Module - 2

Multivariable Calculus - Differentiation

Limit of a function of two variables

A function of two independent variables $f(x,y)$ approaches the limit L as (x,y) approaches (x_0,y_0) and write $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if the value of $f(x,y)$ can be made as close as we like to the number L by restricting the point (x,y) to be sufficiently close to (but different from) the point (x_0,y_0).

Formal definition:

Let f be a function of two variables, and assume that f is defined at all points of some open disk centered at (x_0, y_0), except possibly at (x_0,y_0). We will write $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if given any number $\epsilon > 0$, we can find a number $\delta > 0$ such that $f(x,y)$ satisfies $|f(x,y) - L| < \epsilon$ whenever the distance between (x,y) and (x_0,y_0) satisfies

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta.$$

Stated informally, if $f(x,y)$ has a limit L as (x,y) approaches (x_0,y_0), then the value of $f(x,y)$ gets closer and closer to L as the distance between (x,y) and (x_0,y_0) approaches zero.

→ If the limit of $f(x,y)$ fails to exist as $(x,y) \to (x_0,y_0)$ along some smooth curve, or if $f(x,y)$ has different limits as $(x,y) \to (x_0,y_0)$ along two different smooth curves, then the limit of $f(x,y)$ does not exist as $(x,y) \to (x_0,y_0)$.

Problems:

Limits by substitution

1. Find $\lim_{(x,y)\to(1,4)} \left[5x^3y^2 - 9\right]$

$$= \lim_{(x,y)\to(1,4)} 5x^3y^2 - \lim_{(x,y)\to(1,4)} 9$$

$$= 5 \times 1^3 \times 4^2 - 9 = \underline{71}$$

2. Find $\lim_{(x,y)\to(3,4)} \sqrt{x^2+y^2-1}$

$$= \sqrt{3^2+4^2-1} = \sqrt{24}$$

3. Find $\lim_{(x,y)\to(0,\pi/4)} \sec x \tan y$

$$= \sec 0 \tan \pi/4$$

$$= 1 \times 1 = \underline{1}$$

4. $$\lim_{(x,y)\to(1,0)} \frac{x \sin y}{x^2+1} = \frac{1 \times \sin 0}{1^2+1} = \frac{0}{2} = \underline{0}$$

Limits by eliminating zero denominator algebraically :-

5. $$\lim_{(x,y)\to(0,0)} \frac{x^2-4xy}{\sqrt{x}-2\sqrt{y}}$$

Here denominator approaches zero as $(x,y) \to (0,0)$. So we have to eliminate it by using some algebraic method.

$$\frac{x^2-4xy}{\sqrt{x}-2\sqrt{y}} = \frac{x^2-4xy}{\sqrt{x}-2\sqrt{y}} \times \frac{\sqrt{x}+2\sqrt{y}}{\sqrt{x}+2\sqrt{y}}$$

$$= \frac{(x^2-4xy)(\sqrt{x}+2\sqrt{y})}{x-4y}$$

$$= \frac{x\cancel{(x-4y)}(\sqrt{x}+2\sqrt{y})}{\cancel{x-4y}}$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2-4xy}{\sqrt{x}-2\sqrt{y}} = \lim_{(x,y)\to(0,0)} x(\sqrt{x}+2\sqrt{y})$$

$$= 0(0+2\times 0) = \underline{0}$$

6. $$\lim_{(x,y)\to(1,1)} \frac{x^2-2xy+y^2}{x-y}$$

$$\frac{x^2-2xy+y^2}{x-y} = \frac{(x-y)^2}{x-y} = x-y$$

$$\therefore \lim_{(x,y)\to(1,1)} \frac{x^2-2xy+y^2}{x-y} = \lim_{(x,y)\to(1,1)} (x-y)$$

$$= 1-1 = 0$$

7. $$\lim_{(x,y)\to(1,1)} \frac{xy-y-2x+2}{x-1}$$

$$\frac{xy-y-2x+2}{x-1} = \frac{y(x-1)-2(x-1)}{x-1}$$

$$= \frac{(x-1)(y-2)}{x-1} = y-2$$

$$\therefore \lim_{(x,y)\to(1,1)} \frac{xy-y-2x+2}{x-1} = \lim_{(x,y)\to(1,1)} y-2$$

$$= 1-2 = \underline{\underline{-1}}$$

8. $$\lim_{(x,y)\to(0,0)} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}$$

$$\frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}} \times \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}$$

$$= \frac{\left[(x-y)+2(\sqrt{x}-\sqrt{y})\right](\sqrt{x}+\sqrt{y})}{x-y}$$

$$= \frac{(x-y)(\sqrt{x}+\sqrt{y})+2(x-y)}{x-y}$$

$$\frac{(x-y)\left[(\sqrt{x}+\sqrt{y})+2\right]}{x-y} = (\sqrt{x}+\sqrt{y})+2$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} (\sqrt{x}+\sqrt{y})+2$$

$$= 0+0+2 = \underline{\underline{2}}$$

By considering different paths of approach, show that the following functions have no limit as $(x,y)\to(0,0)$

9. $\lim\limits_{(x,y)\to(0,0)} \frac{-xy}{x^2+y^2}$

Consider the path along $y=kx$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=kx}} \frac{-xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{-x \times kx}{x^2+k^2x^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{-k\cancel{x^2}}{\cancel{x^2}(1+k^2)} = \frac{-k}{1+k^2}$$

ie, the limit depends on k.

