MODULE - V

SERIES REPRESENTATION OF FUNCTIONS

Topics

Taylor Series - Binomial Series and Series representation of exponential, trignometric, logarithmic functions. Fourier Series, Euler formulas, Convergence of Fourier Series, half range Sine and cosine Series, Parseval's theorem

Taylor Series

If f has a derivatives of all orders at x_0, then the Series

$$\sum_{n=1}^{\infty} \frac{f^n(x_0)}{n!}(x-x_0)^n = f(x_0) + \frac{f'(x_0)(x-x_0)}{1!} + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^n(x_0)}{n!}(x-x_0)^n + \cdots$$

is called the Taylor Series for f about $x = x_0$.

Maclaurin Series

The Taylor Series for about $x=0$ is

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^n(0)}{n!}x^n + \cdots$$

and it is called the Maclaurin Series for f.

Power Series

If $c_0, c_1, c_2 \ldots$ are constants and x is a variable, then series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

is called a power series in x.

If x_0 is constant, then the series of the form $\sum_{n=0}^{\infty} c_n (x-x_0)^n$ is called a power series in $(x-x_0)$.

The Taylor series for x about $x=x_0$ is a power series in $(x-x_0)$ and Maclaurin series is a power series in x.

Theorem

For a power series $\sum c_n (x-x_0)^n$ exactly one of the following statements is true.

(a) The series converges only for $x = x_0$

(b) The series converges absolutely (and hence converges) for all real values of x

(c) There is a positive real number R, such that the series diverges for x with $|x-x_0| > R$ and converges absolutely for x with $|x-x_0| < R$. At either of the end points $x = x_0 + R$ and $x = x_0 - R$, the series may converge absolutely, converge conditionally or diverge depending on the particular series.

Problems

I. Find the Maclaurin series for

1. e^x

$f(x) = e^x$

Maclaurin series $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

$f(x) = e^x$ $\quad$ $f(0) = e^0 = 1$

$f'(x) = e^x \qquad f'(0) = e^0 = 1$

$f''(x) = e^x \qquad f''(0) = e^0 = 1$

$f'''(x) = e^x \qquad f'''(0) = e^0 = 1$

$\cdots$

$f(x) = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

2. $f(x) = \sin x$

$f(x) = \sin x = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$

$f(x) = \sin x \qquad f(0) = \sin 0 = 0$

$f'(x) = \cos x \qquad f'(0) = \cos 0 = 1$

$f''(x) = -\sin x \qquad f''(0) = -\sin 0 = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -\cos 0 = -1$

$f(x) = \sin x = 0 + \frac{x}{1!} + 0 + \frac{-x^3}{3!} + \cdots$

$= \frac{x}{1!} - \frac{x^3}{3!} + \cdots$

$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$

3. $f(x) = \cos x$

$f(x) = \cos x = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

U.Q.

$f(x) = \cos x \qquad f(0) = \cos 0 = 1$

$f'(x) = -\sin x \qquad f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x \qquad f''(0) = -\cos 0 = -1$

$f'''(x) = \sin x \qquad f'''(0) = \sin 0 = 0$

$f(x) = \cos x = 1 + 0 - \frac{x^2}{2!} + 0 - \cdots$

$= 1 - \frac{x^2}{2!} + \cdots$

$f(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

4 $f(x) = \frac{1}{1-x}$

$f(x) = \frac{1}{1-x}$ $\qquad f(0) = \frac{1}{1-0} = 1$

uQ

$f'(x) = \frac{-1}{(1-x)^2}(-1)$ $\qquad f'(0) = \frac{1}{(1-0)^2} = 1$

$= \frac{1}{(1-x)^2}$

$f''(x) = -2(1-x)^{-3} \times -1$ $\qquad f''(0) = \frac{2}{1-0} = 2$

$= \frac{2}{(1-x)^3}$

$f'''(x) = 2 \times -3(1-x)^{-4} \times -1$ $\qquad f'''(0) = 6$

$= \frac{6}{(1-x)^4}$

$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

$\frac{1}{1-x} = 1 + \frac{x}{1!} + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 + \cdots$

$[1-x]^{-1} = 1 + x + x^2 + x^3 + \cdots$

$3! = 3 \times 2 \times 1$
$2! = 2 \times 1$
$n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$

5. Find the Maclaurins Series for $\ln(1+x)$

$f(x) = \ln(1+x)$ $\quad f(0) = \ln(1) = 0$

$f'(x) = \frac{1}{1+x}$ $\quad f'(0) = \frac{1}{1+0} = 1$

$f''(x) = \frac{-1}{(1+x)^2}$ $\quad f''(0) = -1$

$f'''(x) = -1 \times -2(1+x)^{-3}$ $\quad f'''(0) = 2$

$= \frac{2}{(1+x)^3}$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$\ln(1+x) = 0 + x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

6. Find the Maclaurin Series for the function xe^x.

$f(x) = xe^x$ $\quad f(0) = 0$

$f'(x) = xe^x + e^x \cdot 1$ $\quad f'(0) = 0 + e^0 = 1$

$f''(x) = xe^x + e^x \cdot 1 + e^x$ $\quad f''(0) = 0 + e^0 + e^0 = 2$

$f'''(x) = xe^x + e^x \cdot 1 + 2e^x$ $\quad f'''(0) = 0 + 3e^0 = 3$

$f(x) =$

u.e

$$f(x) = f(0) + \cancel{x}\frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= 0 + \frac{x}{1!} + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \cdots$$

$$= x + x^2 + \frac{x^3}{2} + \cdots$$

7. Find the Taylor Series expansion of $f(x) = \frac{1}{x+2}$ about $x=1$

U.Q.

$$f(x) = \frac{1}{x+2} \qquad f(1) = \frac{1}{1+2} = 1/3$$

$$f'(x) = \frac{-1}{(x+2)^2} \qquad f'(1) = \frac{-1}{3^2} = -1/9$$

$$f''(x) = -1 \times -2\,(x+2)^{-3} \qquad f''(1) = \frac{2}{3^3} = 2/27$$

$$= \frac{2}{(x+2)^3}$$

$$f'''(x) = 2 \times -3\,(x+2)^{-4} \qquad f'''(1) = \frac{-6}{3^4} = \frac{-2}{27}$$

$$= \frac{-6}{(x+2)^4}$$

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots$$

$$= \frac{1}{3} - \frac{1}{9}(x-1) + \frac{1}{27}(x-1)^2 - \frac{1}{27\times 3}(x-1)^3 + \cdots$$

8. Find the Taylor Series of $\frac{1}{x}$ about $x=1$

U.Q.

$f(x) = \frac{1}{x}$ $\quad f(1)=1$

$f'(x) = \frac{-1}{x^2}$ $\quad f'(1) = -1$

$f''(x) = -1 \times \frac{-2}{x^3} = 2/x^3$ $\quad f''(1) = 2$

$f'''(x) = 2 \times -3 \cdot x^{-4}$ $\quad f'''(1) = \frac{-6}{1} = -6$

$= \frac{-6}{x^4}$

$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots$

$= 1 + \frac{-1}{1}(x-1) + \frac{2}{2!}(x-1)^2 + \frac{-6}{3!}(x-1)^3 + \cdots$

$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$

9. Find the Taylor Series expansion of $x \sin x$ about the point $x = \pi/2$.

U.Q.

$f(x) = x \sin x$ $\quad f(\pi/2) = \frac{\pi}{2}\sin\frac{\pi}{2} = 1$

$f'(x) = x\cos x + \sin x \cdot 1$ $\quad f'(\pi/2) = 0 + \sin\pi/2 = 1$

$f''(x) = x^{-}\sin x + \cos x \cdot 1 + \cos x$

$f''(\pi/2) = -\frac{\pi}{2}\sin\frac{\pi}{2} + 2\cos\pi/2 = -\pi/2$

$f'''(x) = -[x\cos x + \sin x \cdot 1] \cancel{+} 2\sin x$

$$f'''(\pi/2) = -\left[\frac{\pi}{2}\cos\pi/2 + \sin\frac{\pi}{2}\right] + 2\sin\pi/2$$

$$= -1 + 2 = 3$$

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots$$

$$= 1 + 1(x-\pi/2) + \frac{-\pi}{2\times 2!}(x-\pi/2)^2 + \frac{3}{3!}(x-\pi/2)^3 + \cdots$$

$$= 1 + (x-\tfrac{\pi}{2}) - \frac{\pi}{4}(x-\pi/2)^2 + \frac{3}{2}(x-\tfrac{\pi}{2})^3 + \cdots$$

10. Expand $f(x) = \sin\pi x$ into a Taylor Series about $x = 1/2$ up to third derivative.

U.G. $f(x) = \sin\pi x$ $\qquad f(1/2) = \sin\pi/2 = 1$

$f'(x) = \cos\pi x \times \pi$ $\qquad f'(1/2) = \pi\cos\pi/2 = 0$

$f''(x) = \pi\, \bar{} \sin\pi x \times \pi$ $\qquad f''(1/2) = -\pi^2\sin\pi/2 = -\pi^2$

$f'''(x) = -\pi^2\cos\pi x \times \pi$ $\qquad f'''(1/2) = -\pi^3\cos\pi/2 = 0$

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots$$

$$= 1 + 0 - \frac{\pi^2}{2!}(x-1/2)^2 + 0 + \cdots$$

$$= 1 - \frac{\pi^2}{2!}(x-1/2)^2 + \cdots$$

11 Find the Taylor Series expansion of $\log \cos x$ about the point $\pi/3$.
U.Q.

$f(x) = \log \cos x$

$f(\pi/3) = \log \cos \pi/3$
$= \log 1/2$
$= \log 1 - \log 2 = \underline{-\log 2}$

