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The Furuta Pendulum

Technical Report

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Abstract The Furuta Pendulum is an example of a complex non-linear system and therefore of big interest in control system theory. It consists of one controllable arm rotating in the horizontal plane and one pendulum uncontrollably moving in the vertical plane, which is attached to the end of this arm.
The non-linearities result from an interplay between gravitational, coriolis, centrifugal forces and friction.
XX We present an overview over it's technical details and proposed algorithms to solve the control problem. XX

Keywords First keyword · Second keyword · More

1 Introduction

Many examples in the field of control engineering like aircraft landing, aircraft stabilizing and many more can be very well modelled by an inverted pendulum (Akhtaruzzaman and Shafie 2010). As a reaction to problems with the limited movement of the cart from the inverted pendulum, the furuta pendulum (also called rotary inverted pendulum) has been developed by Furuta et al.. The advantages of the furuta pendulum are that it needs less space and one moving arm is directly linked to the motor, therefore, the dynamics is less unmodeled thanks to a power transmission mechanism (Furuta et al. 1992).
The furuta pendulum is an underactuated problem, this means that there are two degrees of freedom ($\phi,\ \theta$), but only one arm is directly controlled by the motor. This

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is the one which changes the angle ϕ in the horizontal plane. The other arm called pendulum is attached to the end of the controlled arm and therefore is moved indirectly by it in the vertical plane with angle θ (Spong 1998; Tedrake 2009). The whole system is highly non-linear as a result of an interplay between gravitational, coriolis, friction and centrifugal forces (Izutsu et al. 2008).

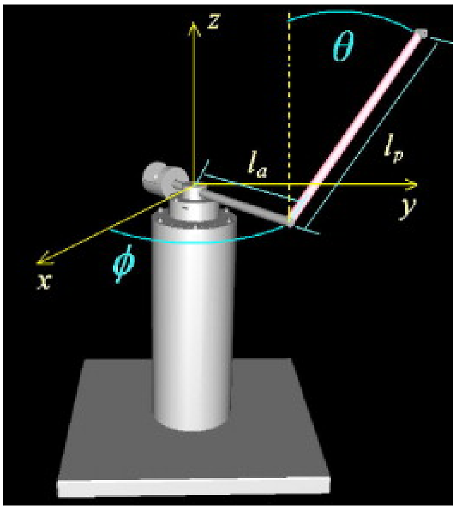

Fig. 1 The furuta pendulum, figure from La Hera et al. (2009)

Finally, there are two different types of control problems. First, bringing the flexible arm from a hanging position to a position which is nearly upright, further called ”swing-up”. Second, if the arm is nearly upright with low enough speed, balancing it to hold this state stable, called ”stabilization”.

1.1 Definitions

The system consists of an arm with length l_a mounted to a DC motor, which is able to apply a torque of τ to it. It has a mass of m_a which is located at l_1 alongside the arm. Another arm with length l_p and mass m_p is attached to the remaining side of the first arm. Both arms have a moment of inertia J_a and J_p respectively. The counter force to the input torque is the viscous damping B_a because of the bearings of the motor. As the pendulum is not controlled directly, the only force it is applied to is damping at the connection to the arm B_p. The angle θ is zero if the pendulum is in an perfectly upright position. The only influence we can take on the system is the voltage V_m we give to the DC motor.

2 Mathematical Modelling[1]

For later derivation of the Lagrangian, the kinetic and potential engergies of the furuta pendulum are needed. Therefore, the kinematics of the pendulum's center of gravity can be described by

$$\begin{cases} x &= l_a \cos\phi - l_p \sin\theta \sin\phi \\ y &= l_a \sin\phi + l_p \sin\theta \cos\phi \\ z &= l_p \cos\theta \end{cases}$$

