Module V

Series representation of functions

Taylor series expansion of a function $f(x)$

If $f(x)$ has derivatives of all orders at x_0, then the series

$$f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \frac{(x-x_0)^3}{3!}f'''(x_0) + \cdots$$

is called Taylor series of $f(x)$ about $x = x_0$.

Note :- Taylor series is a power series in $(x-x_0)$.

Maclaurin series

When $x_0 = 0$, then the series

$$f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

is called Maclaurin series of $f(x)$ about $x=0$

Note :- Maclaurin series is a power series in x.

1. Find the Taylor series of $\log x$ about the point $x=1$

Soln :-

$f(x) = \log x$ $f(1) = \log 1 = 0$

$f'(x) = \frac{1}{x}$ $f'(1) = 1$

$f''(x) = \frac{-1}{x^2}$ $f''(1) = -1$

$f'''(x) = \frac{2}{x^3}$ $f'''(1) = 2$

$\therefore$ Taylor series of $\log x$ is $(x-1) - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} + \cdots$

2. Expand e^x in powers of $(x-1)$ upto 4 terms

Soln :- Here $x_0 = 1$

$f(x) = e^x \qquad f(1) = e$

$f'(x) = e^x \qquad f'(1) = e$

$f''(x) = e^x \qquad f''(1) = e$

Taylor series of e^x about $x_0 = 1$ is

$$e + (x-1)e + \frac{(x-1)^2}{2!}e + \frac{(x-1)^3}{3!}e + \cdots\cdots$$

$$= e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \cdots\cdots\right]$$

3. Find the Maclaurin series expansion of $\cos x$

Soln :- Here $x_0 = 0$

$f(x) = \cos x \qquad f(0) = 1$

$f'(x) = -\sin x \qquad f'(0) = 0$

$f''(x) = -\cos x \qquad f''(0) = -1$

$f'''(x) = \sin x \qquad f'''(0) = 0$

$f^{iv}(x) = \cos x \qquad f^{iv}(0) = 1$

Maclaurin series expansion of $\cos x$ is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\cdots$$

4) Show that $e^{\sin x}$ the series expansion of $e^{\sin x}$ is $1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots\cdots$

<u>Soln</u> :- Here $x_0 = 0$

$f(x) = e^{\sin x}$ $\qquad f(0) = 1$

$f'(x) = e^{\sin x} \cdot \cos x$ $\qquad f'(0) = 1$

$f''(x) = e^{\sin x}\left[\cos^2 x - \sin x\right]$ $\qquad f''(0) = 1$

$f'''(x) = e^{\sin x}(-2\cos x \sin x - \cos x) + (\cos^2 x - \sin x)e^{\sin x} \cdot \cos x$ $\qquad f'''(0) = 0$

$f^{iv}(x) = e^{\sin x}(-3\cos^2 x + 3\sin^2 x + \sin x - 3\cos^2 x \sin x) + (-3\cos x \sin x - \cos x + \cos^3 x)e^{\sin x} \cdot \cos x$ $\qquad f^{iv}(0) = -3$

$\therefore$ Series expansion of $e^{\sin x}$ is $1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots\cdots$

5) Find the Maclaurin series expansion of $(1+x)^m$

<u>Soln</u> :- $f(x) = (1+x)^m$ $\qquad f(0) = 1$

$f'(x) = m(1+x)^{m-1}$ $\qquad f'(0) = m$

$f''(x) = m(m-1)(1+x)^{m-2}$ $\qquad f''(0) = m(m-1)$

$f'''(x) = m(m-1)(m-2)(1+x)^{m-3}$ $\qquad f'''(0) = m(m-1)(m-2)$

$\vdots$

$\therefore$ Maclaurin series expansion of $(1+x)^m$ is

$$1 + mx + m(m-1)\frac{x^2}{2!} + m(m-1)(m-2)\frac{x^3}{3!} + \cdots\cdots$$

<u>Note</u> :- The above series is called binomial series.

6) Find the binomial series for $\frac{1}{(1+x)^2}$.

Soln :- $\frac{1}{(1+x)^2} = (1+x)^{-2} = 1+(-2)x+\frac{(-2)(-3)}{2!}x^2+\frac{(-2)(-3)(-4)}{3!}x^3$

$+ \cdots\cdots$

$= 1-2x+3x^2-4x^3+5x^4-\cdots\cdots$

$= \sum_{k=1}^{\infty}(-1)^{k-1}k\,x^{k-1}$

Problems :-

I Write the Maclaurin series for the function

i) $\tan x$ (ii) $e^x \cos x$ iii) $\log(1+\sin x)$

iv) $\log(1+e^x)$ v) $x\sin x$ vi) xe^x

II Write the Taylor series about $x=x_0$ for the function

i) $\frac{1}{x}$; $x_0 = -1$

ii) $\frac{1}{x+2}$; $x_0 = 1$

iii) $\cos x$; $x_0 = \frac{\pi}{2}$

III Find the binomial series for $\frac{1}{\sqrt{1+x}}$

Interval of convergence (IOC) and radius of convergence (ROC)

We have seen that Taylor series is a power series in $(x-x_0)$. That is $c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \dots\dots$ and Maclaurin series is a power series in x. That is $c_0 + c_1 x + c_2 x^2 + \dots\dots\dots\dots$.

