Chapter 01.02
Measuring Errors

After reading this chapter, you should be able to:
1. *find the true and relative true error,*
2. *find the approximate and relative approximate error,*
3. *relate the absolute relative approximate error to the number of significant digits at least correct in your answers, and*
4. *know the concept of significant digits.*

In any numerical analysis, errors will arise during the calculations. To be able to deal with the issue of errors, we need to
   (A)   identify where the error is coming from, followed by
   (B)   quantifying the error, and lastly
   (C)   minimize the error as per our needs.
In this chapter, we will concentrate on item (B), that is, how to quantify errors.

Q: What is true error?
A: True error denoted by E_t is the difference between the true value (also called the exact value) and the approximate value.

True Error = True value – Approximate value

Example 1

The derivative of a function $f(x)$ at a particular value of x can be approximately calculated by
$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$
of $f'(2)$ For $f(x) = 7e^{0.5x}$ and $h = 0.3$, find
   a) the approximate value of $f'(2)$
   b) the true value of $f'(2)$
   c) the true error for part (a)

Solution

a) $f'(x) \approx \dfrac{f(x+h) - f(x)}{h}$

For $x = 2$ and $h = 0.3$,

$$f'(2) \approx \frac{f(2+0.3) - f(2)}{0.3}$$

$$= \frac{f(2.3) - f(2)}{0.3}$$

$$= \frac{7e^{0.5(2.3)} - 7e^{0.5(2)}}{0.3}$$

$$= \frac{22.107 - 19.028}{0.3}$$

$$= 10.265$$

b) The exact value of $f'(2)$ can be calculated by using our knowledge of differential calculus.

$$f(x) = 7e^{0.5x}$$

$$f'(x) = 7 \times 0.5 \times e^{0.5x}$$

$$= 3.5e^{0.5x}$$

So the true value of $f'(2)$ is

$$f'(2) = 3.5e^{0.5(2)}$$

$$= 9.5140$$

c) True error is calculated as

$$E_t = \text{True value} - \text{Approximate value}$$

$$= 9.5140 - 10.265$$

$$= -0.75061$$

The magnitude of true error does not show how bad the error is. A true error of $E_t = -0.722$ may seem to be small, but if the function given in the Example 1 were $f(x) = 7 \times 10^{-6} e^{0.5x}$, the true error in calculating $f'(2)$ with $h = 0.3$, would be $E_t = -0.75061 \times 10^{-6}$. This value of true error is smaller, even when the two problems are similar in that they use the same value of the function argument, $x = 2$ and the step size, $h = 0.3$. This brings us to the definition of relative true error.

Q: What is relative true error?
A: Relative true error is denoted by $\in_t$ and is defined as the ratio between the true error and the true value.

$$\text{Relative True Error} = \frac{\text{True Error}}{\text{True Value}}$$

Example 2

The derivative of a function $f(x)$ at a particular value of x can be approximately calculated by

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

For $f(x) = 7e^{0.5x}$ and $h = 0.3$, find the relative true error at $x = 2$.

Solution

From Example 1,

E_t = True value – Approximate value

$= 9.5140 - 10.265$

$= -0.75061$

Relative true error is calculated as

$$\in_t = \frac{\text{True Error}}{\text{True Value}}$$

$$= \frac{-0.75061}{9.5140}$$

$= -0.078895$

Relative true errors are also presented as percentages. For this example,

$\in_t = -0.0758895 \times 100\%$

$= -7.58895\%$

Absolute relative true errors may also need to be calculated. In such cases,

$\left| \in_t \right| = \left| -0.075888 \right|$

$= 0.0758895$

$= 7.58895\%$

Q: What is approximate error?
A: In the previous section, we discussed how to calculate true errors. Such errors are calculated only if true values are known. An example where this would be useful is when one is checking if a program is in working order and you know some examples where the true error is known. But mostly we will not have the luxury of knowing true values as why would you want to find the approximate values if you know the true values. So when we are solving a problem numerically, we will only have access to approximate values. We need to know how to quantify error for such cases.

Approximate error is denoted by E_a and is defined as the difference between the present approximation and previous approximation.

Approximate Error $=$ Present Approximation – Previous Approximation

Example 3

The derivative of a function $f(x)$ at a particular value of x can be approximately calculated by

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

For $f(x) = 7e^{0.5x}$ and at $x = 2$, find the following

      a) $f'(2)$ using $h = 0.3$

      b) $f'(2)$ using $h = 0.15$

      c) approximate error for the value of $f'(2)$ for part (b)

Solution

a) The approximate expression for the derivative of a function is

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}.$$

For $x = 2$ and $h = 0.3$,

$$\begin{aligned} f'(2) &\approx \frac{f(2+0.3)-f(2)}{0.3} \\ &= \frac{f(2.3)-f(2)}{0.3} \\ &= \frac{7e^{0.5(2.3)}-7e^{0.5(2)}}{0.3} \\ &= \frac{22.107-19.028}{0.3} \\ &= 10.265 \end{aligned}$$

b) Repeat the procedure of part (a) with $h = 0.15$,

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}$$

For $x = 2$ and $h = 0.15$,

$$\begin{aligned} f'(2) &\approx \frac{f(2+0.15)-f(2)}{0.15} \\ &= \frac{f(2.15)-f(2)}{0.15} \\ &= \frac{7e^{0.5(2.15)}-7e^{0.5(2)}}{0.15} \\ &= \frac{20.50-19.028}{0.15} \\ &= 9.8799 \end{aligned}$$

c) So the approximate error, E_a is

$$\begin{aligned} E_a &= \text{Present Approximation} - \text{Previous Approximation} \\ &= 9.8799 - 10.265 \\ &= -0.38474 \end{aligned}$$

