



Math 4543: Numerical Methods

Lecture 14 — Euler's Method for Ordinary Differential Equations

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Lecture Plan

The agenda for today

- Recap the concept of Ordinary Differential Equations (ODEs)
- Understand the idea behind Euler's method
- Derive the formula for Euler's method
- Approximate the error in the Euler's method using the Taylor Series
- Use Euler's method to solve definite integrals

Euler's Method

What is it?

Euler's method is a numerical technique to solve first-order ordinary differential equations of the form

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0 \quad (1)$$

The diagram illustrates the components of Equation (1). A blue arrow points from the function $f(x, y)$ to a box labeled "Bivariate Function". A red arrow points from the initial condition $y(x_0) = y_0$ to a box labeled "Initial Condition".

Only first-order ordinary differential equations of the form given by Equation (1) can be solved by using Euler's method. So, whatever 1st order differential equation we have, the preliminary step is to manipulate the equation to fit the aforementioned template.

$$69 \frac{dy}{dx} + 420y = 911e^{-x} ; y(0) = 5$$

$$y(x) = ?$$

$$69 \frac{dy}{dx} = 911e^{-x} - 420y$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{911e^{-x} - 420y}{69} ; y(0) = 5}$$

\downarrow
 $f(x, y)$

Euler's Method

Rewriting examples

Rewrite $\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$

in $\frac{dy}{dx} = f(x, y), y(0) = y_0$ form.

Solution

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Rewrite $e^y \frac{dy}{dx} + x^2 y^2 = 2 \sin(3x), y(0) = 5$

in $\frac{dy}{dx} = f(x, y), y(0) = y_0$ form.

Solution

$$e^y \frac{dy}{dx} + x^2 y^2 = 2 \sin(3x), y(0) = 5$$

$$\frac{dy}{dx} = \frac{2 \sin(3x) - x^2 y^2}{e^y}, y(0) = 5$$

In this case

$$f(x, y) = \frac{2 \sin(3x) - x^2 y^2}{e^y}$$

Euler's Method

Deriving the formula

At $x = 0$, we are given the value of $y = y_0$. Let us call $x = 0$ as x_0 . Now since we know the slope of y with respect to x , that is, $f(x, y)$, then at $x = x_0$, the slope is $f(x_0, y_0)$. Both x_0 and y_0 are known from the initial condition $y(x_0) = y_0$.

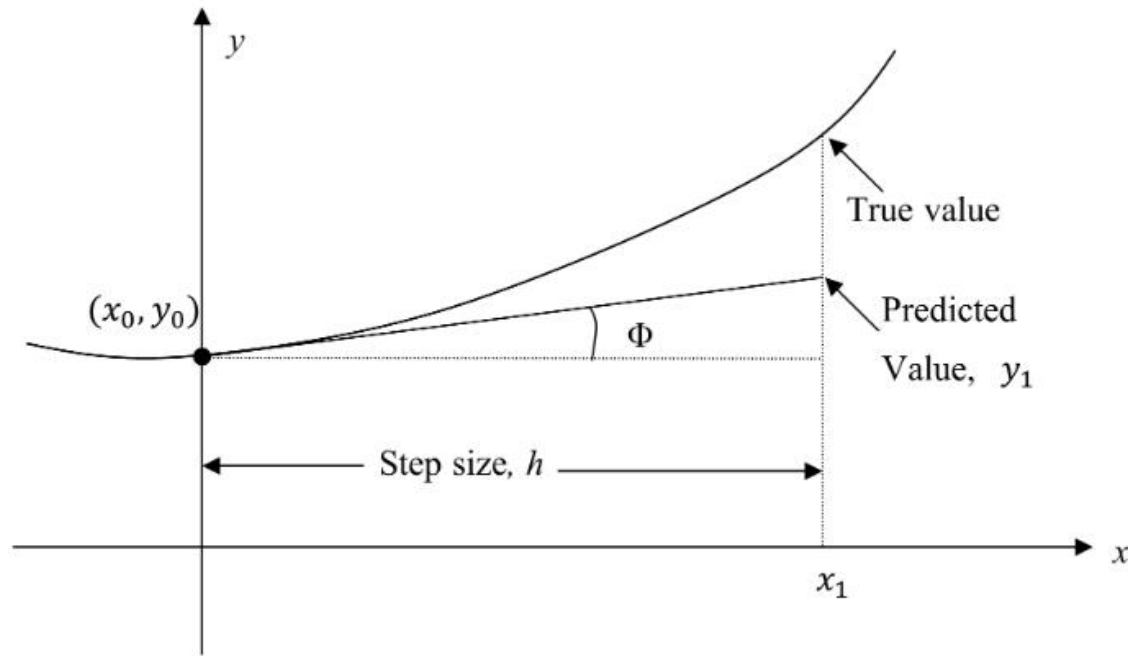


Figure 1 Graphical interpretation of the first step of Euler's method.