For example if we approach $(x,y)\to(0,0)$ along $y=x$, the value of limit $= \frac{-1}{1+1^2} = \frac{-1}{2}$

If we approach along $y=2x$, then value of limit $= \frac{-2}{1+2^2} = \frac{-2}{5}$

ie, function has different limits as $(x,y)\to(0,0)$ along different path

$\therefore$ limit does not exist.

10. $\lim_{(x,y)\to(0,0)} \frac{-x}{\sqrt{x^2+y^2}}$

$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } x=ky}} \frac{-x}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{-ky}{\sqrt{k^2y^2+y^2}}$

$= \lim_{(x,y)\to(0,0)} \frac{-k\cancel{y}}{\cancel{y}\sqrt{k^2+1}} = \frac{-k}{\sqrt{1+k^2}}$,

which depends on k.

$\therefore$ The limit does not exist.

11. $\lim_{(x,y)\to(0,0)} \frac{x^4-y^2}{x^4+y^2}$

$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=kx^2}} \frac{x^4-y^2}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^4-k^2x^4}{x^4+k^2x^4}$

$= \lim_{(x,y)\to(0,0)} \frac{\cancel{x^4}(1-k^2)}{\cancel{x^4}(1+k^2)} = \frac{1-k^2}{1+k^2}$,

which depends on k

$\therefore$ The limit does not exist.

H.W Evaluate the limits (if they exist)

1. $\lim_{(x,y)\to(0,0)} \frac{2x^2-y^2+3}{x^2+2y^2-2}$

2. $\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4}$

3. $\lim_{(x,y)\to(0,0)} \frac{4x^6y^2}{x^{12}+y^4}$

Continuity

A function $f(x,y)$ is said to be continuous at (x_0, y_0) if $f(x_0,y_0)$ is defined and if $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

→ If f is continuous at every point in a domain D, then we say f is continuous on D.

→ If f is continuous at every point in xy-plane, then we say that f is continuous everywhere.

Theorem:

(a) If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x,y) = g(x)h(y)$ is continuous at (x_0,y_0).

(b) If $h(x,y)$ is continuous at (x_0,y_0) and $g(u)$ is continuous at $u = h(x_0,y_0)$, then the composition $g(h(x,y))$ is continuous at (x_0,y_0).

(c) If $f(x,y)$ is continuous at (x_0,y_0) and if $x(t)$ and $y(t)$ are continuous at t_0 with $x(t_0) = x_0$ and $y(t_0) = y_0$, then the composition $f(x(t), y(t))$ is continuous at t_0.

Note: Polynomial functions are continuous everywhere.

Qn 1. Show that the functions $f(x,y) = 3x^2y^5$ and $f(x,y) = \sin(3x^2y^5)$ are continuous everywhere.

Soln. The polynomial functions $g(x) = 3x^2$ and $\cancel{g(y)} = h(y) = y^5$ are continuous everywhere.

$\therefore$ Their product $f(x,y) = g(x)h(y) = 3x^2y^5$ is continuous everywhere.

Also $\sin u$ is also continuous at every real number u.

$\therefore$ The composition $\sin(3x^2y^5)$ is continuous everywhere.

2. Show that $f(x,y) = \begin{cases} \dfrac{4x^2y}{x^3+y^3}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$

is continuous at every point except the origin.

Soln:
$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \lim_{(x,y)\to(x_0,y_0)} \frac{4x^2y}{x^3+y^3}$$
$$= \frac{4x_0^2y_0}{x_0^3+y_0^3} = f(x_0,y_0) \text{ for all } (x_0,y_0) \neq (0,0)$$

Now find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ through the path $y = mx$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{4x^2(mx)}{x^3+(mx)^3}$$

$$= \frac{4m}{1+m^3}$$, which depends on m.

$\therefore$ The limit does not exist as $(x,y)\to(0,0)$

$\therefore$ The function is not continuous at $(0,0)$.

Higher order partial derivatives

If f is a function of x and y, we can define the second order partial derivatives defined by

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

Any higher order partial derivatives (third order, fourth order, etc) can be obtained by successive differentiation.

1. Find all second order partial derivatives of $f(x,y) = x^2y^3 + x^4y$

Soln:
$$\frac{\partial f}{\partial x} = 2xy^3 + 4x^3y$$

$$\frac{\partial f}{\partial y} = 3x^2y^2 + x^4$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(2xy^3 + 4x^3y) = 2y^3 + 12x^2y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(3x^2y^2 + x^4) = 6x^2y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(3x^2y^2 + x^4) = 6xy^2 + 4x^3$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(2xy^3 + 4x^3y) = 6xy^2 + 4x^3$$

2. Let $f(x,y) = y^2e^x + y$, find f_{xyy}

$$f_{xyy} = \frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial^2 f}{\partial y^2}(y^2e^x) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(y^2e^x)\right)$$

$$= \frac{\partial}{\partial y}(2ye^x) = 2e^x$$

Theorem: Let f be a function of two variables x and y. If f_{xy} and f_{yx} are continuous on some open disk then $f_{xy} = f_{yx}$ on that disk.