$f'(x) = \frac{1}{\cos x} \cdot -\sin x = -\tan x$

$f'(\pi/3) = -\tan \pi/3$
$= \underline{-\sqrt{3}}$

$f''(x) = -\sec^2 x$

$f''(\pi/3) = -\sec^2 \pi/3$
$= \underline{-4}$

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$

$$= -\log 2 - \frac{\sqrt{3}}{1!}(x - \pi/3) - \frac{\cancel{4}\,2}{\cancel{2}}(x - \pi/3)^2 + \cdots$$

12 Find the Taylors Series for $f(x) = \cos x$ up to third degree terms about $x = \pi/2$

$f(x) = \cos x$

$f(\pi/2) = \cos \pi/2 = 0$

$f'(x) = -\sin x$

$f'(\pi/2) = -\sin \pi/2 = -1$

$f''(x) = -\cos x$

$f''(\pi/2) = -\cos \pi/2 = 0$

$f'''(x) = \sin x$

$f'''(\pi/2) = \sin \pi/2 = 1$

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots$$

$$= 0 - \frac{1}{1!}(x-\pi/2) + 0 + \frac{1}{3!}(x-\pi/2)^3 + \cdots$$

$$= -(x-\pi/2) + \frac{1}{3!}(x-\pi/2)^3 + \cdots$$

Binomial Series

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$$

The series is called Binomial series which is valid in $|x|<1$.

Problems

1. Find the Binomial Series for $\frac{1}{1-x}$

$$\cancel{f(x)} = \frac{1}{1-x} = [1-x]^{-1}$$

$$= 1 + (-1)^{-}x + \frac{(-1)(-1-1)(-x)^2}{2!} + \frac{(-1)(-1-1)(-1-2)(-x)^3}{3!} + \cdots$$

$$= 1 + x + x^2 + x^3 + \cdots$$

2. Find the Binomial Series for $\frac{1}{(1+x)^2}$

$$f(x) = \frac{1}{(1+x)^2} = [1+x]^{-2}$$

$$= 1 + (-2)x + \frac{(-2)(-3)}{2!}x^2 + \frac{(-2)(-3)(-4)x^3}{3!} + \cdots$$

$$= 1 - 2x + 3x^2 - 4x^3 + \cdots$$

3. Find the Binomial Series for $\frac{1}{(1+x)^3}$

$$\frac{1}{(1+x)^3} = (1+x)^{-3}$$

$$= 1 + (-3)x + \frac{(-3)(-4)}{2!}x^2 + \frac{(-3)(-4)(-5)}{3!}x^3 + \dots$$

$$= 1 - 3x + 6x^2 - 10x^3 + \dots$$

4. Find the Binomial Series for $\frac{1}{\sqrt{1+x}}$

$$f(x) = \frac{1}{\sqrt{1+x}} = (1+x)^{-1/2}$$

$$= 1 + (-1/2)x + \frac{(-1/2)(-3/2)}{2!}x^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}x^3 + \dots$$

$$= 1 - \frac{1}{2}x + \frac{1.3}{2^2\, 2!}x^2 + \frac{1.3.5}{2^3\, 3!}x^3 + \dots$$

Periodic functions

A function $f(x)$ is said to be periodic if the value of $f(x)$ repeats exactly after a regular interval of time.

i.e $f(x) = f(x+T)$

where T is a positive constant.

The least value of $T>0$ is called the period of $f(x)$.

Example

$f(x) = \sin x = \sin(x+2\pi) = \sin(x+4\pi) = \sin(x+6\pi)$

$\therefore$ the function has periods $2\pi, 4\pi, 6\pi \ldots$

However, 2π is the least value and

$\therefore$ is the period of $f(x)$.

Piecewise continuous function

A function $f(x)$ is said to be piecewise continuous in an interval if

(1) the interval can be divided into a finite number of subintervals in each of which $f(x)$ is continuous and

(2) the limits of $f(x)$ as x approaches the end points of each subinterval are finite.

A piecewise continuous function is one that has almost a finite number of finite discontinuities.

Dirichlet's Conditions

(1) $f(x)$ is periodic, single valued and finite.

(2) $f(x)$ has a finite number of discontinuities

(3) $f(x)$ has a finite number of maxima and minima.

Fourier Series

If $f(x)$ is a periodic function with period $2l$ and satisfies Dirichlet's conditions, then it can be represented by an infinite series called Fourier Series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

where a_0, a_n and b_n are called Fourier coefficients.

Euler's Formulae

The Fourier series for the function $f(x)$ in the interval $c < x < c+2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

where

$$a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l} \int_{c}^{c+2l} f(x) \cos\frac{n\pi x}{l} \, dx$$

$$b_n = \frac{1}{l} \int_{c}^{c+2l} f(x) \sin\frac{n\pi x}{l} \, dx$$

The Values a_0, a_n, b_n are known as Euler's formula.

Fundamental theorem on Convergence of Fourier Series

Fourier Series converges at each point x of the interval $[c, c+2l]$ and the sum

$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

Satisfy

(1) $f(x) = S(x)$ if $c < x < c+2l$ and $f(x)$ is Continuous at x.

(2) $S(x) = \dfrac{f(x+0) + f(x-0)}{2}$, if $c < x < c+2l$ and x is a point of discontinuity of $f(x)$

(3) $S(x) = \dfrac{f(c+0) + f(c+2l-0)}{2}$, discontinuity at end points.

Results

I. $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

where $u = x^n$

Dashes denote differentiation and suffixes denote integration.

II. $\sin\theta = 0 \implies \theta = n\pi$

$\cos\theta = 0 \implies \theta = (2n+1)\pi/2$.

III. $\sin(\frac{\pi}{2} - \theta) = \cos\theta$

$\sin(\pi/2 + \theta) = \cos\theta$

$\cos(\pi/2 - \theta) = \sin\theta$

$\cos(\pi/2 + \theta) = -\sin\theta$

$\sin(\pi - \theta) = \sin\theta$

$\sin(\pi + \theta) = -\sin\theta$

$\cos(\pi - \theta) = -\cos\theta$

$\cos(\pi + \theta) = -\cos\theta$

IV. $\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

V. $\sin 2x = 2\sin x\cos x$

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\cos^2 x = \frac{1+\cos 2x}{2}$

VI. $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

VII $\int \sin x \, dx = -\cos x$

$\int \cos x \, dx = \sin x$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos bx + b \sin bx\right]$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]$$

Prarobha

VIII $\sin n\pi = 0$ $\sin 0 = 0$

$\cos n\pi = (-1)^n$

$= 1$, n is even

$= -1$, n is odd

Problems

1. Find the Fourier Series of periodic function $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the interval $(0, 2\pi)$.

Period $2l = 2\pi - 0 = 2\pi$

$\Rightarrow l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\pi} + b_n \sin\frac{n\pi x}{\pi}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi - x)^2\,dx$$

$$= \frac{1}{4\pi}\left[\frac{(\pi - x)^3}{3\times -1}\right]_0^{2\pi} = \frac{-1}{12\pi}\left[(\pi - 2\pi)^3 - \pi^3\right]$$

$$= -\frac{1}{12\pi}\left[-\pi^3 - \pi^3\right] = -\frac{1}{12\pi} \times -2\pi^{3}$$

$$= \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2 \cos nx\, dx$$

$$= \frac{1}{4\pi}\left\{ (\pi-x)^2 \frac{\sin nx}{n} - 2(\pi - x)(-1)\frac{-\cos nx}{n^2} + 2(-1)\frac{\sin nx}{n^3}\right\}_0^{2\pi}$$

$$= \frac{1}{4\pi}\left\{ 0 - 2(\pi - 2\pi)\frac{\cos 2n\pi}{n^2} - 0 - \left[0 - \frac{2\pi \cos 0}{n^2} - 0\right]\right\}$$

$$= \frac{1}{4\pi}\left\{ 2\pi \frac{1}{n^2}\cdot 1 + 2\pi \frac{1}{n^2}\right\}$$

$$= \frac{1}{4\pi} \times \frac{4\pi}{n^2} = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2 \sin nx\, dx$$

$$= \frac{1}{4\pi}\left\{ (\pi-x)^2 \frac{-\cos nx}{n} - 2(\pi - x)(-1)\frac{-\sin nx}{n^2} + 2(-1)\frac{-\cos nx}{n^3}\right\}_0^{2\pi}$$

$$= \frac{1}{4\pi}\left\{ \pi^2 \frac{-\cos 2n\pi}{n} - 0 + 2\frac{\cos 2n\pi}{n^3} - \left[\pi^2 \frac{-\cos 0}{n} - 0 + 2\frac{\cos 0}{n^3}\right]\right\}$$

$$= \frac{1}{4\pi}\left[-\frac{\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right] = 0$$

$$\therefore f(x) = \frac{\pi^2}{6\times 2} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$$

