



Math 4543: Numerical Methods

Lecture 12 — Trapezoidal Rule of Integration

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Lecture Plan

The agenda for today

- Understand the concept of Integration
- Recap the Riemann sum method of integration
- Single-segment Trapezoidal Rule
- Multiple-segment Trapezoidal Rule

Integration

What is it?

The dictionary definition of *integration* is combining parts so that they work together or form a whole.

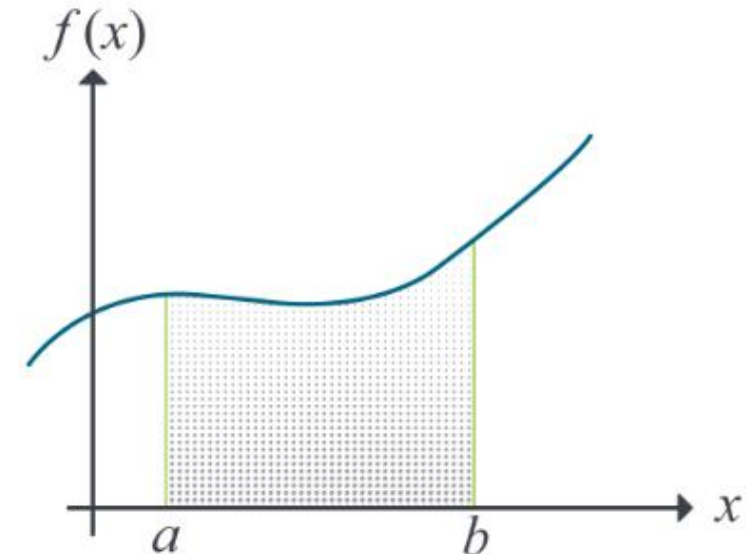
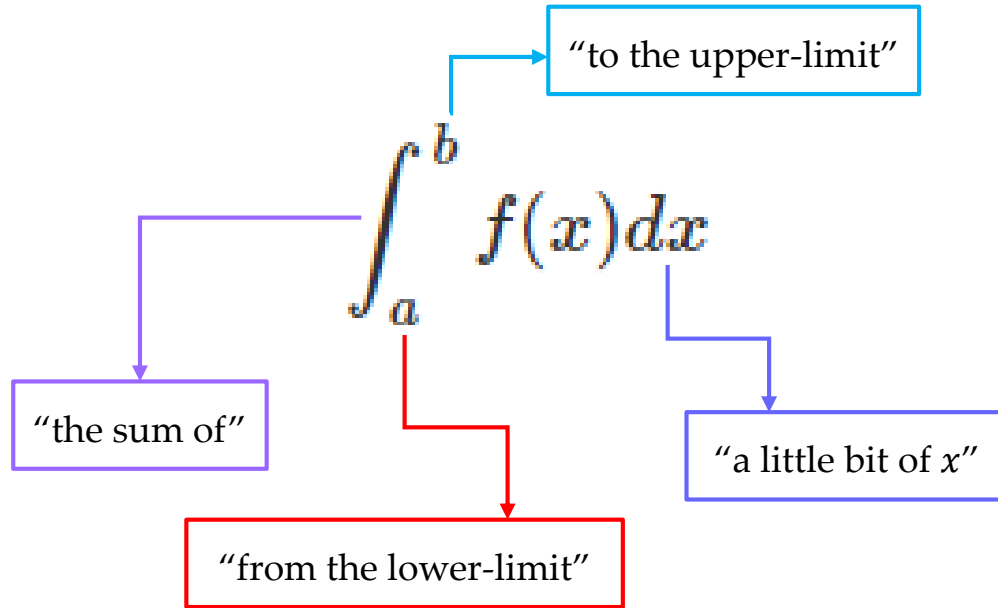
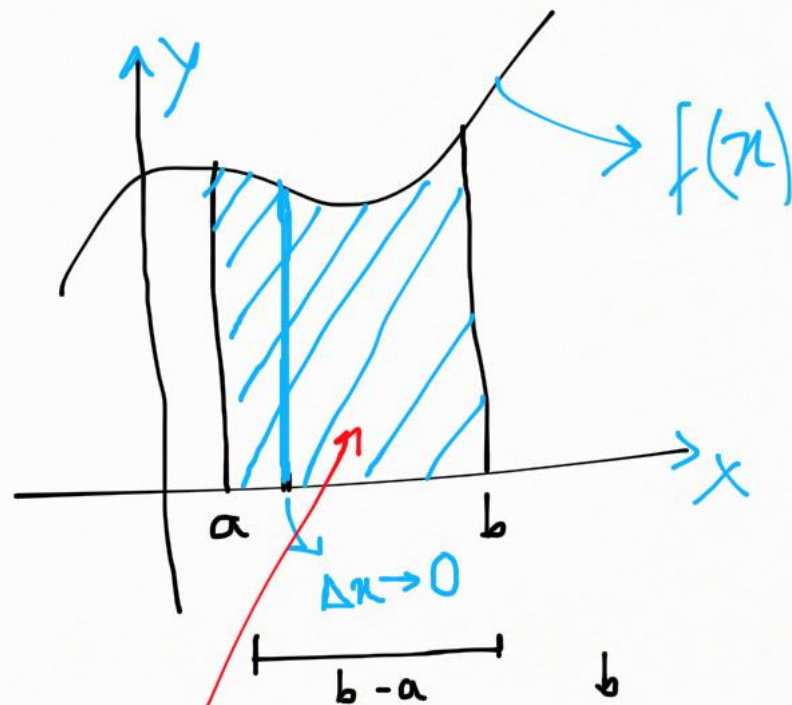
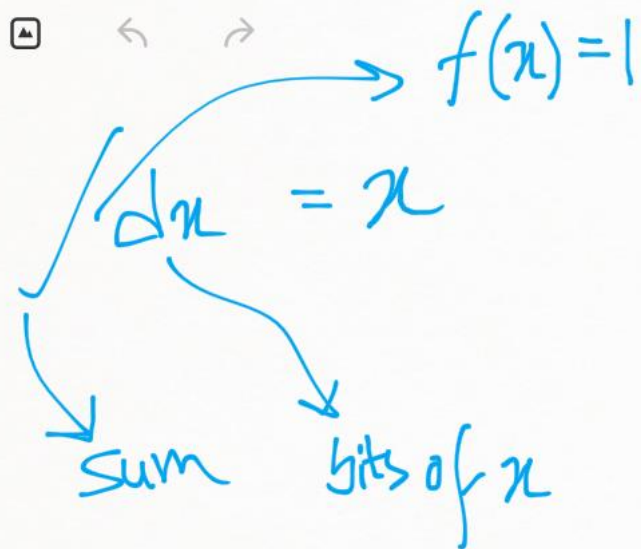


Figure 1 The definite integral as the area of a region under the curve, $\text{Area} = \int_a^b f(x) dx$.

The mean value \bar{f} of a function f in an interval $[a, b]$ is given by

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x) dx$$

Mathematically, integration stands for finding the area under an integrand curve $f(x)$ from one point to another.



$\int_a^b f(x) dx = I$

integration sign

integrand

integral

$$\bar{f} = \frac{\int_a^b f(x) dx}{b-a}$$

Riemann Sum Integration

Simplest way to perform numerical integration

We'll form rectangles of equal widths under the curve and *approximate* the area.

