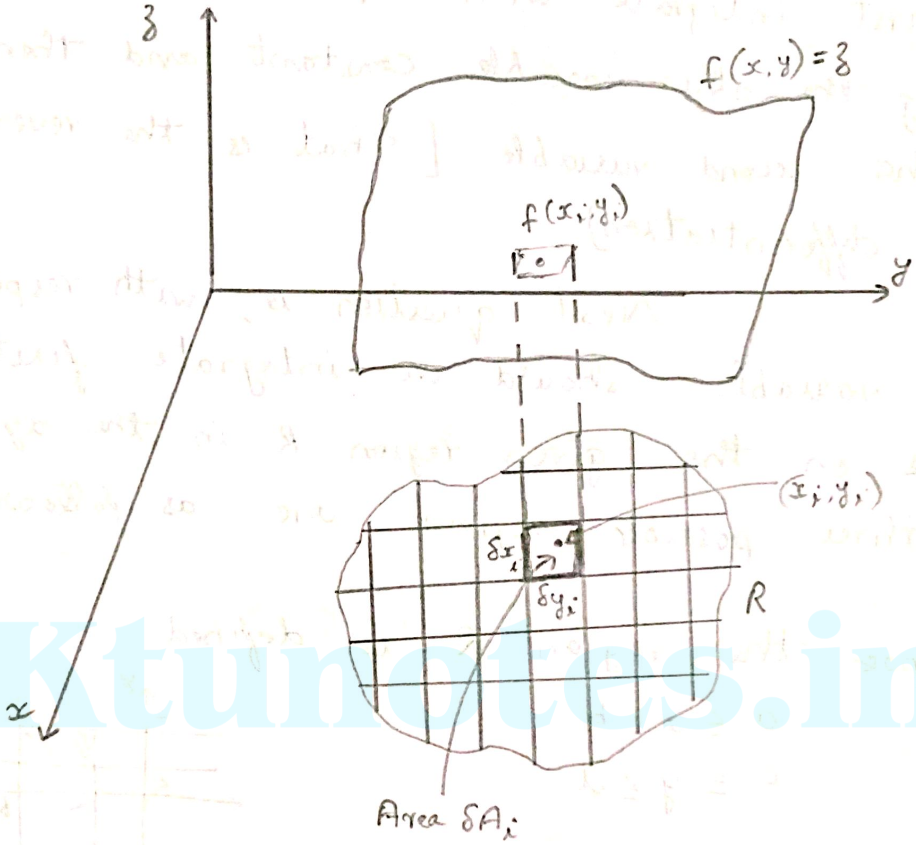

KTU
NOTES

Module III

Multivariable Calculus - Integration

Double Integral



Let $f(x,y)$ be a function of two variables that is continuous on a closed region R in the xy-plane. Divide the region R into n subregions of areas $\delta A_1, \delta A_2, \ldots\ldots, \delta A_n$ by drawing lines parallel to coordinate axes. Let (x_i, y_i) be any point in the i^{th} rectangle whose area is δA_i. Then

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\,\delta A_i \quad \text{OR} \quad \iint_R f(x,y)\,dx\,dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\,\delta x_i\,\delta y_i$$

Geometrically it represents the volume of the solid bounded by the surface $z = f(x,y)$ and the cylinder formed by extending upward the region R.

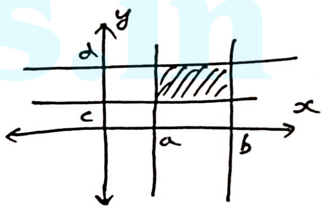
Evaluation of double integral

To evaluate a double integral, we apply single integration twice. That is, we first integrate with respect to one variable, keeping the other variable constant and then with respect to the second variable. [That is, the reverse process of partial differentiation].

Next question is, with respect to which variable should we integrate first? That depends on the given region R in the xy-plane. The three possible regions are as follows

1) Suppose the region R is defined

$$a \le x \le b$$
$$c \le y \le d$$



Here since both variables have constant limits we can integrate either w.r.t x first or w.r.t y first.

1. Evaluate $\displaystyle\int_2^4 \int_1^3 (40 - 2xy)\, dx\, dy$

Soln :-

$$\int_{y=2}^{4} \left[\int_{x=1}^{3} (40 - 2xy)\, dx \right] dy = \int_2^4 \left[40x - \cancel{2}\frac{x^2}{\cancel{2}} y \right]_1^3 dy$$

$$= \int_2^4 (120 - 9y - 40 + y)\, dy$$

$$= \int_2^4 (80 - 8y)\, dy = \left[80y - \frac{8y^2}{2} \right]_2^4$$

$$= 320 - 64 - 160 + 16 = 112$$

OR

$$\int_{x=1}^{3}\left[\int_{y=2}^{4}(40-2xy)\,dy\right]dx = \int_{1}^{3}\left(40y - xy^2\right)_2^4 dx$$

$$= \int_1^3 (160 - 16x - 80 + 4x)\,dx$$

$$= \int_1^3 (80 - 12x)\,dx = \left[80x - 6x^2\right]_1^3$$

$$= 240 - 54 - 80 + 6 = \underline{\underline{112}}$$

2. Evaluate $\int_{-2}^{0}\int_{-1}^{2}(x^2+y^2)\,dx\,dy$

Soln :- $\int_{y=-2}^{0}\left[\int_{x=-1}^{2}(x^2+y^2)\,dx\right]dy = \int_{-2}^{0}\left[\frac{x^3}{3} + xy^2\right]_{-1}^{2} dy$

$$= \int_{-2}^{0}\left(\frac{8}{3} + 2y^2 + \frac{1}{3} + y^2\right)dy$$

$$= \int_{-2}^{0}(3y^2 + 3)\,dy = \left[y^3 + 3y\right]_{-2}^{0} = \underline{\underline{14}}$$

3. Use a double integral to find the volume of the solid that is bounded ~~by~~ above by the plane $z = 4-x-y$ and below by the rectangle $R = \underset{x}{[0,1]} \times \underset{y}{[0,2]}$

Soln :- volume $= \iint_R f(x,y)\,dx\,dy = \int_{y=0}^{2}\left[\int_{x=0}^{1}(4-x-y)\,dx\right]dy$

$$= \int_0^2\left[4x - \frac{x^2}{2} - xy\right]_0^1 dy = \int_0^2\left(\frac{7}{2} - y\right)dy$$

$$= \left[\frac{7}{2}y - \frac{y^2}{2}\right]_0^2 = 7 - 2 = \underline{\underline{5}}$$

Problems

1) $\int_0^1 \int_0^2 (x+5)\, dy\, dx$

2) $\int_2^4 \int_0^2 x^2 y\, dx\, dy$

3) $\int_0^2 \int_0^1 y \cos x\, dy\, dx$

4) $\int_{-1}^0 \int_2^6 dx\, dy$

5) $\int_0^2 \int_0^1 \frac{x}{(xy+1)^2}\, dy\, dx$

6) $\int_0^{\ln 3} \int_0^{\ln 2} e^{x+2y}\, dy\, dx$

7) $\int_{\pi/2}^{\pi} \int_1^2 x \sin(xy)\, dy\, dx$

8) Evaluate double integral over the rectangular region R.

i) $\iint_R 4xy^3\, dA$; $R = \{(x,y) : -1 \le x \le 1,\ -3 \le y \le 3\}$

ii) $\iint_R (x \sin y - y \sin x)\, dA$; $R = \{(x,y) : 0 \le x \le \frac{\pi}{2},\ 0 \le y \le \frac{\pi}{3}\}$

iii) $\iint_R x\sqrt{1-x^2}\, dA$; $R = \{(x,y) : 0 \le x \le 1,\ 2 \le y \le 3\}$

9) Use a double integral to find the volume.

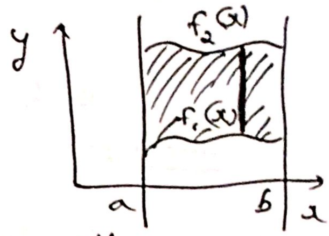
i) The volume under the plane $z = 2x+y$ over the region $R = \{(x,y) : 3 \le x \le 6,\ 1 \le y \le 2\}$

ii) The volume of the solid enclosed by the surface $z = x^3$ and the planes $x=0$, $x=2$, $y=4$, $y=0$ and $z=0$

iii) The volume in the first octant bounded by the coordinate planes, the plane $y=4$ and the plane $\frac{x}{3} + \frac{z}{5} = 1$

2) Suppose the region R is defined by (Type-I region)

$$a \le x \le b$$
$$f_1(x) \le y \le f_2(x)$$



Here we integrate first with respect to y (having varying limits) and then with respect to x.

$$\iint_R f(x,y)\,dx\,dy = \int_{x=a}^{b}\left[\int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dy\right]dx$$

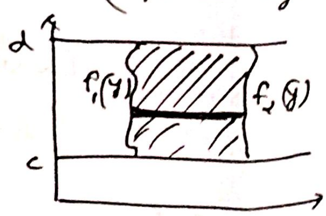
1. Evaluate $\int_0^1 \int_{-x}^{x^2} xy^2 \, dA$

Soln: $\int_{x=0}^{1}\left[\int_{y=-x}^{x^2}(xy^2)\,dy\right]dx = \int_0^1 \left[\frac{xy^3}{3}\right]_{-x}^{x^2} dx = \int_0^1\left(\frac{x^7}{3}+\frac{x^4}{3}\right)dx$

$$= \frac{1}{3}\left[\frac{x^8}{8}+\frac{x^5}{5}\right]_0^1 = \underline{\underline{\frac{13}{120}}}$$

3) Suppose the region R is defined by (Type-II region)

$$f_1(y) \le x \le f_2(y)$$
$$c \le y \le d$$



Here we integrate first with respect to x (having varying limits) and then with respect to y.

$$\iint_R f(x,y)\,dA = \int_{y=c}^{d}\left[\int_{x=f_1(y)}^{f_2(y)} f(x,y)\,dx\right]dy$$

2. Evaluate $\displaystyle\int_0^{\pi/3}\int_0^{\cos y} x \sin y \, dA$

Soln :- $\displaystyle\int_{y=0}^{\pi/3}\left[\int_{x=0}^{\cos y} x \sin y \, dx\right] dy = \int_0^{\pi/3}\left(\sin y \cdot \frac{x^2}{2}\right]_0^{\cos y} dy$

$\displaystyle = \int_0^{\pi/3} \sin y \, \frac{\cos^2 y}{2} \, dy$ Put $t = \cos y$, $dt = -\sin y \, dy$

$\displaystyle = \int_1^{1/2} \frac{-t^2}{2} \, dt = \frac{-1}{2}\left[\frac{t^3}{3}\right]_1^{1/2} = \underline{\underline{\frac{7}{48}}}$

3. Evaluate $\displaystyle\int_0^1\int_{x^2}^{x} xy^2 \, dA$

Soln :- $\displaystyle\int_{x=0}^{1}\left[\int_{y=x^2}^{x} xy^2 \, dy\right] dx = \int_0^1 \left[\frac{xy^3}{3}\right]_{x^2}^{x} dx = \int_0^1\left(\frac{x^4}{3} - \frac{x^7}{3}\right) dx$

$\displaystyle = \left[\frac{x^5}{15} - \frac{x^8}{24}\right]_0^1 = \frac{1}{15} - \frac{1}{24} = \underline{\underline{\frac{1}{40}}}$

4. Evaluate $\displaystyle\int_0^3\int_0^{\sqrt{9-y^2}} y \, dA$

Soln :- $\displaystyle\int_{y=0}^{3}\left[\int_{x=0}^{\sqrt{9-y^2}} y \, dx\right] dy = \int_0^3 \left[xy\right]_0^{\sqrt{9-y^2}} dy = \int_0^3\left(\sqrt{9-y^2}\right) y \, dy$

