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I2

Geometría Diferencial - MAT2305

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Solución problema 1:

(a) Sea S_2 una superficie regular orientable, y sea $S_1 \subset S_2$ una superficie regular. Ahora, sea N la orientación de S_2, y N' la restricción de N a S_1. Si N' no es una orientación de S_1, entonces existe algún punto $p \in S_1$ donde N' no está bien definida, como $p \in S_1$, se tiene que $p \in S_2$, más aún como $\forall q \in S_1 N(q) = N'(q)$ N no está bien definida en $p \in S_2$, pero eso es una contradicción ya que N es una orientación de S_2. Por lo que S_1 es orientable.

(b) Sean S_1, S_2 superficies regulares difeomorfas, además S_1 es orientable. Sea $\varphi : S_1 \to S_2$ un difeomorfismo, luego $\mathrm{d}\varphi : TS_p(S_1) \to TS_p(S_2)$ es un isomorfismo. Sea $p \in S_1$ un punto, ahora sean $\hat{\mathbf{x}}_u, \hat{\mathbf{x}}_v$ los vectores correspondientes normalizados. Se define $N' = \frac{\mathrm{d}\varphi\hat{\mathbf{x}}_u \wedge \mathrm{d}\varphi\hat{\mathbf{x}}_v}{\det(\mathrm{d}\varphi)}$

(c) Sea S una superficie regular tal que la Banda de Möbius es un subconjunto de ella, se asume que S es orientable, por (a) el subconjunto que corresponde a la Banda de Möbius es orientable, pero hay un difeomorfismo natural de ese subconjunto a la Banda de Möbius misma, y por (b) se tendría que la Banda de Möbius es orientable, lo que es una contradicción. Por ende S no es orientable.

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Solución problema 2:

(a) Usando que $\mathbf{x}_u = (\cos v, \sin v, 2u \sin v)$ y que $\mathbf{x}_v = (-u \sin v, u \cos v, u^2 \cos v)$ se llega lo siguiente[1]:

$$\begin{aligned} E &= 4u^2 \sin^2 v + 1 \\ F &= 2u^3 \sin v \cos v \\ G &= u^2 + u^4 \cos^2 v \end{aligned}$$

(b) Usando que $\mathbf{x}_{uu} = (0, 0, 2\sin v)$, $\mathbf{x}_{uv} = (-\sin v, \cos v, 2u \cos v)$, $\mathbf{x}_{vv} = (-u \cos v, -u \sin v, -u^2 \sin v)$,

[1] Los cálculos están en el apendice.

y que $N = \frac{(u\sin v\cos v, -u(1+\sin^2 v), 1)}{\sqrt{u^2(3\sin^2 v+1)+1}}$ se llega[2] a que:

$$
\begin{aligned}
e &= \frac{2\sin v}{\sqrt{u^2(3\sin^2 v+1)+1}} \\
f &= \frac{u^2\cos v}{\sqrt{u^2(3\sin^2 v+1)+1}} \\
e &= \frac{u^3\sin v}{\sqrt{u^2(3\sin^2 v+1)+1}}
\end{aligned}
$$

(c) Usando que $K = \frac{eg-f^2}{EG-F^2}$ y los valores ya calculados se llega[3] a que $K = \frac{u(\sin^2 v - u\cos^v)}{(u^2(3\sin^2 v+1)+1)^2}$.

(d) Usando los coeficientes cálculados anteriormente, se cálcula la matriz de $\mathrm{d}N_{(1,0)}$ en base $\{\mathbf{x}_u, \mathbf{x}_v\}$:

$$
\mathrm{d}N_{(1,0)} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & 0 \end{pmatrix}
$$

Se calculan[4] sus valores propios son $k_1 = \frac{1}{2}, k_2 = -\frac{1}{2}$, y sus vectores propios son $v_1 = (-\sqrt{2}, 1), v_2 = (\sqrt{2}, 1)$. Por definición k_1, k_2 son las curvaturas principales y v_1, v_2 las direcciones principales

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Solución problema 3:

(a) Usando que $\mathbf{x}_r = (\cos\theta, 1+\sin\theta, 5)$ y que $\mathbf{x}_\theta = (-r\sin\theta, r\cos\theta, 0)$, se hacen unos cálculos[5] y se llega a $N = \frac{(-5\cos\theta, -5\sin\theta, 1+\sin\theta)}{\sqrt{(1+\sin^2\theta)^2+25}}$, lo cual no depende de r. Ahora, sea $\alpha(r) = \mathbf{x}(r, \theta_1) = r(\cos\theta, 1+\sin\theta, 5)$, luego $\alpha'(r) = (\cos\theta, 1+\sin\theta, 5)$ y $N'(r) = \mathbf{0} = 0\cdot\alpha'(r)$, por lo que por Olinde Rodrigues α es linea de curvatura.

