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The Furuta Pendulum

Technical Report

Tabea Wilke · **Maximilian Gehrke** · **Yannik Frisch**

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Abstract The Furuta Pendulum is an example of a complex non-linear system and therefore of big interest in control system theory. It consists of one controllable arm rotating in the horizontal plane and one pendulum uncontrollably moving in the vertical plane, which is attached to the end of this arm.
The non-linearities result from an interplay between gravitational, Coriolis and centripetal forces.
XX We present an overview over it's technical details and proposed algorithms to solve the control problem. XX

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1 Introduction

Many examples of the field of control engineering like aircraft landing, aircraft stabilizing and many more can be very well modelled by an inverted pendulum Akhtaruzzaman and Shafie (2010). As a reaction to problems with the limited movement of the cart from the inverted pendulum, the furuta pendulum (also called rotary inverted pendulum) has been developed by Furuta et al.. The advantages of the furuta pendulum are that needs less space and one moving arm is directly linked to the motor, therefore, the dynamics is less unmodeled thanks to a power transmission mechanism Furuta et al. (1992). The furuta pendulum is an underactuated problem, this means that there two degrees of freedom ($\phi,\ \theta$), but only one arm is directly controlled by the motor. This is the one which changes the angle ϕ in the horizontal plane. The

F. Author
first address
Tel.: +123-45-678910
Fax: +123-45-678910
E-mail: fauthor@example.com

S. Author
second address

other arm called pendulum is attached to the end of the controlled arm and therefore is moved indirectly by it in the vertical plane with angle θ Spong (1998); Tedrake (2009). The whole system is highly non-linear as a result of an interplay between gravitational, coriolis, friction and centrifugal forces Izutsu et al. (2008). Finally,

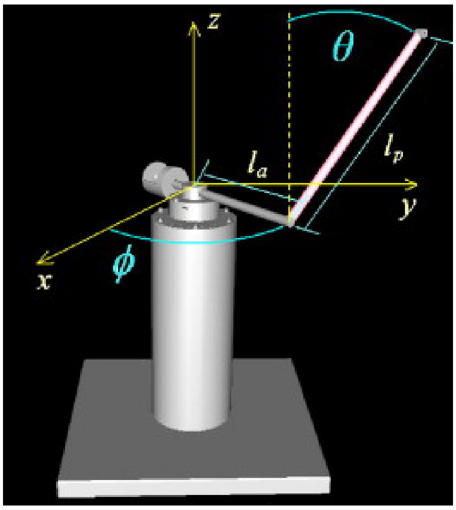

Fig. 1 The furuta pendulum, figure from La Hera et al. (2009)

there are two different types of control problems. First, bringing the flexible arm from a hanging position to a position which is nearly upright, further called "swing-up". Second, if the arm is nearly upright with low enough speed, balancing it to hold this state stable, called "stabilization".

1.1 Definitions

The system consists of an arm with length l_r mounted to a DC motor, which is able to apply a torque of τ to it. It has a mass of m_r which is located at l_1 alongside the arm. Another arm with length l_p and mass m_p is attached to the remaining side of the first arm. Both arms have a moment of inertia J_r and J_p respectively. The counter force to the input torque is the viscous damping B_r because of the bearings of the motor. As the pendulum is not controlled directly, the only force it is applied to is damping at the connection to the arm B_p. The angle θ is zero if the pendulum is in an perfectly upright position. The only influence we can take on the system is the voltage V_m we give to the DC motor.

2 Mathematical Modelling[1]

For later derivation of the Lagrangian, the kinetic and potential engergies of the furuta pendulum are needed. Therefore, the kinematics of the pendulums center of gravity can be described by

$$\begin{cases} x &= l_r \cos\phi - l_p \sin\theta \sin\phi \\ y &= l_r \sin\phi + l_p \sin\theta \cos\phi \\ z &= l_p \cos\theta \end{cases}$$

The relative velocity of the pendulum to the arm can be represented by

$$\begin{cases} \dot{x} &= -l_a \ \sin\phi \dot{\phi} - l_p \ \cos\phi \sin\theta \dot{\phi} - l_p \ \sin\phi \cos\theta \dot{\theta} \\ \dot{y} &= l_a \ \cos\phi \dot{\phi} - l_p \ \sin\phi \sin\theta \dot{\phi} + l_p \ \cos\theta \cos\phi \dot{\theta} \\ \dot{z} &= -l_p \ \sin\theta \dot{\theta} \end{cases}$$

For the energy terms a squared velocity is needed, which is the scalar product of the squared velocities in all directions:

$$\begin{aligned} v^2 &= \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \\ &= (l_a^2 + l_p^2 \ \sin^2\theta)\dot{\phi}^2 + l_p^2\dot{\theta}^2 + 2l_a l_p \dot{\phi}\dot{\theta}\cos\theta \end{aligned}$$

