
KTU
NOTES

MODULE - 1.

LINEAR ALGEBRA

Topics

System of linear equations, Solution by Gauss elimination, row echelon form and rank of a matrix, fundamental theorem for linear systems (homogeneous and non-homogeneous, with out proof) Eigen Values and eigen Vectors. Diagonalization of matrices, orthogonal transformation, quadratic forms and their Canonical forms.

Introduction

Matrices

A system of 'mn' numbers of elements arranged in m rows and n columns in a definite order and enclosed by bracket () or [ ] is called an mxn matrix.

The General form of an mxn matrix A is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

In this matrix the elements $\underline{a_{ij}}$ is in i^{th} row and j^{th} column.

A matrix having only one row is called **row matrix** and a matrix having only one column is called **column matrix**.

eg: $[2, 3, 5, 1]$ row matrix

$\begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$ column matrix.

Square matrix is a matrix with number of rows equal to number of columns.

eg: $\begin{bmatrix} 2 & 4 & 6 \\ 9 & 0 & 2 \\ 3 & 5 & 4 \end{bmatrix}$

A square matrix is said to be **diagonal matrix** if $a_{ij} = 0$ for all $i \neq j$.

eg: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

A diagonal matrix in which the diagonal elements are equal is called **scalar matrix**.

eg: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

A diagonal matrix in which all diagonal elements are equal to 1 is called **unit matrix** or identity matrix.

eg: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A matrix in which all the entries are zero is called a zero matrix or null matrix.

A square matrix $A = (a_{ij})_{n \times n}$ is said to be upper triangular if $a_{ij} = 0$ for all $i > j$ and it is said to be lower triangular if $a_{ij} = 0$ for all $i < j$

eg: $\begin{bmatrix} 2 & 4 & 7 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \end{bmatrix}$ upper triangular matrix

$\begin{bmatrix} 2 & 0 & 0 \\ 9 & 4 & 0 \\ -3 & 0 & 0 \end{bmatrix}$ lower triangular matrix

A square matrix A is said to be a symmetric matrix if $A^T = A$ and is said to be skew symmetric if $A^T = -A$.

eg: $\begin{bmatrix} 1 & -3 & 5 \\ -3 & 2 & 7 \\ 5 & 7 & 3 \end{bmatrix}$ symmetric matrix

$\begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix}$ skew symmetric matrix

Note that the main diagonal elements of a skew symmetric matrix must be all zero.

The conjugate of a matrix is obtained by replacing its elements by the corresponding complex conjugate. The conjugate of the matrix A is denoted by $\bar{A}$.

Eg: $A = \begin{bmatrix} 1+i & 2+3i \\ 2 & -3i \end{bmatrix}$ then $\bar{A} = \begin{bmatrix} 1-i & 2-3i \\ 2 & 3i \end{bmatrix}$

The transpose of the conjugate of the matrix A is denoted by A^*.

ie $A^* = (\bar{A})^T$

A square matrix A is said to be Hermitian if $A^* = A$ and is said to be skew Hermitian if $A^* = -A$.

The main diagonal elements of a Hermitian matrix will be all real numbers and the main diagonal elements of a skew-Hermitian matrix are purely imaginary or zero.

A square matrix A is called an orthogonal matrix if $AA^T = A^TA = I$. where I is the unit matrix of the same order of A.

A complex square matrix A is said to be unitary if $AA^* = A^*A = I$

The square matrix A is said to be singular if $|A| = 0$ and non singular if $|A| \neq 0$.

$$A^{-1} = \frac{\text{adj}\,A}{|A|}$$

Elementary Row operations for Matrices

The following three operations on the row are known as elementary row operations for matrices.

1. Interchanging two rows.

$$R_i \longleftrightarrow R_j$$

2. Multiply a row ~~on the~~ by a non zero constant.

$$R_i \longrightarrow kR_i$$

where k is a non zero constant.

3. Adding a row to another row.

$$R_i \longrightarrow R_i + R_j$$

The last two operations can be combined to get a single operation.

$$R_i \longrightarrow \alpha R_i + \beta R_j$$

where α and β are constants.

Equivalent Matrices

Two matrices are said to be equivalent matrices if one of them can be obtained from the other by a sequence of elementary transformation.

Row Echelon Form of a Matrix

First non-zero element of a row from the left is called leading element or pivot of the row.

A matrix is said to be in its row echelon form or row reduced echelon form if the leading element in each row (if exist) is 1 and the number of zeros before the leading element in each row is greter than the corresponding number of zeros of the preceding rows.

If the leading element is not 1, then the matrix is called row reduced form of the matrix.

eg: $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 4 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 3 & -2 & 0 & -4 & 9 \\ 0 & 0 & -9 & 0 & 3 & 5 & -8 & 2 & 1 \\ 0 & 0 & 0 & 2 & -3 & -4 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & -2 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of a Matrix

Rank of a matrix is defined as the maximum number of linearly independent row vectors or rows ~~matrix~~ of matrix.

Note

The Rank of a matrix is equal to the number of non-zero rows in its equivalent row reduced form (or row reduced echelon form).

Problems

1. Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 2 & 4 & 1 \\ 5 & 6 & 7 & 5 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 2 & 4 & 1 \\ 5 & 6 & 7 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 6 & -5 \\ 0 & -4 & 12 & -10 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 5R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 6 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

Rank = Number of non zero rows in its row reduced form = 2

U.Q.

2. Reduce to row echelon form and find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 3 & 2 & 3 \\ 0 & -15 & -10 & -15 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 8R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 + R_2 \\ R_4 \to R_4 - 5R_2 \end{array}$$

Rank = Number of non zero rows = 2

3. Find the rank of the matrix $\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$

U.Q.

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 7R_1 \end{array}$$

$$\sim \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to 2R_3 + R_2$$

Rank = Number of non-zero rows = 2

4. Find the rank of the matrix $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

U.Q.

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -1 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -1 \\ 0 & 5 & 3 & 1 \\ 0 & 4 & 9 & 5 \\ 0 & 9 & 12 & -1 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 6R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -1 \\ 0 & 5 & 3 & 1 \\ 0 & 0 & 33 & 21 \\ 0 & 0 & 33 & 14 \end{bmatrix} \quad \begin{array}{l} R_3 \to 5R_3 - 4R_2 \\ R_4 \to 5R_3 - 9R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -1 \\ 0 & 5 & 3 & 1 \\ 0 & 0 & 33 & 21 \\ 0 & 0 & 0 & -7 \end{bmatrix} \quad R_4 \to R_4 - R_3$$

Rank = Number of non zero rows = 4

5. Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 5 \end{bmatrix}$$

U.Q.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

Rank = Number of non zero rows = 3

6 Find the rank of $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix}$

U.Q.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 4 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} -1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 4R_2$$

Rank = Number of non zero rows = 2.

Rank of a Matrix and linearly independent Vectors

A set of 'n' vectors $\{v_1, v_2 \cdots v_n\}$ are linearly independent if the rank of the matrix A is equal to the number of rows the matrix, Where A is a matrix obtained by taking each Vector as a row.

The Vectors are linearly dependent if the rank of A is less than the number of rows of A.

problems

1. Are the vectors $(3,-1,4)$ $(6,7,5)$ and $(9,6,9)$ linearly dependent or independent?

$$A = \begin{bmatrix} 3 & -1 & 4 \\ 6 & 7 & 5 \\ 9 & 6 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -1 & 4 \\ 0 & 9 & -3 \\ 0 & 9 & -3 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 3 & -1 & 4 \\ 0 & 9 & -3 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

Rank $= 2$

Number of rows (vectors) $= 3$

Rank $<$ Number of rows $\Rightarrow$ Given Vectors are linearly dependent

2. Find whether the vectors $(1,2,-1,3)$ $(2,-1,3,2)$ and $(-1,8,-9,5)$ are linearly dependent.

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & -1 & 3 & 2 \\ -1 & 8 & -9 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & -4 \\ 0 & 10 & -10 & 8 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + 2R_2$$

Rank = 2

Number of rows (vectors) = 3

Rank < Number of rows $\Rightarrow$ Given Vectors are linearly dependent.

3. Prove that the vectors $(1,1,2)$, $(1,2,5)$, $(5,3,4)$ are linearly dependent.

U.Q.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -6 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 5R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + 2R_2$$

Rank = 2

Number of Vector = 3

Rank < Number of vectors

$\Rightarrow$ Linearly depen

4. Show that the vectors $(1,-1,0)$ $(1,3,-1)$ and $(5,3,-2)$ are linearly dependent.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 3 & -1 \\ 5 & 3 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & -1 \\ 0 & 8 & -2 \end{bmatrix} \quad \begin{matrix} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - 5R_1 \end{matrix}$$

U.Q

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \rightarrow R_3 - 2R_2$$

Rank = 2

Number of vectors = 3

Rank < Number of vectors $\Longrightarrow$ Given vectors are linearly dependent.

5. Prove that the vectors $(2,3,0)$, $(1,2,0)$ and $(8,13,0)$ are linearly dependent in R^3.

U.Q

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 0 \\ 8 & 13 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 8 & 13 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & -3 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 8R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \rightarrow R_3 - 3R_2$$

Rank $= 2$

Number of Vectors $= 3$

Rank $<$ Number of vectors $\Rightarrow$ Given vectors are linearly dependent.

6. Check whether the vectors $(1,2,1)$, $(2,1,4)$, $(4,5,6)$, $(1,8,-3)$ are linearly dependent in R^3.

U.Q.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 4 \\ 4 & 5 & 6 \\ 1 & 8 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & -3 & 2 \\ 0 & 6 & -4 \end{bmatrix} \quad \begin{matrix} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 4R_1 \\ R_4 \rightarrow R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_3 \rightarrow R_3 - R_2 \\ R_4 \rightarrow R_4 + 2R_2 \end{matrix}$$

Rank $= 2$

Number of Vectors $= 4$

... $<$ Number of vectors $\Rightarrow$ linearly dependent

7 Show that the vectors $[3, 4, 0, 1]$, $[2, -1, 3, 5]$ and $[1, 6, -8, -2]$ are linearly independent in R^4.

$$A = \begin{bmatrix} 3 & 4 & 0 & 1 \\ 2 & -1 & 3 & 5 \\ 1 & 6 & -8 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & -8 & -2 \\ 2 & -1 & 3 & 5 \\ 3 & 4 & 0 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 6 & -8 & -2 \\ 0 & -13 & 19 & 9 \\ 0 & -14 & 24 & 7 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 6 & -8 & -2 \\ 0 & -13 & 19 & 9 \\ 0 & 0 & 46 & -35 \end{bmatrix} \quad R_3 \to 13R_3 - 14R_2$$

Rank = 3

Number of vectors = 3

Rank = Number of vectors

$\Longrightarrow$ Given vectors are linearly independent

System of Linear equations

We consider the following system of m linear equations in n unknowns $x_1, x_2 \dots x_n$

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \dots \quad \dots \quad \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad \text{— (1)}$$

Changing to matrix notation, these equations can be written as

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or $A \cdot X = B$

where $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

Here A is called the coefficient matrix of the system of equations.

The matrix [AB] defined by

$$[AB] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

is called the augmented matrix of the given system of equations (1).

In the system $Ax=B$, if $B=0$ then it is called a homogeneous system and if $B \neq 0$ then it is called a non homogeneous system.

Any set of values of $x_1, x_2 \cdots x_n$ which satisfy simultaneously the m equations of the system in (1) is called a solution of the $Ax=B$.

A system of equations having no solutions is called an inconsistent system of equations. A system of equations having one or more solutions is called a consistent system of equations.

The system of equations $Ax=B$ is consistent if and only if the coefficient matrix A and the augmented matrix $[AB]$ are of the same rank. otherwise the system is inconsistent.

Gauss Elimination method

The process of reducing a given matrix to echelon form using a chain of elementary row-operations is called Gauss elimination process. using this process reduce the augmented matrix $[AB]$ to the echelon form. Then there arise the following cases.

(i) If rank $A \neq$ rank $[AB]$, the system is inconsistent.

(ii) If rank A = rank $[AB]$ = the number of unknowns, the system is consistant and has a unique solution.

