Math 4543: Numerical Methods

Lecture 5 — Taylor Theorem Revisited

Syed Rifat Raiyan
Lecturer
Department of Computer Science & Engineering
Islamic University of Technology, Dhaka, Bangladesh
Email: rifatraiyan@iut-dhaka.edu

Lecture Plan

The agenda for today

- Understand the Taylor theorem in simple terms
- Why do we need the Taylor series?
- Apply Taylor's theorem to calculate a function at a point
- Derivation of Maclaurin series of transcendental functions from Taylor series
- Apply Taylor series to calculate a transcendental function at a point
- Bounds of error in Taylor series using the Remainder theorem

Taylor Theorem

What is it?

$$f\left(x+h\right)=f\left(x\right)+f'\left(x\right)h+\frac{f''\left(x\right)}{2!}h^2+\frac{f'''\left(x\right)}{3!}h^3+\ldots$$

We can calculate the value of the function $f\left(x\right)$ at $\left(x+h\right)$, provided the function $f\left(x\right)$ and all the derivatives of $f\left(x\right)$ are given at a single point, x.

- ✓ The derivatives must exist.
- ✓ The function must be continuous between x and $\left(x+h\right)$.

Taylor Series

Applications of the series

- We ***derive numerical methods' formulae*** from Taylor series (*e.g*. Newton-Raphson method).
- We use the Taylor series to derive Maclaurin series of ***transcendental functions*** and ***approximate*** them. (useful for calculators and computers!)

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Taylor Series

An example

Find the value of $f(6)$ given that f$(4) = 125$, $f'(4) = 74$, $f''(4) = 30$, $f'''(4) = 6$ and all other higher derivatives of $f(x)$ at $x = 4$ are zero.

Solution

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + \ldots \qquad (E2.1)$$

$$x = 4$$

$$\begin{aligned} h &= 6 - 4 \\ &= 2 \end{aligned}$$

Since fourth and higher derivatives of $f(x)$ are zero at $x = 4$, from Equation (E2.1)

$$f(4+2) = f(4) + f'(4)2 + f''(4)\frac{2^2}{2!} + f'''(4)\frac{2^3}{3!}$$

$$\begin{aligned} f(6) &= 125 + 74(2) + 30\left(\frac{2^2}{2!}\right) + 6\left(\frac{2^3}{3!}\right) \\ &= 125 + 148 + 60 + 8 \\ &= 341 \end{aligned}$$

Taylor Series

An example

Note that to find $f(6)$ exactly, we only needed the value of the function and all its derivatives at some other point, in this case, $x = 4$. We did not need the expression for the function or for its derivatives. Taylor series application would be redundant if we needed to know the expression for the function, as we could just substitute $x = 6$ in it to get the value of $f(6)$.

Side Note: Actually, the problem posed above was obtained from a known function $f(x) = x^3 + 3x^2 + 2x + 5$ and hence $f(4) = 125$, $f'(4) = 74$, $f''(4) = 30$, $f'''(4) = 6$, and all other higher derivatives are zero.

$$f(6) = 6^3 + 3 \times 6^2 + 2 \times 6 + 5 = 341$$

Transcendental Functions

What are they?

A transcendental function can be defined as a function that is ***not algebraic*** and ***cannot be expressed*** in terms of a ***finite sequence of algebraic operations***.

- **Trigonometric functions** — $\sin(x)$, $\cos(x)$, $\tan(x)$, $\sec(x)$, $\text{cosec}(x)$
- **Inverse trigonometric functions** — $\arcsin(x)$, $\arccos(x)$, $\arctan(x)$, $\text{arcsec}(x)$, $\text{arccosec}(x)$
- **Logarithmic functions** — $\log(x)$, $\ln(x)$
- **Exponential functions** — e^x
- **Hyperbolic functions** — $\sinh(x)$, $\cosh(x)$, $\tanh(x)$
- and many more…

Maclaurin Series

A derivation example

Derive the Maclaurin series of $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

Solution

Maclaurin series is simply a Taylor series around the point $x = 0$.

$$f(x) = e^x,\ f(0) = 1$$

$$f'(x) = e^x,\ f'(0) = 1$$

$$f''(x) = e^x,\ f''(0) = 1$$

$$f'''(x) = e^x,\ f'''(0) = 1$$

$$f''''(x) = e^x,\ f''''(0) = 1$$

$$f'''''(x) = e^x,\ f'''''(0) = 1$$

Using the Taylor series now,

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + f''''(x)\frac{h^4}{4!} + f'''''(x)\frac{h^5}{5!} + \dots$$

Maclaurin Series

A derivation example

For $x = 0$

$$f(0+h) = f(0) + f'(0)\,h + f''(0)\frac{h^2}{2!} + f'''(0)\frac{h^3}{3!} + f''''(0)\frac{h^4}{4!} + f'''''(0)\frac{h^5}{5!} + \ldots$$

$$\begin{aligned} f(h) &= f(0) + f'(0)\,h + f''(0)\frac{h^2}{2!} + f'''(0)\frac{h^3}{3!} + f''''(0)\frac{h^4}{4!} + f'''''(0)\frac{h^5}{5!} + \ldots \\ &= 1 + 1(h) + 1\frac{h^2}{2!} + 1\frac{h^3}{3!} + 1\frac{h^4}{4!} + 1\frac{h^5}{5!} + \ldots \\ &= 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \frac{h^4}{4!} + \frac{h^5}{5!} + \ldots \end{aligned}$$

and can be rewritten as

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

and hence

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

Taylor Series

A transcendental function example

In a way, we know the value of $\sin(x)$ and all its derivatives at $x = \frac{\pi}{2}$ without any need for any major calculations. We do not need to use any calculators; just use our knowledge of differential calculus and trigonometry. Can you use the Taylor series and the information above to find the approximate value of $\sin(2)$?

