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MODULE - II

MULTIVARIABLE CALCULUS - DIFFERENTIATION

Topics

Concept of limit and continuity of functions of two variables - partial derivatives - Differentials - Local linear approximations - chain rule - total derivative - Relative Maxima and Minima - Absolute Maxima and Minima on closed and bounded set.

Functions of two or more Variables

If 3 Variables x, y, z are so related that the values of z depend upon the value of x and y, then z is called a function of two variables x and y and is denoted by $z = f(x, y)$. z is called dependent Variable and x and y are called independent Variables.

The function $z = f(x, y)$ represent a Surface.

Limit of functions of two Variables

A function $f(x,y)$ is said to have a limit at the point (x_0,y_0) if for every given $\epsilon>0$ there is a real number $\delta>0$ such that whenever $|(x,y)-(x_0,y_0)|<\delta$, we have

$$|f(x,y)-L|<\epsilon$$

That is, when the distance between the points (x,y) and (x_0,y_0) is less than δ, the distance between $f(x,y)$ and L is no more than ϵ.

Symbolically this is expressed as

$$\boxed{\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L.}$$

Example

1. $$\lim_{(x,y)\to(1,4)} [5x^3y^2-9] = \lim_{(x,y)\to(1,4)} 5x^3y^2 - \lim_{(x,y)\to(1,4)} 9$$

$$= 5\times1^3\times4^2-9 = 80-9 = \underline{\underline{71}}$$

2. $$\lim_{(x,y)\to(-1,2)} \left[\frac{xy}{x^2+y^2}\right] = \frac{-1\times2}{1^2+2^2} = \frac{-2}{1+4} = \underline{\underline{\frac{-2}{5}}}$$

Continuity

A function $f(x,y)$ is said to be continuous at (x_0,y_0) if $f(x_0,y_0)$ is defined and if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

Example

1. consider the function $f(x,y) = 3x^2y^5$

The polynomials $3x^2$ and y^5 are continuous at every real number, the function $f(x,y) = 3x^2y^5$ is continuous at every point (x,y) in the x-y plane.

2. $f(x,y) = \dfrac{xy}{1-x^2-y^2}$,

is a quotient of continuous function, it is continuous except where $1-x^2-y^2=0$

$\Rightarrow x^2+y^2=1$.

$\Rightarrow f(x,y)$ is continuous everywhere except on the circle $x^2+y^2=1$

Partial Derivatives of Function of Two Variables

The derivative $z = f(x,y)$ when y is kept constant and x is allowed to change (vary) is called first order partial derivative of $z = f(x,y)$ with respect to x and is given by

$$\boxed{\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}},$$

Provided the limit is finite and unique.
$\frac{\partial z}{\partial x}$ is also denoted $f_x(x,y)$ or $\frac{\partial f}{\partial x}$

The derivative of $z = f(x,y)$ when x is kept constant and y is allowed to vary is called first order partial derivative of $z = f(x,y)$ with respect to y and is given by

$$\boxed{\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}},$$

Provided the limit is finite and unique.

$\frac{\partial z}{\partial y}$ also denoted as $f_y(x,y)$ or $\frac{\partial f}{\partial y}$.

Partial derivatives of functions with more than two Variables

If $f(x_1, x_2 \cdots x_n)$ is a function of n Variables, there are 'n' Partial derivatives of f, each of which is obtained by holding n-1 of the Variables fixed and differentiating the function f with respect to the remaining Variables.

If $w = f(x_1, x_2 \cdots x_n)$ then the partial derivatives are denoted by

$$\frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2} \cdots \frac{\partial w}{\partial x_n}$$

where $\frac{\partial w}{\partial x_p}$ is obtained by holding all Variables except V_i fixed and differentiating w. r. to x_i.

Eg A function $f(x,y,z)$ is a 3 Variables there are three partial derivatives,

$f_x(x,y,z)$, $f_y(x,y,z)$, $f_z(x,y,z)$

The Partial derivative f_x is calculated by holding y and z constant and differentiating with respect to x. Similarly f_y and f_z.

Notes:

1. $\frac{\partial z}{\partial x} = f_x(x_0, y_0)$ = Rate of change of z with respect to x along the curve 'c' at the point (x_0, y_0).

2. $\frac{\partial z}{\partial y} = f_y(x_0, y_0)$ = Rate of change of z with respect to y along the curve 'c' at the point (x_0, y_0)

3. $\frac{\partial z}{\partial x} = f_x(x_0, y_0)$ is called the Slope of the Surface $z = f(x,y)$ in the x-direction at (x_0, y_0).

4. $\frac{\partial z}{\partial y} = f_y(x_0, y_0)$ is the Slope of the Surface $z = f(x,y)$ in the y-direction at (x_0, y_0)

Problems

1. Find $f_x(x,y)$ and $f_y(x,y)$ for

$$f(x,y) = 2x^3y^2 + 2y + 4x$$

$$f_x(x,y) = 2y^2(3x^2) + 0 + 4 \cdot 1$$

$$= \underline{6x^2y^2 + 4}$$

$$f_y(x,y) = 2x^3(2y) + 2\cdot 1 + 4\times 0$$
$$= \underline{\underline{4x^3y + 2}}$$

2. If $z = \sin(y^2 - 4x)$, find

(a) the rate of change of z with respect to x at the point $(3,1)$ with y held fixed.

(b) the rate of change of z w.r.to y at the point $(3,1)$ with x held fixed.

$$z = \sin(y^2 - 4x)$$

Rate of change of z w.r.to $x = \frac{\partial z}{\partial x}$

$$= \cos(y^2 - 4x) \times -4$$
$$= \underline{\underline{-4\cos(y^2 - 4x)}}$$

$$\left(\frac{\partial z}{\partial x}\right)_{(3,1)} = -4\cos(1 - 4\times 3)$$
$$= -4\cos(-11)$$
$$= \underline{\underline{-4\cos 11}}$$

$\boxed{\begin{array}{l}\cos(-\theta) = \cos\theta \\ \& \sin(-\theta) = -\sin\theta\end{array}}$

Rate of change of z w.r.to $y = \frac{\partial z}{\partial y}$

$$= \cos(y^2 - 4x)\, 2y$$
$$= 2y\cos(y^2 - 4x)$$

$$\left(\frac{\partial z}{\partial y}\right)_{(3,1)} = 2\times 1\cos(1 - 4\times 3) = 2\cos(-11)$$
$$= \underline{\underline{2\cos 11}}$$

3. Find the rate of change of $f(x,y) = xe^{-y} + 5y$
u.a. with respect to x at the point $(4,0)$ with y held fixed

OR

u.a. Find the slope of the surface $z = xe^{-y} + 5y$ in the y-direction at the point $(4,0)$.

$z = f(x,y) = xe^{-y} + 5y$

Rate of change of z w.r. to x =

Slope of the surface in the y-direction

$= \frac{\partial z}{\partial y} = x e^{-y}(-1) + 5$

$\frac{\partial z}{\partial y} = -xe^{-y} + 5$

$\left(\frac{\partial z}{\partial y}\right)_{(4,0)} = -4e^{0} + 5 = -4 + 5 = \underline{\underline{1}}$

4. Let $f(x,y) = \sqrt{3x+2y}$

(a) Find the slope of the surface $z = f(x,y)$ in the x-direction at the point $(2,5)$

(b) Find the slope of the surface $z = f(x,y)$ in the y-direction at the point $(2,5)$

$z = \sqrt{3x+2y}$

Slope of the surface in the x-direction

$= \frac{\partial z}{\partial x} = \frac{1}{2\sqrt{3x+2y}} \times 3$

$$= \frac{3}{2\sqrt{3x+2y}}$$

$$\left(\frac{\partial z}{\partial x}\right)_{(2,5)} = \frac{3}{2\sqrt{3\times 2+2\times 5}} = \frac{3}{2\sqrt{6+10}} = \frac{3}{2\times 4} = \underline{\underline{3/8}}$$

Slope of the Surface in the y-direction

$$= \frac{\partial z}{\partial y} = \frac{1}{2\sqrt{3x+2y}} \times 2 = \frac{1}{\sqrt{3x+2y}}$$

$$\left(\frac{\partial z}{\partial y}\right)_{(2,5)} = \frac{1}{\sqrt{3\times 2+2\times 5}} = \frac{1}{\sqrt{16}} = \underline{\underline{\frac{1}{4}}}$$

5. Find the slope of the surface $z = xy^2$ in the x-direction at the point $(2,3)$

Slope of the surface in the x-direction

$$= \frac{\partial z}{\partial x} = y^2$$

$$\left(\frac{\partial z}{\partial x}\right)_{(2,3)} = \underline{\underline{3^2 = 9}}$$

6. Find the slope of the function $f(x,y) = x\cos(xy) + y\sin(xy)$ at $(\pi, 1)$ along the x-direction.

Slope of the surface in the x-direction

$$= f_x(x,y) = x\,{-\sin(xy)}y + \cos(xy)\cdot 1 + y\cdot\cos(xy)\cdot y$$

$$= -xy\sin(xy) + \cos(xy) + y^2\cos(xy)$$

$$\left[f_x(x,y)\right]_{(\pi,1)} = -\pi \sin(\pi) + \cos(\pi) + 1\cdot\cos(\pi)$$
$$= 0 - 1 - 1 = \underline{-2}$$

7. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = \cos(xy^3)$

$z = \cos(xy^3)$

$$\frac{\partial z}{\partial x} = -\sin(xy^3)\cdot y^3 = \underline{-y^3 \sin(xy^3)}$$

$$\frac{\partial z}{\partial y} = -\sin(xy^3)(x\,3y^2)$$
$$= \underline{-3xy^2 \sin(xy^3)}$$

8. If $u = \frac{x^3+y^3}{x-y}$, find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$

$$\frac{\partial u}{\partial x} = \frac{(x-y)(3x^2) - (x^3+y^3)\cdot 1}{(x-y)^2}$$

$$\boxed{\frac{\text{Denominator}\ \frac{d}{d}(N_r) - N_r \frac{d}{dx}(D_r)}{D_r^2}}$$

$$= \frac{3x^3 - 3x^2y - x^3 - y^3}{(x-y)^2}$$

$$= \underline{\frac{2x^3 - 3x^2y - y^3}{(x-y)^2}}$$

$$\frac{\partial u}{\partial y} = \frac{(x-y)(3y^2) - (x^3+y^3)(-1)}{(x-y)^2}$$

$$= \frac{3xy^2 - 3y^3 + x^3 + y^3}{(x-y)^2} = \underline{\frac{x^3 + 3xy^2 - 2y^3}{(x-y)^2}}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=\frac{2x^3-3x^2y-y^3}{(x-y)^2}+\frac{x^3+3xy^2-2y^3}{(x-y)^2}$$

$$=\frac{3x^3-3x^2y+3xy^2-3y^3}{(x-y)^2}$$

9. If $z=f(x^2-y^2)$, Show that $y\frac{\partial z}{\partial x}+x\frac{\partial z}{\partial y}=0$

$$z=f(x^2-y^2)$$

$$\frac{\partial z}{\partial x}=f'(x^2-y^2)\,2x$$

$$\frac{\partial z}{\partial y}=f'(x^2-y^2)\,(-2y)$$

$$y\frac{\partial z}{\partial x}+x\frac{\partial z}{\partial y}=2xy\,f'(x^2-y^2)-2xy\,f'(x^2-y^2)$$

$$=\underline{0}$$

10. If $f(x,y,z)=x^3y^2z^4+2xy+z$, find $f_z(-1,1,2)$.

$$f_z(x,y,z)=x^3y^2\,4z^3+0+1=4x^3y^2z^3+1$$

$$f_z(-1,1,2)=4(-1)^3\times1\times2^3+1=-4\times8+1=\underline{\underline{-31}}$$