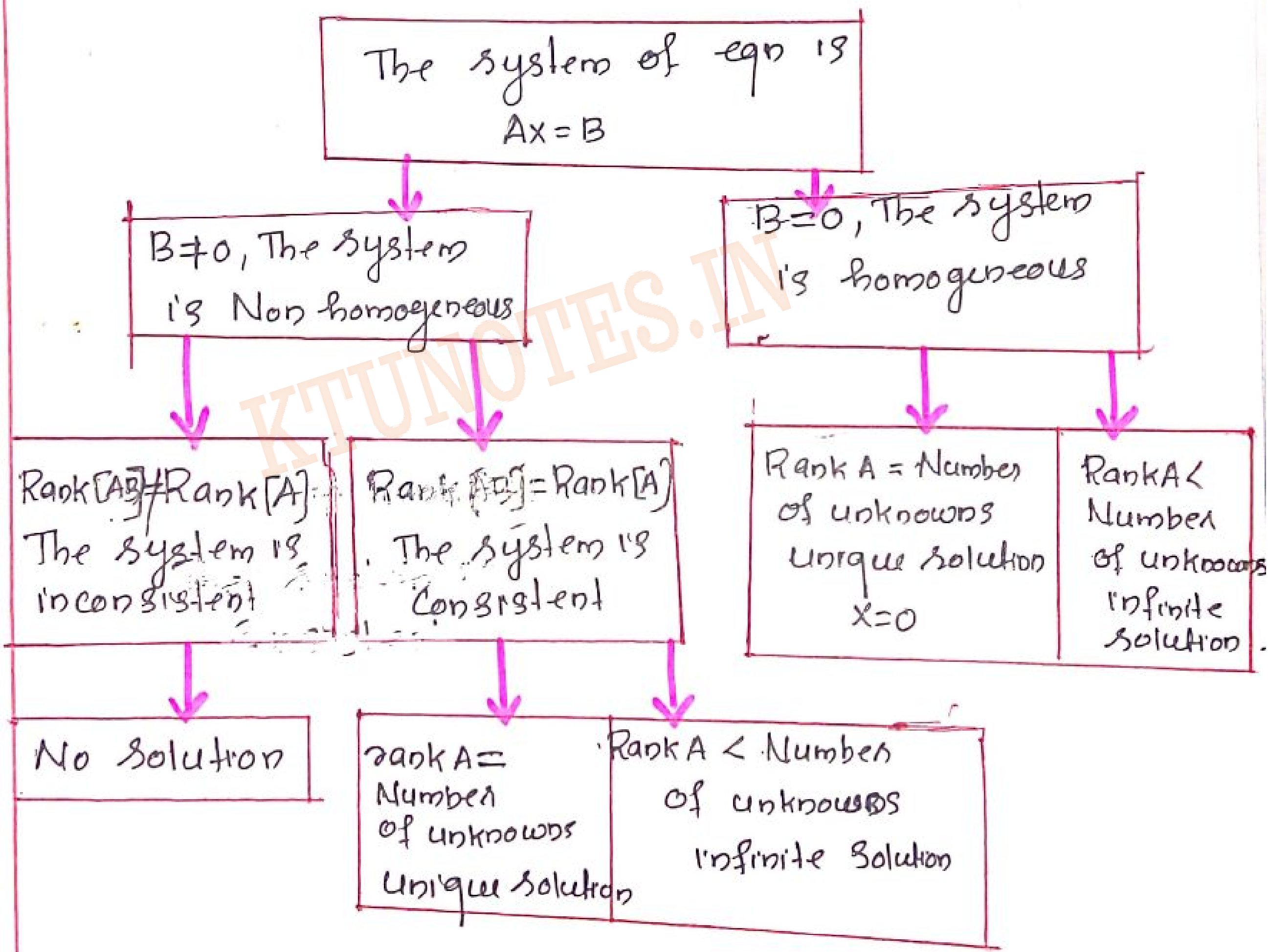
(iii) If rank A = rank $[AB]$ < the number of unknowns, the system is consistent and has infinite number of solutions.

If the system $Ax=B$ is consistent, the solution can be obtained using back substitution.

Homogeneous equations $Ax=0$

(1) Rank = Number of unknowns $\Rightarrow$ the homogeneous system has the trivial solution $x=0$

(2) Rank < Number of unknowns $\Rightarrow$ the homogeneous system has an infinite number of non zero solution.



Problems

1. Solve by Gauss elimination

$$x_1 - x_2 + x_3 = 0$$

$$-x_1 + x_2 - x_3 = 0$$

$$10x_2 + 25x_3 = 90$$

$$20x_1 + 10x_2 = 80$$

$$AX = B$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 10 & 25 \\ 20 & 10 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 90 \\ 80 \end{bmatrix}$$

Augmented matrix $[AB]$

$$[AB] = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_4 \to R_4 - 20R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 95 & 190 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_3 \rightarrow 3R_3 - R_2$$

Rank $[AB] = 3$

Number of unknowns $= 3$

Rank $A = 3$

Rank $[AB]$ = Rank A = Number of unknowns $= 3$

$\therefore \; Ax = B$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 30 & -20 \\ 0 & 0 & 95 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 80 \\ 190 \end{bmatrix}$$

$$x_1 - x_2 + x_3 = 0$$

$$30x_2 - 20x_3 = 80$$

$$95x_3 = 190$$

$$x_3 = 190/95 = 2$$

$$\Rightarrow 30x_2 = 80 + 20x_3 = 80 + 20 \times 2 = 120$$

$$x_2 = 120/30 = 4$$

$$\Rightarrow x_1 = x_2 - x_3 = 4 - 2 = 2 \;//$$

$$\therefore \; X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

2. Solve by Gauss elimination method

$x+y+z=6$, $x+2y-3z=-4$ $-x-4y+9z=18$

$$AX=B \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ -1 & -4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 18 \end{bmatrix}$$

Augmented Matrix

$$[AB] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & -3 & -4 \\ -1 & -4 & 9 & 18 \end{array}\right]$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -10 \\ 0 & -3 & 10 & 24 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 + R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & -2 & -6 \end{bmatrix} \quad R_3 \rightarrow R_3 + 3R_2$$

Rank $[AB] = 3$

Rank $[A] = 3$

Number of unknows $= 3$

Rank $[AB]$ = Rank $[A]$ = Number of unknowns

$\Rightarrow$ the system is consistent and unique solution.

$AX = B \implies \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -10 \\ -6 \end{bmatrix}$

$$x + y + z = 6$$
$$y - 4z = -10$$
$$-2z = -6$$
$$z = \frac{-6}{-2} = 3$$
$$y = -10 + 4z = -10 + 4 \times 3 = 2$$
$$x + y + z = 6 \implies x = -z - y + 6$$
$$= -3 - 2 + 6 = 1$$

$$\therefore X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

3. Using Gauss elimination method Solve the system of equations

$$x + 2y - z = 3$$
$$3x - y + 2z = 1$$
$$2x - 2y + 3z = 2$$
$$x - y + z = -1$$

$AX = B \implies \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}$

Augmented Matrix

$[AB] = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

$$[AB] \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & -1 & -4 \end{bmatrix} \quad \begin{array}{l} R_3 \to 7R_3 - 6R_2 \\ R_4 \to 7R_4 - 3R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to 5R_4 + R_3$$

Rank $[AB] = 3$

Rank $[A] = 3$

Number of unknowns $= 3$

$\Rightarrow$ System consistent and unique solution

$Ax = B \Rightarrow$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ 20 \end{bmatrix}$$

$x + 2y - z = 3$

$-7y + 5z = -8$

$5z = 20 \Rightarrow z = 4$

$\Rightarrow -7y = -8 - 5z = -8 - 5 \times 4 = -28$

$y = -28/-7 = 4$

$\Rightarrow x = 3 - 2y + z = 3 - 2 \times 4 + 4 = -1$

$\therefore \underline{x = -1 \quad y = 4 \quad z = 4}$

4. Examine for consistency the following equations

$x + y + 2z = 2$

$2x - y + 3z = 2$

$5x - y + 8z = 10$

$$AX = B \Longrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 5 & -1 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 10 \end{bmatrix}$$

Augmented Matrix $[AB] = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 2 \\ 5 & -1 & 8 & 10 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & -6 & -2 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 5R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

Rank $[AB] = 3$

Rank $[A] = 2$

Rank $[AB] \neq$ Rank $[A]$

$\therefore$ The system is inconsistent

5. Solve the system of equations using Gauss elimination method

$$4y + 3z = 8$$
$$2x - z = 2$$
$$3x + 2y = 5$$

$$AX = B \Longrightarrow \begin{bmatrix} 0 & 4 & 3 \\ 2 & 0 & -1 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 5 \end{bmatrix}$$

Augmented matrix

$$[AB] = \begin{bmatrix} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 0 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 3 & 2 & 0 & 5 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 0 & 4 & 3 & 4 \end{bmatrix} \quad R_3 \to 2R_3 - 3R_1$$

$$\sim \begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 0 & 0 & 0 & -4 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

Rank $[AB] = 3$

Rank $[A] = 2$

Rank $[AB] \neq$ Rank $[A]$

$\Longrightarrow$ The system is inconsistent and has no solution.

6. Solve the system of equations by Gauss elimination method

$$x + 2y + 3z = 1$$
$$2x + 3y + 2z = 2$$
$$3x + 3y + 4z = 1$$

$$Ax = B \Longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Augmented Matrix

$$[AB] = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 2 & 2 \\ 3 & 3 & 4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -4 & 0 \\ 0 & -3 & -5 & -2 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & 7 & -2 \end{bmatrix} \quad R_3 \rightarrow R_3 - 3R_2$$

Rank $[AB] = 3$

Rank $[A] = 3$

Number of unknowns = 3

Rank $[AB]$ = Rank $[A]$ = Number of unknowns

$\Longrightarrow$ System is consistent and unique solution

$\therefore Ax = B \Longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

$x + 2y + 3z = 1$

$-y - 4z = 0$

$7z = -2 \Longrightarrow z = -2/7$

$-y = 4z$

$= 4 \times -\frac{2}{7} = -8/7$

$y = 8/7$

$x = 1 - 2y - 3z = 1 - 2 \times \frac{8}{7} - 3 \times -\frac{2}{7}$

$= 1 - \frac{16}{7} + \frac{6}{7} = 1 - \frac{10}{7} = \frac{7-10}{7} = -3/7$

$$x = \begin{bmatrix} -3/7 \\ -8/7 \\ -2/7 \end{bmatrix}$$

7. Solve the System of equations by Gauss elimination Method

U.Q.

$$3x+3y+2z=1$$
$$x+2y=4$$
$$10y+3z=-2$$
$$2x-3y-z=5$$

Augmented Matrix

$$[AB] = \begin{bmatrix} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix} \quad R_1 \longleftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix} \quad \begin{matrix} R_2 \longrightarrow R_2 - 3R_1 \\ R_4 \longrightarrow R_4 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & 29 & -116 \\ 0 & 0 & -17 & 68 \end{bmatrix} \quad \begin{matrix} R_3 \longrightarrow 3R_3 + 10R_2 \\ R_4 \longrightarrow 3R_3 - 7R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & 29 & -116 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \rightarrow 29R_4 + 17R_3$$

Rank $[AB] = 3$

Rank $[A] = 3$

Number of unknowns $= 3$

$\Longrightarrow$ System is consistent and unique solution.

$$Ax = B \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 29 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -116 \end{bmatrix}$$

$x + 2y = 4$

$-3y + 2z = -11$

$29z = -116$

$z = \frac{-116}{29} = -4$

$-3y = -11 - 2z = -11 + 8 = -3$

$y = 1$

$x = 4 - 2y = 4 - 2 = 2$

$$X = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$$

8. Solve the following by Gauss

U.Q. elimination method

$$y + z - 2w = 0$$

$$2x - 3y - 3z + 6w = 2$$

$$4x + y + z - 2w = 4$$

$$Ax = B \Rightarrow \begin{bmatrix} 0 & 1 & 1 & -2 \\ 2 & -3 & -3 & 6 \\ 4 & 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$$

Augmented matrix

$$[AB] = \begin{bmatrix} 0 & 1 & 1 & -2 & 0 \\ 2 & -3 & -3 & 6 & 2 \\ 4 & 1 & 1 & -2 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 4 & 1 & 1 & -2 & 4 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 7 & 7 & -14 & 0 \end{bmatrix} \quad R_3 \to R_3 - 2R_1$$

$$\sim \begin{bmatrix} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 7R_2$$

Rank [AB] = 2

Rank [A] = 2

Number of unknowns = 4

Rank [AB] = Rank [A] < Number of unknowns

$\Rightarrow$ The system is consistent and infinitely many solution.

$$Ax = B \Rightarrow \begin{bmatrix} 2 & -3 & -3 & 6 \\ 0 & 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x - 3y - 3z + 6w = 2$$

$$y + z - 2w = 0$$

unknowns $(n) = 4$

rank $(r) = 2$

$\Rightarrow$ free variables $= n - r = 4 - 2 = \underline{\underline{2}}$

Take $w = t_1$ and $z = t_2$

$$\Rightarrow y = 2w - z = 2t_1 - t_2$$

$$\Rightarrow 2x = 2 + 3y + 3z - 6w$$

$$= 2 + 3(2t_1 - t_2) + 3t_2 - 6t_1$$

$$= 2 + \cancel{6t_1} - \cancel{3t_2} + \cancel{3t_2} - \cancel{6t_1}$$

$$= 2$$

$$x = 1$$

$$\therefore X = \begin{bmatrix} 1 \\ 2t_1 - t_2 \\ t_2 \\ t_1 \end{bmatrix}$$

9 Solve the system of equations

$$4y + 4z = 24$$
$$3x - 11y - 2z = -6$$
$$6x - 17y + z = 18$$

$$AX = B \Rightarrow \begin{bmatrix} 0 & 4 & 4 \\ 3 & -11 & -2 \\ 6 & -17 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 24 \\ -6 \\ 18 \end{bmatrix}$$

Augmented matrix

$$[AB] = \begin{bmatrix} 0 & 4 & 4 & 24 \\ 3 & -11 & -2 & -6 \\ 6 & -17 & 1 & 18 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -11 & -2 & -6 \\ 0 & 4 & 4 & 24 \\ 6 & -17 & 1 & 18 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 3 & -11 & -2 & -6 \\ 0 & 4 & 4 & 24 \\ 0 & 5 & 5 & 30 \end{bmatrix} \quad R_3 \rightarrow R_3 - 2R_1$$

$$\sim \begin{bmatrix} 3 & -11 & -2 & -6 \\ 0 & 4 & 4 & 24 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to 4R_3 - 5R_2$$

Rank $[AB] = 2$

Rank $[A] = 2 \to r$

Number of unknowns $= 3 \to n$

Rank $[AB] =$ Rank $[A] <$ Number of unknowns

$\Rightarrow$ The system is consistent and infinitely many solution.