3. If $f(x,y) = x^y$, verify that $f_{xy} = f_{yx}$

Soln: $f = x^y$

Taking logarithm on both sides

$\log f = y \log x$

Differentiating w.r.t x and y bothside

$$\frac{1}{f}\frac{\partial f}{\partial x} = y \times \frac{1}{x} \Rightarrow \frac{\partial f}{\partial x} = f\frac{y}{x}$$

$$\frac{1}{f}\frac{\partial f}{\partial y} = \log x \Rightarrow \frac{\partial f}{\partial y} = f \log x$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(f \log x)$$

$$= f\frac{\partial}{\partial x}(\log x) + \log x \frac{\partial f}{\partial x}$$

$$= f \times \frac{1}{x} + \log x \cdot f\frac{y}{x}$$

$$= \frac{f}{x}(1 + y\log x)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(f \cdot \frac{y}{x}\right)$$

$$= f\frac{\partial}{\partial y}\left(\frac{y}{x}\right) + \frac{y}{x}\frac{\partial f}{\partial y}$$

$$= f \times \frac{1}{x} + \frac{y}{x} f \log x$$

$$= \frac{f}{x}(1 + y\log x)$$

$$\therefore \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

H.W ① If $z = e^x(x\cos y - y\sin y)$, prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

② Check whether $\frac{\partial^3 f}{\partial y \partial x^2} = \frac{\partial^3 f}{\partial x^2 \partial y}$ if $f = y^2 e^x + x^2 y^3 + 16$.

③ If $u = \sin^{-1}\left(\frac{x}{y}\right)$, verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

④ If $w = \ln(2x+3y)$, verify $w_{xy} = w_{yx}$

⑤ Show that $u(x,t) = \sin(x - ct)$, is a solution of $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$

⑥ Find $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial z}$ if ① $w = 2ye^z \sin xz$

② $w = \frac{x^2 - z^2}{y^2 + z^2}$

Differentiability

For a function f of two variables x & y the symbol Δf called the increment of f, denotes the change in the value of $f(x,y)$ that results when (x,y) varies from some initial position (x_0, y_0) to some new position $(x_0+\Delta x, y_0+\Delta y)$

ie, $\Delta f = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$

Defn: A function of two variables x & y, $f(x,y)$ is said to be differentiable at (x_0,y_0), provided $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ both exist and

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta f - f_x(x_0,y_0)\Delta x - f_y(x_0,y_0)\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}} = 0$$

Eg: Prove that $f(x,y) = x^2+y^2$ is differentiable at $(0,0)$.

$$\Delta f = f(0+\Delta x, 0+\Delta y) - f(0,0)$$
$$= (\Delta x)^2+(\Delta y)^2 - (0^2+0^2) = (\Delta x)^2+(\Delta y)^2$$

$f_x(x,y) = 2x \Rightarrow f_x(0,0) = 2\times 0 = 0$

$f_y(x,y) = 2y \Rightarrow f_y(0,0) = 2\times 0 = 0$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta f - f_x(x_0,y_0)\Delta x - f_y(x_0,y_0)\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{((\Delta x)^2+(\Delta y)^2) - 0 - 0}{\sqrt{(\Delta x)^2+(\Delta y)^2}} = \lim_{(\Delta x,\Delta y)\to(0,0)} \sqrt{(\Delta x)^2+(\Delta y)^2}$$

$$= \sqrt{0^2+0^2} = 0$$

$\therefore f$ is differentiable at $(0,0)$.

Note:

* If a function is differentiable at a point, then it is continuous at that point. But its converse need not be true.
* If all first order partial derivatives of f exist and are continuous at a point, then f is differentiable at that point.

Total differential:

If $z = f(x,y)$ is differentiable at (x_0,y_0), then the total differential of z (or f) at (x_0,y_0) is given by

$$dz = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy$$

Similarly for a function of three variable, $w = f(x,y,z)$, the total differential dw at (x_0,y_0,z_0) is given by

$$dw = f_x(x_0,y_0,z_0)\,dx + f_y(x_0,y_0,z_0)\,dy + f_z(x_0,y_0,z_0)\,dz$$

Note: It is common practice to omit the subscripts and write the above equations as

$$\boxed{\begin{aligned} dz &= f_x(x,y)\,dx + f_y(x,y)\,dy \\ dw &= f_x(x,y,z)\,dx + f_y(x,y,z)\,dy + f_z(x,y,z)\,dz \end{aligned}}$$

Qn 1. Approximate the change in $z = xy^2$ from its value at $(0.5, 1.0)$ to its value at $(0.503, 1.004)$. Compare the magnitude of error in this approximation with the distance between the points $(0.5, 1.0)$ and $(0.503, 1.004)$.