First of all, we will derive the equations of motion through the Euler-Lagrange method. The Lagrangian function can be obtained by the difference of the kinetic and potential energies $L = T - V$, with T as the total sum of the kinetic energies in the system and V the total potential energies. The only potential energy which is in the system is the one of the pendulum:

$$V_{total} = V_{pendulum} = m_p g z = m_p g l_p \cos\theta$$

Kinetic energy is available both in the arm and in the pendulum and is composed of the the sum of translation energy and rotational energy:

$$\begin{aligned} T_{arm} &= \frac{1}{2} J_r \dot{\phi}^2 \\ T_{pendulum} &= \frac{1}{2} J_p \dot{\theta}^2 + \frac{1}{2} m_p v^2 \\ &= \frac{1}{2} J_p \dot{\theta}^2 + \frac{1}{2} m_p \left((l_a^2 + l_p^2 \ \sin^2\theta)\dot{\phi}^2 + l_p^2\dot{\theta}^2 + 2l_a l_p \dot{\phi}\dot{\theta}\cos\theta \right) \\ T_{total} &= \frac{1}{2} J_r \dot{\phi}^2 + \frac{1}{2} J_p \dot{\theta_0}^2 + \frac{1}{2} m_p \left((l_a^2 + l_p^2 \ \sin^2\theta)\dot{\phi}^2 + l_p^2\dot{\theta}^2 + 2l_a l_p \dot{\phi}\dot{\theta}\cos\theta \right) \end{aligned}$$

From this the Lagrange Function arises:

$$\begin{aligned} L &= T_{total} - V_{total} \\ &= \frac{1}{2} J_r \dot{\phi}^2 + \frac{1}{2} J_p \dot{\theta}^2 + \frac{1}{2} m_p \left((l_a^2 + l_p^2 \ \sin^2\theta)\dot{\phi}^2 + l_p^2\dot{\theta}^2 + 2l_a l_p \dot{\phi}\dot{\theta}\cos\theta \right) \\ &\quad - m_p g l_p \cos\theta \end{aligned}$$

[1] Akhtaruzzaman and Shafie (2010); Furuta et al. (1992); Gäfvert (2016); Özbek and Efe (2010); Zhang and Zhang (2011); Fairus et al. (2013)

Lagrange's Equation follows

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial F}{\partial \dot{q}_i} = 0$$

Thereby, the variables q_i are called generalized coordinates and F is the friction in the system. The friction in the system can be described by $B_p\dot{\theta}$ and $B_r\dot{\phi}$ For the furuta pendulum the generalized coordinates are $\dot{q}(t)^T = \left[\frac{\partial\phi(t)}{\partial t}\ \frac{\partial\theta(t)}{\partial t}\right]$ which leads to

$$\begin{aligned}
\frac{\partial^2 L}{\partial t \partial \dot{\phi}} - \frac{\partial L}{\partial \phi} &= \tau - B_r\dot{\phi} \\
\frac{\partial^2 L}{\partial t \partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= -B_p\dot{\theta}
\end{aligned}$$

The partial derivations of the Lagrangian for the generalized coordinates are:

$$\begin{aligned}
\frac{\partial L}{\partial \phi} &= 0 \\
\frac{\partial L}{\partial \dot{\phi}} &= \dot{\phi}(J_r + m_p(l_a^2 + l_p^2 \sin^2\theta)) + \dot{\theta}(m_p l_a l_p \cos\theta) \\
\frac{\partial^2 L}{\partial t \partial \dot{\phi}} &= \ddot{\phi}(J_r + m_p(l_a^2 + l_p^2 \sin^2\theta)) + l_p^2 m_p \sin(2\theta)\dot{\phi}\dot{\theta} + m_p l_a l_p(\cos\theta\ddot{\theta} - sin\theta\dot{\theta}^2) \\
\frac{\partial L}{\partial \theta} &= \frac{1}{2}\dot{\phi}^2 m_p l_p^2 \sin(2\theta) - m_p l_a l_p sin\theta\dot{\phi}\dot{\theta} + m_p g l_p sin\theta \\
\frac{\partial L}{\partial \dot{\theta}} &= \dot{\theta}(J_p + m_p l_p^2) + \dot{\phi}(m_p l_a l_p \cos\theta) \\
\frac{\partial^2 L}{\partial t \partial \dot{\theta}} &= \ddot{\theta}(J_p + m_p l_p^2) + \ddot{\phi} m_p l_a l_p \cos\theta - \dot{\phi}\dot{\theta} m_p l_a l_p \sin\theta
\end{aligned}$$

With this equations it is possible to derive the equations for the Euler-Lagrange's Equation:

$$\begin{aligned}
\frac{\partial^2 L}{\partial t \partial \dot{\phi}} - \frac{\partial L}{\partial \dot{\phi}} &= \ddot{\phi}(J_r + m_p(l_a^2 + l_p^2 \sin^2\theta)) + l_p^2 m_p \sin(2\theta)\dot{\phi}\dot{\theta} \\
&\quad + m_p l_a l_p(\cos\theta\ddot{\theta} - sin\theta\dot{\theta}^2) \\
\frac{\partial^2 L}{\partial t \partial \dot{\theta}} - \frac{\partial L}{\partial \dot{\theta}} &= \ddot{\theta}(J_p + m_p l_p^2) + \ddot{\phi} m_p l_a l_p \cos\theta - \frac{1}{2}\dot{\phi}^2 m_p l_p^2 \sin(2\theta) - m_p g l_p sin\theta
\end{aligned}$$

Putting this into a matrix form and filling in the friction, this results in the nonlinear equations of motion (EOM) for the furuta pendulum.

$$\begin{bmatrix} J_r + m_p(l_a^2 + l_p^2\sin^2\theta) & m_p l_a l_p \cos\theta \\ m_p l_a l_p \cos\theta & J_p + m_p l_p^2 \end{bmatrix} \begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \end{bmatrix} +$$

$$\begin{bmatrix} m_p l_p^2 \sin(2\theta)\dot{\theta} - B_r & -m_p l_a l_p \sin\theta\dot{\theta} \\ -\frac{1}{2} m_p l_p^2 \sin(2\theta)\dot{\phi} & -B_p \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ -m_p l_p g \sin\theta \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix}$$

2.1 Linearization

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