$\Rightarrow$ free variables $= n - r = 3 - 2 = 1$

Take $\begin{bmatrix} 3 & -11 & -2 \\ 0 & 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ 24 \end{bmatrix}$

$3x - 11y - 2z = -6$

$4y + 4z = 24$

Take $z = t \Rightarrow 4y = 24 - 4z$

$y = \frac{24 - 4t}{4}$

$= \underline{\underline{6 - t}}$

$\Rightarrow 3x = -6 + 11y + 2z = -6 + 11(6-t) + 2t$

$= -6 + 66 - 11t + 2t$

$3x = 60 - 9t \Rightarrow x = \underline{\underline{20 - 3t}}$

$X = \begin{bmatrix} 20-3t \\ 6-t \\ t \end{bmatrix}$

10. Find the Value of μ for which the System of equations
$$x+y+z=1$$
$$x+2y+3z=\mu$$
$$x+5y+9z=\mu^2$$

U.Q

is consistent. For each Value of μ obtained, find the Solution of the system.

$$Ax=B \Longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 5 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \mu \\ \mu^2 \end{bmatrix}$$

Augmented Matrix

$$[AB] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & \mu \\ 1 & 5 & 9 & \mu^2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & \mu-1 \\ 0 & 4 & 8 & \mu^2-1 \end{bmatrix} \quad \begin{matrix} R_2 \longrightarrow R_2 - R_1 \\ R_3 \longrightarrow R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & \mu-1 \\ 0 & 0 & 0 & \mu^2-4\mu+3 \end{bmatrix} \quad R_3 \longrightarrow R_3 - 4R_2$$

$$\boxed{\begin{matrix} \mu^2-1-4(\mu-1) \\ \mu^2-4\mu+3 \end{matrix}}$$

Given that the System is consistent

$\Rightarrow$

$\Rightarrow$ Rank [AB] = Rank [A]

Rank [A] = 2

$\Rightarrow$ Rank [AB] = 2 is possible only if $\mu^2 - 4\mu + 3 = 0$

$\Rightarrow (\mu-3)(\mu-1) = 0$

$\Rightarrow \mu = 3, 1$

~~Sum = 3~~
product = 3
sum = -4
Numbers
-3, -1

When $\mu = 3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$x + y + z = 1$

$y + 2z = 2$

Rank $(r) = 2$

Unknowns $(n) = 3$

$\Rightarrow$ Infinitely many solutions

$\Rightarrow$ $n-r$ free variables $= 3 - 2 = 1$

put $z = t$ $\Rightarrow$ $y = 2 - 2z$
$= 2 - 2t$

$x = 1 - y - z = 1 - (2 - 2t) - t$
$= 1 - 2 + 2t - t$
$x = t - 1$

$\therefore X = \begin{bmatrix} t-1 \\ 2-2t \\ t \end{bmatrix}$

When $\mu = 1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x + y + z = 1$$
$$y + 2z = 0$$

Free Variables $= 3 - 2 = 1$

put $z = t \implies y = -2z = -2t$

$$x = 1 - y - z = 1 + 2t - t = t + 1$$

$$\therefore X = \begin{bmatrix} t+1 \\ -2t \\ t \end{bmatrix}$$

11. Find the value of λ and μ for which the system of equations

$$2x + 3y + 5z = 9$$

U.Q. $$7x + 3y - 2z = 8$$

$$2x + 3y + \lambda z = \mu$$

has (a) no solution (b) unique solution (c) more than one solution (or infinite solution)

$$Ax = B = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

Augmented Matrix

$$[AB] = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -39 & -47 \\ 0 & 0 & \lambda-5 & \mu-9 \end{bmatrix} \quad \begin{array}{l} R_2 \to 2R_2 - 7R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

(a) No solution

$\Longrightarrow$ Rank $[AB] \neq$ Rank $[A]$

$\Longrightarrow \lambda - 5 = 0$ and $\mu - 9 \neq 0$

$\Longrightarrow \lambda = 5$ and $\mu \neq 9$

(b) Unique Solution

$\Longrightarrow$ Rank $[AB]$ = Rank $[A]$ = Number of unknowns.

Number of unknowns $(x, y, z) = 3$

~~This is possible only~~ i.

$\Longrightarrow \lambda - 5 \neq 0$ ~~and μ take any value~~

$\Longrightarrow \lambda \neq 5$ and μ take any Value.

(C) More than one solution (or infinite solution)

$\Rightarrow$ Rank [AB] = Rank [A] < Number of unknowns (3)

$\Rightarrow$ $\lambda - 5 = 0$ and $\mu - 9 = 0$.

$\Rightarrow$ $\lambda = 5$ and $\mu = 9$

12. Find the values of a and b for which the system of linear equations

U.Q

$x + 2y + 3z = 6$
$x + 3y + 5z = 9$
$2x + 5y + az = b$

has (i) no solution (ii) a unique solution (iii) infinitely many solution.

Augmented Matrix

$$[AB] = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 & 9 \\ 2 & 5 & a & b \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & a-6 & b-12 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a-8 & b-15 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

(i) <u>No Solution</u>

Rank [AB] $\neq$ Rank [A]

$\Longrightarrow$ $a-8=0$ and $b-15 \neq 0$

$\Longrightarrow$ $a=8$ and $b \neq 15$

(ii) <u>unique Solution</u>

Rank [AB] = Rank [A] = Number of unknowns

Number of unknowns $(x, y, z) = 3$

$\Longrightarrow$ $a-8 \neq 0$

$\Longrightarrow$ $a \neq 8$ and b take any Value

(iii) <u>Infinitely many solution</u>

Rank [AB] = Rank [A] < Number of unknowns (3)

$\Longrightarrow$ $a-8=0$ and $b-15=0$

$\Longrightarrow$ $a=8$ and $b=15$

13. Find the Non trivial Solution of the homogeneous system of equations

$x+2y-z=0$

$3x+y-z=0$

$2x-y=0$

Homogeneous System of equation is

$Ax=0$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[A] = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 2 \\ 0 & -5 & 2 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

Rank $A = 2$

Number of unknowns $= 3$

Rank $A <$ Number of unknowns

$\Longrightarrow$ the homogeneous system has a non trivial solution.

$$\Longrightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 2y - z = 0$$

$$-5y + 2z = 0$$

Put $z = t \Longrightarrow -5y = -2z$

$$5y = 2t$$

$$y = \frac{2}{5}t$$

$$x = z - 2y = t - 2 \times \frac{2}{5}t$$

$$= t - \frac{4t}{5} = \frac{1}{5}t$$

$$\therefore X = \begin{bmatrix} \frac{1}{5}t \\ \frac{2}{5}t \\ t \end{bmatrix}$$

14. Determine the value of λ for which the following equations may possess non trivial solution. $3x + y - \lambda z = 0$, $4x - 2y - 3z = 0$
$2\lambda x + 4y + \lambda z = 0$.

If the homogeneous system of equation $AX = 0$ has a non trivial solution, then $|A| = 0$

$$AX = 0 \Rightarrow \begin{bmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{vmatrix} = 0$$

$$3\left[(-2\lambda - (-12)\right] - 1\left[(4\lambda - (-3 \times 2\lambda))\right] - \lambda\left[16 - (2 \times 2\lambda)\right] = 0$$

$$-6\lambda + 36 - 4\lambda - 6\lambda - 16\lambda - 4\lambda^2 = 0$$

$$-4\lambda^2 - 32\lambda + 36 = 0$$

$$\lambda^2 + 8\lambda + 9 = 0$$

$$(\lambda - 1)(\lambda + 9) = 0$$

$$\Rightarrow \lambda = 1, -9$$

Eigen Values and Eigen Vectors

Let $A = [a_{ij}]$ be a given non zero Square matrix of dimension $n \times n$.

Consider the vector equation $Ax = \lambda x$ — ①

The problem of finding non zero x's and λ's that satisfy equation $Ax = \lambda x$ is called eigen value problem. λ is an unknown scalar and x is an unknown vector.

$$Ax = \lambda x \implies (A - \lambda I)x = 0$$

This is a system of linear homogeneous equation and has non trivial solution $(x \neq 0)$ only if

$$|A - \lambda I| = 0$$

This equation is called characteristic equation (latent equation).

The roots of characteristic equation are the eigen values (latent values or characteristic roots).

The solution $x \neq 0$ are called the eigen vectors or characteristic vector of A corresponding to that eigen value λ.

The set of all eigen values of A is called the <u>Spectrum</u> of A.

The largest of the absolute values of the eigen values of the eigen values of A is called the <u>Spectral radius of A</u>.

The set of all solutions of the equation $Ax = \lambda x$ is the <u>eigenspace</u> of A corresponding to λ and is denoted by $E(\lambda)$.

Note

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

characteristic equation $|A - \lambda I| = 0$

$$\Rightarrow \boxed{\lambda^2 - (a_{11} + a_{22})\lambda + |A| = 0}$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

characteristic equation $|A - \lambda I| = 0$

$$\Rightarrow \lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + \left\{ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right\} \lambda - |A| = 0$$

$\Rightarrow \lambda^3 -$ (sum of the diagonal elements) $\lambda^2 +$ (sum of the cofact minors of diagonal elements) $\lambda - |A|$

Problems

1\. Show that the matrix $\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ is symmetric.

U.Q. Find its spectrum.

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \Rightarrow A = A^T$

$\therefore$ A is symmetric

Spectrum $\Rightarrow$ Set of all eigen values.

$\therefore$ the characteristic equation $|A - \lambda I| = 0$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - (a_{11} + a_{22})\lambda + |A| = 0$

$\Rightarrow \lambda^2 - (1-2)\lambda + \begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = 0$

$\Rightarrow \lambda^2 + \lambda + (-2-4) = 0$

$\Rightarrow \lambda^2 + \lambda - 6 = 0$

$\Rightarrow (\lambda - 2)(\lambda + 3) = 0$ Spectrum of $A = \{2, -3\}$

$\Rightarrow \lambda = 2, -3$

2. Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

Characteristic equation is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - (8+2)\lambda + \begin{vmatrix} 8 & -4 \\ 2 & 2 \end{vmatrix} = 0$$

$$\lambda^2 - 10\lambda + (16+8) = 0$$

$$\lambda^2 - 10\lambda + 24 = 0$$

$$(\lambda - 4)(\lambda + 6) = 0$$

$$\underline{\underline{\lambda = 4, 6}}$$

eigen value $= \{4, 6\}$

<u>When $\lambda = 4$</u>

$$(A - 4I)X = 0$$

$$\begin{bmatrix} 8-4 & -4 \\ 2 & 2-4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4x - 4y = 0 \implies x - y = 0$$
$$2x - 2y = 0 \implies x - y = 0$$

$$x - y = 0 \implies x = y$$
$$\implies \frac{x}{1} = \frac{y}{1}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

When $\lambda = 6$

$$(A - 6I)X = 0$$

$$\begin{bmatrix} 8-6 & -4 \\ 2 & 2-6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x - 4y = 0 \implies x - 2y = 0$$
$$2x - 4y = 0$$

$$x - 2y = 0 \implies x = 2y$$
$$\implies \frac{x}{2} = \frac{y}{1}$$

$$X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\therefore$ Eigen Vectors $= \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

3 Find the eigen values and eigen vectors of $\begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

characteristic equation is $|A-\lambda I|=0$

$$|A-\lambda I|=0 \Longrightarrow \begin{vmatrix} 1-\lambda & 1 & 2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix}=0$$

$$\Longrightarrow \lambda^3-(a_{11}+a_{22}+a_{33})\lambda^2+\left\{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right\}\lambda-|A|=0$$

$$\Longrightarrow \lambda^3-(1+2+3)\lambda^2+\left\{\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}+\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix}+\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}\right\}\lambda - \begin{vmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{vmatrix}=0$$

$$\Longrightarrow \lambda^3-6\lambda^2+\{(6-1)+(3-0)+(2+1)\}\lambda - \{1(6-1)-1(-3-0)+2(-1-0)\}=0$$

$$\Longrightarrow \lambda^3-6\lambda^2+(5+3+3)\lambda-(5+3-2)=0$$

$$\lambda^3-6\lambda^2+11\lambda-6=0$$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