Let f be defined on the closed interval $[a, b]$, and let Δ be an arbitrary partition of $[a, b]$ such as:

$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$, where Δx_i is the length of the i^{th} subinterval (Figure 1).

If c_i is any point in the i^{th} subinterval, then the sum

$$\sum_{i=1}^n f(c_i) \Delta x_i, \quad x_{i-1} \leq c_i \leq x_i$$

is called a Riemann sum of the function f for the partition Δ on the interval $[a, b]$. For a given partition Δ , the length of the longest subinterval is called the norm of the partition. It is denoted by $\|\Delta\|$ (the norm of Δ). The following limit is used to define the definite integral.

$$\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i = I$$

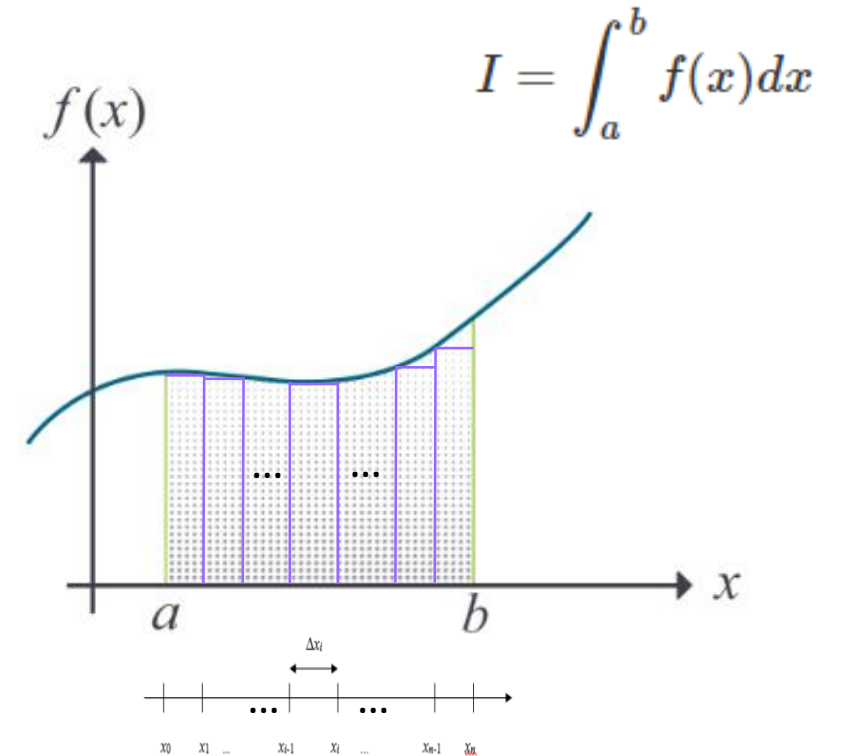
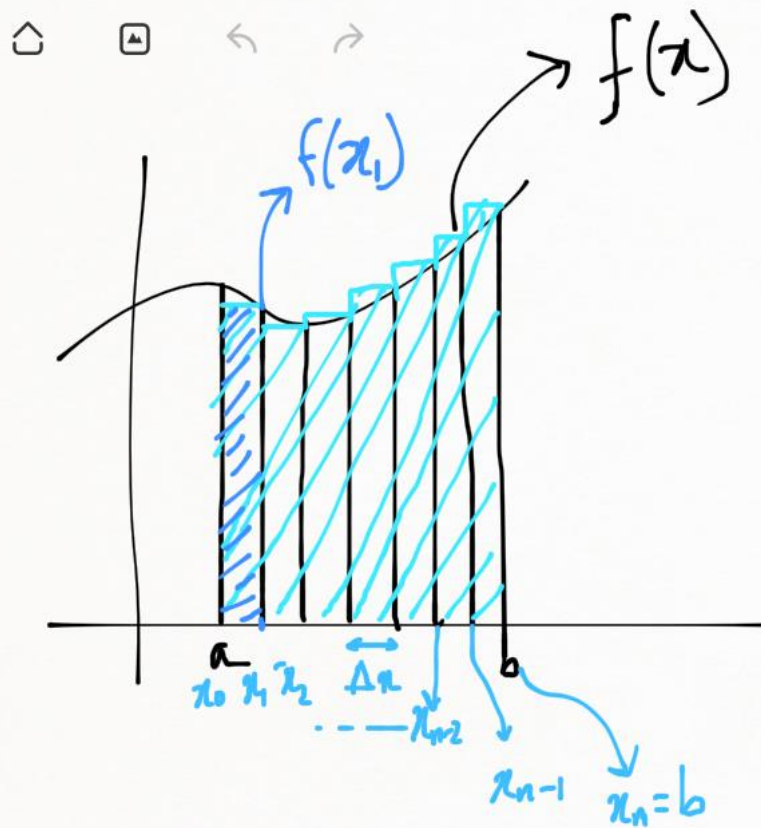


Figure 1 The definite integral as the area of a region under the curve, $\text{Area} = \int_a^b f(x) dx$.



