MATEMATIK 1

Konya Teknik Üniversitesi
Mühendislik ve Doğa Bilimleri Fakültesi
Mühendislik Temel Bilimleri Bölümü

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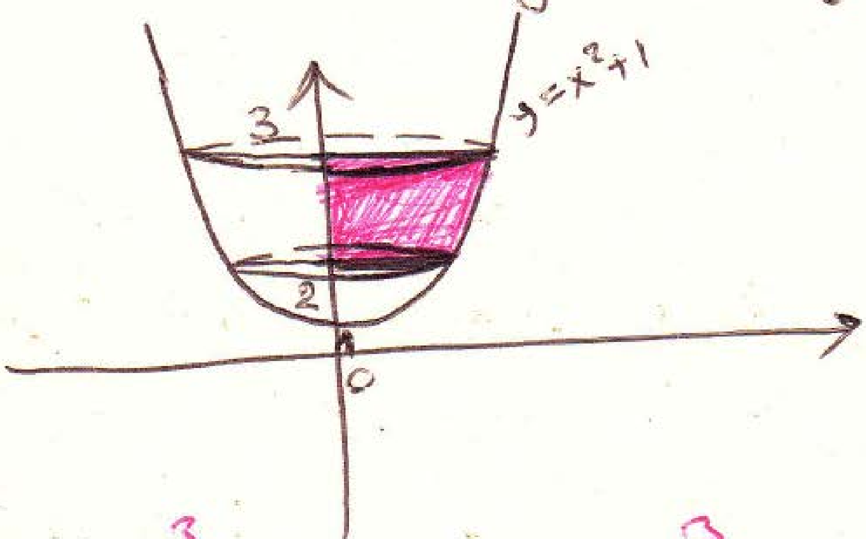
Örnek 4. $y = x^2 + 1$ eğrisinin sağ yarısı, $y=2$, $y=3$ doğruları ve y ekseni tarafından sınırlanan bölgenin y ekseni etrafında döndürülmesiyle oluşan cismin hacmini bulun



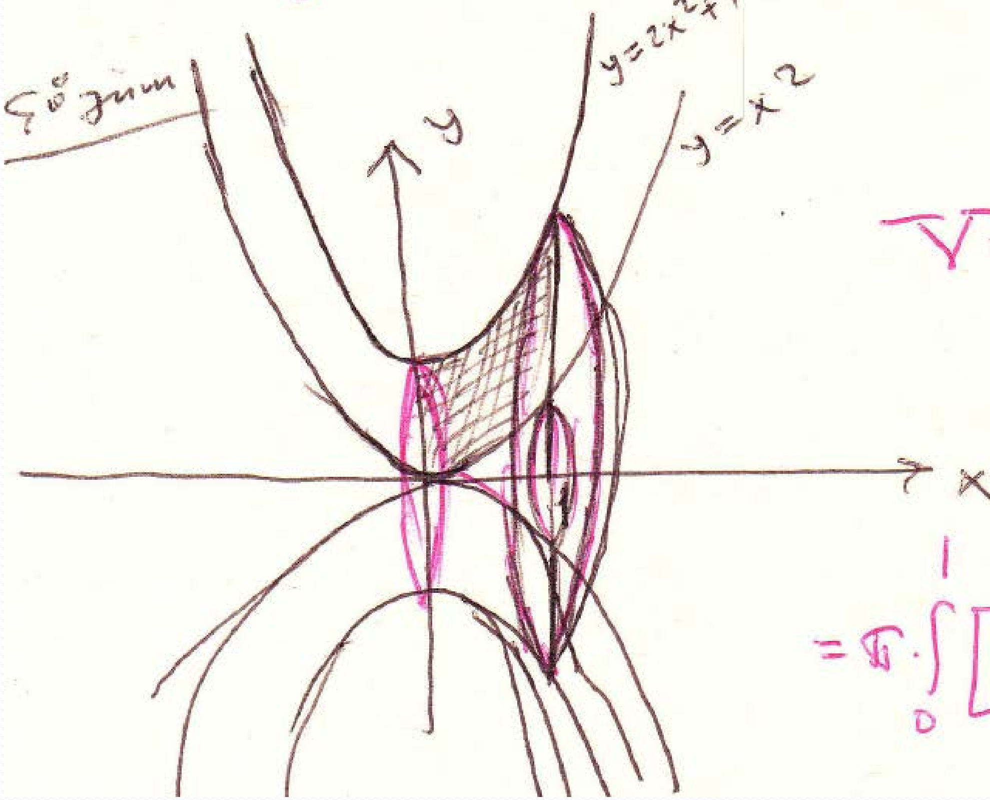
$y = x^2+1 \Rightarrow x^2 = y-1$.

$x = \sqrt{y-1}$ grafiğin oy ekseninin sağ tarafında kalan kısmının denklemidir.

$$V = \pi \int_2^3 \left(\sqrt{y-1}\right)^2 dy = \pi \int_2^3 (y-1)\, dy = \pi \left(\frac{y^2}{2} - y \right) \Big|_2^3 = \pi \left(\frac{3}{2} - 0 \right) = \frac{3\pi}{2}$$

Örnek 5. $y = 2x^2+1$ ve $y = x^2$ eğrileri $x=0$, $x=1$ doğruları tarafından sınırlanan bölgenin Ox ekseni etrafında döndürülmesi ile meydana gelen cismin hacmini bulun

