Math 4543: Numerical Methods

Lecture 10 — Linear Regression

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Lecture Plan

The agenda for today

- Understand the concept of Regression Analysis
- What is Linear Regression?
- Explore the appropriateness of different optimization criteria for the regression model
- Uniqueness of the Least-Square-Errors criterion
- Derive the slope of a Zero-intercept Linear Regression model

Regression Analysis

What is a regression model?

In statistical modeling, regression analysis is a set of statistical processes for ***estimating*** the relationships between a dependent variable and one or more independent variables.

The problem statement for a regression model is as follows. Given n data pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, best fit $y = f(x)$ to the data (Figure 1).

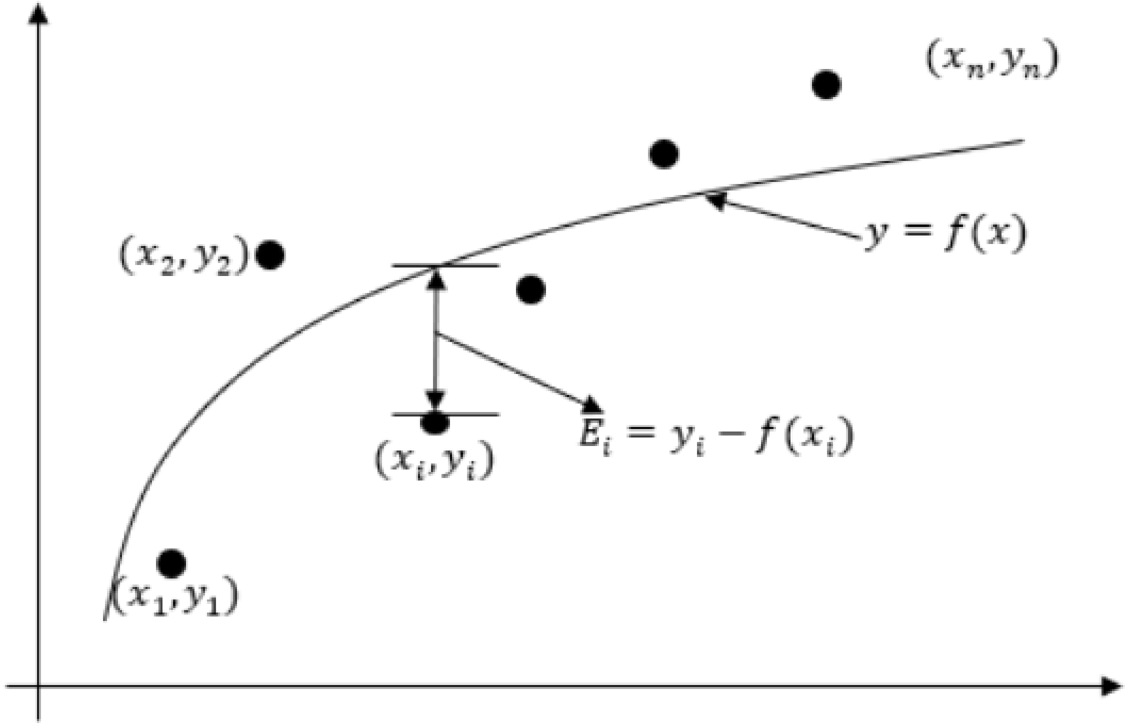


Figure 1. A general regression model for discrete y vs. x data

Linear Regression

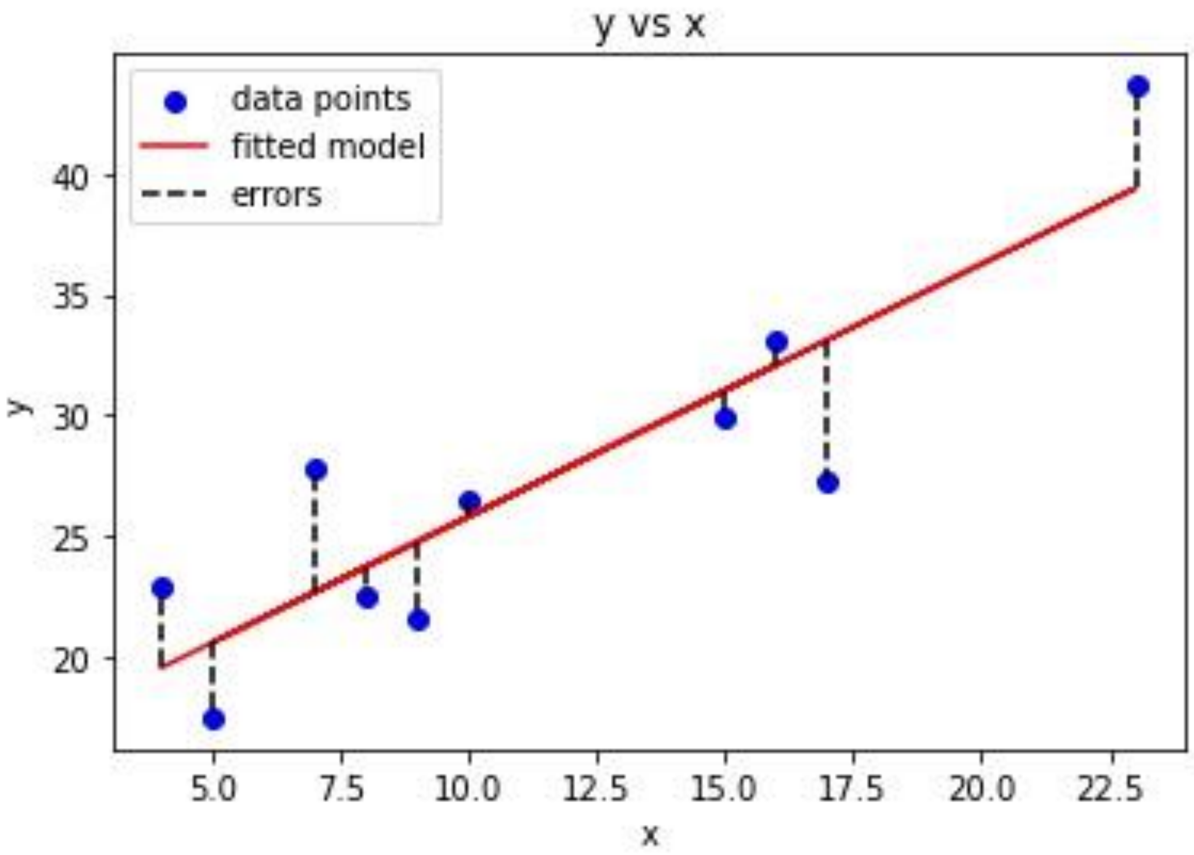
What is it?

In linear regression, the relationships are modeled using ***linear predictor functions*** whose unknown model parameters (slope and y-intercept) are estimated from the data.

Linear regression is the most popular regression model. In this model, we wish to predict response to n data points $(x_1, y_1), (x_2, y_2), \ldots\ldots, (x_n, y_n)$ by a regression model given by

$$y = a_0 + a_1 x \qquad (1)$$

where a_0 and a_1 are the constants of the regression model.



