Date: ______________________________ Name: ______________________________

UPPER CANADA COLLEGE

Answer all questions in the area provided. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum Mark: 11]

In this question you will explore the convergence of infinite sequences and a mathematical test for determining if a specific sequence converges.

Consider the following sequence:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

(a) (1 point) Find the common ratio between consecutive terms of the sequence.

Solution: The common ratio is $\frac{1}{2}$, or 0.5 — A1

(b) (1 point) Hence, find the 4^{th} and 5^{th} terms of the sequence.

Solution: The fourth and fifth terms of the sequence are $\frac{1}{16}$ and $\frac{1}{32}$, respectively. — A1

(c) (1 point) Hence, state whether the sequence converges (i.e. approaches a finite value) or diverges (i.e. tends to infinity).

Solution: The sequence converges. — A1

(d) (1 point) Hence, state a general formula for the k^{th} element of the sequence.

Solution: One possible formula is

$$T_k = \left(\frac{1}{2}\right)^k$$

A1

(e) (1 point) Find a general formula for the following sequence:

$$1, 2, 4, 8 \ldots$$

Solution: One possible formula is

$$T_k = 2^{k-1}$$

A1

(f) (1 point) Hence, state whether the sequence converges or diverges.

Solution: The sequence diverges. A1

(g) (1 point) Calculate the ratio between consecutive terms for terms 1 to 5 in the following sequence in decimal:

$$1, 4, 9, 16 \ldots$$

where $T_k = k^2$

Solution: The ratios are as follows:

$$4, 2.25, 1.777\ldots, 1.5625$$ A1

(h) (2 points) Show that the equation $y = \frac{(x+1)^2}{x^2}$ has a horizontal asymptote at $y = 1$

Solution:

$$y = \frac{(x+1)^2}{x^2} \quad (1)$$

$$= \frac{x^2 + 2x + 1}{x^2} \quad (2)$$

$$= 1 + \frac{2}{x} + \frac{1}{x^2} \quad (3)$$ M1

Since $\frac{2}{x}$ and $\frac{1}{x^2}$ both have asymptotes at $y = 0$, then $1 + \frac{2}{x} + \frac{1}{x^2}$ has an asymptote at $y = 1$. R1

(i) (1 point) Hence, state what the ratio between terms approaches in the sequence in (g)

Solution: The ratio between terms approaches 1. A1

Consider the sequence defined by the following equation:

$$T_k = 2^k + \sin\left(\frac{\pi k}{2}\right)$$

(j) (1 point) Question