(b) Sea $\alpha(\theta) = \mathbf{x}(r_1, \theta)$, se tiene que $\alpha''(\theta) = (-r_1\cos\theta, -r_1\sin\theta, 0)$. Luego, con $n = \frac{\alpha''}{\|\alpha''\|}$ se tiene que $k_n = \frac{5r_1}{\sqrt{(1+\sin^2\theta)^2+25}} > 0$. Ahora, como N no depende de r, se tiene que $\det(\mathrm{d}N) = 0$, por lo que $K = 0$, por lo que dado $k_1 \geq k_2$ y usando continuidad de k_n se tiene que $k_2 = 0$, más aún como $k_n > 0$, para $n = \frac{\alpha''}{\|\alpha''\|}$, se tiene que $k_1 > 0$. Este argumento funciona para todo punto en alguna curva α_{r_1}, y como todo punto está en

[2] Ver 1.
[3] Ver 1.
[4] Ver 1.
[5] Ver 1.

alguna de estas curvas, se tiene que todo punto tiene que $k_1 > k_2 = 0$, por lo que todos los puntos son parabólicos.

(c)

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Solución problema 4:

(a) Dado que $\varphi_x = -2x$ y $\varphi_y = 2y$ se tiene que $\mathbf{x}_x = (1, 0, -2x)$ y que $\mathbf{x}_y = (0, 1, 2y)$. Se usa esto para calcular E, F y G[6]:

$$E = 4x^2 + 1$$
$$F = -4xy$$
$$G = 4y^2 + 1$$

Dado eso se calcula[7] $c = EG - F^2 = 4x^2 + 4y^2 + 1$, y se usa para calcular los símbolos de Christoffel[8]:

$$\Gamma^1_{11} = \frac{4x}{4x^2 + 4y^2 + 1}$$
$$\Gamma^2_{11} = \frac{-4y}{4x^2 + 4y^2 + 1}$$
$$\Gamma^1_{12} = 0$$
$$\Gamma^2_{12} = 0$$
$$\Gamma^1_{22} = \frac{-4x}{4x^2 + 4y^2 + 1}$$
$$\Gamma^2_{22} = \frac{4y}{4x^2 + 4y^2 + 1}$$

(b) Con los símbolos de Christoffel y la siguiente fórmula:

$$K = -\frac{1}{E}\left((\Gamma^2_{12})_x - (\Gamma^2_{11})_y + \Gamma^1_{12} \cdot \Gamma^2_{11} - \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22}\right)$$

Después de unos cálculos[9] se llega a:

$$K = \frac{-4}{(4x^2 + 4y^2 + 1)^2} < 0$$

[6] Ver 1.
[7] Ver 1.
[8] Ver 1.
[9] Ver 1.

Como $K < 0$ se tiene que el paraboloide no puede ser localmente isometrico al plano, ya que la curvatura Gaussiana es invariante a través de isometrías locales[10], y el plano tiene curvatura Gaussiana 0.

(c) Sea $V = \mathbf{x}_x + \mathbf{x}_y$, se calcula[11] V_x y V_y:

$$
\begin{aligned}
V_x &= (\Gamma_{11}^1 + \Gamma_{12}^1)\mathbf{x}_x + (\Gamma_{11}^2 + \Gamma_{12}^2)\mathbf{x}_y + (L_1 + L_2)N \\
V_y &= (\Gamma_{12}^1 + \Gamma_{22}^1)\mathbf{x}_x + (\Gamma_{12}^2 + \Gamma_{22}^2)\mathbf{x}_y + (L_2 + L_3)N
\end{aligned}
$$

Como $\Gamma_{11}^1 + 2\Gamma_{12}^1 + \Gamma_{22}^1 = \Gamma_{11}^2 + 2\Gamma_{12}^2 + \Gamma_{22}^2 = L_1 + 2L_2 + L_3 = 0$[12], se tiene que $V_x + V_y = 0$.