In the Maclaurin series, if a numerical value is substituted for x, the resulting series may either converge or diverge. The set of x-values for which a power series converge is called its convergence set. (or interval of convergence)

Theorem:- For any power series in x, exactly one of the following is true.

(i) The series converges only for $x = 0$. In this case, the series has ROC 0

(ii) The series converges for all real values of x $(-\infty, \infty)$ In this case, the series has ROC ∞

(iii) The series converges for all x in some finite open interval $(-R, R)$. In this case the series has ROC R.

Theorem:- For any power series in $(x-x_0)$, exactly one of the following is true.

i) The series converges only for $x = x_0$. (ROC is 0)

ii) The series converges for all real values of x $(-\infty, \infty)$ (ROC is ∞)

iii) The series converges for all x in some finite open interval $(x_0 - R, x_0 + R)$. [Here ROC is R]

1. Find the IOC and ROC of $\sum_{k=0}^{\infty} x^k$

Soln :- $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \ldots\ldots$

$$\underset{k\to\infty}{Lt} \frac{|u_{k+1}|}{|u_k|} = \underset{k\to\infty}{Lt} \frac{|x^{k+1}|}{|x^k|} = \underset{k\to\infty}{Lt} |x| = |x|$$

By ratio test, the series converges for $|x|<1$ (That is $-1<x<1$)

diverges for $|x|>1$

test fails when $|x|=1$ That is $x=\pm 1$

When $x=+1$, $\sum x^k = 1+1+\ldots\ldots$ diverges

$x=-1$, $\sum x^k = 1-1+1-1\ldots\ldots$

$= 0$ or 1 diverges

(not unique)

$\therefore$ Interval of convergence (IOC) is $(-1,1)$

Radius of convergence (ROC) is 1

2. Find the IOC and ROC of $\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{3^k (k+1)}$

Soln :- $\underset{k\to\infty}{Lt} \frac{|u_{k+1}|}{|u_k|} = \underset{k\to\infty}{Lt} \left| \frac{x^{k+1}}{3^{k+1}(k+2)} \cdot \frac{3^k (k+1)}{x^k} \right|$

$$= \underset{k\to\infty}{Lt} \left| \frac{x}{3} \frac{(k+1)}{(k+2)} \right|$$

$$= \frac{|x|}{3} \underset{k\to\infty}{Lt} \frac{k(1+\frac{1}{k})}{k(1+\frac{2}{k})} = \frac{|x|}{3}$$

By ratio test, the series converges if $\frac{|x|}{3} < 1$

$\Rightarrow |x| < 3 \Rightarrow -3 < x < 3$

test fails when $|x|=3$ That is $x = \pm 3$

When $x = +3$ $\quad \sum u_k = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots$

This is an alternating series. By Leibnitz test the series converges.

When $x = -3$, $\quad \sum u_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots$

This is a p-series with $p = 1$. By p-series test the series diverges.

IOC is $(-3, 3]$ and ROC $= 3$

3. Find the IOC and ROC of $\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2}$

Solution :

$$\underset{k\to\infty}{Lt} \frac{|u_{k+1}|}{|u_k|} = \underset{k\to\infty}{Lt} \left| \frac{(x-5)^{k+1}}{(k+1)^2} \cdot \frac{k^2}{(x-5)^k} \right|$$

$$= \underset{k\to\infty}{Lt} |x-5| \frac{k^2}{(k+1)^2} = \underset{k\to\infty}{Lt} |x-5| \frac{k^2}{k^2\left(1+\frac{1}{k}\right)^2}$$

$$= |x-5|$$

By ratio test, $\sum u_k$ converges if $|x-5| < 1$

$\Rightarrow -1 < (x-5) < 1$

$\Rightarrow 4 < x < 6$

test fails if $|x-5| = 1 \Rightarrow (x-5) = \pm 1 \Rightarrow x = 6$ or 4

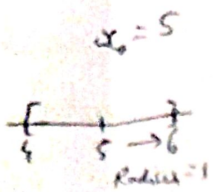
When $x = 6$, the series is $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots$

This is a p-series with $p = 2 > 1$. Hence converges.

When $x = 4$, the series is $-1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \cdots\cdots$

By Leibnitz test, the series converges.

IOC is $[4, 6]$ and ROC $= 1$



Problems

Find the interval of convergence and radius of convergence of the series.

i) $\sum_{k=0}^{\infty} \frac{x^k}{2k+3}$ (ii) $\sum_{k=0}^{\infty} \frac{k!\, x^k}{2^k}$ (iii) $\sum_{k=1}^{\infty} \frac{x^k}{k(k+1)}$

iv) $\sum_{k=0}^{\infty} \frac{(x-3)^k}{3^k}$ (v) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x+1)^k}{k}$ (vi) $\sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k (x+5)^k$

Convergence of Taylor series

Taylor series for a function $f(x)$ about x_0 converges to that function at all points x in the interval of convergence $(x_0 - R, x_0 + R)$ of that series.