$\displaystyle = \int_9^0 \sqrt{t}\left(\frac{-dt}{2}\right)$ Put $9-y^2 = t$, $-2y\,dy = dt$

$\displaystyle = \frac{1}{2}\int_0^9 \sqrt{t} \, dt = \frac{1}{2}\left[\frac{2}{3} t^{3/2}\right]_0^9$

$\displaystyle = \frac{1}{3}\left[27\right] = \underline{\underline{9}}$

Problems

Evaluate

1) $\displaystyle\int_{-1}^{1}\int_{x^2}^{x} xy^2\, dy\, dx$

2) $\displaystyle\int_{1}^{2}\int_{y}^{3-y} y\, dx\, dy$

3) $\displaystyle\int_{1/4}^{4}\int_{x^2}^{x} \sqrt{\frac{x}{y}}\, dA$

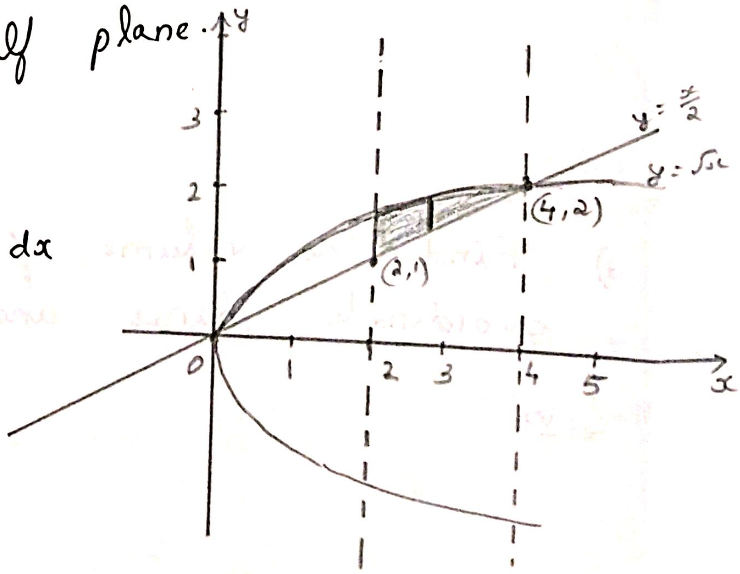
4) $\displaystyle\int_{\sqrt{\pi}}^{\sqrt{2\pi}}\int_{0}^{x^3} \sin\left(\frac{y}{x}\right) dA$

5) $\displaystyle\int_{-2}^{2}\int_{-x^2}^{x^2} (x^2-y)\, dA$

6) $\displaystyle\int_{0}^{2}\int_{0}^{x} y\sqrt{x^2-y^2}\, dy\, dx$

7) $\displaystyle\int_{0}^{2}\int_{0}^{y^2} e^{\frac{x}{y^2}}\, dx\, dy$

1. Evaluate $\displaystyle\iint_R xy\, dA$ over the region R enclosed between $y = \frac{x}{2}$, $y = \sqrt{x}$, $x = 2$ and $x = 4$ which lies in the upper half plane.



Soln :-

$$\iint_R xy\, dA = \int_{x=2}^{4}\int_{y=\frac{x}{2}}^{\sqrt{x}} xy\, dy\, dx$$

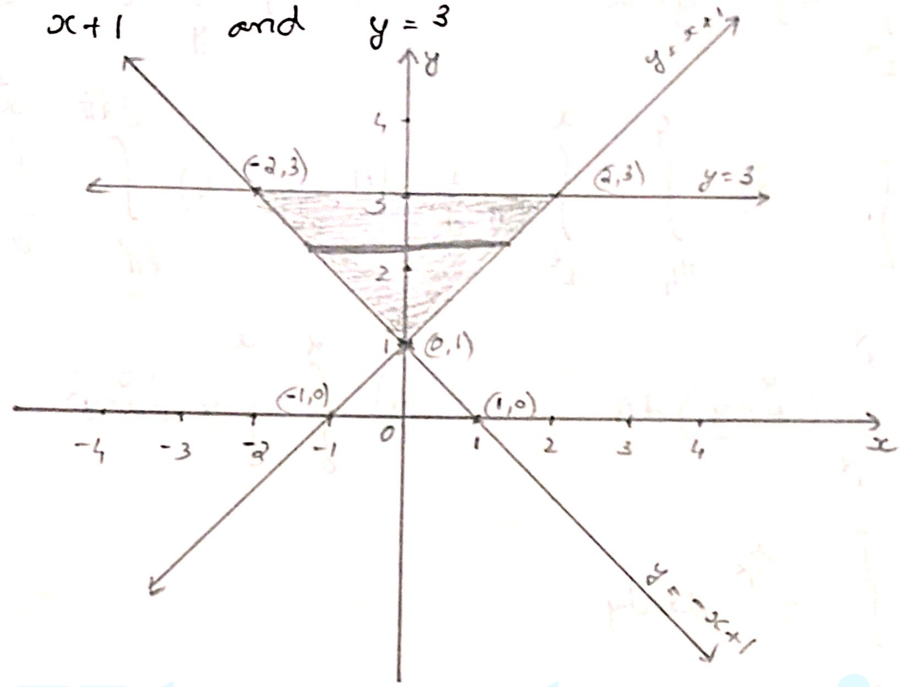
$$= \int_2^4 \left[\frac{xy^2}{2}\right]_{\frac{x}{2}}^{\sqrt{x}} dx$$

$$= \int_2^4 \left(\frac{x^2}{2} - \frac{x^3}{8}\right) dx$$

$$= \left[\frac{x^3}{6} - \frac{x^4}{32}\right]_2^4 = \frac{11}{6}$$

2) Evaluate $\iint_R (2x-y^2)\,dA$ over the triangular region R enclosed between the lines $y=-x+1$, $y=x+1$ and $y=3$

<u>Soln</u> :-



OR

(To get the limits of y, put $x=0$ we get $y=1$ So y varies from 1 to 3 and x varies from $1-y$ to $y-1$)

solve $y=-x+1$ and $y=x+1$ we get $x=0$

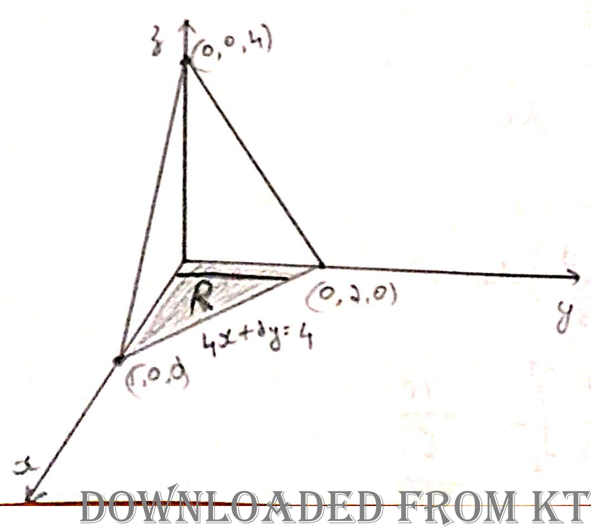
$$\iint_R (2x-y^2)\,dA = \int_{y=1}^{3}\int_{x=1-y}^{y-1} (2x-y^2)\,dx\,dy = \int_1^3 \left(x^2 - xy^2\right)_{1-y}^{y-1} dy$$

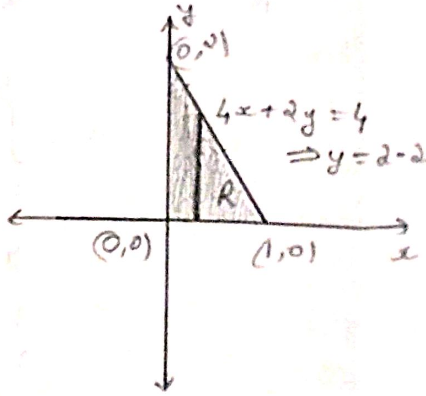
$$= \int_1^3 (2y^2-2y^3)\,dy = \left[\frac{2y^3}{3} - \frac{y^4}{2}\right]_1^3$$

$$= \frac{-68}{3}$$

3) Find the volume of the tetrahedron bounded by the coordinate planes and the plane $z = 4-4x-2y$

<u>Soln</u>:-



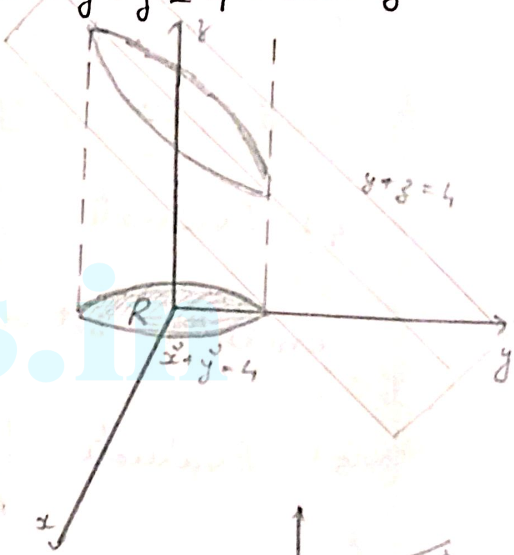


$$\text{Volume} = \iint_R f(x,y)\,dA = \int_{x=0}^{1}\int_{y=0}^{2-2x} (4-4x-2y)\,dy\,dx$$

$$= \int_0^1 \left[4y - 4xy - y^2\right]_0^{2-2x} dx = \int_0^1 (4 - 8x + 4x^2)\,dx$$

$$= \left[4x - 4x^2 + \frac{4x^3}{3}\right]_0^1 = \underline{\underline{\frac{4}{3}}}$$

4) Find the volume of the solid bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$



Soln :-

$$y+z=4 \Rightarrow z = 4-y$$

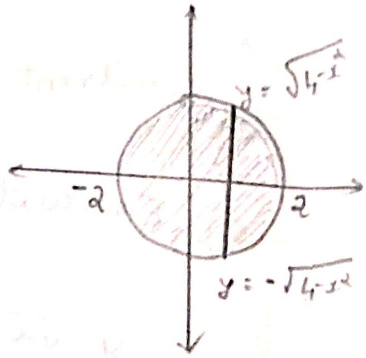
To get the limits of y,

$$x^2+y^2=4 \Rightarrow y = \pm\sqrt{4-x^2}$$

To get the limits of x,

put $y=0$ in $x^2+y^2=4$

$$\Rightarrow x^2=4 \Rightarrow x = \pm 2.$$



$$\text{Volume} = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\,dy\,dx$$

$$= \int_{-2}^{2}\left[4y - \frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx = 8\int_{-2}^{2}\sqrt{4-x^2}\,dx$$

Put $x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

$$= 8\int_{\theta=-\pi/2}^{\pi/2} 2\cos\theta\, 2\cos\theta\, d\theta$$

$$= 32\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= 16\left[\theta + \frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2} = 16\left[\frac{\pi}{2}+\frac{\pi}{2}\right] = \underline{\underline{16\pi}}$$

Problems

1) Evaluate $\iint\limits_R xy^2 \, dA$; R is the region enclosed by $y=1$, $y=2$, $x=0$ and $x=y$.

