
KTU
NOTES

Module IV

Sequences and series

Sequence :-

An ordered set of real numbers $\{u_1, u_2, \ldots\ldots u_n, \ldots\ldots\}$ is called a sequence denoted by $\{u_n\}_{n=1}^{\infty}$

eg:- $\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots \frac{1}{n}, \ldots\ldots\right\}$

$\{2n-1\}_{n=1}^{\infty} = \{1, 3, 5, \ldots\ldots, (2n-1), \ldots\ldots\}$

$\{n^2\}_{n=1}^{\infty} = \{1, 4, 9, 16, \ldots\ldots\ldots\}$

Series :-

The sum of the elements of a sequence is called a series. That is, if $\{u_1, u_2, \ldots\ldots u_n, \ldots\ldots\}$ is an infinite sequence of real numbers, then $u_1 + u_2 + \ldots\ldots + u_n + \ldots\ldots$ is called an infinite series.

It is denoted by $\sum_{n=1}^{\infty} u_n$ or $\sum_{k=1}^{\infty} u_k$

eg:- $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n} + \ldots\ldots$

$\sum_{k=1}^{\infty} (2k-1) = 1 + 3 + 5 + \ldots\ldots + (2k-1) + \ldots\ldots$

The sum of its first k terms $u_1 + u_2 + \ldots\ldots + u_k$ is called k^{th} partial sum denoted by S_k.

Convergence of a series :- If the sum of all the terms in the series converges to a finite value, the series is said to be convergent. Otherwise divergent.

$S_1 = u_1$

$S_2 = u_1 + u_2$

$S_3 = u_1 + u_2 + u_3$

$\vdots$

$S_k = u_1 + u_2 + u_3 + \ldots\ldots + u_k$

$S_{k+1} = u_1 + u_2 + u_3 + \ldots\ldots + u_k + u_{k+1}$

$\vdots$

If the sequence $\{S_k\}$ converges to a limit S as $k \to \infty$, then the series $\sum_{k=1}^{\infty} u_k$ is said to be convergent and S is called the sum of the series.

1. Examine for convergence the series $1 + 2 + \ldots\ldots + n + \ldots\ldots$

Soln :- $S_k = 1 + 2 + \ldots\ldots + k = \frac{k(k+1)}{2}$ (closed form → no: of terms does not vary with k)

$$\underset{k \to \infty}{Lt}\, S_k = \underset{k \to \infty}{Lt}\, \frac{k(k+1)}{2} = \infty$$

$\therefore$ The series is divergent.

2. Check the convergence of the series $\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \ldots\ldots$

Soln :- $S_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \ldots\ldots + \frac{3}{10^n}$ ——(1)

Multiply both sides by $\frac{1}{10}$ (common ratio)

$$\frac{1}{10} S_n = \frac{3}{10^2} + \frac{3}{10^3} + \ldots\ldots + \frac{3}{10^{n+1}} \quad \text{——(2)}$$

(1) - (2) $\Rightarrow$ $\frac{9}{10} S_n = \frac{3}{10} - \frac{3}{10^{n+1}} = \frac{3}{10}\left(1 - \frac{1}{10^n}\right)$

$\Rightarrow$ $S_n = \frac{1}{3}\left(1 - \frac{1}{10^n}\right)$ (closed form)

$\underset{n \to \infty}{Lt}\, S_n = \underset{n \to \infty}{Lt}\, \frac{1}{3}\left(1 - \frac{1}{10^n}\right) = \underline{\underline{\frac{1}{3}}}$ $\therefore$ The series is convergent.

3. Find the exact values for the first four partial sums. Find a closed form for the k^{th} partial sum and determine whether the series converges by calculating the limit of k^{th} partial sum.

a) $2 + \frac{2}{5} + \frac{2}{5^2} + \dots\dots + \frac{2}{5^{k-1}} + \dots\dots$

Soln: $S_1 = 2$; $S_2 = 2 + \frac{2}{5} = \frac{12}{5}$; $S_3 = 2 + \frac{2}{5} + \frac{2}{5^2} = \frac{62}{25}$

$S_4 = 2 + \frac{2}{5} + \frac{2}{5^2} + \frac{2}{5^3} = \frac{312}{125}$

$$S_k = 2 + \frac{2}{5} + \frac{2}{5^2} + \dots\dots + \frac{2}{5^{k-1}} \quad \text{—(1)}$$

$$\frac{1}{5} S_k = \frac{2}{5} + \frac{2}{5^2} + \dots\dots + \frac{2}{5^k} \quad \text{—(2)}$$

(1) − (2) $\Rightarrow$ $\frac{4}{5} S_k = 2 - \frac{2}{5^k} = 2\left(1 - \frac{1}{5^k}\right)$

$\Rightarrow S_k = \frac{5}{2}\left(1 - \frac{1}{5^k}\right)$

$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{5}{2}\left(1 - \frac{1}{5^k}\right) = \frac{5}{2}$. $\therefore$ The series is convergent.

b) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots\dots$

Soln: $S_1 = \frac{1}{2}$; $S_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$; $S_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$

$S_4 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{4}{5}$

$$S_k = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots\dots + \frac{1}{k(k+1)}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots\dots + \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= 1 - \frac{1}{k+1}$$

$\lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(1 - \frac{1}{k+1}\right) = \underline{\underline{1}}$ $\therefore$ The series is convergent.