Put $\lambda = 1$ $\quad 1 - 6 + 11 - 6 = 0$

$\therefore$ $\lambda = 1$ is a root

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & +0 & 1 & -5 & 6 \\ \hline \times & 1 & -5 & 6 & \boxed{0} \end{array}$$

$\Longrightarrow \lambda^2 - 5\lambda + 6 = 0$

$\Longrightarrow (\lambda - 2)(\lambda - 3) = 0$

$\Longrightarrow \lambda = 2, 3$

Eigen values $= \underline{\{1, 2, 3\}}$

When $\lambda = 1$

$(A - I)x = 0$

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} 0x + y + 2z = 0 \\ -x + y + z = 0 \end{array}\right\}$$ consider first two different equation

$0x + y + 2z = 0$

$$\frac{x}{1-2} = \frac{-y}{0+2} = \frac{z}{0+1}$$

$$\Longrightarrow \frac{x}{-1} = \frac{y}{-2} = \frac{z}{1}$$

$$X_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

When $\lambda = 2$

$(A - 2I)X = 0$

$$\begin{bmatrix} -1 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -x + y + 2z &= 0 \\ -x + 0y + z &= 0 \end{aligned}\right\} \text{First two equation}$$

$$0x + y + z = 0$$

$$\frac{x}{1-0} = \frac{-y}{-1+2} = \frac{z}{0+1}$$

$$\frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$$

$$X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

When $\lambda = 3$

$(A - 3I)X = 0$

$$\begin{bmatrix} -2 & 1 & 2 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -2x + y + 2z &= 0 \\ -x - y + z &= 0 \end{aligned}\right\} \text{First two eqns}$$

$$0x + y + z = 0$$

$$\frac{x}{1+2} = \frac{-y}{-2+2} = \frac{z}{2+1} \qquad \therefore X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\frac{x}{3} = \frac{y}{0} = \frac{z}{3} \implies \frac{x}{1} = \frac{y}{0} = \frac{z}{1}$$

4 Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

characteristic equation $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - (1+2+3)\lambda^2 + \left\{ \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} \right\} \lambda$$

$$- \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix} = 0$$

$$\lambda^3 - 6\lambda^2 + \{(6-2) + (3+2) + (2+0)\}\lambda -$$

$$\{1(6-2) - 0(3-2) + (-1)(2-4)\} = 0$$

$$\lambda^3 - 6\lambda^2 + (4+5+2)\lambda - (4-0+2) = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

put $\lambda = 1 \implies 1 - 6 + 11 - 6 = 0$

$\lambda = 1$ is a root.

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda = 2, 3$$

Eigen values $= \{1, 2, 3\}$

<u>When $\lambda = 1$</u> $\quad (A - I)x = 0$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 0x + 0y - z &= 0 \\ x + y + z &= 0 \end{aligned}\right\}$$

$$2x + 2y + 2z = 0$$

$$\frac{x}{0+1} = \frac{-y}{0+1} = \frac{z}{0-0}$$

$$\frac{x}{1} = \frac{y}{-1} = \frac{z}{0}$$

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

<u>When $\lambda = 2$</u>

$$(A - 2I)x = 0$$

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x + 0y - z = 0 \checkmark$$

$$x + 0y + z = 0$$

$$2x + 2y + z = 0 \checkmark$$

Take ~~First~~ Second and last equation

$$\frac{x}{0-2} = \frac{-y}{1-2} = \frac{z}{1-0}$$

$$\frac{x}{-2} = \frac{y}{1} = \frac{z}{1}$$

$$X_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

When $\lambda = 3$

$(A - 3I)X = 0$

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -2x + 0y - z &= 0 \\ x - y + z &= 0 \end{aligned}\right\}$$

$$2x + 2y + 0z = 0$$

$$\frac{x}{0-1} = \frac{-y}{-2+1} = \frac{z}{2-0}$$

$$\Longrightarrow \frac{x}{-1} = \frac{y}{1} = \frac{z}{2}$$

$$\Longrightarrow X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

2. Find the eigen values and eigen vector of the matrix $A = \begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

The characteristic equation $|A - \lambda I| = 0$

$$|A - \lambda I| = 0 \Longrightarrow \begin{vmatrix} 3-\lambda & 5 & 3 \\ 0 & 4-\lambda & 6 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - (3+4+1)\lambda^2 + \left\{ \begin{vmatrix} 4 & 6 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 3 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 5 \\ 0 & 4 \end{vmatrix} \right\} \lambda$$

$$- \left\{ 3 \begin{vmatrix} 4 & 6 \\ 0 & 1 \end{vmatrix} - 5 \begin{vmatrix} 0 & 6 \\ 0 & 1 \end{vmatrix} + 3 \begin{vmatrix} 0 & 4 \\ 0 & 0 \end{vmatrix} \right\} = 0$$

$$\lambda^3 - 8\lambda^2 + \{(4-0) + (3-0) + (12-0)\}\lambda -$$

$$\{3(4-0) - 5(0-0) + 3(0-0)\} = 0$$

$$\lambda^3 - 8\lambda^2 + 19\lambda - 12 = 0$$

Put $\lambda = 1 \implies 1 - 8 + 19 - 12 = 0$

$\therefore$ $\lambda = 1$ is a root

1	1	−8	19	−12
		1	−7	12
	1	−7	12	0

$$\lambda^2 - 7\lambda + 12 = 0$$

$$(\lambda - 3)(\lambda - 4) = 0$$

$$\lambda = 3, 4$$

$\therefore$ Eigen Values are $\lambda = 1, 3, 4$

When $\lambda = 1$

$(A - I)X = 0$

$$\begin{bmatrix} 2 & 5 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x + 5y + 3z = 0$

$0x + 3y + 6z = 0$

$$\frac{x}{30-9} = \frac{-y}{12-0} = \frac{z}{6-0}$$

$$\frac{x}{21} = \frac{y}{-12} = \frac{z}{6}$$

$$\frac{x}{7} = \frac{y}{-4} = \frac{z}{2}$$

$$X_1 = \begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix}$$

When $\lambda = 3$ $\quad (A - 3I)x = 0$

$$\begin{bmatrix} 0 & 5 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 0x + 5y + 3z &= 0 \\ 0x + y + 6z &= 0 \\ 0x + 0y - 2z &= 0 \end{aligned}\right\}$$

$$\frac{x}{30-3} = \frac{-y}{0-0} = \frac{z}{0-0}$$

$$\frac{x}{27} = \frac{y}{0} = \frac{z}{0}$$

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$$

$$\therefore X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

When $\lambda = 4$

$(A - 4I)x = 0$

$$\begin{bmatrix} -1 & 5 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\left.\begin{aligned} -x + 5y + 3z &= 0 \\ 0x + 0y + 6z &= 0 \end{aligned}\right\}$$
$$0x + 0y - 3z = 0$$

$$\frac{x}{30-0} = \frac{-y}{-6-0} = \frac{z}{0-0}$$

$$\frac{x}{30} = \frac{y}{6} = \frac{z}{0}$$

$$\frac{x}{5} = \frac{y}{1} = \frac{z}{0}$$

$$X_3 = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ eigen vectors are

$$\left\{ \begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} \right\}$$

6. Find the eigen Values and the corresponding eigen vectors of $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

The characteristic equation

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - (1+2+(-1))\lambda^2 + \left\{ \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} \right\}\lambda$$

$$- \left\{ 1 \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} - 1 \begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} + (-2) \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} \right\} = 0$$

$$\lambda^3 - 2\lambda^2 + \{(-2-1) + (-1-0) + (2+1)\}\lambda -$$

$$\{(-2-1) - (1-0) - 2(-1-0)\} = 0$$

$$\lambda^3 - 2\lambda^2 + (-3-1+3)\lambda - (-3-1+2) = 0$$

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

Put $\lambda = 1$ $\quad 1-2-1+2 = 0$

$\therefore \lambda = 1$ is a root.

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -1 & 2 \\ & 0 & 1 & -1 & -2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda+1)(\lambda-2) = 0$$

$$\lambda = -1, 2$$

$\therefore$ Eigen Values $= \{1, -1, 2\}$

When $\lambda = 1$

$(A - I)X = 0$

$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 0x + y - 2z &= 0 \\ -x + y + z &= 0 \\ 0x + y - 2z &= 0 \end{aligned}\right\} \qquad \frac{x}{1+2} = \frac{-y}{0-2} = \frac{z}{0+1}$$

$$\frac{x}{3} = \frac{y}{2} = \frac{z}{1}$$

$$\therefore X_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

When $\lambda = -1$

$(A + I)X = 0$

$$\begin{bmatrix} 2 & 1 & -2 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 2x + y - 2z &= 0 \\ -x + y + z &= 0 \\ 0x + y + 0z &= 0 \end{aligned}\right\} \qquad \frac{x}{1+2} = \frac{-y}{2-2} = \frac{z}{2+1}$$

$$\frac{x}{3} = \frac{y}{0} = \frac{z}{3}$$

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{1}$$

$$\therefore X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

When $\lambda = 2$

$(A - 2I)X = 0$

$$\begin{bmatrix} -1 & 1 & -2 \\ -1 & 0 & 1 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -x + y - 2z &= 0 \\ -x + 0y + z &= 0 \end{aligned}\right\} \quad \frac{x}{1-0} = \frac{-y}{-1-2} = \frac{z}{0+1}$$

$$\frac{x}{1} = \frac{y}{3} = \frac{z}{1}$$

$$0x + y - 3z = 0$$

$$X_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Eigen Vectors $= \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}$

7. Find the eigen values and eigen vectors of

$$\begin{bmatrix} 4 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 3 \end{bmatrix}$$

The characteristic equation $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 2 & -2 \\ 2 & 5-\lambda & 0 \\ -2 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - (4+5+3)\lambda^2 + \left\{ \begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 4 & -2 \\ -2 & 3 \end{vmatrix} + \begin{vmatrix} 4 & 2 \\ 2 & 5 \end{vmatrix} \right\} \lambda$$

$$- \left\{ 4(15-0) - 2(6-0) - 2(0+10) \right\} = 0$$

$$= \lambda^3 - 12\lambda^2 + \{(15-0) + (12-4) + (20-4)\}\lambda - \{60 - 12 - 20\} = 0$$

$$\lambda^3 - 12\lambda^2 + 39\lambda - 28 = 0$$

Put $\lambda = 1$ $\quad 1 - 12 + 39 - 28 = 0$

$\therefore$ $\lambda = 1$ is a root

$$\begin{array}{c|cccc} 1 & 1 & -12 & 39 & -28 \\ & 0 & 1 & -11 & 28 \\ \hline & 1 & -11 & 28 & \boxed{0} \end{array}$$

$$\lambda^2 - 11\lambda + 28 = 0$$

$$(\lambda - 4)(\lambda - 7) = 0$$

$$\lambda = 4, 7$$

Eigen Values $= \{1, 4, 7\}$

When $\lambda = 1$

$$\begin{bmatrix} 3 & 2 & -2 \\ 2 & 4 & 0 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 3x + 2y - 2z &= 0 \\ 2x + 4y + 0z &= 0 \end{aligned}\right\} \qquad \frac{x}{0+8} = \frac{-y}{0+4} = \frac{z}{12-4}$$

$$-2x + 0y + 2z = 0$$

$$\frac{x}{8} = \frac{y}{-4} = \frac{z}{8}$$

$$\frac{x}{2} = \frac{y}{-1} = \frac{z}{2}$$

$$\therefore \quad X_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

When $\lambda = 4$

$$\begin{bmatrix} 0 & 2 & -2 \\ 2 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 0x + 2y - 2z &= 0 \\ 2x + y + 0z &= 0 \end{aligned}\right\} \qquad \frac{x}{0+2} = \frac{-y}{0+4} = \frac{z}{0-4}$$

$$-2x + 0y - z = 0$$

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{-4}$$

$$\frac{x}{-1} = \frac{y}{2} = \frac{z}{2}$$

$$\therefore X_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

When $\lambda = 7$

$$\begin{bmatrix} -3 & 2 & -2 \\ 2 & -2 & 0 \\ -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -3x + 2y - 2z &= 0 \\ 2x - 2y + 0z &= 0 \end{aligned}\right\} \qquad \frac{x}{0-4} = \frac{-y}{0+4} = \frac{z}{6-4}$$

$$-2x + 0y + 4z = 0$$

$$\frac{x}{-4} = \frac{y}{-4} = \frac{z}{2}$$

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{-1}$$

$$\therefore X_3 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

Eigen Vectors $= \left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \right\}$

