \delta, \vec{v}_i, \vec{v}_o) \rightsquigarrow (\hat{\sigma}, \pi) \wedge \mathsf{SAT}_{\theta,s}(\pi) \wedge \theta, s, (\sigma_f \cdot \sigma_\delta) \models \hat{\sigma}$

Case $\delta \in \Delta_{\mathbb{S}}$**:**

(H3) From (H1) and the definition of $\models_\Delta$, we have $\exists \hat{i}, i, \sigma_{\delta,i}. \vec{v}_i = \hat{i} :: \vec{v}'_i \wedge [\![\hat{i}]\!]_{\theta,s} = i \wedge \theta, s, \sigma_{\delta,i} \models_\Delta \langle\delta\rangle(\vec{v}'_i; \vec{v}_o) \wedge \sigma_\delta = ([i \mapsto \sigma_{\delta,i}], \bot)$.

(H4) Given the fact $\sigma_f \# \sigma_\delta$ and $\sigma_\delta \models_\Delta \langle\delta\rangle(\hat{i} :: \vec{v}'_i; \vec{v}_o)$, it means the index i is compatible with σ_f, so the state located at i is obtainable via get. We thus have $\exists \sigma_{f,i}. \mathsf{get}(\sigma_f, i) = \sigma_{f,i}$.

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(H5) From (H4), (H3) and (H1), we can use [Get OX Soundness](), giving us $\exists \hat{\sigma}_{f,i}, \hat{i}', \pi'. \, \texttt{get}(\hat{\sigma}_f, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_{f,i}, \pi) \wedge \theta, s, \sigma_{f,i} \models \hat{\sigma}_{f,i} \wedge [\![\hat{i}']\!]_{\theta,s} = i \wedge \mathsf{SAT}_{\theta,s}(\pi')$.

(H6) From (H1), because $\sigma_{f,i}$ and $\sigma_{\delta,i}$ are at the same index and $\sigma_f \, \# \, \sigma_\delta$, from the rules for composition we have $\sigma_{f,i} \, \# \, \sigma_{\delta,i}$.

(H7) From (H3), (H5) and (H6) we can apply [Produce: OX Soundness]() for $\mathbb{S}$, giving us $\exists \hat{\sigma}_i. \, \texttt{produce}(\hat{\sigma}_{f,i}, \delta, \vec{\hat{v}}'_i, \vec{\hat{v}}_o) \rightsquigarrow (\hat{\sigma}_i, \pi'$ $\mathsf{SAT}_{\theta,s}(\pi'') \wedge \theta, s, (\sigma_{f,i} \cdot \sigma_{\delta,i}) \models \hat{\sigma}_i$.

(H8) We may define $\hat{\sigma}$ such that $\texttt{set}(\hat{\sigma}_f, \hat{i}', \hat{\sigma}_i) = \hat{\sigma}$.

(H9) We have $\theta, s, \sigma_f \models \hat{\sigma}_f$ and $\theta, s, (\sigma_{f,i} \cdot \sigma_{\delta,i}) \models \hat{\sigma}_i$ from (H1) and (H7). Furthermore, we know $\sigma_\delta = ([i \mapsto \sigma_{\delta,i}], \bot)$ from (H3). From (H8) and the rules for $\texttt{set}$ we know no state is lost and only the state at i is extended by $\sigma_{\delta,i}$, it thus follows that $\theta, s, (\sigma_f \cdot \sigma_\delta) \models \hat{\sigma}$.

(H10) From and (H5), (H7) and this we can apply [PMapProd](), giving us $\texttt{produce}(\hat{\sigma}_f, \delta, \vec{\hat{v}}_i, \vec{\hat{v}}_o) \rightsquigarrow (\hat{\sigma}, \pi' :: \pi'')$.

(H11) From (H5) and (H7), we have $\mathsf{SAT}_{\theta,s}(\pi' :: \pi'')$. This, along with (H9) and (H10), gives our goal (G1).

Case $\delta = \texttt{domainset}$:

(H12) From (H1) and the definition of $\models_\Delta$ we have $\sigma_\delta = (\emptyset, d)$ such that $\vec{\hat{v}}_i = [\,]$, $\vec{\hat{v}}_o = [\hat{d}]$ and $[\![\hat{d}]\!]_{\theta,s} = d$.

(H13) From (H12) and (H1), by the definition of composition, we must have $\hat{\sigma}_f = (\hat{h}, \bot)$, $\sigma_f = (h, \bot)$ and $\mathrm{dom}(h) \subseteq d$.

(H14) From (H12) and (H13) we may apply [PMapProdDomainSet](), resulting in $\texttt{produce}(\hat{\sigma}_f, \texttt{domainset}, \vec{\hat{v}}_i, \vec{\hat{v}}_o) \rightsquigarrow ((\hat{h}, \hat{d}), [\mathrm{dom}(\hat{h}) \subseteq \hat{d}])$.

(H15) From (H1) and (H13) we have $\mathsf{SAT}_{\theta,s}([\mathrm{dom}(\hat{h}) \subseteq \hat{d}])$.

(H16) From (H12), (H13) and composition rules we have we have $\sigma_f \cdot \sigma_\delta = (h, d)$. It follows from (H14) that we have $\hat{\sigma} = (\hat{h}, \hat{d})$ such that $\theta, s, (\sigma_f \cdot \sigma_\delta) \models \hat{\sigma}$. This along with (H14) and (H15) gives our goal (G1).

$\square$

B.3 Syntactic Partial Map

Proof.

Proposition 1: Get OX Soundness
Assume

(H1) $\theta, s, \sigma \models \hat{\sigma} \wedge \texttt{get}(\sigma, i) = \sigma_i \wedge [\![\hat{i}]\!]_{\theta,s} = i$

To prove

(G1) $\exists \hat{\sigma}_i, \hat{i}', \pi. \, \texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \wedge \theta, s, \sigma_i \models \hat{\sigma}_i \wedge [\![\hat{i}']\!]_{\theta,s} = i \wedge \mathsf{SAT}_{\theta,s}(\pi)$

We proceed by proving the property holds for all rules of the concrete $\texttt{get}$, resulting in three cases.