H.W

c) $\frac{1}{4} + \frac{2}{4} + \frac{2^2}{4} + \cdots\cdots + \frac{2^{k-1}}{4} + \cdots\cdots$

d) $\frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots\cdots + \frac{1}{(k+1)(k+2)} + \cdots\cdots$

e) $\sum_{k=1}^{\infty} \left(\frac{1}{k+3} - \frac{1}{k+4} \right)$

f) $\ln\left(\frac{1}{2}\right) + \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3}{4}\right) + \cdots\cdots + \ln\left(\frac{k}{k+1}\right) + \cdots\cdots$

4. Show that $\sum_{k=1}^{\infty} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k^2+k}} = 1$

Geometric series test

The geometric series $a + ar + ar^2 + \cdots\cdots$

(i) converges if $|r| < 1$

(ii) diverges if $|r| \geq 1$

If the series is convergent, then the sum of the series is $S = \frac{a}{1-r}$

1. Check the convergence of the series and if so, find its sum.

a) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\cdots$

Soln :- $\sum u_n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\cdots$

The given series is a geometric series with first term $a = 1$ and common ratio $r = \frac{1}{2}$

$|r| = \frac{1}{2} < 1$. Hence the series is convergent.

Sum, $S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = \underline{2}$

b) $\sum_{k=0}^{\infty} \frac{5}{4^k}$

<u>Soln</u> : $\Sigma u_n = 5 + \frac{5}{4} + \frac{5}{4^2} + \frac{5}{4^3} + \ldots\ldots\ldots$

It is a geometric series with $a = 5$ and $r = \frac{1}{4}$

$|r| = \frac{1}{4} < 1$. Hence the series is convergent.

Sum, $S = \frac{a}{1-r} = \frac{5}{1-\frac{1}{4}} = \frac{20}{3}$

c) $\sum_{k=1}^{\infty} 3^{2k} \, 5^{1-k}$

<u>Soln</u> :- $\Sigma u_n = 3^2 + \frac{3^4}{5} + \frac{3^6}{5^2} + \ldots\ldots$

It is a geometric series with $a = 9$ and $r = \frac{9}{5}$

$|r| = \frac{9}{5} > 1$. Hence the series is divergent.

d) $\frac{3}{5} + \frac{4}{5^2} + \frac{3}{5^3} + \frac{4}{5^4} + \ldots\ldots\ldots$

<u>Soln</u> : $\Sigma u_n = \underbrace{\left(\frac{3}{5} + \frac{3}{5^3} + \frac{3}{5^5} + \ldots\ldots\ldots\right)}_{I_1} + \underbrace{\left(\frac{4}{5^2} + \frac{4}{5^4} + \frac{4}{5^6} + \ldots\right)}_{I_2}$

I_1 is a geometric series with $a = \frac{3}{5}$ and $r = \frac{1}{5^2}$

$|r| = \frac{1}{5^2} < 1$. Hence convergent.

I_2 is a geometric series with $a = \frac{4}{5^2}$ and $r = \frac{1}{5^2}$

$|r| = \frac{1}{5^2} < 1$. Hence convergent.

Sum of two convergent series is convergent.

Hence $\Sigma u_n = I_1 + I_2$ is convergent.

Sum, $S = \frac{\frac{3}{5}}{1-\frac{1}{5^2}} + \frac{\frac{4}{5^2}}{1-\frac{1}{5^2}} = \frac{19}{24}$

e) $\sum_{k=0}^{\infty} x^k$

Soln :- $\sum u_n = 1 + x + x^2 + \ldots\ldots$

It is a G.S with $a = 1$ and $r = x$

The given series is convergent if $|x| < 1$

Sum $= \dfrac{1}{1-x}$

2. Find the rational number represented by the repeating decimal $0.784784784\ldots\ldots$

Soln :- $0.\underline{784}\,\underline{784}\,\underline{784}\ldots\ldots$

$= 0.784 + 0.000784 + 0.000000784 + \ldots\ldots$

It is a G.S with $a = 0.784$ and $r = 0.001$

$|r| = 0.001 < 1$. Hence convergent.

Sum $= \dfrac{a}{1-r} = \dfrac{0.784}{1-0.001} = \dfrac{0.784}{0.999} = \dfrac{784}{999}$

Problems

Check the convergence. If so, find its sum.

1) $\sum_{k=1}^{\infty} \left(\frac{-3}{4}\right)^{k-1}$ 2) $\sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^{k}$ 3) $\sum_{k=1}^{\infty} 3^{k-1}$

4) $3 - \frac{3x}{2} + \frac{3x^2}{4} - \frac{3x^3}{8} + \ldots\ldots$

5) Use geometric series to show that

a) $\sum_{k=0}^{\infty} (-1)^k x^k = \dfrac{1}{1+x}$ if $-1 < x < 1$

b) $\sum_{k=0}^{\infty} (x-4)^k = \dfrac{1}{5-x}$ if $3 < x < 5$

6) $\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^4}+\ldots\ldots\ldots$

7) Express the repeating decimal as a fraction

a) $0.9999\ldots\ldots\ldots\ldots$

b) $5.646464\ldots\ldots\ldots$

c) $0.45114141414\ldots\ldots$

8) $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{7}{9^{k-1}}$

Hyper harmonic or p-series test

The series $\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots\ldots\ldots$

i) converges if $p>1$

ii) diverges if $p\le 1$

1. Check the convergence

a) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\ldots\ldots$

Soln :- $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\ldots\ldots=\sum_{n=1}^{\infty}\frac{1}{n}$

Here $p=1$. Hence the series is divergent.

b). $1+\frac{1}{4^{2/3}}+\frac{1}{9^{2/3}}+\frac{1}{16^{2/3}}+\ldots\ldots\ldots$

Soln :- $1+\frac{1}{4^{2/3}}+\frac{1}{9^{2/3}}+\frac{1}{16^{2/3}}+\ldots\ldots\ldots$

$=1+\frac{1}{2^{4/3}}+\frac{1}{3^{4/3}}+\frac{1}{4^{4/3}}+\ldots\ldots\ldots=\sum_{n=1}^{\infty}\frac{1}{n^{4/3}}$

Here $p=\frac{4}{3}>1$. Hence the series is ~~diverge~~ convergent.

Problems

1. Check the convergence

a) $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots\dots\dots$

b) $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\dots\dots\dots$

Comparison Test

First form

a) If two positive term series Σu_n and Σv_n be such that (i) Σv_n converges (ii) $u_n \le v_n$ for all n then Σu_n also converges.

b) If two positive term series Σu_n and Σv_n be such that (i) Σv_n diverges (ii) $u_n \ge v_n$ for all n then Σu_n also diverges.