The relative velocity of the pendulum to the arm can be represented by

$$\begin{cases} \dot{x} &= -l_a \ \sin\phi\dot{\phi} - l_p \ \cos\phi \sin\theta\dot{\phi} - l_p \ \sin\phi \cos\theta\dot{\theta} \\ \dot{y} &= l_a \ \cos\phi\dot{\phi} - l_p \ \sin\phi \sin\theta\dot{\phi} + l_p \ \cos\theta \cos\phi\dot{\theta} \\ \dot{z} &= -l_p \ \sin\theta\dot{\theta} \end{cases}$$

For the energy terms a squared velocity is needed, which is the scalar product of the squared velocities in all directions:

$$\begin{aligned} v^2 &= \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \\ &= (l_a^2 + l_p^2 \ \sin^2\theta)\dot{\phi}^2 + l_p^2\dot{\theta}^2 + 2l_a l_p \dot{\phi}\dot{\theta}\cos\theta \end{aligned}$$

First of all, we will derive the equations of motion through the Euler-Lagrange method. The Lagrangian function can be obtained by the difference of the kinetic and potential energies $L = T - V$, with T as the total sum of the kinetic energies in the system and V the total potential energies.
The only potential energy which is in the system is the one of the pendulum:

$$V_{total} = V_{pendulum} = m_p g z = m_p g l_p \cos\theta$$

Kinetic energy is available both in the arm and in the pendulum and is composed of the the sum of translation energy and rotational energy:

$$\begin{aligned} T_{arm} &= \frac{1}{2}J_a\dot{\phi}^2 \\ T_{pendulum} &= \frac{1}{2}J_p\dot{\theta}^2 + \frac{1}{2}m_p v^2 \\ &= \frac{1}{2}J_p\dot{\theta}^2 + \frac{1}{2}m_p\left((l_a^2 + l_p^2 \ \sin^2\theta)\dot{\phi}^2 + l_p^2\dot{\theta}^2 + 2l_a l_p\dot{\phi}\dot{\theta}\cos\theta\right) \\ T_{total} &= \frac{1}{2}J_a\dot{\phi}^2 + \frac{1}{2}J_p\dot{\theta_0}^2 + \frac{1}{2}m_p\left((l_a^2 + l_p^2 \ \sin^2\theta)\dot{\phi}^2 + l_p^2\dot{\theta}^2 + 2l_a l_p\dot{\phi}\dot{\theta}\cos\theta\right) \end{aligned}$$

From this the Lagrange Function arises:

$$\begin{aligned} L &= T_{total} - V_{total} \\ &= \frac{1}{2}J_a\dot{\phi}^2 + \frac{1}{2}J_p\dot{\theta}^2 + \frac{1}{2}m_p\left((l_a^2 + l_p^2 \ \sin^2\theta)\dot{\phi}^2 + l_p^2\dot{\theta}^2 + 2l_a l_p\dot{\phi}\dot{\theta}\cos\theta\right) \\ &\quad - m_p g l_p \cos\theta \end{aligned}$$

[1] Akhtaruzzaman and Shafie (2010); Furuta et al. (1992); Gäfvert (2016); Özbek and Efe (2010); Zhang and Zhang (2011); Fairus et al. (2013)

Lagrange's Equation follows

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial F}{\partial \dot{q}_i} = 0$$

Thereby, the variables q_i are called generalized coordinates and F is the friction in the system. The friction in the system can be described by $B_p\dot{\theta}$ and $B_a\dot{\phi}$ For the furuta pendulum the generalized coordinates are $\dot{q}(t)^T = \left[\frac{\partial\phi(t)}{\partial t}\ \frac{\partial\theta(t)}{\partial t}\right]$ which leads to

$$\begin{aligned}
\frac{\partial^2 L}{\partial t \partial \dot{\phi}} - \frac{\partial L}{\partial \phi} &= \tau - B_a\dot{\phi} \\
\frac{\partial^2 L}{\partial t \partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= -B_p\dot{\theta}
\end{aligned}$$