Çözüm



$$V = \pi \int_0^1 \left[(2x^2+1)^2 - (x^2)^2 \right] dx$$

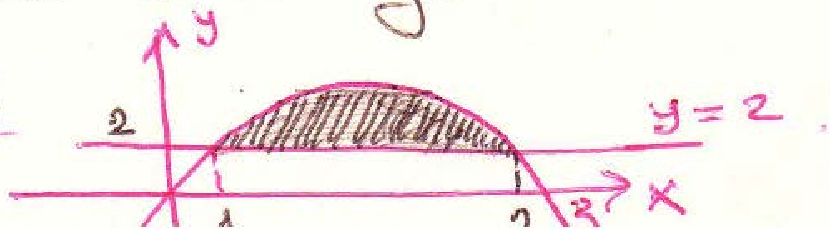
$$= \pi \cdot \int_0^1 \left[4x^4 + 4x^2 + 1 - x^4 \right] dx =$$

$$= \pi \cdot \int_0^1 (3x^4+4x^2+1)\,dx = \pi\left[\frac{3}{5}x^5+\frac{4}{3}x^3+x\right]\Big|_0^1 =$$

$$= \pi\cdot\left(\frac{3}{5}+\frac{4}{3}+1\right) = \pi\cdot\frac{9+20+15}{15} = \frac{44}{15}\pi \text{ br}^3 \text{ olur}$$

Örnek: 6 $y=-x^2+3x$ ve $y=2$ eğri ve doğrusu tarafından sınırlanan bölgenin $y=2$ doğrusu etrafında döndürülmesiyle oluşan cismin hacmini bulunuz.

Çözüm:



$-x^2+3x=2 \Rightarrow$

$x^2-3x+2=0 \Rightarrow x_1=1,$

$x_2=2$

Obținem,

$$V = \pi \int_a^b (f(x) - k)^2 dx \quad \text{din}$$

$$V = \pi \int_1^2 \left[(-x^2 + 3x) - 2\right]^2 dx =$$

$$= \pi \int_1^2 \left[(-x^2+3x)^2 - 4(-x^2+3x) + 4\right] dx =$$

$$= \pi \int_1^2 \left[x^4 - 6x^3 + 9x^2 + 4x^2 - 12x + 4\right] dx =$$

$$= \pi \cdot \int_1^2 \left(x^4 - 6x^3 + 13x^2 - 12x + 4\right) dx =$$

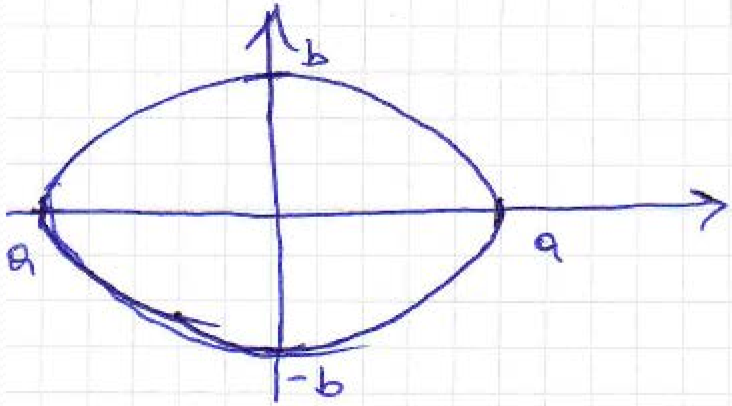
$$= \pi \cdot \left[\frac{x^5}{5} - \frac{6}{4} x^4 + \frac{13}{3} x^3 - 6x^2 + 4x \right] \Big|_1^2 =$$

$$= \pi \left[\frac{32}{5} - \frac{3 \cdot 16}{2} + \frac{13 \cdot 8}{3} - 24 + 8 - \frac{1}{5} + \frac{3}{2} - \frac{13}{3} + 6 - 4 \right]$$

$$= \pi \cdot \left[\frac{192 + 13 \cdot 80 - 6 + 45 - 130}{30} - 38 \right] =$$

$$= \pi \cdot \left[\frac{1142 - 38 \cdot 30}{30} \right] = \pi \cdot \frac{1142 - 1140}{30} = \frac{2\pi}{30} = \frac{\pi}{15} \text{ br}$$

Örnek: Elipsoidin hacmini veren formülü bulunuz. $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ elips} \right)$.



$$y^2 = b^2 \cdot \left[1 - \frac{x^2}{a^2} \right].$$

$$y = \pm b \sqrt{1 - \frac{x^2}{a^2}}$$

$$y = \pm \frac{b}{a} \cdot \sqrt{a^2 - x^2};$$

$$y = \frac{b}{a} \sqrt{a^2 - x^2};$$

$$V = \pi \cdot \int_{-a}^{a} \frac{b^2}{a^2} (a^2 - x^2)\, dx = \frac{\pi b^2}{a^2} \int_{-a}^{a} (a^2 - x^2)\, dx =$$

$$= \frac{\pi b^2}{a^2}\left[a^2 x - \frac{x^3}{3}\right]\Big|_{-a}^{a} = \frac{\pi b^2}{a^2}\left[2a^2 \cdot a - \frac{a^3}{3} - \frac{a^3}{3}\right] =$$

$$= \pi b^2 \cdot a\left[2 - \frac{2}{3}\right] = \frac{4}{3}\pi \cdot b^2 \cdot a \quad br^3 \text{. olcer}$$