Case [CPMapGetMatch]():

(H2) Assume $(h, d) = \mathit{unwrap}(\sigma) \wedge i \in \mathrm{dom}(h) \wedge \sigma_i = h(i)$.

(H3) It follows from (H1) and the definition of $\models$ that $\hat{\sigma} = (\hat{h}, \hat{d})$.

(H4) Either $\hat{i} \in \hat{h}$ or not. If $\hat{i} \in \hat{h}$, we apply SYNTACTICPMAPGETMATCH, giving us $\mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \hat{h}(\hat{i}), [])$. It also follows from the definition of $\models$ that $\theta, s, \sigma_i \models \hat{h}(\hat{i})$, completing our goal (G1).

(H5) Otherwise $\hat{i} \notin \hat{h}$. From the definition of $\models$ we know $\exists \hat{i}'. [\![\hat{i}']\!]_{\theta,s} = i \wedge \hat{i}' \in \hat{h} \wedge \theta, s, \sigma_i \models \hat{h}(\hat{i}')$. We can apply SYNTACTICPMAPGETBRANCH, giving us $\mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, [\hat{i} = \hat{i}'])$. It follows that $\mathsf{SAT}_{\theta,s}([\hat{i} = \hat{i}'])$, which completes our goal (G1).

Case CPMAPGETADD**:** The rule SYNTACTICPMAPGETADD is exactly the same as PMAPGETADD – we omit the proof.

Case CPMAPGETBOTDOMAIN**:** The rule SYNTACTICPMAPGETBOTDOMAIN is exactly the same as PMAPGETBOTDOMAIN – we omit the proof.

Proposition 2: Get UX Soundness

Assume

(H1) $\mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \wedge [\![\hat{i}]\!]_{\theta,s} = [\![\hat{i}']\!]_{\theta,s} = i \wedge \mathsf{SAT}_{\theta,s}(\pi)$

(H2) $\theta, s, \sigma \models \hat{\sigma}$.

To prove

(G1) $\exists \sigma_i. \mathtt{get}(\sigma, i) = \sigma_i \wedge \theta, s, \sigma_i \models \hat{\sigma}_i$

We proceed by proving the property holds for all rules of the symbolic $\mathtt{get}$, resulting in four cases.

Case SYNTACTICPMAPGETMATCH**:**

(H3) Assume $(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \wedge \hat{i} \in \mathrm{dom}(\hat{h}) \wedge \hat{\sigma}_i = \hat{h}(\hat{i})$.

(H4) From (H3), (H1) and SYNTACTICPMAPGETMATCH we have $\pi = []$ and $\hat{i}' = \hat{i}$.

(H5) From (H3), we know $\hat{\sigma} \neq \bot$, thus σ is of the form $\sigma = (h, d)$.

(H6) From (H2), (H3) and the definition of $\models$, we have $i \in h$ such that $\sigma_i = h(i)$ and $\theta, s, \sigma_i \models \hat{\sigma}_i$.

(H7) From (H6) we can apply CPMAPGETMATCH, thus $\mathtt{get}(\sigma, i) = \sigma_i$. This completes our goal (G1).

Case SYNTACTICPMAPGETBRANCH**:**

(H8) Assume $(\hat{h}, \hat{d}) = \mathit{unwrap}(\hat{\sigma}) \wedge \hat{i}' \in \mathrm{dom}(\hat{h}) \wedge \hat{\sigma}_i = \hat{h}(\hat{i}')$

(H9) From (H8), (H1) and PMAPGETMATCH, we have $\pi = [\hat{i} = \hat{i}']$

(H10) From (H8), we have $\hat{\sigma} \neq \bot$, thus from (H2) we have $\sigma = (h, d)$

(H11) From (H2), (H8) and the definition of $\models$, we have $i \in h$ such that $\sigma_i = h(i)$ and $\theta, s, \sigma_i \models \hat{\sigma}_i$

(H12) From (H11) we can apply CPMAPGETMATCH, thus $\mathtt{get}(\sigma, i) = \sigma_i$. This completes our goal (G1).

Case SYNTACTICPMAPGETADD**:** The rule SYNTACTICPMAPGETADD is exactly the same as PMAPGETADD – we omit the proof.

Case SYNTACTICPMAPGETBOTDOMAIN**:** The rule SYNTACTICPMAPGETBOTDOMAIN is exactly the same as PMAPGETBOTDOMAIN – we omit the proof. $\qquad\square$

B.4 Split Partial Map

Proof.

Proposition 1: Get OX Soundness
Assume

(H1) $\theta, s, \sigma \models \hat{\sigma} \wedge \texttt{get}(\sigma, i) = \sigma_i \wedge [\![\hat{i}]\!]_{\theta,s} = i$

To prove

(G1) $\exists \hat{\sigma}_i, \hat{i}', \pi. \, \texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \wedge \theta, s, \sigma_i \models \hat{\sigma}_i \wedge [\![\hat{i}']\!]_{\theta,s} = i \wedge \mathsf{SAT}_{\theta,s}(\pi)$

We proceed by proving the property holds for all rules of the concrete $\texttt{get}$, resulting in three cases.

Case CPMapGetMatch**:**

(H2) Assume $(h, d) = \mathit{unwrap}(\sigma) \wedge i \in \mathrm{dom}(h) \wedge \sigma_i = h(i)$.

(H3) It follows from (H1) and the definition of $\models$ that $\hat{\sigma} = (\hat{h}_c, \hat{h}_s, \hat{d})$. From here we have three cases: $\hat{i}$ is present directly in $\hat{h}_c$, or in $\hat{h}_s$, or neither.