■

[10] Por Theorema Egregium
[11] Ver 1.
[12] Ver 1.

Apéndice

$x(u,v) = (u\cos v, u\sin v, u^2\sin v)$

$x_u = (\cos v, \sin v, 2u\sin v)$

$x_v = (-u\sin v, u\cos v, u^2\cos v)$

$x_{uu} = (0, 0, 2\sin v)$

$x_{uv} = (-\sin v, \cos v, 2u\cos v)$

$x_{vv} = (-u\cos v, -u\sin v, -u^2\sin v) = -x$

$E = \langle x_u, x_u\rangle = \cos^2 v + \sin^2 v + 4u^2\sin^2 v$
$= 4u^2\sin^2 v + 1$

$F = \langle x_u, x_v\rangle = -u\cos v\sin v + u\cos v\sin v + 2u^3\sin v\cos v$
$= 2u^3\sin v\cos v$

$G = \langle x_v, x_v\rangle = u^2\sin^2 v + u^2\cos^2 v + u^4\cos^2 v$
$= u^2 + u^4\cos^2 v$

$$x_u \wedge x_v = u \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 2u\sin v \\ -\sin v & \cos v & u\cos v \end{vmatrix}$$

$= u\left[\vec{i}(\sin v \cdot u\cos v - 2u\sin v\cos v) + \vec{j}(u\cos^2 v + 2u\sin^2 v) + \vec{k}(\cos^2 v + \sin^2 v)\right]$

$= u\left[\vec{i}(-u\sin v\cos v) + \vec{j}(u + u\sin^2 v) + \vec{k}\right]$

$= -\vec{i}(u^2\sin v\cos v) + \vec{j}(u^2 + u^2\sin^2 v) + \vec{k}\,u$

$$N = \frac{x_u \wedge x_v}{|x_u \wedge x_v|}$$

$|x_u \wedge x_v| = \sqrt{u^4\sin^2 v\cos^2 v + u^4(1+\sin^2 v)^2 + u^2}$

$= u \cdot \sqrt{u^2(\sin^2 v\cos^2 v + 1 + 2\sin^2 v + \sin^4 v) + 1}$

$= u \cdot \sqrt{u^2(\sin^2 v - \cancel{\sin^4 v} + 1 + 2\sin^2 v + \cancel{\sin^4 v}) + 1}$

$= u \cdot \sqrt{u^2(3\sin^2 v + 1) + 1}$

$$N = \frac{(u\sin v\cos v, +u(1+\sin^2 v), 1)}{\sqrt{u^2(3\sin^2 v+1)+1}}$$

$$e = \langle N, x_{uu}\rangle = \frac{2\sin v}{\sqrt{u^2(3\sin^2 v+1)+1}}$$

~~$f = \langle N, x_{uv}\rangle$~~

$$f = \langle N, x_{uv}\rangle = \frac{1}{\sqrt{u^2(3\sin^2 v+1)+1}} \cdot \left[+u\sin v\cos v + u\cos v(1+\sin^2 v) + 2u\cos v\right]$$

$$= \frac{1}{\sqrt{u^2(3\sin^2 v+1)+1}} \left(\cancel{2u\sin^2 v\cos v + u\cos v} \right)$$

$g = \langle N, x_{vv}\rangle = \left[\right.$

$$= -\langle N, x\rangle = \frac{-1}{\sqrt{u^2(3\sin^2 v+1)+1}} \left[u^2\sin v\cos^2 v + u^2\sin v(1+\sin^2 v) + u^2\sin v\right]$$

$$= \frac{-1}{\sqrt{u^2(3\sin^2 v+1)+1}} \left[u^2\sin v(1-\sin^2 v) + 2u^2\sin v + u^2\sin^3 v\right]$$

$$= \frac{-1}{\sqrt{u^2(3\sin^2 v+1)+1}} \left(\right.$$

$$f = \langle N, x_{uv} \rangle = \frac{1}{c}\left[u^2\sin^2 v\cos v - u^2\cos v - u^2\sin^2 v\cos v + 2u^2\cos v\right]$$

$$= \frac{1}{c}\, u^2\cos v$$

$$c = \sqrt{u^2(3\sin^2 v+1)+1}$$

$$g = \langle N, x_{vv} \rangle = \frac{1}{c}\left[u^3\sin v\cos^2 v + u^3\sin v + u^3\sin^3 v - u^3\sin v\right]$$