11. If $f(x,y,z)=z\ln(x^3y\cos z)$, find f_x, f_y, f_z.

$$f_x=z\frac{1}{x^3y\cancel{\cos z}}3x^2y\cancel{\cos z}=\frac{3z}{x}$$

$$f_y=z\frac{1}{x^3y\cos z}x^3\cos z=\frac{z}{y}$$

$$f_z = z \frac{1}{x^3 y \cos z} x^3 y(-\sin z) + \ln(x^3 y \cos z)$$

$$= -z \tan z + \ln(x^3 y \cos z)$$

12. If $w = \log(\tan x + \tan y + \tan z)$ then prove that $\sin 2x \frac{\partial w}{\partial x} + \sin 2y \frac{\partial w}{\partial y} + \sin 2z \frac{\partial w}{\partial z} = 2$.

$$\frac{\partial w}{\partial x} = \frac{1}{\tan x + \tan y + \tan z} \sec^2 x$$

$$\boxed{\frac{d}{dx} \tan x = \sec^2 x}$$

$$\sin 2x \frac{\partial w}{\partial x} = \sin 2x \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

$$= \frac{2 \sin x \cancel{\cos x} \frac{1}{\cos^{\cancel{2}} x}}{\tan x + \tan y + \tan z}$$

$$= \frac{2 \tan x}{\tan x + \tan y + \tan z}$$

Similarly

$$\sin 2y \frac{\partial w}{\partial y} = \frac{2 \tan y}{\tan x + \tan y + \tan z}$$

$$\sin 2z \frac{\partial w}{\partial z} = \frac{2 \tan z}{\tan x + \tan y + \tan z}$$

$$\sin 2x \frac{\partial w}{\partial x} + \sin 2y \frac{\partial w}{\partial y} + \sin 2z \frac{\partial w}{\partial z} = \frac{2(\cancel{\tan x + \tan y + \tan z})}{\cancel{\tan x + \tan y + \tan z}}$$

$$= 2$$

Implicit Partial Differentiation

If the equation $f(x,y,z)=c$ define z implicitly as a differentiable function of x and y and if $\frac{\partial f}{\partial z} \neq 0$ then

$$\boxed{\frac{\partial z}{\partial x} = \frac{-\partial f/\partial x}{\partial f/\partial z} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{-\partial f/\partial y}{\partial f/\partial z}}$$

Problems

1. Consider the sphere $x^2+y^2+z^2=1$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$

$$f(x,y,z) = x^2+y^2+z^2=1$$

$$\frac{\partial z}{\partial x} = \frac{-\partial f/\partial x}{\partial f/\partial z} = \frac{-2x}{2z} = \frac{-x}{z}$$

$$\left(\frac{\partial z}{\partial x}\right)_{\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)} = \frac{-2/3}{2/3} = -1$$

$$\frac{\partial z}{\partial y} = \frac{-\partial f/\partial y}{\partial f/\partial z} = \frac{-2y}{2z} = \frac{-y}{z}$$

$$\left(\frac{\partial z}{\partial y}\right)_{\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)} = \frac{-1/3}{2/3} = -1/2$$

2. Find the slope of the sphere $x^2+y^2+z^2=1$ in the y-direction at the points $(\frac{2}{3}, \frac{1}{3}, \frac{-2}{3})$

Slope of the sphere in the y-direction $= \frac{\partial z}{\partial y}$

$x^2+y^2+z^2=1$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{-\partial f/\partial y}{\partial f/\partial z} = -\frac{2y}{2z} = -y/z$$

$$\left(\frac{\partial z}{\partial y}\right)_{(\frac{2}{3}, 1/3, -2/3)} = -\frac{1/3}{-2/3} = \underline{\underline{1/2}}$$

3. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(0,0,0)$ if

$x^3+z^2+ye^{xz}+z\cos y=0$

$f(x,y,z) = x^3+z^2+ye^{xz}+z\cos y=0$

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} = -\frac{3x^2+ye^{xz}\cdot z}{2z+ye^{xz}\cdot x+\cos y}$$

$$= -\frac{3x^2+yze^{xz}}{2z+xye^{xz}+\cos y}$$

$$\left(\frac{\partial z}{\partial x}\right)_{(0,0,0)} = \frac{0}{\cos 0} = \frac{0}{1} = \underline{\underline{0}}$$

$$\frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z} = -\frac{[e^{xz}+z(-\sin y)]}{2z+ye^{xz}\cdot x+\cos y}$$

$$\left(\frac{\partial z}{\partial y}\right)_{(0,0,0)} = -\frac{[e^0+0]}{0+0+\cos 0} = \underline{\underline{-1}}$$

Higher Order Partial Derivatives

The first order partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are the functions of x and y, they can be again differentiated partially with respect to x as well as y. These are called second order partial derivatives. They are denoted by

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$$

The f_{xy} and f_{yx} are called the mixed second order partial derivatives.

Note

1. Let f be a function of two Variables. If f_{xy} and f_{yx} are Continuous on Some open disk, then $f_{xy} = f_{yx}$ on that disk.

2. Third order, fourth order and higher

Order partial derivatives can be obtained by successive differentiation.

Some possibilities are

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial x^3} = f_{xxx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial^2 y \partial x} = f_{xyy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial y^3}\right) = \frac{\partial^4 f}{\partial y^4} = f_{yyyy}$$

<u>Problems</u>

1. Find the second order partial derivatives of $f(x,y) = x^2y^3 + x^4y$

$$\frac{\partial f}{\partial x} = f_x = 2xy^3 + 4x^3y$$

$$\frac{\partial f}{\partial y} = f_y = 3x^2y^2 + x^4$$

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = 2y^3.(1) + 4y(3x^2)$$

$$= \underline{2y^3 + 12x^2y}$$

$$\frac{\partial^2 f}{\partial y^2} = f_{yy} = 3x^2(2y) + 0 = 6x^2y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 3y^2(2x) + 4x^3$$

$$= 6xy^2 + 4x^3$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2x(3y^2) + 4x^3$$

$$= \underline{6xy^2 + 4x^3}$$

2. Prove that $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$ where $z = x^2 y$

$z = x^2 y$

$\frac{\partial z}{\partial x} = 2xy$

$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(2xy)$

$= 2x$

$\frac{\partial z}{\partial y} = x^2$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$

$= \frac{\partial}{\partial x}(x^2) = 2x$

$\therefore \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$

3. Show that $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$ if $z = x^2 y + 5y^3$

$\frac{\partial z}{\partial x} = 2xy$

$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$

$= 2x$

$\frac{\partial z}{\partial y} = x^2 + 15y^2$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$

$= 2x$

$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$

4. If $u = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$, find $\frac{\partial^2 u}{\partial y \partial x}$

$$\frac{\partial u}{\partial x} = x^2 \frac{1}{1+(y/x)^2}(-y/x^2) + \tan^{-1}(y/x)\,2x - y^2 \frac{1}{1+\left(\frac{x}{y}\right)^2}(1/y)$$

$$\boxed{\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}}$$

$$= \frac{-y}{\frac{x^2+y^2}{x^2}} + 2x\tan^{-1}(y/x) - \frac{y}{\frac{y^2+x^2}{y^2}}$$

$$= -\frac{x^2 y}{x^2+y^2} + 2x\tan^{-1}(y/x) - \frac{y^3}{x^2+y^2}$$

$$= -\frac{(x^2+y^2)y}{x^2+y^2} + 2x\tan^{-1}(y/x)$$

$$= -y + 2x\tan^{-1}(y/x)$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = -1 + 2x\frac{1}{1+\left(\frac{y}{x}\right)^2}(1/x)$$

$$= -1 + 2\cdot\frac{x^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

5. Show that the equation $u(x,t) = \sin(x-ct)$, satisfies wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$

$$u(x,t) = \sin(x-ct)$$

$$\frac{\partial u}{\partial x} = \cos(x-ct)$$

$$\frac{\partial^2 u}{\partial x^2} = -\sin(x-ct)$$

$$\frac{\partial u}{\partial t} = -c\cos(x-ct)$$

$$\frac{\partial^2 u}{\partial t^2} = -c\left[-\sin(x-ct)\right](-c) = -c^2\sin(x-ct)$$

$$= c^2\left[-\sin(x-ct)\right]$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

6. Show that the function $f(x,y) = e^x\sin y + e^y\sin x$ satisfies the Laplace equation $f_{xx} + f_{yy} = 0$

$$f_x = e^x\sin y + e^y\cos x$$

$$f_y = e^x\cos y + e^y\sin x$$

$$f_{xx} = e^x\sin y - e^y\sin x$$

$$f_{yy} = e^x - \sin y + e^y\sin x$$

$$f_{xx} + f_{yy} = e^x\sin y - e^y\sin x - e^x\sin y + e^y\sin x$$

$$= 0$$

7. If $u = \log(x^3+y^3+z^3-3xyz)$, show that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz}(3y^2-3xz)$$

$$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz}(3z^2-3xy)$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3x^2-3yz+3y^2-3xz+3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-xz)}{(x+y+z)(x^2+y^2+z^2-xy-yz-xz)}$$

$$= \frac{3}{x+y+z}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) = 3\left[\frac{-1}{(x+y+z)^2}\cdot 1\right]$$

$$= \frac{-3}{(x+y+z)^2}$$

Similarly $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right) = \frac{-3}{(x+y+z)^2}$

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right) = \frac{-3}{(x+y+z)^2}$$

$$\therefore \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$= \frac{-3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}$$

$$= -9/(x+y+z)^2$$

8. Show that the function $f(x,y) = 2\tan^{-1}(y/x)$ satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

$$\frac{\partial f}{\partial x} = 2 \cdot \frac{1}{1+\frac{y^2}{x^2}} \left(-y/x^2\right)$$

$$\left[\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}\right]$$

$$= \frac{2}{\frac{x^2+y^2}{\cancel{x^2}}} \frac{-y}{\cancel{x^2}} = \underline{\underline{\frac{-2y}{x^2+y^2}}}$$

$$\frac{\partial f}{\partial y} = 2 \frac{1}{1+y^2/x^2} \frac{1}{x} = \frac{2}{\frac{x^2+y^2}{\cancel{x^2}}} \frac{1}{\cancel{x}} = \frac{2x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = -2y\left(\frac{-1}{(x^2+y^2)^2} \cdot 2x\right) = \frac{4xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = 2x\left(\frac{-1}{(x^2+y^2)^2} 2y\right) = \frac{-4xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y_2} = \frac{4xy - 4xy}{(x^2+y^2)^2} \underline{\underline{= 0}}$$

9. If $z = (3x-2y)^4$, find $\frac{\partial^4 z}{\partial x \partial y^3}$

$$\frac{\partial z}{\partial y} = 4(3x-2y)^3 \times (-2)$$

$$\frac{\partial^2 z}{\partial y^2} = -8 \times 3(3x-2y)^2 \times (-2)$$

$$= 48(3x-2y)^2$$

$$\frac{\partial^3 z}{\partial y^3} = 48 \times 2\,(3x - 2y)(-2)$$
$$= -192\,(3x - 2y)$$

$$\frac{\partial^4 z}{\partial x\,\partial y^3} = -192 \times 3 = \underline{\underline{-576}}$$

10. If $z = x^y$, then find $\frac{\partial^2 z}{\partial x \partial y}$

$z = x^y$

$$\frac{\partial z}{\partial y} = x^y \log x$$

Diff: w. r. to $y \Rightarrow x$ constant
$\therefore \frac{d}{dx}(a^x) = a^x \log a$

$$\frac{\partial^2 z}{\partial x \partial y} = x^y \times \frac{1}{x} + \log x \; y x^{y-1}$$
$$= x^y \times x^{-1} + y x^{y-1} \log x$$
$$= x^{y-1} + y x^{y-1} \log x$$
$$= \underline{\underline{x^{y-1}\left[1 + y \log x\right]}}$$

Diff: w.r.to x
$\frac{d}{dx}(x^n) = n x^{n-1}$

11. Given $f = e^x \sin y$, Show that the function Satisfies the Laplace equation $f_{xx} + f_{yy} = 0$

$f = e^x \cdot \sin y$

$f_x = e^x \sin y \qquad f_{xx} = e^x \sin y$

$f_y = e^x \cos y \qquad f_{yy} = e^x(-\sin y)$

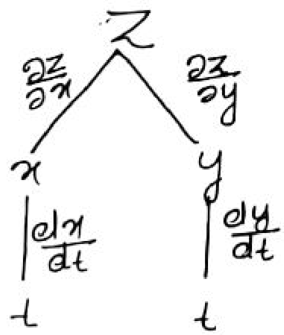
$f_{xx} + f_{yy} = e^x \sin y - e^x \sin y \; \underline{\underline{= 0}}$

Chain Rules For Derivatives

I Function of two Variables

If $x = x(t)$ and $y = y(t)$ are differentiable at t and if $z = f(x,y)$ is differentiable at the point $(x,y) = (x(t), y(t))$, then $z = f(x(t), y(t))$ is differentiable at t and

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$



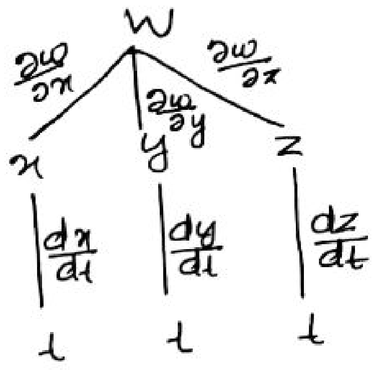
where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x,y).

