MODULE III

MULTIVARIABLE CALCULUS - INTEGRATION

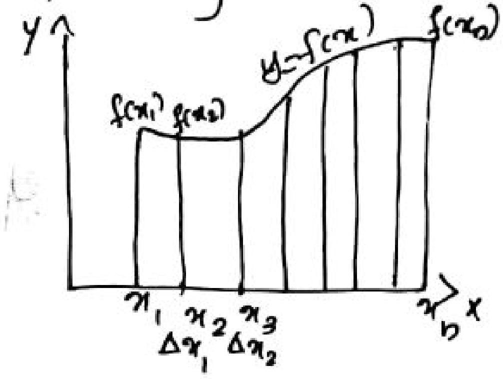
Topics

Double integrals (cartesian), reversing the order of integration, change of coordinates (cartesian to polar), finding areas and volume using double integrals, mass and center of gravity of inhomogeneous laminas using double integral. Triple integrals, volume calculated as triple integral, triple integral in cylindrical and spherical coordinates (computations involving spheres, cylinders)

Definite Integral

The definite integral $\int_a^b f(x)\,dx$ is defined as the limits of the sum $f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots + f(x_n)\Delta x_n$ when $n \to \infty$ and each of the lengths $\Delta x_1, \Delta x_2 \cdots \Delta x_n \to 0$. Here $\Delta x_1, \Delta x_2 \cdots \Delta x_n$ are n subintervals into which the range $b-a$ has been divided and $x_1, x_2 \cdots x_n$ are values of x lying respectively in first, second and nth subinterval.

[Definite integral of a function can be interpreted in terms of area]



Double Integrals

Let R be a bounded region in the x-y plane and a function $f(x,y)$ be defined at every point of R. Divide the region R into small rectangles by drawing lines parallel to the co-ordinate axes. Let us denote the area of the k^{th} rectangle by ΔA_K. Let (x_K, y_K) be any point in the k^{th} rectangle. Form the sum

$$f(x_1,y_1)\Delta A_1 + f(x_2,y_2)\Delta A_2 + \dots + f(x_K,y_K)\Delta A_K + \dots + f(x_n,y_n)\Delta A_n$$

$$= \sum_{K=1}^{n} f(x_K,y_K)\Delta A_K$$

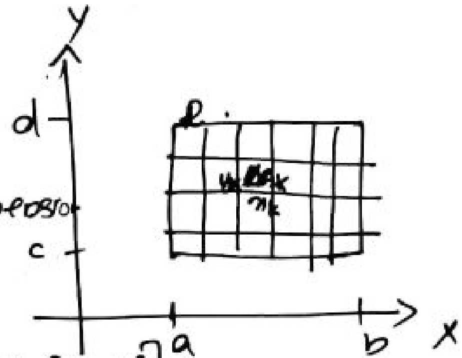
Then $\lim\limits_{n\to\infty} \sum_{K=1}^{n} f(x_K,y_K)\Delta A_K$, if exists,

is denoted by

$$\iint_R f(x,y)\, dA$$

Here $n \to \infty$ is taken to mean that the number of rectangles increases indefinitely in such away that the greatest length of all the rectangles approaches zero so that each $\Delta A_k \to 0$.

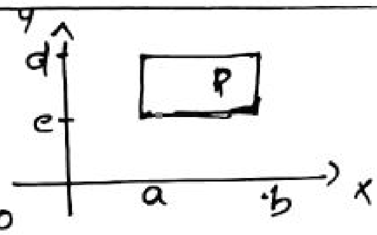
[Double integral of f over R as the volume of the 3-dimensional solid region over xy plane bounded below by R and above surface $z = f(x,y)$]



Double integrals over rectangular regions

If the region R is a rectangle given by $a \le x \le b$ and $c \le y \le d$ then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$



[$\int_a^b f(x,y)\,dx$ is called the partial definite integral w.r.to x, is evaluated by holding y fixed and integrating w.r.to x and $\int_c^d f(x,y)\,dy$ is called partial definite integral w.r.to y, is evaluated by holding x fixed and integrating w.r.to y]

Problems

1. Evaluate (a) $\int_1^3 \int_2^4 (40 - 2xy)\, dy\, dx$

(b) $\int_2^4 \int_1^3 (40 - 2xy)\, dx\, dy$.

(a) $$\int_1^3 \int_2^4 (40-2xy)\,dy\,dx = \int_1^3 \left[40y - 2x\frac{y^2}{2}\right]_2^4 dx$$

$$= \int_1^3 \left\{\left[40\times 4 - x\times 16\right] - \left[80 - x4\right]\right\} dx$$

$$= \int_1^3 (160 - 16x - 80 + 4x)\, dx = \int_1^3 (80 - 12x)\, dx$$

$$= \left[80x - 12\frac{x^2}{2}\right]_1^3 = \left[80\times 3 - 6\times 9\right] - \left[80 - 6\right]$$

$$= 240 - 54 - 74 = \underline{112}$$

(b) $$\int_2^4 \int_1^3 (40-2xy)\,dx\,dy = \int_2^4 \left[40x - 2\frac{x^2}{2}y\right]_1^3 dy$$

$$= \int_2^4 \left[120 - 9y - 40 + y\right] dy = \int_2^4 (80 - 8y)\, dy$$

$$= \left[80y - \frac{8y^2}{2}\right]_2^4 = 320 - 4\times 16 - 160 + 4\times 4$$

$$= \underline{112}$$

2. Evaluate $\int_0^{\ln 3}\int_0^{\ln 2} e^{x+2y}\,dy\,dx$

$$\int_0^{\ln 3}\int_0^{\ln 2} e^{x}e^{2y}\,dy\,dx = \int_0^{\ln 3} e^{x}\left[\frac{e^{2y}}{2}\right]_0^{\ln 2} dx$$

$$= \int_0^{\ln 3} \frac{e^{x}}{2}\left[e^{2\ln 2} - e^{0}\right] dx = \frac{1}{2}\int_0^{\ln 3} e^{x}\left[e^{\ln 2^2} - 1\right] dx$$

$$= \frac{1}{2}\int_0^{\ln 3} e^{x}\left[4-1\right] dx = \frac{3}{2}\int_0^{\ln 3} e^{x}\,dx$$

$$= \frac{3}{2}\left[e^{x}\right]_0^{\ln 3} = \frac{3}{2}\left[e^{\ln 3} - e^{0}\right] = \frac{3}{2}\left[3-1\right]$$

$$= \frac{3}{2}\times 2 = 3$$

$$\boxed{\begin{array}{l} a\log b = \log b^{a} \\ e^{\ln x} = x \end{array}}$$

3. Evaluate $\int_0^{2}\int_0^{1} \frac{x}{(1+xy)^2}\,dy\,dx$

$$\int_0^{2}\int_0^{1} \frac{x}{(1+xy)^2}\,dy\,dx = \int_0^{2}\left[x\cdot\frac{-1}{(1+xy)}\cdot\frac{1}{x}\right]_0^{1} dx$$

$$= \int_0^{2} -\left[\frac{1}{1+x} - \frac{1}{1}\right] dx = \int_0^{2}\left(\frac{-1}{1+x} + 1\right) dx$$

$$\boxed{\begin{array}{l} \int x^n dx = \frac{x^{n+1}}{n+1} \\ \int \frac{1}{x} dx = \log x \end{array}}$$

$$= \left[-\log(1+x) + x\right]_0^{2}$$

$$\boxed{\log 1 = 0}$$

$$= \left[-\log(1+2) + 2\right] - \left[-\log(1+0) + 0\right]$$

$= -\log 3 + 2 + 0$

$= \underline{\underline{2 - \log 3}}$

4. Evaluate $\int_1^a \int_1^b x^2 y \, dx dy$

$$\int_1^a \int_1^b x^2 y \, dx dy = \int_1^a y \left(\frac{x^3}{3}\right)_1^b dy = \frac{1}{3}\int_1^a y \, (b^3 - 1) dy$$

$$= \frac{(b^3-1)}{3} \left[y^2/2 \right]_1^a = \underline{\underline{\frac{(b^3-1)}{6} (a^2-1)}}$$

5. Evaluate $\int_1^a \int_1^b \frac{dy \, dx}{xy}$

$$\int_1^a \int_1^b \frac{1}{xy} dy dx = \int_1^a \frac{1}{x} (\log y)_1^b dx$$

$$= \int_1^a \frac{1}{x} (\log b - \log 1) dx$$

$\boxed{\log 1 = 0}$

$$= \log b \int_1^a \frac{1}{x} dx$$

$$= \log b \left[\log x \right]_1^a$$

$$= \log b \left[\log a - \log 1 \right]$$

$= \underline{\underline{\log a \log b}}$

6. Evaluate $\int_0^1\int_0^1 \frac{1}{\sqrt{(1-x^2)(1-y^2)}}\,dx\,dy$

$$\int_0^1\int_0^1 \frac{1}{\sqrt{1-x^2}\,\sqrt{1-y^2}}\,dx\,dy = \int_0^1 \frac{1}{\sqrt{1-y^2}}\left[\sin^{-1}x\right]_0^1 dy$$

$$= \int_0^1 \frac{1}{\sqrt{1-y^2}}\left[\sin^{-1}(1)-\sin^{-1}(0)\right] dy$$

$$= \int_0^1 \frac{1}{\sqrt{1-y^2}}\left(\pi/2 - 0\right) dy$$

$$\boxed{\int \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a)}$$

$$= \pi/2\left[\sin^{-1} y\right]_0^1$$

$$= \frac{\pi}{2}\left[\sin^{-1}(1)-\sin^{-1}(0)\right]$$

$$= \frac{\pi}{2}\left[\frac{\pi}{2}-0\right]$$

$$= \frac{\pi^2}{4}$$

7. Evaluate $\iint_R y\sin xy\,dA$ where $R=[1,2]\times[0,\pi]$

$\iint_R$ $R: [1,2]\times[0,\pi] = X\times Y$

$x: 1 \longrightarrow 2$ $\quad y: 0 \longrightarrow \pi$

$$\iint_R y\sin xy\,dA = \int_0^{\pi}\int_1^2 y\sin xy\,dx\,dy$$

$$= \int_0^{\pi} y\left[\frac{-\cos xy}{y}\right]_1^2 dy$$

$$= \int_0^{\pi} -\left[\cos 2y - \cos y\right] dy$$

$$= \int_0^{\pi} \left(-\cos 2y + \cos y\right) dy$$

$\boxed{\int \sin x\,dx = -\cos x}$

$\boxed{\sin n\pi = 0}$

$$= \left(-\sin 2y + \sin y\right)_0^{\pi}$$

$$= \left[\sin 2\pi + \sin \pi\right] - \left[-\sin 0 + \sin 0\right]$$

$$= \underline{\underline{0}}$$

8. Use double integral to find the volume of the solid bounded above by the plane $z = 4-x-y$ and below by the rectangle $R = [0,1] \times [0,2]$

$\boxed{z = f(x,y)}$

Volume $V = \iint_R f(x,y)\,dA$

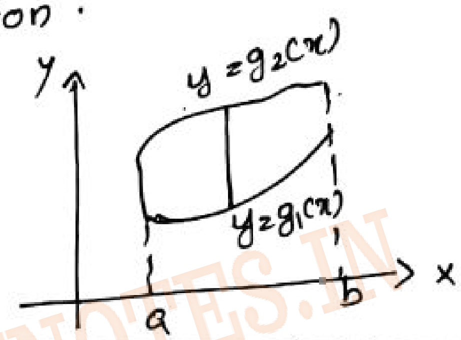
$$= \int_0^1 \int_0^2 (4-x-y)\,dy\,dx$$

$$= \int_0^1 \left[4y - xy - \frac{y^2}{2}\right]_0^2 dx = \int_0^1 \left[8-2x-2-0\right] dx$$

$$= \int_0^1 (6-2x)\,dx = \left[6x - \frac{2x^2}{2}\right]_0^1 = 6-1 = \underline{\underline{5}}$$

Double Integrals over non rectangular Regions

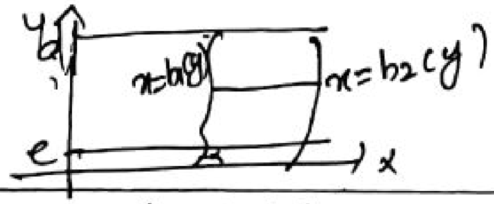
The double integration over non rectangular regions bounded on the left and right by $x=a$ and $x=b$ and bounded below and above by continuous curves $y=g_1(x)$ and $y=g_2(x)$. This region is called type I region.



Type I region.

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

The type II region is bounded ~~then~~ above and below by horizontal lines $y=c$ and $y=d$ and bounded on the left and right by continuous curves $x=h_1(y)$ and $x=h_2(y)$.



$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

Problems

1. Evaluate $\int_0^3 \int_0^{\sqrt{9-y^2}} 2y \, dx\,dy$

$$\int_0^3 \int_0^{\sqrt{9-y^2}} 2y\,dx\,dy = \int_0^3 2y\,[x]_0^{\sqrt{9-y^2}}\,dy$$

$$= \int_0^3 2y\left[\sqrt{9-y^2}-0\right]dy$$

$$= \int_0^3 2y\sqrt{9-y^2}\,dy$$

put $9-y^2=t$

$-2y\,dy = dt$

$$= \int \sqrt{t}\,(-dt)$$

$\boxed{\sqrt{x} = x^{1/2} = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}}$

$$= \left[\frac{t^{3/2}}{3/2}\right] = \cancel{\frac{2}{3}\left[(3^{3/2})-0\right]}$$

$$= \cancel{\frac{2}{3}} = \frac{2}{3}\left[\left(9-y^2\right)^{3/2}\right]_0^3$$

$\boxed{x^{3/2} = (\sqrt{x})^3}$

$$= \frac{2}{3}\left[9-9\right]^{3/2} - (9-0)^{3/2}$$

$$= \frac{2}{3}\left[0-(\sqrt{9})^3\right] = \frac{2}{3}\times 3^{3} = \underline{18}$$

2. Evaluate $\int_1^2 \int_0^x \frac{dy\,dx}{x^2+y^2}$

$$\int_0 \int_0 \frac{dy\,dx}{x^2+y^2} = \int_0^2 \left[\frac{1}{x}\tan^{-1}(y/x)\right]_0^x dx$$

$$= \int_0^2 \frac{1}{x}\left[\tan^{-1}(x/x) - \tan^{-1}(0/x)\right] dx$$

$$\boxed{\int \frac{1}{x^2+a^2} = \frac{1}{a}\tan^{-1}(x/a)}$$

$$= \int_0^2 \frac{1}{x}\left[\tan^{-1}(1) - \tan^{-1}(0)\right] \cdot dx$$

$$= \int_0^2 \frac{1}{x}\left(\frac{\pi}{4} - 0\right) dx = \frac{\pi}{4}\int_0^2 \frac{1}{x}\,dx$$

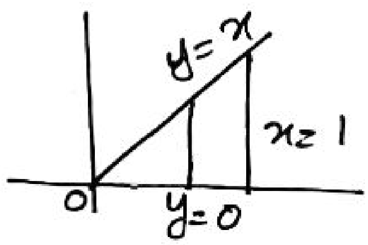
$$= \frac{\pi}{4}\left[\log x\right]_0^2 = \frac{\pi}{4}\left[\log 2 - \log 0\right]$$

$$= \underline{\underline{\frac{\pi}{4}\log 2}}$$

$$\boxed{\log 0 = 1}$$

3 . Evaluate $\iint_R \frac{\sin x}{x}\,dA$ where R is the triangular region bounded by the x-axis $y=x$ and $x=1$

$$\iint_R \frac{\sin x}{x}\,dA = \int_0^1 \int_0^x \frac{\sin x}{x}\,dy\,dx$$



$$= \int_0^1 \frac{\sin x}{x}\left[y\right]_0^x dx$$

$x : 0 \longrightarrow 1$

$y : 0 \longrightarrow x$

$$= \int_0^1 \frac{\sin x}{\cancel{x}}\,\cancel{x}\;dx$$

$$= \int_0 \sin x \, dx$$

$$= \left[-\cos x\right]_0^1 = -\left[\cos 1 - \cos 0\right]$$

$$= -\cos 1 + 1 = \underline{\underline{1 - \cos 1}}$$

4. Evaluate $\iint_R y \, dA$ where R is the region in the first quadrant enclosed between the circle $x^2+y^2=25$ and the line $x+y=5$.