$z = xy^2$

$f_x(x,y) = y^2 \qquad f_y(x,y) = 2xy$

$\therefore dz = y^2 dx + 2xy\, dy$

$(x,y) = (0.5, 1.0)$

$dx \approx \Delta x = 0.503 - 0.5 = 0.003$

$dy \approx \Delta y = 1.004 - 1.0 = 0.004$

$\therefore dz = 1^2 \times 0.003 + 2 \times 0.5 \times 1 \times 0.004$
$= 0.007$

Now to find the actual value Δz

At $(x,y) = (0.5, 1)$, $z = xy^2 = 0.5 \times 1^2 = 0.5$

At $(x,y) = (0.503, 1.004)$

$z = (0.503)(1.004)^2 = 0.507032048$

$\therefore \Delta z = 0.507032048 - 0.5$
$= 0.007032048$

$\therefore$ The error in approximating Δz by dz has magnitutde $|dz - \Delta z|$

$= |0.007 - 0.007032048|$

$= 0.000032048$

Distance between $(0.5, 1)$ and $(0.503, 1.004)$ is

$$\sqrt{(0.503-0.5)^2+(1.004-1)^2} = \sqrt{0.003^2+0.004^2}$$

$$= 0.005$$

∴ To compare the magnitude of error with the distance, take the ratio,

$$\frac{|dz-\Delta z|}{\text{distance}} = \frac{0.000032048}{0.005}$$

$$= 0.0064096$$

∴ The magnitude of error in our approximation is about 0.0064096 times the distance between the points.

2. The length, width and height of a rectangular box are measured with an error of atmost 5%. Use a total differential to estimate the maximum percentage error that results if these quantities are used to calculate the diagonal of the box.

Soln: The diagonal D of the box with length x, width y and height z is

$$D = \sqrt{x^2+y^2+z^2}$$

$$\therefore D_x = \frac{1}{2\sqrt{x^2+y^2+z^2}} \times 2x = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$D_y = \frac{y}{\sqrt{x^2+y^2+z^2}} \quad \text{and} \quad D_z = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

Let x_0, y_0, z_0 and $D_0 = \sqrt{x_0^2+y_0^2+z_0^2}$ denote the actual value of length, width, height and diagonal of box. Then total differential dD of D at (x_0, y_0, z_0) is

$$dD = \frac{x_0}{\sqrt{x_0^2+y_0^2+z_0^2}}dx + \frac{y_0}{\sqrt{x_0^2+y_0^2+z_0^2}}dy + \frac{z_0}{\sqrt{x_0^2+y_0^2+z_0^2}}dz$$

If x, y, z and $D = \sqrt{x^2+y^2+z^2}$ are the measured and computed values of length, width, height and diagonal respectively, then, $\Delta x = x - x_0$
$\Delta y = y - y_0$, $\Delta z = z - z_0$.

Also given $\left|\frac{\Delta x}{x_0}\right| \leq 0.005$, $\left|\frac{\Delta y}{y_0}\right| \leq 0.05$
$\left|\frac{\Delta z}{z_0}\right| \leq 0.05$

We have to find an estimate ~~of~~ for the maximum size of $\frac{\Delta D}{D_0}$

$$\frac{\Delta D}{D_0} \simeq \frac{dD}{D_0} = \frac{\frac{x_0\Delta x + y_0\Delta y + z_0\Delta z}{\sqrt{x_0^2+y_0^2+z_0^2}}}{\sqrt{x_0^2+y_0^2+z_0^2}}$$

$$= \frac{1}{x_0^2+y_0^2+z_0^2}\left[x_0\Delta x + y_0\Delta y + z_0\Delta z\right]$$

$$= \frac{1}{x_0^2+y_0^2+z_0^2}\left[x_0^2\frac{\Delta x}{x_0} + y_0^2\frac{\Delta y}{y_0} + z_0^2\frac{\Delta z}{z_0}\right]$$

$$\left|\frac{dD}{D_0}\right| = \frac{1}{x_0^2+y_0^2+z_0^2}\left|x_0^2\frac{\Delta x}{x_0}+y_0^2\frac{\Delta y}{y_0}+z_0^2\frac{\Delta z}{z_0}\right|$$

$$\leq \frac{1}{x_0^2+y_0^2+z_0^2}\left(x_0^2\left|\frac{\Delta x}{x_0}\right|+y_0^2\left|\frac{\Delta y}{y_0}\right|+z_0^2\left|\frac{\Delta z}{z_0}\right|\right)$$

$$\leq \frac{1}{x_0^2+y_0^2+z_0^2}\left(x_0^2(0.05)+y_0^2(0.05)+z_0^2(0.05)\right)$$

$$\leq \frac{(x_0^2+y_0+z_0^2)(0.05)}{x_0^2+y_0^2+z_0^2} = 0.05$$

$$\therefore \left|\frac{dD}{D_0}\right| \leq 0.05$$

$\therefore$ Maximum % error in D is 5%

1. Use a total differential to approximate the change in the value of f from P to Q. Campare your ~~y~~estimate with actual change in f.

ⓐ $f(x,y) = x^2+2xy-4x$
$P=(1,2)$, $Q=(1.01, 2.03)$

ⓑ $f(x,y) = \ln\sqrt{1+xy}$,
$P=(0,2)$ $Q=(-0.09, 1.98)$

ⓒ $f(x,y,z) = 2xy^2z^3$
$P(1,-1,2)$, $Q=(0.99,-1.02,2.02)$

2. The volume $V=\pi r^2 h$ of a right circular cylinder is to be calculated from measured values of r and h. Suppose that r

is measured with an error of not more than 2% and h with an error of not more that 0.5%. Estimate the resulting possible percentage error in the calculation of V.