8. Find the eigen Values and Eigen Space of

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

characteristic equation

$$|A-\lambda I| = \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - (-2+1+0)\lambda^2 + \left\{ \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} \right\} \lambda$$

$$- \left\{ -2(0-12) - 2(0-6) - 3(-4+1) \right\} = 0$$

$$\lambda^3 + \lambda^2 + \{0-12+0-3-2-4\}\lambda -$$

$$\{24+12+9\} = 0$$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

put $\lambda = 1$ $\quad 1+1-21-45 \neq 0$

$\lambda = 2$ $\quad 8+4-42-45 \neq 0$

$\lambda = 3$ $\quad 27+9-63-45 \neq 0$

$\lambda = 4$ $\quad 64+16-84-45 \neq 0$

$\lambda = 5$ $\quad 125+25-105-45 = 0$

$\therefore$ $\lambda = 5$ is a root

$$\lambda^2 + 6\lambda + 9 = 0$$

$$(\lambda+3)^2 = 0$$

$$\lambda = -3, -3$$

$$\begin{array}{c|cccc} 5 & 1 & 1 & -21 & -15 \\ & & 5 & 30 & 45 \\ \hline & 1 & 6 & 9 & \boxed{0} \end{array}$$

Eigen Values $= \{5, -3, -3\}$

When $\lambda = 5$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -7x + 2y - 3z &= 0 \\ 2x - 4y - 6z &= 0 \\ -x - 2y - 5z &= 0 \end{aligned}\right\}$$

$$\frac{x}{-12-12} = \frac{-y}{42+6} = \frac{z}{28-4}$$

$$\frac{x}{-24} = \frac{y}{-48} = \frac{z}{24}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

When $\lambda = -3$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} x + 2y - 3z &= 0 \\ (1\times 2) \Rightarrow 2x + 4y - 6z &= 0 \\ 1\times(-1) \Rightarrow -x - 2y + 3z &= 0 \end{aligned}\right\}$$

Three eqns same

$\Longrightarrow x + 2y - 3z = 0$

Put $z = t_1$, $y = t_2$, $x = 3t_1 - 2t_2$

$$\therefore X = \begin{bmatrix} 3t_1 - 2t_2 \\ t_2 \\ t_1 \end{bmatrix} = t_1 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore X_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \quad X_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Properties of Eigen Values

1. Eigen Values of A and A^T are Same.

2. If λ is an eigen Value of A, then λ^n is an Eigen Value of A^n.

3. If λ is an eigen Value of A, then $K\lambda$ is an Eigen Value of KA

4. If λ is an Eigen Value of A, then $\lambda - k$ is an eigen Value of $A - KI$

5. If λ is an Eigen Value of A, then $1/\lambda$ is an eigen Value of A^{-1}.

6. If λ is an Eigen Value of A, then $\frac{|A|}{\lambda}$ is an eigen Value of adj A.

7. If λ is an eigen Value of an orthogonal matrix then $1/\lambda$ is also its Eigen Value.

8. Eigen Values of triangular matrix and diagonal matrix are its diagonal elements.

9. The Sum of the eigen Values of a matrix is the sum of its diagonal elements.

10. The product of the eigen Values of a matrix is equal to its determinant Value.

11. The eigen Values of a Symmetric matrix are real.

12. The eigen Values of a skew symmetric matrix are purely imaginary or zero.

13. Let $\lambda_1, \lambda_2 \cdots \lambda_K$ be distinct eigen Values of an $n \times n$ matrix. Then corresponding eigen Vectors $x_1, x_2 \cdots x_K$ form a linearly independent Set.

Problems

1. If 2 is an eigen Value of $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ without using its characteristic equation find the other eigen Values. Also find the eigen Values of $A^3, A^T, A^{-1}, 5A, A-3I$ and adj A.

Let the eigen Values are $\lambda_1, \lambda_2, \lambda_3$

Take $\lambda_1 = 2$

Sum of the eigen Values = Sum of the diagonal elements.

$$\therefore \lambda_1 + \lambda_2 + \lambda_3 = 3+5+3$$

$$2 + \lambda_2 + \lambda_3 = 11$$

$$\lambda_2 + \lambda_3 = 9 \longrightarrow (1)$$

Product of the eigen Values = $|A|$

$$\lambda_1 \lambda_2 \lambda_3 = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$$2\lambda_2\lambda_3 = 3(15-1) - (-1)(-3+1) + 1(1-5)$$

$$= 3 \times 14 - 2 - 4$$

$$= 36$$

$$\lambda_2\lambda_3 = \frac{36}{2} = 18 \quad \text{—} (2)$$

$$\lambda_3 = \frac{18}{\lambda_2}$$

$$\therefore (1) \Rightarrow \lambda_2 + \frac{18}{\lambda_2} = 9$$

$$\lambda_2^2 + 18 = 9\lambda_2 \Rightarrow \lambda_2^2 - 9\lambda_2 + 18 = 0$$

$$(\lambda_2 - 3)(\lambda_2 - 6) = 0$$

$$\lambda_2 = 3, 6$$

$$\lambda_2 = 3 \qquad \lambda_3 = \frac{18}{3} = 6$$

$$\lambda_2 = 6 \qquad \lambda_3 = \frac{18}{6} = 3$$

$$\therefore \lambda_1 = 2 \qquad \lambda_2 = 3 \qquad \lambda_3 = 6$$

(1) The eigen Values of A^3 are $2^3, 3^3, 6^3$
$= 8, 27, 216$

(2) The eigen Values of A^T are $2, 3, 6$

(3) The eigen Values of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

(4) The eigen values of $5A$ are $5\times 2, 5\times 3, 5\times 6$
$= 10, 15, 30$

(5) The eigen values of $A-3I$ are
$\lambda_1 - 3, \lambda_2 - 3, \lambda_3 - 3 = 2-3, 3-3, 6-3$
$= -1, 0, 3$

(6) The eigen values of adj A are $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}$
$= \frac{36}{2}, \frac{36}{3}, \frac{36}{6} = \underline{\underline{18, 12, 6}}$

2. If one eigen values of the matrix
$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ is 5, find the other eigen values without finding the characteristic equation. What are the eigen values of A^2 and A^{-1}.

Eigen values are $\lambda_1, \lambda_2, \lambda_3$

Take $\lambda_1 = 5$

Sum of the eigen values = Sum of the diagonal elements

$\therefore \lambda_1 + \lambda_2 + \lambda_3 = -2+1+0$
$5 + \lambda_2 + \lambda_3 = -1$

$$\lambda_2 + \lambda_3 = -1-5$$

$$\lambda_2+\lambda_3 = -6 \quad \text{—— (1)}$$

Product of the eigen values $= |A|$

$$\lambda_1 \lambda_2 \lambda_3 = |A| = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$$

$$5 \cdot \lambda_2 \lambda_3 = -2(0-12) - 2(0-6) - 3(-4+1)$$

$$= 24 + 12 + 9$$

$$= 45$$

$$\lambda_2 \lambda_3 = \frac{45}{5} = 9 \quad \text{—— (2)}$$

$$\lambda_3 = \frac{9}{\lambda_2}$$

$$(1) \Longrightarrow \lambda_2 + \frac{9}{\lambda_2} = -6$$

$$\lambda_2^2 + 6\lambda_2 + 9 = 0$$

$$(\lambda_2 + 3)^2 = 0$$

$$\lambda_2 = -3, -3 \qquad \left(\lambda_3 = \frac{9}{-3} = -3\right)$$

$$\therefore \lambda_1 = 5 \quad \lambda_2 = -3, \quad \lambda_3 = -3$$

eigen values of $A^2 \Longrightarrow 5^2, (-3)^2, (-3)^2$

$$= 25, 9, 9$$

eigen values of $A^{-1} \Longrightarrow \frac{1}{5}, -\frac{1}{3}, -\frac{1}{3}$

Diagonalization

Two matrices A and B are said to be Similar if there exists an inventible matrix P such that

$$P^{-1}AP = B$$

Diagonalization is a process by which the given matrix A is reduced to a diagonal matrix such that the eigen values are same and both are Similar matrices.

A matrix A is said to be diagonalizable if there is an inventiable matrix P, called modal matrix such that

$$\boxed{P^{-1}AP = D}$$

where D is a diagonal matrix and D is called the Spectral matrix of A.

The diagonal elements of D are called the Spectral values of A.

Note

$$P^{-1}AP = D \implies A = PDP^{-1}$$

$\therefore$ For any positive integer m

$$\boxed{A^m = PD^mP^{-1}}$$

Problems

UQ 1. Diagonalize the matrix $A = \begin{bmatrix} 6 & 0 & 0 \\ 12 & 2 & 0 \\ 21 & -6 & 9 \end{bmatrix}$

The characteristic equation is

$$|A - \lambda I| = \begin{vmatrix} 6-\lambda & 0 & 0 \\ 12 & 2-\lambda & 0 \\ 21 & -6 & 9-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - (6+2+9)\lambda^2 + \left\{ \begin{vmatrix} 2 & 0 \\ -6 & 9 \end{vmatrix} + \begin{vmatrix} 6 & 0 \\ 21 & 9 \end{vmatrix} + \begin{vmatrix} 6 & 0 \\ 12 & 2 \end{vmatrix} \right\} \lambda$$

$$- 6(18+0) - 0 - 0 = 0$$

$$\lambda^3 - 17\lambda^2 + (18+0+54-0+12-0)\lambda - 108 = 0$$

$$\lambda^3 - 17\lambda^2 + 84\lambda - 108 = 0$$

Put $\lambda = 1$ $\quad 1 - 17 + 84 - 108 \neq 0$

$\lambda = 2$ $\quad 8 - 17 \times 4 + 84 \times 2 - 108$

$8 - 68 + 168 - 108 = 0$

$\therefore \lambda = 2$ is a root.

$$\begin{array}{c|cccc} 2 & 1 & -17 & 84 & -108 \\ & & 2 & -30 & 108 \\ \hline & 1 & -15 & 54 & 0 \end{array}$$

$\lambda^2 - 15\lambda + 54 = 0$

$(\lambda - 6)(\lambda - 9) = 0$

$\lambda = 6, 9$

$\therefore \lambda = 2, 6, 9$

When $\lambda = 2$

$(A - 2I)X = 0$

$$\begin{bmatrix} 4 & 0 & 0 \\ 12 & 0 & 0 \\ 21 & -6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$4x + 0y + 0z = 0$ — (1) First and third eqn

$12x + 0y + 0z = 0$

$21x - 6y + 7z = 0$ — (3)

$$\frac{x}{0-0} = \frac{-y}{28-0} = \frac{z}{-24-0}$$

$$\frac{x}{0} = \frac{y}{-28} = \frac{z}{-24}$$

$$\frac{x}{0} = \frac{y}{7} = \frac{z}{6}$$

$$X_1 = \begin{bmatrix} 0 \\ 7 \\ 6 \end{bmatrix}$$

When $\lambda = 6$

$(A - 6I)X = 0$

$$\begin{bmatrix} 0 & 0 & 0 \\ 12 & -4 & 0 \\ 21 & -6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 12x - 4y + 0z &= 0 \\ 21x - 6y + 3z &= 0 \end{aligned}\right\} \quad \frac{x}{-12-0} = \frac{-y}{36-0} = \frac{z}{-72+84}$$

$$\frac{x}{-12} = \frac{y}{-36} = \frac{z}{12} \implies \frac{x}{1} = \frac{y}{3} = \frac{z}{-1}$$

$$X_2 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

<u>When $\lambda = 9$</u>

$$\begin{bmatrix} -3 & 0 & 0 \\ 12 & -7 & 0 \\ 21 & -6 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -3x + 0y + 0z &= 0 \\ 12x - 7y + 0z &= 0 \end{aligned}\right\} \quad \frac{x}{0-0} = \frac{-y}{0-0} = \frac{z}{21-0}$$

$$21x - 6y + 0z = 0 \qquad \frac{x}{0} = \frac{y}{0} = \frac{z}{21}$$

$$\frac{x}{0} = \frac{y}{0} = \frac{z}{1}$$

$$X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ The Modal Matrix $P = [X_1 \ \ X_2 \ \ X_3]$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 7 & 3 & 0 \\ 6 & -1 & 1 \end{bmatrix}$$

Diagonal Matrix $D = P^{-1}AP$

$$|P| = 0(3+0) - 1(7-0) + 0(-7-18)$$

$$= -7$$

$a_{11} = (-1)^{1+1}(3-0) = 3$

$a_{12} = (-1)^{1+2}(7-0) = -7$

$a_{13} = (-1)^{1+3}(-7-18) = -25$

$a_{21} = (-1)^{2+1}(1-0) = -1$

$a_{22} = (-1)^{2+2}(0-0) = 0$

$a_{23} = (-1)^{2+3}(0-6) = 6$

$a_{31} = (-1)^{3+1}(0-0) = 0$

$a_{32} = (-1)^{3+2}(0-0) = 0$

$a_{33} = (-1)^{3+3}(0-7) = -7$

$$\text{adj } P = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ -7 & 0 & 0 \\ -25 & 6 & -7 \end{bmatrix}$$

$$P^{-1} = \frac{\text{adj } P}{|P|} = -\frac{1}{7}\begin{bmatrix} 3 & -1 & 0 \\ -7 & 0 & 0 \\ -25 & 6 & -7 \end{bmatrix}$$

$$\therefore P^{-1}AP = -\frac{1}{7}\begin{bmatrix} 3 & -1 & 0 \\ -7 & 0 & 0 \\ -25 & 6 & -7 \end{bmatrix}\begin{bmatrix} 6 & 0 & 0 \\ 12 & 2 & 0 \\ 21 & -6 & 9 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 7 & 3 & 0 \\ 6 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix} = D$$