$$= \frac{1}{c}\left[u^3\sin v(1-\sin^2 v) + u^3\sin^3 v\right]$$

$$= \frac{1}{c}\left[u^3\sin v\right]$$

$$K = \frac{1}{c^4}\cdot\left(u\sin^2 v - u^2\cos^2 v\right)$$

$$= \frac{u}{c^4}\left(\sin^2 v - u\cos^2 v\right)$$

$$K = \frac{eg-f^2}{EG-F^2} \qquad eg - f^2 = \frac{1}{c^2}\left[2\sin v\cdot u^3\sin v - u^4\cos^2 v\right]$$

$$EG-F^2 = (4u^2\sin^2 v+1)(u^2+u^4\cos^2 v) - (2u^3\sin v\cos v)^2$$

$$= u^2\left[4u^4\sin^2 v\cos^2 v + u^2\cos^2 v + 4u^2\sin^2 v + 1 - 4u^4\sin^2 v\cos^2 v\right]$$

$$= u^2\left[u^2(3\sin^2 v+1)+1\right] = u^2\cdot c^2$$

$x(1,0)$ $\qquad c(1,0) = \sqrt{2}$

$x_u(1,0) = (1,0,0)$ $\qquad K(1,0) = \frac{-1}{4}$

$x_v(1,0) = (0,1,1)$ $\qquad H(1,0) = 0$

$k_1 = \frac{1}{2}$ $\quad k_2 = -\frac{1}{2}$

$E(1,0) = 1$ $\qquad a_{11} = 0$

$a_{22} = 0$

$F(1,0) = 0$ $\qquad a_{12} = -\frac{1}{\sqrt{2}}\cdot\frac{1}{2} = -\frac{1}{\sqrt{2}}$

$G(1,0) = 2$ $\qquad a_{21} = -\frac{1}{\sqrt{2}}\cdot\frac{1}{2}$

$e(1,0) = 0$

$f(1,0) = \frac{1}{\sqrt{2}}$ $\qquad dN_{(1,0)} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$

$g(1,0) = 0$ $\qquad v_1 = (-\sqrt{2}, 1)$ $\quad e_1 = \frac{1}{\sqrt{3}}v_1$

$v_2 = (\sqrt{2}, 1)$ $\quad e_2 = \frac{1}{\sqrt{3}}v_2$

$$K = \frac{eg-f^2}{EG-F^2} = \frac{1}{u^2(3\sin^2 v+1)+1}\cdot\frac{2\sin v\cdot 3u^3\sin v - (3u\cos v)^2}{(4u^2\sin^2 v+1)(u^2+u^4\cos^2 v) - (2u^3\sin v\cos v)^2}$$

$$= \frac{6\sin^2 v - 9\cos^2 v}{u^2(3\sin^2 v+1)+1}\cdot\frac{1}{(4u^2\sin^2 v+1)(u^2\cos^2 v+1) - 4u^4\sin^2 v\cos^2 v}$$

$$= \frac{3(2\sin^2 v - 3 + 3\sin^2 v)}{u^2(3\sin^2 v+1)+1}\cdot\frac{1}{4u^4\sin^2 v\cos^2 v + 4u^2\sin^2 v + u^2\cos^2 v + 1 - 4u^4\sin^2 v\cos^2 v}$$

$$= \frac{3(5\sin^2 v - 3)}{u^2(3\sin^2 v+1)+1}\cdot\frac{1}{4u^2\sin^2 v + u^2\cos^2 v + 1}$$

$$= \frac{3(5\sin^2 v - 3)}{(u^2(3\sin^2 v+1)+1)^2}$$

$$dN_{(1,0)} = \frac{3}{\sqrt{2}}\begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

$$\frac{3}{\sqrt{2}}\begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{3}{2}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\frac{y}{2} = \frac{x\sqrt{2}}{2}$ $\quad y = 1$