Local Linear Approximation

If a function f is differentiable at a point, then it can be very closely approximated by a linear function near that point. This is called local linear approximation to f at that point.

For a function $f(x,y)$, that is differentiable at (x_0, y_0), the local linear approximation to f at (x_0, y_0) is

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

For a function $f(x,y,z)$, differentiable at (x_0, y_0, z_0),

$$L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0).$$

Qn. Find the local linear approximation to f at P. Compare the error in approximating f by L at Q with the distance between P and Q.

1. $f(x,y) = \sqrt{x^2+y^2}$ $P(3,4)$, $Q(3.04, 3.98)$

Ans: $f_x(x,y) = \frac{1}{2\sqrt{x^2+y^2}} \times 2x = \frac{x}{\sqrt{x^2+y^2}}$

$f_y(x,y) = \frac{1}{2\sqrt{x^2+y^2}} \times 2y = \frac{y}{\sqrt{x^2+y^2}}$

$f_x(3,4) = \frac{3}{\sqrt{3^2+4^2}} = \frac{3}{5}$

$f_y(3,4) = \frac{4}{\sqrt{3^2+4^2}} = \frac{4}{5}$

$f(3,4) = \sqrt{3^2+4^2} = 5$

$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$= 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4)$

$\therefore$ Local linear approximation L at Q is

$L(3.04, 3.98) = 5 + \frac{3}{5}(3.04-3) + \frac{4}{5}(3.98-4)$

$= 5.008$

$f(3.04, 3.98) = \sqrt{(3.04)^2 + (3.98)^2} = 5.00819$

$\therefore$ Error in approximating f by L

$= |5.008 - 5.00819| = 0.00019$

Distance between P and Q

$= \sqrt{(3.04-3)^2 + (3.98-4)^2} = 0.045$

$\frac{\text{Error}}{\text{distance}} = \frac{0.00019}{0.045} = 0.00422$

$\therefore$ Error in approximating f by L at Q is 0.00422 times the distance between P and Q.

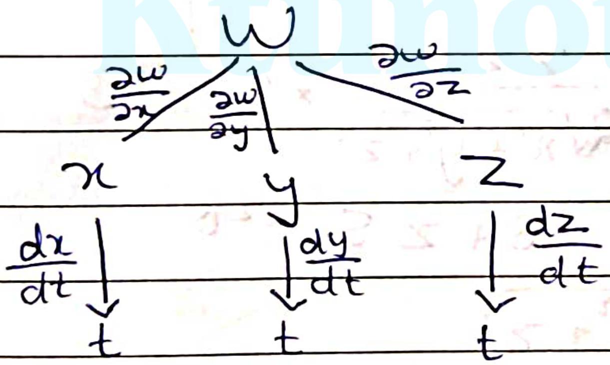
H.W 2. $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$, $P(4,3)$, $Q(3.92, 3.01)$

3. $f(x,y,z) = xyz$; $P(1,2,3)$, $Q(1.001, 2.002, 3.003)$

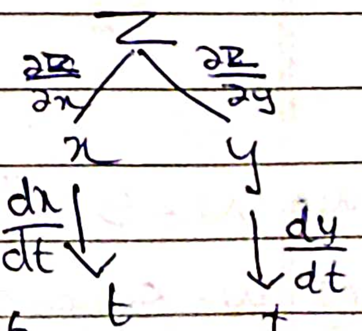
Chain rule:

If $x = x(t)$, $y = y(t)$ and $z = z(t)$ are differentiable at t and if $w = f(x,y,z)$ is differentiable at the point (x,y,z) then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \times \frac{dx}{dt} + \frac{\partial w}{\partial y} \times \frac{dy}{dt} + \frac{\partial w}{\partial z} \times \frac{dz}{dt}$$



1. Suppose $z = x^2 y$, $x = t^2$, $y = t^3$. Use the chain rule to find $\frac{dz}{dt}$ and also check the result by expressing z as a function of t and differentiating directly.



$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

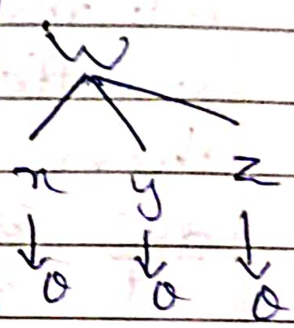
$$= (2xy) \times 2t + (x^2) \times 3t^2$$

$$= 2t^2 \times t^3 \times 2t + t^4 \times 3t^2 = 7t^6$$

$$z = x^2 y = t^4 \times t^3 = t^7$$

$$\therefore \frac{dz}{dt} = \underline{\underline{7t^6}}$$

2. Suppose $w = \sqrt{x^2+y^2+z^2}$, $x = \cos\theta$, $y = \sin\theta$, $z = \tan\theta$. Using chain rule find $\frac{dw}{d\theta}$ when $\theta = \pi/4$



$$\frac{dw}{d\theta} = \frac{\partial w}{\partial x}\frac{dx}{d\theta} + \frac{\partial w}{\partial y}\frac{dy}{d\theta} + \frac{\partial w}{\partial z}\frac{dz}{d\theta}$$

$$= \frac{2x}{2\sqrt{x^2+y^2+z^2}}(-\sin\theta) + \frac{2y}{2\sqrt{x^2+y^2+z^2}}\cos\theta + \frac{2z}{2\sqrt{x^2+y^2+z^2}} \times \sec^2\theta$$