So the slope at $x = x_0$, as shown in Figure 1, is

$$\begin{aligned} \text{Slope, } \left. \frac{dy}{dx} \right|_{x_0, y_0} &= \frac{\text{Rise}}{\text{Run}} = \frac{y_1 - y_0}{x_1 - x_0} \\ &= f(x_0, y_0) \end{aligned}$$

gives

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

Calling $x_1 - x_0$ the step size h , we get

$$y_1 = y_0 + f(x_0, y_0)h \quad (2)$$

One can now use the value of y_1 (an approximate value of y at $x = x_1$) to calculate y_2 , and that would be the predicted value at x_2 , given by

$$y_2 = y_1 + f(x_1, y_1)h$$

$$x_2 = x_1 + h$$

Euler's Method

Deriving the formula

Based on the above equations, if we now know the estimated value of y at x_i as y_i , then

$$y_{i+1} = y_i + f(x_i, y_i) h \quad (3)$$

Equation (3) is known as Euler's method and is illustrated graphically in Figure 2. In some textbooks, it is also called the Euler-Cauchy method.

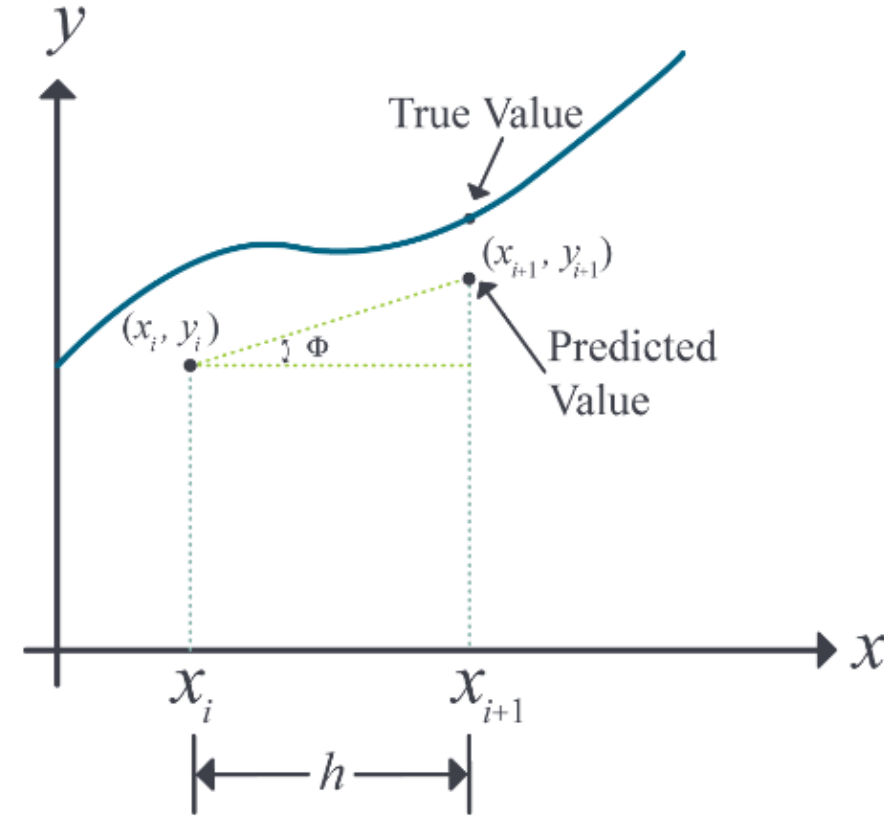
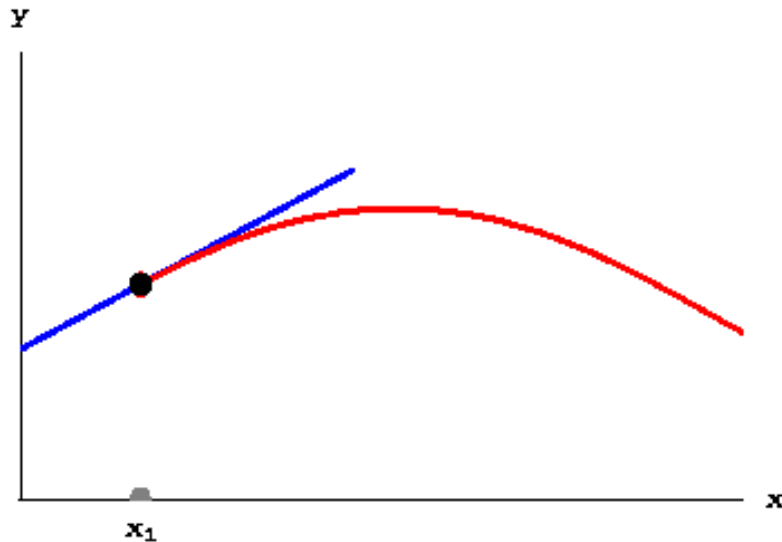