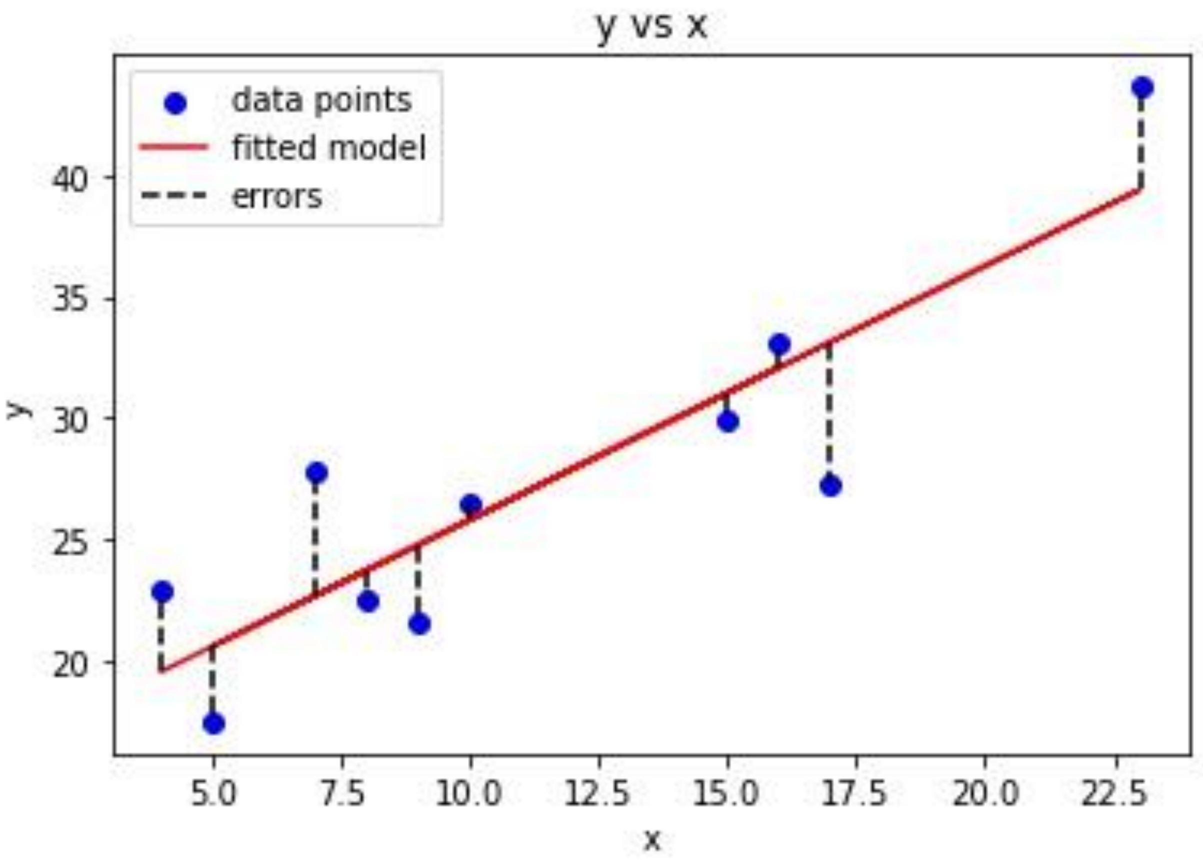
Linear Regression

How to quantify the *goodness* of fit?

A measure of goodness of fit, that is, how well $a_0 + a_1 x$ predicts the response variable y is the magnitude of the residual E_i at each of the n data points.

$$E_i = y_i - (a_0 + a_1 x_i) \qquad (2)$$

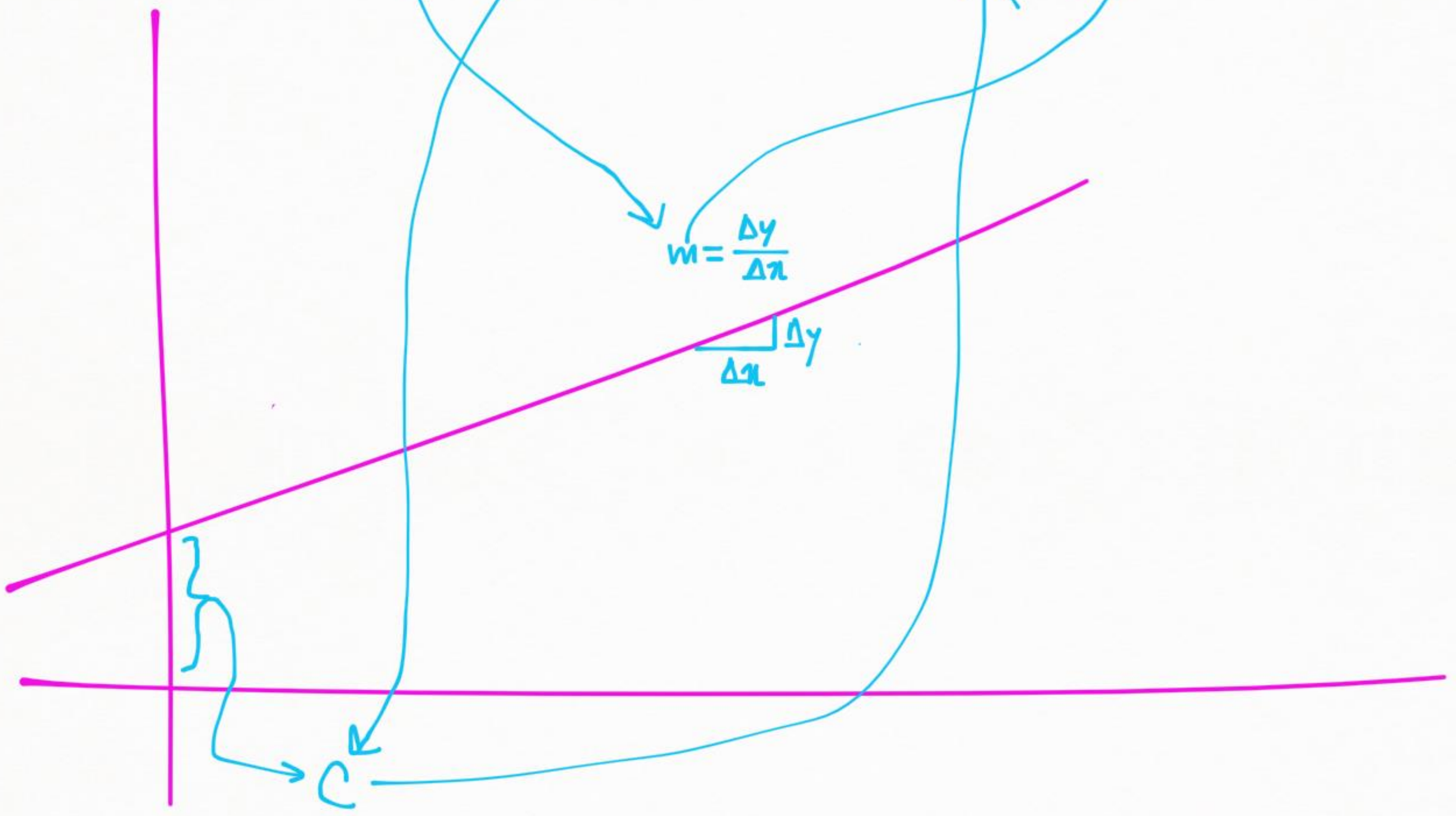
Ideally, if all the residuals E_i are zero, one has found an equation in which all the points lie on the model. Thus, minimization of the residuals is an objective of obtaining regression coefficients.