Imp

$$\iint_R y\,dA = \int_0^5 \int_{5-y}^{\sqrt{25-y^2}} y \, dx\,dy$$

$$= \int_0^5 y \,(x)_{5-y}^{\sqrt{25-y^2}} \, dy$$

$$= \int_0^5 y\left[\sqrt{25-y^2} - (5-y)\right] dy$$

$$= \int_0^5 y\sqrt{25-y^2}\,dy - \int_0^5 y(5-y)\,dy$$

$$= \int \sqrt{t}\left(\frac{-dt}{2}\right) - \int_0^5 (5y-y^2)\,dy$$

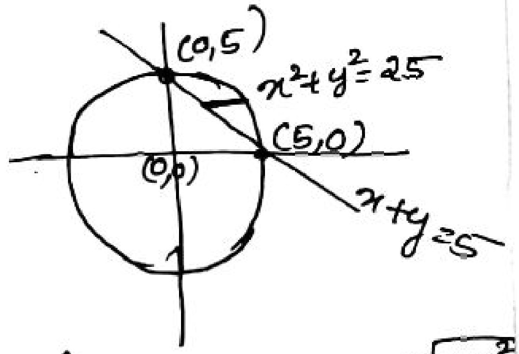
$$= \frac{-1}{2}\left(\frac{t^{3/2}}{3/2}\right) - \left[\frac{5y^2}{2} - \frac{y^3}{3}\right]_0^5$$

$\underline{x^2+y^2=25}$, circle

center $(0,0)$ radius 5

$x+y=5$

x	0	5
y	5	0



limit $x: 5-y \to \sqrt{25-y^2}$

$y: 0 \to 5$

Put $25-y^2=t$

$-2y\,dy = dt$

$y\,dy = \frac{-dt}{2}$

$$= -\frac{1}{\cancel{2}}\times\frac{\cancel{2}}{3}\left[(25-y^2)^{3/2}\right]_0^5 - \left[\frac{5}{2}y^2-\frac{y^3}{3}\right]_0^5$$

$$= -\frac{1}{3}\left[(25-25)^{3/2}-(25-0)^{3/2}\right]-\left[\frac{5}{2}\times 25-\frac{1}{3}\times 125\right]$$

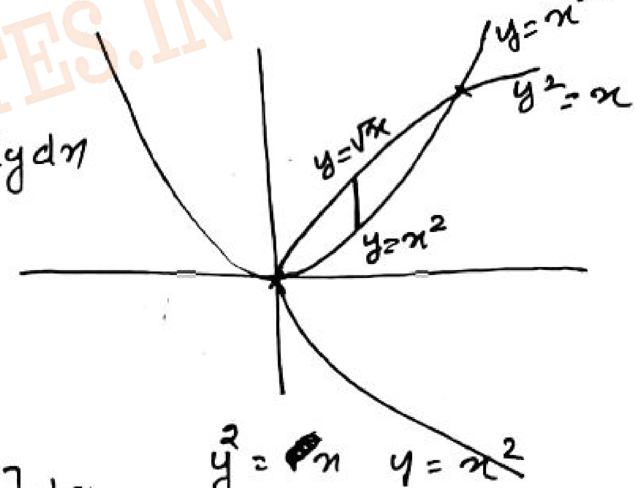
$$= -\frac{1}{3}\left[0-5^3\right]-\left[125\left(\frac{1}{2}-\frac{1}{3}\right)\right]$$

$$= -\frac{1}{3}\times -125 - 125\times\frac{1}{6}$$

$$= \frac{125}{3}-\frac{125}{6} = 125\left(\frac{2-1}{6}\right)=\underline{\underline{\frac{125}{6}}}$$

Imp

5. If R is the region bounded by the parabolas $y=x^2$ and $y^2=x$ in the first quadrant. Evaluate $\iint_R (x+y)\,dA$



$y^2=x \quad y=x^2$

$x^4=x \Rightarrow x^4-x=0$

$x(x^3-1)=0$

$x=0 \quad x=1$

limit: $x: 0\longrightarrow 1$

$y: x^2\longrightarrow\sqrt{x}$

$$\iint_R (x+y)\,dA = \int_0^1\int_{x^2}^{\sqrt{x}}(x+y)\,dy\,dx$$

$$= \int_0^1\left(xy+\frac{y^2}{2}\right)_{x^2}^{\sqrt{x}}dx$$

$$= \int_0^1 x\sqrt{x}+\frac{x}{2}-\left[x^3+\frac{x^4}{2}\right]dx$$

$$= \int_0^1\left[x^{3/2}+\frac{x}{2}-x^3-\frac{x^4}{2}\right]dx$$

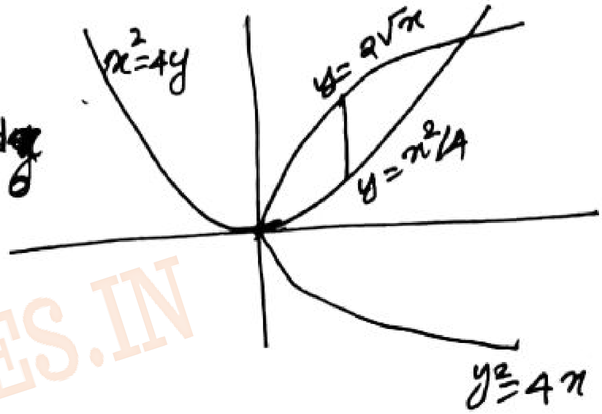
$$= \left[\frac{x^{5/2}}{5/2}+\frac{x^2}{4}-\frac{x^4}{4}-\frac{x^5}{10}\right]_0^1$$

$$= \frac{2}{5}\, 1^{5/2} + \frac{1}{4} - \frac{1}{4} - \frac{1}{10} - 0$$

$$= \frac{2}{5} - \frac{1}{10} = \frac{4-1}{10} = \underline{\frac{3}{10}}$$

6. Evaluate $\iint_R y\, dx\, dy$ where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$.

Imp



$$\iint_R y\, dx\, dy = \int_0^4 \int_{x^2/4}^{2\sqrt{x}} y\, dy\, dx$$

$$= \int_0^4 \left[\frac{y^2}{2}\right]_{x^2/4}^{2\sqrt{x}} dx$$

$$= \int_0^4 \frac{1}{2}\left[4x - \frac{x^4}{16}\right] dx$$

$$= \frac{1}{2}\left[\cancel{4}^{2}\,\frac{x^2}{\cancel{2}} - \frac{1}{16}\,\frac{x^5}{5}\right]_0^4$$

$$= \frac{1}{2}\left[2 \times 16 - \frac{1}{\cancel{80}_{5}} \times \cancel{1024}^{64} - 0\right]$$

$$= \frac{1}{2}\left[\frac{160-64}{5}\right] = \frac{\cancel{96}^{48}}{\cancel{2} \times 5}$$

$$= \underline{\frac{48}{5}}$$

$y = \frac{x^2}{4}$ $\quad y = 2\sqrt{x}$

$\frac{x^2}{4} = 2\sqrt{x}$

$x^4 = 64x$

$x^4 - 64x = 0$

$x(x^3 - 64) = 0$

$x = 0 \quad x^3 = 64$

$x = 4$

Limit: $x: 0 \to 4$

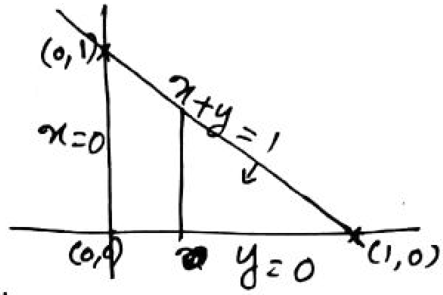
$y: \frac{x^2}{4} \to 2\sqrt{x}$

7. Evaluate $\iint_R (x^2+y^2)\,dx\,dy$ where R is the region taken over the first quadrant for which $x+y \le 1$.

First quadrant $\Rightarrow$ $x=0$ $y=0$

$x+y=1$

x	0	1
y	1	0



$$\iint_R (x^2+y^2)\,dx\,dy = \int_0^1 \int_0^{1-x} (x^2+y^2)\,dy\,dx$$

limit.
$y : 0 \longrightarrow 1-x$
$x : 0 \longrightarrow 1$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1 \left[x^2(1-x) + \frac{1}{3}(1-x)^3 - 0\right] dx$$

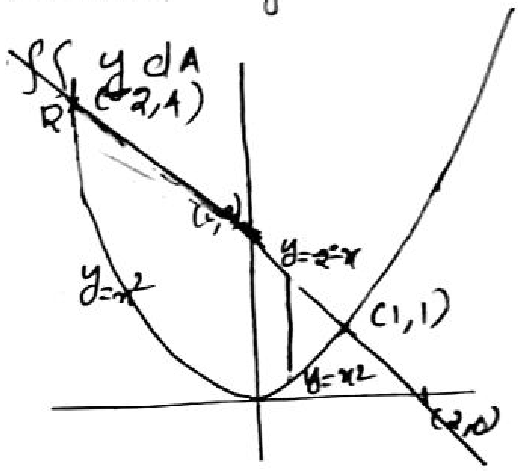
$$= \int_0^1 \left[x^2 - x^3 + \frac{1}{3}(1-x)^3\right] dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{1}{3}\,\frac{(1-x)^4}{4\times(-1)}\right]_0^1$$

$$= \left[\frac{1}{3} - \frac{1}{4} - \frac{1}{12}(1-1)^4\right] - \left[0 - 0 - \frac{1}{12}\times 1\right]$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{12} = \frac{4-3+1}{12} = \frac{2}{12} = \underline{\underline{1/6}}$$

8 The line $y = 2 - x$ and the parabola $y = x^2$ intersect at the points $(-2, 4)$ and $(1, 1)$. If R is the region enclosed by $y = 2 - x$ and $y = x^2$, then find $\iint y \, dA$



$y = 2 - x$

x	0	2
y	2	0

$x : -2 \longrightarrow 1$

$y : x^2 \longrightarrow 2 - x$

$$\iint_R y \, dA = \int_{-2}^{1} \int_{x^2}^{2-x} y \, dy \, dx$$

$$= \int_{-2}^{1} \left[\frac{y^2}{2} \right]_{x^2}^{2-x} dx = \int_{-1}^{2} \frac{1}{2} \left[(2-x)^2 - x^4 \right] dx$$

$$= \left[\frac{1}{2} \left[\frac{(2-x)^3}{3 \times -1} - \frac{x^5}{5} \right] \right]_{-1}^{2}$$

$$= \frac{1}{2} \left\{ -\frac{1}{3} (2-2)^3 - \frac{2^5}{5} - \left[-\frac{1}{3} (2+1)^3 - \frac{(-1)^5}{5} \right] \right\}$$

$$= \frac{1}{2} \left\{ -\frac{32}{5} + \frac{1}{\cancel{3}} \times 3^{\cancel{3}\,2} - \frac{1}{5} \right\} = 1$$

$$= \frac{1}{2} \left\{ -\frac{33}{5} + 9 \right\} = \frac{1}{2} \left[\frac{-33 + 45}{5} \right] = \frac{1}{\cancel{2} \times 5} \times \cancel{12}^{6}$$

$$= \underline{\underline{6/5}}$$

9. Sketch the region of integration in

$$\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\, dy\, dx$$

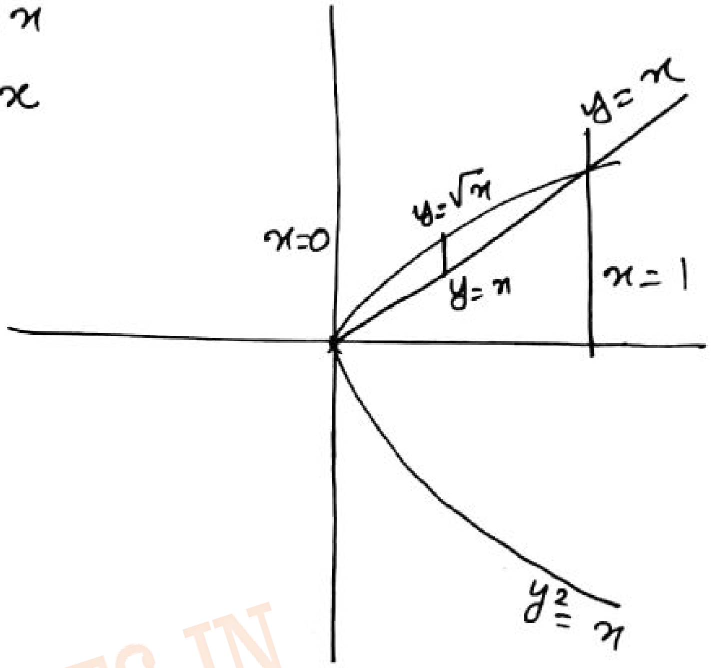
$y: x \longrightarrow \sqrt{x}$

$y = x$ and $y = \sqrt{x}$

$y^2 = x$

~~Sm~~ $x: 0 \longrightarrow 1$

$\Longrightarrow x = 0 \quad x = 1$



10 Use a double integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane $z = 4 - 4x - 2y$.

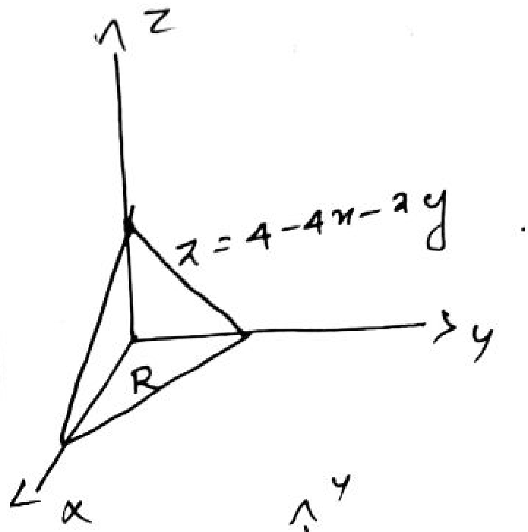
The tetrahedron is bounded above by the plane

$$z = 4 - 4x - 2y = f(x,y)$$

and below by the rectangular region R

$$\therefore \text{Volume} = \iint_R f(x,y)\, dA$$

$$V = \iint_R (4-4x-2y)\, dA$$



Put $z=0$ $\quad 4-4x-2y=0$

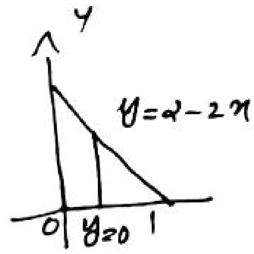
$$2y = 4-4x$$
$$y = 2(1-x)$$

$\therefore\ y : 0 \longrightarrow 2-2x$

Put $y=0$ $\quad 2-2x=0$

$$2x = 2$$
$$x = 1$$

$\therefore\ x : 0 \longrightarrow 1$



$$V = \int_0^1 \int_0^{2-2x} (4-4x-2y)\, dy\, dx$$

$$= \int_0^1 \left[4y - 4xy - \cancel{2}\frac{y^2}{\cancel{2}}\right]_0^{2-2x} dx$$

$$= \int_0^1 \left[4(2-2x) - 4x(2-2x) - (2-2x)^2 - 0\right] dx$$

$$= \int_0^1 \left[4(2-2x) - 4(2x-2x^2) - (2-2x)^2\right] dx$$

$$= \left\{4\left[2x - \frac{2x^2}{2}\right] - 4\left[\frac{2x^2}{2} - \frac{2x^3}{3}\right] - \frac{(2-2x)^3}{3}\right\}_0^1$$

$$= \left\{4[2-2] - 4\left(1 - \frac{2}{3}\right) - \left(\frac{2-2}{3}\right)^3 + \frac{8}{3}\right\}$$

$$= -\frac{4}{3} + \frac{8}{3} = \frac{4}{3}$$

Area calculated as Double Integral

The Volume bounded by a Surface $z = f(x,y)$ with the region R in the x-y plane is