Figure 2 General graphical interpretation of Euler's method.

$$y'(x) = f(x, y)$$
$$y_{n+1} = y_n + f(x_n, y_n) \cdot (x_{n+1} - x_n)$$

Link:

https://phys23p.sl.psu.edu/~mrg3/mathanim/diff_equ/eulera.html



$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(x)}{6}h^3 + \dots$$

$$y_{i+1} = y_i + \left. \frac{dy}{dx} \right|_{x_i, y_i} h + \left. \frac{d^2y}{dx^2} \right|_{x_i, y_i} \times \frac{1}{2} h^2 + \dots$$

$$\Rightarrow y_{i+1} = \underbrace{y_i + \left. \frac{dy}{dx} \right|_{x_i, y_i} (x_{i+1} - x_i)}_{\text{Consider}} + \underbrace{\left. \frac{d^2y}{dx^2} \right|_{x_i, y_i} \times \frac{1}{2} (x_{i+1} - x_i)^2 + \dots}_{\text{Truncation Error}}$$

$$\Rightarrow y_{i+1} = y_i + f(x_i, y_i)(x_{i+1} - x_i)$$

$$\Rightarrow y_{i+1} = y_i + f(x_i, y_i)h$$

Euler's Method

Deriving the formula (*from the Taylor Series*)

Euler's method can be derived from the Taylor series as follows.

$$y_{i+1} = y_i + \left. \frac{dy}{dx} \right|_{x_i, y_i} (x_{i+1} - x_i) + \frac{1}{2!} \left. \frac{d^2y}{dx^2} \right|_{x_i, y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!} \left. \frac{d^3y}{dx^3} \right|_{x_i, y_i} (x_{i+1} - x_i)^3 + \dots$$

Since $\frac{dy}{dx} = f(x, y)$

$$y_{i+1} = y_i + f(x_i, y_i)(x_{i+1} - x_i) + \frac{1}{2!} f'(x_i, y_i)(x_{i+1} - x_i)^2 + \frac{1}{3!} f''(x_i, y_i)(x_{i+1} - x_i)^3 + \dots$$

As you can see, using only the first two terms of the Taylor series in Equation (6)

$$\boxed{y_{i+1} = y_i + f(x_i, y_i) h} \quad (3 - \text{repeated})$$

is the Euler's method.

Euler's Method

Truncation error in Euler's method

There are *two* sources of error (not counting round-off) in Euler's method:

- The error committed in approximating the integral curve by the tangent line over the interval $[x_i, x_{i+1}]$.
- The error committed in using the approximation y_i instead of the exact value $y(x_i)$ to compute y_{i+1} .

The true error in the Euler's method hence is given by

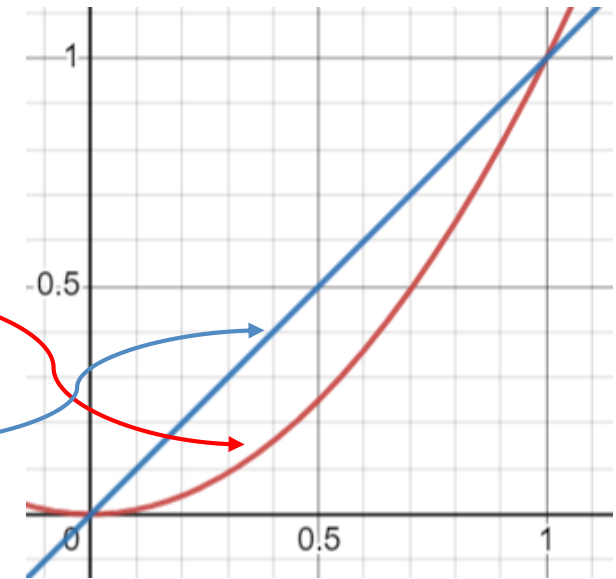
$$E_t = \frac{f'(x_i, y_i)}{2!} h^2 + \frac{f''(x_i, y_i)}{3!} h^3 + \dots$$

Local Truncation Error: The truncation error at the i^{th} step.

$$E_{t_i} = y(x_{i+1}) - y(x_i) - hf(x_i, y(x_i))$$

Global Truncation Error: The cumulative effect of the local truncation errors E_{t_i} committed at each step.

Local Truncation Error,
 $E_t \propto \mathbf{O}(h^2)$
Global Truncation Error,
 $E_{t_{global}} \propto \mathbf{O}(h)$
in general, $h \in (0,1]$



[Click here](#) if you are interested in the detailed analysis of the Truncation Error.

Euler's Method

An example

A ball at 1200 K is allowed to cool down in air at an ambient temperature of 300 K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12} (\theta^4 - 81 \times 10^8), \theta(0) = 1200 \text{ K} \quad (E1.1)$$

where θ is in K and t in seconds. Find the temperature at $t = 480$ seconds using Euler's method. Assume a step size of $h = 240$ seconds. Compare with the exact value.