$$I = \int_a^b f(x) dx \approx \sum_{i=1}^n f(x_i) \Delta x$$

$$\lim_{\Delta x \rightarrow 0} \sum_{i=1}^n f(x_i) \Delta x = I$$

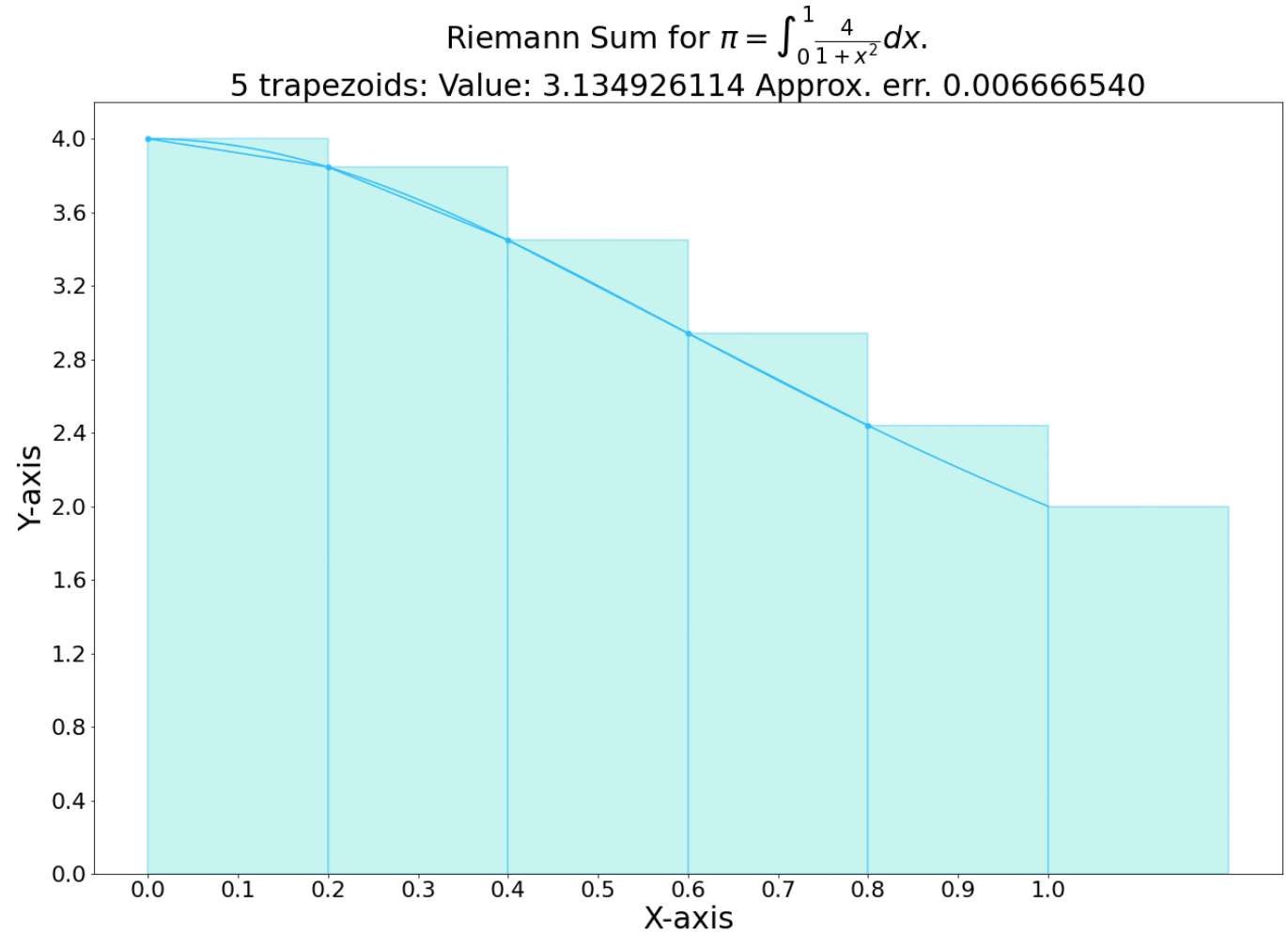
$\searrow n \rightarrow \infty$

Riemann Sum Integration

Simplest way to perform numerical integration

$$I = \int_a^b f(x) dx$$

$$\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i = I$$



Single-segment Trapezoidal Rule

Using trapezoids instead of rectangles

The idea is to form a *single trapezoid* under the curve and *approximate* the area.

The trapezoidal rule is based on the Newton-Cotes formula that if one approximates the integrand by an n^{th} order polynomial, then the integral of the function is approximated by the integral of that n^{th} order polynomial.

$$f_n(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1} + a_nx^n.$$

where $f_n(x)$ is a n^{th} order polynomial. The trapezoidal rule assumes $n = 1$, that is, approximating the integral by a linear polynomial (straight line),

$$\int_a^b f(x)dx \approx \int_a^b f_1(x)dx$$

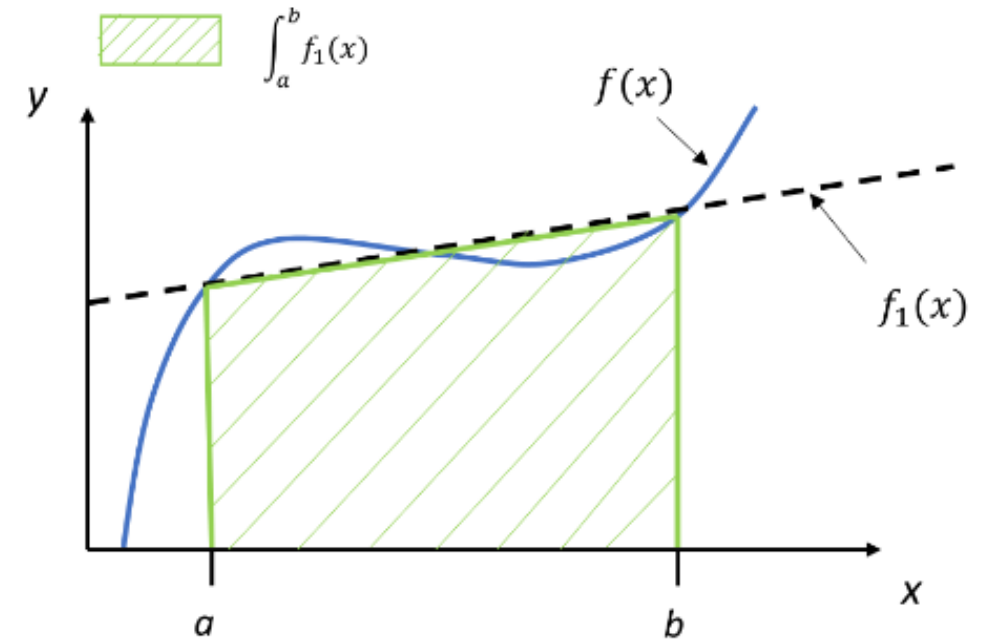


Figure 2. Approximating the function by a first-order polynomial to derive the trapezoidal rule

Single-segment Trapezoidal Rule

Method 1: Derivation from Calculus

Approximating the integrand $f(x)$ by a first-order polynomial $f_1(x)$, that is, $f_1(x) = a_0 + a_1x$,

$$\begin{aligned}\int_a^b f(x)dx &\approx \int_a^b f_1(x)dx = \int_a^b (a_0 + a_1x)dx \\ &= a_0(b-a) + a_1 \left(\frac{b^2 - a^2}{2} \right)\end{aligned}\quad (5)$$

But what are a_0 and a_1 ? Now if one chooses, $(a, f(a))$ and $(b, f(b))$ as the two points to approximate $f(x)$ by a straight line from a to b ,

$$\begin{aligned}f(a) &= f_1(a) = a_0 + a_1a \\ f(b) &= f_1(b) = a_0 + a_1b\end{aligned}\quad (6a, b)$$

Solving Equations (6a) and (6b) for a_0 and a_1 , we get

$$a_1 = \frac{f(b) - f(a)}{b - a} \quad a_0 = \frac{f(a)b - f(b)a}{b - a}\quad (7a, b)$$