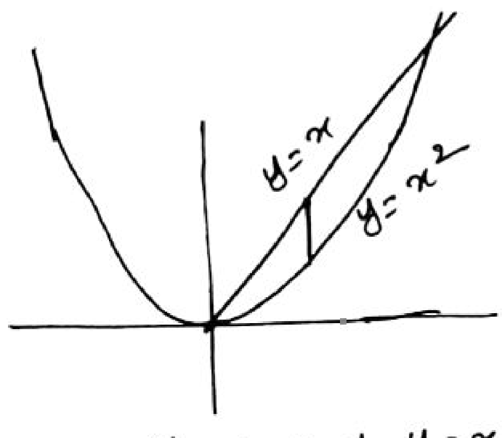
$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy.$$

choose $f(x,y)=1$, then the double integral becomes

$$\boxed{\text{Area} = \iint_R dA = \iint_R dx\,dy}$$

Problems

1. Find the area of the region enclosed by $y=x^2$ and $y=x$



$$\text{Area} = \iint_R dA$$

$$= \int_0^1 \int_{x^2}^{x} dy\,dx$$

$$= \int_0^1 (y)_{x^2}^{x}\,dx$$

$$= \int_0^1 (x-x^2)\,dx$$

$y=x$ and $y=x^2$

$x^2 = x \Rightarrow x^2 - x = 0$

$x(x-1)=0$

$x=0 \quad x=1$

Limit

$x: 0 \longrightarrow 1$

$y: x^2 \longrightarrow x$

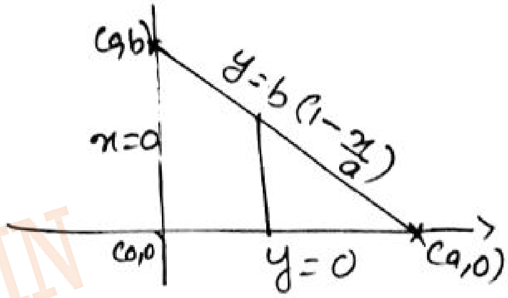
$$= \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} - 0 = \frac{3-2}{6}$$

$$= \underline{1/6}$$

2. Using double integration, evaluate the area enclosed by the lines $x=0, y=0, \frac{x}{a}+\frac{y}{b}=1$.

$$\text{Area} = \iint_R dA$$



$$= \int_0^a \int_0^{b(1-\frac{x}{a})} dy\, dx$$

$$= \int_0^a \left(y \right)_0^{b(1-\frac{x}{a})} dx$$

$\frac{x}{a} + \frac{y}{b} = 1$

x	0	a
y	b	0

$y = b(1-\frac{x}{a})$

limit: $y: 0 \longrightarrow b(1-\frac{x}{a})$

$x: 0 \longrightarrow a$

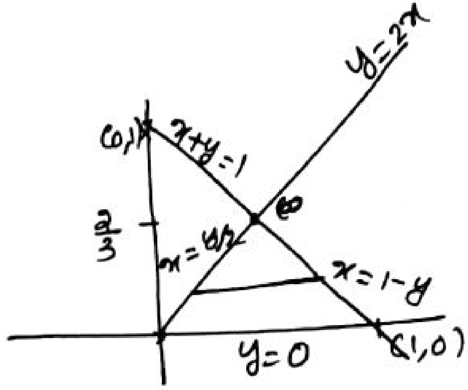
$$= \int_0^a b\left(1 - \frac{x}{a}\right) dx$$

$$= b\left[x - \frac{1}{a}\frac{x^2}{2} \right]_0^a = b\left[a - \frac{1}{a}\frac{a^2}{2} - 0 \right]$$

$$= b\left[\frac{2a-a}{2} \right] = \underline{\frac{ab}{2}}$$

3. Find the area bounded by the x-axis, $y=2x$ and $x+y=1$ using double integration

$$\text{Area} = \iint_R dA$$



$$= \int_0^{2/3}\int_{y/2}^{1-y} dx\,dy$$

$x+y=1$

x	0	1
y	1	0

$$= \int_0^{2/3} (x)_{y/2}^{1-y}\, dy$$

$x = \frac{y}{2} \quad x = 1-y$

$\frac{y}{2} = 1-y \Rightarrow y = 2-2y$

$3y = 2$

$y = 2/3$

$$= \int_0^{2/3} \left(1-y-\frac{y}{2}\right)dy$$

$y : 0 \longrightarrow 2/3$

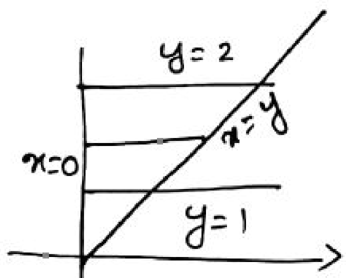
$x : \frac{y}{2} \longrightarrow 1-y$.

$$= \int_0^{2/3}\left(1-\frac{3}{2}y\right)dy$$

$$= \left[y - \frac{3}{2}\frac{y^2}{2}\right]_0^{2/3}$$

$$= \left[\frac{2}{3} - \frac{3}{4}\times\frac{4}{9_3} - 0\right] = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

4. Find the area of the region R enclosed by $y=1 \quad y=2 \quad x=0 \quad x=y$



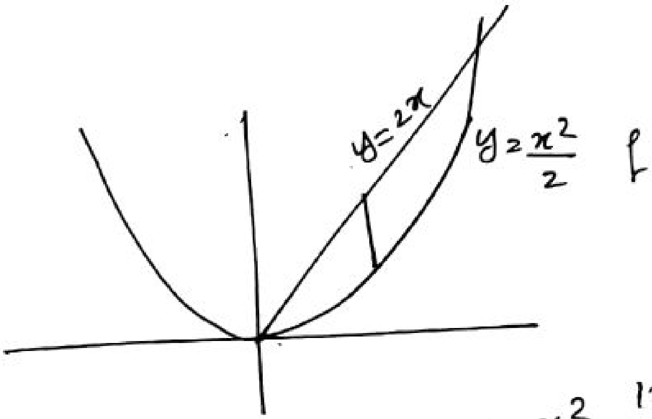
$$\text{Area} = \iint_R dA = \int_1^2\int_0^y dx\,dy$$

$$= \int_1^2 [x]_0^y\, dy = \int_1^2 y\,dy = \left(\frac{y^2}{2}\right)_1^2$$

$$= \frac{1}{2}[4-1] = \frac{3}{2}$$

5\. Find the area of the region enclosed between the parabola $y = \frac{x^2}{2}$ and the line $y = 2x$

Imp.



$$\text{Area} = \iint_R dA$$

$$= \int_0^4 \int_{\frac{x^2}{2}}^{2x} dy\,dx$$

$$= \int_0^4 [y]_{\frac{x^2}{2}}^{2x}\,dx$$

$$= \int_0^4 \left[2x - \frac{x^2}{2}\right] dx$$

$$= \left[2\frac{x^2}{2} - \frac{1}{2}\frac{x^3}{3}\right]_0^4$$

$$= 16 - \frac{1}{6}\times 64 - 0 = \frac{48-32}{3}$$

$$= \frac{16}{3}$$

$y = 2x$ and $y = \frac{x^2}{2}$

$\frac{x^2}{2} = 2x \Rightarrow x^2 = 4x$

$x^2 - 4x = 0$

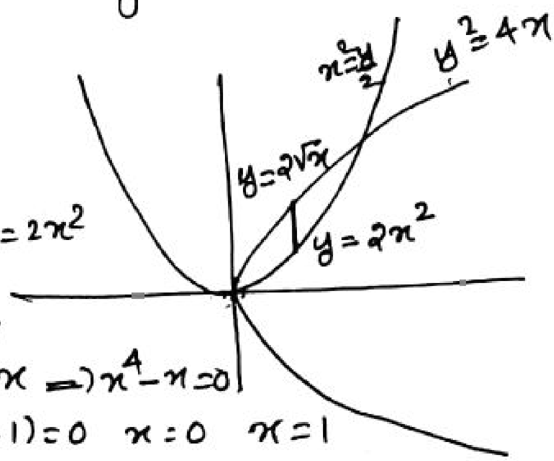
$x(x-4) = 0$

$x = 0 \quad x = 4$

limit $\quad x : 0 \to 4$

$y : \frac{x^2}{2} \to 2x$

6\. Find the area bounded by the parabolas $y^2 = 4x$ and $x^2 = y/2$



$$\text{Area} = \iint_R dA$$

$$= \int_0^1 \int_{2x^2}^{2\sqrt{x}} dy\,dx$$

$y = 2\sqrt{x} \quad y = 2x^2$

$2x^2 = 2\sqrt{x}$

$4x^4 = 4x \Rightarrow x^4 - x = 0$

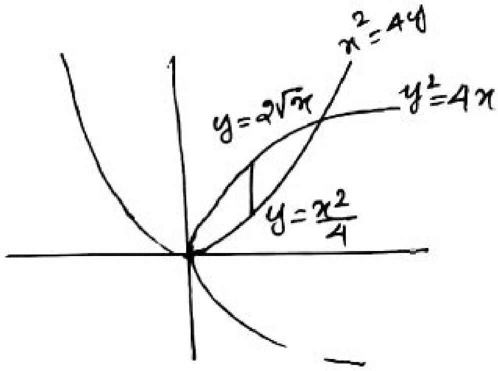
$x(x^3 - 1) = 0 \quad x = 0 \quad x = 1$

$$= \int_0^1 [y]_{2x^2}^{2\sqrt{x}} \, dx = \int_0^1 \left[2\sqrt{x} - 2x^2\right] dx$$

$$= \left[2\,\frac{x^{3/2}}{3/2} - 2\,\frac{x^3}{3}\right]_0^1 = \left[\frac{4}{3} - \frac{2}{3} - 0\right] = 2/3$$

7. Use double integral to find the area of the plane enclosed by $y^2 = 4x$ and $x^2 = 4y$

$$\text{Area} = \iint_R dA = \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dy\,dx$$



$$= \int_0^4 [y]_{\frac{x^2}{4}}^{2\sqrt{x}} \, dx$$

$\frac{x^2}{4} = 2\sqrt{x}$

$x^2 = 8\sqrt{x}$

$x^4 = 64x$

$x(x^3 - 64) = 0$

$x = 0 \quad x^3 = 64$

$x = 4$

$x : 0 \to 4$

$y : \frac{x^2}{4} \to 2\sqrt{x}$

$$= \int_0^4 \left[2\sqrt{x} - \frac{x^2}{4}\right] dx$$

$$= \left[2\,\frac{x^{3/2}}{3/2} - \frac{1}{4}\,\frac{x^3}{3}\right]_0^4$$

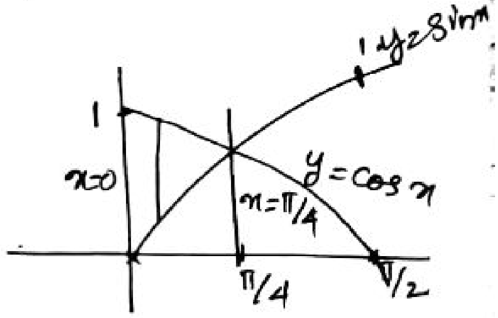
$$= \frac{4}{3}\,\cancel{8}\; 4^{3/2} - \frac{1}{12}\,4^3 - 0$$

$$= \frac{4}{3} \times 2^3 - \frac{1}{3}\,4^2 = \frac{32 - 16}{3}$$

$$= 16/3$$

8.

8. Use double integration to find the area of the plane region enclosed by the given curves $y = \sin x$ and $y = \cos x$ for $0 \le x \le \pi/4$



$$\text{Area} = \int_0^{\pi/4} \int_{\sin x}^{\cos x} dy\, dx$$

$$= \int_0^{\pi/4} [y]_{\sin x}^{\cos x}\, dx$$

$$= \int_0^{\pi/4} [\cos x - \sin x]\, dx$$

$$= [\sin x - -\cos x]_0^{\pi/4}$$

$$= \left[\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right] - [\sin 0 + \cos 0]$$

$$= \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right]$$

$$= \frac{2}{\sqrt{2}} - 1 = \frac{\sqrt{2} \times \sqrt{2}}{\sqrt{2}} - 1$$

$$= \underline{\underline{\sqrt{2} - 1}}$$

Converting Double Integrals from Rectangular to polar coordinates

Double Integrals are sometimes easier to evaluate if we change to polar co-ordinate.

The rectangular co-ordinate (x,y) and polar co-ordinate (r,θ) are connected by $x = r\cos\theta$, $y = r\sin\theta$

$$\therefore \iint_R f(x,y)\,dA = \iint_R f(r\cos\theta, r\sin\theta)\,dA$$

$$= \iint_{\text{appropriate limit}} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta .$$

Problems

1. Evaluate $\int_0^2 \int_0^{\sqrt{4-x^2}} y(x^2+y^2)\,dx\,dy$ using polar co ordinates

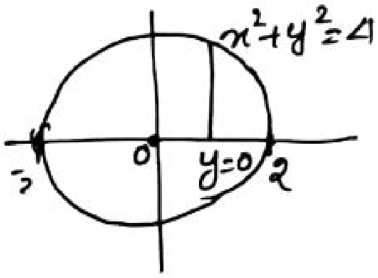
$y: 0 \longrightarrow \sqrt{4-x^2}$ $\Longrightarrow y=0$ and $y=\sqrt{4-x^2}$

$y^2 = 4-x^2$

$x: 0 \longrightarrow 2$

$x^2+y^2 = 4$

The region of integration in the First quadrant of the circle $x^2+y^2=4$.