2. Find a matrix B which transform $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ in to the diagonal form.

Diagonal matrix $D = B^{-1}AB$

B = Modal Matrix $= [X_1 \quad X_2 \quad X_3]$

characteristic equation $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - (1+2+3)\lambda^2 + \left\{ \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} \right\} \lambda$$

$$- \{ 1(6-2) - 0(3-2) - 1(2-4) \} = 0$$

$$\lambda^3 - 6\lambda^2 + \{ (6-2) + (3+2) + (2-0) \} \lambda - [4 - 0 + 2] = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

Put $\lambda = 1 \Rightarrow 1 - 6 + 11 - 6 = 0$

$\therefore \lambda = 1$ is a root

1	1	-6	11	-6
		1	-5	6
	1	-5	6	$\boxed{0}$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$$\underline{\lambda = 2, 3}$$

Eigen Values $= \{1, 2, 3\}$

<u>When $\lambda = 1$</u>

$$(A - I)X = 0$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 0x + 0y - z &= 0 \\ x + y + z &= 0 \end{aligned}\right\}$$

$$2x + 2y + 2z = 0$$

$$\frac{x}{0-(-1)} = \frac{-y}{0-(-1)} = \frac{z}{0-0}$$

$$\frac{x}{1} = \frac{y}{-1} = \frac{z}{0}$$

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

When $\lambda = 2$

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -x + 0y - z &= 0 \\ x + 0y + z &= 0 \\ 2x + 2y + z &= 0 \end{aligned}\right\}$$

Take 2nd and 3rd eqn

$$\frac{x}{0-2} = \frac{-y}{1-2} = \frac{z}{2-0}$$

$$\frac{x}{-2} = \frac{y}{1} = \frac{z}{2}$$

$$X_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

When $\lambda = 3$

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -2x + 0y - z &= 0 \\ x - y + z &= 0 \end{aligned}\right\}$$

$$2x + 2y + 0z = 0$$

$$\frac{x}{0-1} = \frac{-y}{-2+1} = \frac{z}{2-0}$$

$$\frac{x}{-1} = \frac{y}{1} = \frac{z}{2}$$

$$X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore$ Modal Matrix $B = [X_1 \quad X_2 \quad X_3]$

$$B = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

3. Diagonalize $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

The characteristic equation of the matrix A

$|A-\lambda I|=0$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix}=0$$

$$\lambda^3-(-2+1+0)\lambda^2+\left\{\begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix}+\begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix}+\begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}\right\}\lambda$$

$$-\left\{-2(0-12)-2(0-6)-3(-4+1)\right\}=0$$

$$\lambda^3+\lambda^2+\left\{(0-12)+(0-3)+(-2-4)\right\}\lambda$$

$$-\left[24+12+9\right]=0$$

$$\lambda^3+\lambda^2-21\lambda-45=0$$

Put $\lambda=1$ $\quad 1+1-21-45\neq 0$

$\lambda=2$ $\quad 8+4-21\times 2-45\neq 0$

$\lambda=3$ $\quad 27+9-21\times 3-45\neq 0$

$\lambda=4$ $\quad 64+16-21\times 4-45\neq 0$

$\lambda=5$ $\quad 125+25-21\times 5-45=0$

$\lambda=5$ is a root.

$$\begin{array}{r|rrrr} 5 & 1 & 1 & -21 & -45 \\ & & 5 & 30 & 45 \\ \hline & 1 & 6 & 9 & \boxed{0} \end{array}$$

$\lambda^2+6\lambda+9=0$

$(\lambda-3)^2=0 \qquad \lambda=-3,-3$

When $\lambda = 5$

$(A - 5I)X = 0$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -7x + 2y - 3z &= 0 \\ 2x - 4y - 6z &= 0 \end{aligned}\right\} \qquad \frac{x}{-12-12} = \frac{-y}{42+6} = \frac{z}{28-4}$$

$$-x - 2y - 5z = 0 \qquad \frac{x}{-24} = \frac{y}{-48} = \frac{z}{24}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

When $\lambda = -3$

$(A - 3I)X = 0$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 2y - 3z = 0$$

put $z = t_1 \quad y = t_2 \quad x = 3z - 2y$
$= 3t_1 - 2t_2$

$$X = \begin{bmatrix} 3t_1 - 2t_2 \\ t_2 \\ t_1 \end{bmatrix} = t_1 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \qquad X_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Modal Matrix $P = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

$D = P^{-1} A P$

$P^{-1} = \frac{\operatorname{adj} P}{|P|}$

$$|P| = \begin{vmatrix} 1 & 3 & -2 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix} = 1(0-1) - 3(0+1) - 2(2-0)$$

$$= -1 - 3 - 4 = \underline{\underline{-8}}$$

$a_{11} = (-1)^{1+1}(0-1) = -1$

$a_{12} = (-1)^{1+2}(0+1) = -1$

$a_{13} = (-1)^{1+3}(2-0) = 2$

$a_{21} = (-1)^{2+1}(0+2) = -2$

$a_{22} = (-1)^{2+2}(0+2) = 2$

$a_{23} = (-1)^{2+3}(1-3) = -4$

$a_{31} = (-1)^{3+1}(3-0) = 3$

$a_{32} = (-1)^{3+2}(1+4) = -5$

$a_{33} = (-1)^{3+3}(0-6) = -6$

$$\operatorname{adj} P = \begin{bmatrix} -1 & -2 & 3 \\ -1 & -2 & -5 \\ 2 & -4 & -6 \end{bmatrix}$$

$$P^{-1} = \frac{-1}{8} \begin{bmatrix} -1 & -2 & 3 \\ -1 & -2 & -5 \\ 2 & -4 & -6 \end{bmatrix}$$

$$P^{-1} A P = -\frac{1}{8} \begin{bmatrix} -1 & -2 & 3 \\ -1 & -2 & -5 \\ 2 & -4 & -6 \end{bmatrix} \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$= \underline{\begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}}$$

Diagonalization of real Symmetric matrices

Every real symmetric matrices are diagonalizable. The eigen values of real symmetric matrices are orthogonal. ∴ we can form orthonormal vectors from the eigen vectors.

Consider a ~~real~~ 3×3 real symmetric matrix. First we compute the eigen values λ_1, λ_2 and λ_3 of A. Corresponding to two eigen values λ_1 and λ_2, we find orthogonal eigen vectors x_1 and x_2. To find the third eigen vector corresponding to λ_3, compute $x_3 = x_1 \times x_2$, cross product of x_1 and x_2.

Normalize these vectors using the relation $x_i^* = \frac{x_i}{\|x_i\|}$, $i = 1, 2, 3$

Construct the modal matrix

$$P = \left[x_1^* \quad x_2^* \quad x_3^* \right]$$

P is an orthonormal matrix

$$\therefore P^{-1} = P^T$$

$$\therefore \boxed{D = P^{-1} A P = P^T A P}$$

$$\boxed{\begin{aligned} X &= (x_1, x_2, x_3) \\ \|X\| &= \sqrt{x_1^2 + x_2^2 + x_3^2} \end{aligned}}$$

Problems

1. Diagonalize the symmetric matrix

u.a

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

The characteristic equation is $|A-\lambda I|=0$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix}=0$$

$$\lambda^3-(8+7+3)\lambda^2+\left\{\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix}+\begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix}+\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}\right\}\lambda$$

$$-\left\{8(21-16)+6(-18+8)+2(24-14)\right\}=0$$

$$\lambda^3-18\lambda^2+(21-16+24-4+56-36)\lambda$$

$$-(40-60+20)=0$$

$$\lambda^3-18\lambda^2+45\lambda=0$$

$$\lambda(\lambda^2-18\lambda+45)=0$$

$$\lambda(\lambda-3)(\lambda-15)=0$$

$$\lambda=0,3,15$$

Eigen Values $=0,3,15$

When $\lambda = 0$

$(A - 0I)x = 0 \implies Ax = 0$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 8x - 6y + 2z &= 0 \\ -6x + 7y - 4z &= 0 \end{aligned}\right\} \qquad \frac{x}{24-14} = \frac{-y}{-32-12} = \frac{z}{56-36}$$

$$2x - 4y + 3z = 0 \qquad \frac{x}{10} = \frac{y}{20} = \frac{z}{20}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

When $\lambda = 3$

$(A - 3I)x = 0$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 5x - 6y + 2z &= 0 \\ -6x + 4y - 4z &= 0 \end{aligned}\right\} \qquad \frac{x}{24-8} = \frac{-y}{-20+12} = \frac{z}{20-36}$$

$$2x - 4y + 0z = 0 \qquad \frac{x}{16} = \frac{y}{8} = \frac{z}{-16}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$$

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

When $\lambda = 15$

$$X_3 = X_1 \times X_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{vmatrix} = i(-4-2) - j(-2-4) + k(1-4)$$

$$= -6i + 6j - 3k$$

$$= 2i - 2j + k$$

$$x_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Normalized Eigen Vectors are $\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}, \frac{x_3}{\|x_3\|}$

$$\frac{x_1}{\|x_1\|} = \frac{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}{\sqrt{1^2+2^2+2^2}} = \frac{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}{3} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\frac{x_2}{\|x_2\|} = \frac{\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}}{\sqrt{2^2+1^2+(-2)^2}} = \frac{\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}}{3} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$\frac{x_3}{\|x_3\|} = \frac{\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}}{\sqrt{2^2+(-2)^2+1}} = \frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Modal Matrix $P = \begin{bmatrix} \frac{x_1}{\|x_1\|} & \frac{x_2}{\|x_2\|} & \frac{x_3}{\|x_3\|} \end{bmatrix}$

$$P = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Diagonal Matrix $D = P^TAP$

$$P^TAP = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} = D$$

2. Diagonalize the symmetric matrix

$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ and hence find A^4.

The characteristic equation $|A - \lambda I| = 0$

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - (2+2+2)\lambda^2 + \left\{ \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} \right\} \lambda$$

$$- \left\{ 2(4-0) - 0 + 1(0-2) \right\} = 0$$

$$\lambda^3 - 6\lambda^2 + (4-0+4-1+4-0)\lambda - (8-0-2) = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

put $\lambda = 1$ $\quad 1 - 6 + 11 - 6 = 0$

$\lambda = 1$ is a root

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 11 & -6 \\ & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda = 2, 3$$

eigen values are $1, 2, 3$

<u>When $\lambda = 1$</u> $\quad (A - I)x = 0$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} x+0y+z=0 \\ 0x+y+0z=0 \end{array}\right\} \quad \frac{x}{0-1}=\frac{-y}{0-0}=\frac{z}{1-0}$$

$$x+0y+z=0 \qquad \frac{x}{-1}=\frac{y}{0}=\frac{z}{1}$$

$$X_1=\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

<u>When $\lambda=2$</u>

$(A-2I)X=0$

$$\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left.\begin{array}{l} 0x+0y+z=0 \\ x+0y+z=0 \end{array}\right\} \quad \frac{x}{0-0}=\frac{-y}{0-1}=\frac{z}{0-0}$$

$$\frac{x}{0}=\frac{y}{1}=\frac{z}{0}$$

$$X_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$$

<u>When $\lambda=3$</u>

$$X_3=X_1\times X_2=\begin{vmatrix}i&j&k\\-1&0&1\\0&1&0\end{vmatrix}=i(0-1)-j(0-0)+k(-1-0)$$

$$=-i+0j-k$$

$$X_3=\begin{bmatrix}-1\\0\\-1\end{bmatrix}=\begin{bmatrix}1\\0\\1\end{bmatrix}$$

Now $\frac{X_1}{\|X_1\|}=\frac{1}{\sqrt{(-1)^2+0+1}}\begin{bmatrix}-1\\0\\1\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}$

$$\frac{X_2}{\|X_2\|}=\frac{1}{\sqrt{0^2+1^2+0}}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\frac{X_3}{\|X_3\|}=\frac{1}{\sqrt{1^2+0+1^2}}\begin{bmatrix}1\\0\\1\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}$$

Modal Matrix $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 1/\sqrt{2} \end{bmatrix}$

$$P^T A P = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$$

3. $A^m = P D^m P^T$

$A^4 = P D^4 P^T$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 41 & 0 & 40 \\ 0 & 16 & 0 \\ 40 & 0 & 41 \end{bmatrix}$$