II Three Variable Chain rule

If $x = x(t)$, $y = y(t)$ and $z = z(t)$ are differentiable at t and if $w = f(x,y,z)$ is differentiable at the point $(x,y,z) = (x(t), y(t), z(t))$ then the function w is differentiable at t and

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}}$$

where the ordinary derivatives are evaluated at t and partial derivatives are evaluated at x, y, z.

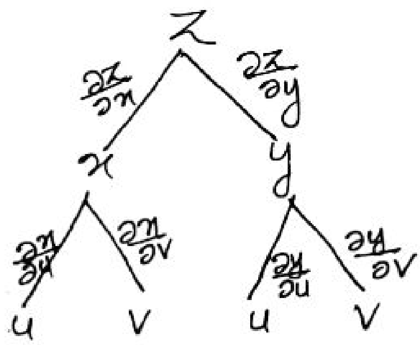


Chain rule for Partial derivatives

Functions of two Variables

If $x = x(u,v)$ and $y = y(u,v)$ have first order partial derivatives at the point (u,v) and if $z = f(x,y)$ is differentiable at the point $(x,y) = (x(u,v), y(u,v))$, then z has first order partial derivatives at the point (u,v) given by

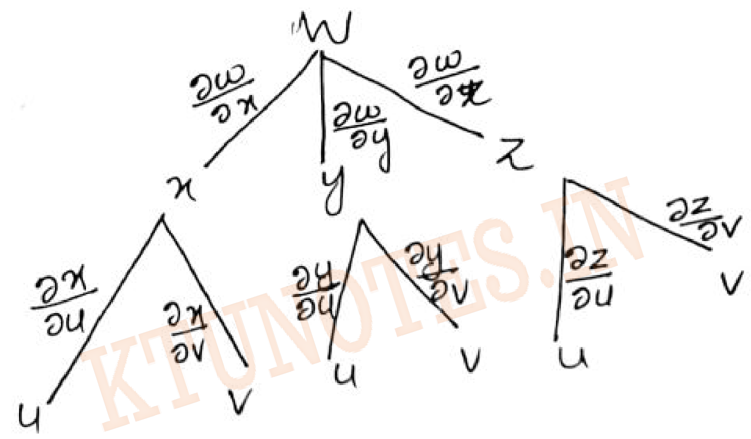
$$\boxed{\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \end{aligned}}$$



Functions of three Variables

If $x = x(u,v)$, $y = y(u,v)$ and $z = z(u,v)$ has first order partial derivatives at the point (u,v) and if the function $w = f(x,y,z)$ is differentiable at the point $(x,y,z) = (x(u,v), y(u,v), z(u,v))$ then w has first order partial derivatives at the point (u,v) given by

$$\boxed{\begin{aligned} \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \\ \frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} \end{aligned}}$$



n- Variable Chain rule

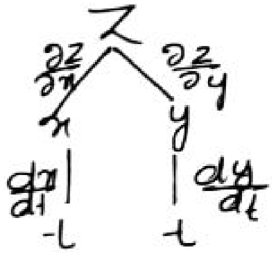
If $w = f(x_1, x_2 \cdots x_n)$ is a differentiable function of 'n' Variables $x_1, x_2 \cdots x_n$ and $x_1, x_2 \cdots x_n$ are differentiable functions of t Then

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial w}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial w}{\partial x_n}\frac{dx_n}{dt}}$$

Problems

1. Suppose that $z = x^2 y$, $x = t^2$ $y = t^3$ use the chain rule to find $\frac{dz}{dt}$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

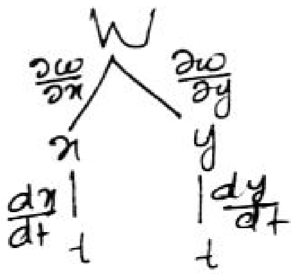


$$= 2xy\, 2t + x^2\, 3t^2$$

$$= 2t^2 t^3\, 2t + 3t^4\, t^2$$

$$= \underline{\underline{7t^6}}$$

2. Find the derivative of $W = x^2 + y^2$ with respect to t along the path $x = at^2$ $y = 2at$.



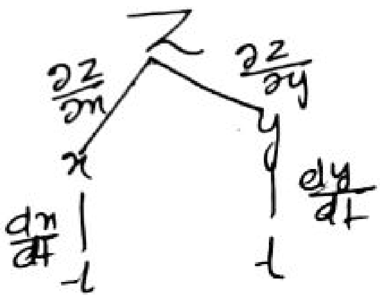
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= 2x\, 2at + 2y\, 2a$$

$$= 4at\,(at^2) + 4a\,(2at)$$

$$= \underline{\underline{4a^2t^3 + 8a^2 t}}$$

3. Find the derivative of $z = \sqrt{1 + x - 2xy^4}$ with respect to t along the path $x = \log t$ $y = 2t$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \frac{1}{2\sqrt{1+x-2xy^4}}(1-2y^4)\frac{1}{t} + \frac{1}{2\sqrt{1+x-2xy^4}}(-8xy^3)(2)$$

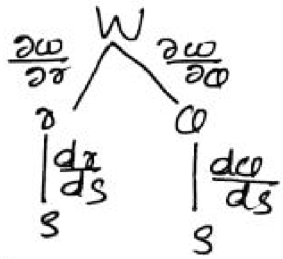
$$= \frac{1}{2\sqrt{1+x-2xy^4}}\left[\frac{1}{t}(1-2y^4) - 16xy^3\right]$$

$$= \frac{1}{2\sqrt{1+x-2xy^4}}\left[\frac{1-2y^4-16xy^3t}{t}\right]$$

$$= \frac{1}{2t\sqrt{1+\log t-2\log t\, 16t^4}}\left[1-2\times 16t^4-16\log t\, 8t^3 t\right]$$

$$= \frac{1-32t^4-128t^4\log t}{2t\sqrt{1+\log t-32t^4\log t}}$$

4. If $w = r^2 - r\tan\theta$, $r=\sqrt{s}$, $\theta = \pi s$, evaluate $\frac{dw}{ds}$ at $s = 1/4$.



$$\frac{dw}{ds} = \frac{\partial w}{\partial r}\frac{dr}{ds} + \frac{\partial w}{\partial \theta}\frac{d\theta}{ds}$$

$$= (2r-\tan\theta)\frac{1}{2\sqrt{s}} + (-r\sec^2\theta)\pi$$

$$= (2\sqrt{s}-\tan\pi s)\frac{1}{2\sqrt{s}} - \sqrt{s}\,\pi\sec^2(\pi s)$$

$$= 1 - \frac{\tan\pi s}{2\sqrt{s}} - \pi\sqrt{s}\sec^2(\pi s)$$

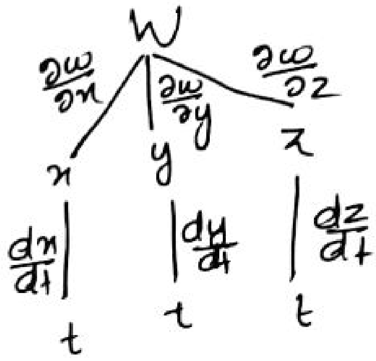
$$\left(\frac{dw}{ds}\right)_{s=1/4} = 1 - \frac{1}{2\sqrt{s}}\tan\frac{\pi}{4} - \frac{\pi}{2}\sec^2\left(\frac{\pi}{4}\right)$$

$$= 1 - \frac{1}{2\sqrt{1/4}} - \frac{\pi}{2}\times(\sqrt{2})^2 = 1 - 1 - \frac{\pi}{2}\times 2$$

$$= \underline{-\pi}$$

5. Find $\frac{dw}{dt}$ if $W = xy + z$, $x = \cos t$

$y = \sin t$ $\quad z = t$



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= y(-\sin t) + x\cos t + 1 \cdot 1$$

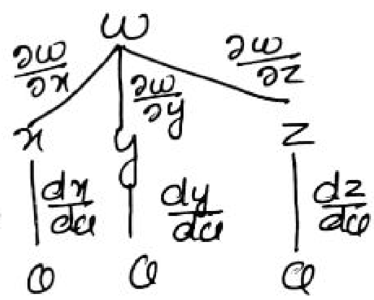
$$= -\sin^2 t + \cos^2 t + 1$$

$$= \underline{\underline{\cos 2t + 1}}$$

$$\boxed{\cos^2 t - \sin^2 t = 1}$$

6. Let $w = \sqrt{x^2 + y^2 + z^2}$, $x = \cos\theta$, $y = \sin\theta$

$z = \tan\theta$ find $\frac{dw}{d\theta}$ at $\theta = \frac{\pi}{4}$.



$$\frac{dw}{d\theta} = \frac{\partial w}{\partial x}\frac{dx}{d\theta} + \frac{\partial w}{\partial y}\frac{dy}{d\theta} + \frac{\partial w}{\partial z}\frac{dz}{d\theta}$$

$$= \frac{1}{2\sqrt{x^2+y^2+z^2}}\, 2x\,(-\sin\theta) +$$

$$\frac{1}{2\sqrt{x^2+y^2+z^2}}\, 2y\,\cos\theta +$$

$$\frac{1}{2\sqrt{x^2+y^2+z^2}}\, 2z\,\sec^2\theta$$

$$= \frac{\cancel{2}}{2\sqrt{x^2+y^2+z^2}}\left[-\cancel{2}x\sin\theta + \cancel{2}y\cos\theta + z\sec^2\theta\right]$$

$$= \frac{1}{\sqrt{\cos^2\theta + \sin^2\theta + \tan^2\theta}}\left[-\cos\theta\sin\theta + \sin\theta\cos\theta + \tan\theta\sec^2\theta\right]$$

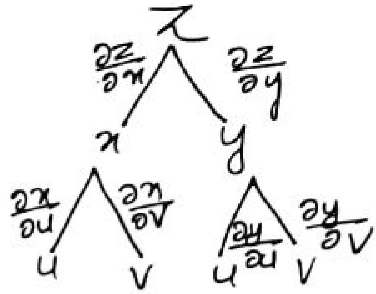
$$= \underline{\underline{\frac{1}{\sqrt{1+\tan^2\theta}}}}\tan\theta\sec^2\theta = \frac{1}{\sqrt{\sec^2\theta}}\cancel{\tan\theta\sec^2\theta}$$

$$= \cancel{\sec\theta}$$

$$\left(\frac{dw}{d\theta}\right)_{\theta=\pi/4} = \frac{1}{\sqrt{1+\tan^2\pi/4}} \tan\pi/4 \ \sec^2\pi/4$$

$$= \frac{1}{\sqrt{1+1}} \ 1\cdot(\sqrt{2})^2 = \frac{1}{\sqrt{2}} \times 2 = \underline{\underline{\sqrt{2}}}$$