2. Find the Fourier Series of periodic function $f(x) = x \sin x$, $0 < x < 2\pi$ with period 2π.

Period $2l = 2\pi - 0 = 2\pi \implies l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{\pi} + b_n \sin\frac{n\pi x}{\pi}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x\,dx$$

$$= \frac{1}{\pi}\left\{x \cdot {}^-\cos x - 1 \cdot {}^-\sin x\right\}_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{{}^-2\pi \cos 2\pi + 0 - 0\right\}$$

$$= \frac{1}{\pi}\, {}^-2\pi = -2$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\left[\cos nx \sin x\right] dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\left[\cos nx \sin x\right]dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \, \frac{1}{2}\left[\sin(nx+x) - \sin(nx-x)\right]dx$$

$$\boxed{2\cos A \sin B = \sin(A+B) - \sin(A-B)}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{2\pi}\left\{ x\left[\frac{-\cos(n+1)x}{n+1} - \frac{-\cos(n-1)x}{n-1}\right] - 1\left[-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]\right\}_0^{2\pi}$$

$$= \frac{1}{2\pi}\left\{ 2\pi\left[\frac{-1}{n+1}\cos(n+1)2\pi + \frac{1}{n-1}\cos(n-1)2\pi\right] - 0 - \left[0-0\right]\right\}$$

$$= \frac{1}{2\pi}\times 2\pi\left[\frac{-1}{n+1}\times 1 + \frac{1}{n-1}\,1\right]$$

$$= \frac{-[n-1]+n+1}{n^2-1} = \frac{-n+1+n+1}{n^2-1} = \cancel{\frac{2}{n^2-1}}$$

$$= \frac{2}{n^2-1}, \quad n \neq 1$$

When $n=1$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx$$

$$\left[2\sin x \cos x = \sin 2x\right]$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \, \frac{1}{2}\sin 2x \, dx$$

$$= \frac{1}{2\pi} \left\{ \int_0^{2\pi} x \sin 2x \, dx \right.$$

$$= \frac{1}{2\pi} \left\{ x \frac{-\cos 2x}{2} - 1 \frac{-\sin 2x}{4} \right\}_0^{2\pi}$$

$$= \frac{1}{2\pi} \left\{ -\frac{2\pi}{2} \cos 2\pi + 0 - 0 \right\} = \frac{-1}{2\pi}\pi$$

$$= \underline{\underline{-1/2}}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \sin x \sin nx \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x (\sin nx \sin x) \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \frac{1}{2} \left[\cos(nx - x) - \cos(nx + x) \right] dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x \left[\cos(n-1)x - \cos(n+1)x \right] dx$$

$$= \frac{1}{2\pi} \left\{ x \left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1} \right] - 1 \left[-\frac{\cos(n-1)x}{(n-1)^2} - -\frac{\cos(n+1)}{(n+1)^2} \right] \right\}_0^{2\pi}$$

$$= \cancel{\frac{1}{2\pi}} \left\{ \cancel{2\pi \left[\frac{\sin(n-1)2\pi}{n-1} - \frac{\sin(n+1)2\pi}{n+1} \right]} \right.$$

$$= \frac{1}{2\pi} \left\{ 0 + \frac{\cos(n-1)2\pi}{(n-1)^2} - \frac{\cos(n+1)2\pi}{(n+1)^2} - \left[0 + \frac{1}{(n-1)^2} \cos 0 - \frac{1}{(n+1)^2} \cos 0 \right] \right.$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$$= 0 \quad , \quad n \neq 1$$

$\underline{n=1}$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin nx \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin^2 x \, dx = \frac{1}{\pi}\int_0^{2\pi} x \left[\frac{1-\cos 2x}{2}\right] dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \left[1-\cos 2x\right] dx$$

$$= \frac{1}{2\pi}\left\{ x\left[x - \frac{\sin 2x}{2}\right] - 1\left[\frac{x^2}{2} - \frac{\cos 2x}{2^2}\right]\right\}_0^{2\pi}$$

$$= \frac{1}{2\pi}\left\{2\pi\left[2\pi - 0\right] - \left[\frac{4\pi^2}{2} + \frac{\cos 2\pi}{4}\right] - \left[0 - 0 - \frac{\cos 0}{4}\right]\right\}$$

$$= \frac{1}{2\pi}\left\{4\pi^2 - 2\pi^2 - \frac{1}{4} + \frac{1}{4}\right\}$$

$$= \frac{1}{2\pi} 2\pi^2 \quad \underline{\underline{= \pi}}$$

$$f(x) = \underline{-1 - \frac{1}{2}\cos x} + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx$$

3. Find the Fourier Series of periodic function $f(x) = e^{-x}$, $0 < x < 2\pi$ with period 2π.

Period $2l = 2\pi \Longrightarrow l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\pi}\right) + b_n \sin\left(\frac{n\pi x}{\pi}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{-1}\right]_0^{2\pi} = \frac{-1}{\pi}\left[e^{-2\pi} - e^{0}\right]$$

$$= \frac{1}{\pi}\left[1 - e^{-2\pi}\right]$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\frac{e^{-x}}{1+n^2}\left(-\cos nx + n\sin nx\right)\right\}_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{\frac{e^{-2\pi}}{1+n^2}\left[-\cos 2n\pi + \overset{0}{n\sin 2n\pi}\right] - \frac{e^0}{1+n^2}\left[-\cos 0 + 0\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{e^{-2\pi}}{1+n^2}(-1) - \frac{1}{1+n^2}(-1)\right\}$$

$$= \cancel{\frac{1}{\pi(1+n^2)}} = \frac{1}{\pi(1+n^2)}\left[1-e^{-2\pi}\right]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x} \sin nx \, dx$$

$$= \frac{1}{\pi}\left\{ \frac{e^{-x}}{1+n^2}\left[-\sin nx - n\cos nx\right]\right\}_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{ \frac{e^{-2\pi}}{1+n^2}\left[0 - n\cos 2n\pi\right] - \frac{e^0}{1+n^2}\left[0 - n\cos 0\right]\right\}$$

$$= \frac{1}{\pi}\left\{ \frac{e^{-2\pi}}{1+n^2}(-n) - \frac{1}{1+n^2}(-n)\right\}$$

$$= \frac{n}{\pi(1+n^2)}\left[-e^{-2\pi}+1\right] = \frac{n(1-e^{-2\pi})}{\pi(1+n^2)}$$

$$f(x) = \frac{1-e^{-2\pi}}{2\pi} + \sum_{n=1}^{\infty} \frac{1-e^{-2\pi}}{\pi(1+n^2)}\cos nx + \frac{n(1-e^{-2\pi})}{\pi(1+n^2)}\sin nx$$

4. Find the Fourier Series representation of the function $f(x)$ given by

$$f(x) = \begin{cases} x & \text{for } 0 \le x \le \pi \\ 2\pi - x & \text{for } \pi \le x \le 2\pi \end{cases}$$

Period $2l = 2\pi \Rightarrow l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\pi}\right) + b_n \sin\left(\frac{n\pi x}{\pi}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{x^2}{2}\right]_0^{\pi} + \left[2\pi x - \frac{x^2}{2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left(\frac{\pi^2}{2} - 0\right) + \left(4\pi^2 - \frac{\cancel{4}\pi^2}{\cancel{2}} - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right)\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi^2}{2} + 2\pi^2 - 2\pi^2 - \frac{\pi^2}{2}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\cancel{2}\pi^2}{\cancel{2}}\right\}$$

$$= \underline{\underline{\pi}}$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cdot\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[x\frac{\sin nx}{n} - 1\cdot\frac{-\cos nx}{n^2}\right]_0^{\pi} + \left[(2\pi - x)\frac{\sin nx}{n} - (-1)\frac{-\cos nx}{n^2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[0 + \frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right] + \left[0 - \frac{\cos 2n\pi}{n^2} - \left[0 - \frac{\cos n\pi}{n^2}\right]\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{1}{n^2}\cos n\pi - \frac{1}{n^2} - \frac{1}{n^2} + \frac{\cos n\pi}{n^2}\right\}$$

$$= \frac{1}{n^2\pi}\left[2\cos n\pi - 2\right] = \underline{\underline{\frac{2}{n^2\pi}\left[(-1)^n - 1\right]}}$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[x\,\frac{-\cos nx}{n} - 1\cdot\frac{-\sin nx}{n^2}\right]_0^{\pi} + \right.$$

$$\left.\left[(2\pi - x)\frac{-\cos nx}{n} - (-1)\frac{-\sin nx}{n^2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\pi\frac{\cos n\pi}{n} + 0 - 0\right] + \left[0 - 0 - \left[\pi\frac{-\cos n\pi}{n} - 0\right]\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi}{n}\cos n\pi + \frac{\pi}{n}\cos n\pi\right\}$$

$$= \underline{0}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}\left[(-1)^n - 1\right]\cos nx$$