The partial derivations of the Lagrangian for the generalized coordinates are:

$$\begin{aligned}
\frac{\partial L}{\partial \phi} &= 0 \\
\frac{\partial L}{\partial \dot{\phi}} &= \dot{\phi}(J_a + m_p(l_a^2 + l_p^2 \sin^2\theta)) + \dot{\theta}(m_p l_a l_p \cos\theta) \\
\frac{\partial^2 L}{\partial t \partial \dot{\phi}} &= \ddot{\phi}(J_a + m_p(l_a^2 + l_p^2 \sin^2\theta)) + l_p^2 m_p \sin(2\theta)\dot{\phi}\dot{\theta} + m_p l_a l_p(\cos\theta\ddot{\theta} - sin\theta\dot{\theta}^2) \\
\frac{\partial L}{\partial \theta} &= \frac{1}{2}\dot{\phi}^2 m_p l_p^2 \sin(2\theta) - m_p l_a l_p sin\theta\dot{\phi}\dot{\theta} + m_p g l_p sin\theta \\
\frac{\partial L}{\partial \dot{\theta}} &= \dot{\theta}(J_p + m_p l_p^2) + \dot{\phi}(m_p l_a l_p \cos\theta) \\
\frac{\partial^2 L}{\partial t \partial \dot{\theta}} &= \ddot{\theta}(J_p + m_p l_p^2) + \ddot{\phi} m_p l_a l_p \cos\theta - \dot{\phi}\dot{\theta} m_p l_a l_p \sin\theta
\end{aligned}$$

With this equations it is possible to derive the equations for the Euler-Lagrange's Equation:

$$\begin{aligned}
\frac{\partial^2 L}{\partial t \partial \dot{\phi}} - \frac{\partial L}{\partial \dot{\phi}} &= \ddot{\phi}(J_a + m_p(l_a^2 + l_p^2 \sin^2\theta)) + l_p^2 m_p \sin(2\theta)\dot{\phi}\dot{\theta} \\
&\quad + m_p l_a l_p(\cos\theta\ddot{\theta} - sin\theta\dot{\theta}^2) \\
\frac{\partial^2 L}{\partial t \partial \dot{\theta}} - \frac{\partial L}{\partial \dot{\theta}} &= \ddot{\theta}(J_p + m_p l_p^2) + \ddot{\phi} m_p l_a l_p \cos\theta - \frac{1}{2}\dot{\phi}^2 m_p l_p^2 \sin(2\theta) - m_p g l_p sin\theta
\end{aligned}$$

Putting this into a matrix form and filling in the friction, this results in the nonlinear equations of motion (EOM) for the furuta pendulum.

$$\begin{bmatrix} J_a + m_p(l_a^2 + l_p^2 \sin^2\theta) & m_p l_a l_p \cos\theta \\ m_p l_a l_p \cos\theta & J_p + m_p l_p^2 \end{bmatrix} \begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \end{bmatrix} +$$
$$\begin{bmatrix} m_p l_p^2 \sin(2\theta)\dot{\theta} - B_a & -m_p l_a l_p \sin\theta\dot{\theta} \\ -\frac{1}{2} m_p l_p^2 \sin(2\theta)\dot{\phi} & -B_p \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ -m_p l_p g \sin\theta \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix}$$

2.1 Linearization of the state-soace model

As all equations of motion contain a part of a trigonometric function the equations are non-linear. There are several different methods to change the non-linear equations to linear ones. For a linearization in the state-space the equation $\dot{x} = Ax + Bu + \tilde{d}$ is used. The linearization is done around the operating point (Rigatos et al. 2018) which is the vertical inverted state and therefore, uses $\theta = 0$ and $\dot{\theta} = 0$. This results in the following equations: The result is an linear form of the equations of motion which are very close to the system's description by the non-linear model, up to the first 25 degree which Kurode et al. showed.