$$= \frac{-x\sin\theta + y\cos\theta + z\sec^2\theta}{\sqrt{x^2+y^2+z^2}}$$

At $\theta = \pi/4$, $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}$, $z = 1$

$$\therefore \left.\frac{dw}{d\theta}\right|_{\theta=\pi/4} = \frac{-\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + 1 \times 2}{\sqrt{\frac{1}{2}+\frac{1}{2}+1}}$$

$$= \frac{2}{\sqrt{2}} = \underline{\underline{\sqrt{2}}}$$

H.W. (1) $w = 5x^2y^3z^4$, $x = t^2$, $y = t^3$, $z = t^4$

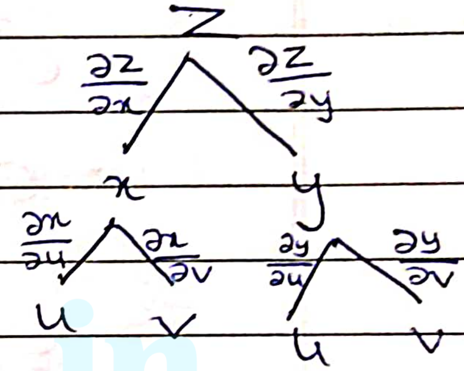
(2) $w = 5\cos xy - \sin xz$, $x = \frac{1}{t}$, $y = t$, $z = t^2$

Chain rules for partial derivatives

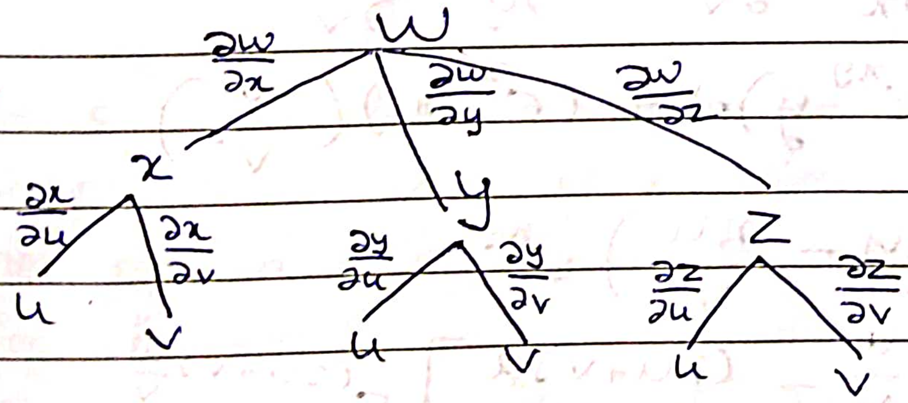
If $x = x(u,v)$ and $y = y(u,v)$ have first order partial derivatives at the point (u,v) and if $z = f(x,y)$ is differentiable at the point (x,y), then $z = f(x,y)$ has first order partial derivatives at (u,v) given by

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$



Similarly if $x = x(u,v)$, $y = y(u,v)$ & $z = z(u,v)$ and $w = f(x,y,z)$

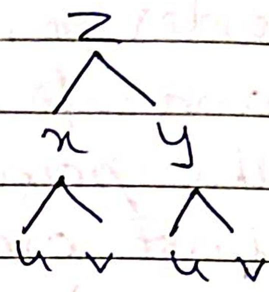


$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$$

Qn 1. Given $z = e^{xy}$, $x = 2u + v$, $y = u/v$

Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ using chain rule.



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= (e^{xy} \cdot y) \times 2 + (e^{xy} \cdot x) \times \frac{1}{v}$$

$$= e^{xy} \left[2y + \frac{x}{v} \right]$$

$$= e^{(2u+v)(\frac{u}{v})} \left[2\frac{u}{v} + \frac{2u+v}{v} \right]$$

$$= \left(\frac{4u+v}{v} \right) e^{\frac{2u^2+uv}{v}}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= (e^{xy} \cdot y) \times 1 + (e^{xy} \cdot x) \left(\frac{-u}{v^2} \right)$$

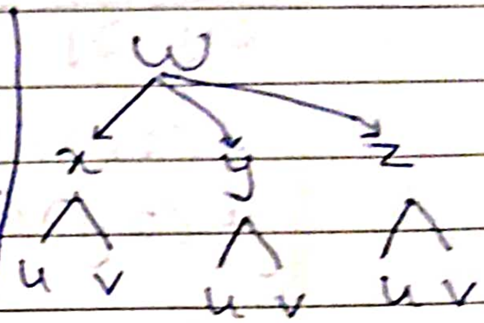
$$= \left(y - \frac{xu}{v^2} \right) e^{xy}$$

$$= \left[\frac{u}{v} - \frac{(2u+v)u}{v^2} \right] e^{(2u+v)\frac{u}{v}}$$

$$= \left[\frac{uv - 2u^2 - uv}{v^2} \right] e^{\frac{2u^2+uv}{v}}$$

$$= \frac{-2u^2}{v^2} e^{\frac{2u^2+uv}{v}}$$

2. Suppose $w = e^{xyz}$, $x = 3u+v$, $y = 3u-v$, $z = u^2 v$. Use chain rule to find $\cancel{\frac{\partial w}{\partial x}}$ $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$