1. Check the convergence of $\sum_{k=1}^{\infty} \frac{1}{2k^2+k}$

Soln :- $\Sigma u_k = \Sigma \frac{1}{2k^2+k}$

Choose $\Sigma v_k = \Sigma \frac{1}{k^2}$

$\frac{1}{2k^2+k} < \frac{1}{k^2}$ and $\Sigma v_k = \Sigma \frac{1}{k^2}$ is convergent since it is a p-series with $p=2>1$

$\therefore$ By comparison test, $\Sigma u_k = \Sigma \frac{1}{2k^2+k}$ is convergent.

Note :- For a successful application of this test, we first take an estimate of the magnitude of the general term u_k and then select the series Σv_k.

eg:- If $u_k = \frac{2k}{k^2+1}$, choose $v_k = \frac{k}{k^2} = \frac{1}{k}$

If $u_n = \sqrt{\frac{n^2+1}{n^3+2}}$, choose $v_n = \sqrt{\frac{n^2}{n^3}} = \sqrt{\frac{1}{n}} = \frac{1}{\sqrt{n}}$

2. Check the convergence of $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k} - \frac{1}{2}}$

Soln :- $\Sigma u_k = \Sigma \frac{1}{\sqrt{k} - \frac{1}{2}}$

Choose $\Sigma v_k = \Sigma \frac{1}{\sqrt{k}}$

$\frac{1}{\sqrt{k} - \frac{1}{2}} > \frac{1}{\sqrt{k}}$ and $\Sigma v_k = \Sigma \frac{1}{\sqrt{k}} = \Sigma \frac{1}{k^{1/2}}$ is divergent since it is a p-series with $p = \frac{1}{2} < 1$

$\therefore$ By comparison test, $\Sigma u_k = \Sigma \frac{1}{\sqrt{k} - \frac{1}{2}}$ is divergent.

Second form

If two positive term series Σu_n and Σv_n be such that $\lim_{n\to\infty} \frac{u_n}{v_n} = l$ where l is a non-zero finite number, then Σu_n and Σv_n converge or diverge together.

1. Check the convergence of $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$

Soln :- $\Sigma u_n = \Sigma \frac{n}{n^3+1}$. Choose $\Sigma v_n = \Sigma \frac{1}{n^2}$

$$\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt}\left[\frac{\left(\frac{n}{n^3+1}\right)}{\left(\frac{1}{n^2}\right)}\right] = \underset{n\to\infty}{Lt}\left(\frac{n^3}{n^3+1}\right) = \underset{n\to\infty}{Lt}\frac{\cancel{n^3}}{\cancel{n^3}\left(1+\frac{1}{n^3}\right)}$$

$= 1$, non-zero finite number

By comparison test, both series converge or diverge together.

$\Sigma v_n = \Sigma \frac{1}{n^2}$ is convergent [$\because$ p-series with $p = 2 > 1$]

Hence $\Sigma u_n = \Sigma \frac{n}{n^3+1}$ is also convergent.

2) Check the convergence of $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$

Soln :- $\Sigma u_n = \Sigma \frac{1}{\sqrt{n}+\sqrt{n+1}}$

Choose $\Sigma v_n = \Sigma \frac{1}{\sqrt{n}}$

$$\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt}\left[\frac{\left(\frac{1}{\sqrt{n}+\sqrt{n+1}}\right)}{\left(\frac{1}{\sqrt{n}}\right)}\right] = \underset{n\to\infty}{Lt}\left(\frac{\sqrt{n}}{\sqrt{n}+\sqrt{n+1}}\right) = \underset{n\to\infty}{Lt}\frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\left(1+\frac{\sqrt{n+1}}{\sqrt{n}}\right)}$$

$$= \underset{n\to\infty}{Lt}\frac{1}{1+\sqrt{1+\frac{1}{n}}} = \frac{1}{2},$$

non-zero finite number

$\Sigma v_n = \Sigma \frac{1}{\sqrt{n}}$ diverges since it is a p-series with $p = \frac{1}{2} < 1$

Hence by comparison test, $\Sigma u_n = \Sigma \frac{1}{\sqrt{n}+\sqrt{n+1}}$ converges.

3) Check the convergence of $\frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \dots\dots$

Soln :- $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{2n-1}{n(n+1)(n+2)}$

Choose $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt}\left[\frac{\frac{(2n-1)}{n(n+1)(n+2)}}{\left(\frac{1}{n^2}\right)}\right] = \underset{n\to\infty}{Lt}\left[\frac{(2n-1)n^2}{n(n+1)(n+2)}\right] = \underset{n\to\infty}{Lt}\frac{\left(2-\frac{1}{n}\right)\cancel{n^3}}{\cancel{n^3}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$= 2$, non-zero finite no:

$\sum v_n = \sum \frac{1}{n^2}$ converges since it is a p-series with $p = 2 > 1$

Hence by comparison test, $\sum u_n = \frac{2n-1}{n(n+1)(n+2)}$ converges.

4) Check the convergence of $\frac{1}{1\cdot 2} + \frac{2}{3\cdot 4} + \frac{3}{5\cdot 6} + \cdots\cdots$

__Soln__ :- $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{n}{(2n-1)\,2n}$

Choose $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n}$

$$\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt}\left[\frac{\frac{n}{(2n-1)(2n)}}{\frac{1}{n}}\right] = \underset{n\to\infty}{Lt} \frac{\cancel{n^2}}{2\cancel{n^2}\left(2-\frac{1}{n}\right)} = \frac{1}{4}, \text{ non-zero finite number}$$

$\sum v_n = \sum \frac{1}{n}$ diverges since it is a p-series with $p = 1$

Hence by comparison test, $\sum u_n = \sum \frac{n}{(2n-1)\,2n}$ diverges.