7. If $z = e^{xy}$, $x = 2u+v$, $y = \frac{u}{v}$, find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= e^{xy}(y)\,2 + e^{xy}(x)\frac{1}{v}$$

$$= e^{xy}\left[2y + \frac{x}{v}\right]$$

$$= e^{(2u+v)\frac{u}{v}}\left[2\frac{u}{v} + \frac{2u+v}{v}\right]$$

$$= \underline{\underline{e^{\frac{(2u+v)u}{v}}\left[\frac{4u}{v}+1\right]}}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= e^{xy}(y)\cdot 1 + e^{xy}(x)\,u\left(-\frac{1}{v^2}\right)$$

$$= e^{xy}\left[y - \frac{xu}{v^2}\right]$$

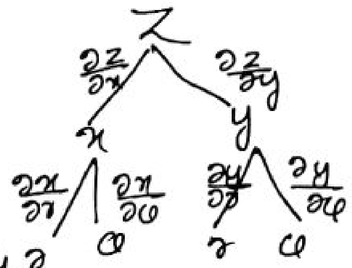
$$= e^{\left(\frac{2u+v}{v}\right)u}\left[\frac{u}{v} - \frac{u}{v^2}(2u+v)\right]$$

$$= e^{(2u+v)u/v}\left[\frac{\cancel{uv} - 2u^2 - \cancel{uv}}{v^2}\right]$$

$$= \underline{\underline{e^{(2u+v)u/v}\left[\frac{-2u^2}{v^2}\right]}}$$

8 Let $z = xy\, e^{x/y}$, $x = r\cos\theta$, $y = r\sin\theta$

find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ at $r = 2$ and $\theta = \pi/6$



$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$= y\left[x e^{x/y}\cdot\frac{1}{y} + e^{x/y}\cdot 1\right]\cos\theta$$

$$+\; x\left[y\, e^{x/y}\cdot x\left(-\frac{1}{y^2}\right) + e^{x/y}\right]\sin\theta$$

$$= e^{x/y}\left\{\left[x + y\right]\cos\theta + \left[-\frac{x^2}{y} + x\right]\sin\theta\right\}$$

$$= e^{r\cos\theta/r\sin\theta}\left\{(r\cos\theta + r\sin\theta)\cos\theta + \left[\frac{-r^2\cos^2\theta}{r\sin\theta} + r\cos\theta\right]\sin\theta\right\}$$

$$= e^{\cot\theta}\left\{r\cos^2\theta + r\cos\theta\sin\theta - r\cos^2\theta + r\cos\theta\sin\theta\right\}$$

$$= e^{\cot\theta}\, 2r\cos\theta\sin\theta$$

$$= 2r\cos\theta\sin\theta\, e^{\cot\theta}$$

$$\left(\frac{\partial z}{\partial r}\right)_{r=2,\ \theta=\pi/6} = 2\times 2\cos\frac{\pi}{6}\sin\frac{\pi}{6}\, e^{\cot \pi/6}$$

$$= 4\times\frac{\sqrt{3}}{2}\times\frac{1}{2}\, e^{\sqrt{3}}$$

$$= \underline{\sqrt{3}\, e^{\sqrt{3}}}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= y\left[x e^{x/y}\frac{1}{y} + e^{x/y}\right] r(-\sin\theta) + x\left[y\, e^{x/y}\, x\left(\frac{-1}{y^2}\right) + e^{x/y}\right] r\cos\theta$$

$$= e^{x/y}\, r \left[(x+y)^{-} \sin\theta + \left(-\frac{x^2}{y} + x\right) \cos\theta \right]$$

$$= e^{r\cos\theta/r\sin\theta}\, r \left[(r\cos\theta + r\sin\theta)^{-} \sin\theta + \left(-\frac{r^2\cos^2\theta}{r\sin\theta} \cancel{r\cos\theta} \right) \cos\theta \right]$$

$$= e^{\cot\theta}\, r^2 \left[-\cos\theta\sin\theta - \sin^2\theta - \cos^2\theta\cot\theta + \cos^2\theta \right]$$

$$\left(\frac{\partial z}{\partial \theta}\right)_{r=2,\ \theta=\pi/6} = e^{\cot\frac{\pi}{6}}\, 4 \left[-\cos\frac{\pi}{6}\sin\frac{\pi}{6} - \sin^2\frac{\pi}{6} - \cos^2\frac{\pi}{6}\cot\pi/6 + \cos^2\pi/6 \right]$$

$$= 4e^{\sqrt{3}} \left[-\frac{\sqrt{3}}{2} \times \frac{1}{2} - \frac{1}{4} - \frac{3}{4} \times \sqrt{3} + \frac{3}{4} \right]$$

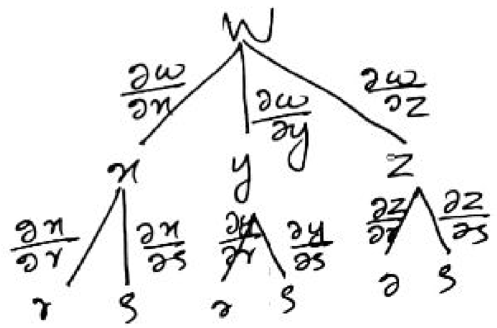
$$= \frac{4e^{\sqrt{3}}}{4} \left[-\sqrt{3} - 1 - 3\sqrt{3} + 3 \right]$$

$$= e^{\sqrt{3}} \left[2 - 4\sqrt{3} \right]$$

9. Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if $W = x + 2y + z^2$

$x = \frac{r}{s}$ $\quad y = r^2 + \ln s$ $\quad z = 2r$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}.$$



$$\frac{\partial w}{\partial r} = 1 \cdot \frac{1}{s} + 2 \cdot 2r + 2x \cdot 2$$

$$= \frac{1}{s} + 4r + 4 \times 2r$$

$$= \frac{1}{s} + 12r$$

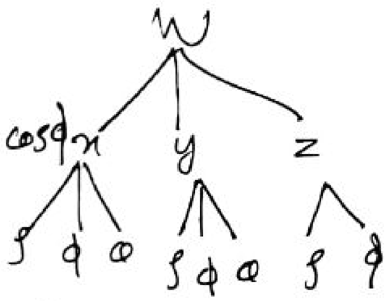
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= 1 \cdot \left(r - \frac{1}{s^2}\right) + 2\left(\frac{1}{s}\right) + 2z \cdot 0$$

$$= \frac{2}{s} - \frac{r}{s^2}$$

10. If $W = x^2 + y^2 - z^2$ and $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$

hom $z = \rho \cos\phi$ Find $\frac{\partial w}{\partial \rho}$ and $\frac{\partial w}{\partial \theta}$



$$\frac{\partial w}{\partial \rho} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \rho}$$

$$= 2x \sin\phi \cos\theta + 2y \sin\phi \sin\theta - 2z\cos\phi$$

$$= 2\rho \sin^2\phi \cos^2\theta + 2\rho \sin^2\phi \sin^2\theta - 2\rho\cos^2\phi$$

$$= 2\rho \sin^2\phi (\cos^2\theta + \sin^2\theta) - 2\rho \cos^2\phi$$

$$= 2\rho(\sin^2\phi - \cos^2\phi) = 2\rho^-(\cos^2\phi - \sin^2\phi)$$

$$\boxed{\cos^2\theta - \sin^2\theta = \cos 2\theta}$$

$$= -2\rho \cos 2\phi$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta}.$$

$$= 2x\, \rho \sin\phi\, {}^-\sin\theta + 2y\, \rho \sin\phi \cos\theta - 2z \times 0$$

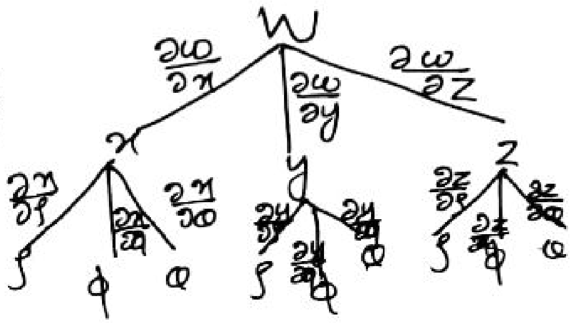
$$= {}^-2\rho^2 \sin^2\phi \cos\theta \sin\theta + 2\rho^2 \sin^2\phi \cos\theta \sin\theta$$

$$= 0$$

11. If $W = 4x^2 + 4y^2 + z^2$, $x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$.

Find $\frac{\partial w}{\partial \rho}$, $\frac{\partial w}{\partial \theta}$ and $\frac{\partial w}{\partial \phi}$



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$$\frac{\partial W}{\partial \rho} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \rho}$$

$= 8x \sin\phi \cancel{\sin\theta}\cos\theta +$

$8y \sin\phi \sin\theta + 2z \cos\phi$

$= 8\rho \sin^2\phi \cancel{\sin\theta}\cos^2\theta + 8\rho \sin^2\phi \sin^2\theta + 2\rho \cos^2\phi$

$= 8\rho \sin^2\phi (\cos^2\theta + \sin^2\theta) + 2\rho \cos^2\phi)$

$= 8\rho \sin^2\phi + 2\rho \cos^2\phi$

$= 6\rho \sin^2\phi + 2\rho \sin^2\phi + 2\rho \cos^2\phi$

$= 6\rho \sin^2\phi + 2\rho(\sin^2\phi + \cos^2\phi)$

$= \underline{6\rho \sin^2\phi + 2\rho}$

$$\frac{\partial W}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta}$$

$= \cancel{8x \sin\phi \cos\theta}$

$= 8x \rho \sin\phi \, {}^{-}\sin\theta + 8y \rho \sin\phi \cos\theta + 2z \times 0$

$= -8\rho^2 \sin^2\phi \sin\theta \cos\theta + 8\rho^2 \sin^2\phi \sin\theta \cos\theta$

$= \underline{0}$

$$\frac{\partial w}{\partial \phi} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \phi}$$

$= 8x \rho \cos\theta \cos\phi + 8y \rho \sin\theta \cos\phi + 2z \rho \, {}^{-}\sin\phi$

$= 8\rho^2 \sin\phi \cos^2\theta \cos\phi + 8\rho^2 \sin\phi \sin^2\theta \cos\phi - 2\rho^2 \sin\phi \cos\phi$

$= 8\rho^2 \sin\phi \cos\phi (\cos^2\theta + \sin^2\theta) - 2\rho^2 \sin\phi \cos\phi$

$= 6\rho^2 \sin\phi \cos\phi$

12. Let f be a differentiable function of three variables and suppose that

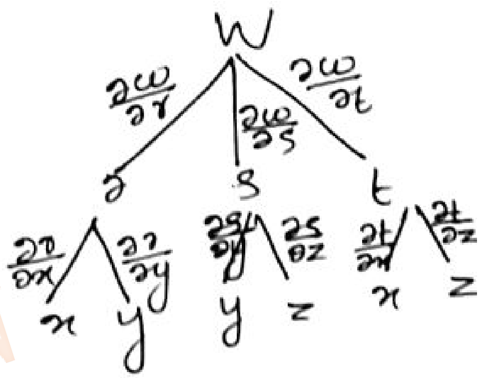
Imp

$w = f(x-y,\ y-z,\ z-x)$

Show that $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 0$

$$w = f(x-y,\ y-z,\ z-x) = f(r, s, t)$$

where $r = x-y \quad s = y-z \quad t = z-x$



$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x}$$

$$= \frac{\partial w}{\partial r}\cdot 1 + \frac{\partial w}{\partial t}\cdot(-1)$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r} - \frac{\partial w}{\partial t} \quad \text{—— (1)}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\cdot\frac{\partial s}{\partial y}$$

$$= \frac{\partial w}{\partial r}(-1) + \frac{\partial w}{\partial s}\cdot 1$$

$$\frac{\partial w}{\partial y} = -\frac{\partial w}{\partial r} + \frac{\partial w}{\partial s} \quad \text{—— (2)}$$