Put $x = r\cos\theta$ $\quad y = r\sin\theta$ $\quad dx\,dy = r\,dr\,d\theta$

$x^2+y^2 = r^2$

$r : 0 \longrightarrow 2$ (radius)

$\theta : 0 \longrightarrow \pi/2$ (First quadrant)

$$\int_0^2 \int_0^{\sqrt{4-x^2}} y\,(x^2+y^2)\,dx\,dy = \int_0^{\pi/2} \int_0^{2} r\sin\theta \; r^2 \; r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \int_0^{2} r^4 \sin\theta \,dr\,d\theta = \int_0^{\pi/2} \left(\frac{r^5}{5}\right)_0^2 \sin\theta\,d\theta$$

$$= \int_0^{\pi/2} \frac{1}{5} 2^5 \sin\theta\,d\theta = \frac{32}{5}\left[-\cos\theta\right]_0^{\pi/2}$$

$$= -\frac{32}{5}\left[\cos\pi/2 - \cos 0\right] = -\frac{32}{5}\left[0-1\right]$$

$$= \frac{32}{5}$$

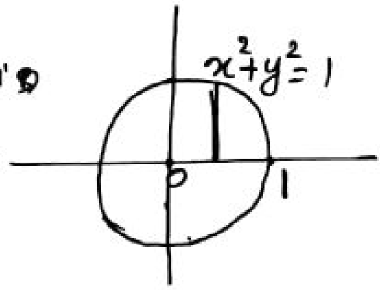
2. Evaluate the double integral by converting to polar co-ordinates $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$

$y : 0 \longrightarrow \sqrt{1-x^2} \implies y = 0 \qquad y = \sqrt{1-x^2}$

$y^2 = 1 - x^2$

$x : 0 \longrightarrow 1 \qquad \implies x^2 + y^2 = 1$

The region of integration is the first quadrant of the circle $x^2+y^2=1$



Put $x = r\cos\theta \quad y = r\sin\theta \quad dx\,dy = r\,dr\,d\theta$

$x^2+y^2 = r^2$

$r : 0 \longrightarrow 1$

$\theta : 0 \longrightarrow \pi/2$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx = \int_0^{\pi/2}\int_0^1 r^2 \, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 r^3\,dr\,d\theta = \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^1 d\theta$$

$$= \int_0^{\pi/2} \frac{1}{4}(1-0)\,d\theta = \frac{1}{4}\left[\theta\right]_0^{\pi/2}$$

$$= \frac{1}{4} \times \frac{\pi}{2} = \underline{\pi/8}$$

3. Evaluate $\iint_R e^{x^2+y^2}\,dy\,dx$ where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$

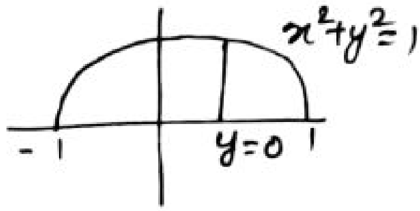
$y : 0 \longrightarrow \sqrt{1-x^2} \implies y=0 \quad y=\sqrt{1-x^2}$

$x : -1 \longrightarrow 1 \qquad y^2 = 1-x^2$

$\implies x^2+y^2=1$

Put $x = r\cos\theta \quad y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta \quad x^2+y^2=r^2$



$$\iint_R e^{x^2+y^2}\,dy\,dx = \int_0^{\pi}\int_0^{1} e^{r^2}\,r\,dr\,d\theta$$

Put $r^2 = t$

$2r\,dr = dt$

$$= \int_0^{\pi}\left(e^t \frac{dt}{2}\right)d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\left[e^t\right]d\theta = \frac{1}{2}\int_0^{\pi}\left[e^{r^2}\right]_0^1 d\theta$$

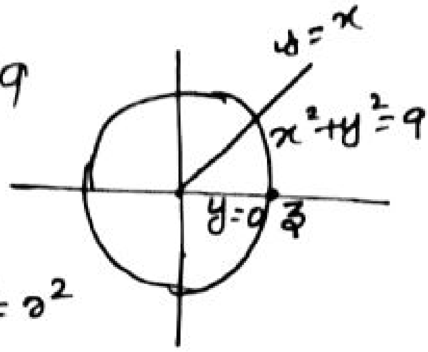
$$= \frac{1}{2}\int_0^{\pi}(e^1-e^0)\,d\theta = \frac{1}{2}\int_0^{\pi}(e-1)\,d\phi$$

$$= \frac{1}{2}(e-1)\left[\theta\right]_0^{\pi} = \frac{1}{2}(e-1)\pi$$

$$= \underline{\frac{\pi}{2}(e-1)}$$

4. Use polar co-ordinates to evaluate $\iint_R \frac{1}{1+x^2+y^2}\,dA$ where R is the sector in the first quadrant bounded by $y=0$, $y=x$ and $x^2+y^2=9$.

$y=0 \quad y=x$ and $x^2+y^2=9$



Put $x=r\cos\theta \quad y=r\sin\theta$

$dx\,dy = r\,dr\,d\theta \quad x^2+y^2=r^2$

$r: 0 \longrightarrow 3$

$\theta: 0 \longrightarrow \pi/4$.

$$\iint_R \frac{1}{1+x^2+y^2}\,dA = \int_0^{\pi/4}\int_0^3 \frac{1}{1+r^2}\,r\,dr\,d\theta$$

Put $1+r^2=t$

$2r\,dr=dt$

$$= \int_0^{\pi/4}\left(\frac{1}{t}\,\frac{dt}{2}\right)d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \log t \; d\theta = \frac{1}{2}\int_0^{\pi/4}\left[\log(1+r^2)\right]_0^3 d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4}(\log 10-\log 0)\,d\theta = \frac{1}{2}\int_0^{\pi/4}\log 10\,d\theta$$

$$= \frac{1}{2}\log 10\,\left[\theta\right]_0^{\pi/4} = \frac{1}{2}\log 10 \times \frac{\pi}{4}$$

$$= \underline{\underline{\frac{\pi}{8}\log 10}}$$

5. Evaluate $\iint_R \sin\theta\,dA$ where R is the region in the first quadrant that is outside the circle $r=2$ and inside the cardioid $r=2(1+\cos\theta)$.

$$\iint_R \sin\theta \, dA = \int_0^{\pi/2} \int_2^{2(1+\cos\theta)} \sin\theta \, r \, dr \, d\theta$$

$\theta : 0 \to \pi/2$ [First quadrant]
$r : 2 \to 2(1+\cos\theta)$

$$= \int_0^{\pi/2} \sin\theta \left[\frac{r^2}{2}\right]_2^{2(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\sin\theta \left[4(1+\cos\theta)^2 - 4\right] d\theta$$

$$= 2\int_0^{\pi/2} \sin\theta(1+\cos\theta)^2 d\theta - \int_0^{\pi/2} (\sin\theta)\, d\theta$$

$$= 2\left\{\int -t^2\, dt + (\cos\theta)_0^{\pi/2}\right.$$

put
$1+\cos\theta = t$
$-\sin\theta \, d\theta = dt$

$$= 2\left\{\left[-\frac{t^3}{3}\right] + \left[\cos\pi/2 - \cos 0\right]\right.$$

$$= 2\left\{-\frac{1}{3}\left[(1+\cos\theta)^3\right]_0^{\pi/2} - 1\right\}$$

$$= 2\left\{-\frac{1}{3}\left[1-(1+1)^3\right] - 1\right\}$$

$$= 2\left\{\frac{7}{3} - 1\right\} = 2\left\{\frac{7-3}{3}\right\}$$

$$= 2 \times \frac{4}{3}$$

$$= 8/3$$

Reversing the order of Integration

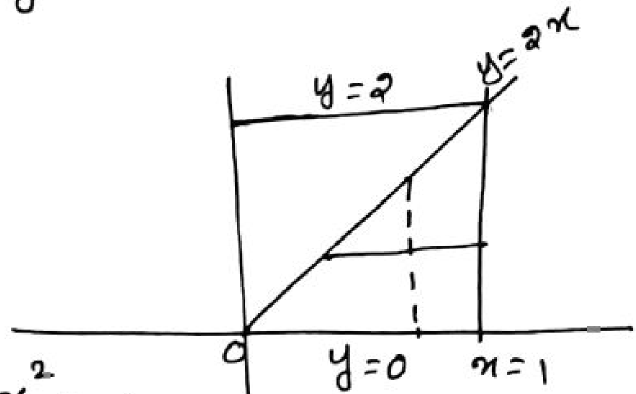
In a double integral, if the limit of integration are constant, then the order of integration is immaterial. But if the limit of integration are variable, a change in the order of integration must change in the limit of integration. Sometimes evaluation of double integrals will be simplified if the order of integration is reversed.

Problems

1. Evaluate the integral by first reversing the order of integration $\int_0^2 \int_{y/2}^1 e^{x^2}\, dx\, dy$.

$x : \frac{y}{2} \longrightarrow 1 \implies x = \frac{y}{2}$ or $y = 2x$ and $x = 1$

$y : 0 \longrightarrow 2 \implies y = 0$ and $y = 2$



Limit

$y : 0 \longrightarrow 2x$

$x : 0 \longrightarrow 1$

$$\int_0^2 \int_{y/2}^1 e^{x^2}\, dx\, dy = \int_0^1 \int_0^{2x} e^{x^2}\, dy\, dx$$

$$= \int_0^1 e^{x^2} [y]_0^{2x}\, dx$$

$$= \int_0^1 e^{x^2}\, 2x\,dx$$ put $x^2 = t$, $2x\,dx = dt$

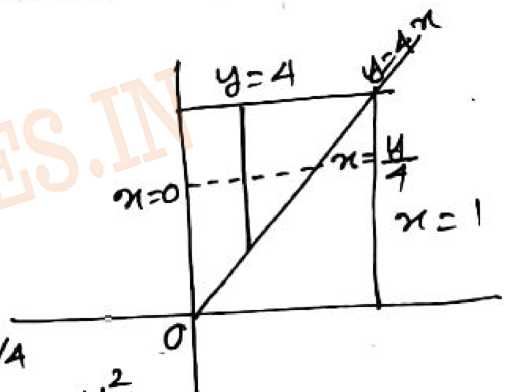
$$= \int e^t\,dt = \left[e^t\right] = \left[e^{x^2}\right]_0^1$$

$$= e^1 - e^0 = \underline{\underline{e-1}}$$

2. Evaluate the integral $\int_0^1\int_{4x}^{4} e^{-y^2}\,dy\,dx$ by reversing the order of integration.

$y : 4x \longrightarrow 4 \implies y = 4x$ and $y = 4$

$x : 0 \longrightarrow 1 \implies x = 0$ and $x = 1$



<u>Limit</u>

$x : 0 \longrightarrow \frac{y}{4}$

$y : 0 \longrightarrow 4$

$$\int_0^1\int_{4x}^{4} e^{-y^2}\,dy\,dx = \int_0^4\int_0^{y/4} e^{-y^2}\,dx\,dy$$

$$= \int_0^4 e^{-y^2}\left[x\right]_0^{y/4} dy = \int_0^4 e^{-y^2}\left[y/4\right] dy$$

$$= \frac{1}{4}\int e^{-t}\,\frac{dt}{2} = \frac{1}{8}\left[\frac{e^{-t}}{-1}\right]$$ put $y^2 = t$, $2y\,dy = dt$

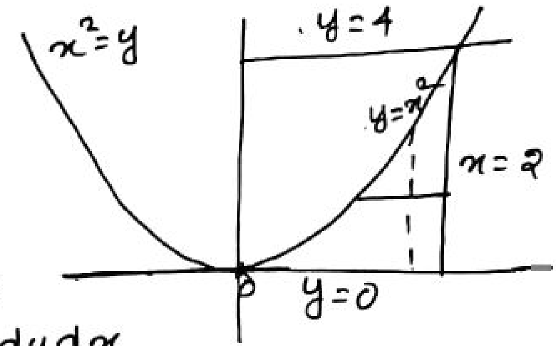
$$= -\frac{1}{8}\left[e^{-y^2}\right]_0^4 = -\frac{1}{8}\left[e^{-16} - e^{0}\right]$$

$$= \underline{\underline{\frac{1}{8}\left[1 - e^{-16}\right]}}$$

3. Evaluate the integral $\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\,dx\,dy$ by changing the order of integration.

$x: \sqrt{y} \longrightarrow 2 \implies x = \sqrt{y} \implies x^2 = y$ and $x = 2$

$y: 0 \longrightarrow 4 \implies y = 0$ and $y = 4$



Limit

$y: 0 \longrightarrow x^2$

$x: 0 \longrightarrow 2$

$$\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\,dx\,dy = \int_0^2 \int_0^{x^2} e^{x^3}\,dy\,dx$$

$$= \int_0^2 e^{x^3} [y]_0^{x^2}\,dx = \int_0^2 e^{x^3} x^2\,dx$$

put $x^3 = t$

$3x^2\,dx = dt$

$$= \int e^t \frac{dt}{3} = \frac{1}{3}\left[e^t\right]$$

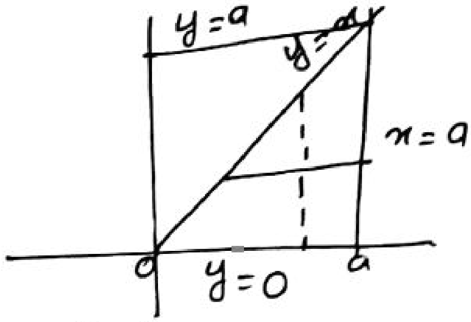
$$= \frac{1}{3}\left[e^{x^3}\right]_0^2$$

$$= \frac{1}{3}\left[e^8 - e^0\right] = \underline{\underline{\frac{1}{3}(e^8 - 1)}}$$

4. Evaluate the integral $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$ by changing the order of integration.

$x : y \longrightarrow a \implies x = y$ and $x = a$

$y : 0 \longrightarrow a \implies y = 0$ and $y = a$



Limit

$y : 0 \longrightarrow x$

$x : 0 \longrightarrow a$

$$\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy = \int_0^a \int_0^x \frac{x}{x^2+y^2}\, dy\, dx$$

$$= \int_0^a x \cdot \left[\frac{1}{x} \tan^{-1}(y/x) \right]_0^x dx$$

$$= \int_0^a \tan^{-1}(x/x) - \tan^{-1}(0) \Big] dx$$

$$= \int_0^a \left[\tan^{-1}(1) - \tan^{-1}(0)\right] dx = \int_0^a \frac{\pi}{4}\, dx$$

$$= \frac{\pi}{4} \left[x\right]_0^a = \underline{\underline{\frac{\pi a}{4}}}$$

5. Reverse the order of integration and evaluate

$$\int_0^1 \int_x^1 \frac{x}{x^2+y^2}\, dy\, dx$$

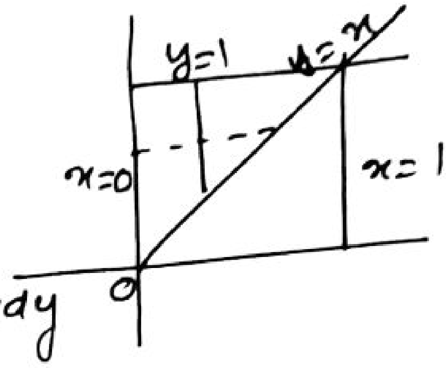
$y : x \longrightarrow 1 \implies y = x$ and $y = 1$