$$y = \underline{m}x + \underline{c}$$

$$y = a_0 + a_1 x$$

$$m = \frac{\Delta y}{\Delta x}$$



Linear Regression

Choosing the right criterion for optimization

$$E_i = y_i - (a_0 + a_1 x_i) \qquad (2)$$

The most popular method to minimize the residual is the least-squares method, where the estimates of the constants of the models are chosen such that the sum of the squared residuals is minimized, that is, minimize

$$S_r = \sum_{i=1}^{n} {E_i}^2 \qquad (3)$$

Why minimize the sum of the square of the residuals, S_r?

$$\boxed{\text{Why not } \sum_{i=1}^{n} E_i \text{ or } \sum_{i=1}^{n} |E_i| \text{ ?}}$$

Linear Regression

The case *against* the $\sum_{i=1}^{n} E_i$ criterion

Table 1 Data points.

x	y
2.0	4.0
3.0	6.0
2.0	6.0
3.0	8.0

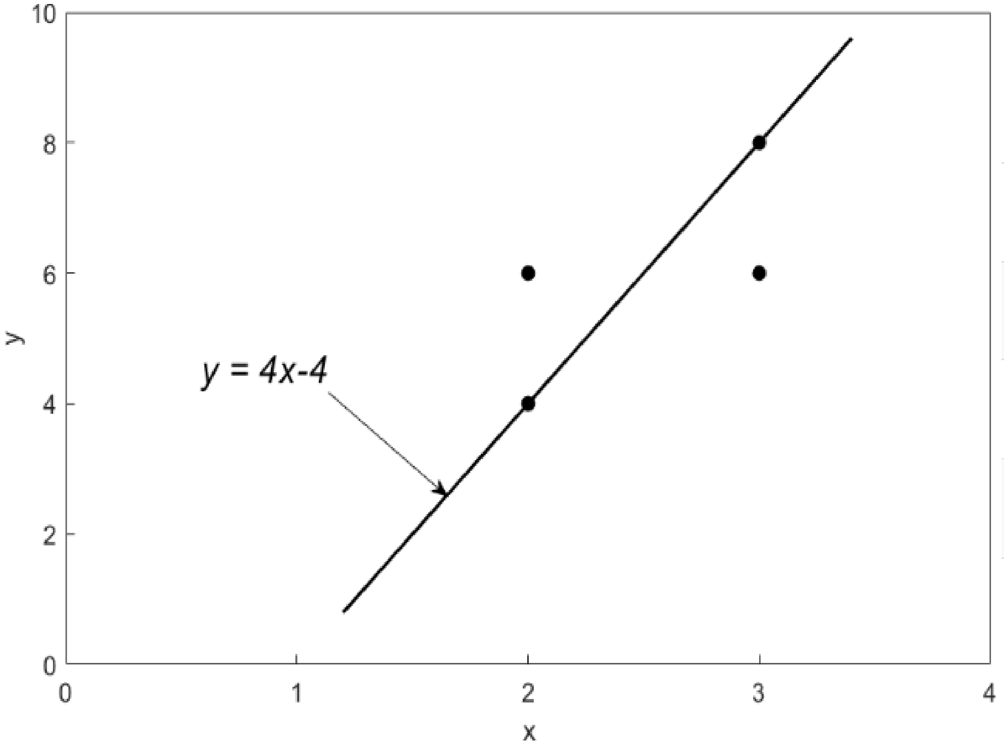
To explain this data by a straight line regression model,

$$y = a_0 + a_1 x \qquad (4)$$

Let us use minimizing $\sum_{i=1} E_i$ as a criterion to find a_0 and a_1. Assume randomly that

$$y = 4x - 4 \qquad (5)$$

as the resulting regression model (Figure 2).



x	y	$y_{predicted}$	$E = y - y_{pred}$
2.0	4.0	4.0	0.0
3.0	6.0	8.0	-2.0
2.0	6.0	4.0	2.0
3.0	8.0	8.0	0.0

$$\sum_{i=1}^{4} E_i = 0$$

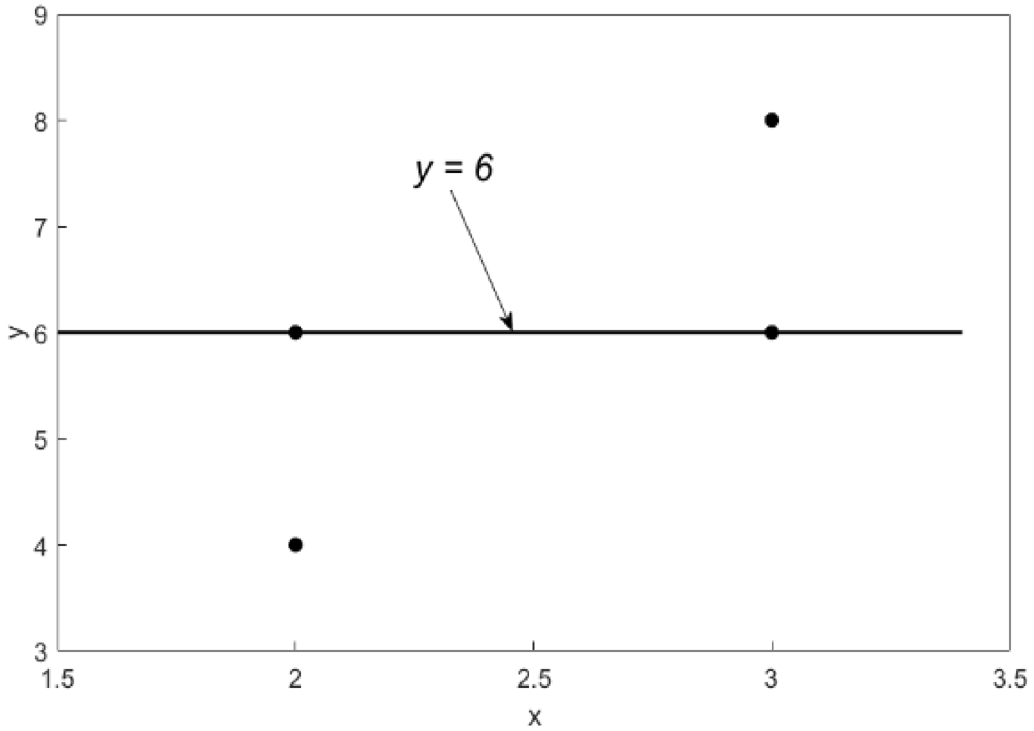
Linear Regression

The case *against* the $\sum_{i=1}^{n} E_i$ criterion

So does this give us the smallest possible sum of residuals? For this data, it does as $\sum_{i=1}^{4} E_i = 0$, and it cannot be made any smaller. But does it give unique values for the parameters of the regression model? No, because, for example, a straight-line model (Figure 3)

$$y = 6 \qquad (6)$$

also gives $\sum_{i=1}^{4} E_i = 0$ as shown in Table 3.



x	y	$y_{\text{predicted}}$	$E = y - y_{pred}$
2.0	4.0	6.0	-2.0
3.0	6.0	6.0	0.0
2.0	6.0	6.0	0.0
3.0	8.0	6.0	2.0

$$\sum_{i=1}^{4} E_i = 0$$

Non-unique regression models. So, bad criterion!