3. Diagonalise the symmetric matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

The characteristic equation of A is

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - (6+3+3)\lambda^2 + \left\{ \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} \right\}\lambda$$

$$- \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 0$$

$$\lambda^3 - 12\lambda^2 + [9-1+18-4+18-4]\lambda - \{6(9-1)+2(-6+2)+2(2-6)\} = 0$$

$$\lambda^3 - 12\lambda^2 + 36\lambda - [48-8-8] = 0$$

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

Put $\lambda = 1$ $\quad 1-12+36-32 \neq 0$

$\lambda = 2$ $\quad 8 - 12\times 4 + 36\times 2 - 32 = 0$

$\therefore \lambda = 2$ is a root.

$$\begin{array}{c|cccc} 2 & 1 & -12 & 36 & -32 \\ & & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \boxed{0} \end{array}$$

$$\lambda^2 - 10\lambda + 16 = 0$$

$$(\lambda-2)(\lambda-8) = 0$$

$$\underline{\lambda = 2, 8}$$

$$\underline{\lambda = 8, 2, 2}$$

<u>When $\lambda = 8$</u>

$(A-8I)X = 0$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned}-2x-2y+2z&=0\\-2x-5y-z&=0\end{aligned}\right\}\quad \frac{x}{2+10}=\frac{-y}{2+4}=\frac{z}{10-4}$$

$$2x-y-5z=0 \qquad \frac{x}{12}=\frac{y}{-6}=\frac{z}{6}$$

$$\frac{x}{2}=\frac{y}{-1}=\frac{z}{1}$$

$$X_1=\begin{bmatrix}2\\-1\\1\end{bmatrix}$$

<u>When $\lambda=2$</u>

$(A-2I)X=0$

$$\begin{bmatrix}4&-2&2\\-2&1&-1\\2&-1&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left.\begin{aligned}4x-2y+2z&=0\\-2x+y-z&=0\\2x-y+z&=0\end{aligned}\right\}\Longrightarrow 2x-y+z=0$$

Put $x=0 \quad y=1 \Longrightarrow z=-2x+y=0+1=1$

$$X_2=\begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$X_3=X_1\times X_2=\begin{vmatrix}i&j&k\\2&-1&1\\0&1&1\end{vmatrix}=$$

$$=i(-1-1)-j(2-0)+k(2-0)$$

$$=-2i-2j+2k$$

$$=i+j-k \qquad X_3=\begin{bmatrix}1\\1\\-1\end{bmatrix}$$

$$\frac{X_1}{\|X_1\|} = \frac{1}{\sqrt{2^2+(-1)^2+1^2}} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\frac{X_2}{\|X_2\|} = \frac{1}{\sqrt{0^2+1^2+1^2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\frac{X_3}{\|X_3\|} = \frac{1}{\sqrt{1^2+1^2+(-1)^2}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Modal Matrix = $P = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix}$

$$P^T A P = \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 0 & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

4. Diagonalize the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

The characteristic equation $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - (2+2+2)\lambda^2 + \left\{ \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \right\} \lambda$$

$$- \left\{ 2(4-1) + 1(-2+1) + 1(1-2) \right\} = 0$$

$$\lambda^3 - 6\lambda^2 + [4-1+4-1+4-1]\lambda - [6-1-1] = 0$$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

Put $\lambda = 1$ $\quad 1 - 6 + 9 - 4 = 0$

$\therefore$ $\lambda = 1$ is a root

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 9 & -4 \\ & & 1 & -5 & 4 \\ \hline & 1 & -5 & 4 & \boxed{0} \end{array}$$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$(\lambda - 1)(\lambda - 4) = 0$$

$$\lambda = 1, 4$$

$\therefore$ Eigen values $= 1, 1, 4$

When $\lambda = 4$

$$(A - 4I)x = 0$$

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

~~when λ=...~~

$$-2x-y+z=0$$
$$-x-2y-z=0$$
$$x-y-2z=0$$

$$\frac{x}{1+2} = \frac{-y}{2+1} = \frac{z}{4-1}$$

$$\frac{x}{3} = \frac{y}{-3} = \frac{z}{3}$$

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

When $\lambda = 1$

$(A-I)X = 0$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} x-y+z&=0 \\ -x+y-z&=0 \\ x-y+z&=0 \end{aligned}\right\} \text{Three eqns same} \implies x-y+z=0.$$

Put $x=0$, $y=1$, $z=-x+y=0+1=1$

$$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$X_3 = X_1 \times X_2 = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = i(-1-1) - j(1-0) + k(1-0)$$

$$= -2i - j + k$$

$$= 2i + j - k$$

$$X_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\frac{X_1}{\|X_1\|} = \frac{1}{\sqrt{1^2+(-1)^2+1^2}} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\frac{X_2}{\|X_2\|} = \frac{1}{\sqrt{0^2+1^2+1^2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\frac{X_3}{\|X_3\|} = \frac{1}{\sqrt{2^2+1^2+(-1)^2}} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

~~[illegible]~~

Modal Matrix $P = \begin{bmatrix} 2/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$

$$P^T A P = \begin{bmatrix} 2/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5 Diagonalize $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

The characteristic equation $|A - \lambda I| = 0$.

$$|A-\lambda I|=0 \Rightarrow \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix}=0$$

$$\lambda^3-(3+3+3)\lambda^2+\left\{\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}+\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}+\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}\right\}\lambda$$

$$-\{3(9-1)+1(-3+1)+1(1-3)\}=0$$

$$\lambda^3-9\lambda^2+\{(9-1+9-1+9-1)\}\lambda-\{24-2-2\}=0$$

$$\lambda^3-9\lambda^2+24\lambda-20=0$$

put $\lambda=1$ $\quad 1-9+24-20\neq 0$

$\lambda=2$ $\quad 8-9\times4+24\times2-20=0$

$\therefore \lambda=2$ is a root

$$\begin{array}{c|cccc} 2 & 1 & -9 & 24 & -20 \\ & & 2 & -14 & 20 \\ \hline & 1 & -7 & 10 & \boxed{0} \end{array}$$

$$\lambda^2-7\lambda+10=0$$

$$(\lambda-2)(\lambda-5)=0$$

$$\lambda=2,5$$

$$\therefore \lambda=2,2,5$$

When $\lambda=5$ $\quad (A-5I)x=0$

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} -2x-y+z=0 \\ -x-2y-z=0 \end{array}\right\} \quad \frac{x}{1+2}=\frac{-y}{2+1}=\frac{z}{4-1}$$

$$x-y-2z=0$$

$$\frac{x}{3}=\frac{y}{-3}=\frac{z}{3}$$

$$\frac{x}{1}=\frac{y}{-1}=\frac{z}{1}$$

$$X_1=\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

When $\lambda = 2$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} x - y + z &= 0 \\ -x + y - z &= 0 \\ x - y + z &= 0 \end{aligned}\right\} \Rightarrow x - y + z = 0$$

Put $x = 0$ $y = 1$ $z = -x + y = 1$

$$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad X_3 = X_1 \times X_2 = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = i(-1-1) - j(1-0) + k(1-0)$$

$$= -2i - j + k$$
$$= 2i + j - k$$

$$X_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\frac{X_1}{\|X_1\|} = \frac{1}{\sqrt{1^2 + (-1)^2 + 1^2}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\frac{X_2}{\|X_2\|} = \frac{1}{\sqrt{0^2 + 1^2 + 1^2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\frac{X_3}{\|X_3\|} = \frac{1}{\sqrt{2^2 + 1^2 + (-1)^2}}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

Modal Matrix $P = \begin{bmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$

$$P^T A P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Quadratic forms

A homogeneous polynomial of second degree in any number of variables is called a quadratic form. A quadratic form in 'n' variables $x_1, x_2 \cdots x_n$ is an expression of the form

$$q = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i x_j$$

$$= a_{11} x_1^2 + a_{12} x_1 x_2 + \cdots + a_{1n} x_1 x_n$$
$$+ a_{21} x_2 x_1 + a_{22} x_2^2 + \cdots + a_{2n} x_2 x_n$$
$$+ \cdots \cdots$$
$$+ a_{n1} x_n x_1 + a_{n2} x_n x_2 + \cdots + a_{nn} x_n^2$$

In Matrix Notation, this can be written as

$$q = [x_1 \; x_2 \cdots x_n] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\boxed{q = X^T A X}$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Rule for finding the Matrix associated with a quadratic form q in 3 Variables x_1, x_2 and x_3.

Let $A = a_0 x_1^2 + b_0 x_2^2 + c_0 x_3^2 + a_1 x_1 x_2 + b_1 x_2 x_3 + c_1 x_1 x_3$ be a quadratic form.

$$q = X^T A X$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

a_{11} = coeff of x_1^2

a_{22} = coeff of x_2^2

a_{33} = coeff of x_3^2

$a_{12} = a_{21} = \frac{1}{2}$ coeff of $x_1 x_2$

$a_{13} = a_{31} = \frac{1}{2}$ coeff of $x_1 x_3$

$a_{23} = a_{32} = \frac{1}{2}$ coeff of $x_2 x_3$

Reduction to canonical form by orthogonal transformation

Let $q = X^T A X$ where A is a symmetric matrix.

We can diagonalise it by the orthogonal transformation P, whose coloumns are the orthonormal eigen vectors of A as

$$P^T A P = \text{diag}(\lambda_1, \lambda_2 \cdots \lambda_n)$$

Where $\lambda_1, \lambda_2 \cdots \lambda_n$ are the eigen values of A.

If we introduce the new variable vector Y by the orthogonal transformation X∈R

$X = PY$, then

$$q = X^T A X = (PY)^T A (PY)$$

[illegible]

$$= Y^T (P^T A P) Y$$

$$= Y^T (\text{diag}\ \lambda_1 \lambda_2 \cdots \lambda_n) Y$$

$$= [y_1, y_2 \cdots y_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

This form of q is known as canonical form or principal axes form.

$$\boxed{\begin{array}{l} \text{Canonical form} \\ q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \end{array}}$$

Rank, Index and Signature of a Quadratic form

Rank of a quadratic form = Number of terms in the canonical form

Index = Number of positive terms in the canonical form

Signature = Number of positive terms − Number of negative terms in the canonical form.

Nature of a quadratic form

A real quadratic form $q = x^T A x$ is

1. positive definite if and only if all the eigen values of A are positive.
2. Negative definite if and only if all the eigen values of A are negative.
3. Positive Semi definite if and only if all the eigen values of A are ≥ 0 and atleast one eigen value is zero.
4. Negative Semidefinite if and only if all the eigen Values of A are ≤ 0 and atleast one eigen Value is zero.
5. Indefinite if and only if A has both positive and negative eigen Values.

Principal Minors

Let $q = x^T A x$ be a quadratic form, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$ is symmetric.

Then $D_1 = |a_{11}|$

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$\cdots$

$$D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

are the principal minors of the symmetric matrix A representing the quadratic form.

Definiteness

A real quadratic form $q = x^T A x$ is

(1) positive definite if and only if $D_n > 0$ for all n.

(2) Positive Semi definite iff $D_n \geq 0$ and atleast one $D_i = 0$

(3) Negative definite iff $D_1 < 0$, $D_2 > 0$, $D_3 < 0$ etc.

(4) Negative Semi definite iff $D_1 < 0$, $D_2 > 0$ $D_3 < 0$ and some of the determinant are zero.

(5) Indefinite in all the other cases.

Application

If a quadratic form is equated to a constant, it represent a quadratic surface. To know the nature of the surface, we reduce the equation to canonical form.