5) Check the convergence of $\sum_{k=1}^{\infty} \frac{3k^3 - 2k^2 + 4}{k^7 - k^3 + 2}$

__Soln__ :- Choose $\sum v_k = \sum \frac{1}{k^4}$

$$\underset{k\to\infty}{Lt}\left(\frac{u_k}{v_k}\right) = \underset{k\to\infty}{Lt}\left(\frac{\left(\frac{3k^3-2k^2+4}{k^7-k^3+2}\right)}{\left(\frac{1}{k^4}\right)}\right) = \underset{k\to\infty}{Lt}\left[\frac{(3k^3-2k^2+4)\,k^4}{(k^7-k^3+2)}\right]$$

$$= \underset{k\to\infty}{Lt} \frac{\left[3 - \frac{2}{k} + \frac{4}{k^3}\right]\cancel{k^7}}{\left[1 - \frac{1}{k^4} + \frac{2}{k^7}\right]\cancel{k^7}}$$

$= 3$, non-zero finite number

$\sum v_k = \sum \frac{1}{k^4}$ converges since it is a p-series with $p = 4 > 1$.

Hence by comparison test $\sum u_k = \sum \frac{3k^3 - 2k^2 + 4}{k^7 - k^3 + 2}$ converges

6) Check the convergence of $\sum_{n=1}^{\infty} \sqrt{n^2+1} - n$

Soln:- $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \sqrt{n^2+1} - n = \sum_{n=1}^{\infty} \frac{(\sqrt{n^2+1} - n)(\sqrt{n^2+1} + n)}{(\sqrt{n^2+1} + n)}$

$= \sum_{n=1}^{\infty} \frac{n^2+1-n^2}{(\sqrt{n^2+1}+n)} = \sum \frac{1}{\sqrt{n^2+1}+n}$

Choose $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n}$

$\underset{n\to\infty}{Lt} \left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt} \left[\frac{\left(\frac{1}{\sqrt{n^2+1}+n}\right)}{\left(\frac{1}{n}\right)}\right] = \underset{n\to\infty}{Lt} \left[\frac{n}{\sqrt{n^2+1}+n}\right] = \underset{n\to\infty}{Lt} \frac{\cancel{n}}{\cancel{n}\left[\sqrt{1+\frac{1}{n^2}}+1\right]}$

$= \frac{1}{2}$, non-zero finite number

$\sum v_n = \sum \frac{1}{n}$ diverges since it is a p-series with $p=1$

Hence by comparison test, $\sum u_n = \sum \sqrt{n^2+1} - n$ diverges.

7) Check the convergence of $\sum_{k=1}^{\infty} \frac{1}{3^k+11}$

Soln:- $\sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} \frac{1}{3^k+11}$

Choose $\sum_{k=1}^{\infty} v_k = \sum_{k=1}^{\infty} \frac{1}{3^k}$

$\underset{k\to\infty}{Lt} \left(\frac{u_k}{v_k}\right) = \underset{k\to\infty}{Lt} \left[\frac{\left(\frac{1}{3^k+11}\right)}{\left(\frac{1}{3^k}\right)}\right] = \underset{k\to\infty}{Lt} \frac{\cancel{3^k}}{\cancel{3^k}\left(1+\frac{11}{3^k}\right)} = 1$, non-zero finite number

$\sum v_k = \sum \frac{1}{3^k}$ converges since it is a geometric series with $|r| = \frac{1}{3} < 1$

Hence by comparison test, $\sum u_k = \sum \frac{1}{3^k+11}$ converges.

Problems

Check the convergence of

i) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+1}$ (ii) $\sum_{n=1}^{\infty} \frac{2n^3+5}{4n^5+1}$

(iii) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ iv) $\frac{1}{1\cdot 3} + \frac{2}{3\cdot 5} + \frac{3}{5\cdot 7} + \cdots\cdots$

v) $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \cdots\cdots$

vi) $\sum_{n=1}^{\infty} \sqrt{n^4+1} - \sqrt{n^4-1}$ vii) $\sum_{k=1}^{\infty} \frac{1}{9k^2+3k-2}$

viii) $\sum_{k=1}^{\infty} \frac{2}{k^4+k^2}$ ix) $\sum_{k=1}^{\infty} \frac{5}{4^k+1}$

x) $\sum_{k=1}^{\infty} \frac{k(k+3)}{(k+1)(k+2)(k+5)}$

D'Alembert's Ratio test

In a positive term series Σu_n, if $\underset{n\to\infty}{Lt} \left(\frac{u_{n+1}}{u_n}\right) = \lambda$, then the series i) converges if $\lambda < 1$

ii) diverges if $\lambda > 1$

iii) test fails when $\lambda = 1$

1) Discuss the convergence of $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$

Soln :- Here $u_n = \frac{n^2}{3^n}$; $u_{n+1} = \frac{(n+1)^2}{3^{n+1}}$

$$\underset{n\to\infty}{Lt}\left(\frac{u_{n+1}}{u_n}\right) = \underset{n\to\infty}{Lt}\left[\frac{(n+1)^2}{3^{n+1}} \times \frac{3^n}{n^2}\right] = \underset{n\to\infty}{Lt}\left[\frac{(n+1)^2}{3n^2}\right] = \underset{n\to\infty}{Lt}\left[\frac{\cancel{n^2}\left(1+\frac{1}{n}\right)^2}{3\cancel{n^2}}\right]$$

$$= \frac{1}{3} < 1$$

$\therefore$ By D'Alembert's ratio test, $\Sigma u_n = \Sigma \frac{n^2}{3^n}$ converges.

2) Check the convergence of $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \ldots\ldots$

Soln :- Here $u_n = \frac{n^2}{n!}$; $u_{n+1} = \frac{(n+1)^2}{(n+1)!}$

$$\underset{n\to\infty}{Lt}\left(\frac{u_{n+1}}{u_n}\right) = \underset{n\to\infty}{Lt}\left[\frac{(n+1)^2}{(n+1)!} \times \frac{n!}{n^2}\right] = \underset{n\to\infty}{Lt}\left[\frac{(n+1)^2}{(n+1)\,n^2}\right]$$

$$= \underset{n\to\infty}{Lt}\left[\frac{n+1}{n^2}\right] = \underset{n\to\infty}{Lt}\left(\frac{1}{n} + \frac{1}{n^2}\right)$$

$$= 0 < 1$$

$\therefore$ By D'Alembert's ratio test, $\Sigma u_n = \Sigma \frac{n^2}{n!}$ converges.