Linear Regression

The case *against* the $\sum_{i=1}^{n} |E_i|$ criterion

Non-unique regression models. Also bad criterion!

Table 1 Data points.

x	y
2.0	4.0
3.0	6.0
2.0	6.0
3.0	8.0

To explain this data by a straight line regression model,

$$y = a_0 + a_1 x \qquad (4)$$

You may think that the reason the criterion of minimizing $\sum_{i=1}^{n} E_i$ does not work is because negative residuals cancel with positive residuals. So, is minimizing the sum of absolute values of the residuals, that is, $\sum_{i=1}^{n} |E_i|$ better? Let us look at the same example data given in Table 1. For the regression

$$y = 4x - 4$$

x	y	$y_{predicted}$	$E = y - y_{pred}$
2.0	4.0	4.0	0.0
3.0	6.0	8.0	2.0
2.0	6.0	4.0	2.0
3.0	8.0	8.0	0.0

$$\sum_{i=1}^{4} |E_i| = 4$$

$$y = 6$$

x	y	$y_{predicted}$	$E = y - y_{pred}$
2.0	4.0	6.0	2.0
3.0	6.0	6.0	0.0
2.0	6.0	6.0	0.0
3.0	8.0	6.0	2.0

$$\sum_{i=1}^{4} |E_i| = 4$$

No other straight-line model that you may choose for this data has $\sum_{i=1}^{4} |E_i| < 4$.

Linear Regression

The case *for* the $\sum_{i=1}^{n} E_i^2$ criterion

To get a unique regression model, the least-squares criterion where we minimize the sum of the square of the residuals

$$S_r = \sum_{i=1}^{n} E_i^2$$
$$= \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2 \qquad (7)$$

is recommended.

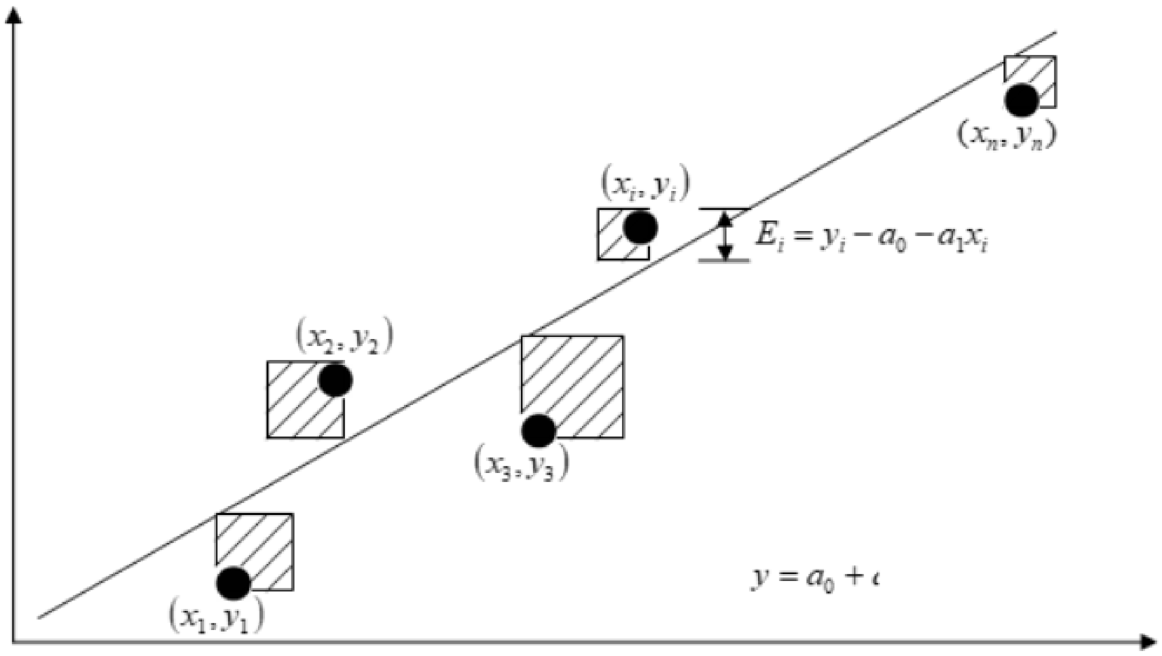


Figure 1. Linear regression of y vs. x data showing residuals and square of residual at a typical point, x_i

To find a_0 and a_1, we need to calculate where the sum of the square of the residuals, S_r is the absolute minimum. We start this process of finding the absolute minimum first by

1) taking the partial derivative of S_r with respect to a_0 and a_1 and set them equal to zero, and

2) conducting the second derivative test.

$$S_r = \sum_i E_i^2$$

$$= \sum_i (y_i - a_0 - a_1 x_i)^2$$

$$\frac{\partial S_r}{\partial a_0} = 2 \sum_i (y_i - a_0 - a_1 x_i) \times (-1)$$

$$= -2 \left(\sum_i (y_i - a_0 - a_1 x_i) \right)$$

$$\frac{\partial S_r}{\partial a_1} = 2 \sum_i (y_i - a_0 - a_1 x_i) \times (-x_i)$$

$$= -2 \sum_i x_i y_i + 2 \sum_i a_0 x_i + 2 \sum_i a_1 x_i^2$$

$$\frac{\partial S_r}{\partial a_0} = 0$$

$$\Rightarrow -2 \sum_i y_i + 2 \sum_{i=1}^{n} a_0 + 2 \sum_i a_1 x_i = 0$$

$$\Rightarrow -\sum_i y_i + n a_0 + \sum_i a_1 x_i = 0$$

$$\Rightarrow \boxed{a_0 = \underbrace{\frac{\sum_i y_i}{n}}_{\bar{y}} - a_1 \underbrace{\frac{\sum_i x_i}{n}}_{\bar{x}}} \quad \text{—— (i)}$$

$$\frac{\partial S_r}{\partial a_1} = 0$$

$$\Rightarrow -2 \sum x_i y_i + 2 \sum a_0 x_i + 2 \sum a_1 x_i^2 = 0$$

$$\Rightarrow \boxed{-\sum x_i y_i + \sum a_0 x_i + \sum a_1 x_i^2 = 0} \quad \text{—— (ii)}$$