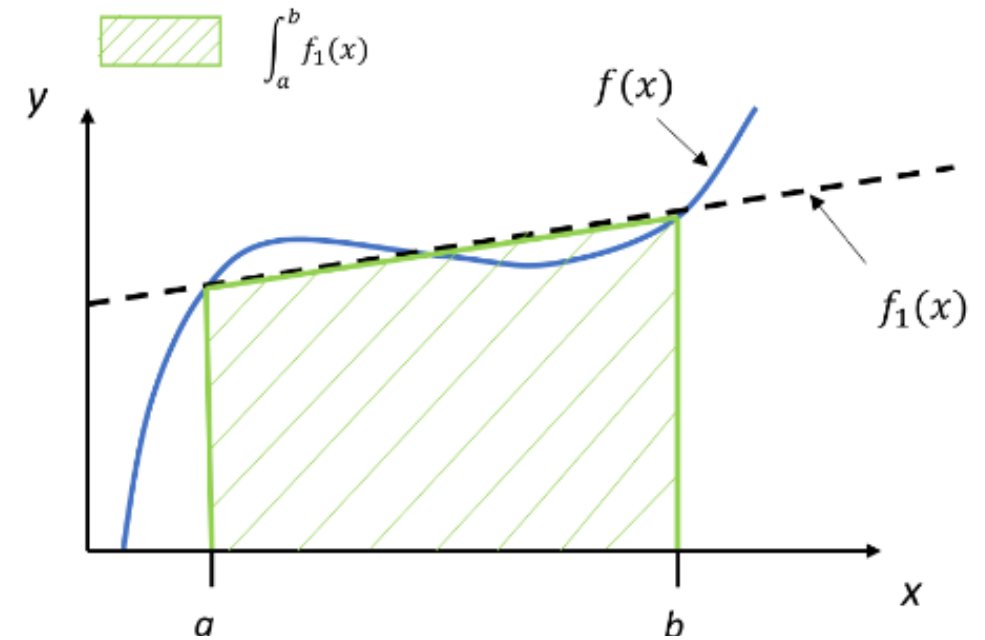
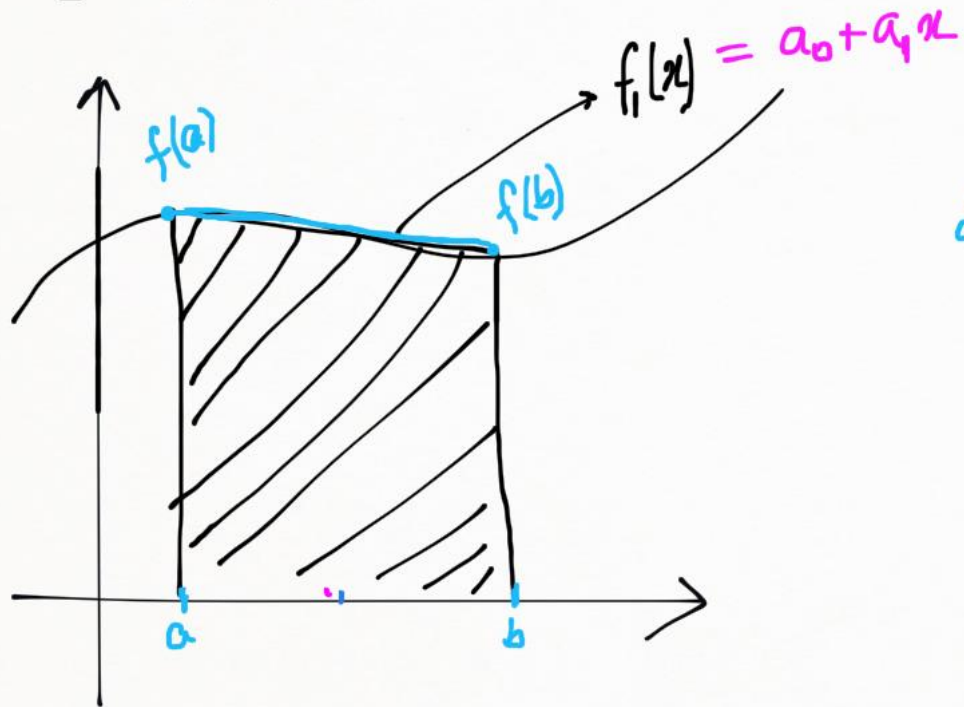


Figure 2. Approximating the function by a first-order polynomial to derive the trapezoidal rule

Substituting values of a_0 and a_1 from Equations (7a) and (7b) in Equation (5) gives,

$$\begin{aligned}\int_a^b f(x)dx &\approx \frac{f(a)b - f(b)a}{b - a}(b - a) + \frac{f(b) - f(a)}{b - a} \frac{b^2 - a^2}{2} \\ &= (b - a) \left[\frac{f(a) + f(b)}{2} \right]\end{aligned}\quad (8)$$



$$\begin{aligned} \int_a^b f(x) dx &\approx \int_a^b f_1(x) dx = \int_a^b (a_0 + a_1 x) dx \\ &= a_0 [x]_a^b + a_1 \left[\frac{x^2}{2} \right]_a^b \\ &= \underline{a_0} (b-a) + \underline{a_1} \left(\frac{b^2 - a^2}{2} \right) \end{aligned}$$

$$f(a) = a_0 + a_1 a$$

$$f(b) = a_0 + a_1 b$$

\Downarrow

$$f(b) - f(a) = a_1 (b-a) \Rightarrow a_1 = \frac{f(b) - f(a)}{b-a}$$

$$a_1 = \frac{f(b) - f(a)}{b-a}$$

$$\begin{aligned} f(a) &= a_0 + \frac{f(b) - f(a)}{b-a} a \\ \Rightarrow f(a) &= a_0 + \frac{a f(b) - a f(a)}{b-a} \end{aligned}$$

$$\Rightarrow f(a) = \frac{a_0 (b-a) + a f(b) - a f(a)}{b-a}$$

$$\Rightarrow b f(a) - \cancel{a f(a)} - \cancel{a f(b)} + \cancel{a f(a)} = a_0 (b-a)$$

$$\Rightarrow \underline{a_0 = \frac{b f(a) - a f(b)}{b-a}}$$

$$\begin{aligned}
 \int_a^b f(x) dx &= \frac{bf(a) - af(b)}{\cancel{(b-a)}} \cancel{(b-a)} + \frac{f(b) - f(a)}{\cancel{(b-a)}} \left(\frac{\cancel{(b^2 - a^2)}^{b+a}}{2} \right) \\
 &= \{bf(a) - af(b)\} + \{f(b) - f(a)\} \frac{b+a}{2} \\
 &= \frac{2bf(a) - 2af(b) + bf(b) + af(b) - bf(a) - af(a)}{2} \\
 &= \frac{bf(a) - af(b) + bf(b) - af(a)}{2} \\
 &= \frac{b\{f(a) + f(b)\} - a\{f(a) + f(b)\}}{2} = (b-a) \left[\frac{f(a) + f(b)}{2} \right]
 \end{aligned}$$

Single-segment Trapezoidal Rule

Method 2: Derivation from Calculus (using NDD polynomial)

$f_1(x)$ can also be approximated by using Newton's divided difference polynomial as

$$f_1(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \quad (A.1)$$

Hence

$$\begin{aligned} \int_a^b f(x) dx &\approx \int_a^b f_1(x) dx = \int_a^b \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right] dx \\ &= \left[f(a)x + \frac{f(b) - f(a)}{b - a} \left(\frac{x^2}{2} - ax \right) \right]_a^b \\ &= f(a)b - f(a)a + \left(\frac{f(b) - f(a)}{b - a} \right) \left(\frac{b^2}{2} - ab - \frac{a^2}{2} + a^2 \right) \\ &= f(a)b - f(a)a + \left(\frac{f(b) - f(a)}{b - a} \right) \left(\frac{b^2}{2} - ab + \frac{a^2}{2} \right) \\ &= f(a)b - f(a)a + \left(\frac{f(b) - f(a)}{b - a} \right) \frac{1}{2} (b - a)^2 \end{aligned}$$