5. Find the Fourier Series with period 3 to represent $f(x) = 2x - x^2$ in $(0,3)$.

period $2l = 3 - 0 = 3 \implies l = 3/2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{3/2} + b_n\sin\frac{n\pi x}{3/2}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{3} + b_n\sin\frac{2n\pi x}{3}\right)$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx = \frac{1}{3/2}\int_0^3 (2x - x^2)\,dx$$

$$= \frac{2}{3}\left[\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3}\right]_0^3 = \frac{2}{3}\left[9 - \frac{\cancel{3^3}}{3} - 0\right]$$

$$= \underline{\underline{0}}$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{2n\pi x}{3}\,dx$$

$$= \frac{1}{3/2}\int_0^3 (2x - x^2)\cos\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left\{(2x - x^2)\frac{\sin\frac{2n\pi x}{3}}{2n\pi/3} - (2 - 2x)\frac{-\cos\frac{2n\pi x}{3}}{(2n\pi/3)^2}\right.$$

$$\left. + (-2)\frac{-\sin\frac{2n\pi x}{3}}{(2n\pi/3)^3}\right\}_0^3$$

$$= \frac{2}{3}\left\{0 + -4\times\left(\frac{3}{2n\pi}\right)^2\cos 2n\pi + 0 - \left[0\right.\right.$$

$$\left.\left. + 2\times\left(\frac{3}{2n\pi}\right)^2\cos 0 + 0\right]\right\}$$

$$= \frac{2}{3}\left\{-4\times\frac{9}{4n^2\pi^2} - 2\times\frac{9}{4n^2\pi^2}\right\}$$

$$= \frac{\cancel{2}}{\cancel{3}}\times\frac{\cancel{9}^{3}}{\cancel{4}_{2}n^2\pi^2}\left[-4-2\right] = \frac{3}{2n^2\pi^2}\times -\cancel{6}^{3}$$

$$= \underline{\underline{\frac{-9}{n^2\pi^2}}}$$

6 Obtain Fourier Series for the function

$$f(x) = \begin{cases} \pi x & \text{when } 0 \le x \le 1 \\ \pi(2-x) & \text{when } 1 \le x \le 2 \end{cases}$$

Period $2l = 2-0 = 2 \implies l = 1$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{1} + b_n \sin \frac{n\pi x}{1}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\pi x + b_n \sin n\pi x\right)$$

$$a_0 = \frac{1}{l} \int_c^{c+2l} f(x)\,dx = \frac{1}{1}\int_0^2 f(x)\,dx$$

$$= \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$= \pi \left\{ \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2 \right\}$$

$$= \pi \left\{ \frac{1}{2} - 0 + 4 - 2 - 2 + \frac{1}{2} \right\}$$

$$= \pi \times 1 = \underline{\pi}$$

$$a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos n\pi x\,dx$$

$$= \frac{1}{1} \int_0^2 f(x) \cdot \cos n\pi x\,dx$$

$$= \int_0^1 \pi x \cos n\pi x\,dx + \int_1^2 \pi(2-x) \cos n\pi x\,dx$$

$$= \pi \left\{ \left[x \frac{\sin n\pi x}{n\pi} - -1 \cdot \frac{-\cos n\pi x}{(n\pi)^2} \right]_0^1 + \left[(2-x)\frac{\sin n\pi x}{n\pi} - (-1)\frac{-\cos n\pi x}{(n\pi)^2} \right]_1 \right.$$

$$= \pi \left\{ \left[\frac{\cos n\pi}{(n\pi)^2} - \frac{\cos 0}{(n\pi)^2} \right] + \left[-\frac{\cos 2n\pi}{(n\pi)^2} + \frac{\cos n\pi}{(n\pi)^2} \right] \right\}$$

$$= \pi \left\{ \frac{\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2} + \frac{\cos n\pi}{n^2\pi^2} \right\}$$

$$= \frac{\pi}{n^2\pi^2} \left[2\cos n\pi - 2 \right] = \underline{\underline{\frac{2}{n^2\pi} \left[(-1)^n - 1 \right]}}$$

$$b_n = \frac{1}{1} \int_0^2 f(x) \sin n\pi x \, dx$$

$$= \int_0^1 \pi x \sin n\pi x \, dx + \int_1^2 \pi (2-x) \sin n\pi x \, dx$$

$$= \pi \left\{ \left[x \frac{-\cos n\pi x}{n\pi} - 1 \cdot \frac{-\sin n\pi x}{(n\pi)^2} \right]_0^1 + \left[(2-x) \frac{-\cos n\pi x}{n\pi} - (-1) \frac{-\sin n\pi x}{(n\pi)^2} \right]_1^2 \right\}$$

$$= \pi \left\{ \left[-\frac{\cos n\pi}{n\pi} + 0 - (0) \right] + \left[0 - 0 - \left(-\frac{\cos n\pi}{n\pi} - 0 \right) \right] \right\}$$

$$= \pi \cdot \left[-\frac{\cos n\pi}{n\pi} + \frac{\cos n\pi}{n\pi} \right] = \underline{\underline{0}}$$

$$\underline{\underline{f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi} \left[(-1)^n - 1 \right] \cos n\pi x}}$$

Odd and Even function

A function $f(x)$ is said to be odd, if $f(-x) = -f(x)$.

eg: (1) $f(x) = x$

$f(-x) = -x = -f(x)$

(2) $f(x) = x^3$

$f(-x) = (-x)^3 = -x^3 = -f(x)$

(3) $f(x) = \sin x$

$f(-x) = \sin(-x) = -\sin x = -f(x)$

A function $f(x)$ is said to be even if $f(-x) = f(x)$.

eg: (1) $f(x) = x^2$

$f(-x) = (-x)^2 = x^2 = f(x)$

(2) $f(x) = \cos x$

$f(-x) = \cos(-x) = \cos x = f(x)$

Results

(1) even ± even = even

odd ± odd = odd

(2) even x even = even

odd x odd = even

even x odd = odd

(3) If $f(x)$ is even function on $[-a, a]$ then

$$\boxed{\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx}$$

(4) If $f(x)$ is an odd function on $[-a,a]$ then,

$$\boxed{\int_{-a}^{a} f(x)\,dx = 0}$$

Fourier Series expansion of an even function in $(-l, l)$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{2}{l}\int_{0}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} \overset{e \times e = e}{f(x)\cos\frac{n\pi x}{l}}\,dx = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} \overset{e \times o = o}{f(x)\sin\frac{n\pi x}{l}}\,dx = 0$$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l},}$$

This series is called Fourier Cosine Series

Fourier Series expansion of an odd function in $(-l, l)$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = 0$$

$$a_n = \frac{1}{l}\int_{-l}^{l} \overset{o \times e = o}{f(x)\cos\frac{n\pi x}{l}}\,dx = 0$$

$$b_n = \frac{1}{l}\int_{-l}^{l} \overset{o \times o = e}{f(x)\sin\frac{n\pi x}{l}}\,dx = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{l}}$$, is called Fourier Sine Series.

Problems

1. Develop the Fourier Series of $f(x) = x^2$ in $-2 \le x \le 2$.

$$f(x) = x^2 \implies \text{even function}$$
$$\implies b_n = 0.$$

Period $2l = 2 - (-2) = 4$
$$\implies l = 2$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{2}\int_0^2 x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{2}{2}\int_0^2 x^2 \cos\frac{n\pi x}{2}\,dx$$

$$= \left\{ x^2 \frac{\sin\frac{n\pi x}{2}}{n\pi/2} - 2x\,\frac{-\cos\frac{n\pi x}{2}}{(n\pi/2)^2} + 2\,\frac{-\sin\frac{n\pi x}{2}}{(n\pi/2)^3} \right\}_0^2$$

$$= \left[0 + \left(\frac{2}{n\pi}\right)^2 2\times 2 \cos n\pi - 0 - 0 \right]$$

$$= \frac{4}{n^2\pi^2} \times 4\,(-1)^n = \frac{16}{n^2\pi^2}(-1)^n$$

$$f(x) = \frac{8}{3\times 2} + \sum_{n=1}^{\infty} \frac{16}{n^2\pi^2}(-1)^n \cos\frac{n\pi x}{2}$$

2. Prove that in the interval $-\pi < x < \pi$ Fourier Series of $f(x) = x\cos x$ with period 2π is $f(x) = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty} \frac{n(-1)^n}{n^2-1}\sin nx$.

$f(x) = x \cdot \cos x$ (o × e = odd) $\implies$ odd function

$\implies a_0 = 0$ and $a_n = 0$

Period $2l = \pi - (-\pi) = 2\pi$

$\implies l = \pi$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\pi}$$

$$= \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos x \sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\left[\sin nx \cos x\right]dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left[\sin(nx+x) + \sin(nx-x)\right]dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left[\sin(n+1)x + \sin(n-1)x\right]dx$$

$$= \frac{1}{\pi}\left\{ x\left[-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right] - 1\left[-\frac{\sin(n+1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2}\right]\right\}_0^{\pi}$$

$$= \frac{1}{\pi}\left\{\pi\left[-\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} - 0 - 0\right]\right.$$

$$= -\frac{1}{n+1}(-1)^{n+1} - \frac{1}{n-1}(-1)^{n-1}$$

$$= -\frac{1}{n+1}(-1)^n\cdot(-1)^1 - \frac{1}{n-1}(-1)^n\cdot(-1)^{-1}$$

$$= \frac{1}{n+1}(-1)^n + \frac{1}{n-1}(-1)^n$$

$$= (-1)^n\left[\frac{1}{n+1}+\frac{1}{n-1}\right] = (-1)^n\left[\frac{\cancel{n-1}+\cancel{n+1}}{n^2-1}\right]$$

$$= \underline{\underline{\frac{2n(-1)^n}{n^2-1}}}, \quad n\neq 1$$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} x\cos x\,\sin x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left\{x\,\frac{-\cos 2x}{2} - 1\,\frac{-\sin 2x}{2^2}\right\}_0^{\pi}$$

$$= \frac{1}{\cancel{\pi}}\left[-\cancel{\pi}\,\frac{\cos 2\pi}{2} + 0 - 0\right]$$

$$= \underline{\underline{-1/2}}$$

$$f(x) = -\frac{1}{2}\sin x + \sum_{n=2}^{\infty}\frac{2n(-1)^n}{n^2-1}\sin nx$$

$$= \underline{\underline{-\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty} n\frac{(-1)^n}{n^2-1}\sin nx}}$$