Solution

$$x = \frac{\pi}{2}$$

$$x + h = 2$$

$$\begin{aligned} h &= 2 - x \\ &= 2 - \frac{\pi}{2} \quad 3.1416 \\ &= 0.42920 \end{aligned}$$

$$x = \frac{\pi}{2}$$

$$h = 0.42920$$

$$f(x) = \sin(x),\ f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

$$f'(x) = \cos(x),\ f'\left(\frac{\pi}{2}\right) = 0$$

$$f''(x) = -\sin(x),\ f''\left(\frac{\pi}{2}\right) = -1$$

$$f'''(x) = -\cos(x),\ f'''\left(\frac{\pi}{2}\right) = 0$$

$$f''''(x) = sin(x),\ f''''\left(\frac{\pi}{2}\right) = 1$$

$$f\left(\frac{\pi}{2} + h\right) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)h + f''\left(\frac{\pi}{2}\right)\frac{h^2}{2!} + f'''\left(\frac{\pi}{2}\right)\frac{h^3}{3!} + f''''\left(\frac{\pi}{2}\right)\frac{h^4}{4!} + \ldots$$

$$\begin{aligned} f\left(\frac{\pi}{2} + 0.42920\right) &= 1 + 0\,(0.42920) - 1\frac{(0.42920)^2}{2!} + 0\frac{(0.42920)^3}{3!} \\ &\quad + 1\frac{(0.42920)^4}{4!} + \ldots \\ &= 1 + 0 - 0.092106 + 0 + 0.00141393 + \ldots \\ &\cong 0.90931 \end{aligned}$$

Using the Taylor series

$$f(x+h) = f(x) + f'(x)\,h + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + f''''(x)\frac{h^4}{4!} + \ldots$$

Error in Taylor Series

Establishing bounds for error using Remainder Theorem

As you have noticed, the Taylor series has infinite terms. Only in special cases such as a polynomial of finite order does it have a finite number of terms. So, whenever you are using a Taylor series to calculate the value of a general function, it is being calculated approximately.

The Taylor polynomial of order n of a function $f(x)$ with $(n+1)$ continuous derivatives in the domain $[x, x+h]$ is given by

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + \ldots + f^{(n)}(x)\frac{h^n}{n!} + R_n(x+h)$$

where the remainder is given by

$$R_n(x+h) = \frac{(h)^{n+1}}{(n+1)!} f^{(n+1)}(c).$$

where

$$x < c < x+h$$

that is, c is some point in the domain $(x, x+h)$.

Error in Taylor Series

An example

The Taylor series for e^x at point $x = 0$ is given by $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$

a) What is the truncation (true) error in the representation of e^1 if only four terms of the series are used?

b) Use the remainder theorem to find the bounds of the truncation error.

Solution

a) If only four terms of the series are used, then

$$
\begin{aligned}
e^x &\approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \\
e^1 &\approx 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} \\
&= 2.66667
\end{aligned}
$$

The truncation (true) error would be the unused terms of the Taylor series, which then are

$$
\begin{aligned}
E_t &= \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \\
&= \frac{1^4}{4!} + \frac{1^5}{5!} + \ldots \\
&\cong 0.0516152
\end{aligned}
$$

$$e = 2.718281 \ldots$$

Error in Taylor Series

An example

b) But is there any way to know the bounds of this error other than calculating it directly? Yes.

We know that

$$f(x+h) = f(x) + f'(x)\,h + \ldots + f^{(n)}(x)\,\frac{h^n}{n!} + R_n(x+h)$$

where

$$R_n(x+h) = \frac{(h)^{n+1}}{(n+1)!} f^{(n+1)}(c)\,,\ x < c < x+h, \text{ and}$$

c is some point in the domain $(x, x+h)$. So, in this case, if we are using four terms of the Taylor series, the remainder is given by $(x = 0, n = 3)$

$$\begin{aligned} R_3(0+1) &= \frac{(1)^{3+1}}{(3+1)!} f^{(3+1)}(c) \\ &= \frac{1}{4!} f^{(4)}(c) \\ &= \frac{e^c}{24} \end{aligned}$$

Error in Taylor Series

An example

Since

$$x < c < x + h$$

$$0 < c < 0 + 1$$

$$0 < c < 1$$

The error is bound between

$$\frac{e^0}{24} < R_3(1) < \frac{e^1}{24}$$

$$\frac{1}{24} < R_3(1) < \frac{e}{24}$$

$$0.041667 < R_3(1) < 0.113261$$

So, the upper bound of the error is 0.113261, which concurs with the calculated error of 0.0516152.