3) Check the convergence of $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Soln :- Here $u_n = \frac{n^n}{n!}$; $u_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$

$$\underset{n\to\infty}{Lt}\left(\frac{u_{n+1}}{u_n}\right) = \underset{n\to\infty}{Lt}\left[\frac{(n+1)^{n+1}}{(n+1)!} \; \frac{n!}{n^n}\right] = \underset{n\to\infty}{Lt}\left[\frac{(n+1)^{n+1}}{(n+1)\,n^n}\right] = \underset{n\to\infty}{Lt}\left[\frac{(n+1)^n}{n^n}\right]$$

$$= \underset{n\to\infty}{Lt}\left(\frac{\cancel{n^n}\,(1+\frac{1}{n})^n}{\cancel{n^n}}\right)$$

$$= e > 1 \qquad \left[\because \underset{n\to\infty}{Lt}\left(1 + \frac{a}{n}\right)^n = e^a\right]$$

$\therefore$ By D'Alembert's ratio test, Σu_n converges.

4) Check the convergence of $\sum_{n=1}^{\infty} \frac{n^3+a}{2^n+a}$

Soln :- Here $u_n = \frac{n^3+a}{2^n+a}$; $u_{n+1} = \frac{(n+1)^3+a}{2^{n+1}+a}$

$$\underset{n\to\infty}{Lt}\left(\frac{u_{n+1}}{u_n}\right) = \underset{n\to\infty}{Lt}\left[\frac{(n+1)^3+a}{2^{n+1}+a} \times \frac{2^n+a}{n^3+a}\right] = \underset{n\to\infty}{Lt}\; \frac{n^3\left[(1+\frac{1}{n})^3 + \frac{a}{n^3}\right] 2^n\left[1+\frac{a}{2^n}\right]}{2^{n+1}\left[1 + \frac{a}{2^{n+1}}\right] n^3\left[1+\frac{a}{n^3}\right]}$$

$= \frac{1}{2} < 1$

$\therefore$ By D'Alembert's ratio test, $\sum u_n = \sum \frac{n^3+a}{2^n+a}$ converges.

5) Check the convergence of $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \cdots\cdots + \frac{x^n}{n^2+1} + \cdots\cdots$

provided $x > 0$

Soln:- Since $x>0$, the given series is a positive term series.

Here $u_n = \frac{x^n}{n^2+1}$; $u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$

$$\underset{n\to\infty}{Lt}\left(\frac{u_{n+1}}{u_n}\right) = \underset{n\to\infty}{Lt}\left[\frac{x^{n+1}}{(n+1)^2+1} \times \frac{n^2+1}{x^n}\right] = \underset{n\to\infty}{Lt}\left[\frac{x \cdot \cancel{n^2}\left(1+\frac{1}{n^2}\right)}{\cancel{n^2}\left[\left(1+\frac{1}{n}\right)^2 + \frac{1}{n^2}\right]}\right]$$

$= x$

$\therefore$ By D'Alembert's ratio test, $\sum u_n$ converges if $x < 1$

diverges if $x > 1$

test fails when $x = 1$

When $x = 1$, $\sum u_n = 1 + \frac{1}{2} + \frac{1}{5} + \cdots\cdots + \frac{1}{n^2+1} + \cdots\cdots$

$\sum u_n = \sum \frac{1}{n^2+1}$ Choose $\sum v_n = \sum \frac{1}{n^2}$

$$\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt}\left[\frac{\left(\frac{1}{n^2+1}\right)}{\left(\frac{1}{n^2}\right)}\right] = \underset{n\to\infty}{Lt}\left(\frac{n^2}{n^2+1}\right) = \underset{n\to\infty}{Lt}\frac{\cancel{n^2}}{\cancel{n^2}\left(1+\frac{1}{n^2}\right)}$$

$= 1$, non-zero finite number

$\sum v_n = \sum \frac{1}{n^2}$ is convergent since it is a p-series with $p = 2 > 1$

Hence by comparison test, $\sum u_n = \sum \frac{1}{n^2+1}$ converges

Therefore $\sum u_n$ converges if $x \le 1$

Problems

Check the convergence of

i) $1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \cdots\cdots$

ii) $\sum_{n=1}^{\infty} \frac{n!\, 2^n}{n^n}$

iii) $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \cdots\cdots$

iv) $1 + x + x^2 + x^3 + \cdots\cdots$ provided $x > 0$

v) $\sum_{k=1}^{\infty} \frac{99^k}{k!}$ vi) $\sum_{k=1}^{\infty} \frac{4^k}{k^2}$

Cauchy's root test

In a positive term series $\sum u_n$, if $\underset{n\to\infty}{Lt} (u_n)^{1/n} = \lambda$, then the series $\sum u_n$ i) converges if $\lambda < 1$

ii) diverges if $\lambda > 1$

iii) test fails when $\lambda = 1$

1. Check the convergence of $\sum_{k=2}^{\infty} \left(\frac{4k-5}{2k+1}\right)^k$

Soln :- Here $u_k = \left(\frac{4k-5}{2k+1}\right)^k$

$$\underset{k\to\infty}{Lt} (u_k)^{1/k} = \underset{k\to\infty}{Lt} \left[\left(\frac{4k-5}{2k+1}\right)^k\right]^{1/k} = \underset{k\to\infty}{Lt} \left(\frac{4k-5}{2k+1}\right) = \underset{k\to\infty}{Lt} \frac{\cancel{k}(4-\frac{5}{k})}{\cancel{k}(2+\frac{1}{k})}$$

$$= 2 > 1$$

Hence by Cauchy's root test, $\sum u_k = \sum \left(\frac{4k-5}{2k+1}\right)^k$ diverges