3. Find the Fourier Series of the function $f(x) = x$ in the range $-\pi < x < \pi$

$f(x) = x \Longrightarrow$ odd function

$\Longrightarrow a_0 = 0 \quad a_n = 0$

$2l = \pi - (-\pi) = 2\pi \Longrightarrow l = \pi$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\pi}\right) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left\{ x\frac{-\cos nx}{n} - 1\cdot\frac{-\sin nx}{n^2}\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi\frac{\cos n\pi}{n} + 0 - 0\right]$$

$$= \frac{2}{\cancel{\pi}}\cdot -\cancel{\pi}\,\frac{(-1)^n}{n} = \underline{\underline{\frac{2}{n}(-1)^{n+1}}}$$

$$\underline{\underline{f(x) = 2\sum_{n=1}^{\infty}\frac{1}{n}(-1)^{n+1}\sin nx}}$$

4. Expand the function $f(x) = x\sin x$ as a Fourier Series in the interval $-\pi < x < \pi$.

$$f(x) = \overset{o\times o = even}{x\sin x} \Longrightarrow \text{even function}.$$

$$\Longrightarrow b_n = 0$$

period $2l = \pi - (-\pi) = 2\pi \Longrightarrow l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{\pi} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx$$

$$= \frac{2}{\pi}\left\{x\,{-\cos x} - 1\cdot{-\sin x}\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{-\pi\cos\pi + 0 - 0\right\} = \frac{2}{\cancel{\pi}}(-\cancel{\pi})(-1)$$

$$= 2$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\left[\cos nx\ \sin x\right]dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left[\sin(nx+x) - \sin(nx-x)\right]dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{\pi}\left\{x\left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right] - 1\left[-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]\right\}_0^{\pi}$$

$$= \frac{1}{\pi}\left\{\pi\left[\frac{-1}{n+1}\cos(n+1)\pi + \frac{1}{n-1}\cos(n-1)\pi - 0 - 0\right]\right\}$$

$$= \frac{-1}{n+1}(-1)^{n+1} + \frac{1}{n-1}(-1)^{n-1}$$

$$= \frac{-1}{n+1}(-1)^{n}(-1) + \frac{1}{n-1}(-1)^{n}(-1)^{-1}$$

$$= (-1)^{n}\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= (-1)^{n}\left[\frac{\cancel{n}-1-\cancel{n}-1}{n^2-1}\right]$$

$$= \frac{-2(-1)^{n}}{n^2-1} = \frac{(-1)^{n+1}}{n^2-1},\quad n\neq 1$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x\cos x\ \sin x\,dx$$

$$= \frac{1}{\pi}\int x\ \sin 2x\,dx$$

$$= \frac{1}{\pi}\left\{ x \cdot \frac{-\cos 2x}{2} - 1 \cdot \frac{-\sin 2x}{4} \right\}_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{2}\cos 2\pi + 0 - 0 \right]$$

$$= \frac{1}{\cancel{\pi}} \times \frac{-\cancel{\pi}}{2} \cdot 1 = -\frac{1}{2}$$

$$f(x) = -\frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1} \cos nx$$

5. Find the Fourier Series for $f(x) = |x|$, $-\pi < x < \pi$

OR

Obtain the Fourier Series for the function

$$f(x) = \begin{cases} -x & \text{when } -\pi < x < 0 \\ x & \text{when } 0 < x < \pi \end{cases}$$

$f(x) = |x| \Rightarrow$ even

$\Rightarrow b_n = 0$

period $2l = \pi - (-\pi) = 2\pi \Rightarrow l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\cancel{\pi}x}{\cancel{\pi}}$$

$$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{\cancel{2}}{\pi}\left(\frac{x^2}{\cancel{2}}\right)_0^{\pi}$$

$$= \frac{1}{\pi}\pi^2 = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx\,dx$$

$$= \frac{2}{\pi}\left\{ x\frac{\sin nx}{n} - 1\cdot\frac{-\cos nx}{n^2}\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{ 0 + \frac{\cos n\pi}{n^2} - \left[0 + \frac{\cos 0}{n^2}\right]\right\}$$

$$= \frac{2}{\pi n^2}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}\left[(-1)^n - 1\right]\cos nx$$

6 Obtain the Fourier Series expansion of $f(x) = |\sin x|$ in $-\pi < x < \pi$.

$$f(x) = |\sin x| = \begin{cases} -\sin x & \text{in } -\pi < x < 0 \\ \sin x & \text{in } 0 < x < \pi \end{cases}$$

Ans $f(x) = |\sin x| =$ even

$\Longrightarrow b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\pi} \qquad [l = \pi]$$

$$a_0 = \frac{2}{l}\int_0^{l} f(x)dx = \frac{2}{\pi}\int_0^{\pi}\sin x\,dx$$

$$= \frac{2}{\pi}\left[-\cos x\right]_0^{\pi} = -\frac{2}{\pi}\left[\cos\pi - \cos 0\right]$$

$$= -\frac{2}{\pi}[-1-1] = \frac{4}{\pi}$$

$$a_n = \frac{2}{l}\int_0^l f(x) \cdot \cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx = \frac{2}{\pi}\int_0^{\pi} \cos nx \sin x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} [\sin (n+1)x - \sin (n-1)x] \, dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos (n+1)x}{n+1} + \frac{\cos (n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[-\frac{1}{n+1}\cos (n+1)\pi + \frac{1}{n-1}\cos (n-1)\pi\right] - \left[-\frac{1}{n+1}\cos 0 + \frac{1}{n-1}\cos 0\right]\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n+1}(-1)^{n+1} + \frac{1}{n-1}(-1)^{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-1}{n+1}(-1)^n(-1) + \frac{1}{n-1}(-1)^n(-1) + \frac{1}{n+1} - \frac{1}{n-1}\right\}$$

$$a_n = \frac{1}{\pi}\left[\frac{1}{n+1}(-1)^n - \frac{1}{n-1}(-1)^n + \frac{1}{n+1} - \frac{1}{n-1}\right], \; n \neq 1$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cdot \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin 2x = \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi}$$

$$= \frac{-1}{2\pi}[\cos 2\pi - \cos 0] = -\frac{1}{2\pi}[1-1]$$

$$= 0$$

7. Obtain the Fourier Series expansion of

$f(x) = |\cos x|$ in $-\pi < x < \pi$

$f(x) = |\cos x| \Rightarrow$ even function

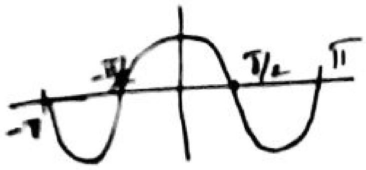
$\Rightarrow b_n = 0$

period $2l = \pi - (-\pi) = 2\pi \Rightarrow l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\pi}$$

$$|\cos x| = \begin{cases} -\cos x & -\pi < x < -\pi/2 \\ \cos x & -\frac{\pi}{2} < x < \pi/2 \\ -\cos x & \pi/2 < x < \pi \end{cases}$$

$$a_0 = \frac{2}{l} \int_0^l f(x)\,dx$$



$$= \frac{2}{\pi} \left\{ \int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^{\pi} f(x)\,dx \right\}$$

$$= \frac{2}{\pi} \left\{ \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx \right\}$$

$$= \frac{2}{\pi} \left\{ (\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi} \right\}$$

$$= \frac{2}{\pi} \left[\sin\frac{\pi}{2} - 0 - [\sin\pi - \sin\pi/2] \right]$$

$$= \frac{2}{\pi} [1 + 1] = \underline{4/\pi}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2}\cos x\cos nx\,dx + \int_{\pi/2}^{\pi} -\cos x\cos nx\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2}\cos nx\cos x\,dx - \int_{\pi/2}^{\pi}\cos nx\cos x\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi/2}\left[\cos(n+1)x + \cos(n-1)x\right]dx - \int_{\pi/2}^{\pi}\left[\cos(n+1)x + \cos(n-1)x\right]dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_0^{\pi/2} - \left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} - 0\right] - \left[0+0-\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right]\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{1}{n+1}\sin\left(\frac{n\pi}{2}+\pi/2\right) + \frac{1}{n-1}\sin\left(\frac{n\pi}{2}-\pi/2\right) + \frac{1}{n+1}\sin\left(n\pi/2+\pi/2\right) + \frac{1}{n-1}\sin\left(\frac{n\pi}{2}-\pi/2\right)\right\}$$

$$= \cancel{\frac{1}{\pi}}\left\{\cancel{\frac{1}{n+1}}\right.$$

$\sin(90+\theta) = \cos\theta$

$\sin(90-\theta) = \cos\theta$

$\sin(n+1)\pi/2 = \sin\left(\frac{n\pi}{2}+\pi/2\right)$

$= \cos n\pi/2$

$\sin(n-1)\pi/2 = \sin\left[\left(\frac{n\pi}{2}-\pi/2\right)\right]$

$-\sin\left[\left(\pi/2-n\pi/2\right)\right] = -\cos n\pi/2$

$$= \frac{1}{\pi}\left\{ \frac{1}{n+1}\cos\frac{n\pi}{2} + \frac{1}{n-1}\cos n\pi/2 + \frac{1}{n+1}\cos\frac{n\pi}{2} - \frac{1}{n-1}\cos\frac{n\pi}{2} \right\}$$