Problems

1. Find out what type of conic section the quadratic form $q = 3x_1^2 + 22x_1x_2 + 3x_2^2 = 0$ represents.

$$q = 3x_1^2 + 22x_1x_2 + 3x_2^2 = 0$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

a_{11} = Coeff of $x_1^2 = 3$

a_{22} = Coeff of $x_2^2 = 3$

$a_{12} = a_{21} = \frac{1}{2}$ coeff of $x_1x_2 = \frac{1}{2} \times 22 = 11$

$$A = \begin{bmatrix} 3 & 11 \\ 11 & 3 \end{bmatrix}$$

characteristic equation $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 11 \\ 11 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - (3+3)\lambda + \begin{vmatrix} 3 & 11 \\ 11 & 3 \end{vmatrix} = 0$$

$$\lambda^2 - 6\lambda + (9 - 121) = 0$$

$\lambda^2 - 6\lambda - 112 = 0$

$(\lambda - 14)(\lambda + 8) = 0$

$\underline{\underline{\lambda = 14, -8}}$

Canonical form $q = \lambda_1 y_1^2 + \lambda_2 y_2^2$

$q = 14 y_1^2 - 8 y_2^2$

Given $q = 0 \implies 14 y_1^2 - 8 y_2^2 = 0,$

Which represent $\underline{\underline{\text{pair of}}}$ Straight line

2. What kind of conic section is given by the quadratic form $7x_1^2 + 6x_1x_2 + 7x_2^2 = 200$.
Also find its equation.

$q = 7x_1^2 + 6x_1x_2 + 7x_2^2 = 200$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

a_{11} = coeff of $x_1^2 = 7$

a_{22} = Coeff of $x_2^2 = 7$

$a_{12} = a_{21} = \frac{1}{2}$ Coeff of $x_1x_2 = 3$

$A = \begin{bmatrix} 7 & 3 \\ 3 & 7 \end{bmatrix}$

Characteristic eqn $|A - \lambda I| = 0$

$\begin{vmatrix} 7-\lambda & 3 \\ 3 & 7-\lambda \end{vmatrix} = 0 \implies \lambda^2 - (7+7)\lambda + \begin{vmatrix} 7 & 3 \\ 3 & 7 \end{vmatrix} = 0$

$$\lambda^2 - 14\lambda + 40 = 0$$

$$(\lambda - 4)(\lambda - 10) = 0$$

$$\lambda = 4, 10$$

canonical form $q = \lambda_1 y_1^2 + \lambda_2 y_2^2$

$$q = 4y_1^2 + 10y_2^2 = 200$$

$$\frac{4}{200} y_1^2 + \frac{10}{200} \cdot y_2^2 = 1$$

$\frac{y_1^2}{50} + \frac{y_2^2}{20} = 1$, which represent ellipse

3. Show that $17x^2 - 30xy + 17y^2 = 128$ represents an ellipse. Also find the equations of the major and minor axes of the ellipse in terms of x and y.

$$q = 17x^2 - 30xy + 17y^2 = 128$$

$x = x_1$
$y = x_2$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{bmatrix}$$

a_{11} = Coeff of x^2 = 17

a_{22} = Coeff of y^2 = 17

$a_{12} = a_{21}$ = ½ coeff of xy = −15

$$A = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

The characteristic equation of A is

$$|A-\lambda I| = \begin{vmatrix} 17-\lambda & -15 \\ -15 & 17-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - (17+17)\cdot\lambda + \begin{vmatrix} 17 & -15 \\ -15 & 17 \end{vmatrix} = 0$$

$$\lambda^2 - 34\lambda + 64 = 0$$

$$(\lambda-2)(\lambda-32) = 0 \implies \lambda = 2, 32$$

canonical form is $q = \lambda_1 y_1^2 + \lambda_2 y_2^2$

$$q = 2y_1^2 + 32y_2^2 = 128$$

$$\frac{2}{128}y_1^2 + \frac{32}{128}y_2^2 = 1$$

$$\frac{y_1^2}{64} + \frac{y_2^2}{4} = 1$$

$$\frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1,$$ represent ellipse.

Equation of major axis is

$y=0$; $-8 \le x \le 8$

Equation of minor axis is

$x=0$; $-2 \le y \le 2$.

4. Write the canonical form of the quadratic form $Q(x,y,z) = 3x^2+5y^2+3z^2-2xy+2xz-2yz$ and hence show that $Q(x,y,z) > 0$ for all non-zero values of x, y, z

$$Q = 3x^2+5y^2+3z^2-2xy+2xz-2yz$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\boxed{\begin{matrix} 1 \to x \\ 2 \to y \\ 3 \to z \end{matrix}}$$

a_{11} = coeff of $x^2 = 3$

a_{22} = coeff of $y^2 = 5$

a_{33} = coeff of $z^2 = 3$

$a_{12} = a_{21} = \frac{1}{2}$ coeff of $xy = \frac{1}{2} \times -2 = -1$

$a_{13} = a_{31} = \frac{1}{2}$ coeff of $xz = \frac{1}{2} \times 2 = 1$

$a_{23} = a_{32} = \frac{1}{2}$ coeff of $yz = \frac{1}{2} \times -2 = -1$

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic equation $|A-\lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - (3+5+3)\lambda^2 + \left\{ \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix} \right\} \lambda$$

$$- \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 0$$

$$\lambda^3 - 11\lambda^2 + \{15-1+9-1+15-1\}\lambda$$

$$- \{3(15-1)+1(-3+1)+1(1-5)\} = 0$$

$$\lambda^3 - 11\lambda^2 + 36\lambda - (42-2-4) = 0$$

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

put $\lambda = 1$ $\quad 1-11+36-36 \neq 0$

$\lambda = 2$ $\quad 8 - 11\times 2 + 36\times 2 - 36 = 0$

$\therefore$ $\lambda = 2$ is a root

$$\begin{array}{c|cccc} 2 & 1 & -11 & 36 & -36 \\ & & 2 & -18 & 36 \\ \hline & 1 & -9 & 18 & 0 \end{array}$$

$$\lambda^2 - 9\lambda + 18 = 0$$

$$(\lambda - 3)(\lambda - 6) = 0$$

$$\lambda = 3, 6$$

$$\therefore \lambda = 2, 3, 6$$

Canonical form $q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$

$$q = 2y_1^2 + 3y_2^2 + 6y_3^2$$

All eigen values are positive.

$\therefore$ $Q > 0$

5. Find the nature, rank and signature of the quadratic form

$3x^2+5y^2+3z^2-2xy+2xz-2yz$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic eqn $|A-\lambda I|=0$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\Longrightarrow \lambda^3-11\lambda^2+36\lambda-36=0$

$\lambda = 2, 3, 6$

Canonical form

$q: \lambda_1 y_1^2+\lambda_2 y_2^2+\lambda_3 y_3^2 = 2y_1^2+3y_2^2+6y_3^2$

Nature $\Longrightarrow$ All Eigen Values are positive

$q. \Longrightarrow$ positive definite

Rank $\Longrightarrow$ Number of terms in the canonical form $= 3$

Signature $\Longrightarrow$ Number of positive terms − Number of −ve terms

$= 3-0 \underline{= 3}$

6. Find the nature, index, rank and Signature of the quadratic form $x_1^2+2x_2^2+3x_3^2+2x_1x_2-2x_1x_3+2x_2x_3$.

$$q = x_1^2+2x_2^2+3x_3^2+2x_1x_2-2x_1x_3+2x_2x_3$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

a_{11} = Coeff of $x_1^2 = 1$

a_{22} = Coeff of $x_2^2 = 2$

a_{33} = Coeff of $x_3^2 = 3$

$a_{12} = a_{21} = \frac{1}{2}$ coeff of $x_1x_2 = 1$

$a_{13} = a_{31} = \frac{1}{2}$ Coeff of $x_1x_3 = -1$

$a_{23} = a_{32} = \frac{1}{2}$ Coeff of $x_2x_3 = 1$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

The characteristic equation $|A-\lambda I|=0$

$$\begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & 2-\lambda & 1 \\ -1 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 6\lambda^2 + \left\{ \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \right\} \lambda$$

$$- \left\{ \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{vmatrix} \right\} = 0$$

$$\lambda^3 - 6\lambda^2 + 8\lambda + 2 = 0$$

$$\lambda = -0.2143,\ 3.675,\ 2.539$$

The eigen values are both positive and negative. $\Longrightarrow$ Indefinite.

Canonical form $q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$

$$q = -0.2143\, y_1^2 + 3.675\, y_2^2 + 2.539\, y_3^2$$

Rank = Number of terms in the canonical form
= 3

Index = Number of positive terms in the Canonical form = 2

Signature = Number of positive terms − Negative terms in the canonical form
= $\underline{\underline{2 - 1 = 1}}$

7. Show that the quadratic form $4x_1^2 + 12x_1x_2 + 13x_2^2 = 16$ is positive definite

$$q = 4x_1^2 + 12x_1x_2 + 13x_2^2$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

a_{11} = coeff of $x_1^2 = 4$ $\quad$ $a_{12} = a_{21} = \frac{1}{2}$ coeff x_1x_2

a_{22} = coeff of $x_2^2 = 13$ $\quad$ $= 6$

$$A = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$$

$D_1 = |a_{11}| = 4 > 0$

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 4 & 6 \\ 6 & 13 \end{vmatrix} = 16 > 0$$

$D_1 > 0$, $D_2 > 0 \implies$ positive definite

8. Show that $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3x_1$ in positive Semi definite.

$$q = 8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3x_1$$

a_{11} = coeff of $x_1^2 = 8$

a_{22} = coeff of $x_2^2 = 7$

a_{33} = coeff of $x_3^2 = 3$

$a_{12} = a_{21} = \frac{1}{2}$ coeff of $x_1x_2 = -6$

$a_{13} = a_{31} = \frac{1}{2}$ coeff of $x_1x_3 = 2$

$a_{23} = a_{32} = \frac{1}{2}$ coeff of $x_2x_3 = -4$

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$D_1 = |a_{11}| = 8 > 0$

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = 56 - 36 = 20 > 0$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 8(21-16) + 6(-18+8) + 2(24-14)$$

$$= 40 - 60 + 20$$

$$= \underline{0}$$

Since $D_1 > 0$, $D_2 > 0$ and $D_3 = 0$

$\Longrightarrow$ quadratic form is positive semidefinite.

9. Find out what type of conic section, the quadratic form $17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$ and transform it to principal axes.

$$q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

Principal axes means canonical form

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

Characteristic equation $|A-\lambda I|=0$

$$\begin{vmatrix} 17-\lambda & -15 \\ -15 & 17-\lambda \end{vmatrix}=0$$

$$\Rightarrow \lambda^2-(17+17)\lambda+\begin{vmatrix} 17 & -15 \\ -15 & 17 \end{vmatrix}=0$$

$$\lambda^2-34\lambda+64=0$$

$$(\lambda-2)(\lambda-32)=0$$

$$\Rightarrow \underline{\underline{\lambda=2,32}}$$

The canonical form $q=\lambda_1 y_1^2+\lambda_2 y_2^2$

$$q=2y_1^2+32y_2^2=128$$

$$\frac{2}{128}y_1^2+\frac{32}{128}y_2^2=1$$

$$\frac{y_1^2}{64}+\frac{y_2^2}{4}=1$$

$$\underline{\underline{\frac{y_1^2}{8^2}+\frac{y_2^2}{2^2}=1}}$$, which represent ellipse.

When $\lambda=2$

$$(A-2I)x=0$$

$$\begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 15x-15y&=0 \\ -15x+15y&=0 \end{aligned}\right\} \Rightarrow x-y=0$$

$$\Rightarrow x=y$$

$$\frac{x}{1}=\frac{y}{1}$$

$$X_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

When $\lambda = 32$

$(A - 32I)X = 0$

$$\begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-15x - 15y = 0 \implies x + y = 0$$

$$x = -y \implies \frac{x}{-1} = \frac{y}{1}$$

$$X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\frac{X_1}{\|X_1\|} = \frac{1}{\sqrt{1^2+1^2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{X_2}{\|X_2\|} = \frac{1}{\sqrt{(-1)^2+1^2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The modal matrix is $P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

The orthogonal transformation is

$$X = PY$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

10. Reduce $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4zx$ in to canonical form (or sum of squares) by orthogonal reduction (transformation). Examine for definiteness.

$q = 6x^2+3y^2+3z^2-4xy-2yz+4zx$

a_{11} = coeff of $x^2 = 6$

a_{22} = coeff of $y^2 = 3$

a_{33} = coeff of $z^2 = 3$

$a_{12} = a_{21} = \frac{1}{2}$ coeff of $xy = -2$

$a_{13} = a_{31} = \frac{1}{2}$ coeff of $xz = 2$

$a_{23} = a_{32} = \frac{1}{2}$ coeff of $yz = -1$

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

The characteristic equation of A is

$$|A-\lambda I| = 0 \implies \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - (6+3+3)\lambda^2 + \left\{ \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} \right\}\lambda$$

$$- \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 0$$

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

put $\lambda = 1$ $\quad 1-12+36-32 \neq 0$

$\lambda = 2$ $\quad 8-12\times 4+36\times 2-32 = 0$

$\lambda = 2$ is a root

$$\begin{array}{c|cccc} 2 & 1 & -12 & 36 & -32 \\ & & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & 0 \end{array}$$

$\lambda^2 - 10\lambda + 16 = 0$

$(\lambda-2)(\lambda-8) = 0$

$\lambda = 2, 8$

$\lambda = 8, 2, 2$

When $\lambda = 8$

$(A - 8I)X = 0$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\left.\begin{aligned} -2x - 2y + 2z &= 0 \\ -2x - 5y - z &= 0 \\ 2x - y - 5z &= 0 \end{aligned}\right\} \quad \frac{x}{2} = \frac{y}{-1} = \frac{z}{1}$$

$$X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

When $\lambda = 2$

$(A - 2I)X = 0$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 4x - 2y + 2z &= 0 \\ -2x + y - z &= 0 \\ 2x - y + z &= 0 \end{aligned}\right\} \Rightarrow 2x - y + z = 0$$

put $x = 0$ $y = 1$ $z = 1$

$$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$X_3 = X_1 \times X_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$\|X_1\| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$ $\quad \|X_3\| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$

$\|X_2\| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$

Modal matrix $P = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix}$

Orthogonal transformation $X = PY$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

~~All $\lambda > 0$~~
+ve definite

$q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = 8y_1^2 + 2y_2^2 + 2y_3^2$