The magnitude of approximate error does not show how bad the error is . An approximate error of $E_a = -0.38300$ may seem to be small; but for $f(x) = 7 \times 10^{-6} e^{0.5x}$, the approximate error in calculating $f'(2)$ with $h = 0.15$ would be $E_a = -0.38474 \times 10^{-6}$. This value of approximate error is smaller, even when the two problems are similar in that they use the same value of the function argument, $x = 2$, and $h = 0.15$ and $h = 0.3$. This brings us to the definition of relative approximate error.

Q: What is relative approximate error?
A: Relative approximate error is denoted by $\in_a$ and is defined as the ratio between the approximate error and the present approximation.

$$\text{Relative Approximate Error} = \frac{\text{Approximate Error}}{\text{Present Approximation}}$$

Example 4

The derivative of a function $f(x)$ at a particular value of x can be approximately calculated by

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

For $f(x) = 7e^{0.5x}$, find the relative approximate error in calculating $f'(2)$ using values from $h = 0.3$ and $h = 0.15$.

Solution

From Example 3, the approximate value of $f'(2) = 10.263$ using $h = 0.3$ and $f'(2) = 9.8800$ using $h = 0.15$.

E_a = Present Approximation – Previous Approximation

$= 9.8799 - 10.265$

$= -0.38474$

The relative approximate error is calculated as

$$\in_a = \frac{\text{Approximate Error}}{\text{Present Approximation}}$$

$$= \frac{-0.38474}{9.8799}$$

$$= -0.038942$$

Relative approximate errors are also presented as percentages. For this example,

$\in_a = -0.038942 \times 100\%$

$= -3.8942\%$

Absolute relative approximate errors may also need to be calculated. In this example

$|\in_a| = |-0.038942|$

$= 0.038942$ or 3.8942%

Q: While solving a mathematical model using numerical methods, how can we use relative approximate errors to minimize the error?

A: In a numerical method that uses iterative methods, a user can calculate relative approximate error $\in_a$ at the end of each iteration. The user may pre-specify a minimum acceptable tolerance called the pre-specified tolerance, $\in_s$. If the absolute relative approximate error $\in_a$ is less than or equal to the pre-specified tolerance $\in_s$, that is, $|\in_a| \leq \in_s$, then the acceptable error has been reached and no more iterations would be required. Alternatively, one may pre-specify how many significant digits they would like to be correct in their answer. In that case, if one wants at least m significant digits to be correct in the answer, then you would need to have the absolute relative approximate error, $|\in_a| \leq 0.5 \times 10^{2-m} \%$.

Example 5

If one chooses 6 terms of the Maclaurin series for e^x to calculate $e^{0.7}$, how many significant digits can you trust in the solution? Find your answer without knowing or using the exact answer.

Solution

$$e^x = 1 + x + \frac{x^2}{2!} + \text{.................}$$

Using 6 terms, we get the current approximation as

$$e^{0.7} \cong 1 + 0.7 + \frac{0.7^2}{2!} + \frac{0.7^3}{3!} + \frac{0.7^4}{4!} + \frac{0.7^5}{5!}$$

$$= 2.0136$$

Using 5 terms, we get the previous approximation as

$$e^{0.7} \cong 1 + 0.7 + \frac{0.7^2}{2!} + \frac{0.7^3}{3!} + \frac{0.7^4}{4!}$$

$$= 2.0122$$

The percentage absolute relative approximate error is

$$\left| \in_a \right| = \left| \frac{2.0136 - 2.0122}{2.0136} \right| \times 100$$

$$= 0.069527\%$$

Since $\left| \in_a \right| \leq 0.5 \times 10^{2-2}\%$, at least 2 significant digits are correct in the answer of

$$e^{0.7} \cong 2.0136$$

Q: But what do you mean by significant digits?

A: Significant digits are important in showing the truth one has in a reported number. For example, if someone asked me what the population of my county is, I would respond, "The population of the Hillsborough county area is 1 million". But if someone was going to give me a $100 for every citizen of the county, I would have to get an exact count. That count would have been 1,079,587 in year 2003. So you can see that in my statement that the population is 1 million, that there is only one significant digit, that is, 1, and in the statement that the population is 1,079,587, there are seven significant digits. So, how do we differentiate the number of digits correct in 1,000,000 and 1,079,587? Well for that, one may use scientific notation. For our data we show

$$1{,}000{,}000 = 1 \times 10^6$$

$$1{,}079{,}587 = 1.079587 \times 10^6$$

to signify the correct number of significant digits.

Example 5

Give some examples of showing the number of significant digits.

Solution

a) 0.0459 has three significant digits
b) 4.590 has four significant digits
c) 4008 has four significant digits
d) 4008.0 has five significant digits

e) 1.079×10^3 has four significant digits

f) 1.0790×10^3 has five significant digits

g) 1.07900×10^3 has six significant digits

INTRODUCTION, APPROXIMATION AND ERRORS	
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