# Assignment 2 Solutions

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## 1 Exercise 1.9

Each of the following two procedures defines a method for adding two positive integers in terms of the procedures `inc`, which increments its argument by 1, and `dec`, which decrements its argument by 1.

```
(define (+ a b)
  (if (= a 0)
      b
      (inc (+ (dec a) b))))

(define (+ a b)
  (if (= a 0)
      b
      (+ (dec a) (inc b))))
```

### 1.1 Solution

```
(inc (+ (dec 4) 5))
(inc (+ 3 5))
(inc (inc (+ (dec 3) 5)))
(inc (inc (+ 2 5)))
(inc (inc (inc (+ (dec 2) 5))))
(inc (inc (inc (+ 1 5))))
(inc (inc (inc (inc (+ (dec 1) 5)))))
(inc (inc (inc (inc (+ 0 5)))))
(inc (inc (inc (inc 5))))
(inc (inc (inc 6)))
(inc (inc 7))
(inc 8)
```

```
9

(+ (dec 4) (inc 5))
(+ 3 6)
(+ (dec 3) (inc 6))
(+ 2 7)
(+ (dec 2) (inc 7))
(+ 1 8)
(+ (dec 1) (inc 8))
(+ 0 9)
9
```

The first process is recursive while the second process is iterative.

## 2 Exercise 1.10

The following procedure computes a mathematical function called Ackermann's function.

```
(define (A x y)
  (cond ((= y 0) 0)
        ((= x 0) (* 2 y))
        ((= y 1) 2)
        (else (A (- x 1)
                 (A x (- y 1))))))
```

What are the values of the following expressions?

```
(A 1 10)
(A 2 4)
(A 3 3)
```

Consider the following procedures, where `A` is the procedure defined above:

```
(define (f n) (A 0 n))
(define (g n) (A 1 n))
(define (h n) (A 2 n))
(define (k n) (* 5 n n))
```

Give concise mathematical definitions for the functions computed by the procedures `f`, `g`, and `h` for positive integer values of `n`. For example, `(k n)` computes $5n^2$

## 2.1 Solution

```
(A 1 10)
(A 0 (A 1 9))
(A 0 (A 0 (A 1 8)))
(A 0 (A 0 (A 0 (A 1 7))))
(A 0 (A 0 (A 0 (A 0 (A 1 6)))))
(A 0 (A 0 (A 0 (A 0 (A 0 (A 1 5))))))
(A 0 (A 0 (A 0 (A 0 (A 0 (A 0 (A 1 4)))))))
(A 0 (A 0 (A 0 (A 0 (A 0 (A 0 (A 0 (A 1 3))))))))
(A 0 (A 0 (A 0 (A 0 (A 0 (A 0 (A 0 (A 0 (A 1 2)))))))))
(A 0 (A 0 (A 0 (A 0 (A 0 (A 0 (A 0 (A 0 (A 0 (A 1 1))))))))))
(A 0 (A 0 (A 0 (A 0 (A 0 (A 0 (A 0 (A 0 (A 0 2)))))))))
(A 0 (A 0 (A 0 (A 0 (A 0 (A 0 (A 0 (A 0 4))))))))
(A 0 (A 0 (A 0 (A 0 (A 0 (A 0 (A 0 8)))))))
(A 0 (A 0 (A 0 (A 0 (A 0 (A 0 16))))))
(A 0 (A 0 (A 0 (A 0 (A 0 32)))))
(A 0 (A 0 (A 0 (A 0 64))))
(A 0 (A 0 (A 0 128)))
(A 0 (A 0 256))
(A 0 512)
1024

(A 2 4)
(A 1 (A 2 3))
(A 1 (A 1 (A 2 2)))
(A 1 (A 1 (A 1 (A 2 1))))
(A 1 (A 1 (A 1 2)))
(A 1 (A 1 (A 0 (A 1 1))))
(A 1 (A 1 (A 0 2)))
(A 1 (A 1 4))
(A 1 (A 0 (A 1 3)))
(A 1 (A 0 (A 0 (A 1 2))))
(A 1 (A 0 (A 0 (A 0 (A 1 1)))))
(A 1 (A 0 (A 0 (A 0 2))))
(A 1 (A 0 (A 0 4)))
(A 1 (A 0 8))
(A 1 16)
(A 0 (A 1 15))
(A 0 (A 0 (A 1 14)))
```

```
(A 0 (A 0 (A 0 (A 1 13))))
(A 0 (A 0 (A 0 (A 0 (A 1 12)))))
(A 0 (A 0 (A 0 (A 0 (A 0 (A 1 11))))))
(A 0 (A 0 (A 0 (A 0 (A 0 (A 0 (A 1 10)))))))
(A 0 (A 0 (A 0 (A 0 (A 0 (A 0 1024))))))
(A 0 (A 0 (A 0 (A 0 (A 0 (A 0 1024))))))
(A 0 (A 0 (A 0 (A 0 (A 0 2048)))))
(A 0 (A 0 (A 0 (A 0 4096))))
(A 0 (A 0 (A 0 8192)))
(A 0 (A 0 16384))
(A 0 32768)
65536

(A 3 3)
(A 2 (A 3 2))
(A 2 (A 2 (A 3 1)))
(A 2 (A 2 2))
(A 2 (A 1 (A 2 1)))
(A 2 (A 1 2))
(A 2 4)
65536
```