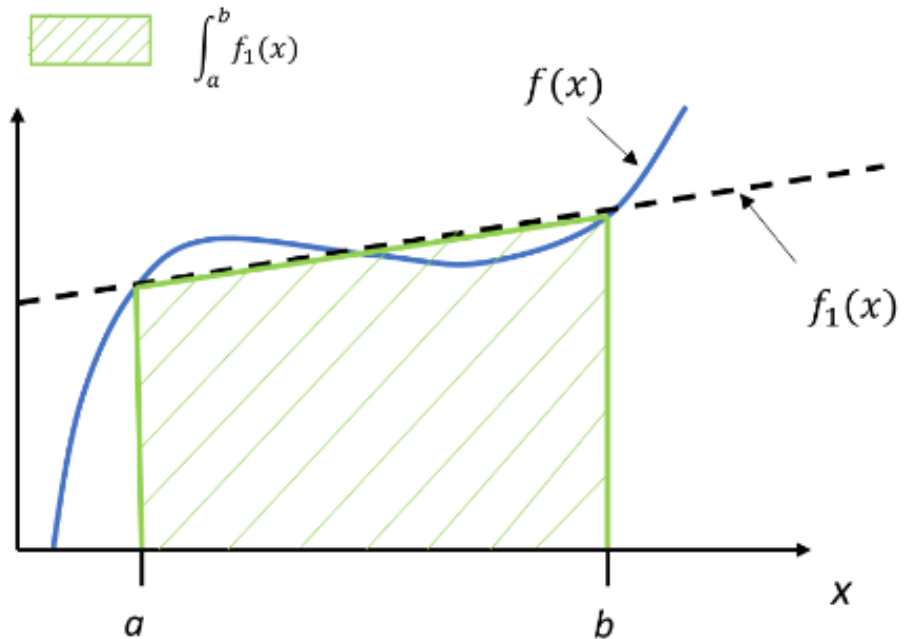


Figure 2. Approximating the function by a first-order polynomial to derive the trapezoidal rule

$$\begin{aligned} &= f(a)b - f(a)a + \frac{1}{2} (f(b) - f(a)) (b - a) \\ &= f(a)b - f(a)a + \frac{1}{2} f(b)b - \frac{1}{2} f(b)a - \frac{1}{2} f(a)b + \frac{1}{2} f(a)a \\ &= \frac{1}{2} f(a)b - \frac{1}{2} f(a)a + \frac{1}{2} f(b)b - \frac{1}{2} f(b)a \\ &= (b - a) \left[\frac{f(a) + f(b)}{2} \right] \quad (A.2) \end{aligned}$$

$$f_1(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

$$\begin{aligned} \int_a^b f_1(x) dx &= \left[x f(a) + \frac{f(b)-f(a)}{b-a} \left(\frac{x^2}{2} - ax \right) \right]_a^b \\ &= [b f(a) - a f(a)] + \left[\left(\frac{f(b)-f(a)}{b-a} \right) \left(\frac{b^2}{2} - ab - \frac{a^2}{2} + a^2 \right) \right] \\ &= [b f(a) - a f(a)] + \left[\left(\frac{f(b)-f(a)}{b-a} \right) \left(\frac{b^2}{2} - ab + \frac{a^2}{2} \right) \right] \\ &= [b f(a) - a f(a)] + \left[\left(\frac{f(b)-f(a)}{b-a} \right) \left(\frac{b^2 - 2ab + a^2}{2} \right) \right] \\ &= [b f(a) - a f(a)] + \left[\frac{f(b)-f(a)}{\cancel{b-a}} \frac{\cancel{b-a}^2}{2} \right] \rightarrow [b-a] \end{aligned}$$

$$\begin{aligned} &= [b f(a) - a f(a)] + \frac{f(b)-f(a)}{2} (b-a) \\ &= b f(a) - a f(a) + \frac{b f(b)}{2} - \frac{a f(b)}{2} - \frac{b f(a)}{2} + \frac{a f(a)}{2} \\ &= \frac{b f(b)}{2} - \frac{a f(b)}{2} + \frac{b f(a)}{2} - \frac{a f(a)}{2} \\ &= b \left(\frac{f(b)}{2} + \frac{f(a)}{2} \right) - a \left(\frac{f(b)}{2} + \frac{f(a)}{2} \right) \\ &= (b-a) \left(\frac{f(a) + f(b)}{2} \right) \end{aligned}$$

Single-segment Trapezoidal Rule

Method 3: Derivation from Geometry

The trapezoidal rule can also be derived from geometry. Look at Figure 2. The area under the curve $f_1(x)$ is the area of a trapezoid.

The integral

$$\begin{aligned}\int_a^b f(x)dx &\approx \text{Area of trapezoid} \\ &= \frac{1}{2} (\text{Sum of the length of parallel sides}) \\ &\quad \times (\text{Perpendicular distance between the parallel sides}) \\ &= \frac{1}{2} (f(b) + f(a)) (b - a) \\ &= (b - a) \left[\frac{f(a) + f(b)}{2} \right] \quad (A.3)\end{aligned}$$

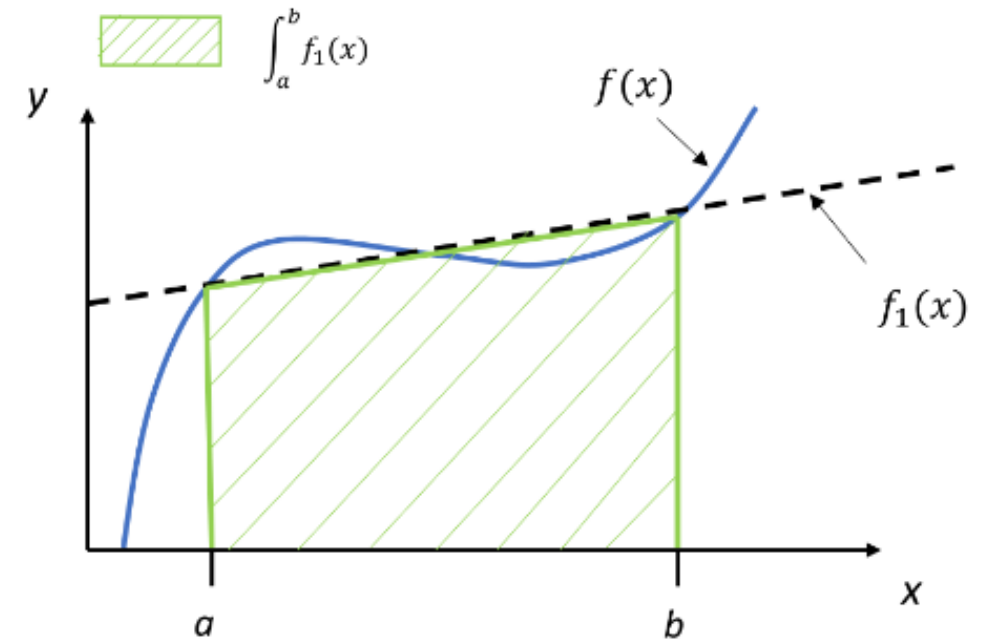


Figure 2. Approximating the function by a first-order polynomial to derive the trapezoidal rule

Single-segment Trapezoidal Rule

Method 4: Derivation from Method of Coefficients

Choose the integral $\int_a^b f(x)dx$ approximated as follows.

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b) \quad (A.4)$$

The coefficients c_1 and c_2 are undetermined. We will find these coefficients such that the right-hand side is exact for integrals of a straight line $a_0 + a_1 x$.

