Computational Physics Homework Assignment #1

November 11, 2024; Due November 18, 2024

Reading Assignment

1. Read lecture notes and references; Recall your previous learned; Get familiar with computational resources, some useful software, and various operating system commands.
2. Study the sample programs either in the textbook or lecture notes, and write your own computer programs for coffee-cooling, falling objects, oscillatory motion, etc., use any programming language as you prefer. **You may ask teachers for sample programs.**

Laboratory Assignments (Total Points: 160; Choice: Problems 1+2, 3+4; or 1+2, 5)

1. (Problem 2.1) **Coffee cooling program** (55 points: 20, 10, 10, 10, 5)
   (a) **Determine Cooling Constant** r: based on Newton's law of cooling, find approximated values of r for black coffee and coffee with cream that describe the experimental results shown in the following Table (is r a constant from the following Table). What is your implicit criterion for determining the best value of r? Give two ways of determining the best value of r and argue which is better and why it is better. Because time is measured in minutes, the unit of the cooling constant r is min^{-1}.
   (b) Use the value of r found in part (a) and make a graph showing the dependence of temperature on time. Plot the data given in the Table on the same graph and compare them with your results. You may use any graphic package to make the plot.
   (c) Does the time step Δt have any physical significance? Make sure your choice of Δt is sufficiently small so that it does not affect your results. You should estimate the error of your results.
   (d) The initial difference in temperature between the black coffee and its surroundings is approximately $64^{\circ}C$. How long does it take for the coffee to cool so that the difference is $64/2 = 32^{\circ}C$? How long does it take the difference to become $64/4$ and $64/8$ Try to understand your results in simple terms without first using a computer.
   (e) Refer to the Table, discuss if Newton's law of cooling is applicable to the cooling of a cup of coffee, and what modifications can one make?

Table The temperature of coffee in a glass. The temperature was recorded with an estimated accuracy of $0.1^{\circ}C$. The air temperature was $17^{\circ}C$. The second column corresponds to black coffee and the third column corresponds to coffee with heavy cream.

time (min)	$T^{\circ}C$ (black)	$T^{\circ}C$ (cream)	time (min)	$T^{\circ}C$ (black)	$T^{\circ}C$ (cream)
0	82.3	68.8	24	51.2	45.9
2	78.5	64.8	26	49.9	44.8
4	74.3	62.1	28	48.6	43.7
6	70.7	59.9	30	47.2	42.6
8	67.6	57.7	32	46.1	41.7
10	65.0	55.9	34	45.0	40.8
12	62.5	53.9	36	43.9	39.9
14	60.1	52.3	38	43.0	39.3
16	58.1	50.8	40	41.9	38.6
18	56.1	49.5	42	41.0	37.7
20	54.3	48.1	44	40.1	37.0
22	52.8	46.8	46	39.5	36.4

2. (Problem 2.3) **Accuracy and stability of the Euler method** (25 points: 5, 10, 10)

   (a) Show that the analytical solution of (2.1) can be written in the form

   $$T(t) = T_s - (T_s - T_0)e^{-rt} \qquad (1)$$

   Note that $T(t = 0) = T_0$ and that $T(t \rightarrow \infty) = T_s$.

   (b) Use your program to compute the temperature at $t = 1.60$ min with $\Delta t = 0.1, 0.05, 0.025, 0.01$, and 0.005. Use the value of r of the Table. Make a table showing the difference between the exact solution (1) and your numerical solution as a function of Δt. Is the difference a decreasing function of Δt? If Δt is decreased by a factor of two, how does the difference change? Plot the difference as a function of Δt. If your points fall approximately on a straight line, then the difference is proportional to Δt, for $\Delta t << 1$. A numerical method is called *nth* order, if the difference between the analytical solution and the numerical solution is proportional to $(\Delta t)^n$ at a fixed value of t. What is the order of the Euler method?

   (c) One way to determine the accuracy of a numerical solution is to repeat the calculation with a smaller step size and compare the results. If the two calculations agree to p decimal places, we can reasonably assume that the results are correct to p decimal places. What value of Δt is necessary for 0.1% accuracy at $t = 1.60$? What value of Δt is necessary for 0.1% accuracy at $t = 5.5$?

3. **Comparison of algorithm.** (40 Points: 10, 5, 10, 15)

   (a) Use your programs to determine the time dependence of the velocity and position of a freely falling body near the earth's surface. Assume the values $y = 32m$ and $v = 0$ at $t = 0s$. The coordinate system is shown in the slide 8 of lecture-3 note. Compare your output to the exact results given by $v = v_0 - gt$, $y(t) = y_0 + v_0t - gt^2/2$. To reach the accuracy of 10^{-5}, what is a suitable value of Δt for the Euler, the Euler-Cromer, and the Euler-Richardson algorithms? (Make a table of comparison.)

   (b) Use any graphic tool to plot y and v of falling object as functions of time.

   (c) Apply your program to a simple harmonic oscillator for which $F = -kx$, taking units such that $k = 1$ and $m = 1$. Assuming $x(t = 0) = 1.1$ and $v(t = 0) = 0.0$, determine $x(t)$ by using the three algorithms and compare them with the exact result (I hope you remember how to get it) at $t = n\pi/4$, $n = 1, 2, \cdots, 32$. What happens if you run for longer time, say $n = 256$, or 512? Try different values of Δt such that the accuracy as compared to the exact solution at $t = \pi$ is of 10^{-5}.

