Row Space, Column Space, and Nullspace

Linear Algebra

MATH 2010

- **Terminology:** Let A be the 2x4 matrix
$$A = \begin{bmatrix} 2 & 3 & -1 & 0 \\ 4 & 5 & 6 & 2 \end{bmatrix}$$
The *row vectors of* A are
$$\begin{bmatrix} 2 & 3 & -1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 5 & 6 & 2 \end{bmatrix}$$
(the rows of A) in $\Re^4$.
The *column vectors of* A are
$$\begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$
(the columns of A) in $\Re^2$
- **Definition:** Let A be a mxn matrix (recall m is the number of rows and n is the number of columns), then
  - The *row space* of A is the subspace of $\Re^n$ spanned by the row vectors of A
  - The *column space* of A is the subspace of $\Re^m$ spanned by the column vectors of A.
- **Theorem:** If a mxn matrix A is row-equivalent to a mxn matrix B, then the row space of A is equal to the row space of B. (NOT true for the column space)
- **Theorem:** If a matrix A is row-equivalent to a matrix B in row-echelon form, then the nonzero row vectors of B form a basis for the row space of A.
- **Example - Finding a Basis for Row Space** Let
$$A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$$
Find a basis for the row space of A.

  We must reduce A:
$$\begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & -2 & -4 & -1 & 0 \\ 0 & -1 & -2 & -2 & -3 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$w_1 = [1, 1, 4, 1, 2] \;\; w_2 = [0, 1, 2, 1, 1] \;\; w_3 = [0, 0, 0, 1, 2]$$

form a basis for the row space of A.

- **Example- Finding a basis spanned by a set S:** Let $S = \{v_1, v_2, v_3, v_4\}$ where

$$\begin{aligned} v_1 &= [1, -2, 0, 3, -4] \\ v_2 &= [3, 2, 8, 1, 4] \\ v_3 &= [2, 3, 7, 2, 3] \\ v_4 &= [-1, 2, 0, 4, -3] \end{aligned}$$

Find a basis for the subspace of $\Re^5$ spanned by S.

If we look the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix}$$

then a basis for the row space of A gives a basis for the subspace of $\Re^5$ spanned by S.

$$\begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 0 & 8 & 8 & -8 & 16 \\ 0 & 7 & 7 & -4 & 11 \\ 0 & 0 & 0 & 7 & -7 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 7 & 7 & -4 & 11 \\ 0 & 0 & 0 & 7 & -7 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 7 & -7 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$\begin{aligned} w_1 &= [1, -2, 0, 3, -4] \\ w_2 &= [0, 1, 1, -1, 2] \\ w_3 &= [0, 0, 0, 1, -1] \end{aligned}$$

form a basis for the subspace spanned by S. The dimension of the row space is 3.

- **Example - Finding a basis for the column space of A:** There are two ways to find a basis for the column space:

  1. Find the row-echelon form of A:

$$\begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The columns from the *original matrix* which have leading ones when reduced form a basis for the column space of A. In the above example, columns 1, 2, and 4 have leading ones. Therefore, columns 1, 2, and 4 of the original matrix form a basis for the column space of A.So,

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

form a basis for the column space of A. The dimension of the column space of A is 3.

2. The second way to find a basis for the column space of A is to recognize that the column space of A is equal to the row space of A^T. Finding a basis for the row space of A^T is the same as finding a basis for the column space of A.

$$A^T = \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & -1 & 1 \\ 4 & 2 & 0 & 0 & 6 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 2 & 1 \end{bmatrix}$$

Reducing A^T,we get

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & -1 & 1 \\ 4 & 2 & 0 & 0 & 6 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 2 & 0 & -4 & -2 \\ 0 & 1 & 1 & -1 & -2 \\ 0 & 1 & 2 & 0 & -3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & 2 & -2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$\begin{aligned} w_1 &= (1, 0, 0, 1, 2) \\ w_2 &= (0, 1, 0, -2, -1) \\ w_3 &= (0, 0, 1, 1, 1) \end{aligned}$$

form a basis for the column space of A.

- **Example:** Let $S = \{v_1, v_2, v_3, v_4\}$ from above where

$$\begin{aligned} v_1 &= [1, -2, 0, 3, -4] \\ v_2 &= [3, 2, 8, 1, 4] \\ v_3 &= [2, 3, 7, 2, 3] \\ v_4 &= [-1, 2, 0, 4, -3] \end{aligned}$$

Find a basis for the subspace of $\Re^5$ spanned by S *that is a subset of the vectors in* S. To do this, we set the columns of a matrix A as the vectors v_1, v_2, v_3 and v_4:

$$A = \begin{bmatrix} 1 & 3 & 2 & -1 \\ -2 & 2 & 3 & 2 \\ 0 & 8 & 7 & 0 \\ 3 & 1 & 2 & 4 \\ -4 & 4 & 3 & -3 \end{bmatrix}$$