2) Check the convergence of $\sum_{k=1}^{\infty} \frac{1}{[\ln(k+1)]^k}$

Soln :- Here $u_k = \frac{1}{[\ln(k+1)]^k}$

$$\lim_{k\to\infty} u_k^{1/k} = \lim_{k\to\infty} \left[\frac{1}{[\ln(k+1)]^k}\right]^{1/k} = \lim_{k\to\infty} \frac{1}{\ln(k+1)} = 0 < 1$$

Hence by Cauchy's root test, $\sum u_k$ converges

3) Discuss the convergence of $\sum_{n=1}^{\infty} \frac{n^{n^2}}{(n+1)^{n^2}}$

Soln :- Here $u_n = \frac{n^{n^2}}{(n+1)^{n^2}}$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left[\frac{n^{n^2}}{(n+1)^{n^2}}\right]^{1/n} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \frac{\cancel{n^n}}{\cancel{n^n}(1+\frac{1}{n})^n}$$

$$= \frac{1}{e} < 1$$

Hence by Cauchy's root test, $\sum u_n$ converges

4) Check the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^n}$

Soln :- Here $u_n = \frac{1}{n^n}$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1$$

Hence by Cauchy's root test, $\sum u_n$ converges.

5) Check the convergence of $\sum_{n=1}^{\infty} \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

Soln :- Let $u_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \left[\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}\right]^{1/n} = \lim_{n\to\infty} \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{1/2}} = \lim_{n\to\infty} \left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}}$$

$$= \underset{n\to\infty}{Lt} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \frac{1}{e} < 1$$

Hence by Cauchy's root test, Σu_n converges

6) Check the convergence of $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \dots$

Soln:- Here $u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-n}$

$$\underset{n\to\infty}{Lt} (u_n)^{1/n} = \underset{n\to\infty}{Lt} \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-1}$$

$$= \underset{n\to\infty}{Lt} \left[\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$= (e-1)^{-1} = \frac{1}{e-1} < 1$$

Hence by Cauchy's root test, Σu_n converges

Problems

Check the convergence of

i) $\sum_{n=1}^{\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$

ii) $\sum_{k=1}^{\infty} \left(\frac{3k+2}{2k-1}\right)^{k}$

Alternating Series

A series in which the terms are alternately positive or negative is called an alternating series, $u_1 - u_2 + u_3 - u_4 + \ldots\ldots\ldots$

Leibnitz rule :- An alternating series $u_1 - u_2 + u_3 - u_4 + \ldots\ldots$ converges if
i) $u_{n+1} < u_n$ for every n
ii) $\lim_{n\to\infty} u_n = 0$

1. Check the convergence of $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots\ldots$

Solution :- Here $u_n = \frac{1}{n}$; $u_{n+1} = \frac{1}{n+1}$

$n+1 > n \Rightarrow \frac{1}{n+1} < \frac{1}{n} \Rightarrow u_{n+1} < u_n$ for every n

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

Hence by Leibnitz test, $\Sigma u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots$ converges.

2. Check the convergence of $\frac{5}{2} - \frac{7}{4} + \frac{9}{6} - \frac{11}{8} + \ldots\ldots$

Soln :- Here $u_n = \frac{2n+3}{2n}$; $u_{n+1} = \frac{2n+5}{2n+2}$

$$u_{n+1} - u_n = \frac{2n+5}{2n+2} - \frac{2n+3}{2n}$$

$$= \frac{\cancel{4n^2} + \cancel{10n} - \cancel{4n^2} - \cancel{10n} - 6}{(2n+2)2n}$$

$$< 0$$

or

$u_1 = \frac{5}{2} = 2.5$

$u_2 = \frac{7}{4} = 1.75$

$u_3 = \frac{9}{6} = 1.5$

$\vdots$

$u_{n+1} < u_n$

$\Rightarrow u_{n+1} < u_n$ for every n.

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{2n+3}{2n} = \lim_{n\to\infty} \frac{\cancel{n}\left(2 + \frac{3}{n}\right)}{2\cancel{n}} = 1 \neq 0$$

Hence by Leibnitz test, Σu_n is not convergent.

3. Check the convergence of $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+1}{4k+11}$

Soln :- $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(k+1)}{4k+11} = \frac{2}{15} - \frac{3}{19} + \frac{4}{23} - \dots\dots$

Here $u_k = \frac{k+1}{4k+11}$ $\quad u_{k+1} = \frac{k+2}{4k+15}$

$$u_{k+1} - u_k = \frac{k+2}{4k+15} - \frac{k+1}{4k+11}$$

$$= \frac{(4k+11)(k+2) - (k+1)(4k+15)}{(4k+15)(4k+11)}$$

$$= \frac{(\cancel{4k^2} + \cancel{19k} + 22) - (\cancel{4k^2} + \cancel{19k} + 15)}{(4k+15)(4k+11)}$$

$$= \frac{7}{(4k+15)(4k+11)} > 0$$

$u_{k+1} > u_k$

Hence by Leibnitz test $\sum u_k$ diverges.

4) Check the convergence of $\sum_{k=1}^{\infty} (-1)^{k+1} e^{-k}$

Soln :- Here $u_k = e^{-k} = \frac{1}{e^k}$; $u_{k+1} = e^{-(k+1)} = \frac{1}{e^{k+1}}$

$\frac{1}{e^{k+1}} < \frac{1}{e^k} \Rightarrow u_{k+1} < u_k$ $\quad$ [$\because e^x$ is an increasing fn]

$\underset{k\to\infty}{Lt}\, u_k = \underset{k\to\infty}{Lt}\, \frac{1}{e^k} = 0$. Hence by Leibnitz test, $\sum u_k$ converges.