Euler's Method

An example

Solution

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12} (\theta^4 - 81 \times 10^8)$$

$$f(t, \theta) = -2.2067 \times 10^{-12} (\theta^4 - 81 \times 10^8)$$

Euler's method formula is

$$\theta_{i+1} = \theta_i + f(t_i, \theta_i) h$$

For $i = 0$, $t_0 = 0$, $\theta_0 = 1200$

$$\begin{aligned}\theta_1 &= \theta_0 + f(t_0, \theta_0) h \\ &= 1200 + f(0, 1200) \times 240 \\ &= 1200 + (-2.2067 \times 10^{-12} (1200^4 - 81 \times 10^8)) \times 240 \\ &= 1200 + (-4.5579) \times 240 \\ &= 106.09 \text{ K}\end{aligned}$$

θ_1 is the approximate temperature at

$$t = t_1 = t_0 + h = 0 + 240 = 240 \text{ s} \quad \theta_1 = \theta(240) \approx 106.09 \text{ K}$$

For $i = 1$, $t_1 = 240$, $\theta_1 = 106.09$

$$\begin{aligned}\theta_2 &= \theta_1 + f(t_1, \theta_1) h \\ &= 106.09 + f(240, 106.09) \times 240 \\ &= 106.09 + (-2.2067 \times 10^{-12} (106.09^4 - 81 \times 10^8)) \times 240 \\ &= 106.09 + (0.017595) \times 240 \\ &= 110.32 \text{ K}\end{aligned}$$

θ_2 is the approximate temperature at

$$t = t_2 = t_1 + h = 240 + 240 = 480 \text{ s}$$

$$\theta_2 = \theta(480) \approx 110.32 \text{ K}$$

The exact solution of the ordinary differential Equation (E1.1) is given without proof as the solution of the nonlinear equation

$$\begin{aligned}&0.92593 \ln \frac{\theta - 300}{\theta + 300} - 1.8519 \tan^{-1}(0.333 \times 10^{-2} \theta) \\ &= -0.22067 \times 10^{-3} t - 2.9282\end{aligned}$$

$$\theta_{exact} = 647.57 \text{ K at } t = 480 \text{ s}$$

Euler's Method

An example

Figure 1 compares the exact solution (647.57 K) with the numerical solution from Euler's method for the step size of $h = 240$.

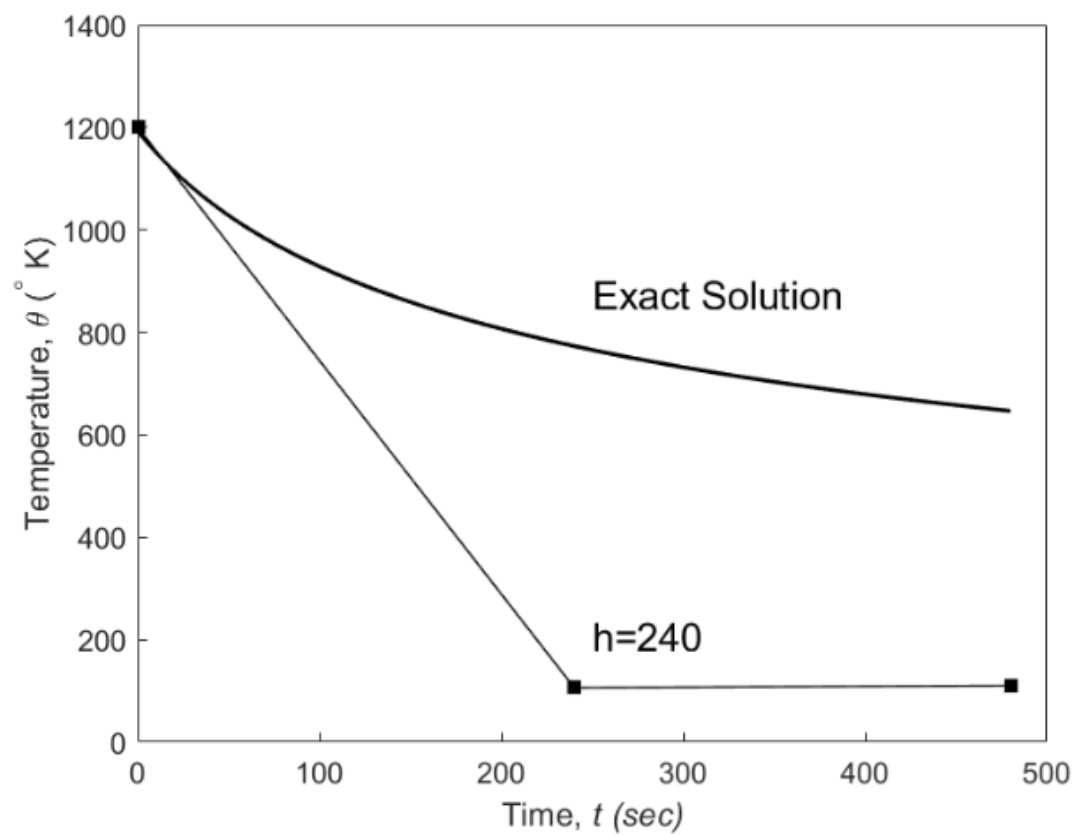


Figure 1 Comparing the exact and Euler's method solutions.