(H4) From (H3), assume the binding is in the concrete part of the heap: $\hat{i} \in \hat{h}_c$. From the definition of $\texttt{set}$, we know that if an entry is in the concrete part it must be concrete; thus $\texttt{is\_concrete} \ \hat{i}$. From this, we can apply SplitPMapGetMatchCon, giving us $\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \hat{h}_c(\hat{i}), [])$. From the definition of $\models$, it also follows that $\theta, s, \sigma_i \models \hat{h}_c(\hat{i})$. This, along with the fact $\mathsf{SAT}_{\theta,s}([])$, gives our goal (G1).

(H5) Assume now from (H3) that the binding is in the symbolic part of the heap, $\hat{h}_s$. From this, we can apply SplitPMapGetMatchSym, giving us $\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \hat{h}_s(\hat{i}), [])$. From the definition of $\models$, it also follows that $\theta, s, \sigma_i \models \hat{h}_s(\hat{i})$. This again gives our goal (G1).

(H6) Finally, it may be that $\hat{i} \notin \hat{h}_c \wedge \hat{i} \notin \hat{h}_s$. From the definition of $\models$, there must however still exist a $\hat{i}'$ such that $[\![\hat{i}']\!]_{\theta,s} = i$ and given $\hat{h}_{all} = \hat{h}_c \cup \hat{h}_s$, $\theta, s, \sigma_i \models \hat{h}_{all}(\hat{i}')$. From this we may apply SplitPMapGetBranch, giving us $\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, [\hat{i} = \hat{i}'])$. Because we have $[\![\hat{i}]\!]_{\theta,s} = i$, it follows that $\mathsf{SAT}_{\theta,s}([\hat{i} = \hat{i}'])$, completing our goal (G1).

Case CPMapGetAdd**:**

(H7) Assume $(h, d) = \mathit{unwrap}(\sigma) \wedge i \notin \mathrm{dom}(h) \wedge d \neq \bot \wedge i \in d$.

(H8) It follows from (H1) and the definition of $\models$ that $\hat{\sigma} = (\hat{h}_c, \hat{h}_s, \hat{d})$ such that $[\![\hat{i}]\!]_{\theta,s} = i$, $\hat{i} \in \hat{d}$ and $\hat{i} \notin \mathrm{dom}(\hat{h}_c \cup \hat{h}_s)$.

(H9) We can then apply SplitPMapGetAdd, giving us $\texttt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \bot, [\hat{i} \notin \mathrm{dom}(\hat{h}_c \cup \hat{h}_s) \wedge \hat{i} \notin \hat{d}])$.

(H10) From (H8) it follows that $\mathsf{SAT}_{\theta,s}([\hat{i} \notin \mathrm{dom}(\hat{h}) \wedge \hat{i} \notin \hat{d}])$, which completes our goal (G1).

Case CPMapGetBotDomain**:**

(H11) Assume $(h, d) = \mathit{unwrap}(\sigma) \wedge i \notin \mathrm{dom}(h) \wedge d = \bot$.

(H12) From (H11), (H1) and CPMapGetBotDomain, we have $\sigma_i = \bot$.

(H13) It follows from (H1) and the definition of $\models$ that $\hat{\sigma} = (\hat{h}_c, \hat{h}_s, \bot)$ such that $[\![\hat{i}]\!]_{\theta,s} = i$ and $\hat{i} \notin \text{dom}(\hat{h}_c \cup \hat{h}_s)$.

(H14) We can then apply SPLITPMAPGETBOTDOMAIN, giving us $\text{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}, \bot, [\hat{i} \notin \text{dom}(\hat{h}_c \cup \hat{h}_s)])$.

(H15) From (H13) it follows that $\text{SAT}_{\theta,s}([\hat{i} \notin \text{dom}(\hat{h}_c \cup \hat{h}_s)])$, which completes our goal (G1).

Proposition 2: Get UX Soundness
Assume

(H1) $\text{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \wedge [\![\hat{i}]\!]_{\theta,s} = [\![\hat{i}']\!]_{\theta,s} = i \wedge \text{SAT}_{\theta,s}(\pi)$

(H2) $\sigma.\,\theta, s, \sigma \models \hat{\sigma}$

To prove

(G1) $\exists \sigma_i.\, \text{get}(\sigma, i) = \sigma_i \wedge \theta, s, \sigma_i \models \hat{\sigma}_i$

We proceed by proving the property holds for all rules of the symbolic get, resulting in five cases.

Case SPLITPMAPGETMATCHCON**:**

(H3) Assume $(\hat{h}_c, \hat{h}_s, \hat{d}) = \text{unwrap}(\hat{\sigma}) \wedge \text{is\_concrete } \hat{i} \wedge \hat{i} \in \text{dom}(\hat{h}_c) \wedge \hat{\sigma}_i = \hat{h}_c(\hat{i})$.

(H4) From (H3), (H1) and SPLITPMAPGETMATCHCON we have $\pi = [\,]$ and $\hat{i}' = \hat{i}$.

(H5) From (H3), we know $\hat{\sigma} \neq \bot$, thus σ is of the form $\sigma = (h, d)$.

(H6) From (H2), (H3) and the definition of $\models$, we have $i \in \text{dom}(h)$ such that $\sigma_i = h(i)$ and $\theta, s, \sigma_i \models \hat{\sigma}_i$.

(H7) From (H6) we can apply CPMAPGETMATCH, thus $\text{get}(\sigma, i) = \sigma_i$. This completes our goal (G1).

Case SPLITPMAPGETMATCHSYM**:** This is proved analogously to the above.

Case SPLITPMAPGETBRANCH**:**

(H8) Assume $(\hat{h}_c, \hat{h}_s, \hat{d}) = \text{unwrap}(\hat{\sigma}) \wedge \hat{h}_{all} = \hat{h}_c \cup \hat{h}_s \wedge \hat{i} \notin \text{dom}(\hat{h}_{all}) \wedge \hat{i}' \in \text{dom}(\hat{h}_{all}) \wedge \hat{\sigma}_i = \hat{h}_{all}(\hat{i}')$.

(H9) From (H8), (H1) and SPLITPMAPGETBRANCH we have $\pi = [\hat{i} = \hat{i}']$.