Then find the column space of A:

$$\begin{bmatrix} 1 & 3 & 2 & -1 \\ -2 & 2 & 3 & 2 \\ 0 & 8 & 7 & 0 \\ 3 & 1 & 2 & 4 \\ -4 & 4 & 3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 1 & 7/8 & 0 \\ 0 & 0 & 1 & 7/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are leading ones in columns 1,2, and 3, so columns 1,2, and 3 from the original matrix form a basis for the column space of A. This corresponds to the vectors

$$\begin{aligned} v_1 &= [1, -2, 0, 3, -4] \\ v_2 &= [3, 2, 8, 1, 4] \\ v_3 &= [2, 3, 7, 2, 3] \end{aligned}$$

- **Theorem:** If A is an mxn matrix, then the row space and column space of A have the *same* dimension.
- **Definition:** The dimension of the row (or column) space of a matrix A is called the **rank** of A; denoted rank(A).
- **Example:** Let

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ 7 & 1 & 8 \end{bmatrix}$$

Then

$$\begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ 7 & 1 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/3 & 2/3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, rank$(A) = 2$ because there are 2 leading ones.

- **Definition:** If A is a mxn matrix, then the set of all solutions of the homogeneous system of linear equations

$$Ax = 0$$

is a subspace of $\Re^n$ called the *nullspace* of A and denoted $N(A)$.

$$N(A) = \{x \in \Re^n : Ax = 0\}$$

The dimension of $N(A)$ is called the *nullity* of A.

- **Example: Finding a basis for the nullspace of A:** Let

$$A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$$

We need to solve the system $Ax = 0$:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 4 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 1 & -1 & 0 & 0 & 2 & 0 \\ 2 & 1 & 6 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, $x_3 = s$ and $x_5 = t$ are free parameters. The solution to the system is given by

$$x = \begin{bmatrix} -2s - t \\ -2s + t \\ s \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} t$$

Therefore, the basis for $N(A)$ is given by

$$\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

We have $dim(N(A)) = 2$, i.e, nullity$(A) = 2$. Note that dim(rowspace(A)) $= 3$ and nullity$(A) = 2$. 3+2 $= 5 = n$ (the number of columns of A).

- **Theorem:** If A is a mxn matrix of rank A (r), then the dimension of the solution space of $Ax = 0$ is $n - r$, i.e.

$$n = rank(A) + nullity(A)$$

- **Example:** Let

$$A = \begin{bmatrix} 2 & 4 & -3 & -6 \\ 7 & 14 & -6 & -3 \\ -2 & -4 & 1 & -2 \\ 2 & 4 & -2 & -2 \end{bmatrix}$$

Find

1. basis for row space of A
2. basis for column space of A that is a subset of the column vectors of A
3. basis for nullspace of A
4. rank(A)
5. nullity(A)

Answers:

1. basis for row space of A: $\{[1, 2, -1, -1], [0, 0, 1, 4]\}$
2. basis for column space of A that is a subset of the column vectors of A:

$$\left\{ \begin{bmatrix} 2 \\ 7 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 1 \\ -2 \end{bmatrix} \right\}$$

3. basis for nullspace of A:

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$$

4. rank(A): 2
5. nullity(A): 2

- **Solutions of Systems of Linear Equations:** The solution x to

$$Ax = b$$

can be written as

$$x = x_p + x_h$$

where x_p is called a particular solution of $Ax = b$ and x_h is called the homogeneous solution of $Ax = 0$.

- **Example:** Consider the system

$$\begin{array}{rcrcrcrcr} x_1 & & & - & 2x_3 & + & x_4 & = & 5 \\ 3x_1 & + & x_2 & - & 5x_3 & & & = & 8 \\ x_1 & + & 2x_2 & & & - & 5x_4 & = & -9 \end{array}$$

Solving the system we have

$$\left[\begin{array}{rrrrcr} 1 & 0 & -2 & 1 & | & 5 \\ 3 & 1 & -5 & 0 & | & 8 \\ 1 & 2 & 0 & -5 & | & -9 \end{array}\right] \rightarrow \left[\begin{array}{rrrrcr} 1 & 0 & -2 & 1 & | & 5 \\ 0 & 1 & 1 & -3 & | & -7 \\ 0 & 2 & 2 & -6 & | & -14 \end{array}\right] \rightarrow \left[\begin{array}{rrrrcr} 1 & 0 & -2 & 1 & | & 5 \\ 0 & 1 & 1 & -3 & | & -7 \\ 0 & 0 & 0 & 0 & | & 0 \end{array}\right]$$

Therefore, we get the solution

$$x = \begin{bmatrix} 5+2s-t \\ -7-s+3t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

The particular solution is given by

$$x_p = \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix}$$

and the homogeneous solution is given by

$$x_h = s \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

- **Theorem:** A system of linear equations $Ax = b$ is consistent (has a solution) if and only if b is in the column space of A (i.e., b can be written as a linear combination of the columns of A).

- **Equivalent Statements:** If A is an nxn matrix, then the following are equivalent:

  (a) A is invertible

  (b) $Ax = 0$ has only the trivial solution

  (c) The reduced row-echelon form of A is I_n

  (d) A is expressible as a product of elementary matrices

  (e) $Ax = b$ is consistent for every nx1 matrix b

  (f) $Ax = b$ has exactly one solution for every nx1 matrix b

  (g) $|A| \neq 0$

  (h) $\lambda = 0$ is not an eigenvalue of A

  (i) rank(A)=n

  (j) n row vectors of A are linearly independent

  (k) n column vectors of A are linearly independent