1. Show that the Maclaurin series for e^x converges to e^x for all x.

Soln :- Here $x_0 = 0$

$f(x) = e^x$ $\quad f(0) = 1$

$f'(x) = e^x$ $\quad f'(0) = 1$

$f''(x) = e^x$ $\quad f''(0) = 1$

$\vdots$

Maclaurin series for e^x is $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$

To find the IOC,

$$\underset{k\to\infty}{Lt} \frac{|u_{k+1}|}{|u_k|} = \underset{k\to\infty}{Lt} \left| \frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k} \right| = |x| \underset{k\to\infty}{Lt} \left|\frac{1}{k+1}\right|$$

$$= |x| \times 0 = 0 < 1$$

$\therefore$ The series converges to e^x for all values of x.

2) Show that the Maclaurin series of $\ln(1+x)$ converges to $\ln(1+x)$ for all x in $(-1, 1]$

Soln :- Here $x_0 = 0$

$f(x) = \ln(1+x)$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{1+x}$ $\qquad f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2}$ $\qquad f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3}$ $\qquad f'''(0) = 2$

$\vdots$ $\qquad \vdots$

Maclaurin series of $\ln(1+x)$ is

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdots$$

To find the IOC

$$\lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} \left| \frac{x^{k+1}}{k+1} \cdot \frac{k}{x^k} \right| = |x| \lim_{k\to\infty} \left| \frac{\cancel{k}}{\cancel{k}(1+\frac{1}{k})} \right|$$

$$= |x|$$

By ratio test, series converges if $|x| < 1 \Rightarrow -1 < x < 1$

test fails when $|x| = 1 \Rightarrow x = \pm 1$

when $x = +1$, the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

This is an alternating series. By Leibnitz test, the series converges.

when $x = -1$, the series is $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} \cdots\cdots$

$$\Rightarrow -\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots\right)$$

It is a p-series with $p = 1$. The series diverges.

$\therefore \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots\cdots$ in the interval $(-1, 1]$

Problem

1. Show that the Maclaurin series for $\sin x$ converges to $\sin x$ for all x.

Fourier Series

Periodic functions

A function $f(x)$ which satisfies the relation $f(x+T) = f(x)$ for all real x and some fixed T is called a periodic function of period T.

eg:- $\sin x$, $\cos x$, $\sec x$ and $\operatorname{cosec} x$ are periodic functions with period 2π.

$\tan x$, $\cot x$ are periodic functions with period π.

The functions $\sin nx$ and $\cos nx$ are periodic with period $\frac{2\pi}{n}$.

Fourier series :- Periodic functions are of common occurence in many physical and engineering problems.

Any periodic functions can be expressed in terms of sine and cosine function as follows

$$\frac{1}{2}a_0 + a_1\cos\left(\frac{\pi x}{l}\right) + a_2\cos\left(\frac{2\pi x}{l}\right) + a_3\cos\left(\frac{3\pi x}{l}\right) + \dots\dots + b_1\sin\left(\frac{\pi x}{l}\right) + b_2\sin\left(\frac{2\pi x}{l}\right) + b_3\sin\left(\frac{3\pi x}{l}\right) + \dots\dots$$

Such a series is known as Fourier series

i.e $$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right)$$

where a_0, a_n, b_n are the Fourier coefficients of $f(x)$.

To find a_0, a_n, b_n we use Euler's formula.

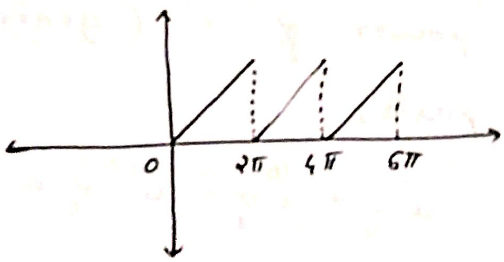
The Fourier series expansion of $f(x)$ in the interval $c < x < c+2l$ is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$ where

$$\left.\begin{aligned} a_0 &= \frac{1}{l}\int_c^{c+2l} f(x)\,dx \\ a_n &= \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \\ b_n &= \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx. \end{aligned}\right\} \text{Euler's formula}$$

1. Find the Fourier series representing $f(x) = x$ in the interval $0 < x < 2\pi$

<u>Soln</u> :-



Here $2l = 2\pi \Rightarrow l = \pi$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = \underline{\underline{2\pi}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx = \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \int\left(\frac{\sin nx}{n}\right)dx\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[0 + \frac{\cos(2n\pi)}{n^2} - 0 - \frac{1}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^{2n}}{n^2} - \frac{1}{n^2}\right] \qquad [\because \cos n\pi = (-1)^n]$$

$$= \frac{1}{\pi n^2}[1-1] = \underline{\underline{0}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin nx\,dx = \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \int\left(\frac{-\cos nx}{n}\right)dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-2\pi(-1)^{2n}}{n} + 0 - 0 - 0\right] = \frac{-2}{n}.$$

$$\therefore x = \pi + \sum_{n=1}^{\infty}\left(\frac{-2}{n}\right)\sin nx$$

$$= \pi - 2\left[\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\ldots\ldots\right]$$

Note 1 :- To integrate product of two functions, one of which is a power of x (generalised rule of integration by parts)

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\ldots$$

eg:- 1) $\int x^3 e^{-2x}\,dx = x^3\left(\frac{e^{-2x}}{-2}\right) - 3x^2\left[\frac{e^{-2x}}{(-2)^2}\right] + 6x\left[\frac{e^{-2x}}{(-2)^3}\right] - 6\left[\frac{e^{-2x}}{(-2)^4}\right]$