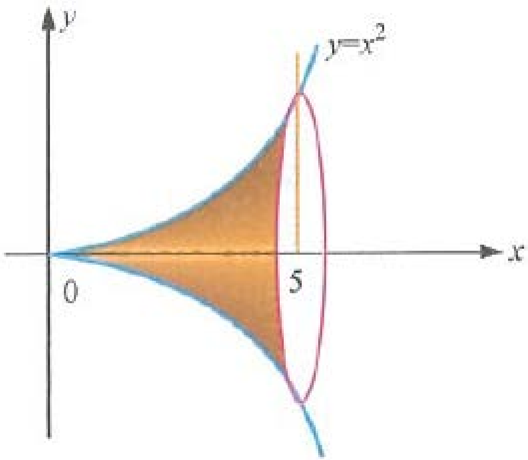
$$\boxed{V = \frac{4}{3}\pi \cdot b^2 \cdot a \quad br^3}$$

Aşağıda denklemleri verilen eğri ve doğrular tarafından sınırlanan bölgelerin Ox– ekseni etrafında döndürülmesiyle oluşan dönel cisimlerin hacimlerini bulunuz.

a.	$y = x^2$,	$x = 5$,	$y = 0$	
b.	$y = 1 - x^2$,	$y = 0$,		
c.	$y = x^3$,	$y = 0$,	$x = 1$,	$x = 2$
ç.	$y = e^x$,	$x = 0$,	$x = 1$	
d.	$y = \sqrt{a^2 - x^2}$,	$y = 0$		
e.	$y = \sqrt{x}$,	$y = 0$,	$x = 1$	
f.	$xy = 4$,	$x = 1$,	$x = 4$,	$y = 0$

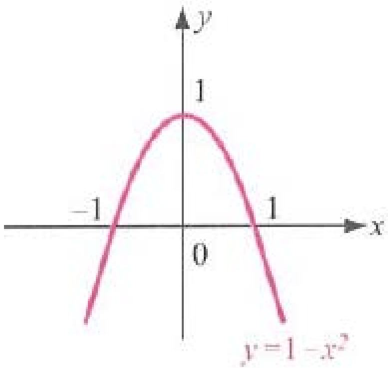
Çözüm

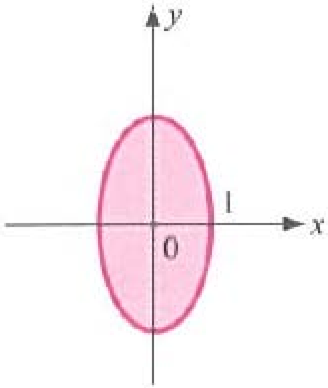
a.



$$V=\pi\int_0^5 y^2\,dx=\pi\int_0^5 x^4\,dx=\pi\frac{x^5}{5}\Big|_0^5=625\pi$$

b.



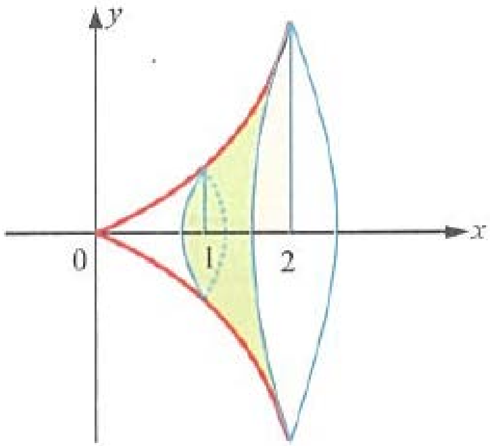


$$V=\pi\int_{-1}^{1} y^2\,dx=\pi\int_{-1}^{1}(1-x^2)^2\,dx$$

$$=2\pi\int_0^1(1-2x^2+x^4)\,dx$$

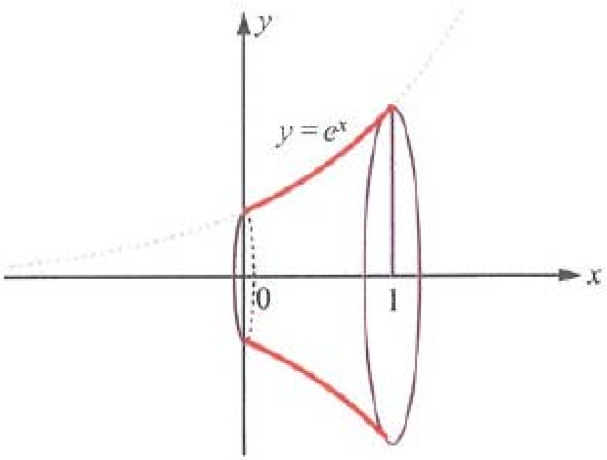
$$=2\pi(x-\frac{2}{3}x^3+\frac{1}{5}x^5)\Big|_0^1=\frac{16}{15}\pi$$

c.



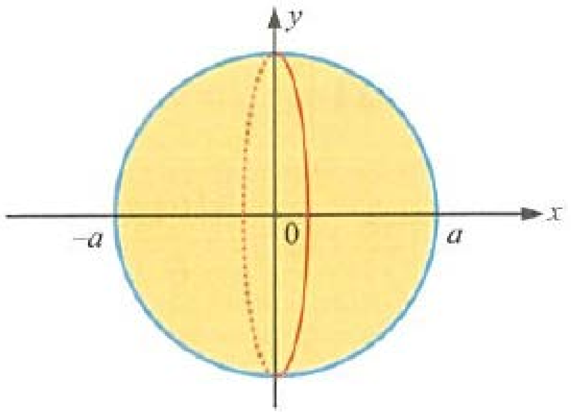
$$V = \pi\int_1^2 y^2\,dx = \pi\int_1^2 (x^3)^2\,dx = \pi\frac{x^7}{7}\Bigg|_1^2 = \frac{127}{7}\pi$$