2) Evaluate $\iint\limits_R (3x-2y) \, dA$; R is the region enclosed by the circle $x^2+y^2=1$

3) Evaluate $\iint\limits_R x^2 \, dA$; R is the region bounded by $y=\frac{16}{x}$, $y=x$ and $x=8$

4) Evaluate $\iint\limits_R y \, dA$, R is the region in the first quadrant enclosed between the circle $x^2+y^2=25$ and the line $x+y=5$

5) Evaluate $\iint\limits_R x(1+y^2)^{-1/2} \, dA$; R is the region in the first quadrant enclosed by $y=x^2$, $y=5$ and $x=0$

6) Evaluate $\iint\limits_R xy \, dA$; R is the region enclosed by $y=\sqrt{x}$, $y=6-x$ and $y=0$

7) Evaluate $\iint\limits_R x^2 \, dA$; R is the region in the first quadrant enclosed by $xy=1$, $y=x$ and $y=3x$

8) Evaluate $\iint\limits_R x \cos y \, dA$; R is the triangular region bounded by the lines $y=x$, $y=0$ and $x=\frac{\pi}{2}$

9) Find the volume of the solid bounded by the cylinder $x^2+y^2=16$ and the planes $z=0$ and $z=4-x$

10) Find the volume of the solid in the first octant bounded above by the paraboloid $z=x^2+3y^2$, below by the plane $z=0$ and laterally by $y=x^2$ and $y=x$.

11) Find the volume of the solid bounded above by the paraboloid $z=9x^2+y^2$, below by the plane $z=0$ and laterally by the planes $x=0$, $y=0$, $x=3$ and $y=2$.

12) Find the volume of the solid enclosed by $y^2=x$ $z=0$ and $x+z=1$

13) Find the volume of the solid bounded by the cylinder $4x^2+y^2=9$ and the planes $z=0$ and $z=y+4$

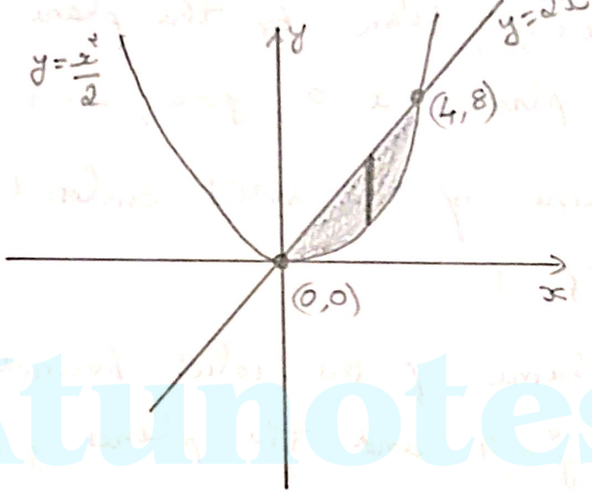
14) Find the volume of the solid in the first octant bounded above by $z=9-x^2$, below by $z=0$ and laterally by $y^2=3x$.

Area calculated as double integral

Area of the region $R = \iint_R dA = \iint_R dx\,dy$

1. Use double integral to find the area of the region R enclosed between the parabola $y = \frac{x^2}{2}$ and the line $y = 2x$.

Soln :-



Solving $y = \frac{x^2}{2}$ and $y = 2x$, we get $y = \frac{x^2}{2}$

$\Rightarrow 2x = \frac{x^2}{2}$

$\Rightarrow x^2 - 4x = 0$

$\Rightarrow x(x-4) = 0$

$\Rightarrow x = 0 \quad x = 4$

when $x = 0,\ y = 0$

$x = 4,\ y = 8$

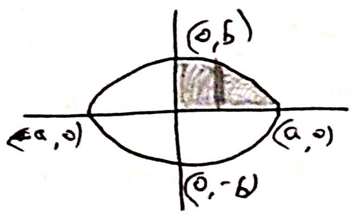
$$\text{Area} = \iint_R dA = \int_{x=0}^{4} \int_{y=\frac{x^2}{2}}^{2x} dy\,dx = \int_0^4 [y]_{\frac{x^2}{2}}^{2x} dx$$

$$= \int_0^4 \left(2x - \frac{x^2}{2}\right) dx = \left[x^2 - \frac{x^3}{6}\right]_0^4$$

$$= \frac{16}{3}$$

2. Find the area of the plate in the form of a quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Soln :-



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\Rightarrow y^2 = \frac{b^2(a^2 - x^2)}{a^2} \Rightarrow y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

$$\text{Area} = \iint_R dA = \int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dy\, dx$$

$$= \int_0^a [y]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx = \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\, dx$$

$$= \frac{b}{a} \int_{\theta=0}^{\pi/2} \sqrt{a^2 - a^2\sin^2\theta}\; a\cos\theta\, d\theta$$

Put $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$

$$= \frac{b}{a} \times a^2 \int_0^{\pi/2} \cos^2\theta\, d\theta = ab \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{ab}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{ab}{2}\left[\frac{\pi}{2}\right] = \frac{\pi ab}{4}$$

3) Using double integration, find the area of the plane region enclosed by $y = \sin x$ and $y = \cos x$ for $0 \le x \le \frac{\pi}{4}$.

Soln :-
$$\text{Area} = \iint_R dA = \int_{x=0}^{\pi/4} \int_{y=\sin x}^{\cos x} dy\, dx = \int_0^{\pi/4} [y]_{\sin x}^{\cos x} dx$$

$$= \int_0^{\pi/4} (\cos x - \sin x)\, dx$$

$$= [\sin x + \cos x]_0^{\pi/4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 = \sqrt{2} - 1$$

Problems :-

1) Use double integration, to find the area of the region enclosed by the curves

i) $y^2 = -x$ and $4y - x = 5$

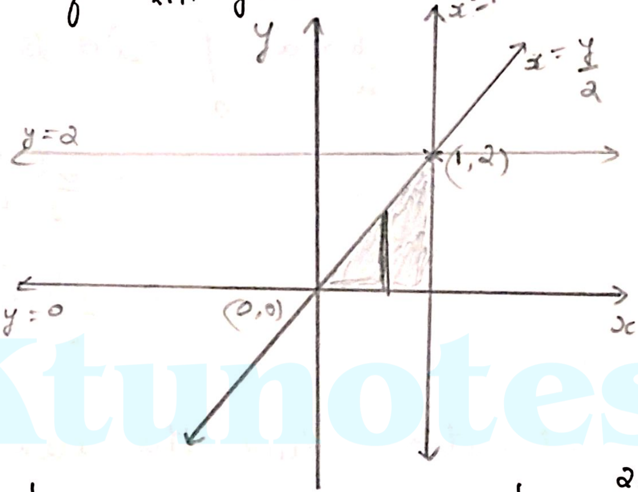
ii) $y^2 = 9 - x$ and $y^2 = 9 - 9x$

Reversing the order of integration

Sometimes the evaluation of the integral can be simplified by reversing the order of integration.

1. Evaluate $\int_0^2 \int_{y/2}^1 e^{x^2}\, dx\, dy$ by reversing the order of integration.

Soln:



$$\int_{y=0}^{2} \int_{x=y/2}^{1} e^{x^2}\, dx\, dy = \int_{x=0}^{1} \int_{y=0}^{2x} e^{x^2}\, dy\, dx$$

$$= \int_0^1 \left[e^{x^2} y \right]_0^{2x} dx = \int_0^1 \left(e^{x^2} \cdot 2x \right) dx$$

Put $x^2 = u$, $2x\,dx = du$

$$= \int_{u=0}^{1} e^{u}\, du$$

$$= \left[e^{u} \right]_0^1 = \underline{\underline{e-1}}$$

2) Change the order of integration and evaluate

$$\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\, dx$$

$y^2 = 4ax$

$\Rightarrow \left(\frac{x^2}{4a}\right)^2 = 4ax$

$\Rightarrow x^4 = 64a^3x$

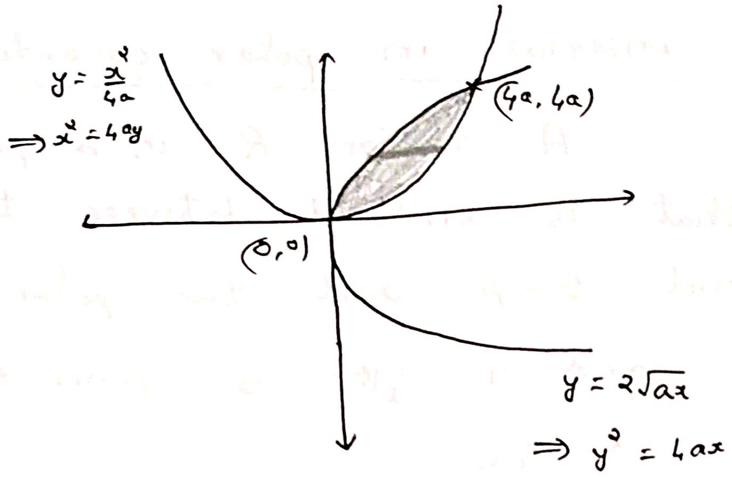
$\Rightarrow x(x^3 - 64a^3) = 0$

$\Rightarrow x = 0 \quad x = 4a$

when $x=0 \quad y=0$

$x = 4a \quad y = 4a$

Soln:



$$\int_{y=0}^{4a}\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy = \int_0^{4a}\left[x\right]_{y^2/4a}^{2\sqrt{ay}} dy = \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right)dy$$

$$= \left[2\sqrt{a}\,\frac{2}{3}\,y^{3/2} - \frac{y^3}{12a}\right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{(4a)^3}{12a}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}$$

Problems

Evaluate the integral by reversing the order of integration.

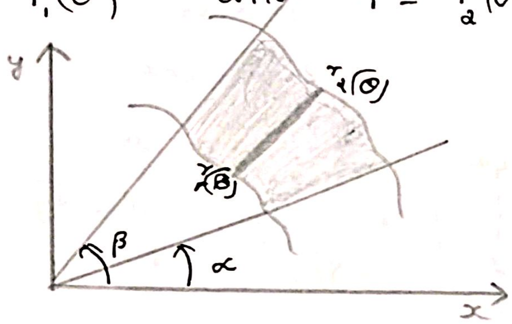
1) $\int_0^1\int_{4x}^4 e^{-y^2}\,dy\,dx$

2) $\int_0^2\int_{y/2}^1 \cos(x^2)\,dx\,dy$

3) $\int_0^4\int_{\sqrt{y}}^2 e^{x^3}\,dx\,dy$

Double integral in polar co-ordinates (r, θ)

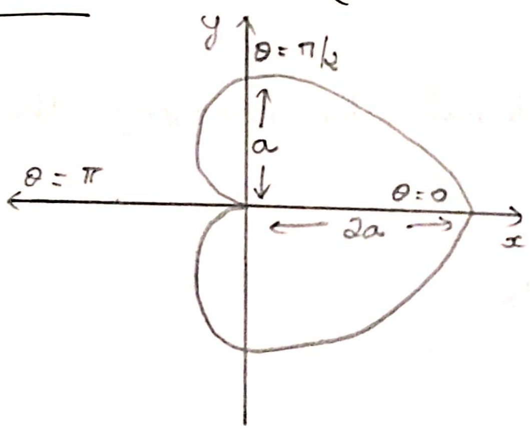
A region R in a polar co-ordinate system that is enclosed between two rays $\theta = \alpha$ and $\theta = \beta$ and two polar curves $r = r_1(\theta)$ and $r = r_2(\theta)$ is given by



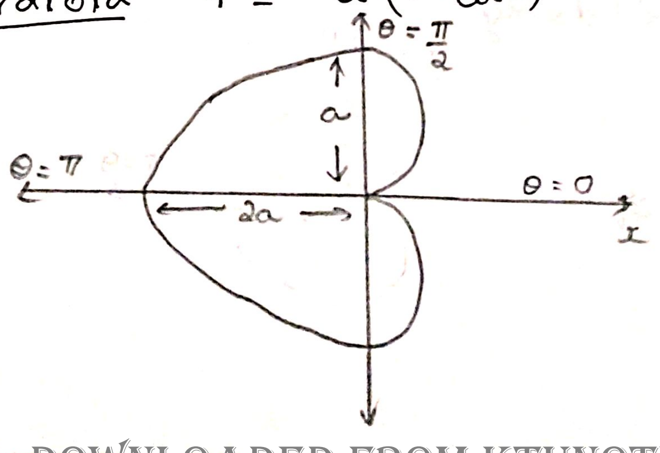
$$\iint_R f(r,\theta)\, dA = \int_{\theta=\alpha}^{\beta} \int_{r=r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