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial z}$$

$$= \frac{\partial w}{\partial s}(-1) + \frac{\partial w}{\partial t}\cdot 1$$

$$\frac{\partial w}{\partial z} = -\frac{\partial w}{\partial s} + \frac{\partial w}{\partial t} \quad \text{—— (3)}$$

(1) + (2) + (3) $\Rightarrow$

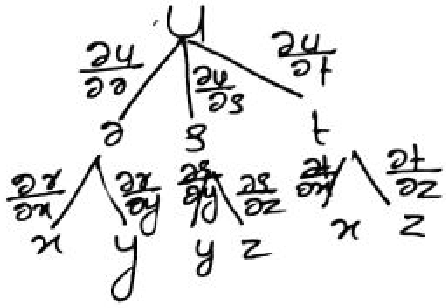
$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial w}{\partial r} - \frac{\partial w}{\partial t} - \frac{\partial w}{\partial r} + \frac{\partial w}{\partial s} - \frac{\partial w}{\partial s} + \frac{\partial w}{\partial t} = 0$$

Imp

13. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ Prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

$u = f(r, s, t)$ where $r = \frac{x}{y}$ $s = \frac{y}{z}$ $t = \frac{z}{x}$



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}\cdot\frac{1}{y} + \frac{\partial u}{\partial t}\left(z\cdot\frac{-1}{x^2}\right)$$

$$x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\frac{\partial u}{\partial t} \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y}$$

$$= \frac{\partial u}{\partial r}\left(x\cdot\frac{-1}{y^2}\right) + \frac{\partial u}{\partial s}\left(\frac{1}{z}\right)$$

$$y\frac{\partial u}{\partial y} = \frac{-x}{y}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s} \quad \text{--- (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial s}\left(y\times\frac{-1}{z^2}\right) + \frac{\partial u}{\partial t}\cdot\frac{1}{x}$$

$$z\frac{\partial u}{\partial z} = \frac{-y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t} \quad \text{--- (3)}$$

(1) + (2) + (3) $\Rightarrow$ $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} =$

$$\frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\frac{\partial u}{\partial t} - \frac{x}{y}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s} - \frac{y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t}$$

$$= 0$$

Differentials

If $z = f(x,y)$ is differentiable at (x_0, y_0), then the total differential of z at (x_0, y_0),

$$\boxed{\begin{gathered} dz = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy \\ \text{OR} \\ dz = f_x(x,y)\,dx + f_y(x,y)\,dy \end{gathered}}$$

The differentials dx and dy are respectively the increment of Δx and Δy.

If $w = f(x,y,z)$ is differentiable at (x_0, y_0, z_0), then the total differentiable of w at (x_0, y_0, z_0)

$$\boxed{\begin{gathered} dw = f_x(x_0,y_0,z_0)\,dx + f_y(x_0,y_0,z_0)\,dy + f_z(x_0,y_0,z_0)\,dz \\ \text{OR} \\ dw = f_x(x,y,z)\,dx + f_y(x,y,z)\,dy + f_z(x,y,z)\,dz \end{gathered}}$$

Note

Error = Actual value − Approximate value = Δx $\quad \boxed{\Delta x = x - x_0}$

Relative error or Proportional error = $\dfrac{\text{Error}}{\text{Actual value}} = \left|\dfrac{\Delta x}{x}\right|$

Percentage error = Relative error × 100 = $\dfrac{\Delta x}{x} \times 100$

Problems

1. Find the differential dz of the function

$z = xe^{y^2}$

$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$

$= e^{y^2}\cdot 1\,dx + x\,e^{y^2}(2y)\,dy$

$dz = e^{y^2}dx + 2xy\,e^{y^2}dy$

2. Find the differential dz of the function

$z = \tan^{-1}(x^2y)$

$z = f(x,y) = \tan^{-1}(x^2y)$

$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$

$= \frac{1}{1+(x^2y)^2} \times 2xy\,dx + \frac{1}{1+(x^2y)^2}x^2\,dy$

$= \frac{2xy}{1+x^4y^2}dx + \frac{x^2}{1+x^4y^2}dy$

$= \frac{2xy\,dx + x^2dy}{1+x^4y^2}$

3. Find the differential dw of the function

$w = x \sin(yz)$

$dw = f_x(x,y,z)\, dx + f_y(x,y,z)\, dy + f_z(x,y,z)\, dz$

$= \sin(yz)\, dx + x \cos(yz)\, z\, dy + x \cos(yz) \cdot y\, dz$

$= \sin(yz)\, dx + xz \cos(yz)\, dy + xy \cos(yz)\, dz$

4. The length and width of a rectangle are measured with errors of almost 3% and 4% respectively. Use differentials to approximate the maximum percentage error in the calculated area.

Let 'x' be the length and 'y' be the width of the rectangle.

Area $(A) = x \cdot y$

Total differential of A is

$dA = f_x(x,y)\, dx + f_y(x,y)\, dy$

$= y\, dx + x\, dy$

$$\frac{dA}{A} = \frac{y}{xy} dx + \frac{x}{xy} dy$$

$$\frac{dA}{A} = \frac{dx}{x} + \frac{dy}{y}$$

Relative error $= \left|\frac{\Delta A}{A}\right| \approx \left|\frac{dA}{A}\right|$

Given that $\left|\frac{\Delta x}{x}\right| \approx \left|\frac{dx}{x}\right| \leq 0.03$

$$\left|\frac{\Delta y}{y}\right| \approx \left|\frac{dy}{y}\right| \leq 0.04$$

$\therefore \left|\frac{dA}{A}\right| \leq \left|\frac{dx}{x}\right| + \left|\frac{dy}{y}\right|$ $\quad \boxed{|A+B| \leq |A|+|B|}$

$\leq 0.03 + 0.04$

≤ 0.07

Percentage error = Relative error $\times 100$

$= \left|\frac{dA}{A}\right| \times 100 = 0.07 \times 100$

$= 7\%$

2. The radius and height of a right circular cone are measured with errors of ~~atmost~~ atmost 2% and 4% respectively use differentials to approximate the maximum percentage error in the calculated volume.

Volume$(V) = \frac{1}{3}\pi r^2 h$

Total differential $dV = f_r(r,h)dr + f_h(r,h)dh$

$$dV = \frac{1}{3}\pi\, 2rh\, dr + \frac{1}{3}\pi r^2\, dh$$

$$dV = \frac{1}{3}\pi \left[2rh\, dr + r^2\, dh\right]$$

$$\frac{dV}{V} = \frac{\cancel{\frac{1}{3}\pi}\left[2rh\,dr + r^2\,dh\right]}{\cancel{\frac{1}{3}\pi} r^2 h}$$

$$= \frac{2\cancel{r}\cancel{h}}{r^{\cancel{2}}\cancel{h}}\, dr + \frac{\cancel{r^2}}{\cancel{r^2}h}\, dh$$

$$\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h}$$

Given that $\left|\frac{\Delta r}{r}\right| \approx \left|\frac{dr}{r}\right| \leq 0.02$

$$\left|\frac{\Delta h}{h}\right| \approx \left|\frac{dh}{h}\right| \leq 0.04$$

Relative error $\left|\frac{\Delta V}{V}\right| = \left|2\frac{\Delta r}{r} + \frac{\Delta h}{h}\right|$

$$\leq 2\left|\frac{\Delta r}{r}\right| + \left|\frac{\Delta h}{h}\right|$$

$$\leq 2 \times 0.02 + 0.04$$

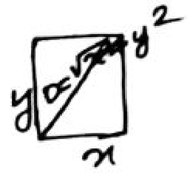
$$\leq \underline{\underline{0.08}}$$

Percentage error $= 0.08 \times 100 = \underline{\underline{8\%}}$

6. The length and width of a rectangle are measured with errors of at most $r\%$ where r is small. Use differentials to approximate the maximum percentage error in the calculated length of the diagonal.

Let x = length

y = width



Diagonal $= \sqrt{x^2+y^2}$

The total differential of D is

$$dD = f_x(x,y)\,dx + f_y(x,y)\,dy$$

$$= \frac{1}{2\sqrt{x^2+y^2}}\,2x\,dx + \frac{1}{2\sqrt{x^2+y^2}}\,2y\,dy$$

$$dD = \frac{x}{\sqrt{x^2+y^2}}\,dx + \frac{y}{\sqrt{x^2+y^2}}\,dy$$

$$\frac{dD}{D} = \frac{x}{\sqrt{x^2+y^2}\sqrt{x^2+y^2}}\,dx + \frac{y}{\sqrt{x^2+y^2}\sqrt{x^2+y^2}}\,dy$$

$$= \frac{x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy$$

$$= \frac{1}{x^2+y^2}\left[x\,dx + y\,dy\right]$$

$$\frac{dD}{D} = \frac{1}{x^2+y^2}\left[x^2\frac{dx}{x} + y^2\frac{dy}{y}\right]$$

Given that $\left|\frac{\Delta x}{x}\right| \approx \left|\frac{dx}{x}\right| \leq \frac{r}{100}$

$$\left|\frac{\Delta y}{y}\right| \approx \left|\frac{dy}{y}\right| \leq \frac{r}{100}$$

Relative error $\left|\frac{\Delta d}{d}\right| \leq \frac{1}{x^2+y^2}\left[x^2\left|\frac{\Delta x}{x}\right| + y^2\left|\frac{\Delta y}{y}\right|\right.$

Let x = length

y = width

$$\left|\frac{\Delta d}{d}\right| \leq \frac{1}{x^2+y^2}\left[x^2\frac{r}{100}+y^2\frac{r}{100}\right]$$

$$\leq \frac{r}{100}\frac{\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}}$$

$$\leq \frac{r}{100}$$

Percentage error = $\underline{\underline{r\%}}$

7. The length, width and height of a rectangular box are measured with an error of at most 5%. Use a total differential to estimate the maximum percentage error that results if these quantities are used to calculate the diagonal of the box.

length $= x$ width $= y$ height $= z$

Diagonal $D = \sqrt{x^2+y^2+z^2}$

The total differential

$$dD = f_x(x,y,z)\,dx + f_y(x,y,z)\,dy + f_z(x,y,z)\,dz$$

$$= \frac{1}{2\sqrt{x^2+y^2+z^2}}2x\,dx + \frac{1}{2\sqrt{x^2+y^2+z^2}}2y\,dy + \frac{1}{2\sqrt{x^2+y^2+z^2}}2z\,dz$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}}\left[x\,dx + y\,dy + z\,dz\right]$$

$$\frac{dD}{D} = \frac{1}{(\sqrt{x^2+y^2+z^2})(\sqrt{x^2+y^2+z^2})}\left[x\,dx+y\,dy+z\,dz\right]$$

$$= \frac{1}{x^2+y^2+z^2}\left[x^2\, dx + y^2\, dy + z^2\, dz\right]$$

$$\frac{dD}{D} = \frac{1}{x^2+y^2+z^2}\left[x^2 \frac{dx}{x} + y^2 \frac{dy}{y} + z^2 \frac{dz}{z}\right]$$

$$\left|\frac{\Delta x}{x}\right| \approx \frac{dx}{x} \leq 0.05$$

$$\left|\frac{\Delta y}{y}\right| \approx \frac{dy}{y} \leq 0.05$$

$$\left|\frac{\Delta z}{z}\right| \approx \frac{dz}{z} \leq 0.05$$

$$\left|\frac{\Delta D}{D}\right| \leq \frac{1}{x^2+y^2+z^2}\left[x^2\left|\frac{\Delta x}{x}\right| + y^2\left|\frac{\Delta y}{y}\right| + z^2\left|\frac{\Delta z}{z}\right|\right]$$

$$\leq \frac{1}{x^2+y^2+z^2}\left[x^2 \times 0.05 + y^2 \times 0.05 + z^2 \times 0.05\right]$$

$$\leq \frac{0.05\,(x^2+y^2+z^2)}{x^2+y^2+z^2}$$

$$\left|\frac{\Delta D}{D}\right| \leq 0.05$$

Percentage error $= 0.05 \times 100 = \underline{5\%}$

8 The period T of a simple pendulum with small oscillations is calculated by the formula $T = 2\pi\sqrt{\frac{L}{g}}$, where L is the

length of the pendulum and g is the acceleration due to gravity. Suppose that measured values of L and g have errors of almost 0.5% and 0.1% respectively. Use differentials to approximate the maximum percentage error in the calculated value of T.