So from exact integration

$$\begin{aligned} \int_a^b (a_0 + a_1 x) dx &= \left[a_0 x + a_1 \frac{x^2}{2} \right]_a^b \\ &= a_0 (b - a) + a_1 \left(\frac{b^2 - a^2}{2} \right) \end{aligned} \quad (A.5)$$

But we want the right-hand side formula to give the same result as Equation (A.5) for $f(x) = a_0 + a_1 x$ which is

$$\begin{aligned} c_1 f(a) + c_2 f(b) &= c_1 (a_0 + a_1 a) + c_2 (a_0 + a_1 b) \\ &= a_0 (c_1 + c_2) + a_1 (c_1 a + c_2 b) \end{aligned} \quad (A.6)$$

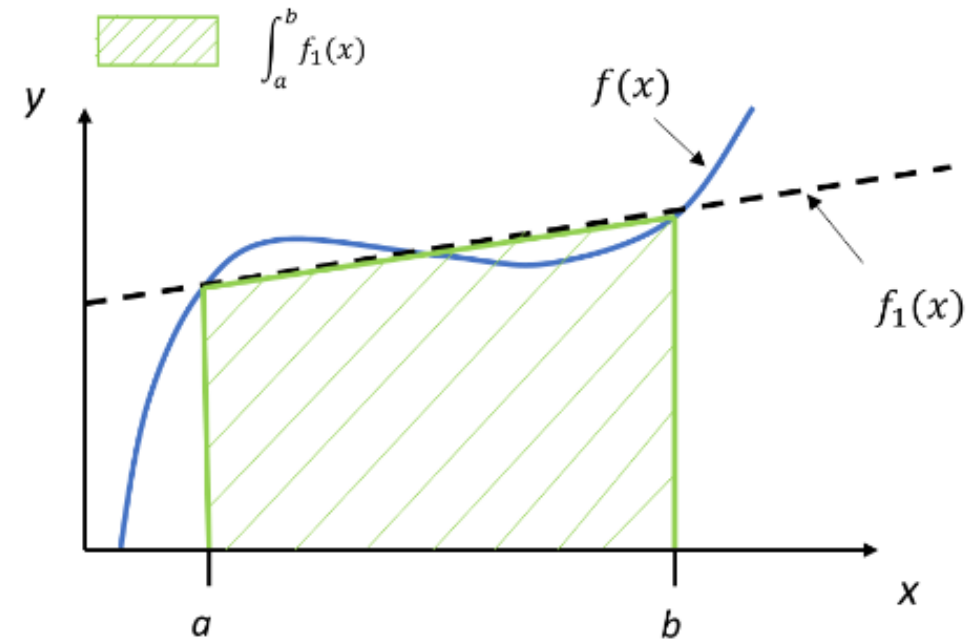


Figure 2. Approximating the function by a first-order polynomial to derive the trapezoidal rule

Hence from Equations (A.5) and (A.6),

$$a_0 (b - a) + a_1 \left(\frac{b^2 - a^2}{2} \right) = a_0 (c_1 + c_2) + a_1 (c_1 a + c_2 b) \quad (A.7)$$

Single-segment Trapezoidal Rule

Method 4: Derivation from Method of Coefficients

Since a_0 and a_1 are arbitrary constants for the chosen general straight line, the coefficients of a_0 and a_1 need to be equal. That gives

$$c_1 + c_2 = b - a \quad (A.8a)$$

$$c_1 a + c_2 b = \frac{b^2 - a^2}{2} \quad (A.8b)$$

Multiplying Equation (A.8a) by a and subtracting from Equation (A.8b) gives

$$c_2 = \frac{b - a}{2} \quad (A.9a)$$

Substituting the value of c_2 from Equation (A.9a) in Equation (A.8a) gives

$$c_1 = \frac{b - a}{2} \quad (A.9b)$$

Therefore, from Equation (A.4), (A.9a), and (A.9b),

$$\begin{aligned} \int_a^b f(x) dx &\approx c_1 f(a) + c_2 f(b) \\ &= \frac{b - a}{2} f(a) + \frac{b - a}{2} f(b) \end{aligned} \quad (A.10)$$

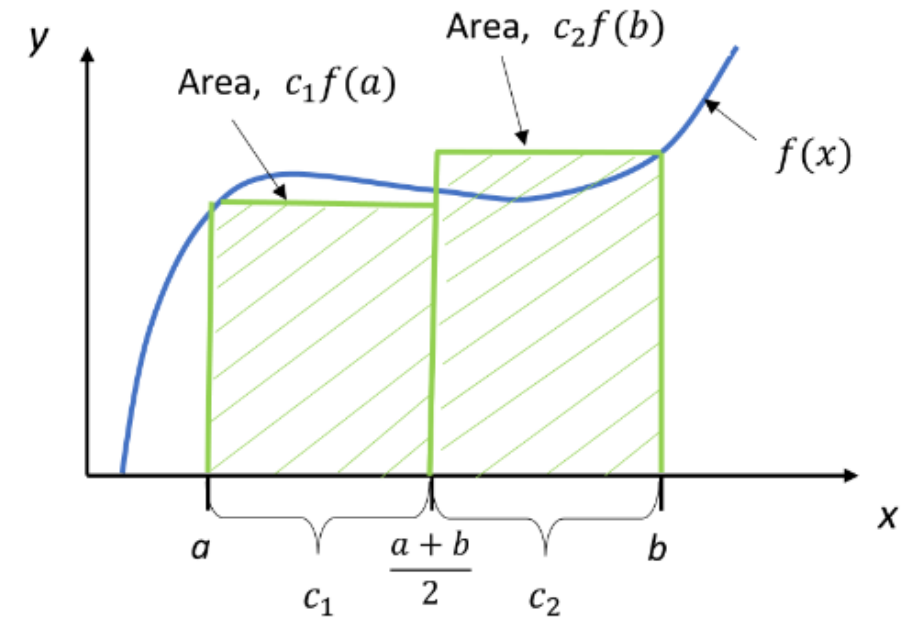
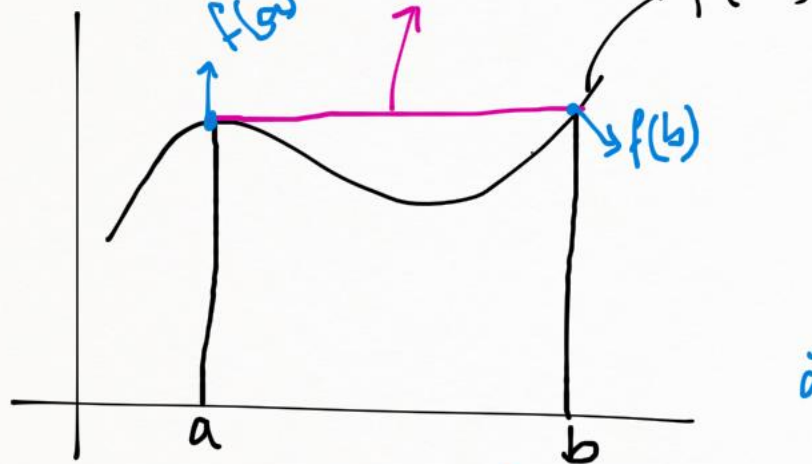


Figure 3 Area by the method of coefficients.