Some important polar curves

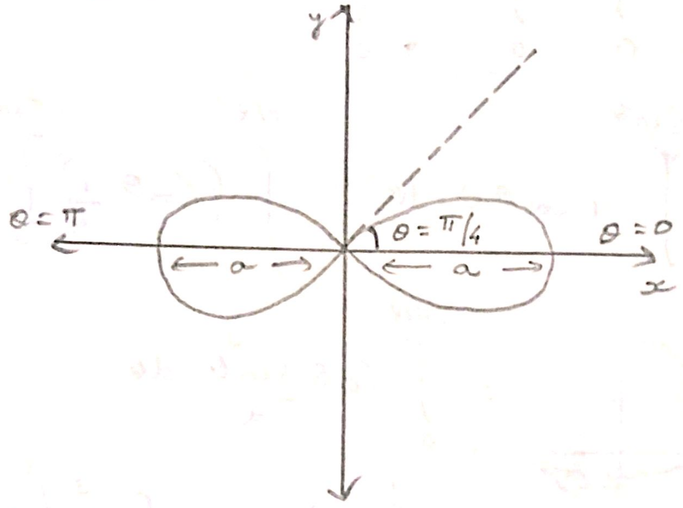
1) **Cardioid** $r = a(1+\cos\theta)$



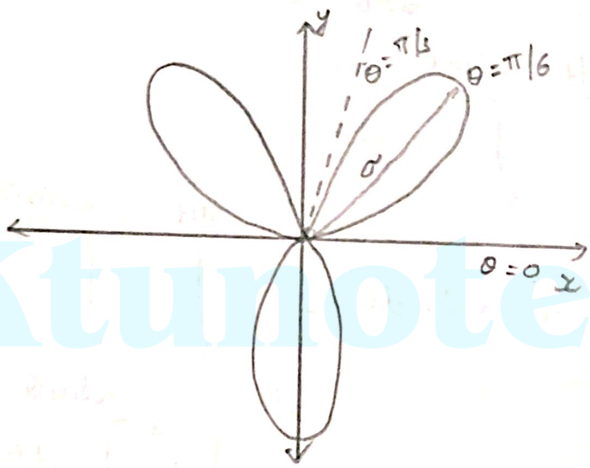
2) **Cardioid** $r = a(1-\cos\theta)$



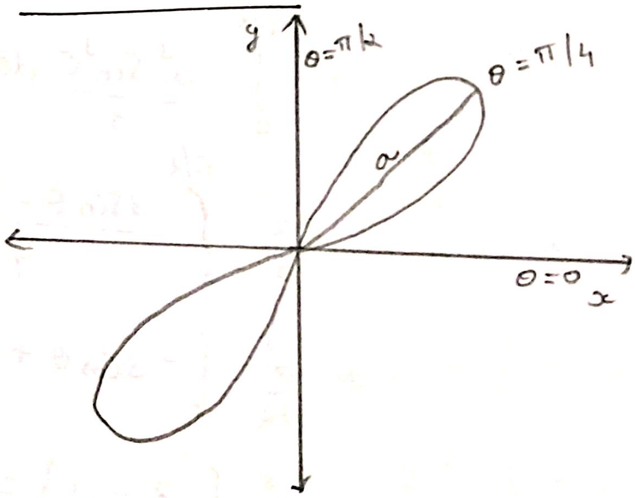
3) Lemniscate $r^2 = a^2 \cos 2\theta$



4) Three leaved rose $r = a \sin 3\theta$



5) Two leaved rose $r = a \sin 2\theta$



1. Evaluate $\int_0^{\pi/2}\int_0^{\sin\theta} r\cos\theta \, dr\, d\theta$

Soln :- $\int_{\theta=0}^{\pi/2}\int_{r=0}^{\sin\theta} r\cos\theta\, dr\, d\theta = \int_0^{\pi/2} \left[\cos\theta \frac{r^2}{2}\right]_0^{\sin\theta} d\theta$

$= \int_0^{\pi/2} \cos\theta \frac{\sin^2\theta}{2} d\theta$ Put $u = \sin\theta$
$du = \cos\theta\, d\theta$

$= \int_{u=0}^{1} \frac{u^2}{2} du = \frac{1}{2}\left[\frac{u^3}{3}\right]_0^1 = \frac{1}{6}$

2. Evaluate $\int_0^{\pi/2}\int_0^{a\sin\theta} r\, dA$

Soln :- $\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sin\theta} r\,(r\,dr\,d\theta) = \int_0^{\pi/2}\int_0^{a\sin\theta} r^2\, dr\, d\theta$

$= \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^{a\sin\theta} d\theta$

$= \int_0^{\pi/2} \frac{a^3\sin^3\theta}{3} d\theta$

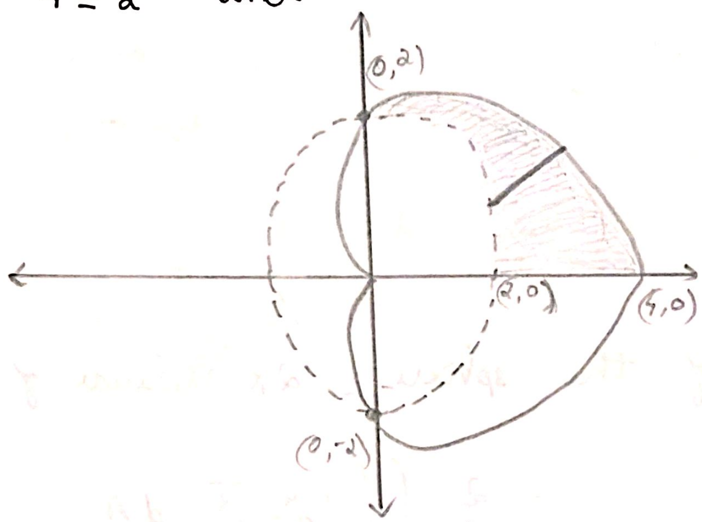
$= \frac{a^3}{3}\int_0^{\pi/2} \frac{3\sin\theta - \sin 3\theta}{4} d\theta$

$= \frac{a^3}{12}\left[-3\cos\theta + \frac{\cos 3\theta}{3}\right]_0^{\pi/2}$

$= \frac{a^3}{12}\left[3 - \frac{1}{3}\right] = \frac{2a^3}{9}$

3. Evaluate $\iint_R \sin\theta \, dA$ where R is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1+\cos\theta)$

Soln



or solving the 2 eqn

$r = 2(1+\cos\theta)$

$\Rightarrow 2 = 2(1+\cos\theta)$

$\Rightarrow 1+\cos\theta = 1$

$\Rightarrow \cos\theta = 0$

$\Rightarrow \theta = \frac{\pi}{2}$ ($\because$ first quadrant)

$\therefore \theta$ varies from 0 to $\pi/2$

$$\iint_R \sin\theta \, dA = \int_{\theta=0}^{\pi/2} \int_{r=2}^{2(1+\cos\theta)} \sin\theta \, r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \left(\sin\theta \, \frac{r^2}{2} \right)_2^{2(1+\cos\theta)} d\theta = \int_0^{\pi/2} \left[2\sin\theta (1+\cos\theta)^2 - 2\sin\theta \right] d\theta$$

Put $1+\cos\theta = u$

$-\sin\theta \, d\theta = du$

$$= 2\int_{u=2}^{1} -u^2 du - 2\int_0^{\pi/2} \sin\theta \, d\theta$$

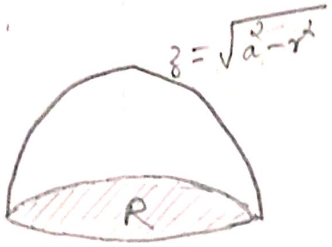
$$= 2\int_1^2 u^2 du - 2\int_0^{\pi/2} \sin\theta \, d\theta$$

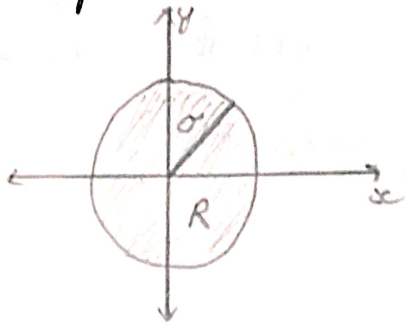
$$= 2\left[\frac{u^3}{3} \right]_1^2 - 2\left[-\cos\theta \right]_0^{\pi/2}$$

$$= 2\left[\frac{8}{3} - \frac{1}{3} + 0 - 1 \right] = \frac{8}{3}$$

4) The equation of a sphere of radius 'a' centered at the origin is $r^2 + z^2 = a^2$. Using double integral, find the volume of the sphere.

<u>Soln</u> :-





Volume of the sphere = 2 x Volume of hemisphere

$$= 2 \iint_R \sqrt{a^2 - r^2}\, dA$$

$$= 2 \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \sqrt{a^2 - r^2}\, r\, dr\, d\theta$$ Put $a^2 - r^2 = u$, $-2r\,dr = du$

$$= \cancel{2} \int_0^{2\pi} \int_{u=a^2}^{0} \sqrt{u}\, \frac{-du}{\cancel{2}}\, d\theta$$

$$= \int_0^{2\pi} \int_0^{a^2} \sqrt{u}\, du\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{2}{3} u^{3/2}\right]_0^{a^2} d\theta = \int_0^{2\pi} \frac{2}{3} a^3\, d\theta$$

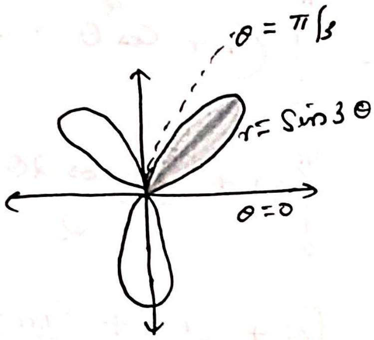
$$= \frac{2}{3} a^3 \left[\theta\right]_0^{2\pi} = \frac{4\pi a^3}{3}$$

Finding areas using polar double integral

Area of the region, R $= \iint_R dx\,dy = \iint_R r\,dr\,d\theta$

1. Use a polar double integral to find the area enclosed by the three-petaled rose $r = \sin 3\theta$

Soln :-



Area = 3 x Area of one petal

$$= 3\int_{\theta=0}^{\pi/3}\int_{r=0}^{\sin 3\theta} r\,dr\,d\theta = 3\int_0^{\pi/3}\left[\frac{r^2}{2}\right]_0^{\sin 3\theta} d\theta$$