ç.



$$V = \pi\int_0^1 y^2\,dx = \pi\int_0^1 (e^x)^2\,dx = \pi\frac{e^{2x}}{2}\Bigg|_0^1 = \frac{e^2-1}{2}\pi$$

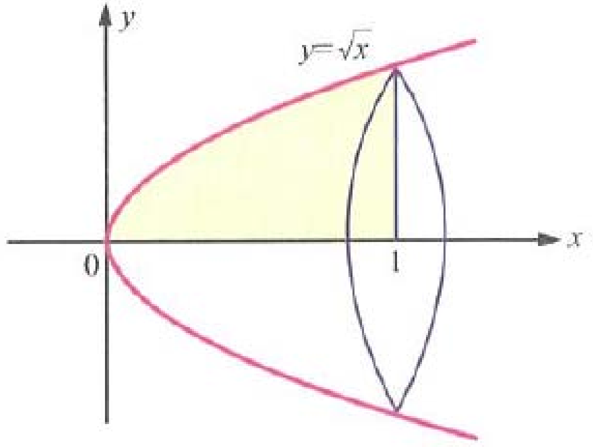
d.



$$V=\pi\int_{-a}^{a} y^2\,dx=\pi\int_{-a}^{a}(a^2-x^2)\,dx=\pi\left(a^2x-\frac{1}{3}x^3\right)\Big|_{-a}^{a}$$

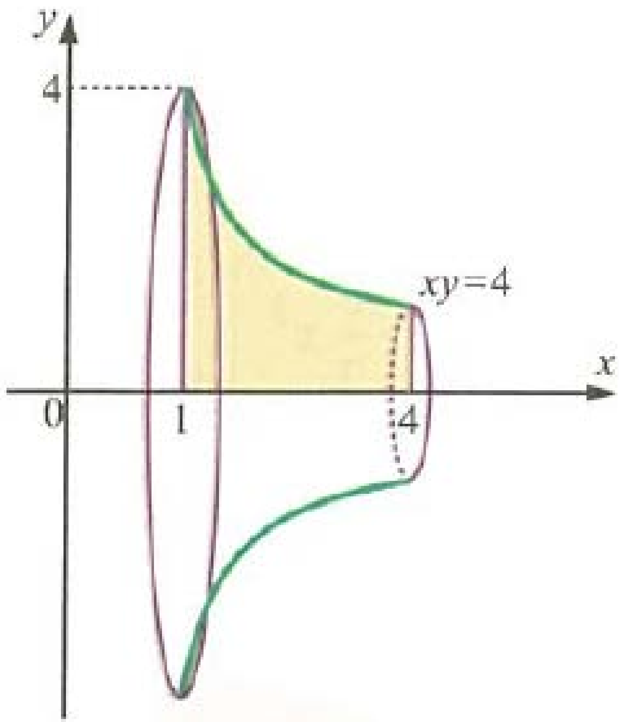
$$=\pi\left(a^3-\frac{a^3}{3}+a^3-\frac{1}{3}a^3\right)=\frac{4}{3}\pi a^3$$

e.



$$V=\pi\int_{0}^{1} y^2\,dx=\pi\int_{0}^{1} x\,dx=\frac{\pi}{2}$$

f.



$$V=\pi\int_1^4\left(\frac{4}{x}\right)^2 dx=16\pi\int_1^4 x^{-2}\,dx=16\pi\left(-\frac{1}{x}\right)\Big|_1^4=12\pi$$

8. Aşağıda denklemleri verilen eğri ve doğrular arasında kalan bölgelerin Ox– ekseni etrafında döndürülmesiyle oluşan dönel cisimlerin hacmini bulunuz.

a. $y = x,\quad y = \sqrt{x}$

b. $y = 2x,\quad y = x,\quad x = 1$

c. $y = x^2 + 3,\quad y = 4,$

d. $y = x^2 + 1\quad y = x + 3$

e. $y = 4 - x^2,\quad y = 2 - x$

f. $y = \sec x,\quad y = \tan x,\quad x = 0,\quad x = 1$

g. $y = 3x - x^2,\quad y = x$

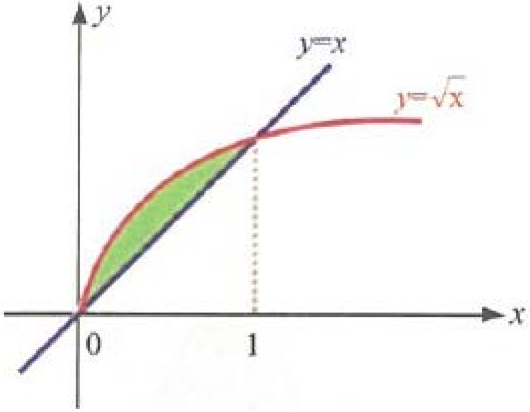
h. $y^2 = 2px,\quad x = h$

ı. $(y - a)^2 = ax,\quad x = 0,\quad y = 2a$

i. $y = x^2 + 2,\quad y = -x^2 + 10$

Çözüm

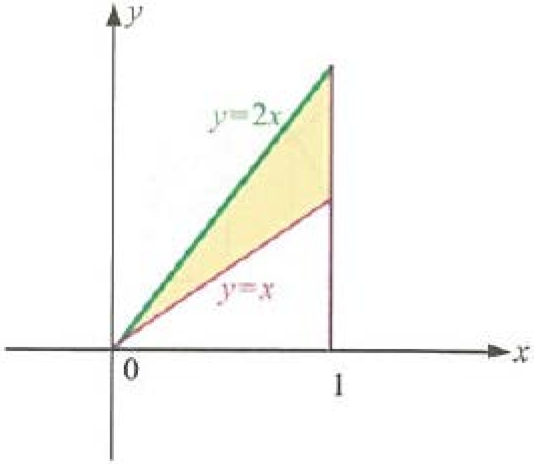
a.