`(f n)` computes $2n$. `(g n)` computes $2^n$. `(h n)` computes ${}^n2$.

# 3 Exercise 1.11

A function $f$ is defined by the rule that

$$f(n) = \begin{cases} n & n < 3 \\ f(n-1) + 2f(n-2) + 3f(n-3) & n \geq 3 \end{cases} \tag{1}$$

Write a procedure that computes $f$ by means of a recursive procedure. Write a procedure that computes $f$ by means of an iterative procedure

## 3.1 Solution

```
(define (f-recur n)
  (if (< n 3)
      n
      (+ (f-recur (- n 1))
```

```
         (* 2 (f-recur (- n 2)))
         (* 3 (f-recur (- n 3))))))

(define (f n)
  (define (iter count a b c)
    (if (>= count n)
        a
        (iter (+ count 1)
              b
              c
              (+ (* 3 a)
                 (* 2 b)
                 c))))
  (iter 0 0 1 2))
```

# 4 Exercise 1.12

The following pattern of numbers is called *Pascal's triangle.*

```
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
   ...
```

The numbers at the edge of the triangle are all 1, and each number inside the triangle is the sum of the two numbers above it. Write a procedure that computes elements of Pascal's triangle by means of a recursive process.

## 4.1 Solution

```
(define (pascal row column)
  (cond ((or (< column 0)
             (> column row))
         0)
        ((or (= row 0)
             (= column 0)
             (= row column))
         1)
```

```
(else
 (+ (pascal (- row 1)
            (- column 1))
    (pascal (- row 1)
            column)))))
```

# 5 Exercise 1.13

Prove that $\mathrm{Fib}(n)$ is the closest integer to $\phi^n/\sqrt{5}$, where $\phi = (1+\sqrt{5})/2$. Hint: Let $\psi = (1-\sqrt{5})/2$. Use induction and the definition of the Fibonacci numbers to prove that $\mathrm{Fib}(n) = (\phi^n - \psi^n)/\sqrt{5}$.

## 5.1 Solution

This solution is incomplete. The formula for the nth fibonacci number is

$$\mathrm{Fib}(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \mathrm{Fib}(n-1) + \mathrm{Fib}(n-2) & \text{otherwise} \end{cases}$$

Setting $n = 0$, we have $\frac{\phi^0}{\sqrt{5}} = \frac{1}{\sqrt{5}}$. Since $\sqrt{5} > 2$, we must have that $\frac{1}{\sqrt{5}} < \frac{1}{2}$ and so is closer to zero than it is to one. Thus the claim holds for the case $n = 0$. Similarly, for $n = 1$ we have $\frac{\phi^1}{\sqrt{5}} \approx 0.7236$ which is closer to one than it is to zero as needed. For the inductive step, we assume that $\mathrm{Fib}(k)$ is the closest integer to $\frac{\phi^k}{\sqrt{5}}$ and show that $\mathrm{Fib}(k+1)$ is the closest integer to $\frac{\phi^{k+1}}{\sqrt{5}}$.