$x : 0 \longrightarrow 1 \implies x = 0$ and $x = 1$

Limit

$x : 0 \longrightarrow y$

$y : 0 \longrightarrow 1$



$$\int_0^1 \int_x^1 \frac{x}{x^2+y^2}\, dy\, dx = \int_0^1 \int_0^y \frac{x}{x^2+y^2}\, dx\, dy$$

put $x^2+y^2 = t$

$2x\,dx = dt$

$$= \int_0^1 \int \int \left[\frac{dt/2}{t}\right] dy$$

$$= \int_0^1 \frac{1}{2}\,[\log t]\, dy = \frac{1}{2}\int_0^1 \left[\log(x^2+y^2)\right]_0^y dy$$

$$= \frac{1}{2}\int_0^1 \left[\log(2y^2) - \log(y^2)\right] dy$$

$$= \frac{1}{2}\int_0^1 \log\left[\frac{2y^2}{y^2}\right] dy = \frac{1}{2}\int_0^1 \log 2\, dy$$

$$= \frac{1}{2}\log 2\,[y]_0^1 = \underline{\frac{1}{2}\log 2}$$

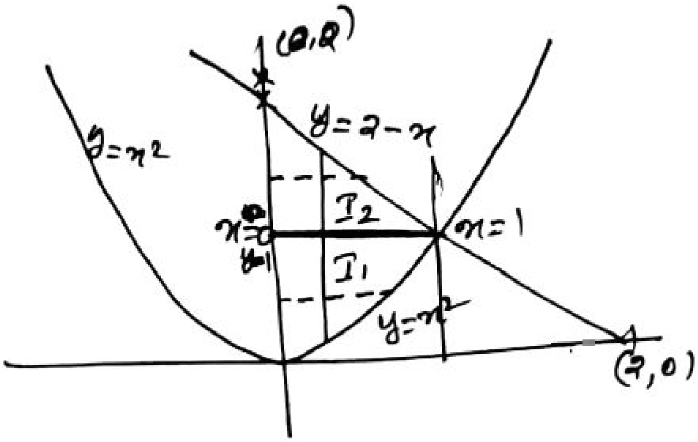
6. Change the order of integration and hence evaluate $\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$

$y : x^2 \longrightarrow 2-x \implies y = x^2$ and $y = 2-x$

$x : 0 \longrightarrow 1 \implies x = 0$ and $x = 1$

$y = 2 - x$

x	0	2
y	2	0



Limit

$I_1 : x : 0 \longrightarrow \sqrt{y}$

$\quad y : 0 \longrightarrow 1$

$I_2 : x : 0 \longrightarrow 2 - y$

$\quad y : 1 \longrightarrow 2$

$$\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx = \int_0^1 \int_0^{\sqrt{y}} xy\, dx\, dy + \int_1^2 \int_0^{2-y} xy\, dx\, dy$$

$$= \int_0^1 y \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 y \left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$= \int_0^1 \frac{1}{2} y \times y\, dy + \int_1^2 \frac{1}{2} y\, (2-y)^2\, dy$$

$$= \frac{1}{2} \left\{ \int_0^1 y^2\, dy + \int_1^2 y\, [4 - 4y + y^2]\, dy \right\}$$

$$= \frac{1}{2} \left\{ \left[\frac{y^3}{3}\right]_0^1 + \int_1^2 (4y - 4y^2 + y^3)\, dy \right\}$$

$$= \frac{1}{2} \left\{ \frac{1}{3} + \left[\cancel{4}^{2} \frac{y^2}{\cancel{2}} - 4\frac{y^3}{3} + \frac{y^4}{4} \right]_1^2 \right\}$$

$$= \frac{1}{2} \left\{ \frac{1}{3} + \left[2 \times 4 - \frac{4}{3} \times 8 + \frac{1}{\cancel{4}} \times \cancel{16}^{4} - \left(2 - \frac{4}{3} + \frac{1}{4}\right) \right\}$$

$$= \frac{1}{2} \left\{ \frac{1}{3} + \left[\frac{24 - 32 + 12}{3} \right] + \left[\frac{-6 + 4}{3} \right] - \frac{1}{4} \right\}$$

$$= \frac{1}{2}\left\{\frac{1}{3} + \frac{4}{3} - \frac{2}{3} - \frac{1}{4}\right\}$$

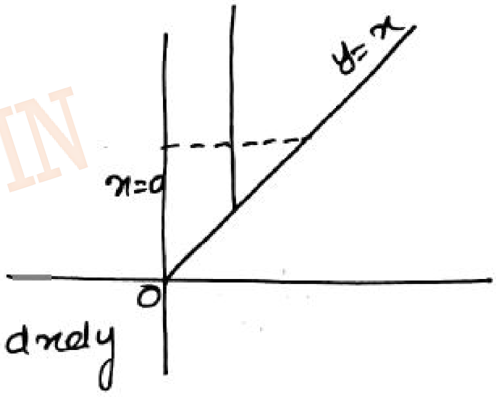
$$= \frac{1}{2}\left[1 - \frac{1}{4}\right] = \frac{1}{2} \times \frac{3}{4}$$

$$= 3/8$$

7 By changing the order of integration evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$

$y : x \longrightarrow \infty \implies y = x$ and $y = \infty$

$x : 0 \longrightarrow \infty \implies x = 0$ and $x = \infty$



Limit

$x : 0 \longrightarrow y$

$y : 0 \longrightarrow \infty$

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} \left[x\right]_0^y dy = \int_0^\infty \frac{e^{-y}}{\cancel{y}} \cancel{y}\, dy$$

$$= \int_0^\infty e^{-y}\, dy$$

$$= \left[\frac{e^{-y}}{-1}\right]_0^\infty = -\left[e^{-\infty} - e^{0}\right] = -\left[0 - 1\right] = 1$$

8 Change the order of integration in

$$\int_0^1 \int_{y^2}^{\sqrt{2-y^2}} f(x,y)\,dx\,dy .$$

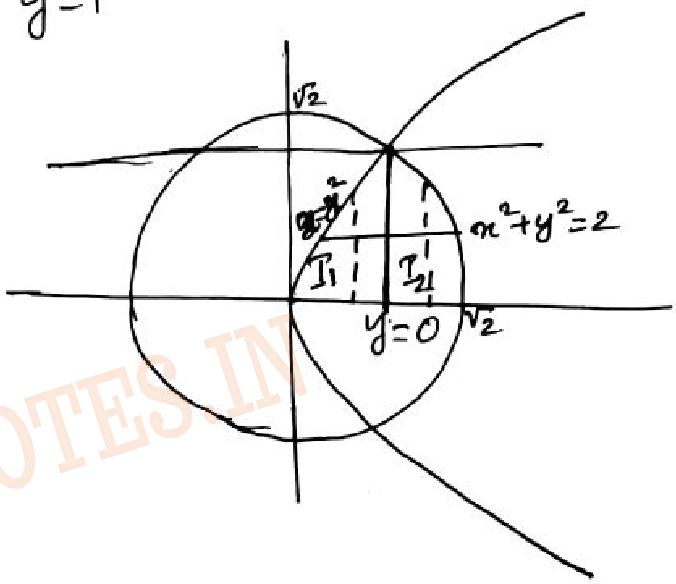
$x: y^2 \longrightarrow \sqrt{2-y^2} \implies x = y^2$ and $x = \sqrt{2-y^2}$

$x^2 = 2-y^2$

$x^2+y^2 = 2$

$y: 0 \longrightarrow 1$

$\implies y=0$ and $y=1$



Limit

I_1

$y: 0 \longrightarrow \sqrt{x}$

$x: 0 \longrightarrow 1$

I_2

$y: 0 \longrightarrow \sqrt{2-x^2}$

$x: 1 \longrightarrow \sqrt{2}$

$$\int_0^1 \int_{y^2}^{\sqrt{2-y^2}} f(x,y)\,dx\,dy = \int_0^1 \int_0^{\sqrt{x}} f(x,y)\,dy\,dx + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} f(x,y)\,dy\,dx$$

Centre of Mass using Double Integrals

When a system is in rest or in motion in space, at every instant of time, there is always a unique point which is the average position of the mass of the system. This point is called the centre of mass or centre of gravity of the system.

A lamina is a thin two-dimensional plane region. A lamina is said to be homogeneous if the distribution of materials is uniform throughout the region. The density ρ of the lamina at each point is defined as the mass per unit area surrounding the point. Since the distribution of material is uniform, density of homogeneous lamina having mass M and area A will be the same at every point and is given by $\rho = \frac{M}{A}$.

In an inhomogeneous lamina, the density at a point (x,y) can be specified by a function $\rho(x,y)$ called the density function. If a lamina with a continuous density function $\rho(x,y)$ occupies a region R in the x-y plane, then its total mass M is

$$\boxed{M = \iint_R \rho(x,y)\, dA}$$

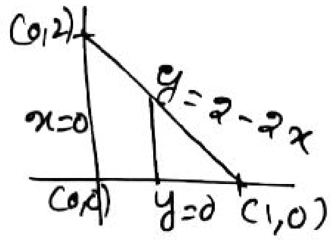
The centre of gravity (mass) of an inhomogeneous lamina is $(\bar{x}, \bar{y})$.

Where,

$$\boxed{\begin{aligned} \bar{x} &= \frac{1}{M} \iint_R x\, \rho(x,y)\, dA \\ \bar{y} &= \frac{1}{M} \iint_R y\, \rho(x,y)\, dA \end{aligned}}$$

Problems

1. Find the mass and centre of mass of a triangular lamina with vertices $(0,0)$, $(1,0)$ and $(0,2)$ if the density function is $\rho(x,y) = 1+3x+y$.



eqn of S.L joining $(1,0)$ and $(0,2)$

$$\frac{y-0}{2-0} = \frac{x-1}{0-1}$$

$$y = -2(x-1)$$

$$y = -2x+2$$

Limit: $y : 0 \longrightarrow 2-2x$

$x : 0 \longrightarrow 1$

$$\text{Mass} \quad M = \iint_R \rho(x,y)\,dA$$

$$= \int_0^1 \int_0^{2-2x} (1+3x+y)\,dy\,dx$$

$$= \int_0^1 \left(y + 3xy + \frac{y^2}{2}\right)_0^{2-2x}$$

$$= \int_0^1 \left[2-2x+3x(2-2x)+\frac{1}{2}(2-2x)^2 - 0\right] dx$$

$$= \int_0^1 2-2x+6x-6x^2+\frac{1}{2}\left[4-8x+4x^2\right] dx$$

$$= \int_0^1 \left(2+\cancel{4x}-6x^2+2-\cancel{4x}+2x^2\right) dx$$

$$= \int_0^1 (4-4x^2)\,dx = 4\int_0^1 (1-x^2)\,dx$$

$$= 4\left[x-\frac{x^3}{3}\right]_0^1 = 4\left[1-1/3\right] = 4 \times \frac{2}{3} = \underline{\underline{8/3}}$$

$$\bar{x} = \frac{1}{M}\iint_R x\,\rho(x,y)\,dA$$

$$= \frac{1}{8/3}\int_0^1\int_0^{2-2x} x\,(1+3x+y)\,dy\,dx$$

$$= \frac{3}{8}\int_0^1 x\left[y+3xy+\frac{y^2}{2}\right]_0^{2-2x} dx$$

$$= \frac{3}{8}\int_0^1 xA(1-x^2)\,dx$$

$$= \frac{3}{2}\int_0^1 (x - x^3)\,dx = \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{3}{2}\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{3}{2}\left[\frac{2-1}{4}\right] = \frac{3}{2}\times\frac{1}{4}$$

$$= \underline{\frac{3}{8}}$$

$$\bar{y} = \frac{1}{M}\iint_R y\,\delta(x,y)\,dA$$

$$= \frac{1}{8/3}\int_0^1\int_0^{2-2x} y\,[1+3x+y]\,dy\,dx$$

$$= \frac{3}{8}\int_0^1\int_0^{2-2x} (y + 3xy + y^2)\,dy\,dx$$

$$= \frac{3}{8}\int_0^1 \left[\frac{y^2}{2} + 3x\frac{y^2}{2} + \frac{y^3}{3}\right]_0^{2-2x}$$

$$= \frac{3}{8}\int_0^1 \left[\frac{2^2(1-x)^2}{2} + 3x\frac{2^2(1-x)^2}{2} + \frac{1}{3}2^3(1-x)^3 - 0\right]$$

$$= \frac{3}{8}\int_0^1 \left[2(1-x)^2 + 6x(1-x)^2 + \frac{8}{3}(1-x)^3\right]dx$$

$\frac{11}{16}$ $$= \frac{3}{8}\int_0^1 (1-x)^2\left[2 + 6x + \frac{8}{3}(1-x)\right]dx$$

$$= \frac{3}{8}\int_0^1 (1-x)^2\left[2 + 6x + \frac{8}{3} - \frac{8}{3}x\right]dx$$

$$= \frac{1}{4}\int_0^1 (7 - 9x - 3x^2 + 5x^3)\, dx$$

$$= \frac{1}{4}\left[7x - \frac{9x^2}{2} - \frac{3x^3}{3} + 5\frac{x^4}{4}\right]_0^1$$

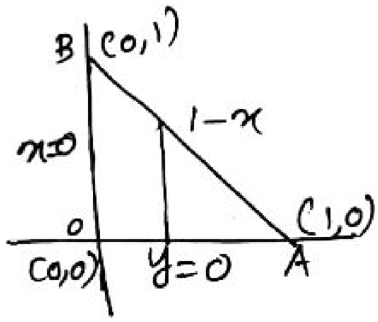
$$= \frac{1}{4}\left\{7 - \frac{9}{2} - 1 + \frac{5}{4} - 0\right\}$$

$$= \underline{\underline{\frac{11}{16}}}$$

2. Find the centre of gravity of the triangular lamina with vertices $(0,0)$, $(0,1)$ and $(1,0)$ and density function $f(x,y) = xy$.

The mass of the lamina is

$$M = \iint_R f(x,y)\, dA = \iint_R xy\, dA$$



eqn of the straight line joining $(0,1)$ to $(1,0)$ is

$$\frac{x-0}{1-0} = \frac{y-1}{0-1}$$

$$x = -(y-1) \implies y = 1 - x$$

$$M = \int_0^1 \int_0^{1-x} xy\, dy\, dx$$

$$= \int_0^1 x \left(\frac{y^2}{2} \right)_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 x \; (1-x)^2 \, dx = \frac{1}{2}\int_0^1 x\,(1-2x+x^2)\,dx$$

$$= \frac{1}{2} \int_0^1 (x - 2x^2 + x^3)\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2} - 2\frac{x^3}{3} + \frac{x^4}{4} \right]_0^1$$

$$= \frac{1}{2}\left\{ \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} - 0 \right) \right\}$$

$$= \frac{1}{2}\left[\frac{6-8+3}{12} \right] = \frac{1}{2} \times \frac{1}{12}$$

$$= \underline{\underline{\frac{1}{24}}}$$

$$\bar{x} = \frac{1}{M} \iint_R x \, \delta(x,y)\, dA$$

$$= \frac{1}{1/24} \int_0^1 \int_0^{1-x} x \cdot xy \; dy\,dx = 24 \int_0^1 \int_0^{1-x} x^2 y \, dy\,dx$$

$$= 24 \int_0^1 x^2 \left(\frac{y^2}{2} \right)_0^{1-x} dx = \frac{24}{2} \int_0^1 x^2 (1-x)^2 dx$$

$$= 12 \int_0^1 x^2 \left[1 - 2x + x^2 \right] dx = 12 \int_0^1 (x^2 - 2x^3 + x^4)\, dx$$

$$= 12\left[\frac{x^3}{3} - \cancel{2}\frac{x^4}{\cancel{4}2} + \frac{x^5}{5}\right]_0^1$$

$$= 12\left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5} - 0\right] = 12\left[\frac{10-15+6}{30}\right]$$

$$= \frac{\cancel{12}2}{\cancel{30}5} \times 1 = \frac{12}{30} = \underline{\underline{\frac{2}{5}}}$$

$$\bar{y} = \frac{1}{M}\iint_R y\, \rho(x,y)\, dA$$

$$= \frac{1}{1/24}\int_0^1\int_0^{1-x} y\, xy\, dy\, dx$$

$$= 24\int_0^1 x\left[\frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \frac{24}{3}\int_0^1 x\,[1-x]^3\, dx$$

$$= 8\int_0^1 x\left[1-3x+3x^2-x^3\right] dx$$

$$= 8\int_0^1 \left(x-3x^2+3x^3-x^4\right) dx$$

$$= 8\left[\frac{x^2}{2} - \frac{\cancel{3}x^3}{\cancel{3}} + 3\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1$$

$$= 8\left[\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5} - 0\right]$$

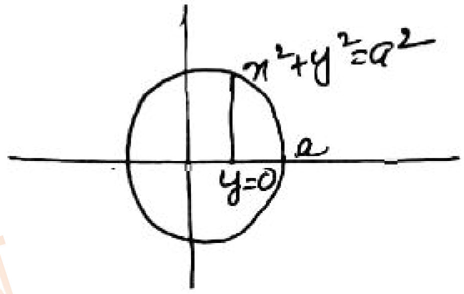
$$= 8\left[\frac{10-20+15-4}{20}\right] = \frac{\cancel{8}2}{\cancel{20}5} \times 1$$

$$= \underline{\underline{2/5}}$$

Centre of gravity $(\bar{x}, \bar{y})$

$= \underline{\underline{\left(\frac{2}{5}, \frac{2}{5}\right)}}$

3. Find the mass and centre of gravity of the lamina with density $\rho(x,y) = xy$ is in the first quadrant and is bounded by the circle $x^2+y^2=a^2$ and the co-ordinate axes.