2) $\int x^2\cos nx\,dx = x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)$

Note 2 :- $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

Note 3 : $\sin(n\pi) = 0$ $\cos(n\pi) = (-1)^n$

~~$\sin\left(n+\frac{1}{2}\right)\pi = (-1)^n$~~ ~~Cos~~

$\sin(2n+1)\frac{\pi}{2} = (-1)^n$ $\cos(2n+1)\frac{\pi}{2} = 0$

2. Obtain the Fourier series to represent $f(x) = \frac{1}{4}(\pi - x)^2$ in the interval $0 < x < 2\pi$

Soln :
$$a_0 = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2 dx = \frac{1}{4\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi}$$

$$= \frac{-1}{12\pi}\left[-\pi^3 - \pi^3\right] = \underline{\underline{\frac{\pi^2}{6}}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2 \cos nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right) - \left(-2(\pi - x)\right)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(0 + \frac{2\pi\cos 2n\pi}{n^2} + 0\right) - \left(0 - \frac{2\pi\cos 0}{n^2} + 0\right)\right]$$

$$= \frac{1}{4\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right] = \underline{\underline{\frac{1}{n^2}}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2 \sin nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(\frac{-\cos nx}{n}\right) - \left(-2(\pi - x)\right)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(\frac{-\pi^2\cos 2n\pi}{n} + 0 + \frac{2\cos 2n\pi}{n^3}\right) - \left(\frac{-\pi^2}{n} + 0 + \frac{2}{n^3}\right)\right]$$

$$= \frac{1}{4\pi}\left[\frac{-\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right] = \underline{\underline{0}}$$

$$\therefore f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$$

$$= \frac{\pi^2}{12} + \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots\cdots$$

3. Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$

<u>Soln</u> :- $a_0 = \frac{1}{\pi} \int_0^{2\pi} e^{-x} dx = \frac{1}{\pi} \left[\frac{e^{-x}}{-1} \right]_0^{2\pi} = \frac{1}{\pi} \left[-e^{-2\pi} + 1 \right]$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} e^{-x} \cos nx \, dx$$

$$= \frac{1}{\pi} \left[\frac{e^{-x}}{1+n^2} \left(-\cos nx + n \sin nx \right) \right]_0^{2\pi}$$

$$\left[\because \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} \left\{ a \cos bx + b \sin bx \right\} \right]$$

$$= \frac{1}{\pi} \left[\frac{e^{-2\pi} (-\cos 2n\pi)}{1+n^2} - \frac{(-1)}{1+n^2} \right] = \frac{1}{\pi(1+n^2)} \left[-e^{-2\pi} + 1 \right] = \frac{1-e^{-2\pi}}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} e^{-x} \sin nx \, dx$$

$$= \frac{1}{\pi} \left[\frac{e^{-x}}{1+n^2} \left(-\sin nx - n \cos nx \right) \right]_0^{2\pi}$$

$$\left[\because \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} \left[a \sin bx - b \cos bx \right] \right]$$

$$= \frac{1}{\pi} \left[\frac{e^{-2\pi}(-n)}{1+n^2} + \frac{n}{1+n^2} \right]$$

$$= \frac{n}{\pi(1+n^2)} \left(1 - e^{-2\pi} \right)$$

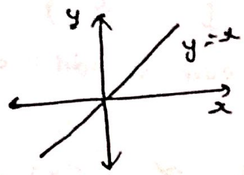
$$\therefore e^{-x} = \frac{1}{2\pi} \left(1 - e^{-2\pi} \right) + \frac{1-e^{-2\pi}}{\pi} \sum_{n=1}^{\infty} \frac{\cos nx}{1+n^2} + \frac{1-e^{-2\pi}}{\pi} \sum_{n=1}^{\infty} \frac{n \sin nx}{1+n^2}$$

Fourier series of odd and even functions in the interval $(-l, l)$

A fn $f(x)$ is said to be an odd fn, if $f(-x) = -f(x)$

eg:- x, x^3, $\sin x$ etc.

The graph of an odd fn is symmetrical about the origin

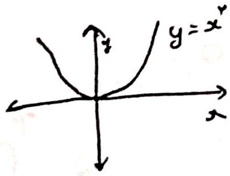


$$\int_{-c}^{c} f(x)\,dx = 0 \quad \text{when } f(x) \text{ is odd.}$$

A fn $f(x)$ is said to be an even fn if $f(-x) = f(x)$

eg:- x^2, $\cos x$, $\sin^2 x$ etc

The graph of an even fn is symmetrical about y-axis



$$\int_{-c}^{c} f(x)\,dx = 2\int_{0}^{c} f(x)\,dx \quad \text{when } f(x) \text{ is even.}$$

Therefore,

When $f(x)$ is odd

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = 0$$

$$a_n = \frac{1}{l}\int_{-l}^{l} \underbrace{f(x)\cos\frac{n\pi x}{l}}_{\text{odd} \times \text{even} = \text{odd}}\,dx = 0$$

$$b_n = \frac{1}{l}\int_{-l}^{l} \underbrace{f(x)\sin\frac{n\pi x}{l}}_{\text{odd} \times \text{odd} = \text{even}}\,dx = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$\therefore$ Fourier series of $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$, where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

When $f(x)$ is even,

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{2}{l}\int_{0}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} \underbrace{f(x)\cos\frac{n\pi x}{l}}_{\text{even} \times \text{even} = \text{even}}\,dx = \frac{2}{\pi}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} \underbrace{f(x)\sin\frac{n\pi x}{l}}_{\text{even} \times \text{odd} = \text{odd}}\,dx = 0$$

$\therefore$ Fourier series of $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$.