Sub (i) into (ii) $\Rightarrow$

$$\sum x_i y_i - \left(\frac{\sum y_i}{n} - a_1 \frac{\sum x_i}{n}\right)\sum x_i - a_1 \sum x_i^2 = 0$$

$$\Rightarrow \sum x_i y_i - \sum x_i \frac{\sum y_i}{n} + a_1 \frac{(\sum x_i)^2}{n} - a_1 \sum x_i^2 = 0$$

$$\Rightarrow a_1 \left(\frac{(\sum x_i)^2}{n} - \sum x_i^2\right) = \frac{\sum x_i \sum y_i}{n} - \sum x_i y_i$$

$$\Rightarrow a_1 = \frac{\frac{\sum x_i \sum y_i}{n} - \sum x_i y_i}{\frac{(\sum x_i)^2}{n} - \sum x_i^2} = \frac{\sum x_i y_i - \frac{\sum x_i \sum y_i}{n}}{\sum x_i^2 - \frac{(\sum x_i)^2}{n}}$$

$$\Rightarrow a_1 = \boxed{\frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2}} \text{ (iii)}$$

$$a_1 = \frac{\frac{\sum x_i y_i}{n} - \frac{\sum x_i}{n}\frac{\sum y_i}{n}}{\frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2}$$

$$\Rightarrow a_1 = \boxed{\frac{\sum x_i y_i - n\bar{x}\bar{y}}{\sum x_i^2 - n\bar{x}^2}}$$

$\sum x_i y_i - n\bar{x}\bar{y} \rightarrow S_{xy}$

$\sum x_i^2 - n\bar{x}^2 \rightarrow S_{xx}$

$$\boxed{a_0 = \frac{\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i}{n \sum x_i^2 - (\sum_i x_i)^2}} \text{ (iv)}$$

Linear Regression

The case *for* the $\sum_{i=1}^{n} E_i^2$ criterion

Taking the partial derivative of S_r with respect to a_0 and a_1 and set them equal to zero

$$\frac{\partial S_r}{\partial a_0} = 2\sum_{i=1}^{n}(y_i - a_0 - a_1 x_i)(-1) = 0 \qquad (2)$$

$$\frac{\partial S_r}{\partial a_1} = 2\sum_{i=1}^{n}(y_i - a_0 - a_1 x_i)(-x_i) = 0 \qquad (3)$$

Dividing both sides by 2 and expanding the summations in Equations (2) and (3) gives,

$$-\sum_{i=1}^{n} y_i + \sum_{i=1}^{n} a_0 + \sum_{i=1}^{n} a_1 x_i = 0$$

$$-\sum_{i=1}^{n} y_i x_i + \sum_{i=1}^{n} a_0 x_i + \sum_{i=1}^{n} a_1 x_i^2 = 0$$

Noting that

$$\sum_{i=1}^{n} a_0 = a_0 + a_0 + \ldots + a_0 = n a_0$$

$$n a_0 + a_1 \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \qquad (4)$$

$$a_0 \sum_{i=1}^{n} x_i + a_1 \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i \qquad (5)$$

Solving the above two simultaneous linear equations (4) and (5) gives

$$\boxed{a_1 = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}} \qquad (6)$$

$$\boxed{a_0 = \frac{\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}} \qquad (7)$$

Linear Regression

The case *for* the $\sum_{i=1}^{n} E_i^2$ criterion

Redefining

$$S_{xy} = \sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y} \qquad \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

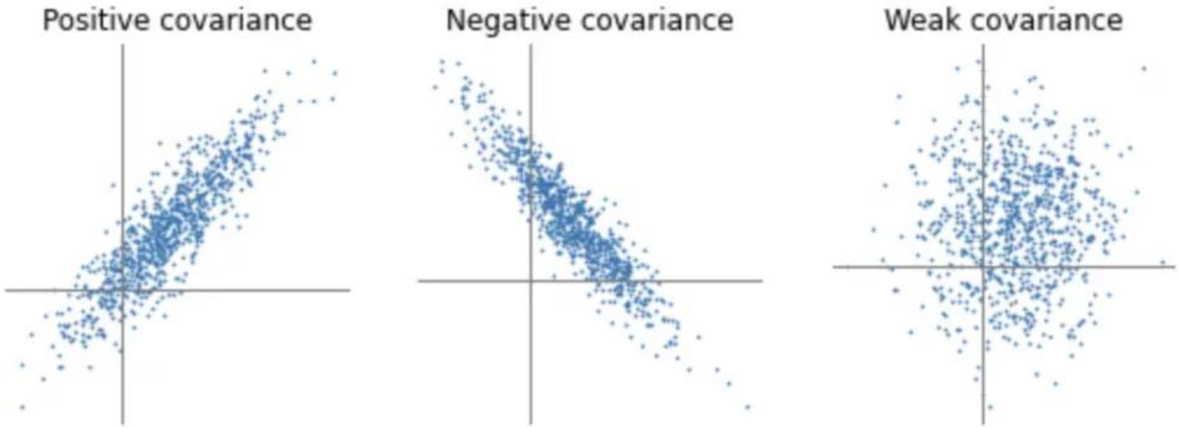
$$S_{xx} = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2 \qquad \bar{y} = \frac{\sum_{i=1}^{n} y_i}{n}$$

we can also rewrite the constants a_0 and a_1 from (6) and (7) as

$$a_1 = \frac{S_{xy}}{S_{xx}} \qquad (8)$$

$$a_0 = \bar{y} - a_1\bar{x} \qquad (9)$$

A small detour on *Variance* and *Covariance*



$Cov(X,Y) > 0$ $\quad$ $Cov(X,Y) < 0$ $\quad$ $Cov(X,Y) \approx 0$

Variance is the measure of dispersion of data points from the mean.

Covariance is the measure of how 2 random variables vary with respect to each other.

Linear Regression

The case *for* the $\sum_{i=1}^{n} E_i^2$ criterion

Putting the first derivative equations equal to zero only gives us a critical point. For a general function, it could be a local minimum, a local maximum, a saddle point, or none of the previous. The second derivative test, though, given in the shows that it is a local minimum.