$y : 0 \longrightarrow \sqrt{a^2-x^2}$

$x : 0 \longrightarrow a$

$$\text{Mass} = \iint_R \rho(x,y)\, dA$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy \, dy\, dx = \int_0^a x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a x\,(a^2-x^2)\, dx = \frac{1}{2}\int_0^a (a^2x - x^3)\, dx$$

$$= \frac{1}{2}\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4} - 0\right]$$

$$= \frac{1}{2}a^4\left[\frac{2-1}{4}\right] = \underline{\underline{\frac{a^4}{8}}}$$

$$\bar{x} = \frac{1}{M}\iint_R x \, f(x,y) \, dA$$

$$= \frac{1}{a^4/8}\int_0^a\int_0^{\sqrt{a^2-x^2}} x \, xy \, dy\,dx$$

$$= \frac{8}{a^4}\int_0^a x^2 \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{\cancel{8}\,4}{a^4}\int_0^a x^2 (a^2-x^2)\,dx = \frac{4}{a^4}\int_0^a (a^2x^2 - x^4)\,dx$$

$$= \frac{4}{a^4}\left[a^2\frac{x^3}{3} - \frac{x^5}{5}\right]_0^a = \frac{4}{a^4}\left[\frac{a^5}{3} - \frac{a^5}{5}\right]$$

$$= \frac{4}{a^4} a^5 \left[\frac{5-3}{15}\right] = 4a \times \frac{2}{15} = \underline{\underline{\frac{8a}{15}}}$$

$$\bar{y} = \frac{1}{M}\iint_R y \, f(x,y) \, dA$$

$$= \frac{1}{a^4/8}\int_0^a\int_0^{\sqrt{a^2-x^2}} y \, xy \, dy\,dx = \frac{8}{a^4}\int_0^a\int_0^{\sqrt{a^2-x^2}} xy^2\,dy\,dx$$

$$= \frac{8}{a^4}\int_0^a x\left(\frac{y^3}{3}\right)_0^{\sqrt{a^2-x^2}} dx = \frac{8}{a^4}\int_0^a x\left(\frac{y^3}{3}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{8}{a^4} \times \frac{1}{3}\int_0^a x\,(a^2-x^2)^{3/2}\,dx$$

put $a^2 - x^2 = t$

$-2x\,dx = dt$

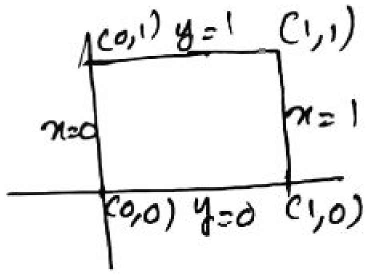
$$= \frac{8}{3a^4}\int t^{3/2}\left(\frac{-dt}{2}\right) = \frac{-4}{3a^4}\left[\frac{t^{5/2}}{5/2}\right]$$

$$= -\frac{\cancel{4}}{3a^4} \times \frac{2}{5}\left[(a^2-x^2)^{5/2}\right]_0^a = -\frac{8}{15a^4}\left[0-(a^2)^{5/2}\right]$$

$$= \frac{-8}{15a^4} - a^5 = \frac{8}{15}a$$

Centre of gravity $(\bar{x},\bar{y}) = \left(\frac{8}{15}a, \frac{8}{15}a\right)$

4. Find the mass of the square lamina with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ and density function x^2y.



Mass $M = \iint_R f(x,y)\,dA =$

$$= \int_0^1\int_0^1 x^2y\,dx\,dy$$

$$= \int_0^1 y\left[\frac{x^3}{3}\right]_0^1 dy = \frac{1}{3}\int_0^1 y\,dy$$

$$= \frac{1}{3}\left[\frac{y^2}{2}\right]_0^1 = \frac{1}{3}\left(\frac{1}{2}-0\right)$$

$$= \frac{1}{6}$$

Triple Integral

Let G be a closed and bounded solid region in an xyz - coordinate system. Let $f(x,y,z)$ be a continuous function defined at each point in the solid region G having volume V. Divide the solid region into small cubes ΔG_K having volume ΔV_K using planes parallel to the coordinate planes.

Let (x_K, y_K, z_K) be an arbitrary point in the cube ΔG_K. Then the triple integral of the continuous function $f(x,y,z)$ over the solid region G is denoted by $\iiint\limits_G f(x,y,z)dv$ and is defined as

$$\iiint\limits_G f(x,y,z)dv = \lim_{n\to\infty} \sum_{K=1}^{n} f(x_K, y_K, z_K)\,\Delta V_K$$

Evaluation of triple integrals over rectangular boxes

Let $f(x,y,z)$ be a continuous function at all points in a rectangular box G defined by $a \le x \le b$, $c \le y \le d$, $e \le z \le f$. Then

$$\boxed{\iiint\limits_H f(x,y,z)\,dv = \int_e^f \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz}$$

Problems

1. Evaluate $\int_{-1}^{2}\int_{0}^{2}\int_{0}^{1}(x^2+y^2+z^2)\,dx\,dy\,dz$

$$\int_{-1}^{2}\int_{0}^{2}\left(\frac{x^3}{3}+y^2x+z^2x\right)_0^1 dy\,dz$$

$$=\int_{-1}^{2}\int_{0}^{2}\left(\frac{1}{3}+y^2+z^2\right)dy\,dz$$

$$=\int_{-1}^{2}\left(\frac{1}{3}y+\frac{y^3}{3}+z^2y\right)_0^2 dz$$

$$=\int_{-1}^{2}\left(\frac{2}{3}+\frac{8}{3}+2z^2\right)dz=\int_{-1}^{0}\left(\frac{10}{3}+2z^2\right)dz$$

$$=\left[\frac{10}{3}z+2\frac{z^3}{3}\right]_{-1}^{2}=\left[\frac{20}{3}+\frac{16}{3}-\left(\frac{10}{3}\times-1+\frac{2}{3}(-1)\right)\right]$$

$$=\frac{36}{3}+\frac{10}{3}+\frac{2}{3}=\frac{48}{3}=\underline{16}$$

2. Evaluate $\int_{0}^{1}\int_{0}^{1}\int_{0}^{1}e^{x+y+z}\,dx\,dy\,dz$

$$\int_{0}^{1}\int_{0}^{1}\int_{0}^{1}(e^x\cdot e^y\cdot e^z)\,dx\,dy\,dz=\int_{0}^{1}\int_{0}^{1}(e^y e^z)(e^x)_0^1\,dy\,dz$$

$$=\int_{0}^{1}\int_{0}^{1}e^y e^z\,(e^1-e^0)\,dy\,dz$$

$$= \int_0^1\int_0^1 e^y e^z (e-1)\, dy\, dz$$

$$= (e-1)\int_0^1 e^z \left(e^y\right)_0^1 dz = (e-1)\int_0^1 e^z (e^1 - e^0)\, dz$$

$$= (e-1)^2 \int_0^1 e^z\, dz = (e-1)^2 \left[e^z\right]_0^1$$

$$= (e-1)^2 (e-1) = \underline{\underline{(e-1)^3}}$$

3. Evaluate $\int_0^3\int_0^2\int_0^1 xyz\, dx\,dy\,dz$

$$\int_0^3\int_0^2 yz \left(\frac{x^2}{2}\right)_0^1 dy\,dz = \int_0^3\int_0^2 yz\,\frac{1}{2}\, dy\,dz$$

$$= \frac{1}{2}\int_0^3 z \left(y^2/2\right)_0^2 dz = \frac{1}{4}\int_0^3 z\, 4\, dz$$

$$= \left[\frac{z^2}{2}\right]_0^3 = \underline{\underline{\frac{9}{2}}}$$

4. Evaluate $\int_0^a\int_0^a\int_0^a (yz+xz+xy)\, dx\,dy\,dz$

$$\int_0^a\int_0^a\int_0^a (yz+xz+xy)\, dx\,dy\,dz = \int_0^a\int_0^a \left[xyz + \frac{x^2}{2}z + \frac{x^2}{2}y\right]_0^a$$

$$= \int_0^a\int_0^a \left(ayz + \frac{a^2}{2}z + \frac{a^2}{2}y\right) dy\,dz = \int_0^a \left[az\frac{y^2}{2} + \frac{a^2}{2}zy + \frac{a^2}{2}\frac{y^2}{2}\right]_0^a$$

$$= \int_0^a \left[\frac{a}{2}z\, a^2 + \frac{a^2}{2}z\, a + \frac{a^2}{4}\, a^2\right] dz$$

$$= \int_0^a \left[\frac{a^3}{2}z + \frac{a^3}{2}z + \frac{a^4}{4}\right] dz$$

$$= \left[\frac{a^3}{2}\frac{z^2}{2} + \frac{a^3}{2} * \frac{z^2}{2} + \frac{a^4}{4}z\right]_0^a$$

$$= \frac{a^3}{2}\frac{a^2}{2} + \frac{a^3}{2}\frac{a^2}{2} + \frac{a^4}{4}a = \frac{a^5}{4} + \frac{a^5}{4} + \frac{a^5}{4}$$

$$= \underline{\underline{3a^5/4}}$$

5 Evaluate $\iiint_G xy \sin yz \, dv$ where G is the rectangular box defined by the inequalities $0 \le x \le \pi/2$, $0 \le y \le 1$, $0 \le z \le \pi/6$.

$$\iiint_G xy \cdot \sin yz \, dv = \int_0^{\pi/2}\int_0^1\int_0^{\pi/6} xy \sin yz \, dz\, dy\, dx$$

$$= \int_0^{\pi/2}\int_0^1 x\cancel{y}\left[\frac{-\cos yz}{\cancel{y}}\right]_0^{\pi/6} dy\, dx = \int_0^{\pi/2}\int_0^1 -xy\left[\cos y\frac{\pi}{6} - \cos\right.$$

$$= \int_0^{\pi/2}\int_0^1 x\left[1 - \cos\frac{y\pi}{6}\right] dy\, dx$$

$$= \int_0^{\pi/2} x\left[y - \frac{\sin\frac{y\pi}{6}}{\pi/6}\right]_0^1 dx = \int_0^{\pi/2} x\left[1 - \frac{6}{\pi}\sin\pi/6 - (0\right.$$

$$= \int_0^{\pi/2} x\left[1 - \frac{\cancel{6}^{3}}{\pi}\frac{1}{\cancel{2}} - 0\right] = \left(1 - \frac{3}{\pi}\right)\int_0^{\pi/2} x\, dx$$

$$= \left(1 - \frac{3}{\pi}\right)\left[\frac{x^2}{2}\right]_0^{\pi/2} = \left(1 - \frac{3}{\pi}\right)\frac{\pi^2}{2} = \frac{\pi - 3}{\cancel{\pi}} \times \frac{\pi^{\cancel{2}}}{2\times 4}$$

$$= \underline{\underline{\frac{\pi(\pi-3)}{8}}}$$

Evaluation of triple integrals over more general regions

Let G be a solid region with lower surface $z = g_1(x,y)$ and upper surface $z = g_2(x,y)$ and let R be the projection of G on the xy plane. Let $f(x,y,z)$ be a continuous functions defined at all points in the region G. Then,

$$\iiint_G f(x,y,z)\,dv = \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\right] dA.$$

Problems

1\. Evaluate $\int_0^1 \int_0^{y^2} \int_{-1}^{z} z\,dx\,dy\,dz$

$$\int_0^1 \int_0^{y^2} \int_{-1}^{z} z\,dx\,dz\,dy = \int_0^1 \int_0^{y^2} z\,[x]_{-1}^{z}\,dz\,dy$$

$$= \int_0^1 \int_0^{y^2} z\,[z+1]\,dz\,dy = \int_0^1 \int_0^{y^2} (z^2+z)\,dz\,dy$$

$$= \int_0^1 \left[\frac{z^3}{3} + \frac{z^2}{2}\right]_0^{y^2} dy = \int_0^1 \left(\frac{y^6}{3} + \frac{y^4}{2}\right) dy$$

$$= \left[\frac{1}{3}\frac{y^7}{7} + \frac{1}{2}\frac{y^5}{5}\right]_0^1 = \frac{1}{21} + \frac{1}{10} = \frac{10+21}{210} = \frac{31}{210}$$

$$= \underline{31/210}$$

2. Evaluate $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x \, dz \, dx \, dy$

$$\int_0^1 \int_{y^2}^1 \int_0^{1-x} x \, dz \, dx \, dy = \int_0^1 \int_{y^2}^1 x \, [z]_0^{1-x} \, dx \, dy$$

$$= \int_0^1 \int_{y^2}^1 x(1-x) \, dx \, dy = \int_0^1 \int_{y^2}^1 (x - x^2) \, dx \, dy$$

$$= \int_0^1 \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{y^2}^1 dy = \int_0^1 \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{y^4}{2} - \frac{y^6}{3}\right) dy$$

$$= \int_0^1 \left(\frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3} \right) dy$$

$$= \left[\frac{1}{6} y - \frac{1}{2} \frac{y^5}{5} + \frac{1}{3} \frac{y^7}{7} \right]_0^1 = \frac{1}{6} - \frac{1}{10} + \frac{1}{21}$$

$$= \frac{35 - 21 + 10}{210} = \frac{\cancel{24}\,4}{\cancel{210}\,35} = \underline{\underline{\frac{4}{35}}}$$

3. Evaluate $\int_0^1 \int_0^z \int_0^{y+z} dz \, dy \, dx$

$$\int_0^1 \int_0^z \int_0^{y+z} dx \, dy \, dz = \int_0^1 \int_0^z (x)_0^{y+z} \, dy \, dz$$

$$= \int_0^1 \int_0^z (y+z) \, dy \, dz = \int_0^1 \left(\frac{y^2}{2} + zy \right)_0^z dz$$

$$\int_0^1 \left(\frac{z^2}{2}+z^2\right)dz = \int_0 3\frac{z^2}{2}dz = \frac{3}{2}\left[\frac{z^3}{3}\right]_0^1$$

$$= \underline{\underline{1/2}}$$

4\. Evaluate $\iiint_V x\,dx\,dy\,dz$ where V is the volume of the tetrahedron bounded by the plane $x=0$, $y=0$, $z=0$, $x+y+z=a$