The problem was solved again using a smaller step size. The results are given below in Table 1.

Table 1 Temperature at 480 seconds as a function of step size, h .

Step size, h	$\theta(480)$	E_t	$ \epsilon_t \%$
480	-987.81	1635.4	252.54
240	110.32	537.26	82.964
120	546.77	100.80	15.566
60	614.97	32.607	5.0352
30	632.77	14.806	2.2864

Euler's Method

An example

Figure 2 shows how the temperature varies as a function of time for different step sizes.

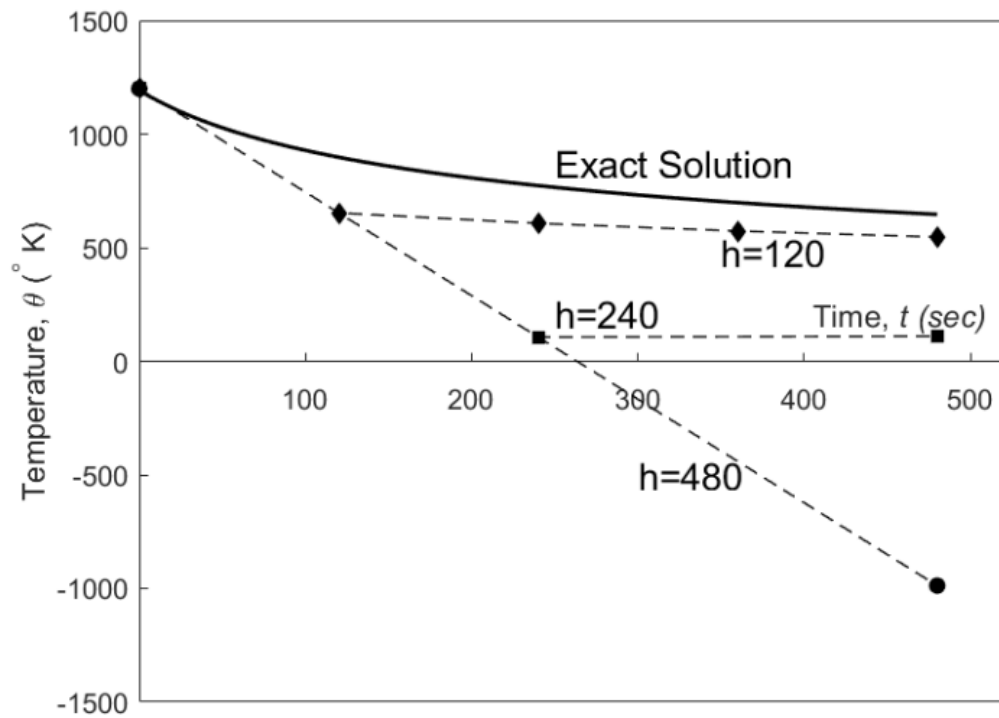


Figure 2 Comparison of Euler's method with the exact solution for different step sizes.

The values of the calculated temperature at $t = 480$ s as a function of step size are plotted in Figure 3.