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Total differential

$$dT = f_L(L,g)\,dL + f_g(L,g)\,dg$$

$$= \frac{2\pi}{\sqrt{g}}\frac{1}{2\sqrt{L}}\,dL + 2\pi\sqrt{L}\,\frac{-1}{2}g^{-3/2}\,dg$$

$$= \frac{\pi}{\sqrt{Lg}}\,dL - \frac{\pi\sqrt{L}}{g\sqrt{g}}\,dg$$

$$\frac{dT}{T} = \frac{\frac{\pi}{\sqrt{Lg}}\,dL}{2\pi\sqrt{\frac{L}{g}}} - \frac{\frac{\pi\sqrt{L}}{g\sqrt{g}}\,dg}{2\pi\sqrt{L/g}}$$

$$= \frac{\cancel{\pi}}{\sqrt{L}\sqrt{g}} \times \frac{\sqrt{g}}{2\cancel{\pi}\sqrt{L}}\,dL - \frac{\cancel{\pi}\cancel{\sqrt{L}}}{g\cancel{\sqrt{g}}} \times \frac{\cancel{\sqrt{g}}}{2\cancel{\pi}\cancel{\sqrt{L}}}\,dg$$

$$= \frac{1}{2L}\,dL - \frac{1}{2g}\,dg$$

Given that

$$\left|\frac{\Delta L}{L}\right| \approx \frac{dL}{L} \leq 0.005$$

$$\left|\frac{\Delta g}{g}\right| \approx \frac{dg}{g} \leq 0.001$$

$\therefore \cancel{\frac{dT}{T} \approx \left|\frac{\Delta T}{T}\right| \leq \frac{1}{2}\left\{\left|\frac{\Delta L}{L}\right| -\right.}$

$$\left|\frac{\Delta T}{T}\right| = \frac{1}{2}\left|\frac{\Delta L}{L} - \frac{\Delta g}{g}\right|$$

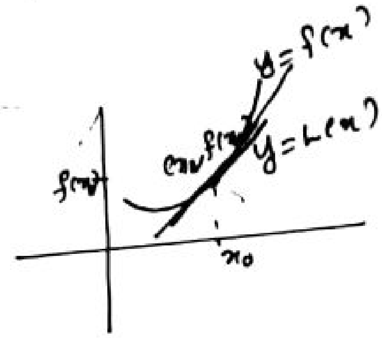
$$\leq \frac{1}{2}\left\{\left|\frac{\Delta L}{L}\right| + \left|\frac{\Delta g}{g}\right|\right\}$$

$$\leq \frac{1}{2}\left\{0.005 + 0.001\right\}$$

$$\leq 0.003$$

Local linear approximation

If $y = f(x)$ is differentiable at $x = x_0$.



$\boxed{\text{Point Slope equation} \quad y - y_0 = m(x - x_0)}$:

$$y - f(x_0) = f'(x_0)(x - x_0)$$

$$y = f(x_0) + f'(x_0)(x - x_0),$$ this tangent line is the graph of linear function.

$\therefore$ The tangent to the curve $y = f(x)$ at $x = x_0$ is the line

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

[If f is differentiable at a point then it can be very closely approximate by a linear function near that point]

If $f(x,y)$ is differentiable at the point (x_0, y_0) then the local linear approximation

$$\boxed{L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)}$$

If $f(x,y,z)$ is differentiable at (x_0, y_0, z_0) then the <u>local linear</u> approximation

$$L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$$

* Distance between the points (x_0, y_0) and (x,y) is $\sqrt{(x-x_0)^2+(y-y_0)^2}$

<u>Problems</u>

1. Find the local linear approximation L to the function $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ at the point $P(4,3)$. Compare the error in approximating f by L at the point $Q(3.92, 3.01)$ with the distance between P and Q.

$$f(x,y) = \frac{1}{\sqrt{x^2+y^2}} \qquad f(4,3) = \frac{1}{\sqrt{16+9}} = \frac{1}{5} = 0.2$$

Local linear approximation

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$L(x,y) = f(4,3) + f_x(4,3)(x-4) + f_y(4,3)(y-3)$$

$$f_x(x,y) = -\frac{1}{2}(x^2+y^2)^{-3/2}(2x) = \frac{-x}{(x^2+y^2)^{3/2}}$$

$$f_x(4,3) = \frac{-4}{(16+9)^{3/2}} = \frac{-4}{5^3} = \frac{-4}{125} = \underline{-0.032}$$

$$f_y(x,y) = -\frac{1}{2}(x^2+y^2)^{-3/2}(2y) = \frac{-y}{(x^2+y^2)^{3/2}}$$

$$f_y(4,3) = \frac{-3}{(16+9)^{3/2}} = \frac{-3}{5^3} = \frac{-3}{125} = \underline{-0.024}$$

$$L(x,y) = 0.2 - 0.032(x-4) - 0.024(y-3)$$

$$L(3.92, 3.01) = 0.2 - 0.032(3.92-4) - 0.024(3.01-3)$$

$$= \underline{0.20232} = \text{Approximate Value}$$

$$f(3.92, 3.01) = \frac{1}{\sqrt{(3.92)^2+(3.01)^2}} = 0.202334238 = \text{Actual Value}$$

Error in approximation = Actual Value − Approximate value

$$= 0.202334238 - 0.20232$$

$$= 0.000014238$$

Distance between the points $(4,3)$ and $(3.92, 3.01)$ is

$$\boxed{\text{Distance} = \sqrt{(x-x_0)^2+(y-y_0)^2}}$$

$$= \sqrt{(3.92-4)^2+(3.01-3)^2} = 0.080622577$$

$$\frac{\text{Error in approximation}}{\text{Distance between the points}} = \frac{0.000014238}{0.080622577}$$

$$= \underline{0.0001766}$$

2. Let $L(x,y)$ denote the local linear approximation to $f(x,y)=\sqrt{x^2+y^2}$ at the point $(3,4)$. Compare the error in approximating $f(3.04,3.98)=\sqrt{(3.04)^2+(3.98)^2}$ by $L(3.04,3.98)$ with the distance between the points $(3,4)$ and $(3.04,3.98)$.

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$L(x,y) = f(3,4) + f_x(3,4)(x-3) + f_y(x,y)(y-4)$$

$$f(x,y) = \sqrt{x^2+y^2}$$

$$f(3,4) = \sqrt{3^2+4^2} = 5$$

$$f_x(x,y) = \frac{1}{2\sqrt{x^2+y^2}}\,2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$f_x(3,4) = \frac{3}{\sqrt{3^2+4^2}} = \frac{3}{5}$$

$$f_y(x,y) = \frac{1}{2\sqrt{x^2+y^2}}\,2y = \frac{y}{\sqrt{x^2+y^2}}$$

$$f_y(3,4) = \frac{4}{\sqrt{3^2+4^2}} = \frac{4}{5}$$

$$L(x,y) = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4)$$

$$L(3.04,3.98) = 5 + \frac{3}{5}(3.04-3) + \frac{4}{5}(3.98-4)$$

$$= 5 + \frac{3}{5} \times 0.04 + \frac{4}{5} \times -0.02$$

$$= \underline{5.008} = \text{Approximate Value}$$

$$f(3.04, 3.98) = \sqrt{(3.04)^2 + (3.98)^2} = 5.00819 = \text{Actual Value}$$

$$\text{Errors} = \text{Actual Value} - \text{Approximate value}$$

$$= 5.00819 - 5.008 = \underline{0.00019}$$

Distance between the points $(3,4)$ and $(3.04, 3.98)$ is

$$\sqrt{(3.04-3)^2 + (3.98-4)^2} = \underline{0.045}$$

$$\frac{\text{Error in approximation}}{\text{Distance between the points}} = \frac{0.00019}{0.045}$$

$$= \underline{0.004222}$$

3. Find the local linear approximation L of $f(x,y) = \ln(xy)$ at the point $P(1,2)$. & Compute the error in approximation f by L at the point $Q(1.01, 2.01)$

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$= f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)$$

$$f(x,y) = \ln(x \cdot y) \qquad f(1,2) = \ln(1 \times 2) = \ln 2$$

$$= \underline{0.69314718}$$

$$f_x(x,y) = \frac{1}{xy} y = \frac{1}{x}$$

$$f_x(1,2) = \frac{1}{1} = 1$$

$$f_y(x,y) = \frac{1}{xy} x = \frac{1}{y}$$

$$f_y(1,2) = \frac{1}{2}$$

$$L(x,y) = 0.69314718 + (x-1) + \frac{1}{2}(y-2)$$

$$L(1.01, 2.01) = 0.69314718 + (1.01-1) + \frac{1}{2}(2.01-2)$$

$$= 0.69314718 + 0.01 + \frac{1}{2} \times 0.01$$

$$= \underline{\underline{0.70814718}} = \text{Approximate Value}$$

$$f(1.01, 2.01) = \ln(1.01 \times 2.01)$$

$$= \ln(2.0301)$$

$$= \underline{\underline{0.708085052}} = \text{Actual Value}$$

$$\text{Error} = \text{Actual Value} - \text{Approximate value}$$

$$= 0.708085052 - 0.70814718$$

$$= \underline{\underline{-0.000062128}}$$

4 Find the local linear approximation L of $f(x,y,z) = \frac{x+y}{y+z}$ at the point $P(-1,1,1)$

$$L(x,y,z) = f(x_0 y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0)$$
$$+ f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0 z_0)(z - z_0)$$

$$f(x,y,z) = \frac{x+y}{y+z}$$

$$f(-1,1,1) = \frac{-1+1}{1+1} = \underline{\underline{0}}$$

$$f_x(x,y,z) = \frac{1}{(y+z)} \cdot 1 = \frac{1}{y+z}$$

$$f_x(-1,1,1) = \frac{1}{1+1} = \frac{1}{2}$$

$$f_y(x,y,z) = \frac{(y+z)\cdot 1 - (x+y)\cdot 1}{(y+z)^2}$$

$$= \frac{y+z-x-y}{(y+z)^2} = \frac{z-x}{(y+z)^2}$$

$$f_y(-1,1,1) = \frac{1+1}{(1+1)^2} = \frac{2}{4} = \underline{\underline{1/2}}$$

$$f_z(x,y,z) = (x+y)\frac{-1}{(y+z)^2}$$

$$f_z(-1,1,1) = (-1+1)\frac{-1}{(1+1)^2} = 0$$

$$L(x,y,z) = 0 + \frac{1}{2}(x - x_0) + \frac{1}{2}(y - y_0)$$

$$= \frac{1}{2}(x+1) + \frac{1}{2}(y-1)$$

$$= \underline{\underline{\frac{x+y}{2}}}$$

5. Find the local linear approximation L of $f(x,y,z) = xyz$ at the point $P(1,2,3)$. Compute the error in approximation f by L at the point $Q(1.001, 2.002, 3.003)$.