$$= \frac{2}{\pi}\left[\frac{1}{n+1}\cos\frac{n\pi}{2} - \frac{1}{n-1}\cos\frac{n\pi}{2} \right]$$

$$= \frac{2}{\pi}\cos\frac{n\pi}{2}\left[\frac{1}{n+1} - \frac{1}{n-1} \right]$$

$$= \frac{2}{\pi}\cos\frac{n\pi}{2}\left[\frac{\cancel{n}-1-\cancel{n}-1}{n^2-1} \right]$$

$$= -\frac{4}{\pi(n^2-1)}\cos\frac{n\pi}{2}, \quad n \neq 1$$

$$a_1 = \frac{2}{\pi}\left\{ \int_0^{\pi/2} \cos^2 x \, dx + \int_{\pi/2}^{\pi} -\cos^2 x \, dx \right\}$$

$$= \frac{1}{\pi}\left\{ \int_0^{\pi/2} (1+\cos 2x)\, dx - \int_{\pi/2}^{\pi} (1+\cos 2x)\, dx \right\}$$

$$= \frac{1}{\pi}\left\{ \left[x + \frac{\sin 2x}{2} \right]_0^{\pi/2} - \left[x + \frac{\sin 2x}{2} \right]_{\pi/2}^{\pi} \right\}$$

$$= \frac{1}{\pi}\left\{ \frac{\pi}{2} + \frac{\sin 2\frac{\pi}{2}}{2} - 0 - \left[\pi + 0 - \frac{\pi}{2} + \frac{\sin 2\frac{\pi}{2}}{2} \right] \right\}$$

$$= \frac{1}{\pi}\left\{ \frac{\pi}{2} - \pi + \frac{\pi}{2} \right\} \underline{\underline{= 0}}$$

$$f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{-4\cos n\pi/2}{(n^2-1)\pi}\cos nx$$

8 Obtain Fourier Series for the function $f(x)$ given by

$$f(x) = \begin{cases} 1+\frac{2x}{\pi} & -\pi < x < 0 \\ 1-\frac{2x}{\pi} & 0 < x < \pi \end{cases}$$

Hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots = \frac{\pi^2}{8}$

Period $2l = \pi - (-\pi) = 2\pi \Longrightarrow l = \pi$

When $\underline{-\pi < x < 0}$

$$f(-x) = 1-\frac{2x}{\pi} = f(x) \quad 0 < x < \pi$$

When $\underline{0 < x < \pi}$ $\quad f(-x) = 1-\frac{2(-x)}{\pi} = 1+\frac{2x}{\pi} = f(x)$

$\Longrightarrow f(x)$ is even

$\Longrightarrow b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\pi}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\left[x - \frac{2}{\pi}\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi - \frac{\pi^2}{\pi} - 0\right]$$

$$= \underline{\underline{0}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{ \frac{1}{n+1}\cos\frac{n\pi}{2} + \frac{1}{n-1}\cos n\pi/2 + \frac{1}{n+1}\cos\frac{n\pi}{2} - \frac{1}{n-1}\cos\frac{n\pi}{2} \right\}$$

$$= \frac{2}{\pi}\left[\frac{1}{n+1}\cos\frac{n\pi}{2} - \frac{1}{n-1}\cos\frac{n\pi}{2} \right]$$

$$= \frac{2}{\pi}\cos\frac{n\pi}{2}\left[\frac{1}{n+1} - \frac{1}{n-1} \right]$$

$$= \frac{2}{\pi}\cos\frac{n\pi}{2}\left[\frac{\cancel{n}-1-\cancel{n}-1}{n^2-1} \right]$$

$$= -\frac{4}{\pi(n^2-1)}\cos\frac{n\pi}{2}, \quad n \neq 1$$

$$a_1 = \frac{2}{\pi}\left\{ \int_0^{\pi/2}\cos^2 x\,dx + \int_{\pi/2}^{\pi} -\cos^2 x\,dx \right\}$$

$$= \frac{1}{\pi}\left\{ \int_0^{\pi/2}(1+\cos 2x)\,dx - \int_{\pi/2}^{\pi}(1+\cos 2x)\,dx \right\}$$

$$= \frac{1}{\pi}\left\{ \left[x + \frac{\sin 2x}{2} \right]_0^{\pi/2} - \left[x + \frac{\sin 2x}{2} \right]_{\pi/2}^{\pi} \right\}$$

$$= \frac{1}{\pi}\left\{ \frac{\pi}{2} + \frac{\sin 2\frac{\pi}{2}}{2} - 0 - \left[\pi + 0 - \frac{\pi}{2} + \frac{\sin 2\frac{\pi}{2}}{2} \right] \right\}$$

$$= \frac{1}{\pi}\left\{ \frac{\pi}{2} - \pi + \frac{\pi}{2} \right\} = 0$$

$$f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{-4\cos n\pi/2}{(n^2-1)\pi}\cos nx$$

$$= \frac{2}{\pi}\left\{ \left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n} - \left(\frac{-2}{\pi}\right)\frac{-\cos nx}{n^2} \right\}_0^{\pi}$$

$$= \cancel{\frac{2}{\pi}\left\{\left(1-\frac{2\pi}{\pi}\right)\frac{\sin n\pi}{n} + 1\right.}$$

$$= \frac{2}{\pi}\left\{ 0 - \frac{2}{\pi}\frac{\cos n\pi}{n^2} - \left[0 - \frac{2}{\pi}\frac{\cos 0}{n^2}\right]\right\}$$

$$= \frac{2\times 2}{\pi\ \pi n^2}\left\{ -\cos n\pi + \cos 0 \right\}$$

$$= \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]\cos nx$$

Put $x=0$ [Point of discontinuity]

$$L.H.S \Rightarrow \frac{f(x+0)+f(x-0)}{2} = \frac{1+\cancel{\frac{2x}{\pi}}+1-\cancel{\frac{2x}{\pi}}}{2} = \frac{2}{2} = 1$$

$$R.H.S \Rightarrow \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left[1-(-1)^n\right]\cos 0$$

$$= \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]$$

$$= \frac{4}{\pi^2}\left[\frac{1}{1^2}2 + \frac{1}{2^2}\times 0 + \frac{1}{3^2}2 + \frac{1}{4^2}\times 0 + \frac{1}{5^2}2 + \cdots\right]$$

$$= \frac{8}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right]$$

$L.H.S = R.H.S$

$$1 = \frac{8}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right]$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}$$

9. Find the Fourier Series of ~~fn~~

$$f(x) = \begin{cases} -1+x, & -\pi < x < 0 \\ 1+x, & 0 < x < \pi \end{cases}$$

Period $2l = \pi - (-\pi) = 2\pi \implies l = \pi$

$\underline{-\pi < x < 0}$

$f(-x) = -1+(-x) = -(1+x) = -f(x) \quad 0 < x < \pi$

$\implies$ odd

$\underline{0 < x < \pi} \quad f(-x) = 1-x = -[-1+x] = -f(x)$

$-\pi < x < 0$

$\implies$ odd

$\therefore f(x)$ is odd function

$\implies a_0 = 0 \quad a_n = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\pi} = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}(1+x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left\{(1+x)\frac{-\cos nx}{n} - (-1)\frac{-\sin nx}{n^2}\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{(1+\pi)\frac{-\cos n\pi}{n} - 0 - \left[1.\frac{-\cos 0}{n} - 0\right]\right\}$$

$$= \frac{2}{\pi}\left[-(1+\pi)\frac{(-1)^n}{n} + \frac{1}{n}\right]$$

$$= \frac{2}{n\pi}\left[(1+\pi)(-1)^{n+1} + 1\right] = \frac{2}{n\pi}\cancel{\{(-1)^{n+1} + \pi(-1)^{n+1}\}}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[(1+\pi)(-1)^{n+1} + 1\right]\sin nx$$

10 Find the Fourier Series for $f(x) = \{$

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \pi & 0 < x < \pi \end{cases}$$

Period $2l = \pi - (-\pi) = 2\pi \implies l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{\pi} + b_n \sin\frac{n\pi x}{\pi}\right)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \pi\,dx$$

$$= \frac{1}{\pi} \times \pi \left[x\right]_0^{\pi}$$

$$= \underline{\underline{\pi}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} \pi \cos nx\,dx$$

$$= \left[\frac{\sin nx}{n}\right]_0^{\pi} = \frac{1}{n}\left\{\sin n\pi - 0\right\} = \underline{\underline{0}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} \pi \sin nx\,dx$$

$$= \left[-\frac{\cos nx}{n}\right]_0^{\pi} = -\frac{1}{n}\left[\cos n\pi - \cos 0\right]$$

$$= -\frac{1}{n}\left[(-1)^n - 1\right] = \underline{\underline{\frac{1}{n}\left[1 - (-1)^n\right]}}$$

$$\underline{\underline{f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{1}{n}\left[1 - (-1)^n\right] \sin nx}}$$

11. Develop $f(x)$ in Fourier Series in the interval $[-2, 2]$ if $f(x) = \begin{cases} 0 & \text{for } -2 < x < 0 \\ 1 & \text{for } 0 < x < 2 \end{cases}$