$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$= (e^{xyz} yz)(3) + (e^{xyz} \cdot xz)(3) + (e^{xyz} xy)(2uv)$$

$$= \underline{\underline{e^{xyz}(3yz + 3xz + 2xyuv)}}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$$

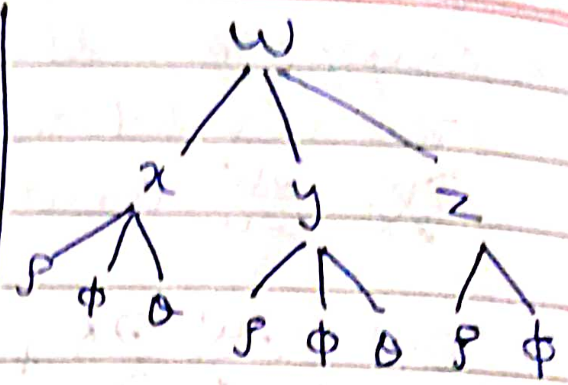
$$= (e^{xyz} \cdot yz)(1) + (e^{xyz} \cdot xz)(-1) + (e^{xyz} xy)(u^2)$$

$$= e^{xyz}(yz - xz + xyu^2)$$

Note: If desired we can express $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ interms of u and v alone by replacing x, y and z by their expressions in terms of u and v.

3. Suppose $w = x^2 + y^2 - z^2$ and $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$. Use ~~appropo~~ appropriate forms of chain rule to find $\frac{\partial w}{\partial \rho}$ and $\frac{\partial w}{\partial \theta}$

$$\frac{\partial w}{\partial \rho} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial \rho}$$



$$= (2x)(\sin\phi\cos\theta) + (2y)(\sin\phi\sin\theta) + (-2z)(\cos\phi)$$

$$= (2\rho\sin\phi\cos\theta)(\sin\phi\cos\theta) + 2(\rho\sin\phi\sin\theta)\cdot(\sin\phi\sin\theta) - 2\rho\cos\phi\cdot(\cos\phi)$$

$$= 2\rho\sin^2\phi\cos^2\theta + 2\rho\sin^2\phi\sin^2\theta - 2\rho\cos^2\phi$$

$$= 2\rho\sin^2\phi(\cos^2\theta + \sin^2\theta) - 2\rho\cos^2\phi$$

$$= 2\rho\sin^2\phi - 2\rho\cos^2\phi$$

$$= 2\rho(\sin^2\phi - \cos^2\phi)$$

$$= -2\rho\cos 2\phi$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial \theta}$$

$$= (2x)(\rho\sin\phi\cdot -\sin\theta) + (2y)(\rho\sin\phi\cos\theta) + (-2z)\times 0$$

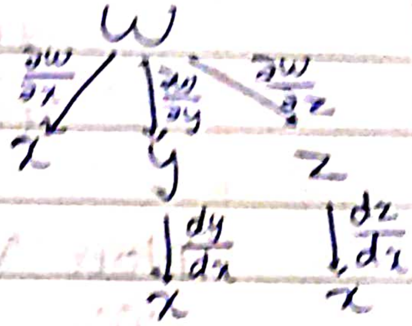
$$= (2\rho\sin\phi\cos\theta)(-\rho\sin\phi\sin\theta) + (2\rho\sin\phi\sin\theta)(\rho\sin\phi\cos\theta) - 0$$

$$= -2\rho^2\sin^2\phi\sin\theta\cos\theta + 2\rho^2\sin^2\phi\sin\theta\cos\theta$$

$$= \underline{0}$$

$\frac{\partial w}{\partial \theta} = 0 \Rightarrow$ w does not vary with θ.

4 Suppose $w = xy + yz$, $y = \sin x$, $z = e^x$
Use appropriate form of chain rule to find $\frac{dw}{dx}$



$$\frac{dw}{dx} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dx}$$

$$= y \times 1 + (x + z)\cos x + y\, e^x$$

$$= \sin x + (x + e^x)\cos x + \sin x\, e^x$$

Note: This result can also be obtained by first expressing w explicitly interms of x as

$$w = x \sin x + \sin x\, e^x$$

$$\therefore \frac{dw}{dx} = [x\cos x + \sin x \times 1] + [\sin x \cdot e^x + e^x \cos x]$$

$$= \sin x + (x + e^x)\cos x + \sin x\, e^x$$

<u>Implicit differentiation</u>

If the function $f(x,y) = c$ defines y implicitly as a differentiable function of x, and if $\frac{\partial f}{\partial y} \neq 0$, then

$$\boxed{\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}} \quad \text{ie } \frac{dy}{dx} = \frac{-f_x}{f_y}$$

classmate

Date

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5. Given that $x^3+y^2x-3=0$. Find $\frac{dy}{dx}$.