   (d) From your simulation results, is the Euler-Cromer algorithm better than the Euler algorithm? Think of a simple modification of either algorithm that yields exact results for the case of a freely falling body without air resistance.

4. **Trajectory of a shot.** (40 Points: 10, 10, 10, 10)

   (a) Use (or modify) your program(s) to compute the two-dimensional trajectory of a ball moving in air and plot y as function of x. Neglect air resistance first so that you can compare your computed results with the exact results. Assume that a ball is thrown from ground level at an angle θ_0 above the horizontal with an initial velocity v_0 =25 m/s. Vary θ_0 (at least 5 θ_0 values) and show that the maximum range occurs at $\theta_0 = \theta_{max} = 45^{o}$. What is R_{max}, the maximum range of the ball, divided by v_0^2/g at corresponding angles?

(b) Suppose that a ball is thrown from a height h at an angle θ_0 above the horizontal with the same initial speed as in part (a). Again neglect air resistance, do you expect θ_{max} to be larger or smaller than 45°? What is θ_{max} for $h = 2.1m$? By what percent is the range R changed if θ is varied by 3.0% from θ_{max}?

(c) Consider the effects of air resistance on the range and optimum angle of the ball in part (b). Assume $F_d(v) = k_2v^2$. What is the unit of k_2? Compute the optimum angle for $h = 3.3m$, $v_0 = 32m/s$, and $C = k_2/m = 0.1$, and compare your answer to the value found in part (b). Is R more or less sensitive to changes in θ_0 from θ_{max} than in part (b)? Determine the optimum launch angle and the corresponding range for the more realistic value of C = 0.002.

(d) Consider the motion of two identical objects that both start from a height h = 15m. One object is dropped vertically from rest and the other is thrown with a horizontal velocity v_x = 30m/s. Answer the following two questions by intuition and then by simulation results.

i. Which object reaches the ground first?

ii. Assume that there is air resistance and that the drag force is proportional to v^2 with $C = k_2/m = 0.03$. Which object reaches the ground first now? What if the drag force is proportional to v with $C = k_1/m = 0.055$?

5. **Motion of a linear oscillator** (80 Points: 10, 10, 15, 15, 10, 20).

(a) Set $\omega_0 = 3$, $\gamma = 0.4$, $\omega = 2$ and the amplitude of the external force $A_0 = 3.1$ for all runs unless otherwise stated. This corresponds to an underdamped oscillator in the absence of an external force.

Plot $x(t)$ versus t for $A_0 = 0$ and $A_0 = 3.2$.

The initial conditions are: $x(t = 0) = 1$, $v(t = 0) = 0$.

Q1: How does the qualitative behavior of $x(t)$ differ when $A_0 = 3.1$ from the non-perturbed ($A_0 = 0$) case?

Q2: What is the period and angular frequency of $x(t)$ after several oscillations have occurred? (*Just estimate it by counting.*)

(b) Repeat (a) for another initial condition $x(t = 0) = 0.5$, $v(t = 0) = 1.0$.

Q3: Does $x(t)$ approach a limiting behavior independently of the initial conditions?

(Note: Try to identify a *transient* part of $x(t)$ that depends on the initial conditions and decays in time, and a *steady state* part that dominates at longer times and is independent of the initial conditions.)

(c) Modify your program so that E_n, the total energy per unit mass, is computed at each time $t_n = t_0 + n\Delta t$. Plot the difference $\Delta E_n = E_n - E_0$ as a function of time for part (a).

Q4: Is E_n a conserved quantity within the algorithm accuracy?

Q5: Try the 4th order Runge-Kutta algorithm for **Q4**.

(d) Compute $x(t)$ for several combinations of ω_0 and ω (3 sets at least).

Q6: What is the period and angular frequency of the steady state in each case?

Q7: What parameters determine the frequency of the steady behavior? Assume that T is proportional to ω^{α} and estimate the exponent α from a log-log plot of T versus ω.

(e) There is an important concept called *Phase Space Trajectory*, which is obtained by treating $x(t)$ and $v(t)$ as two-dimensional coordinates and then plot the point $(x(t), v(t))$ as time increases.

Q8: What is the shape of the phase space trajectory for the initial condition $x(t = 0) = 1$, $v(t = 0) = 0.5$? Do you find a different phase trajectory for other initial conditions?

(f) One measure of the long-term behavior of the driven harmonic oscillator is the amplitude of the steady displacement $A(\omega)$, which is the maximum value of $x(t)$. Verify that the steady state behavior of $x(t)$ is given by

$$x(t) = A(\omega)\cos(\omega t + \delta(\omega)).$$

The quantity $\delta(\omega)$ is the phase difference between the applied force and the steady state motion.

Q9: Compute $A(\omega)$ and $\delta(\omega)$ for $\omega_0 = 2.8$, $\gamma = 0.5$, and $\omega = 0$, 1.0, 2.0, 2.3, 2.6, 2.9, 3.2, 3.5, and 3.8. Choose the initial condition, $x(t = 0) = 1.0$, $v(t = 0) = 0$. Repeat the simulation for $\gamma = 2.0$, and plot $A(\omega)$ and $\delta(\omega)$ versus ω for the two values of γ.

Q10: If $A(\omega)$ has a maximum, determine the angular frequency ω_m at which the maximum of A occurs. Is the value of ω_m close to the natural frequency ω_0? Compare ω_m to ω_0 and to the frequency of the damped linear oscillator in the absence of an external force.