4. Express $f(x) = |x|$ in $-\pi < x < \pi$ as Fourier series

Soln: Here $f(x)$ is an even fn $\left[\because f(x) = |x|\right.$
$f(-x) = |-x| = |x|$
$\left. f(x) = f(-x)\right]$

Hence $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \underline{\underline{\pi}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} |x|\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$\therefore |x| = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2\left[(-1)^n - 1\right]}{\pi n^2}\cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{\pi \times 3^2}\cos 3x - \frac{4}{\pi \times 5^2}\cos 5x + \cdots\cdots$$

5. Find the Fourier series expansion of $f(x) = x - x^2$ in the interval $-1 < x < 1$

Soln :- Here $2l = 2 \Rightarrow l = 1$

$$a_0 = \frac{1}{1}\int_{-1}^{1}(x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{2} - \frac{1}{3} - \frac{1}{2} - \frac{1}{3} = \frac{-2}{3}$$

$$a_n = \frac{1}{1}\int_{-1}^{1}(x - x^2)\cos\left(\frac{n\pi x}{1}\right)dx$$

$$= \left[(x - x^2)\frac{\sin(n\pi x)}{n\pi} - (1 - 2x)\left(\frac{-\cos(n\pi x)}{n^2\pi^2}\right) + (-2)\left(\frac{-\sin(n\pi x)}{n^3\pi^3}\right)\right]_{-1}^{1}$$

$$= \left(0 - \frac{(-1)^n}{n^2\pi^2} + 0\right) - \left(0 + \frac{3(-1)^n}{n^2\pi^2} + 0\right) = \frac{-4(-1)^n}{n^2\pi^2}$$

$$b_n = \frac{1}{1}\int_{-1}^{1}(x - x^2)\sin\left(\frac{n\pi x}{1}\right)dx$$

$$= \left[(x - x^2)\left(\frac{-\cos(n\pi x)}{n\pi}\right) - (1 - 2x)\left(\frac{-\sin(n\pi x)}{n^2\pi^2}\right) + (-2)\frac{\cos(n\pi x)}{n^3\pi^3}\right]_{-1}^{1}$$

$$= \left(0 + 0 - \frac{2(-1)^n}{n^3\pi^3}\right) - \left(\frac{2(-1)^n}{n\pi} + 0 - \frac{2(-1)^n}{n^3\pi^3}\right)$$

$$= \frac{-2(-1)^n}{n\pi}$$

$$\therefore f(x) = \frac{-1}{3} + \frac{-4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi x$$

$$= \frac{-1}{3} - \frac{4}{\pi^2}\left[-\frac{\cos(\pi x)}{1^2} + \frac{\cos(2\pi x)}{2^2} - \frac{\cos(3\pi x)}{3^2} + \cdots\cdots\right]$$

$$-\frac{2}{\pi}\left[-\frac{\sin(\pi x)}{1} + \frac{\sin(2\pi x)}{2} - \frac{\sin(3\pi x)}{3} + \cdots\cdots\right]$$

6) Expand $f(x)$ in Fourier series in the interval $(-2, 2)$ when $f(x) = \begin{cases} 0 & -2 < x < 0 \\ 1 & 0 < x < 2 \end{cases}$

Solution :- Here $2l = 4 \Rightarrow l = 2$

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left[\int_{-2}^{0} 0\,dx + \int_{0}^{2} 1.dx\right]$$

$$= \frac{1}{2}\left[x\right]_0^2 = \underline{\underline{1}}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{0}^{2}\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\left[\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right]_0^2 = \frac{1}{2}\times\frac{2}{n\pi}\left[0-0\right] = \underline{\underline{0}}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{0}^{2}\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\left[\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right]_0^2 = \frac{-1}{2}\times\frac{2}{n\pi}\left[(-1)^n - 1\right]$$

$$= \frac{-1}{n\pi}\left[(-1)^n - 1\right]$$

$$\therefore f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{-1}{n\pi}\left[(-1)^n - 1\right]\sin\left(\frac{n\pi x}{2}\right)$$

7) Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$ and hence show that

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots$$

Soln :- $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\,dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$

$$= \frac{1}{\pi}\left[\cancel{\frac{\pi^2}{2}} - \frac{\pi^3}{3} - \cancel{\frac{\pi^2}{2}} - \frac{\pi^3}{3}\right] = \frac{-2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n} - (1-2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(0 + (1-2\pi)\frac{(-1)^n}{n^2} + 0\right) - \left(0 + (1+2\pi)\frac{(-1)^n}{n^2} + 0\right)\right]$$

$$= \frac{1}{\pi}\frac{(-1)^n}{n^2}\left[1 - 2\pi - 1 - 2\pi\right] = \frac{-4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(\frac{-\cos nx}{n}\right) - (1-2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[(\pi-\pi^2)\left(\frac{-(-1)^n}{n}\right) + 0 + \cancel{(-2)\frac{(-1)^n}{n^3}}\right] - \left[(-\pi-\pi^2)\left(\frac{-(-1)^n}{n}\right) + 0 + \cancel{(-2)\frac{(-1)^n}{n^3}}\right]\right\}$$

$$= \frac{1}{\pi}\frac{(-1)^n}{n}\left[-\pi + \cancel{\pi^2} - \pi - \cancel{\pi^2}\right] = \frac{-2(-1)^n}{n}$$

$$\therefore (x-x^2) = \frac{-\pi^2}{3} + (-4)\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx + (-2)\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$= \frac{-\pi^2}{3} - 4\left[-\cos x + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots\cdots\right]$$

$$-2\left[-\sin x + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + \cdots\cdots\right]$$