Period $2l = 2-(-2) = 4 \implies l = 2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{2} + b_n \sin\frac{n\pi x}{2}\right)$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\int_{0}^{2} 1\,dx$$

$$= \frac{1}{2}\left[x\right]_0^2 = \frac{1}{2}\,2 = \underline{\underline{1}}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_0^2 1\cdot\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\frac{\sin\frac{n\pi x}{2}}{n\pi/2}\right]_0^2 = \frac{1}{2}\times\frac{2}{n\pi}\left[\sin n\pi - \sin 0\right]$$

$$= \underline{\underline{0}}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\int_0^2 1\cdot\sin\frac{n\pi x}{2}\,dx = \frac{1}{2}\left[\frac{-\cos\frac{n\pi x}{2}}{n\pi/2}\right]_0^2$$

$$= \frac{1}{\cancel{2}}\times\frac{-\cancel{2}}{n\pi}\left[\cos n\pi - \cos 0\right]$$

$$= -\frac{1}{n\pi}\left[(-1)^n - 1\right] = \underline{\underline{\frac{1}{n\pi}\left[1-(-1)^n\right]}}$$

$$\underline{\underline{f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{n\pi}\left[1-(-1)^n\right]\sin\frac{n\pi x}{2}}}$$

12. Find the Fourier Series of the period function $f(x)$ of period 4, where

$$f(x) = \begin{cases} 0 & -2 < x \le -1 \\ k & -1 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

period $2l = 2-(-2) = 4 \implies l = 2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{2} + b_n \sin\frac{n\pi x}{2}\right)$$

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\int_{-1}^{1} k\,dx$$

$$= \frac{k}{2}\left[x\right]_{-1}^{1} = \frac{k}{2}\left[1-(-1)\right] = \frac{k}{2}\times 2$$

$$= \underline{\underline{k}}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{k}{2}\left[\frac{\sin\frac{n\pi x}{2}}{n\pi/2}\right]_{-1}^{1} = \frac{k}{\cancel{2}}\times\frac{\cancel{2}}{n\pi}\left[\sin\frac{n\pi}{2} - \sin\frac{-n\pi}{2}\right]$$

$$= \frac{k}{n\pi}\left[\sin\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right]$$

$$= \frac{k}{n\pi}\, 2\sin\frac{n\pi}{2} = \underline{\underline{\frac{2k}{n\pi}\sin\left(\frac{n\pi}{2}\right)}}$$

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-1}^{1} k\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{k}{2}\left[\frac{-\cos\frac{n\pi x}{2}}{n\pi/2}\right]_{-1}^{1} = \frac{-k}{\cancel{2}}\times\frac{\cancel{2}}{n\pi}\left[\cos\frac{n\pi}{2} - \cos\frac{n\pi}{2}\right]$$

$$= -\frac{k}{n\pi}\left[\cos n\pi/2 - \cos n\pi/2\right] = 0$$

$$f(x) = \frac{K}{2} + \sum_{n=1}^{\infty} \frac{2K}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{n\pi x}{2}\right)$$

13 Obtain the Fourier Series for the function $f(x) = x^2$, $-\pi < x < \pi$. Hence show that

1) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

2) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \pi^2/6$

3) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \pi^2/8$.

Period $2l = \pi - (-\pi) = 2\pi \implies l = \pi$

$f(x) = x^2 \implies$ even function
$\implies b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\pi}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{3\pi}\pi^3 = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi} \left\{ x^2 \frac{\sin nx}{n} - 2x \frac{-\cos nx}{n^2} + 2 \frac{-\sin nx}{n^3} \right\}_0^{\pi}$$

$$= \frac{2}{\pi} \left\{ 0 + 2\pi \frac{\cos n\pi}{n^2} + 0 - 0 \right\}$$

$$= \frac{2}{\cancel{\pi}} \, 2\cancel{\pi} \frac{(-1)^n}{n^2}$$

$$= \frac{4}{n^2} (-1)^n$$

$$f(x) = \frac{2\pi^2}{3 \times 2} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cdot \cos nx$$

Half Range expansions

Suppose in a particular situation we want to find the Fourier expansion of a function f(x) which is defined only in a half period say in the interval [0,l]. It is possible to extended the definition of f(x) to the other half [-l,0] of the interval [-l,l]. So that f(x) is either an even or an odd function.

The half range Cosine Series in the interval (0,l) is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

The half range Sine Series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$

Problems

1. Find half range cosine series for $f(x) = x^2$ in $0 \le x \le \pi$.

$l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\pi}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{3\pi}\pi^3 = \frac{2}{3}\pi^2$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left\{x^2\frac{\sin nx}{n} - 2x\frac{-\cos nx}{n^2} + 2\frac{-\sin nx}{n^3}\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 + 2\pi\frac{\cos n\pi}{n^2} + 0 - 0\right]$$

$$= \frac{2}{\cancel{\pi}}\, 2\cancel{\pi}\,\frac{(-1)^n}{n^2}$$

$$= \frac{4}{n^2}(-1)^n$$

$$f(x) = \frac{\cancel{2}\pi^2}{\cancel{2}\times 3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \cos nx$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \cos nx$$

2.~~14~~ Develop the half range cosine series ~~in the interval~~ of $f(x) = \pi x - x^2$ in the interval $0 < x < \pi$.

$l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\pi}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\,dx$$

$$= \frac{2}{\pi}\left[\pi\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2}\pi^2 - \frac{\pi^3}{3} - 0\right]$$

$$= \frac{2}{\cancel{\pi}}\pi^{\cancel{3}\,2}\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= \cancel{2}\pi^2\left[\frac{3-2}{\cancel{6}\,3}\right] = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{(\pi x - x^2)\frac{\sin nx}{n} - (\pi - 2x)\frac{-\cos nx}{n^2} + (-2)\frac{-\sin nx}{n^3}\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{0 + (\pi - 2\pi)\frac{\cos n\pi}{n^2} + 0 - \left[\pi\frac{\cos 0}{n^2}\right]\right\}$$

$$= \frac{2}{\pi}\left\{ -\pi \frac{(-1)^n}{n^2} - \frac{\pi}{n^2} \right\}$$

$$= \frac{-2}{n^2\pi}\left[(-1)^n + 1\right]$$

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} -\frac{2}{n^2\pi}\left[(-1)^n+1\right] \cos nx$$

3. Find the Fourier Cosine Series of

$f(x) = x \sin x, \quad 0 < x < \pi.$

$l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\pi}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x \sin x\,dx$$

$$= \frac{2}{\pi}\left\{ x \,{-\cos x} - 1 \cdot {-\sin x} \right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi \cos\pi + 0 - 0\right]$$

$$= -\frac{2\cancel{\pi}(-1)}{\cancel{\pi}} = 2$$

$$a_n = \frac{2}{l}\int_0^l f(x) \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \cos nx \sin x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left[\sin(n+1)x - \sin(n-1)x\right] dx$$

$$= \frac{1}{\pi}\left\{ x\left[\frac{-\cos(n+1)x}{n+1} - \frac{-\cos(n-1)x}{n-1}\right] - 1\cdot\left[-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]\right\}_0^{\pi}$$

$$= \frac{1}{\cancel{\pi}}\left\{ \cancel{\pi}\left[-\frac{1}{n+1}\cos(n+1)\pi + \frac{1}{n-1}\cos(n-1)\pi\right] - 0\right]\right\}$$

$$= \frac{-1}{n+1}\left\{(-1)^{n+1} + \frac{1}{n-1}\cdot(-1)^{n-1}\right.$$

$$= \frac{-1}{n+1}(-1)^n\cdot(-1) + \frac{1}{n-1}(-1)^n\cdot(-1)^{-1}$$

$$= (-1)^n\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= (-1)^n\left[\frac{n-1-n-1}{n^2-1}\right]$$

$$= \frac{-2(-1)^n}{n^2-1} = \frac{2}{n^2-1}\cdot(-1)^{n+1}$$

4\. Find the Fourier Cosine Series of $f(x) = \cos x$, $0 < x < \pi/2$.

$l = \pi/2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\pi/2}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 2nx$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{\pi/2}\int_0^{\pi/2} \cos x\,dx$$

$$= \frac{4}{\pi}\left[\sin x\right]_0^{\pi/2} = \frac{4}{\pi}\left[\sin \pi/2 - 0\right]$$

$$= \frac{4}{\pi} \times 1 = \underline{4/\pi}$$

$$a_n = \frac{2}{l}\int_0^l f(x) \cos 2nx\,dx$$

$$= \frac{2}{\pi/2}\int_0^{\pi/2} \cos x \cdot \cos 2nx\,dx$$

$$= \frac{4}{\pi}\int_0^{\pi/2} \cos 2nx \cos x\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \left[\cos(2n+1)x + \cos(2n-1)x\right] dx$$

$$= \frac{2}{\pi}\left\{\frac{\sin(2n+1)x}{2n+1} + \frac{\sin(2n-1)x}{2n-1}\right\}_0^{\pi/2}$$

$$= \frac{2}{\pi}\left\{\frac{1}{2n+1}\sin(2n+1)\pi/2 + \frac{1}{2n-1}\sin(2n-1)\pi/2 - 0\right\}$$