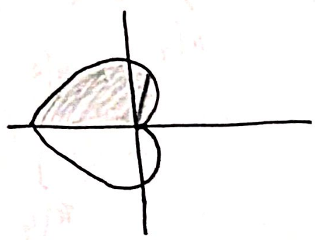
$$= 3\int_0^{\pi/3} \frac{\sin^2 3\theta}{2}\,d\theta = \frac{3}{2}\int_0^{\pi/3}\left(\frac{1-\cos 6\theta}{2}\right)d\theta$$

$$= \frac{3}{4}\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi/3}$$

$$= \frac{3}{4}\left[\frac{\pi}{3}\right] = \frac{\pi}{4}$$

2. Find the area of the region enclosed by the cardioid $r = 1-\cos\theta$

Soln :-



Area = 2 x Area of the region above the initial line

$$= 2\int_{\theta=0}^{\pi}\int_{r=0}^{1-\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{1-\cos\theta} d\theta$$

$$= \int_0^{\pi}(1-\cos\theta)^2\,d\theta$$

$$= \int_0^{\pi}(1+\cos^2\theta - 2\cos\theta)\,d\theta$$

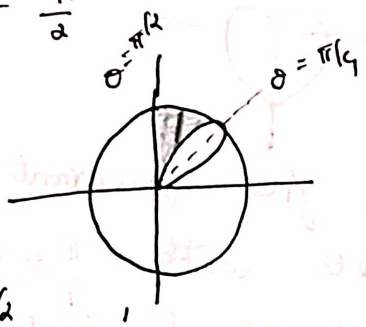
$$= \int_0^{\pi}\left(1+\frac{1+\cos 2\theta}{2} - 2\cos\theta\right)d\theta$$

$$= \left[\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} - 2\sin\theta\right]_0^{\pi}$$

$$= \underline{\underline{\frac{3\pi}{2}}}$$

3. Find the area of the region in the first quadrant bounded by $r=1$ and $r=\sin 2\theta$ with $\frac{\pi}{4}\le\theta\le\frac{\pi}{2}$

Soln :-



$$\text{Area} = \int_{\theta=\pi/4}^{\pi/2}\int_{r=\sin 2\theta}^{1} r\,dr\,d\theta = \int_{\pi/4}^{\pi/2}\left[\frac{r^2}{2}\right]_{\sin 2\theta}^{1} d\theta = \frac{1}{2}\int_{\pi/4}^{\pi/2}(1-\sin^2 2\theta)\,d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/2}(\cos^2 2\theta)\,d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/2}\left(\frac{1+\cos 4\theta}{2}\right)d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{\sin 4\theta}{4}\right]_{\pi/4}^{\pi/2} = \frac{1}{4}\left[\frac{\pi}{2} - \frac{\pi}{4}\right]$$

$$= \frac{\pi}{16}$$

Problems

Evaluate the integral

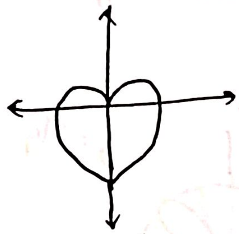
1) $\int_0^{\pi}\int_0^{a\sin\theta} r^2\, dr\, d\theta$ 2) $\int_0^{\pi/6}\int_0^{\sin 3\theta} r\, dr\, d\theta$

3) $\int_0^{\pi/2}\int_0^{1-\sin\theta} r^2\cos\theta\, dr\, d\theta$ 4) $\int_{\pi/2}^{3\pi/2}\int_0^{1+\sin\theta} r\, dr\, d\theta$

5) Evaluate $\iint r\sin\theta\, dr\, d\theta$ over the area of the cardioid $r = a(1+\cos\theta)$ above the initial line

- Use a double integral in polar co-ordinates to find the area of the region described.

6) The region enclosed by the cardioid $r = 1-\sin\theta$

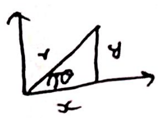


7) The region in the first quadrant bounded by $r = 2$ and $r = \sin 2\theta$ with $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

8) The region enclosed by the lemniscate $r^2 = a^2\cos 2\theta$

Converting double integral from rectangular to polar coordinates

Using the substitution $x = r\cos\theta$; $y = r\sin\theta$ we can convert double integral from rectangular to polar co-ordinates.

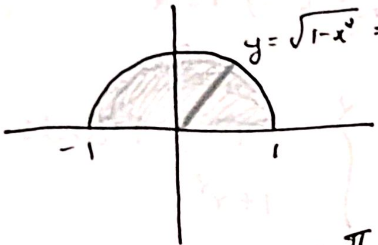


$$\iint_R f(x,y)\, dx\, dy = \iint_R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

1. Use polar coordinates to evaluate

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx$$

Soln: Here y varies from 0 to $\sqrt{1-x^2}$ and x from -1 to 1. So the given region is

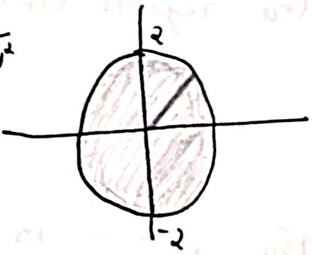


$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx = \int_{\theta=0}^{\pi}\int_{r=0}^{1} r^3\, r\, dr\, d\theta = \int_0^{\pi}\left[\frac{r^5}{5}\right]_0^1 d\theta$$

$$= \left[\frac{1}{5}\theta\right]_0^{\pi} = \underline{\frac{\pi}{5}}$$

2. Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{-(x^2+y^2)}\, dx\, dy$

Soln: Here x varies from $-\sqrt{4-y^2}$ to $\sqrt{4-y^2}$ and y varies from -2 to 2. So the given region is



$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{-(x^2+y^2)}\, dx\, dy = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} e^{-r^2}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_{t=0}^{4} e^{-t}\,\frac{dt}{2}\, d\theta$$

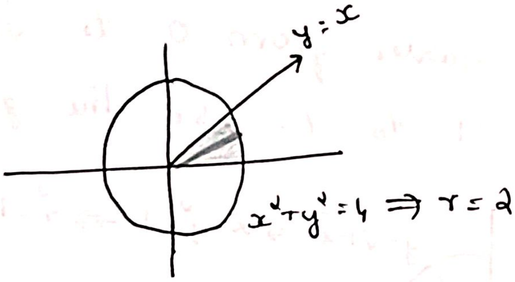
$\boxed{\text{Put } r^2=t \quad 2r\,dr=dt}$

$$= \frac{1}{2}\int_0^{2\pi}\left(\frac{e^{-t}}{-1}\right)_0^4 d\theta = \frac{1}{2}\int_0^{2\pi}(1-e^{-4})\, d\theta$$

$$= \frac{1-e^{-4}}{2}\,[\theta]_0^{2\pi} = \underline{\pi(1-e^{-4})}$$

3. Evaluate $\iint_R \frac{1}{1+x^2+y^2}\, dA$ where R is the sector in the first quadrant bounded by $y=0$, $y=x$ and $x^2+y^2=4$ using polar form

<u>Soln</u> :-



$y = x$
$\Rightarrow r\sin\theta = r\cos\theta$
$\Rightarrow \sin\theta = \cos\theta$
$\Rightarrow \theta = \frac{\pi}{4}$

$$\iint_R \frac{1}{1+x^2+y^2}\, dA = \int_{\theta=0}^{\pi/4} \int_{r=0}^{2} \frac{1}{1+r^2}\, r\, dr\, d\theta$$

Put $1+r^2 = t$
$2r\,dr = dt$

$$= \int_0^{\pi/4} \int_{t=1}^{5} \frac{1}{t} \frac{dt}{2}\, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/4} (\log t)_1^5\, d\theta = \frac{1}{2} \int_0^{\pi/4} (\log 5)\, d\theta$$

$$= \frac{\log 5}{2} [\theta]_0^{\pi/4} = \frac{\log 5}{2}\left(\frac{\pi}{4}\right)$$

<u>Problems</u>

Use polar coordinates to evaluate the double integral

1) $\iint_R e^{-(x^2+y^2)}\, dA$, where R is the region enclosed by the circle $x^2+y^2=1$

2) $\iint_R \sqrt{16-x^2-y^2}\, dA$, where R is the region in the first quadrant within the circle $x^2+y^2=16$.

3) $\int_0^2 \int_0^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx$

4) $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx$

5) $\int_0^1 \int_0^{\sqrt{1-y^2}} \cos(x^2+y^2)\,dx\,dy$

6) $\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{\sqrt{1+x^2+y^2}}\,dx\,dy$

7) $\int_0^4 \int_3^{\sqrt{25-x^2}} dy\,dx$

Mass and centre of gravity of an inhomogeneous lamina

An idealized flat object that is thin enough to be viewed as a two-dimensional plane region is called a lamina. A lamina is called homogeneous if its composition is uniform throughout and inhomogeneous otherwise. The density of a lamina is defined to be its mass per unit area.

Mass of a lamina: If a lamina with a continuous density function $\delta(x,y)$ occupies a region R in the xy-plane, then its total mass M is

$$\boxed{M = \iint_R \delta(x,y)\,dA}$$

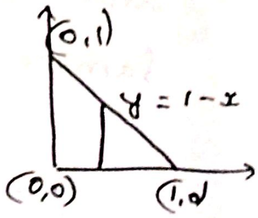
Center of gravity :- of a lamina occupying a region R in the xy-plane is the point $(\bar{x}, \bar{y})$ such that the effect of gravity on the lamina is equivalent to that of a single force acting at $(\bar{x}, \bar{y})$. If $(\bar{x}, \bar{y})$ is in R, then the lamina will balance horizontally on a point of support placed at $(\bar{x}, \bar{y})$.

Center of gravity $(\bar{x}, \bar{y})$ of a lamina is

$$\bar{x} = \frac{\iint_R x\, \delta(x,y)\, dA}{\iint_R \delta(x,y)\, dA} = \frac{\iint_R x\, \delta(x,y)\, dA}{\text{Mass of } R}$$

$$\bar{y} = \frac{\iint_R y\, \delta(x,y)\, dA}{\iint_R \delta(x,y)\, dA} = \frac{\iint_R y\, \delta(x,y)\, dA}{\text{Mass of } R}$$

1. A triangular lamina with vertices $(0,0)$, $(0,1)$ and $(1,0)$ has density function $\delta(x,y) = xy$. Find its total mass and the center of gravity.



Soln :- Mass, $M = \iint_R \delta(x,y)\, dA$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} xy\, dy\, dx$$

$$= \int_0^1 \left[x \frac{y^2}{2} \right]_0^{1-x} dx = \int_0^1 \left(\frac{x}{2} + \frac{x^3}{2} - x^2 \right) dx$$

$$= \left[\frac{x^2}{4} + \frac{x^4}{8} - \frac{x^3}{3} \right]_0^1 = \underline{\underline{\frac{1}{24}}}$$

$$\bar{x} = \frac{\iint_R x\,\delta(x,y)\,dA}{\text{Mass}}$$

$$\iint_R x\,\delta(x,y)\,dA = \int_{x=0}^{1}\int_{y=0}^{1-x} (x^2 y)\,dy\,dx = \int_0^1 \left[\frac{x^2y^2}{2}\right]_0^{1-x} dx$$

$$= \int_0^1 \left(\frac{x^2}{2} + \frac{x^4}{2} - x^3\right) dx$$

$$= \left[\frac{x^3}{6} + \frac{x^5}{10} - \frac{x^4}{4}\right]_0^1 = \frac{1}{60}$$

$$\therefore \bar{x} = \frac{\left(\frac{1}{60}\right)}{\left(\frac{1}{24}\right)} = \frac{2}{5}$$

$$\bar{y} = \frac{\iint_R y\,\delta(x,y)\,dA}{\text{Mass}}$$

$$\iint_R y\,\delta(x,y)\,dA = \int_{x=0}^{1}\int_{y=0}^{1-x} xy^2\,dy\,dx = \int_0^1 \left[\frac{xy^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1 \left(\frac{x}{3} - x^2 + x^3 - \frac{x^4}{3}\right) dx$$

$$= \left[\frac{x^2}{6} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{15}\right]_0^1 = \frac{1}{60}$$

$$\therefore \bar{y} = \frac{\left(\frac{1}{60}\right)}{\left(\frac{1}{24}\right)} = \frac{2}{5}$$

So the center of gravity is $\left(\frac{2}{5}, \frac{2}{5}\right)$

Problems:

Find the mass and center of gravity of

1) A lamina with density $\delta(x,y) = x + 2y$ is bounded by the x-axis, the line $x = 1$ and the curve $y = \sqrt{x}$.