Put $x = 0$

$$0 = \frac{-\pi^2}{3} + 4\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots\right]$$

$$\Rightarrow \frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots$$

8) Find the Fourier series to represent the function $f(x) = \begin{cases} -k & \text{when } -\pi < x < 0 \\ k & \text{when } 0 < x < \pi \end{cases}$

~~Soln~~ Also deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\cdots$

Soln:- $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\,dx + \int_{0}^{\pi} k\,dx\right]$

$$= \frac{1}{\pi}\left[-k[x]_{-\pi}^{0} + k[x]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-k\pi + k\pi\right] = \underline{\underline{0}}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-k\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + k\left[\frac{\sin nx}{n}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi} \times 0 = \underline{\underline{0}}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-k\left(\frac{-\cos nx}{n}\right)_{-\pi}^{0} + k\left(\frac{-\cos nx}{n}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{k}{n} - \frac{k}{n}(-1)^n - \frac{k}{n}(-1)^n + \frac{k}{n}\right] = \frac{2k}{n\pi}\left[1 - (-1)^n\right]$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2k}{n\pi}\left[1-(-1)^n\right] \sin nx$$

$$= \frac{4k}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\cdots\right]$$

Put $x = \frac{\pi}{2}$ $\quad f\left(\frac{\pi}{2}\right) = k$

Therefore $k = \frac{4k}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \cdots\cdots\right]$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots\cdots$$

9) Find the Fourier series expansion of

$$f(x) = \begin{cases} -\pi & , -\pi < x < 0 \\ x & , 0 < x < \pi \end{cases}$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots = \frac{\pi^2}{8}$

Soln :- $a_0 = \frac{1}{\pi}\left\{ \int_{-\pi}^{0} (-\pi)\, dx + \int_{0}^{\pi} x\, dx \right.$

$$= \frac{1}{\pi}\left[-\pi (x)_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi} \right] = \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = \frac{-\pi}{2}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi) \cos nx\, dx + \int_{0}^{\pi} x \cos nx\, dx \right]$$

$$= \frac{1}{\pi}\left[-\pi \left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left[x \frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} \right]$$

$$= \frac{1}{\pi}\left[0 + 0 + \left[0 + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]\right]$$

$$= \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin nx\, dx + \int_{0}^{\pi} x \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left\{\left[-\pi\left(\frac{-\cos nx}{n}\right)_{-\pi}^{0}\right] + \left[x\left(\frac{-\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[+\pi\left(\frac{1}{n} - \frac{(-1)^n}{n}\right) + \frac{-\pi(-1)^n}{n}\right]$$

$$= \frac{1}{\pi} \times \frac{\pi}{n}\left[1 - (-1)^n - (-1)^n\right] = \frac{1-2(-1)^n}{n}$$

$$\therefore f(x) = \frac{-\pi}{4} + \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{\pi n^2} \cos nx + \sum_{n=1}^{\infty} \frac{[1-2(-1)^n]}{n} \sin nx$$

$$= \frac{-\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\cdots\right]$$

$$+ \sum_{n=1}^{\infty} \frac{[1-2(-1)^n]}{n} \sin nx.$$

Put $x = 0$

$$f(0) = \frac{1}{2}[\underset{\text{right hand side}}{f(0^+)} + \underset{\text{left hand side}}{f(0^-)}] = \frac{1}{2}[0-\pi] = \frac{-\pi}{2}$$

$$\therefore \frac{-\pi}{2} = \frac{-\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots\right]$$

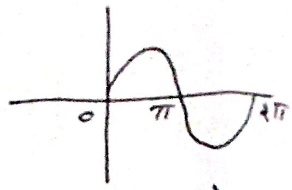
$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots$$

10) Find the Fourier series to represent the function $f(x) = |\sin x|$ for $-\pi < x < \pi$

Soln: $f(x) = |\sin x|$ $f(-x) = |\sin(-x)| = |-\sin x| = |\sin x| = f(x)$

$f(x)$ is even. Hence $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} |\sin x|\, dx$$