Soln: $f(x,y) = x^3+y^2x-3$

$f_x = 3x^2+y^2$, $f_y = 2yx$

$$\therefore \frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-(3x^2+y^2)}{2xy}$$

Note: We can check this by implicit ~~differen~~ differentiation.

$$\frac{d}{dx}(x^3+y^2x-3) = \frac{d}{dx}(0)$$

$$3x^2+\frac{d}{dx}(y^2x)-0=0$$

$$3x^2+(y^2\times 1+x\cdot 2y\frac{dy}{dx})=0$$

$$\therefore \frac{dy}{dx} = \frac{-3x^2-y^2}{2xy}$$

6. Find $\frac{dy}{dx}$ if $x^2+\sin y-2y=0$

Soln: $f(x,y) = x^2+\sin y-2y$

$f_x = 2x$, $f_y = \cos y-2$

$$\therefore \frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-2x}{\cos y-2}$$

Theorem: If $f(x,y,z)=c$ defines z implicitly as a function of x and y, and if $\frac{\partial f}{\partial z} \neq 0$, then

$$\frac{\partial z}{\partial x} = \frac{-\partial f/\partial x}{\partial f/\partial z} \text{ and } \frac{\partial z}{\partial y} = \frac{-\partial f/\partial y}{\partial f/\partial z}$$

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7. Consider the sphere $x^2+y^2+z^2=1$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$.

Soln: $f(x,y,z) = x^2+y^2+z^2$

$$\frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = 2y \quad \frac{\partial f}{\partial z} = 2z$$

$$\therefore \frac{\partial z}{\partial x} = \frac{-\partial f/\partial x}{\partial f/\partial z} = \frac{-2x}{2z} = \frac{-x}{z}$$

$$\frac{\partial z}{\partial y} = \frac{-\partial f/\partial y}{\partial f/\partial z} = \frac{-2y}{2z} = \frac{-y}{z}$$

$\therefore$ At the point $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$,

$$\frac{\partial z}{\partial x} = \frac{-2/3}{2/3} = \underline{\underline{-1}}$$

$$\frac{\partial z}{\partial y} = \frac{-1/3}{2/3} = \underline{\underline{\frac{-1}{2}}}$$

8. If $f(x,y,z) = x^2 - 3yz^2 + xyz - 2 = 0$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ by the formula

$$f(x,y,z) = x^2 - 3yz^2 + xyz - 2$$

$$\frac{\partial f}{\partial x} = 2x + yz, \quad \frac{\partial f}{\partial y} = -3z^2 + xz$$

$$\frac{\partial f}{\partial z} = -6yz + xy$$

$$\frac{\partial z}{\partial x} = \frac{-(2x+yz)}{-6yz+xy}, \quad \frac{\partial z}{\partial y} = \frac{-(-3z^2+xz)}{-6yz+xy}$$

H.W. Find $\frac{dy}{dx}$ using formula and check the result by implicit differentiation

ⓐ $x^2y^3 + \sin y = 0$

ⓑ $x^3 - 3xy^2 + y^4 = 5$

2. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ by the formula

if ⓐ $f(x,y,z) = ye^x - 5\sin 3z - 4z = 0$

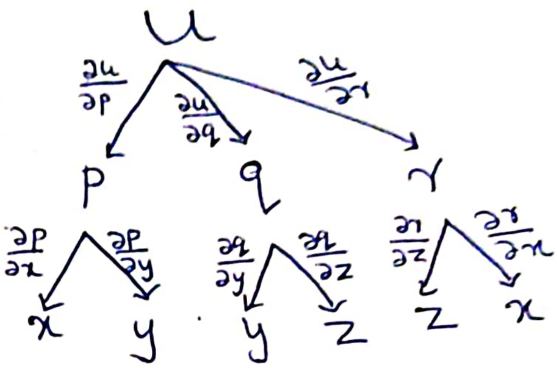
ⓑ $\ln(1+z) + xy^2 + 2z - 1 = 0$

Qn If $u = f(x/y, y/z, z/x)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

Soln: Let $\frac{x}{y} = p$, $\frac{y}{z} = q$, $\frac{z}{x} = r$

$\therefore u = f(p, q, r)$



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\times\frac{\partial p}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$$

$$= \frac{\partial u}{\partial p}\times\frac{1}{y} + \frac{\partial u}{\partial r}\times\frac{-z}{x^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\times\frac{\partial q}{\partial y}$$

$$= \frac{\partial u}{\partial p}\times\frac{-x}{y^2} + \frac{\partial u}{\partial q}\times\frac{1}{z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial q}\times\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\times\frac{\partial r}{\partial z}$$

$$= \frac{\partial u}{\partial q}\times\frac{-y}{z^2} + \frac{\partial u}{\partial r}\times\frac{1}{x}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = x\left(\frac{\partial u}{\partial p}\times\frac{1}{y} + \frac{\partial u}{\partial r}\times\frac{-z}{x^2}\right)$$

$$+ y\left(\frac{\partial u}{\partial p}\times\frac{-x}{y^2} + \frac{\partial u}{\partial q}\times\frac{1}{z}\right) + z\left(\frac{\partial u}{\partial q}\times\frac{-y}{z^2} + \frac{\partial u}{\partial r}\times\frac{1}{x}\right)$$

$$= \frac{\partial u}{\partial p}\left(\frac{x}{y} - \frac{x}{y}\right) + \frac{\partial u}{\partial q}\left(\frac{y}{z} - \frac{y}{z}\right) + \frac{\partial u}{\partial r}\left(-\frac{z}{x} + \frac{z}{x}\right)$$

$$= \underline{\underline{0}}$$