$$V=\pi\int_0^1\left[(\sqrt{x})^2-x^2\right]dx=\pi\int_0^1(x-x^2)\,dx$$

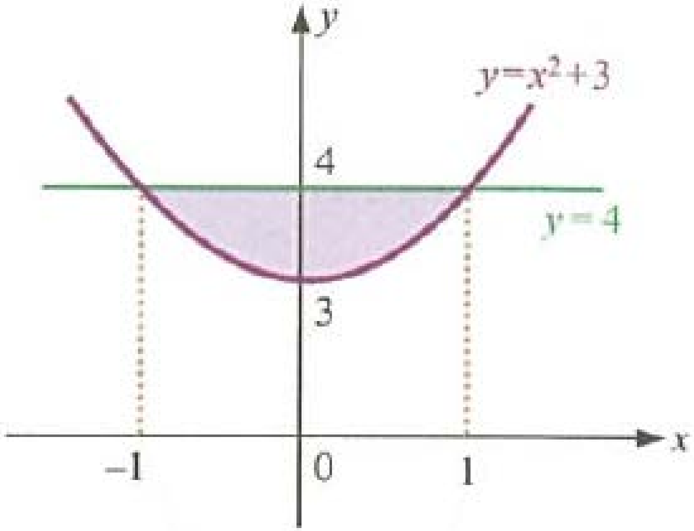
$$=\pi\left(\frac{1}{2}x^2-\frac{1}{3}x^3\right)\Bigg|_0^1=\frac{\pi}{6}$$

b.



$$V=\pi\int_0^1\left[(2x)^2-x^2\right]dx=\pi\int_0^1 3x^2\,dx=\pi x^3\ \Big|_0^1=\pi$$

c.

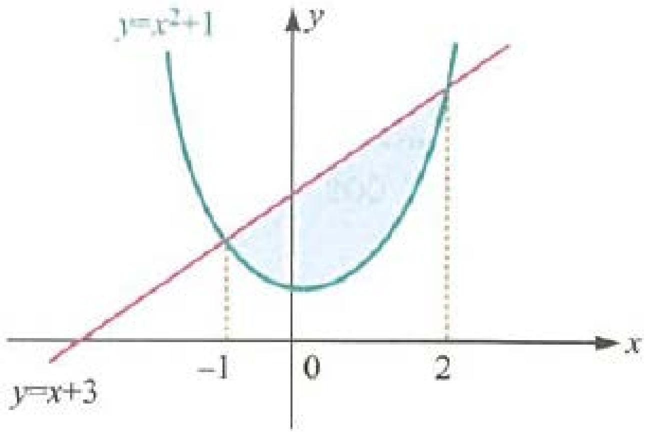


$$x^2+3=4 \Rightarrow x_1=-1, \quad x_2=1$$

$$V=\pi\int_{-1}^{1}\left[4^2-(x^2+3)^2\right]dx=2\pi\int_{0}^{1}(7-6x^2-x^4)\,dx$$

$$=2\pi(7x-2x^3-\frac{1}{5}x^5)\Big|_0^1=\frac{48}{5}\pi$$

d.



$$x^2+1=x+3 \Rightarrow x^2-x-2=0$$

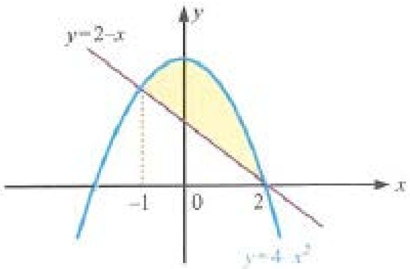
$$\Rightarrow x_1=-1,\ x_2=2$$

$$V=\pi\int_{-1}^{2}\left[(x+3)^2-(x^2+1)^2\right]dx$$

$$=\pi\int_{-1}^{2}(-x^4-x^2+6x+8)\ dx$$

$$=\pi\left(-\frac{1}{5}x^5-\frac{1}{3}x^3+3x^2+8x)\right)\Big|_{-1}^{2}=\frac{117}{5}\pi$$

e.



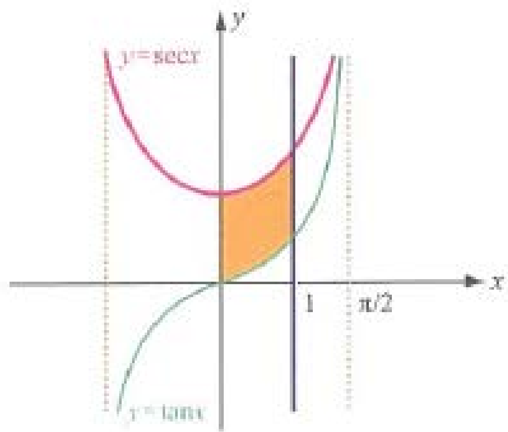
$$4-x^2=2-x \;\Rightarrow\; x^2-x-2=0$$

$$(x-2)(x+1)=0 \;\Rightarrow\; x_1=-1,\;\; x_2=2$$

$$V=\pi\int_{-1}^{2}\left[(4-x^2)^2-(2-)^2\right]dx=\pi\int_{-1}^{2}(x^4-9x^2+4x+12)\,dx$$

$$=\pi\left(\frac{x^5}{5}-3x^3+2x^2+12x\right)\Big|_{-1}^{2}=\frac{108}{5}\pi$$

f.