(H10) From (H8) we know $\hat{\sigma} \neq \bot$, thus σ is of the form $\sigma = (h, d)$.

(H11) From (H8), (H2) and the definition of $\models$, we have $i \in \text{dom}(h)$ such that $\sigma_i = h(i)$ and $\theta, s, \sigma_i \models \hat{\sigma}_i$.

(H12) From (H11) we can apply CPMAPGETMATCH, thus $\text{get}(\sigma, i) = \sigma_i$. This completes our goal (G1).

Case SPLITPMAPGETADD**:**

(H13) Assume $(\hat{h}_c, \hat{h}_s, \hat{d}) = \text{unwrap}(\hat{\sigma}) \wedge \hat{h}_{all} = \hat{h}_c \cup \hat{h}_s \wedge \hat{i} \notin \text{dom}(\hat{h}_{all}) \wedge \hat{d} \neq \bot$.

(H14) From (H13), (H1) and SPLITPMAPGETADD we have $\pi = [\hat{i} \notin \text{dom}(\hat{h}_{all}) \wedge \hat{i} \in \hat{d}]$, $\hat{i}' = \hat{i}$ and $\hat{\sigma}_i = \bot$.

(H15) From (H13) we know $\hat{\sigma} \neq \bot$, thus σ is of the form $\sigma = (h, d)$.

(H16) From (H1) and (H14), we know $\text{SAT}_{\theta,s}(\pi)$, such that from (H2) we have $i \notin \text{dom}(h) \wedge i \in d$.

(H17) From (H16) we can straightforwardly apply CPMAPGETADD, giving us $\text{get}(\sigma, i) = \bot$ – together with (H14) and Empty Memory this completes our goal (G1).

Case SPLITPMAPGETBOTDOMAIN**:**

(H18) Assume $(\hat{h}_c, \hat{h}_s, \hat{d}) = unwrap(\hat{\sigma}) \wedge \hat{h}_{all} = \hat{h}_c \cup \hat{h}_s \wedge \hat{i} \notin dom(\hat{h}_{all}) \wedge \hat{d} = \bot$.

(H19) From (H18), (H1) and SPLITPMAPGETBOTDOMAIN we have $\pi = [\hat{i} \notin dom(\hat{h}_{all})]$ and $\hat{i} = \hat{i}$.

(H20) From (H18) we have $\hat{d} = \bot$, so given $(h, d) = unwrap(\sigma)$ and (H2), we have $d = \bot$.

(H21) From (H1) $\mathsf{SAT}_{\theta,s}(\pi)$, thus from (H19) and (H2), $i \notin dom(h)$.

(H22) From (H21) and (H20) we can apply CPMAPGETBOTDOMAIN, thus $\mathtt{get}(\sigma, i) = \bot$ – together with (H19) and Empty Memory this completes our goal (G1).

$\square$

B.5   Abstract Location Partial Map

Because we extend the signature of get with a matching mode, we proceed with the proofs for `get` and `set` by doing no assumption on the value of the mode m. Interestingly, we will see that *we cannot prove OX and UX soundness* with $m = \mathtt{NO\_MATCH}$, as it only holds for $m = \mathtt{MATCH}$. This is central to the difference in behaviour between $\mathrm{PMAP}(\mathrm{Loc}, \mathbb{S})$ and $\mathrm{PMAP}_{\mathrm{ALoc}}(\mathbb{S})$.
Proof.

Proposition 1: Get OX Soundness
Assume

(H1) $\theta, s, \sigma \models \hat{\sigma} \wedge \mathtt{get}(\sigma, i) = \sigma_i \wedge [\![\hat{i}]\!]_{\theta,s} = i$

To prove

(G1) $\exists \hat{\sigma}_i, \hat{i}', \pi. \, \mathtt{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \wedge \theta, s, \sigma_i \models \hat{\sigma}_i \wedge [\![\hat{i}']\!]_{\theta,s} = i \wedge \mathsf{SAT}_{\theta,s}(\pi)$

We proceed by proving the property holds for all rules of the concrete `get`, resulting in three cases.

Case CPMAPGETMATCH**:**

(H2) Assume $(h, d) = unwrap(\sigma) \wedge i \in dom(h) \wedge \sigma_i = h(i)$.

(H3) It follows from (H1) and the definition of $\models$ that $\hat{\sigma} = (\hat{h}, \hat{d})$ such that $\exists \hat{i}'. \, \hat{i}' = \mathrm{aloc}(a') \wedge a' \in dom(\hat{h})$, $[\![\hat{i}']\!]_{\theta,s} = i$ and $\theta, s, \sigma_i \models \hat{h}(a')$.

(H4) From here, multiple cases are possible: either $\mathtt{to\_aloc} \; \hat{i} = a'$, or $\mathtt{to\_aloc} \; \hat{i} = a$ such that $a \neq a' \wedge [\![\mathrm{aloc}(a)]\!]_{\theta,s} = [\![\mathrm{aloc}(a')]\!]_{\theta,s}$, or $\mathtt{to\_aloc} \; \hat{i} = \bot$. Indeed, $\mathtt{to\_aloc}$ is a best effort function, that may not find a match despite there being one; it can also be that $\hat{i}$ is a fresh symbolic variable, such that in the current path condition nothing binds it to $\mathrm{aloc}(a)$.

(H5) In the first case, $\mathtt{to\_aloc} \; \hat{i} = a'$ we can apply ALOCPMAPGETMATCH, giving us our goal (G1).

(H6) In the second case, $\mathtt{to\_aloc} \; \hat{i} = a \wedge a \neq a' \wedge [\![\mathrm{aloc}(a)]\!]_{\theta,s} = [\![\mathrm{aloc}(a')]\!]_{\theta,s}$, we may apply ALOCPMAPMATCHING, *but only when* $m = \mathtt{MATCH}$. This gives our goal (G1). Otherwise, when $m = \mathtt{NO\_MATCH}$, the only rule that suits is ALOCPMAPGETNOMATCHNOTFOUND, which gives us $\hat{\sigma}_i = \bot$, from which it follows that $\theta, s, \sigma_i \not\models \hat{\sigma}_i$.