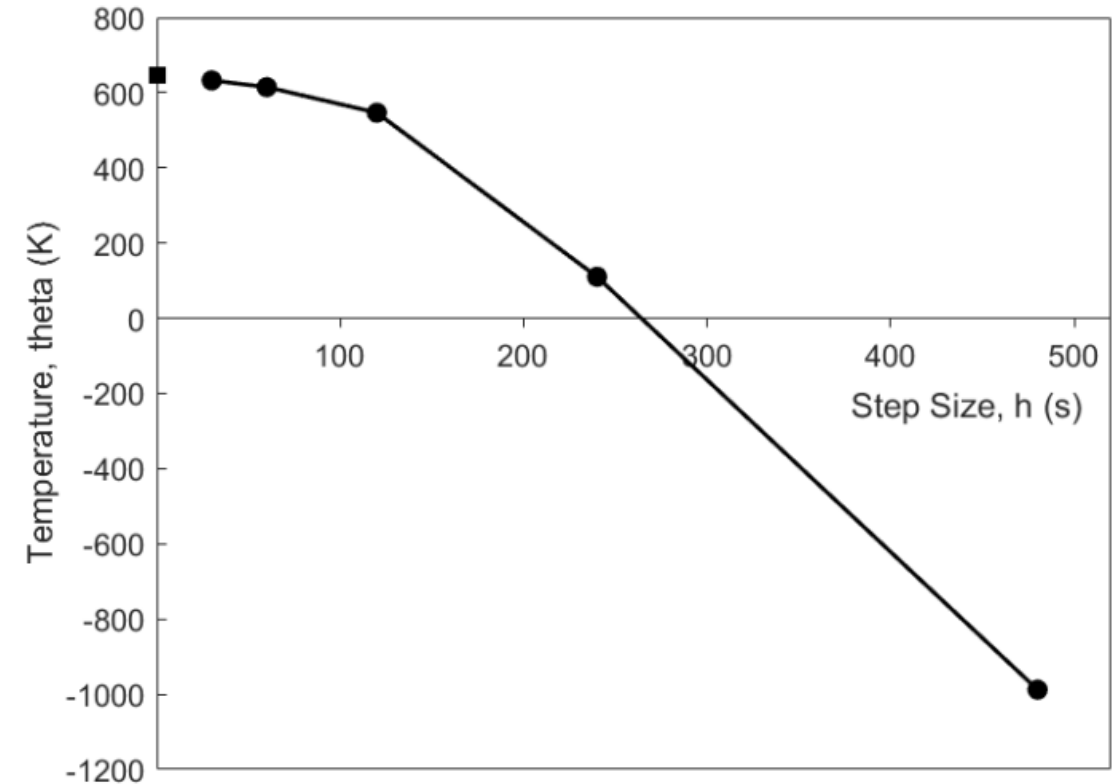


Figure 3 Effect of step size in Euler's method.

Euler's Method

Approximating the value of a Definite Integral

We need to invoke the fundamental theorems of calculus for this —

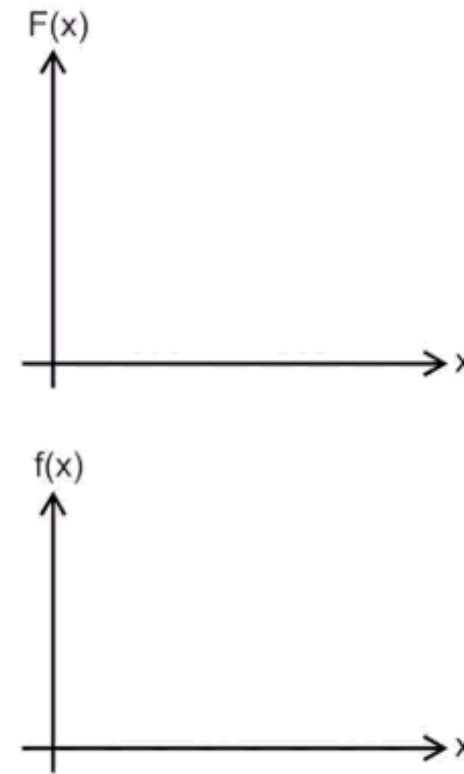
- **First Fundamental Theorem of Calculus:** Let f be a continuous real-valued function defined on a closed interval $[a, b]$. Let F be the function defined, for all x in $[a, b]$ by

$$F(x) = \int_a^x f(t) dt$$

Then F is uniformly continuous on $[a, b]$ and differentiable on the open interval (a, b) , and $F'(x) = f(x)$ for all x in (a, b) so F is an antiderivative of f .