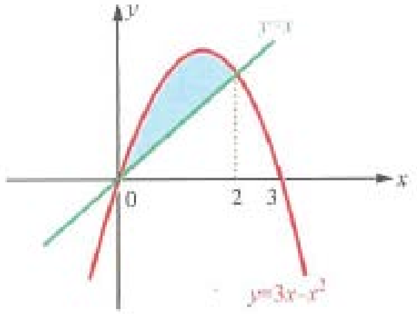


$$V=\pi\int_{0}^{1}\left[(\sec x)^2-(\tan x)^2\right]dx$$

$$=\pi\int_{0}^{1}\left(\frac{1}{\cos^2 x}-\frac{\sin^2 x}{\cos^2 x}\right)dx=\pi\int_{0}^{1}\frac{\cos^2 x}{\cos^2 x}\,dx$$

$$=\pi\int_{0}^{1}1\cdot dx=\pi x\,\Big|_{0}^{1}=\pi$$

g.



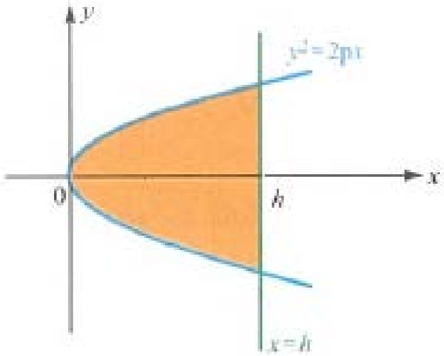
$$3x - x^2 = x \Rightarrow x^2 - 2x = 0$$

$$x(x-2) = 0 \Rightarrow x_1 = 0, \quad x_2 = 2$$

$$V = \pi\int_0^2 \left[(3x-x^2)^2 - x^2\right]dx = \pi\int_0^2 (x^4 - 6x^3 + 8x^2)\,dx$$

$$= \pi\left(\frac{x^5}{5} - \frac{3}{2}x^4 + \frac{8}{3}x^3\right)\Big|_0^2 = \frac{56}{15}\pi$$

h.



$$V = \pi\int_0^h y^2\,dx = \pi\int_0^h 2px\;dx = 2\pi p.\frac{x^2}{2}\Big|_0^h = \pi p h^2$$

Aşağıdaki eğri ve doğrular tarafından sınırlanan bölgelerin Oy– ekseni etrafında döndürülmesiyle meydana gelen dönel cismin hacmini bulunuz.

a. $y = x^2$, $x = 0$, $y = 4$

b. $x = y\,(3 - y)$, $y = 0$,

c. $y = x^3$, $x = -1$, $x = 1$, $y = 0$

d. $y = e^x$, $x = 0$, $x = 1$, $y = 0$

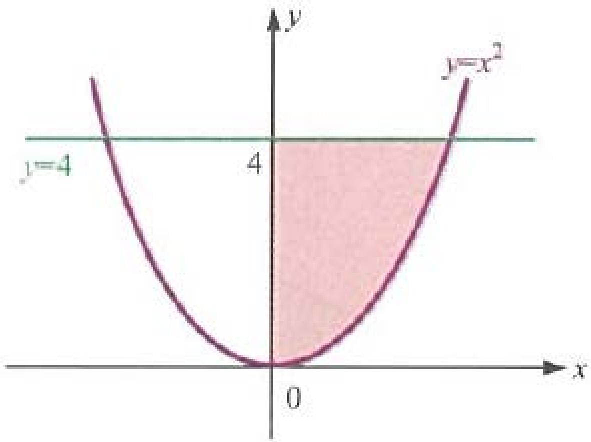
e. $x = \sqrt{4 - y}$, $x = 0$ $y = 0$

f. $x = 1 - y^2$, $x = 0$

g. $x = y^{3/2}$, $x = 0$, $y = 0$

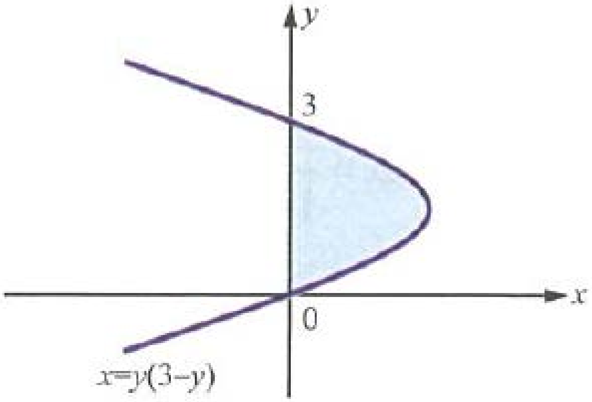
Çözüm

a.



$$V=\pi\int_0^4 x^2\,dy=\pi\int_0^4 y\,dy=\frac{\pi}{2}y^2\Big|_0^4=8\pi$$

b.