5) Check the convergence of $\sum_{k=3}^{\infty} (-1)^k \frac{\ln k}{k}$

Soln :- Here $u_k = \frac{\ln k}{k}$; $u_{k+1} = \frac{\ln(k+1)}{(k+1)}$

$\frac{\ln(k+1)}{(k+1)} < \frac{\ln k}{k}$ $\quad$ $\left[\frac{\ln 3}{3} = 0.3662 ; \frac{\ln 4}{4} = 0.3466 \right.$
$\left. \frac{\ln 5}{5} = 0.3219 \right]$

$\Rightarrow u_{k+1} < u_k$

$$\underset{k\to\infty}{Lt}\ u_k = \underset{k\to\infty}{Lt}\ \frac{\ln k}{k} \quad \left(\frac{\infty}{\infty}\right) \text{ Indeterminate form}$$

$$= \underset{k\to\infty}{Lt}\ \frac{\left(\frac{1}{k}\right)}{1} \qquad \text{[Applying L'Hospital rule]}$$

$$= 0$$

Hence by Leibnitz test, Σu_k converges.

Problems :-

Check the convergence of

i) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots\dots\dots$

ii) $1 - \frac{2}{3} + \frac{3}{5} - \frac{4}{7} + \dots\dots\dots$

iii) $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \dots\dots$

iv) $\frac{2}{1^3} - \frac{3}{2^3} + \frac{4}{3^3} - \frac{5}{4^3} + \dots\dots$

v) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+1}$

vi) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{5k}{3^k}$

Absolute Convergence

A series of arbitrary terms $u_1 + u_2 + u_3 + \cdots$ is said to be absolutely convergent if $|u_1| + |u_2| + |u_3| + \cdots$ is convergent

Every absolutely convergent series will be convergent

[Reason:- $u_n \le |u_n|$ and $\Sigma |u_n|$ is cgt.
Hence by comparison test, Σu_n is cgt.]

But the converse is not true.

If $\Sigma |u_n|$ is divergent, but Σu_n is convergent, then Σu_n is said to be conditionally convergent.

1. Check the convergence of $1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots$

Soln:- $\Sigma |u_n| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ is convergent since it is a p-series with $p = 2 > 1$

$\therefore$ Σu_n is absolutely convergent and hence it is convergent.

2. Check the convergence of $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \cdots$

Soln:- $\Sigma |u_n| = 1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \cdots$ is convergent since it is a p-series with $p = \frac{3}{2} > 1$

$\therefore$ Σu_n is absolutely convergent and hence it is convergent.

3) Check the convergence of $\sum \frac{\cos k}{k^2}$

Soln: $\sum |u_k| = \sum \left|\frac{\cos k}{k^2}\right|$

$\left|\frac{\cos k}{k^2}\right| \le \frac{1}{k^2}$ and $\sum \frac{1}{k^2}$ is cgt [p-series with $p=2>1$]

Hence by comparison test, $\sum \left|\frac{\cos k}{k^2}\right|$ is convergent.

$\therefore$ $\sum u_k$ is absolutely convergent and hence convergent.

4) Check the convergence of $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \cdots\cdots$

Soln: $\sum |u_n| = \sum \left|\frac{\sin nx}{n^3}\right|$

$\left|\frac{\sin nx}{n^3}\right| \le \frac{1}{n^3}$ and $\sum \frac{1}{n^3}$ is cgt [p-series with $p=3>1$]

Hence by comparison test, $\sum \left|\frac{\sin nx}{n^3}\right|$ is convergent.

$\therefore$ $\sum u_n$ is absolutely convergent and hence convergent.

Ratio test and root test for absolute convergence

Ratio test: Let $\sum u_k$ be an arbitrary series

and $\lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lambda$.

a) if $\lambda < 1$, then $\sum u_k$ converges absolutely and hence cgs
b) if $\lambda > 1$, then $\sum u_k$ diverges
c) if $\lambda = 1$, test fails.

Root test: Let $\sum u_k$ be an arbitrary series

and $\lim_{k\to\infty} \left[|u_k|\right]^{1/k} = \lambda$

a) if $\lambda < 1$, then $\sum u_k$ converges absolutely and hence cgs
b) if $\lambda > 1$, then $\sum u_k$ diverges
c) if $\lambda = 1$, test fails.

1. Check the convergence of $\sum_{k=1}^{\infty} (-1)^k \frac{2^k}{k!}$

Soln :- $|u_k| = \frac{2^k}{k!}$; $|u_{k+1}| = \frac{2^{k+1}}{(k+1)!}$

$$\underset{k\to\infty}{Lt} \frac{|u_{k+1}|}{|u_k|} = \underset{k\to\infty}{Lt} \frac{2^{k+1}}{(k+1)!} \frac{k!}{2^k} = \underset{k\to\infty}{Lt} \frac{2}{k+1} = 0 < 1$$

Hence $\sum u_k$ converges absolutely and hence converges

2. Check the convergence of $\sum_{k=1}^{\infty} \left(\frac{-4}{5}\right)^k$

Soln :- $|u_k| = \left(\frac{4}{5}\right)^k$

$$\underset{k\to\infty}{Lt} \left[|u_k|\right]^{1/k} = \underset{k\to\infty}{Lt} \left[\left(\frac{4}{5}\right)^k\right]^{1/k} = \frac{4}{5} < 1$$

Hence $\sum u_k$ converges absolutely and hence converges.