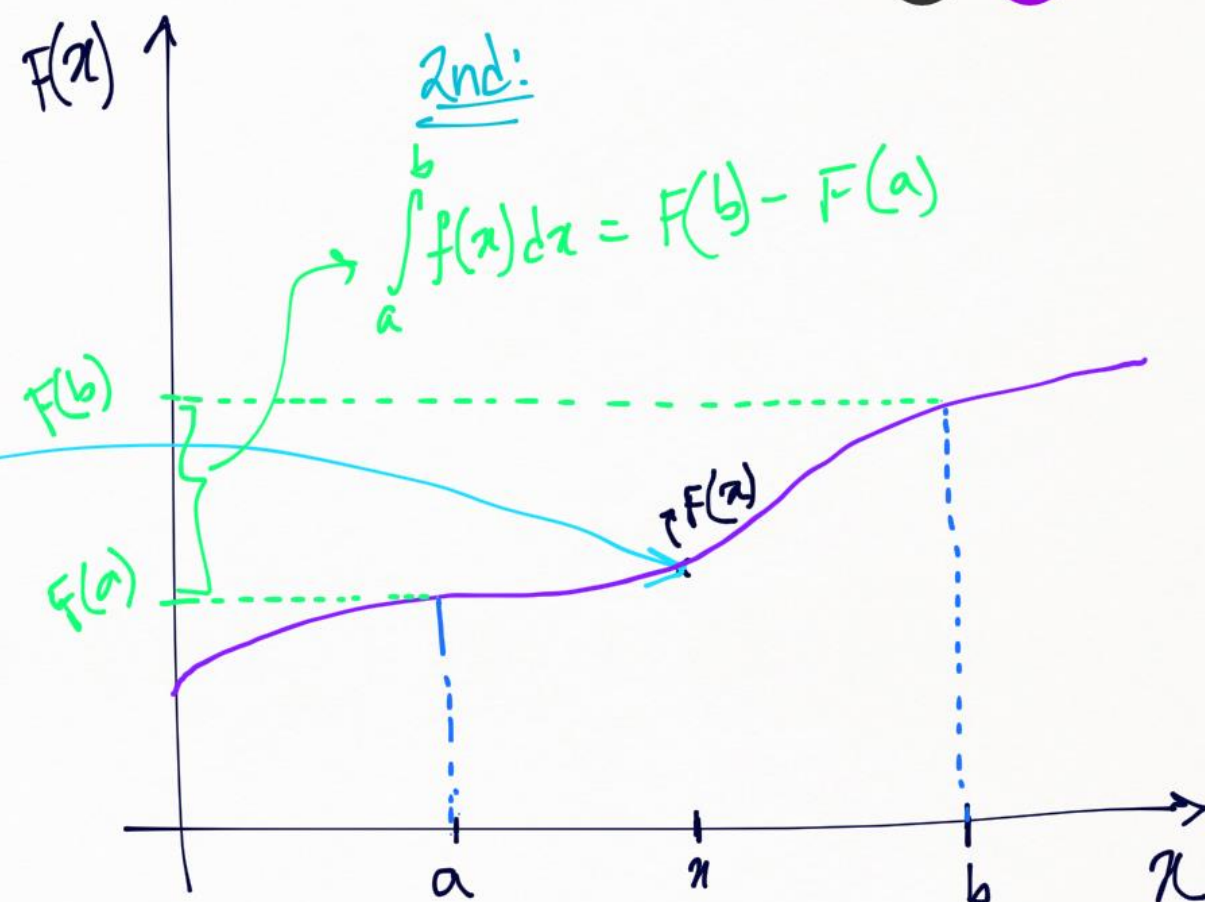
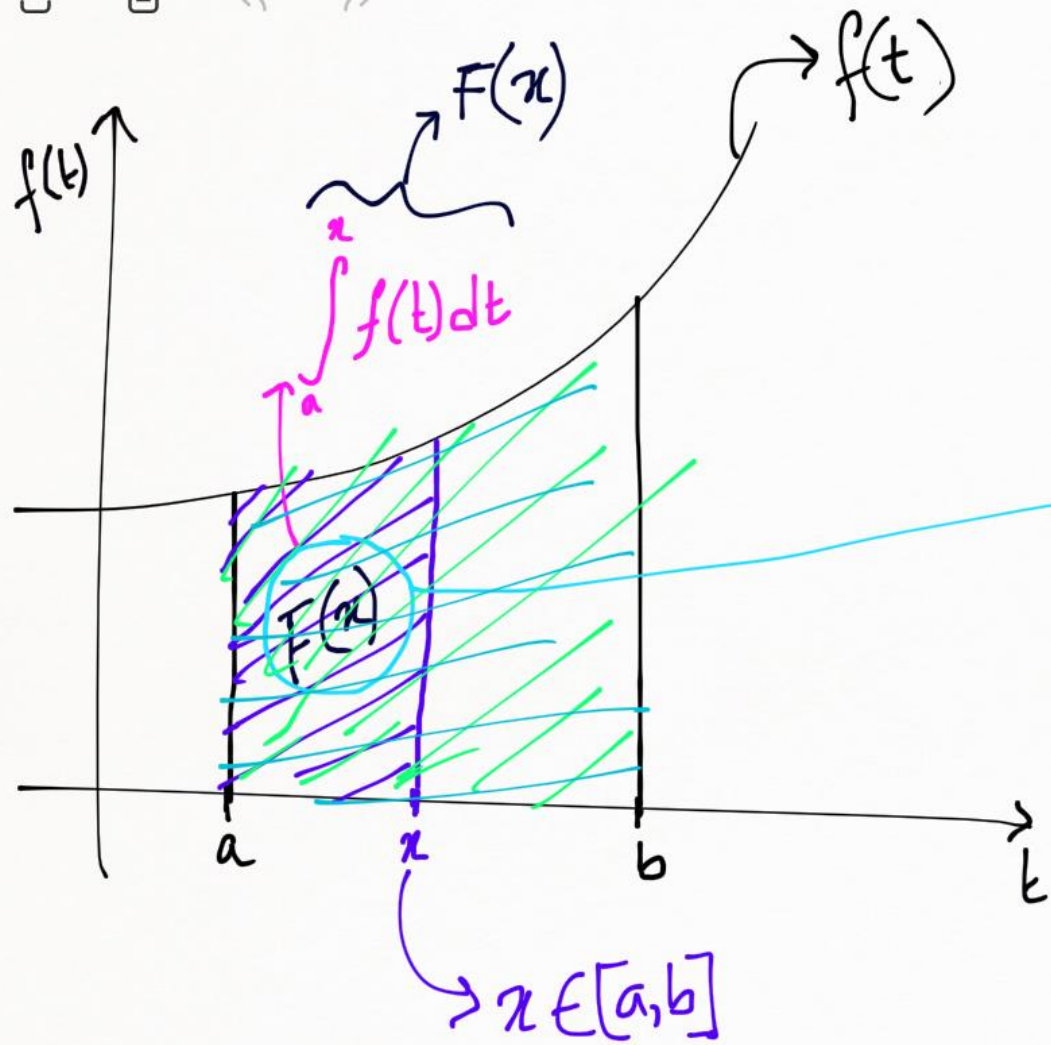
- **Second Fundamental Theorem of Calculus:** If f is Riemann integrable on $[a, b]$ then