2) A lamina with density $\delta(x,y) = 2y$ is bounded by $y = \sin x$, $y = 0$, $x = 0$ and $x = \pi$

3) A lamina with density $\delta(x,y) = xy$ is in the first quadrant and is bounded by the circle $x^2 + y^2 = a^2$ and the co-ordinate axes.

Triple Integral

Consider a function $f(x,y,z)$ which is continuous at every point of a finite solid V of three dimensional space. Divide the solid V into n sub solids of volumes $\delta V_1, \delta V_2 \ldots\ldots \delta V_n$. Let (x_i, y_i, z_i) be an arbitrary point in the i^{th} sub solid. Then

$$\iiint_V f(x,y,z)\, dV = \underset{n\to\infty}{Lt} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\delta V_i$$

Or

$$\iiint_V f(x,y,z)\, dx\,dy\,dz = \underset{n\to\infty}{Lt} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\delta x_i\,\delta y_i\,\delta z_i$$

Evaluating triple integral

1. Evaluate $\iiint_G 12xy^2z^3\, dV$ over the rectangular box G defined by the inequalities $-1 \le x \le 2$, $0 \le y \le 3$, $0 \le z \le 2$.

Soln :-
$$\iiint_G 12xy^2z^3\, dV = \int_{x=-1}^{2}\int_{y=0}^{3}\int_{z=0}^{2} 12xy^2z^3\, dz\, dy\, dx$$

$$= \int_{-1}^{2}\int_{0}^{3}\left(3xy^2z^4\right)_0^2 dy\, dx$$

$$= \int_{-1}^{2}\int_{0}^{3} 48xy^2\, dy\, dx = \int_{-1}^{2}\left(16xy^3\right)_0^3 dx$$

$$= \int_{-1}^{2} 432x\, dx = \left(216x^2\right)_{-1}^{2}$$

$$= \underline{648}$$

2. Evaluate $\int_0^1\int_{-1}^{y^2}\int_{-1}^{z} yz\, dx\, dz\, dy$

Soln :-
$$\int_{y=0}^{1}\int_{z=-1}^{y^2}\left[\int_{x=-1}^{z} yz\, dx\right] dz\, dy = \int_0^1\int_{-1}^{y^2} yz[x]_{-1}^{z}\, dz\, dy$$

$$= \int_0^1\int_{-1}^{y^2}\left(yz^2 + yz\right) dz\, dy$$

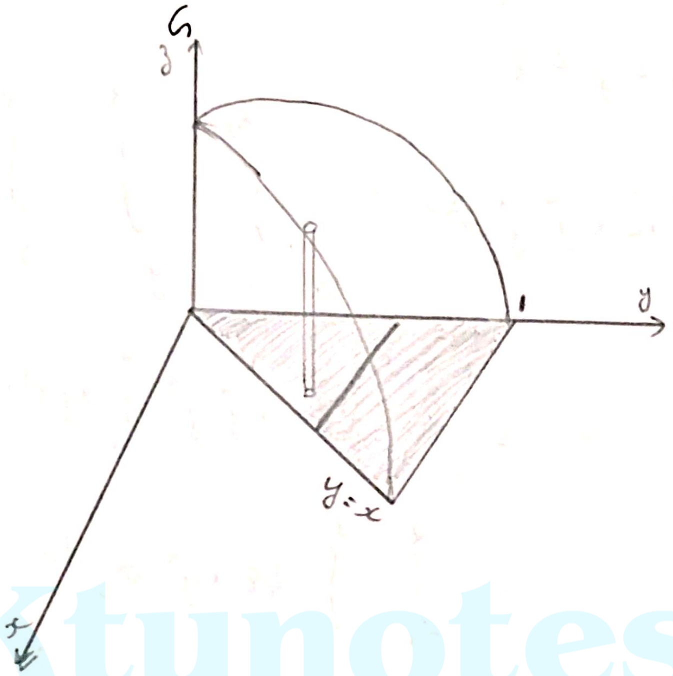
$$= \int_0^1\left[\frac{yz^3}{3} + \frac{yz^2}{2}\right]_{-1}^{y^2} dy = \int_0^1\left(\frac{y^7}{3} + \frac{y^5}{2} - \frac{y}{6}\right) dy$$

$$= \left[\frac{y^8}{24} + \frac{y^6}{12} - \frac{y^2}{12}\right]_0^1 = \underline{\frac{1}{24}}$$

3) Let G be the wedge in the first octant that is cut from the cylindrical solid $y^2+z^2 \le 1$ by the planes $y = x$ and $x=0$.

Evaluate $\iiint z \, dV$

Soln:-



OR Since it is in first octant lower limit of z will be 0. Upper limit is $y^2+z^2=1 \Rightarrow z=\sqrt{1-y^2}$. Limit of x are given 0 to y. To get the limits of y, put $z=0$ in $y^2+z^2=1$ $\Rightarrow y^2=1 \Rightarrow y=1$. Since in first octant, y varies from 0 to 1.

Here z varies from 0 to $\sqrt{1-y^2}$

x varies from 0 to y

y varies from 0 to 1

$$\therefore \iiint_G z\, dV = \int_{y=0}^{1} \int_{x=0}^{y} \int_{z=0}^{\sqrt{1-y^2}} z \, dz \, dx \, dy$$

$$= \int_0^1 \int_0^y \left[\frac{z^2}{2}\right]_0^{\sqrt{1-y^2}} dx\,dy = \int_0^1 \int_0^y \frac{(1-y^2)}{2} \, dx\,dy$$

$$= \int_0^1 \left(\frac{1-y^2}{2}\right)(x)_0^y \, dy = \int_0^1 \left(\frac{y}{2} - \frac{y^3}{2}\right) dy$$

$$= \left[\frac{y^2}{4} - \frac{y^4}{8}\right]_0^1 = \underline{\underline{\frac{1}{8}}}$$

4) Evaluate $\iiint_G y\,dV$, where G is the solid enclosed by the plane $z = y$, the xy-plane and the parabolic cylinder $y = 4 - x^2$.

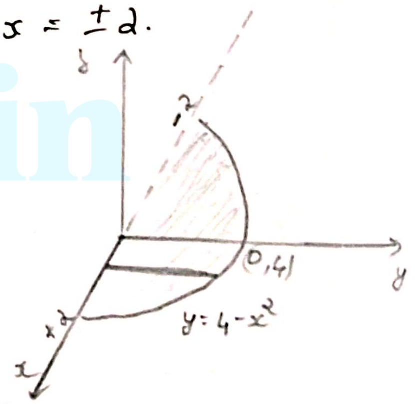
<u>Solution:-</u>

Here z varies from 0 to y

To get the limits of y, take the projection of the surface on the xy-plane. i.e put $z=0$ in $z=y$ which gives $y=0$. So y varies from 0 to $4-x^2$

To get the limits of x, find the intersection of $y=0$ and $y = 4-x^2 \Rightarrow 0 = 4-x^2 \Rightarrow x = \pm 2$.

So x varies from -2 to 2.



$$\therefore \iiint_G y\,dV = \int_{x=-2}^{2}\int_{y=0}^{4-x^2}\int_{z=0}^{y} y\,dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{0}^{4-x^2} y(z)_0^y\,dy\,dx = \int_{-2}^{2}\int_{0}^{4-x^2} y^2\,dy\,dx$$

$$= \int_{-2}^{2}\left[\frac{y^3}{3}\right]_0^{4-x^2} dx = \frac{1}{3}\int_{-2}^{2}(64-48x^2+12x^4-x^6)\,dx$$

$$= \frac{1}{3}\left[64x - 16x^3 + \frac{12x^5}{5} - \frac{x^7}{7}\right]_{-2}^{2}$$

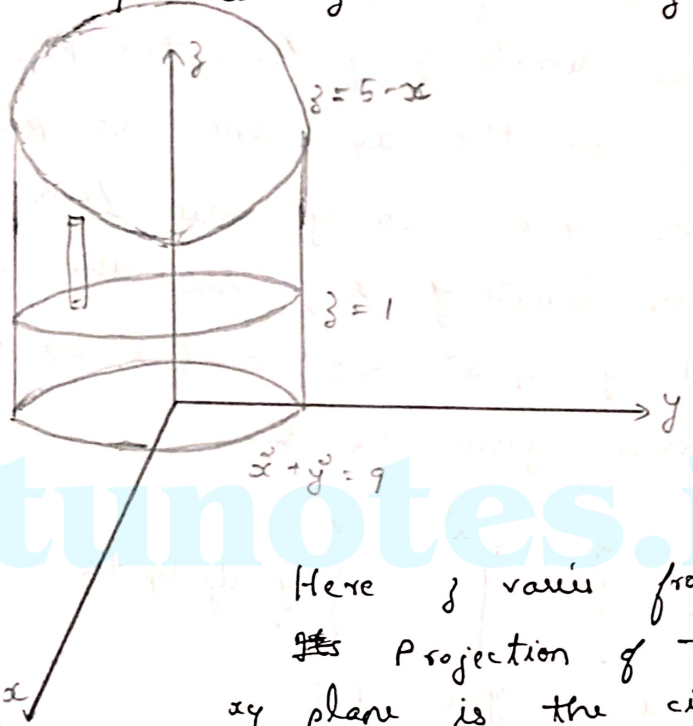
$$= \underline{39.009}$$

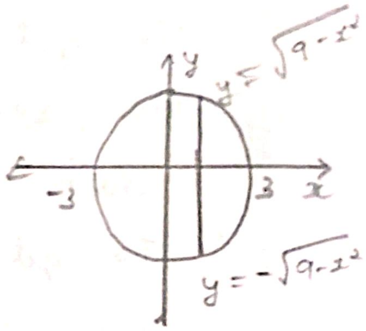
Volume calculated as triple integral

Volume of the solid $G = \iiint_G dV$

1. Use a triple integral to find the volume of the solid within the cylinder $x^2+y^2=9$ and between the planes $z=1$ and $x+z=5$

Soln:





Here z varies from 1 to $5-x$

~~the~~ Projection of the solid in the xy plane is the circle $x^2+y^2=9$

y varies from $-\sqrt{9-x^2}$ to $\sqrt{9-x^2}$

x varies from -3 to 3

$\therefore$ Volume of the solid $= \iiint_G dV$

$$= \int_{x=-3}^{3} \int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{z=1}^{5-x} dz\, dy\, dx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left[z \right]_1^{5-x} dy\, dx = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4-x)\, dy\, dx$$

$$= \int_{-3}^{3} (4-x) \left[y \right]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx = \int_{-3}^{3} (8-2x)\sqrt{9-x^2}\, dx$$

$$= 8\int_{-3}^{3}\sqrt{9-x^2}\,dx - \int_{-3}^{3} 2x\sqrt{9-x^2}\,dx$$

(Put $x = 3\sin\theta$ $\Rightarrow dx = 3\cos\theta\, d\theta$)

(Put $9-x^2 = t$ $\Rightarrow -2x\,dx = dt$)

$$= 8\int_{\theta=-\pi/2}^{\pi/2}(3\cos\theta)(3\cos\theta)\,d\theta + 0$$

$$= 72\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = 36\left[\theta + \frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2}$$

$$= 36\left[\frac{\pi}{2}+\frac{\pi}{2}\right] = \underline{36\pi}$$

Problems

Evaluate

1) $\displaystyle\int_{-1}^{2}\int_{0}^{2}\int_{0}^{1}(x^2+y^2+z^2)\,dx\,dy\,dz$

2) $\displaystyle\int_{0}^{\pi/2}\int_{0}^{1}\int_{0}^{x^2} x\cos y\,dz\,dx\,dy$

3) $\displaystyle\int_{0}^{3}\int_{0}^{\sqrt{9-z^2}}\int_{0}^{x} xy\,dy\,dx\,dz$

4) $\displaystyle\int_{0}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{-1+x^2+y^2}^{3-x^2-y^2} x\,dz\,dy\,dx$