$x+y+z=a \implies z = a-x-y.$

$x+y=a \implies y = a-x$

$x = a$

$\therefore$ $z : 0 \longrightarrow a-x-y$

$y : 0 \longrightarrow a-x$

$x : 0 \longrightarrow a$

$$\iiint_V x\,dx\,dy\,dz = \int_0^a\int_0^{a-x}\int_0^{a-x-y} x\,dz\,dy\,dx$$

$$= \int_0^a\int_0^{a-x} x\,[z]_0^{a-x-y}\,dz\,dy = \int_0^a\int_0^{a-x} x(a-x-y)\,dy\,dx$$

$$= \int_0^a\int_0^{a-x}\left[ax-x^2-xy\right]dy\,dx = \int_0^a \cancel{\left[axz-x^2z-xyz\right]_0^{a-x} dy}$$

$$= \int_0^a \left[axy-x^2y-x\frac{y^2}{2}\right]_0^{a-x} dx = \int_0^a ax(a-x)-x^2(a-x) - \frac{x}{2}(a-x)^2\,dx$$

$$= \int_0^a a^2x - \cancel{ax^2} - \cancel{ax^2} + x^3 - \frac{x}{2}\left(a^2-2ax+x^2\right)\Big]\,dx$$

$$= \int_0^a \left[a^2x - ax^2 - ax^2 + x^3 - \frac{a^2x}{2} + ax^2 - \frac{x^3}{2}\right] dx$$

$$= \int_0^a \left(\frac{1}{2}a^2x - ax^2 + \frac{1}{2}x^3\right) dx$$

$$= \left[\frac{1}{2}a^2\frac{x^2}{2} - a\frac{x^3}{3} + \frac{1}{2}\frac{x^4}{4}\right]_0^a$$

$$= \frac{1}{4}a^2 \times a^2 - \frac{a}{3}a^3 + \frac{1}{8}a^4$$

$$= a^4\left[\frac{1}{4} - \frac{1}{3} + \frac{1}{8}\right] = a^4\left[\frac{6-8+3}{24}\right]$$

$$= \underline{\underline{\frac{a^4}{24}}}$$

5. Let G be the wedge in the first octant that is cut from the cylindrical solid $y^2+z^2 \leq 1$ by the planes $y = x$ and $x = 0$. Evaluate $\iiint_G z\,dv$.

$x = 0 \quad x = y \implies x: 0 \longrightarrow y$

$y^2 + z^2 = 1 \implies z^2 = 1 - y^2$

$\implies z = \sqrt{1-y^2}$

$z: 0 \longrightarrow \sqrt{1-y^2}$

$y^2 = 1 \implies y = 1$

$\therefore y: 0 \longrightarrow 1$

$$\iiint_G z\,dv = \int_0^1\int_0^y\int_0^{\sqrt{1-y^2}} z\,dz\,dx\,dy$$

$$= \int_0^1\int_0^y \left[\frac{z^2}{2}\right]_0^{\sqrt{1-y^2}} dx\,dy = \int_0^1\int_0^y \frac{1}{2}(1-y^2)\,dx\,dy$$

$$= \cancel{\frac{1}{2}\int_0^1 \left(y - \frac{y^3}{3}\right)\Big|_0^y}\ \frac{1}{2}\int_0^1 (1-y^2)\,(x)_0^y\,dy$$

$$= \frac{1}{2}\int_0^1 (1-y^2)\,y\,dy = \frac{1}{2}\int_0^1 (y-y^3)\,dy$$

$$= \frac{1}{2}\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = \frac{1}{2}\left[\frac{1}{2}-\frac{1}{4}\right]$$

$$= \frac{1}{2}\left[\frac{2-1}{4}\right] = \frac{1}{8}$$

6 Evaluate $\iiint_G xyz\,dv$ where G is the solid in the first octant bounded by the parabolic cylinder $z = 3-x^2$ and the planes $z=0$ $y=x$ and $y=0$.

$z : 0 \longrightarrow 3-x^2$

$y : 0 \longrightarrow x$

$x : 0 \longrightarrow \sqrt{3}$

$3-x^2 = 0$

$x^2 = 3$

$x = \sqrt{3}$

$\boxed{\text{First octant lower limit } 0}$

$$\iiint_G xyz\,dv = \int_0^{\sqrt{3}}\int_0^x\int_0^{3-x^2} xyz\,dz\,dy\,dx$$

$$= \int_0^{\sqrt{3}} \int_0^{x} xy \left(\frac{z^2}{2}\right)_0^{3-x^2} dy\,dx$$

$$= \frac{1}{2} \int_0^{\sqrt{3}} \int_0^{x} xy \left[3-x^2\right]^2 dy\,dx$$

$$= \frac{1}{2} \int_0^{\sqrt{3}} x\left[3-x^2\right]^2 \left[\frac{y^2}{2}\right]_0^{x} dx$$

$$= \frac{1}{4} \int_0^{\sqrt{3}} x \left[3-x^2\right]^2 x^2\, dx$$

$$= \frac{1}{4} \int_0^{\sqrt{3}} x^3 \left[9 - 6x^2 + x^4\right] dx$$

$$= \frac{1}{4} \int_0^{\sqrt{3}} \left(9x^3 - 6x^5 + x^7\right) dx$$

$$= \frac{1}{4} \left[\frac{9x^4}{4} - \frac{6x^6}{6} + \frac{x^8}{8} \right]_0^{\sqrt{3}}$$

$$= \frac{1}{4} \left[\frac{9}{4} \times 9 - 27 + \frac{81}{8} \right]$$

$$= \frac{1}{4} \left[\frac{81 \times 2 - 27 \times 8 + 81}{8} \right]$$

$$= \frac{1}{4} \left[\frac{162 - 216 + 81}{8} \right]$$

$$= \frac{27}{32}$$

Volume Calculated as a Triple Integral

Volume of $G = \iiint_G 1 \cdot dv = \iiint_G dv$

$\boxed{\text{Let } f(x,y,z) = 1}$

Problems

1. Find the volume of the solid in the first octant bounded by the coordinate planes and the plane $x+y+z=1$

Volume $= \iiint_G dv$

$x+y+z=1 \implies z = 1-x-y$

$x+y = 1 \implies y = 1-x$

$x = 1$

Limit

$x : 0 \longrightarrow 1$ [First octant $\implies$ lower limit zero]

$y : 0 \longrightarrow 1-x$

$z : 0 \longrightarrow 1-x-y$.

$$V = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\,dy\,dx = \int_0^1 \int_0^{1-x} [z]_0^{1-x-y} dy\,dx$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx = \int_0^1 \left[y - xy - \frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \int_0^1 \left[(1-x) - x(1-x) - \frac{1}{2}(1-x)^2\right] dx$$

$$= \int_0^1 (1-x)\left[1-x-\frac{1}{2}(1-x)\right]dx$$

$$= \int_0^1 (1-x)\left[\frac{1}{2}-\frac{1}{2}x\right]dx = \frac{1}{2}\int_0^1 (1-x)(1-x)\,dx$$

$$= \frac{1}{2}\int_0^1 (1-x)^2\,dx$$

$$= \frac{1}{2}\left[\frac{(1-x)^3}{-3}\right]_0^1 = -\frac{1}{2\times 3}\left[(1-1)^3-(1-0)^3\right]$$

$$= -\frac{1}{6}[0-1] = \underline{1/6}$$

2. Find the volume of the solid in the first octant bounded by the coordinate planes and the plane $x+2y+z=6$.

$$\text{Volume} = \iiint_G dv$$

$x+2y+z=6 \implies z = 6-x-2y$ [First octant]

$x+2y=6 \implies x = 6-2y$

$2y = 6 \implies y = 3$

$z: 0 \longrightarrow 6-x-2y$

$y: 0 \longrightarrow 3$

$x: 0 \longrightarrow 6-2y$

$$V = \int_0^3 \int_0^{6-2y} \int_0^{6-x-2y} dz\,dx\,dy.$$

$$= \int_0^3 \int_0^{6-2y} [z]_0^{6-x-2y} \, dx\,dy$$

$$= \int_0^3 \int_0^{6-2y} (6-x-2y)\, dx \cdot dy = \int_0^3 \left[6x - \frac{x^2}{2} - 2yx\right]_0^{6-2y} dy$$

$$= \int_0^3 \left[6[6-2y] - \frac{[6-2y]^2}{2} - 2y(6-2y)\right] dy$$

$$= \int_0^3 (6-2y)\left[6 - \frac{1}{2}(6-2y) - 2y\right] dy$$

$$= \int_0^3 (6-2y)\left[(6-2y) - \frac{1}{2}(6-2y)\right] dy$$

$$= \int_0^3 (6-2y)\frac{1}{2}(6-2y)\, dy = \frac{1}{2}\int_0^3 (6-2y)^2 dy$$

$$= \frac{1}{2}\left(\frac{[6-2y]^3}{3 \times -2}\right)_0^3 = \frac{-1}{12}\left[0 - 6^3\right]$$

$$= \frac{1}{\cancel{6} \times 2} 6^{\cancel{3}\,2} = \underline{18}$$

3 . Use triple integral to find the volume of the solid bounded by the surface $y = x^2$ and the planes $y + z = 4$, $z = 0$,

$$\text{Volume} = \iiint_G dv$$

$y + z = 4 \implies z = 4 - y \implies z : 0 \longrightarrow 4 - y$

$y = 4$ and $y = x^2 \implies y : x^2 \longrightarrow 4$

$x^2 = y = 4 \implies x^2 = 4 \implies x = \pm 2 \implies x : -2 \longrightarrow 2$

$$V = \int_{-2}^{2} \int_{x^2}^{4} \int_{0}^{4-y} dz\, dy\, dx = \int_{-2}^{2} \int_{x^2}^{4} [z]_0^{4-y}\, dy\, dx$$

$$= \int_{-2}^{2} \int_{x^2}^{4} (4-y)\, dy\, dx = \int_{-2}^{2} \left[4y - \frac{y^2}{2} \right]_{x^2}^{4} dx$$

$$= \int_{-2}^{2} \left[16 - 8 - \left(4x^2 - \frac{x^4}{2} \right) \right] dx = \int_{-2}^{2} \left(8 - 4x^2 + \frac{x^4}{2} \right) dx$$

$$= \left[8x - 4\frac{x^3}{3} + \frac{x^5}{2 \times 5} \right]_{-2}^{2}$$

$$= \left[8 \times 2 - \frac{4}{3} \times 8 + \frac{1}{10} \times 32 - \left(8 \times -2 - \frac{4}{3}(-8) + \frac{1}{10}(-32) \right) \right]$$

$$= 16 - \frac{32}{3} + \frac{32}{10} + 16 - \frac{32}{3} + \frac{32}{10}$$

$$= 32 - \frac{2 \times 32}{3} + \frac{2 \times 32}{10} = \cancel{32 \left[1 - \frac{1}{5} \right]} =$$

$$= 32 \left[1 - \frac{2}{3} + \frac{1}{5} \right] = 32 \left[\frac{15 - 10 + 3}{15} \right]$$

$$= \frac{256}{15}$$

4. Find the Volume of the Paraboloid of revolution $x^2+y^2=4z$ cut off by the plane $z=4$.

$$\text{Volume} = \iiint dx\,dy\,dz$$

$$x^2+y^2=4z \implies z=\frac{1}{4}(x^2+y^2) \qquad z: \frac{1}{4}(x^2+y^2) \longrightarrow 4$$

$$z=4$$

$$x^2+y^2 = 4\times4=16 \implies x^2+y^2=16$$

Put $x=r\cos\theta$ $\quad y=r\sin\theta$ $\quad dx\,dy = r\,dr\,d\theta$

$$x^2+y^2=r^2$$

$$\theta: 0 \longrightarrow 2\pi \qquad r: 0 \longrightarrow 4$$

$$V = \int_0^{2\pi}\int_0^{4}\int_{\frac{1}{4}(x^2+y^2)}^{4} dz\; r\,dr\,d\theta = \iint [z]_{\frac{1}{4}(x^2+y^2)}^{4}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{4}\left[4-\frac{(x^2+y^2)}{4}\right] r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{4}\left[4-\frac{r^2}{4}\right] r\,dr\,d\theta = \int_0^{2\pi}\int_0^{4}\left(4r-\frac{r^3}{4}\right)dr\,d\theta$$

$$= \int_0^{2\pi}\left[\cancel{4}^{2}\frac{r^2}{\cancel{2}}-\frac{1}{4}\frac{r^4}{4}\right]_0^4 d\theta = \int_0^{2\pi}\left[2\times16-\frac{1}{\cancel{16}}\cancel{256}^{16}\right]d\theta$$

$$= \int_0^{2\pi} 16\,d\theta = 16\,[\theta]_0^{2\pi} = 16\,(2\pi-0)$$

$$= 32\pi$$

5. Using triple integral to find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $z=0$ and $y+z=3$.

Imp

$$\text{Volume } V = \iiint_G dv = \iiint_G dx\,dy\,dz$$

$z=0$ and $y+z=3 \Rightarrow z=3-y \Rightarrow z: 0 \to 3-y$

$\underline{x^2+y^2=4}$.

$x = r\cos\theta \quad y = r\sin\theta \quad dx\,dy = r\,dr\,d\theta$

$x^2+y^2 = r^2$

$\therefore r: 0 \longrightarrow 2 \qquad \theta: 0 \longrightarrow 2\pi$

$$V = \int_0^{2\pi}\int_0^{2}\int_0^{3-y} dz\,(r\,dr\,d\theta)$$

$$= \int_0^{2\pi}\int_0^{2} [z]_0^{3-y}\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^{2} (3-y)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} (3 - r\sin\theta)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^{2} (3r - r^2\sin\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[3\frac{r^2}{2} - \frac{r^3}{3}\sin\theta\right]_0^2 d\theta$$

$$= \int_0^{2\pi} \left[\frac{3}{2}\times 2^2 - \frac{1}{3}\times 8\sin\theta\right] d\theta$$

$$= \int_0^{2\pi} \left[6 - \frac{8}{3}\sin\theta\right] d\theta$$

$$= \left[6\theta - \frac{8}{3}{}^{-}\cos\theta\right]_0^{2\pi}$$

$$= \left[6 \times 2\pi + \frac{8}{3}\cos 2\pi\right] - \left[0 + \frac{8}{3}\cos 0\right]$$

$$= 12\pi + \frac{8}{3} - \frac{8}{3}$$

$$= \underline{\underline{12\pi}}$$

Imp

6. Using triple integrals to find the Volume of the solid within the cylinder $x^2+y^2=9$ and between the planes $z=1$ and $x+z=5$.