$$\int_a^b f(x) dx = F(b) - F(a)$$



Link:


https://upload.wikimedia.org/wikipedia/commons/2/2f/Fundamental_theorem_of_calculus_%28animation%29.gif?20200314134940



1st:

$$\frac{d}{dx} (F(x)) = f(x) \quad \forall x \in (a, b)$$

antiderivative


$$y = I = \int_a^b f(x) dx$$

$$\frac{dy}{dx} = f(x) ; y(a) = 0$$

$$y(b) = \int_a^b f(x) dx$$

Euler's Method

Approximating the value of a Definite Integral

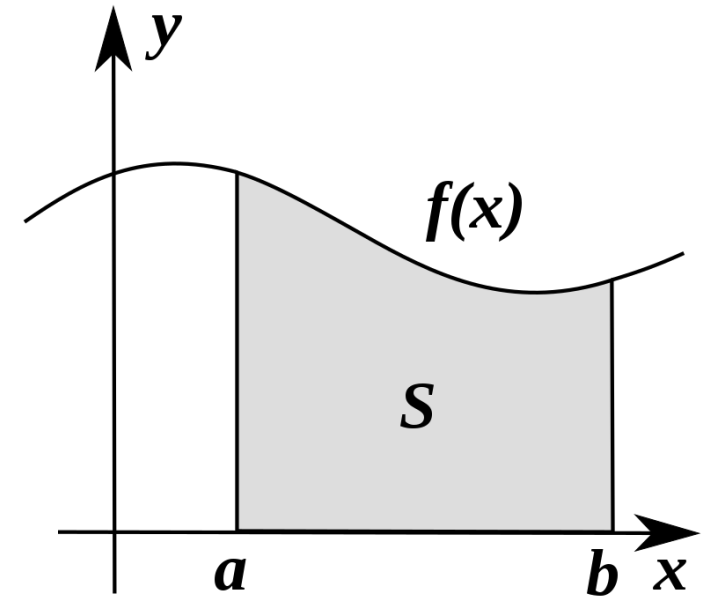
The idea is to rewrite the integral as the solution of an ODE,

$$\frac{dy}{dx} = f(x), \quad y(a) = 0,$$

1st Fundamental Theorem

where then $y(b)$ gives us the value of the integral $\int_a^b f(x) dx$.

2nd Fundamental Theorem



Euler's Method

A definite integral approximation example

Find an approximate value of

$$\int_5^8 6x^3 dx$$

using Euler's method of solving an ordinary differential equation. Use a step size of $h = 1.5$.

Euler's Method

A definite integral approximation example

Solution

Given $\int_5^8 6x^3 dx$, we can rewrite the integral as the solution of an ordinary differential equation

$$\frac{dy}{dx} = 6x^3, y(5) = 0$$

where $y(8)$ gives the value of the integral $\int_5^8 6x^3 dx$.

$$\frac{dy}{dx} = 6x^3 = f(x, y), y(5) = 0$$

The Euler's method formula is

$$y_{i+1} = y_i + f(x_i, y_i) h$$

Step 1

$$i = 0, x_0 = 5, y_0 = 0$$

$$h = 1.5$$

$$\begin{aligned} x_1 &= x_0 + h \\ &= 5 + 1.5 \\ &= 6.5 \end{aligned}$$

$$y_1 = y_0 + f(x_0, y_0) h$$

$$\begin{aligned} y_1 &= 0 + f(5, 0) \times 1.5 \\ &= 0 + f(6 \times 5^3) \times 1.5 \\ &= 1125 \\ &\approx y(6.5) \end{aligned}$$

Step 2

$$i = 1, x_1 = 6.5, y_1 = 1125$$

$$\begin{aligned} x_2 &= x_1 + h \\ &= 6.5 + 1.5 \\ &= 8 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + f(x_1, y_1) h \\ &= 1125 + f(6.5, 1125) \times 1.5 \\ &= 1125 + f(6 \times 6.5^3) \times 1.5 \\ &= 3596.625 \\ &\approx y(8) \end{aligned}$$

Hence

$$\begin{aligned} \int_5^8 6x^3 dx &= y(8) - y(5) \\ &\approx 3596.625 - 0 \\ &= 3596.625 \end{aligned}$$