If you have a function $f(x, y)$ and we found a critical point (a, b) from the first derivative test, then (a, b) is a minimum point if

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0, \text{ and} \qquad (A.6)$$

$$\frac{\partial^2 f}{\partial x^2} > 0 \text{ or } \frac{\partial^2 f}{\partial y^2} > 0 \qquad (A.7)$$

$$\frac{\partial^2 S_r}{\partial a_0^2} = -2\sum_{i=1}^{n} -1$$
$$= 2n \qquad (A.10)$$

$$\frac{\partial^2 S_r}{\partial a_1^2} = 2\sum_{i=1}^{n} x_i^2 \qquad (A.11)$$

$$\frac{\partial^2 S_r}{\partial a_0 \partial a_1} = 2\sum_{i=1}^{n} x_i \qquad (A.12)$$

$$\frac{\partial^2 S_r}{\partial a_0^2}\frac{\partial^2 S_r}{\partial a_1^2} - \left(\frac{\partial^2 S_r}{\partial a_0 \partial a_1}\right)^2 = (2n)\left(2\sum_{i=1}^{n} x_i^2\right) - \left(2\sum_{i=1}^{n} x_i\right)^2$$
$$= 4\left[n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right]$$
$$= 4\sum_{\substack{i=1 \\ i<j}}^{n} (x_i - x_j)^2 > 0 \qquad (A.13)$$

Unique regression model. Good criterion!

$$\frac{\partial S_r}{\partial a_0} = -2\sum y_i + 2na_0 + 2\sum_i a_1 x_i$$

$$\boxed{\Rightarrow \frac{\partial^2 S_r}{\partial a_0^2} = 2n} \rightarrow > 0$$

$$\frac{\partial S_r}{\partial a_1} = -2\sum x_i y_i + 2\sum a_0 x_i + 2\sum a_1 x_i^2$$

$$\boxed{\Rightarrow \frac{\partial^2 S_r}{\partial a_1^2} = 2\sum x_i^2} \rightarrow \boxed{\cancel{\forall_i\ x_i = 0}}$$

$$\rightarrow > 0$$

$$\frac{\partial^2 S_r}{\partial a_0 \partial a_1} = \frac{\partial}{\partial a_1}\left(\frac{\partial S_r}{\partial a_0}\right)$$

$$= \frac{\partial}{\partial a_0}\left(\frac{\partial S_r}{\partial a_1}\right)$$

$$\boxed{\Rightarrow \frac{\partial^2 S_r}{\partial a_0 \partial a_1} = 2\sum x_i}$$

$$\frac{\partial^2 S_r}{\partial a_0^2}\frac{\partial^2 S_r}{\partial a_1^2} - \left(\frac{\partial^2 S_r}{\partial a_0 \partial a_1}\right)^2$$

$$= 2n \times 2\sum x_i^2 - 4\left(\sum x_i\right)^2$$

$$= 4n\sum x_i^2 - 4\left(\sum x_i\right)^2 = 4\left[n\sum x_i^2 - \left(\sum x_i\right)^2\right]$$

$$4\sum_{\substack{i=1 \\ i<j}}^{n} (x_i - x_j)^2$$

$$> 0$$

Linear Regression

An example

Tip: Use the *"Statistics"* mode of your calculator. (**Mode 6** for **Casio fx-991EX**)

The torque T needed to turn the torsional spring of a mousetrap through an angle, θ is given below

Angle, θ Radians	*Torque,* T $\mathrm{N \cdot m}$
0.698132	0.188224
0.959931	0.209138
1.134464	0.230052
1.570796	0.250965
1.919862	0.313707

Find the constants k_1 and k_2 of the regression model

$$T = k_1 + k_2\theta \qquad (E1.1)$$

Solution

For the linear regression model,

$$T = k_1 + k_2\theta$$

the constants of the regression model are given by

$$k_2 = \frac{n\sum_{i=1}^{5}\theta_i T_i - \sum_{i=1}^{5}\theta_i \sum_{i=1}^{5} T_i}{n\sum_{i=1}^{5}\theta_i^2 - \left(\sum_{i=1}^{5}\theta_i\right)^2} \qquad (E1.2)$$

$$k_1 = \bar{T} - k_2\bar{\theta} \qquad (E1.3)$$

Linear Regression

An example

Table 2 shows the summations needed for the calculation of the above two constants k_1 and k_2 of the regression model.

Table 2. Tabulation of data for calculation of needed summations.

i	θ	T	θ^2	$T\theta$
	$Radians$	$N \cdot m$	$Radians^2$	$N \cdot m$
1	0.698132	0.188224	4.87388×10^{-1}	1.31405×10^{-1}
2	0.959931	0.209138	9.21468×10^{-1}	2.00758×10^{-1}
3	1.134464	0.230052	1.2870	2.60986×10^{-1}
4	1.570796	0.250965	2.4674	3.94215×10^{-1}
5	1.919862	0.313707	3.6859	6.02274×10^{-1}
$\sum_{i=1}^{5}$	6.2831	1.1921	8.8491	1.5896

$$k_2 = \frac{n\sum_{i=1}^{5}\theta_i T_i - \sum_{i=1}^{5}\theta_i \sum_{i=1}^{5} T_i}{n\sum_{i=1}^{5}\theta_i^2 - \left(\sum_{i=1}^{5}\theta_i\right)^2}$$
$$= \frac{5(1.5896) - (6.2831)(1.1921)}{5(8.8491) - (6.2831)^2}$$
$$= 9.6091 \times 10^{-2}\text{N-m/rad}$$

$$\bar{T} = \frac{\sum_{i=1}^{5} T_i}{n} = \frac{1.1921}{5} = 2.3842 \times 10^{-1} N - m$$

$$\bar{\theta} = \frac{\sum_{i=1}^{5}\theta_i}{n} = \frac{6.2831}{5} = 1.2566\ radians$$

$$k_1 = \bar{T} - k_2\bar{\theta}$$
$$= 2.3842 \times 10^{-1} - (9.6091 \times 10^{-2})(1.2566)$$
$$= 1.1767 \times 10^{-1}\text{N-m}$$