Volume $V = \iiint_G dx\,dy\,dz$

$z=1 \qquad x+z=5 \Rightarrow z = 5-x \qquad z: 1 \longrightarrow 5-x$

$\underline{x^2+y^2=9}$

$x = r\cos\theta \qquad y = r\sin\theta \qquad dx\,dy = r\,dr\,d\theta$

$x^2+y^2 = r^2 \qquad r: 0 \longrightarrow 3 \qquad \theta: 0 \longrightarrow 2\pi$

$$V = \int_0^{2\pi}\int_0^{3}\int_1^{5-x} dz\; r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{3} [z]_1^{5-x}\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^{3} (5-x-1)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 (4 - r\cos\theta)\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^3 (4r - r^2\cos\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\overset{2}{\cancel{4}} \frac{r^2}{\cancel{2}} - \frac{r^3}{3}\cos\theta \right]_0^3 d\theta = \int_0^{2\pi} \left[2\times 9 - \frac{\cancel{3}^{\,\cancel{3}\,2}}{\cancel{3}}\cos\theta \right] d\theta$$

$$= \int_0^{2\pi} \left[18 - 9\cos\theta \right] d\theta = \left[18\theta - 9\sin\theta \right]_0^{2\pi}$$

$$= 18\times 2\pi - 9\sin 2\pi - 0 = 36\pi - 0 = \underline{\underline{36\pi}}$$

7 Use triple integral to find the volume of the solid within the cylinder $x^2+y^2=9$ and between the planes $z=1$ and $z=5$

Volume $V = \iiint_G dv$

$\underline{x^2+y^2=9} \Longrightarrow x = r\cos\theta \quad y = r\sin\theta \quad dx\,dy = r\,dr\,d\theta$

$x^2+y^2=r^2 \qquad r: 0 \longrightarrow 3 \qquad \theta: 0 \longrightarrow 2\pi$

$$dv = \int_0^{2\pi}\int_0^3\int_1^5 dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^3 [z]_1^5\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^3 4\, r\, dr\, d\theta = \overset{2}{\cancel{4}} \int_0^{2\pi} \left[\frac{r^2}{\cancel{2}} \right]_0^3 d\theta$$

$$= 2\int_0^{2\pi} 9\, d\theta = 18\,[\theta]_0^{2\pi}$$

$$= \underline{\underline{36\pi}}$$

Triple integrals in cylindrical coordinates

Cylindrical coordinates is a generalization of polar coordinates to three dimensions by superimposing a height (z axis). A point in this system is represented by the triplet (r, θ, z), where r, the radial distance, θ the circumferential distance and z, axial distance.

Converting triple Integrals from rectangular to cylinder Co-ordinates

$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$

dv = Area of the sector shaped region × height

$= (r\,dr\,d\theta)\,dz$

$$\therefore \iiint_G f(x,y,z)\,dv = \iiint f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz$$

Problems

1. Use triple integration in cylindrical coordinates to find the volume of the solid G that is bounded above by the hemisphere $z = \sqrt{25 - x^2 - y^2}$, below

by the xy plane, and laterally by the cylinder $x^2+y^2=9$.

$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$

$x^2+y^2 = r^2$

$z = \sqrt{25-x^2-y^2} = \sqrt{25-(x^2+y^2)} = \sqrt{25-r^2}$

$z : 0 \longrightarrow \sqrt{25-r^2}$

$x^2+y^2 = 9 \implies r : 0 \longrightarrow 3 \qquad \theta : 0 \longrightarrow 2\pi$

Volume $V = \iiint\limits_G dV = \iiint\limits_G dz\,dy\,dx$

$$= \int_0^{2\pi}\int_0^3\int_0^{\sqrt{25-r^2}} dz\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^3 r\,[z]_0^{\sqrt{25-r^2}}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 r\sqrt{25-r^2}\,dr\,d\theta \qquad \text{put } 25-r^2=t$$
$$-2r\,dr = dt$$

$$= \int_0^{2\pi}\int \sqrt{t}\left(\frac{-dt}{2}\right)d\theta$$

$$= -\frac{1}{2}\int_0^{2\pi}\left(\frac{t^{3/2}}{3/2}\right)d\theta = -\frac{1}{2}\times\frac{2}{3}\int_0^{2\pi}\left[(25-r^2)^{3/2}\right]_0^3 d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi}\left[(25-9)^{3/2}-(25)^{3/2}\right]d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi}\left[(16)^{3/2}-(25)^{3/2}\right]d\theta = -\frac{1}{3}\int_0^{2\pi}(4^3-5^3)\,d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi} -61\, d\theta = \frac{61}{3}\left[\theta\right]_0^{2\pi}$$

$$= \frac{61}{3} \times 2\pi = \underline{\frac{122\pi}{3}}$$

2. Use Cylindrical Co-ordinates to evaluate

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} x^2\, dz\, dy. dx$$

$z: 0 \longrightarrow 9-x^2-y^2$

$z=0$ and $z = 9-x^2-y^2 = 9-(x^2+y^2) = 9-r^2$

$y: -\sqrt{9-x^2} \longrightarrow \sqrt{9-x^2}$

$y = \sqrt{9-x^2} \Longrightarrow y^2 = 9-x^2 \Longrightarrow x^2+y^2=9$

$r: 0 \longrightarrow 3 \qquad \theta: 0 \longrightarrow 2\pi$

$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$

$dv = dz\, r\,dr\,d\theta$

$z: 0 \longrightarrow 9-r^2 \qquad r: 0 \longrightarrow 3 \qquad \theta: 0 \longrightarrow 2\pi$

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} x^2\, dz\, dy\, dx = \int_0^{2\pi}\int_0^3\int_0^{9-r^2} r^2\cos^2\theta\, dz\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 r^3\cos^2\theta \left[z\right]_0^{9-r^2} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 r^3\cos^2\theta \left[9-r^2\right] dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{3} \cos^2\theta \left[9r^3 - r^5\right] dr\,d\theta$$

$$= \int_0^{2\pi} \cos^2\theta \left[\frac{9r^4}{4} - \frac{r^6}{6}\right]_0^3 d\theta$$

$$= \int_0^{2\pi} \cos^2\theta \left[\frac{9}{4}\times 81 - \frac{1}{\cancel{6}3\times 2} 3^{\cancel{6}5} - 0\right]$$

$$= \int_0^{2\pi} \cos^2\theta \left[\frac{729}{4} - \frac{243}{2}\right] d\theta$$

$$= \int_0^{2\pi} \frac{243}{4} \cos^2\theta\, d\theta = \frac{243}{4}\int_0^{2\pi} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{243}{8}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{243}{8}\left[2\pi + \frac{\sin 2\pi}{2} - 0\right]$$

$$= \frac{243}{\cancel{8}4} \times \cancel{2}\pi = \frac{243\pi}{4}$$

3 Use cylindrical co-ordinates to find the volume of the solid enclosed by the paraboloid $z = x^2+y^2$ and the plane $z=16$.

$$\text{Volume } V = \iiint_G dV = \iiint_G dz\,dy\,dx$$

$z : x^2+y^2 \longrightarrow 16$.

$x = r\cos\theta \qquad y = r\sin\theta \qquad dz\,dy\,dx = dz\,r\,dr\,d\theta$

~~$V = \iiint$ ...~~ $x^2+y^2 = z \Rightarrow x^2+y^2 = 16$

$r : 0 \longrightarrow 4 \qquad \theta : 0 \longrightarrow 2\pi$.

$$V = \int_0^{2\pi}\int_0^{4}\int_{x^2+y^2}^{16} dz\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^{4} r\,[z]_{x^2+y^2}^{16}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{4} r\,[16-(x^2+y^2)]\,dr\,d\theta = \int_0^{2\pi}$$

$$= \int_0^{2\pi}\int_0^{4} r\,[16-r^2]\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{4} (16r-r^3)\,dr\,d\theta = \int_0^{2\pi}\left[\frac{\cancel{16}^{8}\,r^2}{\cancel{2}}-\frac{r^4}{4}\right]_0^4 d\theta$$

$$= \int_0^{2\pi}\left[8\times16-\frac{1}{\cancel{4}}\,4^{\cancel{4}\,3}-0\right]d\theta$$

$$= \int_0^{2\pi}(8\times16-64)\,d\theta = \int_0^{2\pi} 64\,d\theta$$

$$= 64\,[\theta]_0^{2\pi}$$

$$= 64\times 2\pi$$

$$= 128\pi$$

4. Evaluate $\iiint z\,(x^2+y^2+z^2)\,dx\,dy\,dz$ through the volume of the cylinder $x^2+y^2=a^2$ intercepted by the planes $z=0$ and $z=b$.

$z : 0 \longrightarrow h$

$x^2+y^2=a^2 \implies$

$r : 0 \longrightarrow a \qquad \theta : 0 \longrightarrow 2\pi \qquad x^2+y^2=r^2$

$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z \qquad dx\,dy\,dz = dz\,r\,dr\,d\theta$

$$\iiint z(x^2+y^2+z^2)\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{a}\int_0^{h} z\,(r^2+z^2)\,dz\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{a}\int_0^{h} r(r^2 z + z^3)\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{a} r\left[r^2\frac{z^2}{2}+\frac{z^4}{4}\right]_0^h dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{a} r\left[\frac{r^2h^2}{2}+\frac{h^4}{4}\right]dr\,d\theta = \int_0^{2\pi}\int_0^{a}\left(\frac{\cancel{2}r^3h^2+rh^4}{4}\right)dr$$

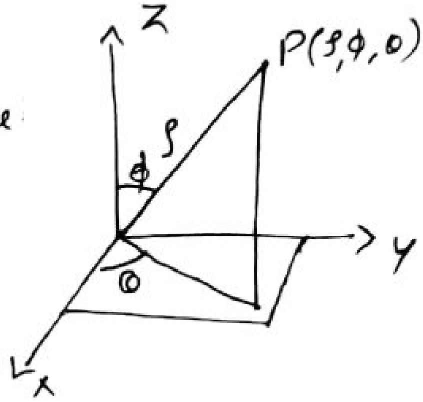
$$= \frac{1}{4}\int_0^{2\pi}\left[\cancel{2}h^2\frac{r^4}{\cancel{4}\,2}+h^4\frac{r^2}{2}\right]_0^a d\theta = \frac{1}{4\times 2}\int_0^{2\pi}\left(h^2a^4+h^4a^2\right)d\theta$$

$$= \frac{1}{8}\left[h^2a^4+h^4a^2\right]\left[\theta\right]_0^{2\pi}$$

$$= \frac{1}{\cancel{8}\,4}a^2h^2(a^2+h^2)\,\cancel{2}\pi$$

$$= \underline{\underline{\frac{\pi a^2h^2(a^2+h^2)}{4}}}$$

Triple Integrals in Spherical co-ordinates



Spherical co-ordinate represent a point P in space by ordered triplets (ρ, ϕ, θ) in which

1) ρ is the distance from P to the origin
2) ϕ is the angle $\vec{OP}$ makes with the +ve z-axis $(0 \le \phi \le \pi)$
3) θ is the angle from cylindrical co-ordinates. $(0 \le \theta \le 2\pi)$

Converting triple integrals from Rectangular to spherical co-ordinates.

$x = \rho \sin\phi \cos\theta \quad y = \rho \sin\phi \sin\theta \quad z = \rho \cos\phi$.

$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$$\iiint_G f(x,y,z)\,dV = \iiint f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Problems

1. Find the volume of the ice cream cone G cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \pi/3$.

$\rho : 0 \longrightarrow 1$

$\phi : 0 \longrightarrow \pi/3$

$\theta : 0 \longrightarrow 2\pi$

Volume $V = \iiint_G dv$

$$= \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \sin\phi \left(\frac{\rho^3}{3}\right)_0^1 d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \sin\phi \, \frac{1}{3} \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \left(-\cos\phi\right)_0^{\pi/3} d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} - \left[\cos\frac{\pi}{3} - \cos 0\right] d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} - \left[\frac{1}{2} - 1\right] d\theta = \frac{-1}{3} \times \frac{-1}{2} \left[\theta\right]_0^{2\pi}$$

$$= \frac{1}{6} \, 2\pi = \underline{\pi/3}$$

2. Use Spherical Coordinates to evaluate

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$$

$z : 0 \longrightarrow \sqrt{4-x^2-y^2}$

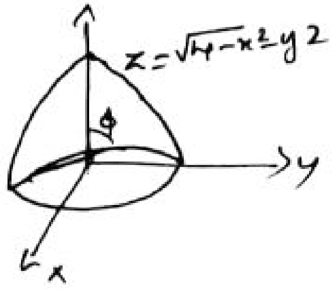
$z=0 \qquad z=\sqrt{4-x^2-y^2} \Longrightarrow z^2=4-x^2-y^2$

$x^2+y^2+z^2=4$

$\rho^2=4 \Longrightarrow \rho=2$

$\rho : 0 \longrightarrow 2 \qquad \phi : 0 \longrightarrow \pi/2$

$x^2+y^2=4 \Longrightarrow \theta : 0 \longrightarrow 2\pi$



$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$$

$$=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2} \rho^2\cos^2\phi\;\rho\;\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2} \rho^5\cos^2\phi\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi/2}\left(\frac{\rho^6}{6}\right)_0^2\cos^2\phi\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi/2}\frac{1}{6}\,2^6\cos^2\phi\sin\phi\,d\phi\,d\theta$$

$$=\frac{64}{6}\int_0^{2\pi}\int t^2(-dt)\,d\phi\,d\theta$$

Put $\cos\phi=t$

$-\sin\phi\,d\phi=dt$

$$= \frac{-64}{6} \int_0^{2\pi} \left[\frac{t^3}{3}\right] d\theta$$

$$= \frac{-64}{6 \times 3} \int_0^{2\pi} \left[\cos^3\phi\right]_0^{\pi/2} d\theta$$

$$= \frac{-32}{9} \int_0^{2\pi} \left[\cos^3\frac{\pi}{2} - \cos 0\right] d\theta$$

$$= \frac{-32}{9} \int_0^{2\pi} (0 - 1)\, d\theta$$

$$= \frac{32}{9} \left[\theta\right]_0^{2\pi}$$

$$= \frac{64\pi}{9}$$

3. Use spherical co-ordinates to find the volume of the solid G bounded above by the sphere $x^2+y^2+z^2=16$ and below by the cone $z = \sqrt{x^2+y^2}$.

$$x^2+y^2+z^2=16$$

$$\rho^2 = 16 \qquad \rho = 4$$

$$\rho : 0 \longrightarrow 4$$

$$z = \sqrt{x^2+y^2} \Longrightarrow \cancel{\rho\cos\phi} = \sqrt{\rho^2\sin^2\phi\cos^2\theta +}$$

$$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta}$$

$$= \sqrt{\rho^2\sin^2\phi\,(\cos^2\theta+\sin^2\theta)}$$

$$\rho\cos\phi = \rho\sin\phi \Longrightarrow \frac{\sin\phi}{\cos\phi} = 1 \longrightarrow$$

$$\tan\phi = 1$$

$\phi = \tan^{-1}(1) = \pi/4$

$\phi : 0 \longrightarrow \pi/4$

~~$x^2+y^2 = z^2$~~

$\theta : 0 \longrightarrow 2\pi$

$$\text{Volume } V = \iiint_G dv = \int_0^{2\pi}\int_0^{\pi/4}\int_0^4 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \sin\phi \left[\frac{\rho^3}{3}\right]_0^4 d\phi \, d\theta$$

$$= \frac{64}{3}\int_0^{2\pi} \left[-\cos\phi\right]_0^{\pi/4} d\theta = -\frac{64}{3}\int_0^{2\pi}\left(\cos\frac{\pi}{4} - \cos 0\right) d\theta$$

$$= -\frac{64}{3}\int_0^{2\pi}\left(\frac{1}{\sqrt{2}} - 1\right) d\theta$$

$$= \frac{64}{3}\left(1 - \frac{1}{\sqrt{2}}\right)\left[\theta\right]_0^{2\pi}$$

$$= \frac{64}{3}\left(1 - 1/\sqrt{2}\right) 2\pi$$

$$= \frac{64}{3}\left(2 - \sqrt{2}\right)$$