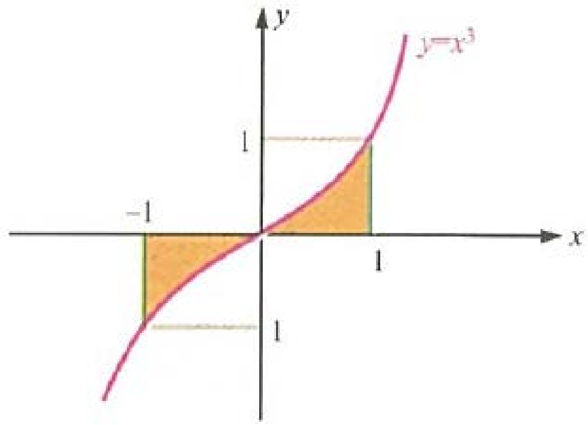


$$V=\pi\int_0^3 x^2\,dy=\pi\int_0^3 y^2(3-y)^2\,dy$$

$$=\pi\int_0^3 (9y^2-6y^3+y^4)\,dy=\pi(3y^3-\frac{3}{2}y^4+\frac{1}{5}y^5)\Big|_0^3$$

$$=\pi(81-\frac{3}{2}.3^4+\frac{1}{5}3^5)=\frac{81}{10}\pi$$

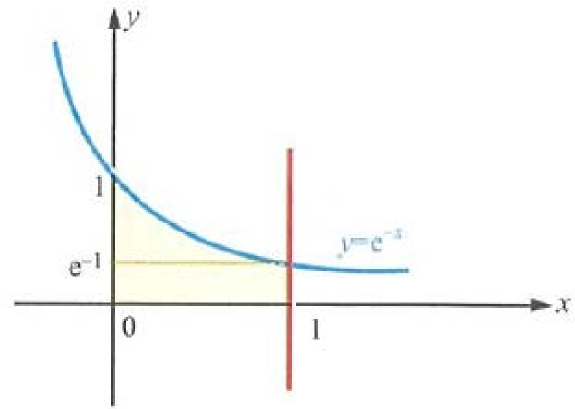
c.



$$V = 2\pi\int_0^1 (1^2 - x^2)\,dy = 2\pi\int_0^1 (1 - y^{2/3})\,dy$$

$$= 2\pi\left(y - \frac{3}{5}y^{5/3} \right)\Big|_0^1 = 2\pi\left(1 - \frac{3}{5}\right) = \frac{4}{5}\pi$$

d.



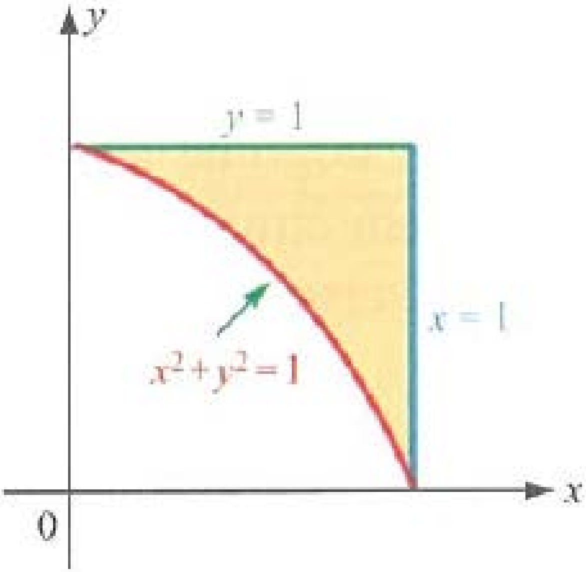
$$V = \pi\int_0^{e^{-1}} 1^2\,dy + \pi\int_{e^{-1}}^{1} x^2\,dy = \pi y\Big|_0^{1/e} + \pi\int_{1/e}^{1} (\ln y)^2\,dy$$

$$= \frac{\pi}{e} + \pi\left[y\ln^2 y\ \Big|_{e^{-1}}^{1} - \int_{e^{-1}}^{1} 2y\ln y \cdot \frac{1}{y}\,dy \right]$$

$$= \frac{\pi}{e} + \pi\left[-\frac{1}{e} - 2\int_{e^{-1}}^{1} \ln y\ dy \right]$$

$$= -2\pi\,(y\ln y - y)\Big|_{e^{-1}}^{1} = 2\pi\,\frac{e-2}{e}$$

11.



Yukarıdaki taralı bölgenin Oy– ekseni etrafında döndürülmesiyle oluşan cismin hacmini bulunuz.

Çözüm

$$V = \pi\int_0^1 \left[1-(1-y^2)\right] dy = \pi\int_0^1 y^2\, dy = \pi\frac{y^3}{3}\Bigg|_0^1 = \frac{\pi}{3}$$

13. $y=\sqrt{x}, \quad y=2, \quad x=0$

tarafından sınırlanan bölgenin

a. Ox– ekseni

b. Oy– ekseni

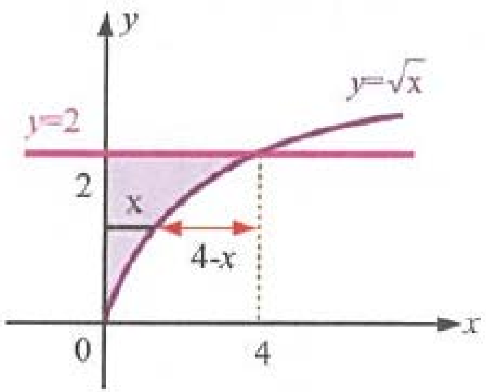
c. $y=2$ doğrusu

d. $x=4$ doğrusu

etrafında döndürülmesiyle meydana gelen cismin hacmini bulunuz.