Imp

$$L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$$

$$= f(1,2,3) + f_x(1,2,3)(x-1) + f_y(1,2,3)(y-2) + f_z(1,2,3)(z-3)$$

$f(x,y,z) = xyz$

$f(1,2,3) = 1\times2\times3 = \underline{\underline{6}}$

$f_x(x,y,z) = yz$

$f_x(1,2,3) = 2\times3 = \underline{\underline{6}}$

$f_y(x,y,z) = xz$ $\qquad f_y(1,2,3) = 1\times3 = \underline{\underline{3}}$

$f_z(x,y,z) = xy$ $\qquad f_z(1,2,3) = 1\times2 = \underline{\underline{2}}$

$L(x,y,z) = 6 + 6(x-1) + 3(y-2) + 2(z-3)$

$L(1.001, 2.002, 3.003) = 6 + 6(1.001-1) + 3(2.002-2) + 2(3.003-3)$

$= \underline{\underline{6.018}}$ = Appro

$f(1.001, 2.002, 3.003) = 1.001\times2.002\times3.003$

$= \underline{6.018018}$

Error = Actual Value − Approx. Value $= 6.018018 - 6.018 = 0.\underline{000018}$

6. A function $f(x,y) = x^2+y^2$ is differentiable at the point $P(x,y)$. If $L(x,y) = 2x+4y-5$ be the local linear approximation to f at P, determine P.

U.Q.

$$L(x,y) = 2x+4y-5$$
$$= 2(x-x_0)+4(y-y_0)+2x_0+4y_0-5$$
$$= f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)+f(x_0,y_0)$$

comparing

$f_x(x_0,y_0) = 2 \qquad f_y(x_0,y_0) = 4$

$f(x_0,y_0) = 2x_0+4y_0-5$

Given that $f(x,y) = x^2+y^2 \implies f(x_0,y_0) = x_0^2+y_0^2$

$\implies f_x(x_0,y_0) = 2x_0 = 2$

$\implies \underline{\underline{x_0 = 1}}$

$f_y(x_0,y_0) = 2y_0 = 4$

$\implies y_0 = 2$

$\therefore$ The point P is $\underline{\underline{(1,2)}}$

7. A function $f(x,y,z) = xy+z^2$ is differentiable at the point $P(x,y,z)$. If

$L(x,y,z) = y + 2z + 5$ be the local linear approximation to f at P, determine P.

$$L(x,y,z) = y + 2z + 5$$
$$= (y-y_0) + 2(z-z_0) + y_0 + 2z_0 + 5$$
$$= f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0)$$
$$+ f_z(x_0,y_0,z_0)(z-z_0) + f(x_0,y_0,z_0)$$

Comparing

$f_x(x_0,y_0,z_0) = 0$ $\quad$ $f_y(x_0,y_0,z_0) = 1$

$f_z(x_0,y_0,z_0) = 2$ $\quad$ $f(x_0,y_0,z_0) = y_0 + 2z_0 + 5$

Given that $f(x,y,z) = xy + z^2$

$\Rightarrow f(x_0,y_0,z_0) = x_0 y_0 + z_0^2$

$f_x(x_0,y_0,z_0) = y_0 = 0$

$f_y(x_0,y_0,z_0) = x_0 = 1$

$f_z(x_0,y_0,z_0) = 2z_0 = 2$

$\Rightarrow z_0 = 1$

$\therefore$ point $P = (x_0,y_0,z_0) = \underline{\underline{(1,0,1)}}$

8. Suppose that a function $f(x,y)$ is differentiable at the point $(2,5)$ with $f_x(2,5)=5$ and $f(2,5)=9$. Let $L(x,y)$ denote the local linear approximation of f at $(2,5)$. If $L(2.1, 4.8) = 9.15$, find the value of $f_y(2,5)$.

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$= f(2,5) + f_x(2,5)(x-2) + f_y(2,5)(y-5)$$

$$L(x,y) = 9 + 5(x-2) + f_y(2,5)(y-5)$$

$$L(2.1, 4.8) = 9 + 5(2.1-2) + f_y(2,5)(4.8-5)$$

$$9.15 = 9 + 0.5 - 0.2\, f_y(2,5)$$

$$9.15 - 9.5 = -0.2\, f_y(2,5)$$

$$-0.35 = -0.2\, f_y(2,5)$$

$$f_y(2,5) = \frac{0.35}{0.2} = \underline{1.75}$$

9. Suppose that a function $f(x,y,z)$ is differentiable at the point $(1,2,2)$ and $L(x,y,z) = 3x - y + 2z + 7$ is the local linear approximation of f at $(1,2,2)$. Find $f(1,2,2)$, $f_x(1,2,2)$, $f_y(1,2,2)$, $f_z(1,2,2)$.

Local linear approximation of f at (1,2,2) is

$$L(x,y,z) = f(1,2,2) + f_x(1,2,2)(x-1) + f_y(1,2,2)(y-2) + f_z(1,2,2)(z-2) \quad \text{--- (1)}$$

Given that

$$L(x,y,z) = 3x - y + 2z + 7$$
$$= 3(x-1) + 3 - (y-2) - 2 + 2(z-2) + 4 + 7$$
$$= 3(x-1) - (y-2) + 2(z-2) + 12 \quad \text{--- (2)}$$

Comparing (1) and (2)

$f(1,2,2) = 12 \qquad f_x(1,2,2) = 3$

$f_y(1,2,2) = -1 \qquad f_z(1,2,2) = 2$

Maxima and Minima of functions of two Variables

A function of two variables is said to have a relative maximum at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that

$$f(x_0, y_0) \geq f(x, y)$$

for all points (x, y) that lie inside the disk and f is said to have an absolute maximum at (x_0, y_0) if

$$f(x_0, y_0) \geq f(x, y)$$

for all points (x, y) in the domain of f.

A function f of two variables is said to have a relative minimum at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that

$$f(x_0, y_0) \leq f(x, y)$$

for all points (x, y) that lie inside the disk and f is said to have an absolute minimum at (x_0, y_0) if

$f(x_0, y_0) \leq f(x, y)$, for all points (x, y) in the domain of f.

If f has either a relative maximum or a relative minimum at (x_0, y_0) then f is said to have a <u>relative extremum</u> at (x_0, y_0).

If f has either an <u>absolute maximum or absolute minimum</u> at (x_0, y_0) then f is said to have an absolute extremum at (x_0, y_0).

Bounded Set

A set of points in 2-space is called bounded if the entire set can be contained with in some rectangle and is called unbounded if there is no rectangle that can contain all the points of the set.

A set of points in 3-space is bounded if the entire set can be contained with in some box.

Critical point

A point (x_0, y_0) in the domain of a function $f(x,y)$ is called a <u>critical</u>

point of the function if $f_x(x_0,y_0)=0$ and $f_y(x_0,y_0)=0$ or if one or both partial derivatives do not exist at (x_0,y_0).

Saddle point

A differentiable function $f(x,y)$ has a neither a relative maximum nor a relative minimum at (x_0,y_0), the point (x_0,y_0) is called a saddle point of f.

The Second partial test

u.a. Let f be a function of two variables with continuous second order partial derivatives in some disk centered at a critical point (x_0,y_0) and let

$$D = f_{xx}(x_0,y_0)\,f_{yy}(x_0,y_0) - f_{xy}^2(x_0,y_0).$$

1) If $D>0$ and $f_{xx}(x_0,y_0)>0$ then f has a relative minimum at (x_0,y_0)

2) If $D>0$ and $f_{xx}(x_0,y_0)<0$ then f has a relative maximum at (x_0,y_0).

3) If $D<0$, then f has a saddle point at (x_0, y_0).

4) If $D=0$, then no conclusion can be drawn.

Problems

1. Locate all relative extrema and saddle points of

$$f(x,y) = 3x^2 - 2xy + y^2 - 8y$$

The critical points of f are

$f_x(x,y)=0$ and $f_y(x,y)=0$

$$f_x(x,y) = 6x - 2y = 0$$

$$f_y(x,y) = -2x + 2y - 8 = 0$$

$$\Rightarrow -2x + 2y = 8$$

$$\begin{aligned} 6x - 2y &= 0 \\ -2x + 2y &= 8 \\ \hline 4x &= 8 \\ x &= 2 \end{aligned} \rightarrow \begin{aligned} 2y &= 6 \times 2 \\ y &= 6 \end{aligned}$$

$\therefore$ $(2, 6)$ is the critical point.

$f_{xx}(x,y) = 6$, $f_{xx}(2,6) = 6$

$f_{yy}(x,y) = 2$ $f_{yy}(2,6) = 2$

$f_{xy}(x,y) = -2$ $f_{yy}(2,6) = -2$

$D = f_{xx}(x_0 y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0 y_0)$

$= 6 \times 2 - (-2)^2 = 8 > 0$

$f_{xx}(2,6) = 6 > 0$

$\therefore$ f has a relative minimum at

$\underline{\underline{(2,6)}}$

2. Find the relative extreme values of the function $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

Critical points $\Longrightarrow f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$

$f_x(x,y) = y - 2x - 2 = 0 \Longrightarrow y - 2x = 2$

$f_y(x,y) = x - 2y - 2 = 0 \Longrightarrow x - 2y = 2$

$$\left.\begin{aligned} y - 2x &= 2 \\ -2y + x &= 2 \end{aligned}\right\} \Longrightarrow \begin{aligned} 2y - 4x &= 4 \\ -2y + x &= 2 \\ \hline -3x &= 6 \\ x &= -2 \end{aligned}$$

$y = 2 + 2 \times -2 \underline{\underline{= -2}}$

$\underline{\underline{(x,y) = (-2, -2)}}$

$f_{xx} = -2 \quad f_{yy} = -2 \quad f_{xy} = 1$

$D = f_{xx} f_{yy} - f_{xy}^2 = 4 - 1 = 3 > 0$

$f_{xx} < 0$

$\therefore$ f has a relative maximum at $(-2,-2)$

3. Find the relative extreme values of

$f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy$

$f_x(x,y) = -6x + 6y = 0 \implies x = y$

$f_y(x,y) = 6y - 6y^2 + 6x = 0$

$\implies 6y - 6y^2 + 6y = 0$

$\implies 12y - 6y^2 = 0$

$\implies 6y(2 - y) = 0$

$\implies 2 - y = 0 \implies y = 2$

or $6y = 0 \implies y = 0$

$\therefore x = y \implies y = 2 \implies x = 2$

$y = 0 \implies x = 0$

Critical points $(0,0)$ and $(2,2)$

$f_{xx} = -6 \quad f_{yy} = 6 - 12y \quad f_{xy} = 6$

$D = f_{xx} f_{yy} - f_{xy}^2$

Critical points	f_{xx}	f_{yy}	f_{xy}	$D = f_{xx}f_{yy} - f_{xy}^2$
$(0,0)$	-6	0	6	$-72 < 0$
$(2,2)$	$-6 < 0$	18	6	$72 > 0$

At $(0,0)$ $D < 0 \implies$ Saddle point

At $(2,2)$ $D > 0 \implies f_{xx} < 0 \implies f$ has Relative maximum at $(2,2)$

4. Locate all relative extrema and saddle points of $f(x,y) = 4xy - x^4 - y^4$

Critical points

$f_x(x,y) = 4y - 4x^3 = 0 \implies y = x^3$

$f_y(x,y) = 4x - 4y^3 = 0 \implies x = y^3$

$x = y^3$

$\implies x = (x^3)^3 = x^9$

$x^9 - x = 0 \implies x(x^8 - 1) = 0$

$x = 0 \quad x^8 = 1$

$x = \pm 1$

$$\left.\begin{array}{ll} x = 0 & y = 0 \\ x = 1 & y = 1 \\ x = -1 & y = -1 \end{array}\right\} \text{critical points are } (0,0), (1,1), (-1,-1)$$

$f_{xx}(x,y) = -12x^2 \qquad f_{yy} = -12y^2$

$f_{xy} = 4$

Critical point	f_{xx}	f_{yy}	f_{xy}	$D = f_{xx}f_{yy} - f_{xy}^2$
$(0,0)$	0	0	4	$-16 < 0$
$(1,1)$	$-12 < 0$	-12	4	$128 > 0$
$(-1,-1)$	$-12 < 0$	-12	4	$128 > 0$

At $(1,1)$ and $(-1,1) \Longrightarrow D>0$ and $f_{xx}<0$

$\therefore$ Relative maximum occur at $(1,1)$ and $(-1,-1)$

At $(0,0)$ $D<0$ $\therefore$ It is a saddle point

5. Find the maximum and minimum values of

$f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

For critical points

$f_x(x,y) = 3x^2 + 3y^2 - 30x + 72 = 0$

$f_y(x,y) = 6xy - 30y = 0$

$6y(x-5) = 0$

$y = 0$ or $x - 5 = 0$

$x = 5$

$y=0 \Rightarrow 3x^2 - 30x + 72 = 0$

$$x^2 - 10x + 24 = 0$$

$$(x-4)(x-6) = 0$$

$$\underline{x = 4, 6}$$

$x = 5 \Rightarrow 75 + 3y^2 - 150 + 72 = 0$

$$3y^2 = 3 \Rightarrow y^2 = 1 \quad \underline{y = \pm 1}$$

$\therefore$ ~~...~~ critical points are

$(4,0)$, $(6,0)$, $(5,1)$, $(5,-1)$

$f_{xx} = 6x - 30 \qquad f_{yy} = 6x - 30$

$f_{xy} = 6y$

C.P	$f_{xx} = 6x-30$	$f_{yy} = 6x-30$	$f_{xy} = 6y$	$D = f_{xx}f_{yy} - f_{xy}^2$	
$(4,0)$	$-6 < 0$	-6	0	$36 > 0$	Relative maximum
$(6,0)$	$6 > 0$	6	0	$36 > 0$	Relative minimum
$(5,1)$	0	0	6	$-36 < 0$	Saddle point
$(5,-1)$	0	0	-6	$-36 < 0$	Saddle point.