(H7) In the last case, $\mathtt{to\_aloc} \; \hat{i} = \bot$. Again, if $m = \mathtt{MATCH}$, we can apply ALOCPMAPMATCHINGBOT and get our goal (G1); otherwise, the only applicable rule is ALOCPMAPGETNOMATCHNEW, which invalidates our goal, as $\theta, s, \sigma_i \not\models \hat{\sigma}_i$.

Case CPMapGetAdd**:**

(H8) Assume $(h, d) = \textit{unwrap}(\sigma) \land i \notin \text{dom}(h) \land d \neq \bot \land i \in d$.

(H9) From (H14), (H1) and CPMapGetAdd, we have $\sigma_i = \bot$.

(H10) It follows from (H1) and the definition of $\models$ that $\hat{\sigma} = (\hat{h}, \hat{d})$ such that $\hat{i} = \text{aloc}(a') \land \hat{i} \in \hat{d}$ and $a' \notin \text{dom}(\hat{h})$.

(H11) We again get three possible cases, depending on to_aloc: either to_aloc $\hat{i} = a'$, or to_aloc $\hat{i} = a$ such that $a \neq a' \land [\![\text{aloc}(a)]\!]_{\theta,s} = [\![\text{aloc}(a')]\!]_{\theta,s}$, or to_aloc $\hat{i} = \bot$.

(H12) In the two first cases, from the definition of $\models$ and to_aloc, it still holds from (H10) that $a' \notin \hat{h}$ and $a \notin \hat{h}$, respectively. Depending on the mode m, we thus apply ALocPMapGetNoMatchNotFound or ALocPMapGetMatchNotFound, giving us $\hat{\sigma}_i = \bot$ and $\pi = [\hat{i} \in^? \hat{d}]$ or $\pi = [\hat{i} \notin \{\text{aloc}(a') : a' \in \text{dom}(\hat{h})\} \land \hat{i} \in^? \hat{d}]$, which from (H10) we know $\text{SAT}_{\theta,s}(\pi)$ in both cases, giving our goal (G1).

(H13) In the last case, we have to_aloc $\hat{i} = \bot$, we thus apply either ALocPMapGetNoMatchNew or ALocPMapGetMatchNew depending on m, giving us $a' = \text{fresh\_aloc}$ (), $\hat{\sigma}_i = \bot$, $\hat{i}' = \text{aloc}(a')$ and $\pi = [\hat{i} = \text{aloc}(a') \land \hat{i} \in^? \hat{d}]$ or $[\hat{i} = \text{aloc}(a') \land \hat{i} \notin \{\text{aloc}(a'') : a'' \in \text{dom}(\hat{h})\} \land \hat{i} \in^? \hat{d}]$. From (H9) and Empty Memory, we have $\theta, s, \sigma_i \models \hat{\sigma}_i$. Because a' is a fresh abstract location, it's equality to one term can always be satisfied; and from (H10) we know $\hat{i} \in^? \hat{d}$ holds – thus $\text{SAT}_{\theta,s}(\pi)$, completing our goal (G1).

Case CPMapGetBotDomain**:**

(H14) Assume $(h, d) = \textit{unwrap}(\sigma) \land i \notin \text{dom}(h) \land d = \bot$.

(H15) From (H14), (H1) and CPMapGetBotDomain, we have $\sigma_i = \bot$.

(H16) It follows from (H1) and the definition of $\models$ that $\hat{\sigma} = (\hat{h}, \bot)$ such that $\hat{i} \notin \text{dom}(\hat{h})$.

(H17) The remainder of the proof is analogous to the above; three cases are possible depending on the result of to_aloc $\hat{i}$ – in all three cases, we can apply one of the four [...]NotFound and [...]New rules, such that $\hat{\sigma}_i = \bot$, $[\![\hat{i}']\!]_{\theta,s} = i$ and $\text{SAT}_{\theta,s}(\pi)$, giving our goal (G1).

Proposition 2: Get UX Soundness
Assume

(H1) $\text{get}(\hat{\sigma}, \hat{i}) \rightsquigarrow (\hat{i}', \hat{\sigma}_i, \pi) \land [\![\hat{i}]\!]_{\theta,s} = [\![\hat{i}']\!]_{\theta,s} = i \land \text{SAT}_{\theta,s}(\pi)$

(H2) $\theta, s, \sigma \models \hat{\sigma}$

To prove

(G1) $\exists \sigma_i. \text{get}(\sigma, i) = \sigma_i \land \theta, s, \sigma_i \models \hat{\sigma}_i$

We proceed by proving the property holds for all rules of the symbolic get, resulting in five cases.

Case ALocPMapGetMatch**:**

(H3) Assume $(\hat{h}, \hat{d}) = \textit{unwrap}(\hat{\sigma}) \land a = \text{to\_aloc } \hat{i} \land a \neq \bot \land a \in \text{dom}(\hat{h}) \land \hat{\sigma}_i = \hat{h}(a)$

(H4) From (H3), (H1) and ALocPMapGetMatch, we have $\pi = [\,]$

(H5) From (H3), we have $\hat{\sigma} \neq \bot$, thus from (H2) we have $\sigma = (h, d)$

(H6) From (H2), (H3) and the definition of $\models$, we have $i \in h$ such that $\sigma_i = h(i)$ and $\theta, s, \sigma_i \models \hat{\sigma}_i$

(H7) From (H6) we can apply CPMAPGETMATCH, thus $\text{get}(\sigma, i) = \sigma_i$. This completes our goal (G1).