Çözüm

a.



$$V_1=\pi\int_0^4(4-x)dx=\pi\left(4x-\frac{1}{2}x^2\right)\Big|_0^4=\pi(16-8)=8\pi$$

b. $$V_2 = \pi\int_0^2 (x^2)\,dy = \pi\int_0^2 y^4\,dy = \pi\left(\frac{y^5}{5}\right)\Bigg|_0^2 = \frac{32}{5}\pi$$

c. $$V_3 = \pi\int_0^4 (2-\sqrt{x})^2\,dx = \pi\int_0^4 (4-4x^{1/2}+x)\,dx$$

$$= \pi\left(4x - \frac{8}{3}x^{3/2} + \frac{1}{2}x^2\right)\Bigg|_0^4 = \pi\left(16 - \frac{64}{3} + 8\right) = \frac{8}{3}\pi$$

d. $$V_4 = \pi\int_0^2 \left[4^2 - (4-x)^2\right]dy$$

$$= \pi\int_0^2 \left[16-(4-y^2)^2\right]dy = \pi\int_0^2 (8y^2 - y^4)\,dy$$

$$= \pi\left(\frac{8}{3}y^3 - \frac{1}{5}y^5\right)\Bigg|_0^2 = \frac{224}{15}\pi$$

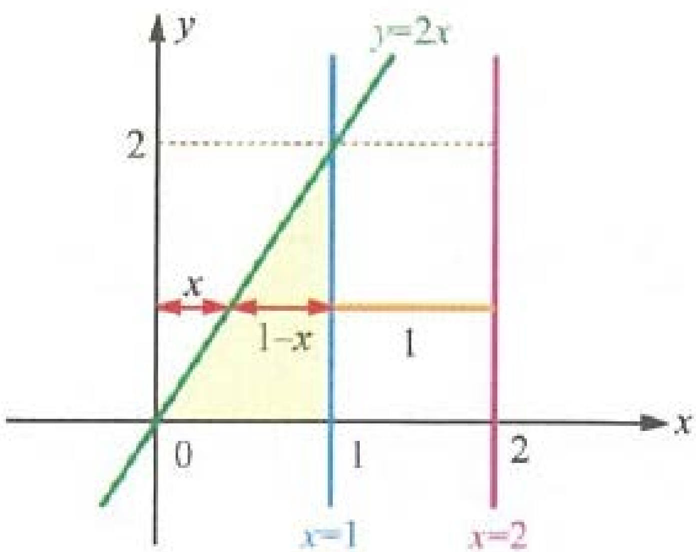
14. $y = 2x$, $y = 0$, $x = 1$ doğruları tarafından sınırlanan bölgenin

a. $x = 1$ doğrusu

b. $x = 2$ doğrusu

etrafında döndürülmesiyle meydana gelen dönel cismin hacmini bulunuz.

Çözüm



a. $$V=\pi\int_0^2\left(1-\frac{y}{2}\right)^2 dy=\pi\int_0^2(1-y+\frac{1}{4}y^2)\,dy$$

$$=\pi\left(y-\frac{1}{2}y^2+\frac{1}{12}y^3\right)\Big|_0^2=\frac{2}{3}\pi$$

b. $$V=\pi\int_0^2\left[(2-x)^2-1^2\right]dy=\pi\int_0^2\left[(2-\frac{y}{2})^2-1\right]dy$$

$$=\pi\int_0^2(3-2y+\frac{1}{4}y^2)\,dy=\pi(3y-y^2+\frac{1}{12}y^3)\Big|_0^2=\frac{8}{3}\pi$$

16. $y = x^2 + 1$ eğrisi ile bu eğriye $x = 1$ apsisli noktasından çizilen teğet ve $x = 0$ doğrusu tarafından sınırlanan bölgenin

a. Ox– ekseni

b. Oy– ekseni

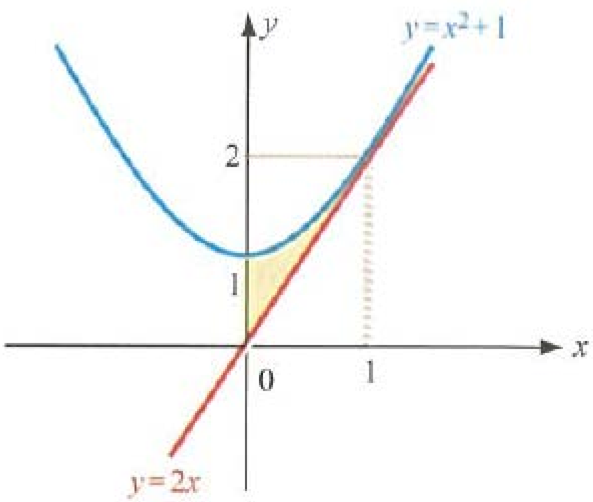
etrafında döndürülmesiyle meydana gelen cismin hacmini bulunuz.

Çözüm

Teğet denklemi

$$y' = 2x \Rightarrow m = 2 \Rightarrow$$

$$y - 2 = 2(x - 1) \Rightarrow y = 2x$$



a. $$V=\pi\int_0^1\left[(x^2+1)^2-(2x)^2\right]dx$$

$$=\pi\int_0^1(x^4-2x^2+1)dx=\pi\left(\frac{1}{5}x^5-\frac{2}{3}x^3+x\right)\Bigg|_0^1$$

$$=\frac{8\pi}{15}$$

$$V=\pi\int_0^1\left(\frac{y}{2}\right)^2dy+\pi\int_1^2\left[\left(\frac{y}{2}\right)^2-(y-1)\right]dy$$

$$=\pi\frac{y^3}{12}\Bigg|_0^1+\pi\left(\frac{y^3}{12}-\frac{y^2}{2}+y\right)\Bigg|_1^2=\frac{\pi}{12}+\frac{\pi}{12}=\frac{\pi}{6}$$

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