$x = y\frac{\sqrt{2}}{2}$ $\quad x = \frac{\sqrt{2}}{2}$

vectors spr... are

$\left(\frac{\sqrt{2}}{2}, 1\right), \left(-\frac{\sqrt{2}}{2}, 1\right)$

$$EG - F^2 = u^2\left(u^2(3\sin^2 v+1)+1\right)$$

$x(1,0) = (1,0,0)$

$K(1,0) = \frac{-9}{4}$

$e(1,0) = 0$

$f(1,0) = \frac{3}{\sqrt{2}}$

$g(1,0) = 0$

$F(1,0) = 0$

$G(1,0) = 2$

$E(1,0) = 1$

$$H = \frac{1}{2}\,\frac{eG - 2fF + gE}{EG-F^2}$$

$\Rightarrow H(1,0) = 0$

$\therefore k_1 > k_2$

$k_i(1,0) = \pm\sqrt{-K(1,0)} = \pm\frac{3}{2}$

$i = 1,2$ $\qquad k_1 = \frac{3}{2}$ $\quad k_2 = -\frac{3}{2}$

$x_u(1,0) = (1,0,0)$

$x_v(1,0) = (0,1,1)$

$a_{11} = 0$

$a_{22} = 0$

$a_{12} = \frac{3}{2\sqrt{2}}$

$a_{21} = \frac{3}{\sqrt{2}}$

$$dN_{(1,0)} = \begin{pmatrix} 0 & \frac{3}{2\sqrt{2}} \\ \frac{3}{\sqrt{2}} & 0 \end{pmatrix}$$

$$e_1 = \frac{\sqrt{2}}{2} \cdot x_u + x_v = \left(\frac{\sqrt{2}}{2}, 1, 1\right)$$

$$e_2 = -\frac{\sqrt{2}}{2} \cdot x_u + x_v = \left(-\frac{\sqrt{2}}{2}, 1, 1\right)$$

$$N = \frac{x_t \wedge x_\theta}{|x_t \wedge x_\theta|} = \frac{(-5\cos\theta, -5\sin\theta, 1+\sin\theta)}{\sqrt{(1+\sin\theta)^2+25}}$$

$$\alpha(t) = x(t, \theta_1) = t(\cos\theta, 1+\sin\theta, 5)$$

$$\alpha'(t) = (\cos\theta, 1+\sin\theta, 5)$$

$$X(t,\theta) = (t\cos\theta, t+t\sin\theta, 5t)$$

$$X_t = (\cos\theta, 1+\sin\theta, 5)$$

$$X_\theta = (-t\sin\theta, t\cos\theta, 0)$$

$$X_t \wedge X_\theta = t\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & 1+\sin\theta & 5 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix}$$

$$= t\left[\hat{k}(\cos^2\theta + \sin^2\theta + \sin\theta) - \hat{j}(5\sin\theta) - \hat{i}\, 5\cos\theta\right]$$

$$= \hat{k}(t + t\sin\theta) - \hat{j}\, 5t\sin\theta - \hat{i}\, 5t\cos\theta$$

$$|x_t \wedge x_\theta| = \sqrt{t^2(1+\sin\theta)^2 + 25t^2\sin^2\theta + 25t^2\cos^2\theta}$$

$$= t\sqrt{(1+\sin\theta)^2 + 25}$$

$$N(\alpha(t)) = N(t) = \frac{(-5\cos\theta_1, -5\sin\theta_1, \ldots}{\sqrt{(1+\sin\theta_1)^2 \ldots}}$$

$$N'(t) = 0$$

$$\boxed{0 = N'(t) = 0 \cdot \alpha'(t)}$$

$$\boxed{k = \det(dN) = 0}$$

Sea $\alpha(\theta) = X(t_1, \theta) = (t_1\cos\theta, t_1 + t_1\sin\theta, 5t_1)$

$$K_n = k\langle u, N\rangle = \langle \alpha''(\theta), N\rangle = \frac{t_1 \cdot [5\cos^2\theta + 5\sin^2\theta]}{\sqrt{(1+\sin\theta)^2+25}} = \frac{5t_1}{\sqrt{(1+\sin\theta)^2+25}} > 0$$

$$u = \frac{\alpha'(\theta)}{\|\alpha'(\theta)\|}$$

$$\alpha''(\theta) = (-t_1\cos\theta, -t_1\sin\theta, 0)$$

$$x_{tt} = (0, 0, 0)$$

$$x_{t\theta} = (-\sin\theta, \cos\theta, 0)$$

$$x_{\theta\theta} = (-t\cos\theta, -t\sin\theta, 0)$$

$$e = \langle N, x_{tt}\rangle = 0$$

$$f = \langle N, x_{t\theta}\rangle = \frac{5\sin\theta\cos\theta - 5\sin\theta\cos\theta}{\sqrt{(1+\sin\theta)^2+25}} = 0$$