$$\int_a^b f_1(x) dx \approx c_1 f(a) + c_2 f(b)$$

$$\int_a^b f_1(x) dx = \int_a^b (a_0 + a_1x) dx = \underline{a_0(b-a)} + a_1 \left(\underline{\frac{b^2 - a^2}{2}} \right)$$

$$c_1 = \frac{(b-a) - c_2}{b-a} = \frac{b-a}{b-a} - \frac{c_2}{b-a} = \frac{b-a}{b-a} - \frac{c_2}{b-a}$$

$$c_1 f(a) + c_2 f(b) = c_1 (a_0 + a_1 a) + c_2 (a_0 + a_1 b) = c_1 a_0 + c_1 a_1 a + c_2 a_0 + c_2 a_1 b = \underline{a_0 (c_1 + c_2)} + a_1 \underline{(c_1 a + c_2 b)}$$

$$c_1 + c_2 = b - a \quad \text{--- (i)}$$

$$c_1 a + c_2 b = \frac{b^2 - a^2}{2} \quad \text{--- (ii)}$$

$$\Rightarrow c_1 a + c_2 a = ab - a^2$$

$$\Rightarrow c_1 a = ab - a^2 - c_2 a$$

$$ab - a^2 - c_2 a + c_2 b = \frac{b^2 - a^2}{2}$$

$$\Rightarrow \frac{b^2 - a^2}{2} = a(b-a) + c_2(b-a)$$

$$\Rightarrow \frac{(b+a)(b-a)}{2} = (b-a)(a + c_2)$$

$$\Rightarrow \frac{b+a}{2} = a + c_2$$

$$\Rightarrow c_2 = \frac{b+a}{2} - a = \frac{b-a}{2}$$

$$C_1 = \frac{b-a}{2}$$

$$C_2 = \frac{b-a}{2}$$

$$\begin{aligned} \int_a^b f(x) dx &\approx C_1 f(a) + C_2 f(b) \\ &= \left(\frac{b-a}{2}\right) f(a) + \left(\frac{b-a}{2}\right) f(b) \\ &= (b-a) \left[\frac{f(a) + f(b)}{2} \right] \end{aligned}$$



Single-segment Trapezoidal Rule

An example

The following integral is given

$$\int_{0.1}^{1.3} 5xe^{-2x} dx$$

- a) Use the trapezoidal rule to estimate the value of the integral.
- b) Find the true error, E_t for part (a).
- c) Find the absolute relative true error, $|\epsilon_t|$ for part (a).

b) The true error is

$$\begin{aligned} E_t &= \text{True Value} - \text{Approximate Value} \\ &= 0.8939 - 0.5353 \\ &= 0.3586 \end{aligned}$$

Solution

a)

where

$$a = 0.1$$

$$b = 1.3$$

$$\int_b^a f(x) dx \approx \frac{(b-a)}{2} [f(a) + f(b)],$$

Then

$$\begin{aligned} \int_{0.1}^{1.3} f(x) dx &\approx \frac{(1.3 - 0.1)}{2} [f(0.1) + f(1.3)] \\ &= 0.6 [f(0.1) + f(1.3)] \\ &= 0.6 [5(0.1)e^{-2(0.1)} + 5(1.3)e^{-2(1.3)}] \\ &= 0.6(0.4094 + 0.4828) \\ &= 0.5353 \end{aligned}$$

c) The absolute relative true error, $|\epsilon_t|$ would then be

$$\begin{aligned} |\epsilon_t| &= \left| \frac{\text{True Error}}{\text{True Value}} \right| \times 100 \\ &= \left| \frac{0.3586}{0.8939} \right| \times 100 \\ &= 40.12\% \end{aligned}$$

Now that's a lotta error!

Multi-segment Trapezoidal Rule

The general idea

A single segment trapezoidal rule seldom gives you acceptable results for an integral. Instead for higher accuracy and its control, we can use the composite (also called multiple-segment) trapezoidal rule where the integral is broken into segments, and the single-segment trapezoidal rule is applied over each segment.

Divide $(b - a)$ into n equal segments, as shown in Figure 1. Then the width of each segment is

$$h = \frac{b - a}{n} \quad (1)$$

The integral I can be broken into n integrals as

$$\begin{aligned} I &= \int_a^b f(x) dx \\ &= \int_a^{a+h} f(x) dx + \int_{a+h}^{a+2h} f(x) dx + \dots + \int_{a+(n-2)h}^{a+(n-1)h} f(x) dx + \int_{a+(n-1)h}^b f(x) dx \quad (2) \end{aligned}$$

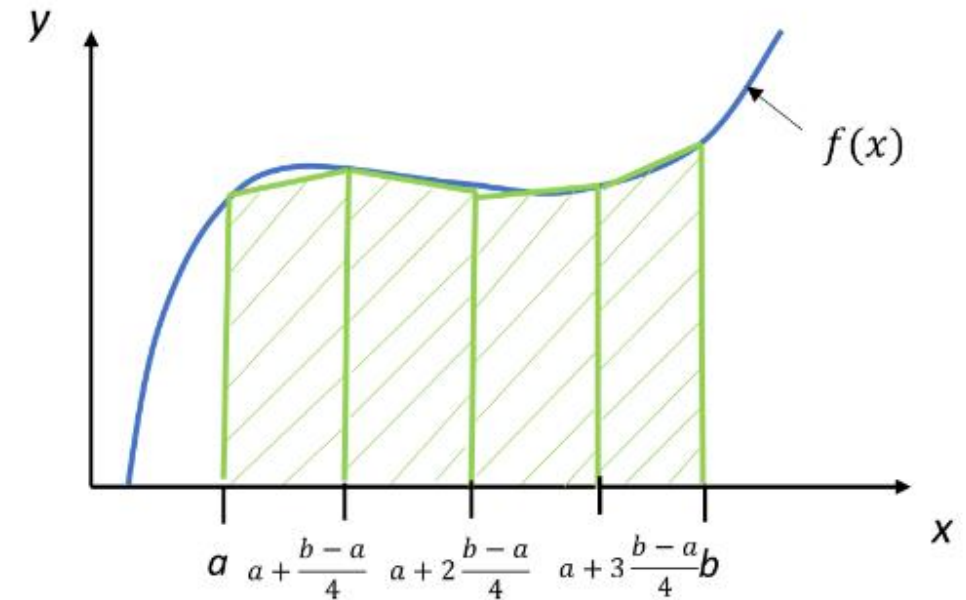


Figure 1 Composite ($n = 4$) trapezoidal rule

Multi-segment Trapezoidal Rule

Deriving the formula

Applying single-segment trapezoidal rule on Equation (2) on each segment gives

$$\begin{aligned}
 \int_a^b f(x)dx &\approx [(a+h) - a] \left[\frac{f(a) + f(a+h)}{2} \right] \\
 &+ [(a+2h) - (a+h)] \left[\frac{f(a+h) + f(a+2h)}{2} \right] \\
 &+ \dots \\
 &+ [(a+(n-1)h) - (a+(n-2)h)] \left[\frac{f(a+(n-2)h) + f(a+(n-1)h)}{2} \right] \\
 &+ [b - (a+(n-1)h)] \left[\frac{f(a+(n-1)h) + f(b)}{2} \right] \\
 &= h \left[\frac{f(a) + f(a+h)}{2} \right] + h \left[\frac{f(a+h) + f(a+2h)}{2} \right] \\
 &+ \dots \\
 &+ h \left[\frac{f(a+(n-2)h) + f(a+(n-1)h)}{2} \right] \\
 &+ h \left[\frac{f(a+(n-1)h) + f(b)}{2} \right]
 \end{aligned}$$