Case ALocPMapGetNoMatchNotFound:

(H8) Assume $(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \wedge a = \texttt{to\_aloc}\ \hat{i} \wedge a \neq \bot \wedge a \notin \text{dom}(\hat{h}) \wedge m = \texttt{NO\_MATCH}$

(H9) From (H21), (H1) and ALocPMapGetNoMatchNotFound, we have $\hat{\sigma}_i = \bot$ and $\pi = [\text{aloc}(a) \in^? \hat{d}]$

(H10) Let $(h, d) = unwrap(\sigma)$.

(H11) From (H1) $\texttt{SAT}_{\theta,s}(\pi)$, thus from (H22), either $d \neq \bot \wedge i \in d$, or $d = \bot$.

(H12) From this, we get two cases: either $i \in \text{dom}(h)$, or $i \notin \text{dom}(h)$.

(H13) If $i \notin h$, we can apply either CPMAPGETADD or CPMAPGETBOTDOMAIN depending on d, in both cases giving us $\text{get}(\sigma, i) = \bot$. Together with Empty Memory, this completes our goal.

(H14) However, if $i \in \text{dom}(h)$, we can only apply CPMAPGETMATCH, which however gives us $\text{get}(\sigma, i) = \sigma_i$ where $\sigma_i \neq \bot$ – this however *is not compatible with $\hat{\sigma}_i$*.

Case ALocPMapGetNoMatchNew:

(H15) Assume $(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \wedge a = \texttt{to\_aloc}\ \hat{i} \wedge a = \bot \wedge a' = \texttt{fresh\_aloc ()} \wedge m = \texttt{NO\_MATCH}$

(H16) From (H26), (H1) and ALocPMapGetNoMatchNew, we have $\hat{i}' = \text{aloc}(a')$, $\hat{\sigma}_i = \bot$ and $\pi = [\hat{i} = \text{aloc}(a') \wedge \hat{i} \in^? \hat{d}]$

(H17) Let $(h, d) = unwrap(\sigma)$.

(H18) From here, we get two cases: either $i \in \text{dom}(h)$, or $i \notin \text{dom}(h)$ – as indeed it may be the case that $[\![\hat{i}]\!]_{\theta,s} = i$, but that due to the nature of $\texttt{to\_aloc}$, $a \notin \hat{h}$.

(H19) If $i \notin h$, we can apply either CPMAPGETADD or CPMAPGETBOTDOMAIN depending on d, in either cases giving us $\text{get}(\sigma, i) = \bot$. Together with Empty Memory, this completes our goal.

(H20) However, if $i \in \text{dom}(h)$, we can only apply CPMAPGETMATCH, which however gives us $\text{get}(\sigma, i) = \sigma_i$ where $\sigma_i \neq \bot$ – this however *is not compatible with $\hat{\sigma}_i$*.

Case ALocPMapGetMatchNotFound:

(H21) Assume $(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \wedge a = \texttt{to\_aloc}\ \hat{i} \wedge a \neq \bot \wedge a \notin \text{dom}(\hat{h}) \wedge m = \texttt{MATCH}$

(H22) From (H21), (H1) and ALocPMapGetNoMatchNotFound, we have $\hat{\sigma}_i = \bot$ and $\pi = [\hat{i} \notin \{\text{aloc}(a') : a' \in \text{dom}(\hat{h})\} \wedge \hat{i} \in^? \hat{d}]$

(H23) Let $(h, d) = unwrap(\sigma)$.

(H24) From (H1) $\texttt{SAT}_{\theta,s}(\pi)$, thus from (H22), either $d \neq \bot \wedge i \in d$, or $d = \bot$. We also have, crucially, that $i \notin \text{dom}(h)$.

(H25) We can thus apply either CPMAPGETADD or CPMAPGETBOTDOMAIN depending on d, in both cases giving us $\text{get}(\sigma, i) = \bot$. Together with Empty Memory, this completes our goal.

Case ALocPMapGetMatchNew:

(H26) Assume $(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \wedge a = \mathtt{to\_aloc}\ \hat{i} \wedge a = \bot \wedge a' = \mathtt{fresh\_aloc}\ () \wedge m = \mathtt{MATCH}$

(H27) Let $(h, d) = unwrap(\sigma)$.

(H28) From (H26), (H1) and ALocPMapGetNoMatchNew, we have $\hat{i}' = \mathrm{aloc}(a'), \hat{\sigma}_i = \bot$ and
$\pi = [\hat{i} = \mathrm{aloc}(a') \wedge \hat{i} \notin \{\mathrm{aloc}(a'') : a'' \in \mathrm{dom}(\hat{h})\} \wedge \hat{i} \in^? \hat{d}]$

(H29) From (H1) $\mathsf{SAT}_{\theta,s}(\pi)$, thus from (H28), either $d \neq \bot \wedge i \in d$, or $d = \bot$. We also have, crucially,
that $[\![\mathrm{aloc}(a')]\!]_{\theta,s} = i$, and thus $i \notin \mathrm{dom}(h)$.

(H30) From (H29), we can apply either CPMapGetAdd or CPMapGetBotDomain depending on d, in
either cases giving us $\mathtt{get}(\sigma, i) = \bot$. Together with Empty Memory, this completes our goal.

Case ALocPMapMatching**:**

(H31) Assume $(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \wedge a = \mathtt{to\_aloc}\ \hat{i} \wedge a \neq \bot \wedge a \notin \mathrm{dom}(\hat{h}) \wedge a' \in \mathrm{dom}(\hat{h}) \wedge \hat{\sigma}_i = \hat{h}(a')$

(H32) From (H31), (H1) and ALocPMapMatching, we have $\hat{i}' = \mathrm{aloc}(a'), \hat{\sigma}_i = \hat{h}(a'), \pi = [\mathrm{aloc}(a) = \mathrm{aloc}(a')]$
and $m = \mathtt{MATCH}$.

(H33) From (H2), (H31) and the definition of $\models$, we have $i \in h$ such that $\sigma_i = h(i)$ and $\theta, s, \sigma_i \models \hat{\sigma}_i$

(H34) From (H33) we can apply CPMapGetMatch, thus $\mathtt{get}(\sigma, i) = \sigma_i$. This completes our goal (G1).