$$g = \langle N, x_{\theta\theta}\rangle = \frac{5t\cos^2\theta + 5t\sin^2\theta}{\sqrt{(1+\sin\theta)^2+25}} = \frac{5t}{\sqrt{(1+\sin\theta)^2+25}}$$

$$k_2 = 0$$

$$E = \langle x_t, x_t\rangle = \cos^2\theta + (1+\sin\theta)^2 + 5^2 = 2\sin\theta + 27$$

$$F = \langle x_t, x_\theta\rangle = -t\sin\theta\cos\theta + t\cos\theta + t\sin\theta\cos\theta = t\cos\theta$$

$$G = \langle x_\theta, x_\theta\rangle = t^2\sin\theta + t^2\cos^2\theta = t^2$$

$$EG - F^2 = 2t^2\sin\theta + 27t^2 - t^2\cos^2\theta = 3t^2\sin\theta + 26t^2$$

$$H = \frac{1}{2} \cdot \frac{5t}{\sqrt{(1+\sin\theta)^2+25}} \cdot \frac{1}{t^2(3\sin\theta+26)} = \frac{1}{2} \cdot \frac{5}{t(3\sin\theta+26)\sqrt{(1+\sin\theta)^2+25}}$$

$$\frac{1}{2}(k_1+k_2) = \frac{1}{2}k_1 \Rightarrow k_1 = \frac{5}{t(3\sin\theta+26)\sqrt{(1+\sin\theta)^2+25}}$$

$\varphi(x,y)$

$\varphi_x = -2x$

$\varphi_y = 2y$

$E = \langle (1,0,-2x),(1,0,-2x)\rangle = 4x^2+1$

$F = \langle (1,0,-2x),(0,1,2y)\rangle = -4xy$

$G = \langle (0,1,2y),(0,1,2y)\rangle = 4y^2+1$

$EG - F^2 = (4x^2+1)(4y^2+1) - (4xy)^2$

$= 16x^2y^2 + 4x^2 + 4y^2 + 1 - 16x^2y^2$

$= 4x^2+4y^2+1 \geq 1 > 0$

$\Gamma^1_{11}(4x^2+1) + \Gamma^2_{11}(-4xy) = \frac{1}{2}8x$

$\Gamma^1_{11}(-4xy) + \Gamma^2_{11}(4y^2+1) = -4y - \frac{1}{2}\cdot 0$

$E_x = 8x$

$E_y = 0$

$F_x = -4y$

$F_y = -4x$

$G_x = 0$

$G_y = 8y$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \Gamma^1_{11} \\ \Gamma^2_{11} \end{pmatrix} = \begin{pmatrix} 4x \\ -4y \end{pmatrix}$$

$$\left(\begin{array}{cc|c} E & F & 4x \\ F & G & -4y \end{array}\right)$$

$$\left(\begin{array}{cc|c} E & F & 4x \\ 0 & G - F^2/E & (-4y) - F/E\,(4x) \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & F/E & \frac{4x}{E} \\ 0 & 1 & \frac{(-4y) - F/E\,(4x)}{G - F^2/E} \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & F/E & \frac{4x}{E} \\ 0 & 1 & \frac{(-4y)E - F(4x)}{EG-F^2} \end{array}\right)$$

$$\Gamma^2_{11} = \frac{(-4y)E - F(4x)}{EG-F^2} = \frac{(-4y)(4x^2+1) + (4xy)(4x)}{4x^2+4y^2+1} = -\frac{16x^2y + 4y - 16x^2y}{4x^2+4y^2+1} = \Gamma^2_{11}$$

$$= \frac{-4y(4x^2+1-4x^2)}{4x^2+4y^2+1} = \frac{-4y(1)}{4x^2+4y^2+1}$$

$$\Gamma^1_{11} = \frac{4x}{E} - \frac{F}{E}\cdot\Gamma^2_{11} = \frac{4x - 4xy\left[\frac{4y}{4x^2+4y^2+1}\right]}{4x^2+1} = \frac{4x\left(1 - \frac{4y^2}{4x^2+4y^2+1}\right)}{4x^2+1} = \frac{4x}{4x^2+4y^2+1}$$

$$\Gamma^2_{12} = \frac{0\cdot E - F\cdot 0}{EG-F^2} = 0$$

$$\Gamma^1_{12} = \frac{0 - F\cdot\Gamma^2_{12}}{E} = \frac{0 - F\cdot 0}{E} = 0$$