Linear Regression

An example

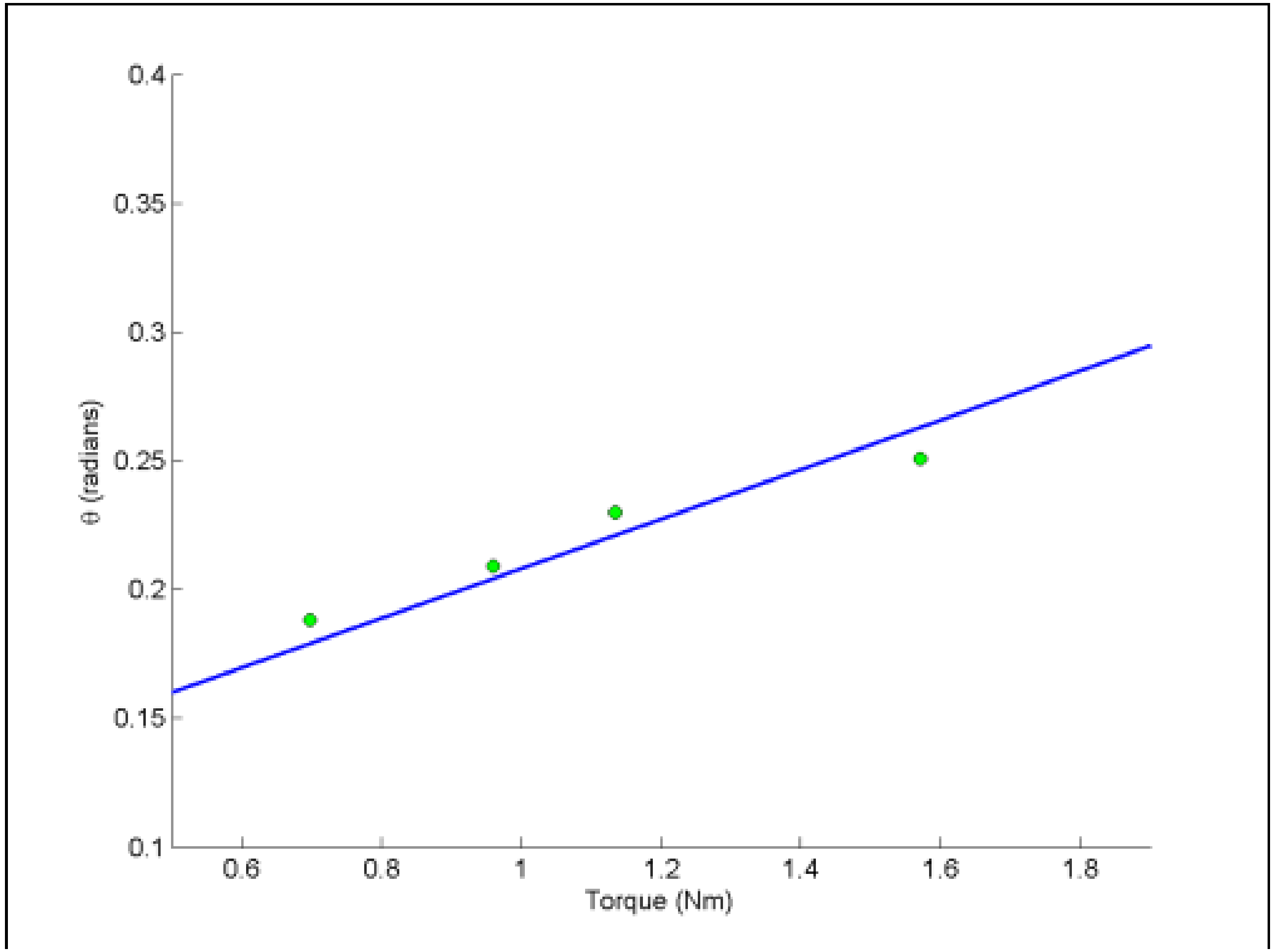


Figure 4 Linear regression of torque vs. angle data

$$k_2 = \frac{n\sum_{i=1}^{5}\theta_i T_i - \sum_{i=1}^{5}\theta_i \sum_{i=1}^{5} T_i}{n\sum_{i=1}^{5}\theta_i^2 - \left(\sum_{i=1}^{5}\theta_i\right)^2}$$

$$= \frac{5(1.5896) - (6.2831)(1.1921)}{5(8.8491) - (6.2831)^2}$$

$$= 9.6091 \times 10^{-2}\text{N-m/rad}$$

$$k_1 = \bar{T} - k_2\bar{\theta}$$

$$= 2.3842 \times 10^{-1} - (9.6091 \times 10^{-2})(1.2566)$$

$$= 1.1767 \times 10^{-1}\text{N-m}$$

Mini Quiz

Forcing the regression line to pass through the origin

If we use $y = a_1 x$ instead of $y = a_0 + a_1 x$ the same formula

doesn't seem to work. Why not?

$$a_1 = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$a_1 = \frac{S_{xy}}{S_{xx}}$$

$$a_0 = \bar{y} - a_1 \bar{x}$$

Linear Regression

The zero y-intercept variant of the model

In this model, we wish to predict response to n data points $(x_1, y_1), (x_2, y_2), \ldots\ldots, (x_n, y_n)$ by a regression model

$$y = a_1 x \qquad (1)$$

where a_1 is the only constant of the regression model.

A measure of goodness of fit, that is, how well $a_1 x$ predicts variable y is the sum of the square of the residuals, S_r

$$S_r = \sum_{i=1}^{n} E_i^{\,2} = \sum_{i=1}^{n} (y_i - a_1 x_i)^2 \qquad (2)$$

To find a_1, we look for the value of a_1 for which S_r is the absolute minimum.

We will begin by conducting the first derivative test. Take the derivative of Equation (2)

$$\frac{dS_r}{da_1} = 2\sum_{i=1}^{n} (y_i - a_1 x_i)(-x_i) = 0 \qquad (3)$$

Now putting $\frac{dS_r}{da_1} = 0$ gives $2\sum_{i=1}^{n} (y_i - a_1 x_i)(-x_i) = 0$

giving

$$-2\sum_{i=1}^{n} y_i x_i + 2\sum_{i=1}^{n} a_1 x_i^2 = 0$$

$$-2\sum_{i=1}^{n} y_i x_i + 2a_1 \sum_{i=1}^{n} x_i^2 = 0$$

Solving the above equation for a_1 gives

$$\boxed{a_1 = \frac{\sum_{i=1}^{n} y_i x_i}{\sum_{i=1}^{n} x_i^2}} \qquad (4)$$

Linear Regression

Is the slope a_1 unique?

Let's conduct the second derivative test.

$$\begin{aligned}\frac{d^2 S_r}{d{a_1}^2} &= \frac{d}{da_1}\left(2\sum_{i=1}^{n}(y_i - a_1 x_i)(-x_i)\right) \\ &= \frac{d}{da_1}\sum_{i=1}^{n}(-2x_i y_i + 2a_1 {x_i}^2) \\ &= \sum_{i=1}^{n} 2{x_i}^2 > 0 \qquad (5)\end{aligned}$$

for at most one $x_i \neq 0$, which is a pragmatic assumption that all the x-values are not zero.

This inequality shows that the Equation (2) value of a_1 corresponds to a location of local minimum. Since the sum of the squares of the residuals, S_r is a continuous function of a_1, that S_r has only one point where $\frac{dS_r}{da_1} = 0$, and at that point, we have $\frac{d^2 S_r}{d{a_1}^2} > 0$, it corresponds not only to a local minimum but an absolute minimum as well. Hence, Equation (4) gives us the value of the constant, a_1 of the regression model, $y = a_1 x$.

Linear Regression

A zero y-intercept example

To find the longitudinal modulus of a composite material, the following data, as given in Table 1, is collected.

Table 1 Stress vs. strain data for a composite material.

Strain (%)	Stress (MPa)
0	0
0.183	306
0.36	612
0.5324	917
0.702	1223
0.867	1529
1.0244	1835
1.1774	2140
1.329	2446
1.479	2752
1.5	2767
1.56	2896

Find the longitudinal modulus E using the regression model.

$$\sigma = E\varepsilon$$

Solution

Rewriting data from Table 1 in the base SI system of units is given in Table 2.

Table 2 Stress vs strain data for a composite in SI system of units

Strain (m/m)	Stress (Pa)
0.0000	0.0000
1.8300×10^{-3}	3.0600×10^{8}
3.6000×10^{-3}	6.1200×10^{8}
5.3240×10^{-3}	9.1700×10^{8}
7.0200×10^{-3}	1.2230×10^{9}
8.6700×10^{-3}	1.5290×10^{9}
1.0244×10^{-2}	1.8350×10^{9}
1.1774×10^{-2}	2.1400×10^{9}
1.3290×10^{-2}	2.4460×10^{9}
1.4790×10^{-2}	2.7520×10^{9}
1.5000×10^{-2}	2.7670×10^{9}
1.5600×10^{-2}	2.8960×10^{9}

Linear Regression

A zero y-intercept example

Using Equation (4) gives

$$E = \frac{\sum_{i=1}^{n} \sigma_i \varepsilon_i}{\sum_{i=1}^{n} {\varepsilon_i}^2} \qquad (E1.1)$$

The summations used in Equation (E1.1) are given in Table 3.