Case ALocPMapMatchingBot**:**

(H35) Assume $(\hat{h}, \hat{d}) = unwrap(\hat{\sigma}) \wedge a = \mathtt{to\_aloc}\ \hat{i} \wedge a = \bot \wedge a' \in \mathrm{dom}(\hat{h}) \wedge \hat{\sigma}_i = \hat{h}(a')$

(H36) From (H35), (H1) and ALocPMapMatchingBot, we have $\hat{i}' = \mathrm{aloc}(a'), \hat{\sigma}_i = \hat{h}(a'), \pi = [\hat{i} = \mathrm{aloc}(a')]$
and $m = \mathtt{MATCH}$.

(H37) From (H2), (H35) and the definition of $\models$, we have $i \in h$ such that $\sigma_i = h(i)$ and $\theta, s, \sigma_i \models \hat{\sigma}_i$

(H38) From (H37) we can apply CPMapGetMatch, thus $\mathtt{get}(\sigma, i) = \sigma_i$. This completes our goal (G1).

$\square$

Appendix C

Proof for Sum Soundness

We first prove that a transition in a sum is not sound when no additional properties are specified. We then show the proof for the properties that must be held by the elements of a sum to allow for OX and UX-sound transitions from one side of the sum to the other *when using PCMs*. This shows that sound constructions using PCMs are non-trivial, even for simple cases.

C.1   Proof of Unsoundness

The sum state model transformer, $\mathbb{S}_1 + \mathbb{S}_2$ with Σ_1 and Σ_2 the carrier sets of each side's RA, is unsound for actions that allow flipping the sum from one side to the other, if the actions does not have additional requirements to preserve soundness. We define the PCM of SUM_{PCM} PCM as:

$$\text{SUM}_{\text{PCM}}(\mathbb{S}_1, \mathbb{S}_2) \triangleq \mathbb{S}_1 + \mathbb{S}_2 \triangleq l(\sigma \colon \Sigma_1) \mid r(\sigma \colon \Sigma_2) \mid 0$$
$$\sigma \cdot 0 = 0 \cdot \sigma = \sigma$$
$$l(\sigma) \cdot l(\sigma') = l(\sigma \cdot \sigma')$$
$$r(\sigma) \cdot r(\sigma') = r(\sigma \cdot \sigma')$$
$$\text{undefined otherwise}$$

We also remind the frame substraction property, defined as:
Proof.

Proposition 1: Sum actions that swap sides are not frame preserving
Assume

(H1) $\mathbb{S}_1$ is a sound state model with a PCM $(\Sigma_1, \cdot, 0_1)$.

(H2) $\mathbb{S}_2$ is a sound state model with a PCM $(\Sigma_2, \cdot, 0_2)$.

(H3) There is an action swap that may flip the side of a sum from one side to the other:

$$\exists \sigma, \sigma', \vec{v}_i, \vec{v}_o. \, \mathsf{swap}(l(\sigma), \vec{v}_i) \rightsquigarrow (\mathsf{Ok}, r(\sigma'), \vec{v}_o)$$

We want to prove [Frame subtraction](#) does not hold with action swap.

(H4) From the definition of composition, $l(\sigma) \cdot l(0_1) = l(\sigma)$

(H5) From (H4) and (H3), $\mathsf{swap}(l(\sigma) \cdot l(0_1), \vec{v}_i) \rightsquigarrow (\mathsf{Ok}, r(\sigma'), \vec{v}_o)$

To satisfy frame substraction, as the outcome in (H??) is Ok, we would need $r(\sigma') = r(\sigma') \cdot l(0_1)$, which is not the case, as per the definition of composition, $r(\sigma') \cdot l(0_1)$ is undefined. Frame substraction thus does not hold for swap. $\qquad\square$

C.2   Sound Sum

To define a sound version of sum that supports flipping the side, we thus need to remove the 0 element of both sides from the allowed states. This ensure the states $l(0_1)$ and $r(0_2)$ aren't allowed, avoiding the problem seen with frame substraction.

$$\mathbb{S}_1 + \mathbb{S}_2 \triangleq l(\sigma \colon \Sigma_1 \setminus \{0_1\}) \mid r(\sigma \colon \Sigma_2 \setminus \{0_2\}) \mid 0$$

Additionally, to allow for a frame-preserving swap action, another requirement must be met: the state must be *exclusively owned* – no other non-empty state can be composed with it. This thus requires for a `is_exclusively_owned` function to be provided, defined as:[1]

$$\texttt{is\_exclusively\_owned } \sigma \triangleq \forall \sigma'. \, \sigma' \neq 0 \implies \neg(\sigma \, \# \, \sigma')$$

Proof.

Proposition 1:
If a valid frame-preserving sum actions swaps sides, then that sum cannot permit empty elements on either sides, swaps of smaller states must miss, and the resulting state is exclusively owned
Assuming

(H1) $\mathbb{S}_1$ is a sound state model with a PCM $(\Sigma_1, \cdot, 0_1)$.

(H2) $\mathbb{S}_2$ is a sound state model with a PCM $(\Sigma_2, \cdot, 0_2)$.

(H3) A sum state model is defined as $\mathbb{S}_1 + \mathbb{S}_2 \triangleq l(\sigma \colon \Sigma'_1) \mid r(\sigma \colon \Sigma'_2) \mid 0$, such that $\Sigma'_1 \subseteq \Sigma_1$ and $\Sigma'_2 \subseteq \Sigma_2$. We re-use the definition of composition shown before.