Relative maximum at $(4,0)$ and Relative minimum at $(6,0)$.

The saddle points are $(5,1)$ and $(5,-1)$

6. Locate all relative extrema and saddle points of $f(x,y) = y^2 + xy + 4y + 2x + 3$

The critical points are

$f_x(x,y) = y + 2 = 0 \implies y = -2$

$f_y(x,y) = 2y + x + 4 = 0 \implies 2x - 2 + x + 4$

$\implies \underline{\underline{x = 0}}$

c.p $(0,-2)$

$f_{xx} = 0 \qquad f_{yy} = 2 \qquad f_{xy} = 1$

$D = f_{xx} f_{yy} - f_{xy}^2 = 0 \times 2 - 1 = -1 < 0$

$\therefore$ $(0,-2)$ $\underline{\underline{\text{is the saddle point}}}$

7. Locate all relative extrema and saddle points of $f(x,y) = 2xy - x^3 - y^2$

Critical points

$f_x(x,y) = 2y - 3x^2 = 0$

$f_y(x,y) = 2x - 2y = 0 \implies x - y = 0 \implies x = y$

$\implies 2x - 3x^2 = 0 \qquad x(2-3x) = 0$

$\implies x = 0$ or $3x = 2$, $x = 2/3$ $\qquad x = y \implies y = 0, 2/3$

Critical points are $(0,0)$ $(2/3, 2/3)$

$f_{xx} = -6x \quad f_{yy} = -2 \quad f_{xy} = 2$

C.P	f_{xx}	f_{yy}	f_{xy}	$D = f_{xx}f_{yy} - f_{xy}^2$	
$(0,0)$	0	2	2	$-4 < 0$	Saddle point
$(2/3, 2/3)$	$-4 < 0$	-2	2	$4 > 0$	Relative maxim

Relative maximum occur at $(2/3, 2/3)$

Saddle point $(0,0)$

8. Locate all relative maxima, relative minima and saddle point, if any, for the function

$x^2 + 2x + y - e^y$

$f(x,y) = x^2 + 2x + y - e^y$

Critical points $\Rightarrow$

$f_x(x,y) = 2x + 2 = 0 \longrightarrow x = -1$

$f_y(x,y) = 1 - e^y = 0 \Longrightarrow e^y = 1 \quad \boxed{e^0 = 1}$

$\Longrightarrow y = 0$

critical point is $(-1, 0)$

$f_{xx} = 2 \quad f_{yy} = -e^y \quad f_{xy} = 0$

$D = f_{xx}f_{yy} - f_{xy}^2 = 2 \times -e^0 - 0^2 = -2 < 0$

$\therefore (-1, 0)$ is a saddle point

Absolute Extrema

Extreme Value theorem

If $f(x,y)$ is continuous on a closed and bounded set R, then f has both an absolute maximum and absolute minimum on R.

Theorem

If a function f of two variables has an absolute extremum at an interior point of its domain, then this extremum occurs at a critical point.

Finding absolute extrema on closed and Bounded Sets

Steps

1. Find the critical points of f that lie in the interior of R.

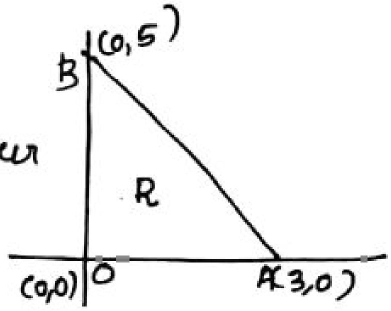
2. Find all boundary points at which the absolute extrema can occur.

3. Evaluate $f(x,y)$ at the points obtained in the preceeding steps. The largest of these values is the absolute maximum and the smallest the absolute minimum.

Problems

1. Find the absolute maximum and minimum value of $f(x,y) = 3xy - 6x - 3y + 7$ on the closed triangular region R with vertices $(0,0)$, $(3,0)$ and $(0,5)$.

$f(x,y) = 3xy - 6x - 3y + 7$



Interior of R $\Rightarrow$ extremum occur at critical point.

$f_x(x,y) = 3y - 6 = 0$

$3y = 6$

$y = 2$

$f_y(x,y) = 3x - 3 = 0 \Rightarrow x = 1$

$\therefore (1,2)$ is the critical points.

Boundary points

on OA $\quad y = 0 \quad f(x,y) = -6x + 7$

$f_x(x,y) = -6 \neq 0 \Rightarrow$ There is no critical point. $\therefore$ the extreme value only occur at end points. ie $(0,0)$ and $(3,0)$ and $(0,5)$

On OB $\quad x = 0 \quad f(x,y) = -3y + 7$

$f_y(x,y) = -3 \neq 0 \Rightarrow$ There is no critical point $\therefore$ the extreme value only occur at end points ie $(0,0)$ and $(0,5)$.

on AB

(3, 0) and (0, 5)

$$\boxed{\text{equation of line joining 2 points} \quad \frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}}$$

$$\frac{y-0}{5-0} = \frac{x-3}{0-3} \implies y = \frac{-5}{3}(x-3)$$

$$y = -\frac{5}{3}x + 5$$

$$f(x,y) = 3x\left(-\frac{5}{3}x+5\right) - 6x - 3\left(-\frac{5}{3}x+5\right) + 7$$

$$= -5x^2 + 15x - 6x + 5x - 15 + 7$$

$$= -5x^2 + 14x - 8$$

$$f_x(x,y) = -10x + 14 = 0 \implies 10x = 14, \quad x = \frac{14}{10} = 7/5$$

$$y = -\frac{5}{3} \times \frac{7}{5} + 5 = -\frac{7}{3} + 5 = \underline{8/3}$$

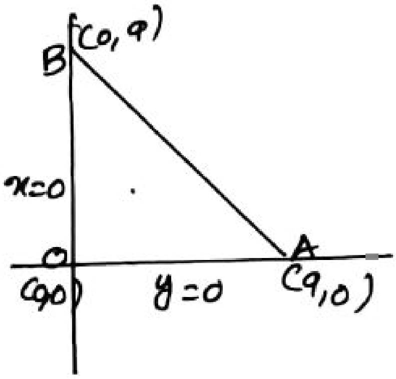
Crona C.P = (7/5, 8/3)

(x, y)	(1, 2)	(0, 0)	(3, 0)	(0, 5)	(7/5, 8/3)	
$f(x,y) = 3xy - 6x - 3y + 7$	1	7	-11	-8	9/5	

Absolute maxium at (0, 0) (height value)

Absolute minimum at (3, 0) (lowest value)

7. Find the absolute maximum and minimum value of $f(x,y) = 2+2x+2y-x^2-y^2$ on the triangular region in the first quadrant bounded by the lines $x=0$ $y=0$ $y=9-x$.



Interior points $\Rightarrow$ extremum occur at critical point.

$$f(x,y) = 2+2x+2y-x^2-y^2$$

$$f_x(x,y) = 2-2x = 0$$
$$\Longrightarrow 2x = 2$$
$$x = 1$$

$$f_y(x,y) = 2-2y = 0 \Longrightarrow y = 1$$

critical point $(1,1)$.

<u>Boundary points</u>

On <u>OA</u> $y=0$

$$f(x,y) = 2+2x-x^2$$

$$f_x(x,y) = 2-2x = 0 \Longrightarrow 2x = 2 \Longrightarrow \underline{x=1}$$

$\therefore$ The critical point $(1,0)$

<u>On OB</u> $x=0$

$$f(x,y) = 2+2y-y^2$$

$$f_y(x,y) = 2-2y = 0 \Longrightarrow 2 = 2y \Rightarrow y = 1$$

critical point $(0,1)$

On AB

Straight line joining $(9,0)$ and $(0,9)$

$$\frac{y-0}{9-0} = \frac{x-9}{0-9} \implies y = -(x-9)$$

$$y = 9-x$$

$$f(x,y) = 2+2x+2(9-x)-x^2-(9-x)^2$$
$$= -61+18x-2x^2$$

$$f_x(x,y) = 18-4x = 0 \implies x = 9/2$$

$$y = 9-\frac{9}{2} = 9/2$$

Critical point $(9/2, 9/2)$

(x,y)	$(0,0)$	$(9,0)$	$(0,9)$	$(1,1)$	$(1,0)$	$(0,1)$	$(9/2,9/2)$
$f(x,y) = 2+2x+2y-x^2-y^2$	2	-61	-61	4	3	3	$-\frac{41}{2}$

Largest Value $\rightarrow 4$

lowest Value $\rightarrow -61$

$\therefore$ Absolute maximum occur at $(1,1)$

Absolute minimum occur at $(9,0)$ and $(0,9)$

3. Determine the dimensions of a rectangular box open at the top having a volume of 32 ft^3 and requiring the least amount of material for the construction.

Let x = length of the box

y = width of the box

z = height of the box

S = Surface area of the box

The box with least surface area requires the least amount of material. So our object is minimize the surface area

$$S = xy + 2xz + 2yz$$

Volume $V = xyz = 32$

$$z = \frac{32}{xy}$$

$$S = xy + 2x\frac{32}{xy} + 2y\frac{32}{xy}$$

$$S = xy + \frac{64}{y} + \frac{64}{x}$$

At extreme points $S_x(x,y) = 0$ and $S_y(x,y) = 0$

$$S_x(x,y) = y - \frac{64}{x^2} = 0 \implies x^2y = 64$$

$$S_y(x,y) = x - \frac{64}{y^2} = 0 \implies xy^2 = 64$$

$$\implies x^2y = xy^2$$

$$x^2y - xy^2 = 0$$

$$\Rightarrow xy(x-y) = 0$$

$$\Rightarrow x - y = 0$$

$$\underline{\underline{x = y}}$$

$$x^2 \times x = 64 \Rightarrow x^3 = 64$$

$$\Rightarrow \underline{\underline{x = 4}}$$

$$\underline{\underline{y = 4}}$$

$$z = \frac{32}{4 \times 4} = 2$$

$\therefore \quad x = 4 \quad y = 4 \quad z = 2$

$$S = xy + 2xz + 2yz = 4 \times 4 + 2 \times 4 \times 2 + 2 \times 4 \times 2$$

$$= \underline{\underline{48}}$$