In $(0,\pi)$, $\sin x$ is +ve

$$= \frac{2}{\pi}\int_0^{\pi} \sin x\, dx$$

$$= \frac{2}{\pi}\left(-\cos x\right)_0^{\pi} = \frac{2}{\pi}\left[-(-1)+1\right] = \underline{\underline{\frac{4}{\pi}}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} \frac{\sin[(1+n)x] + \sin[(1-n)x]}{2}\, dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\cos(1+n)\pi}{1+n} - \frac{\cos(1-n)\pi}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$= \frac{1}{\pi}\left[\frac{-(-1)^{1+n}}{1+n} - \frac{(-1)^{1-n}}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^{2+n}(1-n) + (-1)^{2-n}(1+n)}{1-n^2} + \frac{2}{1-n^2}\right]$$

$$= \frac{1}{\pi}\left[(-1)^{2-n}\frac{\left[(-1)^{2n}(1-n) + (1+n)\right]}{1-n^2} + \frac{2}{1-n^2}\right]$$

$$= \frac{1}{\pi}\left[(-1)^{2-n}\frac{\left[1-\cancel{n}+1+\cancel{n}\right]}{1-n^2} + \frac{2}{1-n^2}\right]$$

$$= \frac{2}{\pi(1-n^2)}\left[(-1)^{2-n} + 1\right] \quad \text{for } n \neq 1$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x\, dx = \frac{2}{\pi}\int_0^{\pi} \frac{\sin 2x}{2}\, dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{-1}{2} + \frac{1}{2}\right]$$

$$= \underline{\underline{0}}$$

$$\therefore |\sin x| = \frac{2}{\pi} + 0 + \sum_{n=2}^{\infty} \frac{2}{\pi(1-n^2)} \left[(-1)^{2-n} + 1\right] \cos nx$$

Problems

1. Find a F.S to represent $f(x) = \frac{\pi - x}{2}$ in the interval $0 < x < 2\pi$

2. Obtain the F.S for the function $f(x) = x^2$ in $-\pi < x < \pi$. Hence show that

   i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$

   ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$

3. Expand $f(x) = x\sin x$ as a Fourier series in the interval $-\pi \le x \le \pi$ and deduce that

   $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \dots = \frac{\pi - 2}{4}$

4. Expand $f(x) = |\cos x|$ as F.S in the interval $-\pi < x < \pi$

5. If $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}$

   prove that $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}$

   Hence show that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} \dots = \frac{1}{4}(\pi - 2)$

6. $f(x) = \begin{cases} 1 & 0 < x < \pi \\ 2 & \pi < x < 2\pi \end{cases}$

   Find F.S for $f(x)$ and from it deduce that

   $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

7) Obtain Fourier series for $f(x) = \begin{cases} \pi x & 0 \le x \le 1 \\ \pi(2-x) & 1 \le x \le 2 \end{cases}$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots = \frac{\pi^2}{8}$

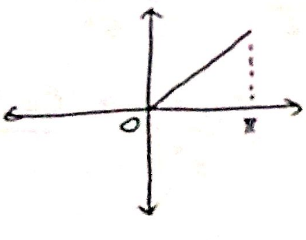
8) Find the F.S expansion of $f(x) = x^2 - 2$ when $-2 \le x \le 2$

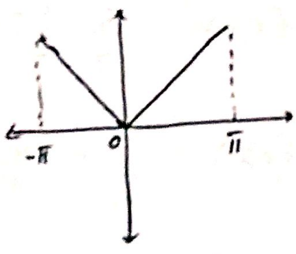
9) Find the F.S for the function $f(t) = \begin{cases} t & 0 < t < 1 \\ 1-t & 1 < t < 2 \end{cases}$

Half Range Series

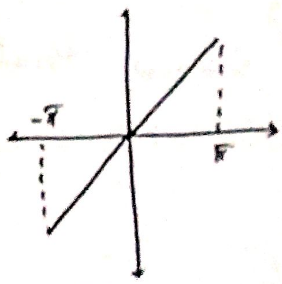
A function $f(x)$ defined over the interval $0 < x < l$ is capable of two distinct half range series

eg:- Consider $f(x) = x$ in $0 < x < \pi$





Cosine series (symmetric about y-axis)



Sine series (symmetric about origin)

The half range cosine series is

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \text{ where}$$

$$a_0 = \frac{2}{l} \int_0^l f(x)\,dx \qquad a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

The half range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{where} \quad b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

1. Expand $\pi x - x^2$ in a half range sine series in the interval $(0,\pi)$

Soln :- $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\sin nx\, dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-2\frac{(-1)^n}{n^3} + \frac{2}{n^3}\right] = \frac{4}{\pi n^3}\left[1-(-1)^n\right]$$

$$\therefore \pi x - x^2 = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{n^3}\sin nx$$

2. If $f(x) = \begin{cases} x & 0 < x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x < \pi \end{cases}$

Show that i) $f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\cdots\right]$

ii) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots\cdots\right]$

Soln :- i) $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right) - \frac{-\sin nx}{n^2}\right]_0^{\pi/2} + \left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\cancel{-\frac{\pi}{2}\frac{\cos\left(\frac{n\pi}{2}\right)}{n}} + \frac{\sin\frac{n\pi}{2}}{n^2} + \cancel{\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n}} + \frac{\sin\frac{n\pi}{2}}{n^2}\right]$$

$$= \frac{4}{\pi n^2}\sin\left(\frac{n\pi}{2}\right)$$

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n^2} \sin nx$$

$$= \frac{4}{\pi} \left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\cdots \right]$$

ii) $f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx = \frac{2}{\pi} \left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx \right]$$

$$= \frac{2}{\pi} \left[\left(\frac{x^2}{2}\right)_0^{\pi/2} + \left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{\pi} \right]$$