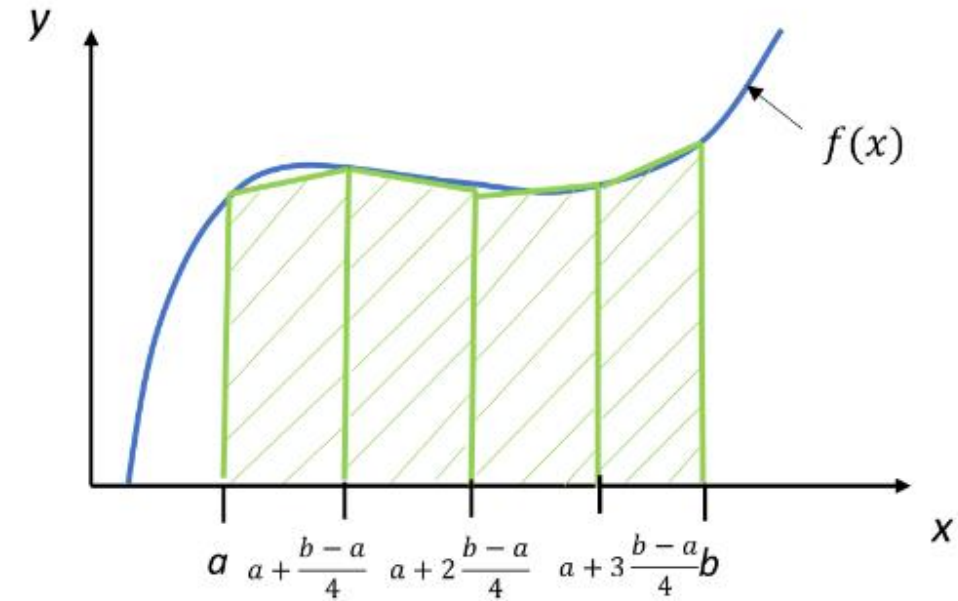


Figure 1 Composite ($n = 4$) trapezoidal rule

$$\begin{aligned}
 &= h \left[\frac{f(a) + 2f(a+h) + 2f(a+2h) + \dots + 2f(a+(n-1)h) + f(b)}{2} \right] \\
 &= \frac{h}{2} \left[f(a) + 2 \left\{ \sum_{i=1}^{n-1} f(a+ih) \right\} + f(b) \right] \\
 &= \frac{b-a}{2n} \left[f(a) + 2 \left\{ \sum_{i=1}^{n-1} f(a+ih) \right\} + f(b) \right] \quad (3)
 \end{aligned}$$

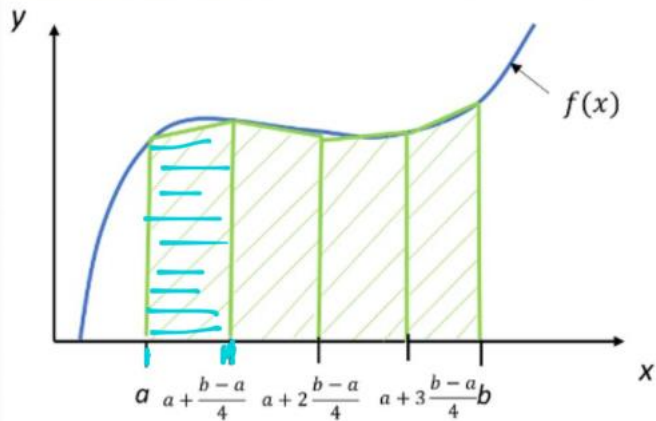


Figure 1 Composite ($n = 4$) trapezoidal rule

$$I = \int_a^b f(x) dx$$

$$= \int_a^{a+h} f(x) dx + \int_{a+h}^{a+2h} f(x) dx + \dots + \int_{a+(n-2)h}^{a+(n-1)h} f(x) dx + \int_{a+(n-1)h}^{a+nh=b} f(x) dx$$

$$= \frac{h}{2} [f(a) + f(a+h)] + \frac{h}{2} [f(a+h) + f(a+2h)] + \dots + \frac{h}{2} [f(a+(n-2)h) + f(a+(n-1)h)] + \frac{h}{2} [f(a+(n-1)h) + f(b)]$$

$$= \frac{h}{2} [f(a) + 2f(a+h) + \dots + 2f(a+(n-1)h) + f(b)]$$

$$= \frac{h}{2} [f(a) + 2 \sum_{i=1}^{n-1} f(a+ih) + f(b)]$$

$$= \frac{b-a}{2n} [f(a) + 2 \sum_{i=1}^{n-1} f(a+ih) + f(b)]$$

Multi-segment Trapezoidal Rule

The same example (now with *multiple segments*)

The following integral is given:

$$\int_{0.1}^{1.3} 5xe^{-2x} dx$$

- a) Use the composite trapezoidal rule to estimate the value of this integral. Use three segments.
- b) Find the true error E_t for part (a).
- c) Find the absolute relative true error $|\varepsilon_t|$ for part (a).

Multi-segment Trapezoidal Rule

The same example (now with *multiple segments*)

Solution

a) The solution using the composite trapezoidal rule with 3 segments is applied as follows.

$$I \approx \frac{b-a}{2n} \left[f(a) + 2 \sum_{i=1}^{n-1} f(a+ih) + f(b) \right]$$

From Equation (3),

$$I \approx \frac{1.3 - 0.1}{6} \left[f(0.1) + 2 \sum_{i=1}^{3-1} f(0.1 + 0.4i) + f(1.3) \right]$$

$$I \approx \frac{1.3 - 0.1}{6} \left[f(0.1) + 2 \sum_{i=1}^2 f(0.1 + 0.4i) + f(1.3) \right]$$

$$= 0.2[f(0.1) + 2f(0.5) + 2f(0.9) + f(1.3)]$$

$$= 0.2[5 \times 0.1 \times e^{-2(0.1)} + 2(5 \times 0.5 \times e^{-2(0.5)}) + 2(5 \times 0.9 \times e^{-2(0.9)}) + 5 \times 1.3 \times e^{-2(1.3)}]$$

$$= 0.84385$$

$$\begin{aligned} n &= 3 & h &= \frac{b-a}{n} \\ a &= 0.1 & &= \frac{1.3 - 0.1}{3} \\ b &= 1.3 & &= 0.4 \end{aligned}$$

Multi-segment Trapezoidal Rule

The same example (now with *multiple segments*)

b) The exact value of the above integral can be found by integration by parts and is

$$\int_{0.1}^{1.3} 5xe^{-2x} dx = 0.89387$$

So the true error is

$$\begin{aligned} E_t &= \text{True Value} - \text{Approximate Value} \\ &= 0.89387 - 0.84385 \\ &= 0.05002 \end{aligned}$$

c) The absolute relative true error is

$$\begin{aligned} |\varepsilon_t| &= \left| \frac{\text{True Error}}{\text{True Value}} \right| \times 100 \\ &= \left| \frac{0.05002}{0.89387} \right| \times 100 \\ &= 5.5959\% \end{aligned}$$

Much better!