5) $\displaystyle\iiint_G xy\sin yz\,dV$, where G is the rectangular box defined by the inequalities $0\le x\le \frac{\pi}{2}$, $0\le y\le 1$, $0\le z\le\frac{\pi}{6}$

6) $\displaystyle\iiint_G xyz\,dV$, where G is the solid in the first octant that is bounded by the parabolic cylinder $z = 3-x^2$ and the planes $z=0$, $y=x$ and $y=0$

7) Use a triple integral to find the volume of the solid

i) The solid in the first octant bounded by the co-ordinate planes and the plane $3x + 6y + 4z = 12$

ii) The solid bounded by the surface $z = \sqrt{y}$ and the planes $x + y = 4$, $x = 0$ and $z = 0$

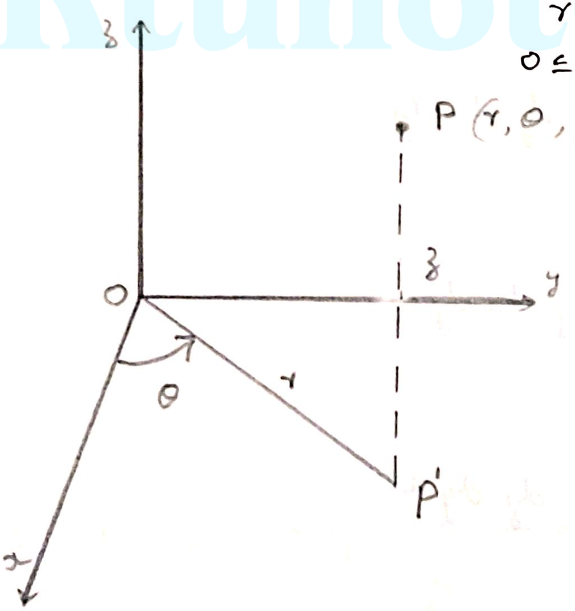
iii) The solid bounded by the surface $y = x^2$ and the planes $y + z = 4$ and $z = 0$

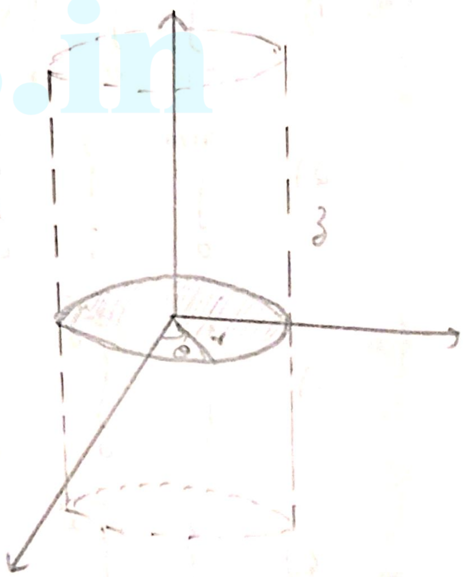
iv) The solid enclosed between the paraboloids $z = 5x^2 + 5y^2$ and $z = 6 - 7x^2 - y^2$ (Solved in text book)

Triple Integral in Cylindrical and Spherical Coordinates

Cylindrical Coordinates (r, θ, z)

$r \geq 0$
$0 \leq \theta \leq 2\pi$





$r \rightarrow$ distance from the origin to the point P' which is the projection of P in the xy-plane.

$\theta \rightarrow$ angle measured in the anticlockwise direction from +ve x-axis to the line OP'.

$z \rightarrow$ height PP'.

Relation between rectangular and cylindrical coordinates

$$x = r\cos\theta \quad ; \quad y = r\sin\theta \quad ; \quad z = z$$

$$r = \sqrt{x^2+y^2} \quad ; \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \quad ; \quad z = z$$

Triple integral in cylindrical coordinates

$$\iiint_G f(r,\theta,z)\,dV = \iiint f(r,\theta,z)\, r\,dr\,d\theta\,dz$$

1. Evaluate $\int_0^{2\pi}\int_0^{2}\int_0^{\sqrt{1-r^2}} zr\,dz\,dr\,d\theta$

Soln: $\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=0}^{\sqrt{1-r^2}} zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{2} r\left[\frac{z^2}{2}\right]_0^{\sqrt{1-r^2}} dr\,d\theta$

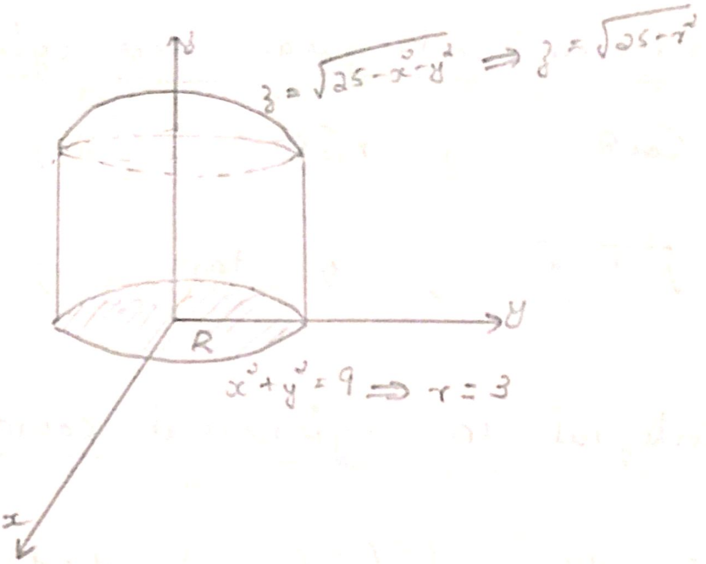
$$= \int_0^{2\pi}\int_0^{2}\left(\frac{r-r^3}{2}\right)dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(\frac{r^2}{2}-\frac{r^4}{4}\right)_0^2 d\theta = \frac{1}{2}\int_0^{2\pi}(-2)\,d\theta$$

$$= [-\theta]_0^{2\pi} = \underline{\underline{-2\pi}}$$

2. Use triple integration in cylindrical coordinates to find the volume of the solid G that is bounded above by the hemisphere $z = \sqrt{25-x^2-y^2}$, below by the xy-plane and laterally by the cylinder $x^2+y^2=9$

Soln :-



Here z varies from 0 to $\sqrt{25-r^2}$

r varies from 0 to 3

θ varies from 0 to 2π.

$\therefore$ Volume of the solid $G = \iiint\limits_G dv = \iiint r\,dr\,d\theta\,dz$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{3}\int_{z=0}^{\sqrt{25-r^2}} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{3} r\,[z]_0^{\sqrt{25-r^2}}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{3} r\sqrt{25-r^2}\,dr\,d\theta \qquad \text{Put } 25-r^2 = t$$
$$-2r\,dr = dt$$

$$= \int_0^{2\pi}\int_{t=25}^{16} \sqrt{t}\left(\frac{-dt}{2}\right)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[\frac{2}{3}t^{3/2}\right]_{16}^{25} d\theta = \frac{1}{3}\int_0^{2\pi}(125-64)\,d\theta$$

$$= \frac{61}{3}[\theta]_0^{2\pi} = \frac{122\pi}{3}$$

3) Use cylindrical coordinates to evaluate

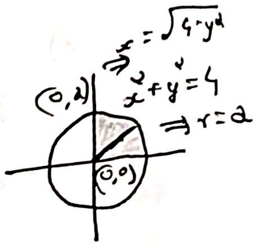
$$\int_0^2 \int_0^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} z^2 \, dz \, dx \, dy$$

<u>Soln</u>: Here z varies from $\sqrt{x^2+y^2}$ to $\sqrt{8-x^2-y^2}$

i.e from r to $\sqrt{8-r^2}$

x varies from 0 to $\sqrt{4-y^2}$.

y varies from 0 to 2



Hence the region R in the xy plane is

r varies from 0 to 2

θ varies from 0 to $\pi/2$

$$\therefore \int_0^2 \int_0^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} z^2 \, dz \, dx \, dy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{2} \int_{z=r}^{\sqrt{8-r^2}} z^2 (r \, dz \, dr \, d\theta)$$

$$= \int_0^{\pi/2} \int_0^2 r \left[\frac{z^3}{3}\right]_r^{\sqrt{8-r^2}} dr \, d\theta$$

$$= \frac{1}{3} \int_0^{\pi/2} \int_0^2 r \left[(8-r^2)^{3/2} - r^3\right] dr \, d\theta$$

Put $8-r^2 = t$

$$= \frac{1}{3} \int_0^{\pi/2} \left[\int_{t=8}^{4} -\frac{t^{3/2}}{2} \, dt - \int_0^2 r^4 \, dr \right] d\theta$$

$$= \frac{1}{3} \int_0^{\pi/2} \left\{ \frac{1}{5} \left[t^{5/2}\right]_4^8 - \left[\frac{r^5}{5}\right]_0^2 \right\} d\theta$$

$$= \frac{1}{15} \int_0^{\pi/2} (128\sqrt{2} - 64) \, d\theta = \frac{64(2\sqrt{2}-1)\left[\theta\right]_0^{\pi/2}}{15} = \frac{32(2\sqrt{2}-1)\pi}{15}$$

Problems

1) Evaluate $\int_0^{\pi/2}\int_0^{\sin\theta}\int_0^{r^2} r\sin\theta \, dz\, dr\, d\theta$

Use cylindrical coordinates to find the volume of the solid

2) The solid enclosed by the paraboloid $z = x^2+y^2$ and the plane $z = 16$

3) The solid that is bounded above and below by the sphere $x^2+y^2+z^2 = 9$ and inside the cylinder $x^2+y^2 = 4$

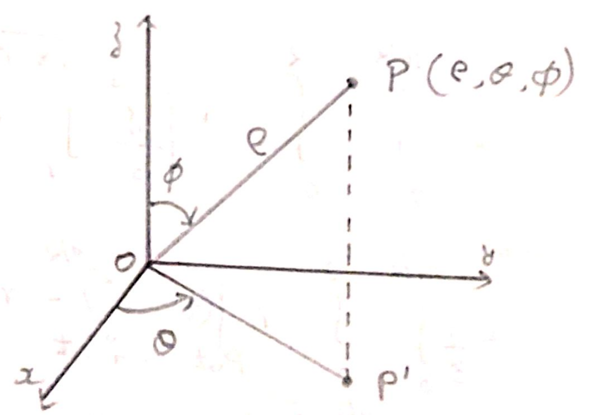
4) Use cylindrical coordinates to evaluate

$$\int_0^{a}\int_0^{\sqrt{a^2-x^2}}\int_0^{a^2-x^2-y^2} x^2 \, dz\, dy\, dx \quad (a>0)$$

Spherical Coordinates (ρ, θ, ϕ)