$$= \frac{2}{\pi} \left[\frac{\pi^2}{8} + \pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8} \right]$$

$$= \frac{2}{\pi} \left[\frac{\pi^2}{4} \right] = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi} \left[\int_0^{\pi/2} x \cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x) \cos nx\,dx \right]$$

$$= \frac{2}{\pi} \left\{ \left[x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right) \right]_0^{\pi/2} + \left[(\pi - x)\frac{\sin nx}{n} - (-1)\left(\frac{-\cos nx}{n^2}\right) \right]_{\pi/2}^{\pi} \right\}$$

$$= \frac{2}{\pi} \left[\cancel{\frac{\pi}{2} \frac{\sin\left(\frac{n\pi}{2}\right)}{n}} + \frac{\cos\left(\frac{n\pi}{2}\right)}{n^2} - \frac{1}{n^2} - \frac{\cos n\pi}{n^2} - \cancel{\frac{\pi}{2} \frac{\sin\left(\frac{n\pi}{2}\right)}{n}} + \frac{\cos\left(\frac{n\pi}{2}\right)}{n^2} \right]$$

$$= \frac{2}{\pi} \left[\frac{2\cos\left(\frac{n\pi}{2}\right)}{n^2} - \frac{1}{n^2} - \frac{(-1)^n}{n^2} \right]$$

$$\therefore f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{2}{\pi} \left[\frac{2\cos\left(\frac{n\pi}{2}\right)}{n^2} - \frac{1}{n^2} - \frac{(-1)^n}{n^2} \right]$$

$$= \frac{\pi}{4} - \frac{2}{\pi} \left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots\cdots \right]$$

Problems

1. Obtain the Cosine series for $f(x) = x$ in the interval $0 \le x \le \pi$. Hence show that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots = \frac{\pi^2}{8}$$

2. Obtain the half range sine series for e^x in $0 < x < 1$

3. If $f(x) = \begin{cases} \frac{\pi}{3} & 0 \le x \le \frac{\pi}{3} \\ 0 & \frac{\pi}{3} \le x \le \frac{2\pi}{3} \\ \frac{-\pi}{3} & \frac{2\pi}{3} \le x \le \pi \end{cases}$

then show that $f(x) = \frac{2}{\sqrt{3}} \left\{ \cos x - \frac{\cos 5x}{5} + \frac{\cos 7x}{7} - \dots \right)$

Convergence of a Fourier series

Dirichlet's theorem

Fourier series of a function $f(x)$ converges to $f(x)$ itself, if the following conditions are satisfied.

i) $f(x)$ is periodic

ii) $f(x)$ has finite no: of discontinuities in any one period

iii) $f(x)$ has finite no: of maxima and minima in any one period.

Parseval's theorem

If the Fourier series of $f(x)$ over an interval $c < x < c+2l$ is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right]$$

then $$\frac{1}{2l}\int_c^{c+2l}[f(x)]^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

1. Find the Fourier sine series for unity in $0<x<\pi$ and hence show that

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots = \frac{\pi^2}{8}$$

Solution :- $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ Here $f(x)=1$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\sin nx\,dx = \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{2}{n\pi}\left[1-(-1)^n\right]$$

$$\therefore\ 1 = \sum_{n=1}^{\infty}\frac{2}{n\pi}\left[1-(-1)^n\right]\sin nx$$

$$\Rightarrow 1 = \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\cdots\right]$$

From Parseval's theorem, $\int_c^{c+2l}[f(x)]^2dx = 2l\left[\frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)\right]$

$$\Rightarrow \int_0^{\pi}(1)^2dx = \pi\left[0+\frac{1}{2}\sum_{n=1}^{\infty}\left(0^2+\left(\frac{2}{n\pi}(1-(-1)^n)\right)^2\right)\right]$$

$$\Rightarrow [x]_0^{\pi} = \pi\left[\frac{1}{2}\sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left[1-(-1)^n\right]^2\right]$$

$$\Rightarrow \pi = \frac{\pi}{2}\times\frac{4}{\pi^2}\left[4+\frac{4}{3^2}+\frac{4}{5^2}+\cdots\cdots\right]$$

$$\Rightarrow \frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+$$

2. Find the Fourier series of x^2 in $(-\pi, \pi)$. Use Parseval's theorem to prove that

$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\cdots$$

Soln :- $f(x) = x^2$ is an even fn.

Hence $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi} \times \frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi(-1)^n}{n^2}\right] = \frac{4(-1)^n}{n^2}$$

By Parseval's theorem, $\int_c^{c+2l} [f(x)]^2\,dx = 2l\left[\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$

$$\Rightarrow \int_{-\pi}^{\pi} (x^2)^2\,dx = 2\pi\left[\frac{\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4}\right]$$

$$\Rightarrow \left[\frac{x^5}{5}\right]_{-\pi}^{\pi} = \frac{2\pi^5}{9} + \pi\sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\Rightarrow \frac{2\pi^5}{5} = \frac{2\pi^5}{9} + \pi\sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\Rightarrow \frac{8\pi^5}{45} = 16\pi\left[1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\cdots\right]$$

$$\Rightarrow \frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\cdots$$

Problem

1. Find the Fourier Cosine series for x in $(0, l)$ and hence show that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\cdots = \frac{\pi^4}{96}$