$= \cancel{\frac{2}{\pi}}$

$$= \frac{2}{\pi}\left\{\frac{1}{2n+1}\cos n\pi + \frac{1}{2n-1}\,-\cos n\pi\right\}$$

$$= \frac{2}{\pi}\cos n\pi\left[\frac{1}{2n+1} - \frac{1}{2n-1}\right]$$

$$= \frac{2}{\pi}(-1)^n\left[\frac{\cancel{2n}-1-\cancel{2n}-1}{4n^2-1}\right]$$

$$= \frac{2}{\pi}(-1)^n\frac{(-2)}{4n^2-1}$$

$$= \frac{4\,(-1)^{n+1}}{\pi(4n^2-1)}$$

$$f(x) = \frac{4}{\pi\times 2} + \sum_{n=1}^{\infty}\frac{4\,(-1)^{n+1}}{\pi(4n^2-1)}\cos 2nx$$

$\sin(2n+1)\pi/2 =$

$\sin\left(2n\frac{\pi}{2} + \pi/2\right)$

$= \cos n\pi$ $\left[\begin{array}{l}\sin(90+\theta)\\ = \cos\theta\end{array}\right]$

$\sin(2n-1)\pi/2$

$= \sin\left(2n\frac{\pi}{2} - \pi/2\right)$

$= \sin -\left[\frac{\pi}{2} - n\pi\right]$

$= -\sin\left[\pi/2 - n\pi\right]$

$= -\cos n\pi$

$\left[\begin{array}{l}\sin(-\theta) = \\ -\sin\theta\end{array}\right]$

$\sin(90-\theta) = \cos\theta$

5. Obtain the cosine and sine series for $f(x) = x$ in the interval $0 \le x \le \pi$.

The half range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\pi}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\pi^2 = \underline{\underline{\pi}}$$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cdot\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{ x\frac{\sin nx}{n} - 1\cdot\frac{-\cos nx}{n^2}\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{ 0 + \frac{1}{n^2}\cos n\pi - \left[0 + \frac{\cos 0}{n^2}\right]\right\}$$

$$= \frac{2}{n^2\pi}\left[\cos n\pi - \cos 0\right]$$

$$= \underline{\underline{\frac{2}{n^2\pi}\left[(-1)^n - 1\right]}}$$

The half range Sine Series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\pi}$$

$$b_n = \frac{2}{l}\int_0^{l} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left\{ x\frac{-\cos nx}{n} - 1\frac{-\sin nx}{n^2}\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{ -\pi \frac{\cos n\pi}{n} + 0 - 0 \right\}$$

$$= \frac{2}{\pi n} - \pi (-1)^n = \frac{2(-1)^{n+1}}{n}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nx$$

6 Find the half range Fourier Sine Series representation of $f(x) = k$ in $(0,\pi)$.

$l = \pi$

The half range Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\pi} = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} k \sin nx \, dx$$

$$= \frac{2k}{\pi} \left[-\frac{\cos nx}{n} \right]_0^{\pi}$$

$$= \frac{-2k}{\pi n} \left[\cos n\pi - \cos 0 \right]$$

$$= \frac{-2k}{n\pi} \left[(-1)^n - 1 \right]$$

$$= \frac{2k}{n\pi} \left[1 - (-1)^n \right]$$

7\. Find the fourier Sine Series of $f(x) = e^x$ in $0 < x < 1$.

$l = 1$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{1}$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin n\pi x \, dx$$

$$= \frac{2}{1} \int_0^1 e^x \sin n\pi x \, dx$$

$$= 2 \left\{ \frac{e^x}{1+n^2\pi^2} \left[\sin n\pi x - n\pi \cos n\pi x \right] \right\}_0^1$$

$$= 2 \left\{ \frac{e}{1+n^2\pi^2} \left[0 - n\pi \cos n\pi \right] - \frac{e^0}{1+n^2\pi^2} \left[0 - n\pi \cos 0 \right] \right\}$$

$$= \frac{2}{1+n^2\pi^2} \left\{ -e\, n\pi (-1)^n + n\pi \right\}$$

$$= \frac{2n\pi}{1+n^2\pi^2} \left[1 - e(-1)^n \right]$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2n\pi}{1+n^2\pi^2} \left[1 - e(-1)^n \right] \sin n\pi x$$

8\. Find the Fourier Sine Series of

$$f(x) = \begin{cases} x, & 0 < x < 2 \\ 4 - x, & 2 < x < 4 \end{cases}$$

$l = 4$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{4}\right)$$

$$b_n = \frac{2}{l}\int_0^4 f(x)\sin\left(\frac{n\pi x}{4}\right)dx$$

$$= \frac{2}{4}\left\{\int_0^2 x \sin\frac{n\pi x}{4}\,dx + \int_2^4 (4-x)\sin\frac{n\pi x}{4}\,dx\right\}$$

$$= \frac{1}{2}\left\{\left[x\frac{-\cos\frac{n\pi x}{4}}{n\pi/4} - 1\frac{-\sin\frac{n\pi x}{4}}{(n\pi/4)^2}\right]_0^2 + \left[(4-x)\frac{-\cos\frac{n\pi x}{4}}{n\pi/4} - (-1)\frac{-\sin\frac{n\pi x}{4}}{(n\pi/4)^2}\right]_2^4\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{-4}{n\pi}2\cos\frac{n\pi}{2} + \left(\frac{4}{n\pi}\right)^2\sin\frac{n\pi}{2} - 0\right] + \left[0 - 0 - \left(\frac{-4}{n\pi}2\cos\frac{n\pi}{2} - \left(\frac{4}{n\pi}\right)^2\sin\frac{n\pi}{2}\right)\right]\right\}$$

$$= \frac{1}{2}\left\{\cancel{\frac{-8}{n\pi}\cos\frac{n\pi}{2}} + \frac{16}{n^2\pi^2}\sin\frac{n\pi}{2} + \cancel{\frac{8}{n\pi}\cos\frac{n\pi}{2}} + \frac{16}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$= \frac{1}{2}\left\{\frac{32}{n^2\pi^2}\sin\frac{n\pi}{2}\right\} = \frac{16}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$f(x) = \sum_{n=1}^{\infty}\frac{16}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\left(\frac{n\pi x}{4}\right)$$

9. Find the Fourier Sine Series of

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \end{cases}$$

$l = 2$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{2}\, dx$$

$$= \frac{2}{2} \left\{ \int_0^1 x \sin \frac{n\pi x}{2}\, dx + \int_1^2 (2-x) \sin \frac{n\pi x}{2}\, dx \right\}$$

$$= \left\{ \left[x \frac{-\cos \frac{n\pi x}{2}}{n\pi/2} - 1 \frac{-\sin \frac{n\pi x}{2}}{(n\pi/2)^2} \right]_0^1 + \left[(2-x) \frac{-\cos \frac{n\pi x}{2}}{n\pi/2} - (-1) \frac{-\sin \frac{n\pi x}{2}}{(n\pi/2)^2} \right]_1^2 \right.$$

$$= \left\{ \left[\frac{2}{n\pi} - \cos \frac{n\pi}{2} + \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} - 0 \right] + \right.$$

$$\left. \left[0 - 0 - \left[\frac{2}{n\pi} - \cos \frac{n\pi}{2} - \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} \right] \right] \right\}$$

$$= -\frac{2}{n\pi} \cancel{\cos \frac{n\pi}{2}} + \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} + \frac{2}{n\pi} \cancel{\cos \frac{n\pi}{2}} + \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2}$$

$$= \underline{\frac{8}{n^2\pi^2} \sin \frac{n\pi}{2}}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{2}$$

Parseval's Identity

Let $f(x)$ be a periodic function with period $2l$ and is piecewise continuous in the interval $(c, c+2l)$. Then

$$\frac{1}{2l}\int_{c}^{c+2l}[f(x)]^2\,dx = \left[\frac{a_0}{2}\right]^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

This is known as parseval's identity for the function $f(x)$ in the interval $(c, c+2l)$

Problems

1. Find the Fourier Series of $f(x) = x^2$ in the interval $(-\pi, \pi)$. Hence deduce that

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

$f(x) = x^2 \implies$ even function $(-\pi, \pi)$

$\implies b_n = 0$

Period $2l = 2\pi \implies l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\pi}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{3\pi}\pi^3 = \underline{\frac{2\pi^2}{3}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left\{ x^2 \frac{\sin nx}{n} - 2x\,\frac{-\cos nx}{n^2} + 2\,\frac{-\sin nx}{n^3}\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{ \cancel{\pi^2}0 + 2\pi \frac{\cos n\pi}{n^2} + 0 - 0 \right\}$$

$$= \frac{2}{\cancel{\pi}} \frac{2\cancel{\pi}}{n^2} (-1)^n$$

$$= \frac{4}{n^2}(-1)^n$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$

By parseval's identity

$$\frac{1}{2\pi}\int_c^{c+2l} [f(x)]^2 dx = \left[\frac{a_0}{2}\right]^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^4\,dx = \left[\frac{\pi^2}{3}\right]^2 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4}(-1)^{2n} + 0$$

$$\frac{1}{2\pi}\left[\frac{x^5}{5}\right]_{-\pi}^{\pi} = \frac{\pi^4}{9} + \frac{1}{2}\times 16 \sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{1}{10\pi}\left[\pi^5 + (\pi)^5\right] = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{1}{5\cancel{10}\cancel{\pi}}\cancel{2}\pi^{\cancel{5}4} - \frac{\pi^4}{9} = 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{9\pi^4 - 5\pi^4}{45} = 8\sum_{n=1}^{\infty}\frac{1}{n^4} \Longrightarrow \frac{\cancel{4}\pi^4}{45\times\cancel{8}_2} = \sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\Longrightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$