(H4) There is an action swap that may flip the side of a sum from the left to then right-hand side: $\exists \sigma, \sigma', \vec{v}_i, \vec{v}_o. \, \texttt{swap}(l(\sigma), \vec{v}_i) \rightsquigarrow (\texttt{Ok}, r(\sigma'), \vec{v}_o)$

This swap action is frame preserving, thus satisfying Frame subtraction and Frame Addition:

$$\begin{aligned} &\sigma \, \# \, \sigma_f \wedge \alpha(\sigma \cdot \sigma_f, \vec{v}_i) \rightsquigarrow (o, \sigma', \vec{v}_o) \implies \\ \textbf{(H5)} \quad &\exists \sigma'', o', \vec{v}'_o. \, \alpha(\sigma, \vec{v}_i) \rightsquigarrow (o', \sigma'', \vec{v}'_o) \wedge \\ &\quad (o' \neq \texttt{Miss} \implies o' = o \wedge \vec{v}'_o = \vec{v}_o \wedge \sigma' = \sigma'' \cdot \sigma_f) \end{aligned}$$

$$\textbf{(H6)} \quad \begin{aligned} &\alpha(\sigma, \vec{v}_i) \rightsquigarrow (o, \sigma', \vec{v}_o) \wedge o \neq \texttt{Miss} \wedge \sigma' \, \# \, \sigma_f \implies \\ &\sigma \, \# \, \sigma_f \wedge \alpha(\sigma \cdot \sigma_f, \vec{v}_i) \rightsquigarrow (o, \sigma' \cdot \sigma_f, \vec{v}_o) \end{aligned}$$

We aim to prove:

(G1) The zero elements of the left side of the sum is not permitted: $0_1 \notin \Sigma'_1$

(G2) The zero elements of the right side of the sum is not permitted: $0_2 \notin \Sigma'_2$

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[1] This definition is for PCMs – the formulation of `is_exclusively_owned` for RAs is simpler: $\texttt{is\_exclusively\_owned } \sigma \triangleq \nexists \sigma'. \, \sigma \, \# \, \sigma'$

(G3) If swap flips the side of the sum, any swap of smaller states will result in a Miss:
$$\text{swap}(l(\sigma), \vec{v}_i) \rightsquigarrow (\text{Ok}, r(\sigma'), \vec{v}_o) \land \sigma_a \preccurlyeq \sigma \land \text{swap}(l(\sigma_a), \vec{v}_i) \rightsquigarrow (o', \sigma'', \vec{v}_o')$$
$$\implies o' = \text{Miss}$$

(G4) If swap flips the side of the sum, the new state must be exclusively owned:
$$\text{swap}(l(\sigma), \vec{v}_i) \rightsquigarrow (\text{Ok}, r(\sigma'), \vec{v}_o) \implies \text{is\_exclusively\_owned } \sigma'$$

We first prove (G1).

(H7) Assume $0_1 \in \Sigma_1'$.

(H8) From composition, (H7) and (H4), $\text{swap}(l(\sigma) \cdot l(0_1), \vec{v}_i) \rightsquigarrow (\text{Ok}, r(\sigma'), \vec{v}_o)$

From (H5) and (H8),

(H9) $\exists \sigma'', o', \vec{v}_o'. \text{swap}(l(\sigma), \vec{v}_i) \rightsquigarrow (o', \sigma'', \vec{v}_o')$

(H10) $o' \neq \text{Miss} \implies o' = o \land \vec{v}_o' = \vec{v}_o \land r(\sigma') = \sigma'' \cdot l(0_1)$

(H11) Because there can only be one action execution for a given state, action and arguments, we know from (H8) that $o' = \text{Ok}$, $\sigma'' = r(\sigma')$, $\vec{v}_o' = \vec{v}_o$.

However, from composition, $r(\sigma') \cdot l(0_1)$ is undefined, thus despite $o' = \text{Ok}$, $r(\sigma') \neq r(\sigma') \cdot l(0_1)$ – our hypothesis (H7) is thus wrong, giving us our goal $0_1 \notin \Sigma_1'$ (G1).
We now prove (G2).

(H12) Assume $0_2 \in \Sigma_2'$.

(H13) Per (H12) and composition, $r(\sigma') \# r(0_2)$, and $r(\sigma') \cdot r(0_2) = r(\sigma')$

Thus per (H4), (H13) and (H6), we have $l(\sigma) \# r(0_2)$. This is however not the case from the definition of composition. Our hypothesis (H12) is thus wrong, giving us our goal (G2).
We may now prove our third goal (G3).

(H14) Let $\text{swap}(l(\sigma), \vec{v}_i) \rightsquigarrow (\text{Ok}, r(\sigma'), \vec{v}_o) \land \sigma_a \preccurlyeq \sigma \land \text{swap}(l(\sigma_a), \vec{v}_i) \rightsquigarrow (o', \sigma'', \vec{v}_o')$.

(H15) Assume $o' \neq \text{Miss}$.

(H16) From the definition of $\preccurlyeq$ and (H14), we have $\exists \sigma_b. \sigma_a \cdot \sigma_b = \sigma$, which also implies $l(\sigma_a) \cdot l(\sigma_b) = l(\sigma)$.

(H17) From (H5), (H14) and (H15), we have $r(\sigma') = \sigma'' \cdot l(\sigma_b)$, and $o' = \text{Ok}$. From composition this is however not possible, as no composition from $l(-)$ yields $r(-)$. Our assumption (H15) is thus false, thus $o' = \text{Miss}$, giving our goal (G3).

We may finally prove our fourth and last goal (G4).

(H18) Let $\text{swap}(l(\sigma), \vec{v}_i) \rightsquigarrow (\text{Ok}, r(\sigma'), \vec{v}_o)$.

(H19) Assume $\neg\text{is\_exclusively\_owned } \sigma'$.

(H20) From (H19), $\exists \sigma''. \sigma'' \neq 0_2 \land \sigma'' \# \sigma'$, thus from composition we also have $r(\sigma'') \# r(\sigma')$.

(H21) From (H6), we must have $\text{swap}(l(\sigma) \cdot r(\sigma''), \vec{v}_i) \rightsquigarrow (\text{Ok}, r(\sigma') \cdot r(\sigma''), \vec{v}_o)$. This is however not possible, as per composition, $l(\sigma) \cdot r(\sigma'')$ is undefined. Thus our hypothesis (H19) is false, giving us our goal (G4).

$\square$