3. Check the convergence of

$$\frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \ldots\ldots + \frac{2^n-2}{2^n+1}x^{n-1} + \ldots\ldots$$

Soln :- $|u_n| = \left|\frac{2^n-2}{2^n+1}x^{n-1}\right|$; $|u_{n+1}| = \left|\frac{2^{n+1}-2}{2^{n+1}+1}x^n\right|$

$$\underset{n\to\infty}{Lt} \frac{|u_{n+1}|}{|u_n|} = \underset{n\to\infty}{Lt} \left| \frac{2^{n+1}-2}{2^{n+1}+1} x^n \frac{2^n+1}{(2^n-2)x^{n-1}} \right|$$

$$= \underset{n\to\infty}{Lt} \left| \frac{\cancel{2^{n+1}}\left(1-\frac{2}{2^{n+1}}\right)}{\cancel{2^{n+1}}\left(1+\frac{1}{2^{n+1}}\right)} . x \frac{\cancel{2^n}\left(1+\frac{1}{2^n}\right)}{\cancel{2^n}\left(1-\frac{2}{2^n}\right)} \right|$$

$$= |x|$$

Hence by ratio test, $\sum u_n$ converges if $|x| < 1$
diverges if $|x| > 1$
test fails if $|x| = 1$
i.e $x = \pm 1$

When $x=1$, $\Sigma u_n = 1 + \frac{2}{5} + \frac{6}{9} + \ldots\ldots + \frac{2^n-2}{2^n+1} + \ldots\ldots$

Here $u_n = \frac{2^n-2}{2^n+1}$; Choose $v_n = 1$

$\therefore \Sigma v_n = 1+1+1+\ldots\ldots$ diverges

$$\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt}\left(\frac{2^n-2}{2^n+1}\right) = \underset{n\to\infty}{Lt}\frac{\cancel{2^n}(1-\frac{2}{2^n})}{\cancel{2^n}(1+\frac{1}{2^n})} = 1,$$ non-zero finite number.

Hence Σu_n also diverges. (by comparison test)

When $x=-1$, $\Sigma u_n = \frac{-2}{5} + \frac{6}{9} - \frac{14}{17} + \ldots\ldots$

$u_1 = 0.4$ $u_2 = 0.667$ $u_{n+1} > u_n$

Hence by Leibnitz test, Σu_n diverges.

4) Check the convergence of $\left(\frac{2}{3}\right)x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \ldots\ldots$

<u>Soln</u>: $|u_n| = \left|\left(\frac{n+1}{n+2}\right)^n x^n\right|$

$$\underset{n\to\infty}{Lt}\left[|u_n|\right]^{1/n} = \underset{n\to\infty}{Lt}\left[\left|\left(\frac{n+1}{n+2}\right)^n x^n\right|\right]^{1/n} = \underset{n\to\infty}{Lt}\left|\frac{n+1}{n+2}x\right|$$

$$= \underset{n\to\infty}{Lt}\left|\frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n}(1+\frac{2}{n})}x\right| = |x|$$

By root test, Σu_n converges if $|x|<1$

diverges if $|x|>1$

test fails if $|x|=1$ i.e. $x=\pm 1$

When $x=1$, $\Sigma u_n = \frac{2}{3} + \left(\frac{3}{4}\right)^2 + \left(\frac{4}{5}\right)^3 + \ldots\ldots$

$u_n = \left(\frac{n+1}{n+2}\right)^n$ Choose $v_n = 1$

$$\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt}\left(\frac{n+1}{n+2}\right)^n = \underset{n\to\infty}{Lt}\frac{\cancel{n}\left(1+\frac{1}{n}\right)^n}{\cancel{n}\left(1+\frac{2}{n}\right)^n} = \frac{e}{e^2} = \frac{1}{e},$$ non-zero finite no.

$\Sigma v_n = 1+1+\ldots\ldots$ diverges. Hence by comparison test, Σu_n diverges.

When $x = -1$, $\Sigma u_n = \frac{-2}{3} + \left(\frac{3}{4}\right)^2 - \left(\frac{4}{5}\right)^3 + \cdots\cdots$

$u_1 = \frac{2}{3} = 0.666$; $u_2 = \frac{9}{16} = 0.562$; $u_3 = 0.384, \ldots$

$u_{n+1} < u_n$

$$\underset{n\to\infty}{Lt}\ u_n = \underset{n\to\infty}{Lt}\left(\frac{n+1}{n+2}\right)^n = \underset{n\to\infty}{Lt}\ \frac{\cancel{n}(1+\frac{1}{n})^n}{\cancel{n}(1+\frac{2}{n})^n} = \frac{e}{e^2} = \frac{1}{e} \neq 0$$

Hence by Leibnitz test, Σu_n diverges.

5) Classify the given series as absolutely convergent, conditionally convergent or divergent.

i) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{7k}$

Soln :- $|u_k| = \frac{1}{7k}$; $|u_{k+1}| = \frac{1}{7k+7}$

$$\underset{k\to\infty}{Lt}\ \frac{|u_{k+1}|}{|u_k|} = \underset{k\to\infty}{Lt}\left(\frac{7k}{7k+7}\right) = \underset{k\to\infty}{Lt}\ \frac{\cancel{7k}}{\cancel{7k}(1+\frac{7}{7k})} = 1,$$ test fails

Apply comparison test, choose $|v_k| = \frac{1}{k}$

$$\underset{k\to\infty}{Lt}\ \frac{|u_k|}{|v_k|} = \underset{k\to\infty}{Lt}\ \frac{k}{7k} = \frac{1}{7},$$ non-zero finite number.

$\Sigma |v_k| = \Sigma \frac{1}{k}$ is divergent (p-series with $p=1$)

Hence $\Sigma |u_k|$ is divergent

The given series is $\frac{1}{7} - \frac{1}{14} + \frac{1}{21} - \cdots\cdots$

$u_k = \frac{1}{7k}$ $\quad u_{k+1} = \frac{1}{7k+7}$

$u_{k+1} < u_k$; $\underset{k\to\infty}{Lt}\ u_k = \underset{k\to\infty}{Lt}\ \frac{1}{7k} = 0$

Hence by Leibnitz test, Σu_k is convergent

$\therefore$ The given series is conditionally convergent.

Problems

1. Check the convergence of

i) $\sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!}{3^k}$

ii) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3^k}{k^2}$

iii) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n$

2. Classify each series as absolutely convergent, conditionally convergent or divergent

i) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\cdots$

ii) $\sum_{k=1}^{\infty} \frac{(-4)^k}{k^3}$

iii) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{4/3}}$

iv) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} (2k)!}{(3k-2)!}$