3 Pendulum Control

The goal of controlling the furuta pendulum is to bring it from a hanging position into a vertical upright position. Therefore, it is necessary to generate enough energy to swing the pendulum up in a nearly upright position where the linear region begins (Kurode et al. 2011). If this region is reached, the controller should change to the balancing mode to stabilize the upright position.
To check whether the controller needs to be switched from the swing-up task to the stabilize task, the switching criteria can for example be defined by:

$$\text{switching criteria}^2 = \begin{cases} \text{stabilization} & \begin{cases} |\theta| < \frac{\pi}{9} & \text{and } \dot{\theta} < 2.62 \text{ rad/sec} \\ |E - E_r| < 0.04 \text{ Joule} & \text{and } \dot{\theta} < 2.62 \text{ rad/sec} \end{cases} \\ \text{swing-up} & \text{otherwise} \end{cases}$$

3.1 Swing-Up

There are many different approaches for solving the swing-up task of the furuta pendulum, linear ones which use the linearized equations of motion, non-linear ones and model-free approaches. The classical way solving the swing-up task is an approach with energy control (Seman et al. 2013). Therefore, the sum of kinetic and potential energy of the pendulum is used to model which amount of energy should be added. The energy of the pendulum can be described by

$$E = \frac{1}{2} J_p \theta^2 + m_p g l_p (\cos\theta - 1)$$

which was already done by Åström and Furuta for the normal pendulum. The selection of the state without energy can be made differently, logically it is the upright state because there is no further need to add energy (Seman et al. 2013).
There are two different directions in which the energy can be supplied either the pendulum seems to be "pulled" or "pushed". To define this, the following term is used:

[2] Hamza et al. (2019)

sign($\dot{\theta}\cos\theta$) The velocity helps to find out whether the pendulum starts to swing in the opposite direction, the $\cos\theta$ defines if the pendulum is in the left or in the right half (Awtar et al. 2002). The magnitude of energy which should be provided is selected by an aggressivity factor and the difference of actual and desired system energy $k_a(E-E_0)$. The aggressivity factor handles the amount of input and thereby the number of oscillations which are needed to achieve the swing-up. Often there is also a saturation function which limits the signal to the maximum acceleration of the pendulum. The whole swing-up controller is then implemented by

$$u = sat(k_a(E-E_0))\mathrm{sign}(\dot{\theta}\cos\theta)$$

3.2 Stabilization

There are many different approaches for solving the stabilization problem as well. The most common one is the linear quadratic regulation (LQR) because it guarantees the optimal solution (Hamza et al. 2019). But also the proportional integral derivative (PID) is often used because of its simplicity and robustness and also because of the usage in the industry (Hassanzadeh and Mobayen 2011). Nevertheless, there are a lot of other algorithms from sliding modes (Izutsu et al. 2008) to particle swarm optimization and other evolutionary algorithms (Hassanzadeh and Mobayen 2011).

4 Reinforcement Learning on the Furuta Pendulum

In reinforcement learning the furuta pendulum has only a very small influence on the simulation and experimental research because of the complexity of the problem. Only a few algorithms were tested on it with divers results. Even if the robustness of the stabilization control should improve (Wang et al. 2004).

Hennig compares Gaussian process optimal learner with Kalman filter, TD(λ) and as baseline full information. He used the same baseline function for all algorithms and collated the cumulated loss. Surprisingly, none of the methods could stabilize the swung up pendulum totally, even not the best method, the Gaussian process optimal learner.

Also an artificial neural network (ANN) was used to model the pendulum (Quyen et al. 2012). They worked with the voltage of the motor as input values and the angles of arm and pendulum as the output. Different numbers of hidden neurons were tested with the result that more neurons in the hidden layer led to a smaller mean squared error. The ANN could foresee the angle of the pendulum very well.

A good approach was done with recurrent neural networks and a Genetic Algorithm (Shojaei et al. 2011). They used a recurrent neural network identifier and controller with a proportional integral derivative controller which was responsible for the feedback and the prediction in the derivative action.

5 Conclusion

The furuta pendulum is a very complex control problem which could be solved by a very wide variety of approaches. The control problem splits into two mayor problems the swing-up and the balancing task.

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