$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi$

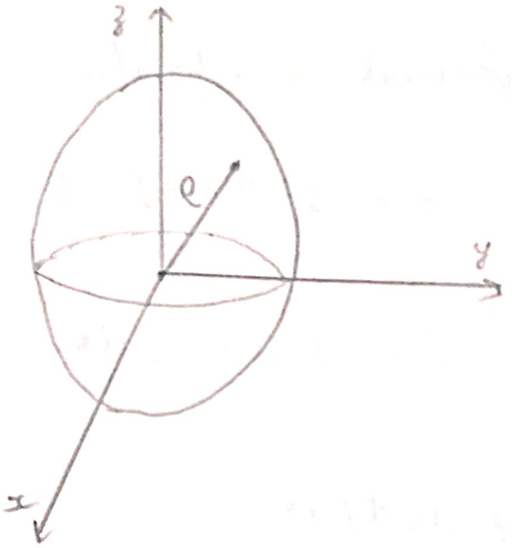
$0 \leq \phi \leq \pi$

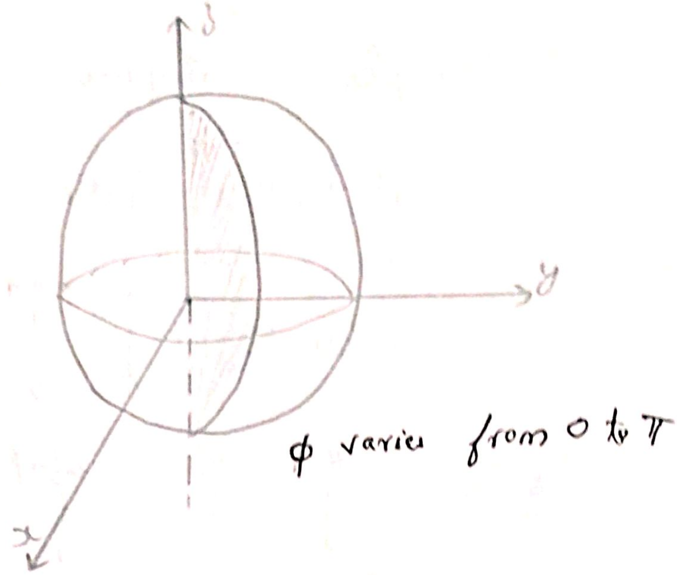


$\rho \rightarrow$ distance between origin and P

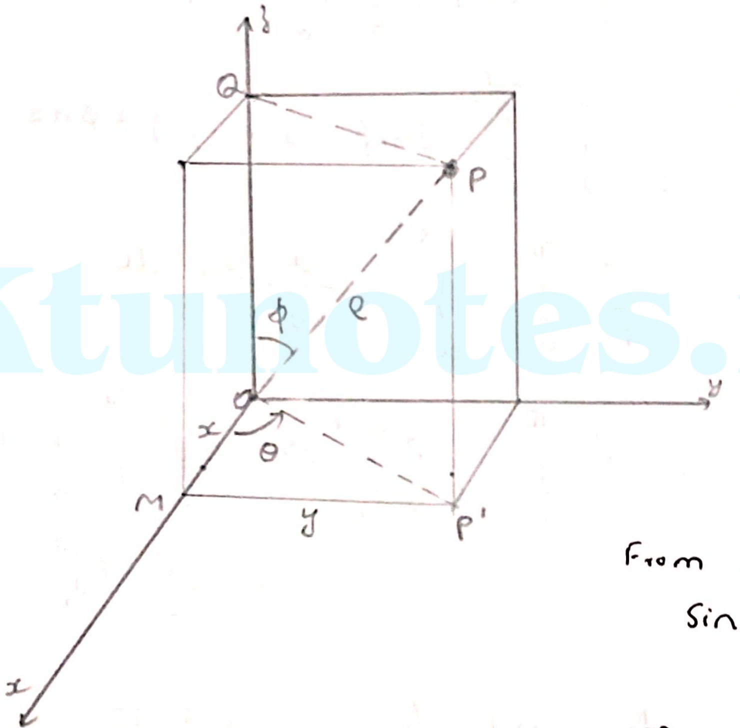
$\theta \rightarrow$ angle measured in the anticlockwise direction from +ve x-axis to the line OP'

$\phi \rightarrow$ angle between z-axis and the line OP.





Relation between rectangular and spherical coordinates



From ΔOQP,

$$\sin\phi = \frac{PQ}{OP} = \frac{PQ}{\rho}$$

$$\Rightarrow PQ = \rho\sin\phi$$

$$\cos\phi = \frac{OQ}{OP} = \frac{z}{\rho}$$

$$\Rightarrow z = \rho\cos\phi$$

From $\Delta OMP'$,

$$\sin\theta = \frac{y}{OP'} = \frac{y}{PQ} = \frac{y}{\rho\sin\phi}$$

$$\Rightarrow y = \rho\sin\phi\sin\theta$$

$$\cos\theta = \frac{x}{OP'} = \frac{x}{PQ} = \frac{x}{\rho\sin\phi}$$

$$\Rightarrow x = \rho\sin\phi\cos\theta$$

$$\therefore \boxed{x = \rho\sin\phi\cos\theta \; ; \quad y = \rho\sin\phi\sin\theta \quad \text{and} \quad z = \rho\cos\phi}$$

From the above, $x^2+y^2+z^2 = \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi$

$$= \rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta) + \rho^2\cos^2\phi$$

$$= \rho^2(\sin^2\phi+\cos^2\phi) = \rho^2$$

$$\Rightarrow \boxed{\rho = \sqrt{x^2+y^2+z^2} \quad \text{similarly} \quad \tan\theta = \frac{y}{x}, \quad \cos\phi = \frac{z}{\sqrt{x^2+y^2+z^2}}}$$

Triple Integral in spherical coordinates

$$\iiint_G f(\rho,\theta,\phi)\,dV = \iiint f(\rho,\theta,\phi)\,\rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

1. Evaluate $\int_0^{\pi}\int_0^{\pi/4}\int_0^{a\sec\phi} \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$

Soln: $\int_{\theta=0}^{\pi}\int_{\phi=0}^{\pi/4}\int_{\rho=0}^{a\sec\phi} \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$

$$= \int_0^{\pi}\int_0^{\pi/4} \sin\phi \left(\frac{\rho^3}{3}\right)_0^{a\sec\phi} d\phi\, d\theta$$

$$= \frac{1}{3}\int_0^{\pi}\int_0^{\pi/4} \sin\phi\,(a^3 \sec^3\phi)\, d\phi\, d\theta$$

Put $u = \sec\phi$

$du = \sec\phi \tan\phi\, d\phi$

$= \sec\phi \frac{\sin\phi}{\cos\phi} d\phi$

$\Rightarrow \sin\phi\, d\phi = \frac{du}{\sec^2\phi}$

$= \frac{du}{u^2}$

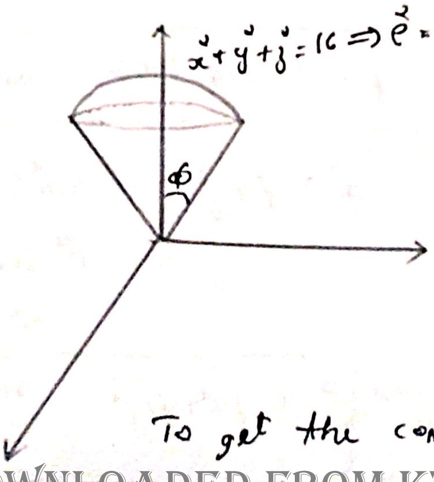
$$= \frac{a^3}{3}\int_0^{\pi}\int_{u=1}^{\sqrt{2}} \frac{u^3}{u^2}\, du\, d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi} \left[\frac{u^2}{2}\right]_1^{\sqrt{2}} d\theta = \frac{a^3}{3}\int_0^{\pi} \frac{1}{2}\, d\theta$$

$$= \frac{a^3}{6}\left[\theta\right]_0^{\pi} = \underline{\underline{\frac{\pi a^3}{6}}}$$

2. Use spherical coordinates to find the volume of the solid G bounded above by the sphere $x^2+y^2+z^2=16$ and below by the cone $z=\sqrt{x^2+y^2}$

Soln:



$x^2+y^2+z^2=16 \Rightarrow \rho^2=16 \Rightarrow \rho=4$. So ρ varies from 0 to 4

$z=\sqrt{x^2+y^2}$

$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta}$

$\rho\cos\phi = \rho\sin\phi$

$\Rightarrow \tan\phi = 1 \Rightarrow \phi = \frac{\pi}{4}$

$\therefore \phi$ varies from 0 to $\frac{\pi}{4}$

To get the complete cone, θ should vary from 0 to 2π

$\therefore$ volume of the solid $G = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{4} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

$$= \int_0^{2\pi} \int_0^{\pi/4} \sin\phi \left[\frac{\rho^3}{3}\right]_0^4 d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi/4} \sin\phi \, \frac{64}{3}\, d\phi\, d\theta$$

$$= \frac{64}{3} \int_0^{2\pi} \left(-\cos\phi\right)_0^{\pi/4} d\theta$$

$$= \frac{64}{3} \int_0^{2\pi} \left(\frac{-1}{\sqrt{2}} + 1\right) d\theta$$

$$= \frac{64}{3} \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right) 2\pi \simeq \underline{39.26}$$

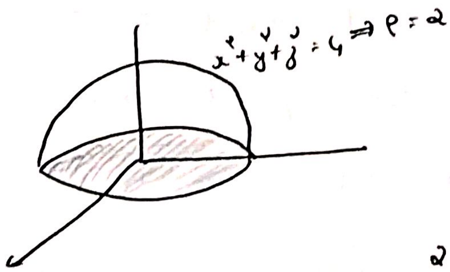
2) Use spherical coordinates to evaluate

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} z^2 \sqrt{x^2+y^2+z^2} \; dz\, dy\, dx$$

<u>Soln</u> :- Here z varies from 0 to $\sqrt{4-x^2-y^2}$ i.e $z^2 = 4-x^2-y^2$
$\Rightarrow x^2+y^2+z^2 = 4$

y varies from $-\sqrt{4-x^2}$ to $\sqrt{4-x^2}$ i.e $y^2 = 4-x^2$
$\Rightarrow x^2+y^2 = 4$

x varies from -2 to 2



ρ varies from 0 to 2

ϕ varies from 0 to $\frac{\pi}{2}$

θ varies from 0 to 2π

$\therefore$ Given integral $= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{2} (\rho^2 \cos^2\phi)(\rho)\, \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$

$$= \int_0^{2\pi} \int_0^{\pi/2} \cos^2\phi \, \sin\phi \left(\frac{\rho^6}{6}\right)_0^2 d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \frac{32}{3} \cos^2\phi \, \sin\phi \; d\phi\, d\theta$$

Put $u = \cos\phi$
$du = -\sin\phi\, d\phi$

$$= \frac{32}{3} \int_0^{2\pi} \int_{u=0}^{1} u^2 \, du \, d\theta = \frac{32}{3} \int_0^{2\pi} \left(\frac{u^3}{3} \right)_0^1 d\theta$$

$$= \frac{32}{9} (2\pi) = \frac{64\pi}{9}$$

Problems

1. Evaluate $\displaystyle\int_0^{\pi} \int_0^{\pi/2} \int_0^{1} \rho^3 \sin\phi \cos\phi \, d\rho \, d\phi \, d\theta$

2. Evaluate $\displaystyle\int_{-1}^{1} \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} e^{-(x^2+y^2+z^2)^{3/2}} \, dz \, dy \, dx$

3. Evaluate $\displaystyle\int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2} \, dz \, dx \, dy$

4. Use spherical coordinates to find the volume of
   i) the solid bounded above by the sphere $\rho = 4$ and below by the cone $\phi = \frac{\pi}{6}$
   ii) the solid within the cone $\phi = \frac{\pi}{4}$ and between the spheres $\rho = 1$ and $\rho = 3$
   iii) the solid enclosed by the sphere $x^2+y^2+z^2 = 4a^2$ and the planes $z = 0$ and $z = a$
   iv) the solid within the sphere $x^2+y^2+z^2 = 9$ and outside the cone $z = \sqrt{x^2+y^2}$ and above the xy-plane.