Table 3 Tabulation for Example 2 for needed summations

i	ε	σ	ε^2	$\varepsilon\sigma$
1	0.0000	0.0000	0.0000	0.0000
2	1.8300×10^{-3}	3.0600×10^{8}	3.3489×10^{-6}	5.5998×10^{5}
3	3.6000×10^{-3}	6.1200×10^{8}	1.2960×10^{-5}	2.2032×10^{6}
4	5.3240×10^{-3}	9.1700×10^{8}	2.8345×10^{-5}	4.8821×10^{6}
5	7.0200×10^{-3}	1.2230×10^{9}	4.9280×10^{-5}	8.5855×10^{6}
6	8.6700×10^{-3}	1.5290×10^{9}	7.5169×10^{-5}	1.3256×10^{7}
7	1.0244×10^{-2}	1.8350×10^{9}	1.0494×10^{-4}	1.8798×10^{7}
8	1.1774×10^{-2}	2.1400×10^{9}	1.3863×10^{-4}	2.5196×10^{7}
9	1.3290×10^{-2}	2.4460×10^{9}	1.7662×10^{-4}	3.2507×10^{7}
10	1.4790×10^{-2}	2.7520×10^{9}	2.1874×10^{-4}	4.0702×10^{7}
11	1.5000×10^{-2}	2.7670×10^{9}	2.2500×10^{-4}	4.1505×10^{7}
12	1.5600×10^{-2}	2.8960×10^{9}	2.4336×10^{-4}	4.5178×10^{7}
$\sum_{i=1}^{12}$			1.2764×10^{-3}	2.3337×10^{8}

$$n = 12$$

$$\sum_{i=1}^{12} \varepsilon_i^2 = 1.2764 \times 10^{-3}$$

$$\sum_{i=1}^{12} \sigma_i \varepsilon_i = 2.3337 \times 10^{8}$$

$$E = \frac{\sum_{i=1}^{12} \sigma_i \varepsilon_i}{\sum_{i=1}^{12} {\varepsilon_i}^2} = \frac{2.3337 \times 10^{8}}{1.2764 \times 10^{-3}} = 1.8284 \times 10^{11} \text{ Pa} = 182.84 \text{ GPa}$$

Linear Regression

A zero y-intercept example

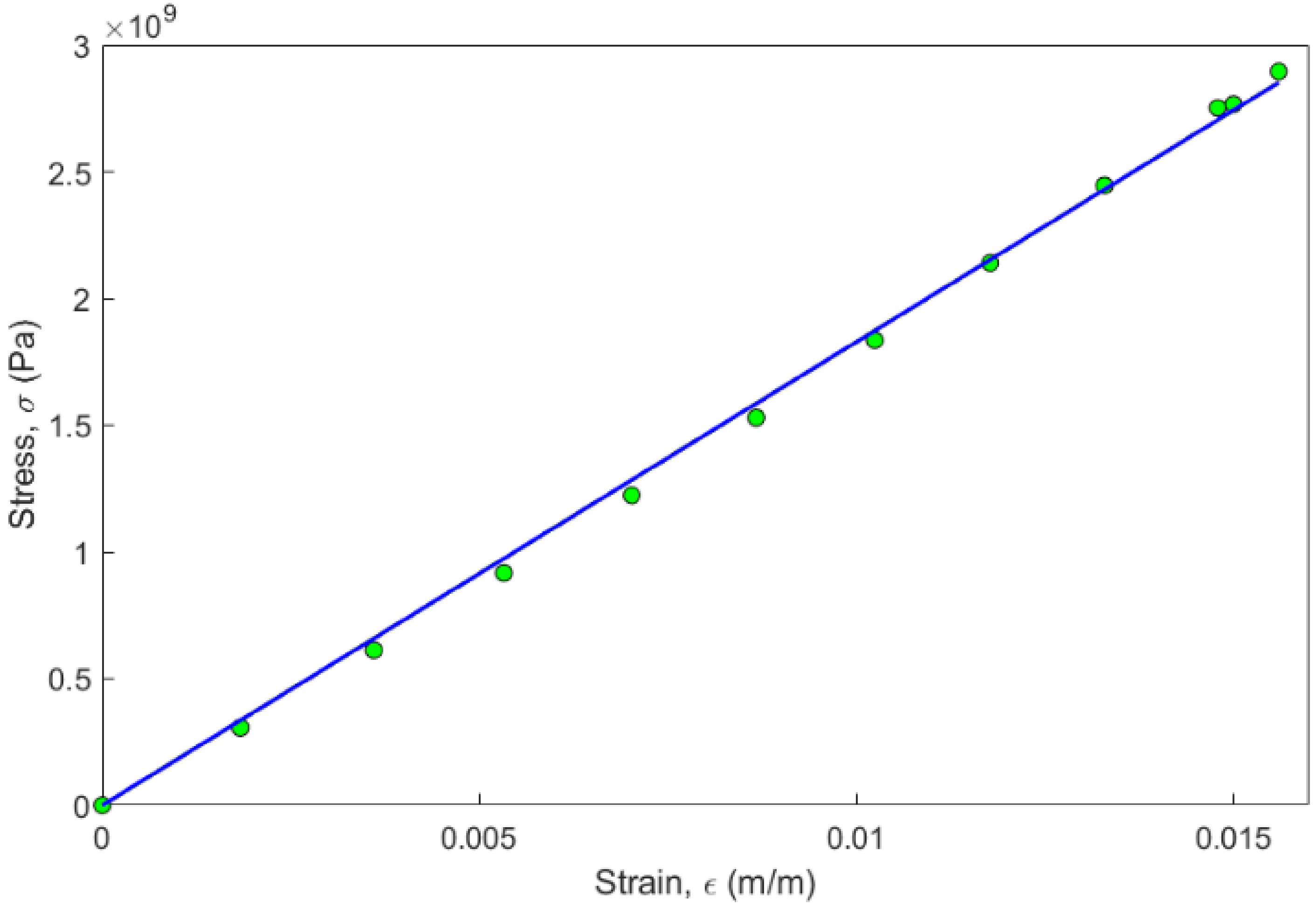


Figure 1. Stress vs strain data and regression model for a composite material uniaxial test

$$E = \frac{\sum_{i=1}^{12} \sigma_i \varepsilon_i}{\sum_{i=1}^{12} {\varepsilon_i}^2}$$

$$= \frac{2.3337 \times 10^8}{1.2764 \times 10^{-3}}$$

$$= 1.8284 \times 10^{11} \text{ Pa}$$

$$= 182.84 \text{ GPa}$$