$$\Gamma^2_{22} = \frac{(4y)E - F(-4x)}{EG-F^2} = \frac{(4y)(4x^2+1) + (4xy)(-4x)}{4x^2+4y^2+1} = \frac{16x^2y + 4y - 16x^2y}{4x^2+4y^2+1} = \frac{4y}{4x^2+4y^2+1}$$

$$\Gamma^1_{22} = \frac{-4x - F\cdot\Gamma^2_{21}}{E} = \frac{-4x}{4x^2+4y^2+1}$$

$$\left(\Gamma^2_{12}\right)_x = 0 \qquad \left(\Gamma^2_{11}\right)_y = \frac{-4\cdot(4x^2+4y^2+1) + 4y(8y)}{(4x^2+4y^2+1)^2} = \frac{16y^2 - 16x^2 - 4}{(4x^2+4y^2+1)^2}$$

$$K = -\frac{1}{E}\left[\left(\Gamma^2_{12}\right)_x - \left(\Gamma^2_{11}\right)_y + \Gamma^1_{12}\Gamma^2_{11} - \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^1_{22}\right]$$

$C = 4x^2+4y^2+1$

$$\Gamma^1_{11} + 2\Gamma^1_{12} + \Gamma^1_{22} = \frac{+4y}{4x^2+4y^2+1} + 2\cdot 0 + \frac{-4x}{4x^2+4y^2+1} = 0$$

$$\Gamma^2_{11} + 2\Gamma^2_{12} + \Gamma^2_{22} = \frac{-4y}{4x^2+4y^2+1} + 2\cdot 0 + \frac{4y}{4x^2+4y^2+1} = 0$$

$$e = \frac{-2}{\sqrt{4x^2+4y^2+1}}, \quad f = 0, \quad g = \frac{2}{\sqrt{4x^2+4y^2+1}}$$

$$L_1 = e \qquad L_1 + 2L_2 + L_3 = \frac{-2}{\sqrt{4x^2+4y^2+1}} + 2\cdot 0 + \frac{2}{\sqrt{4x^2+4y^2+1}} = 0$$

$$L_2 = f$$

$$L_3 = g$$

$$K = -\frac{1}{E}\left[(\Gamma^2_{12})_x - (\Gamma^2_{11})_y + \Gamma^1_{12}\cdot\Gamma^2_{11} - \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22}\right]$$

$$= -\frac{1}{E}\left[0 - \frac{16y^2-16x^2-4}{c^2} + 0\cdot\frac{-4y}{c} - \frac{4x}{c}\cdot 0 + 0\cdot 0 - \frac{-4y}{c}\cdot\frac{4y}{c}\right]$$

$$= +\frac{1}{c^2(4x^2+1)}\left[16y^2 - 16x^2 - 4 - 16y^2\right] = \frac{1}{c^2(4x^2+1)}\cdot(-(16x^2+4)) = -\frac{4}{c^2} = \frac{-4}{(4x^2+4y^2+1)^2}$$

$$K = \frac{-4}{(4x^2+4y^2+1)^2} < 0$$

$$V = x_x + x_y = (1, 1, 2y-2x)$$

$$V_x = x_{xx} + x_{xy} = \Gamma^1_{11}x_x + \Gamma^2_{11}x_y + L_1 N + \Gamma^1_{12}x_x + \Gamma^2_{12}x_y + L_2 N = (\Gamma^1_{11} + \Gamma^1_{12})x_x + (\Gamma^2_{11} + \Gamma^2_{12})x_y + (L_1 + L_2)N$$

$$V_y = x_{xy} + x_{yy} = \Gamma^1_{12}x_x + \Gamma^2_{12}x_y + L_2 N + \Gamma^1_{22}x_x + \Gamma^2_{22}x_y + L_3 N = (\Gamma^1_{12} + \Gamma^1_{22})x_x + (\Gamma^2_{12} + \Gamma^2_{22})x_y + (L_2 + L_3)N$$

$$V_x + V_y = (\Gamma^1_{11} + 2\Gamma^1_{12} + \Gamma^1_{22})x_x + (\Gamma^2_{11} + 2\Gamma^2_{12} + \Gamma^2_{22})x_y + (L_1 + 2L_2 + L_3)N$$