
KTU
NOTES

Module-1

Linear Algebra

Linear System of equations

A linear system of m linear equations in n unknowns $x_1, x_2, \ldots x_n$ is a set of equations of the form

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_n \end{array}\right\} \rightarrow (1)$$

where the coefficients a_{jk} and b_j are real numbers.

* This system is called linear because each variable x_j appears in the first power only, just as in the equation of a straight line.
* $a_{11}, a_{12}, \ldots$ are given numbers, called the coefficient of the system.
  $b_1, b_2, \ldots$ on the right are also numbers.
* The system is said to be homogeneous if all the b_j's are zero.
  If atleast one b_j is non zero, then the system is called non-homogeneous.
* A solution of system (1) is any set of values $x_1, x_2 \cdots x_n$ which satisfy all the m equation.

* When the system has atleast one solution, it is said to be consistent. Otherwise the system is said to be inconsistent.

For eg: the system

$$\left.\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 + 2x_2 &= 3 \end{aligned}\right\}$$ has no solution

$\therefore$ This system is inconsistent.

* If the system ① is homogeneous, then it has atleast the trivial solution $x_1 = 0,\ x_2 = 0, \ldots, x_n = 0$.

* The linear system ① of equations can be expressed as a matrix equation $AX = B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The matrix,

$$\tilde{A} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$ is called the augmented matrix of ①.

<u>Geometric interpretation</u>

If $m = n = 2$, we have 2 equations in 2 unknowns x_1, x_2.

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

Here each of these equations represents a straight line in x_1x_2-plane.
(x_1, x_2) is a solution of this system iff the point $P(x_1, x_2)$ lies on both the lines. Hence there are 3 possible cases.
(i) Precisely one solution if the lines intersects
(ii) Infinitely many solutions if the lines coincides
(iii) No solotion if the lines are parallel

Gauss elimination method

In Gauss elimination method, the solution of linear system ① is obtained by the following steps.

Step 1: First write the system in the matrix form $AX = B$ and write the augmented matrix $[A : B]$

Step 2: By elementary row operation the augmented matrix is brought to upper triangular form.

Step 3: The upper triangular system is solved using back substitution procedure by which we obtain the solution in the order $x_n, x_{n-1}, \dots x_2, x_1$.

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1 Solve the system of equations by Gauss elimination method

$6x_1 + 2x_2 + 8x_3 = 26$

$3x_1 + 5x_2 + 2x_3 = 8$

$8x_2 + 2x_3 = -7$

Soln: This system can be written as

$$\begin{bmatrix} 6 & 2 & 8 \\ 3 & 5 & 2 \\ 0 & 8 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 26 \\ 8 \\ -7 \end{bmatrix}$$

$$\therefore \tilde{A} = \left[\begin{array}{ccc:c} 6 & 2 & 8 & 26 \\ 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \end{array}\right]$$

Now $\tilde{A}$ should brought in to upper triangular form by row operations by keeping first row unchanged.

$$R_2 \to 2R_2 - R_1 \left[\begin{array}{ccc:c} 6 & 2 & 8 & 26 \\ 0 & 8 & -4 & -10 \\ 0 & 8 & 2 & -7 \end{array}\right]$$

Here all the entries below a_{11} is zero. Now we have to make all entries below a_{22} zero by keeping second row unchanged.

$R_3 \to R_3 - R_2$ $\left[\begin{array}{ccc:c} 6 & 2 & 8 & 26 \\ 0 & 8 & -4 & -10 \\ 0 & 0 & 6 & 3 \end{array}\right]$

So we get a triangular system. Now write the corresponding equations and do back substitution

$$6x_1 + 2x_2 + 8x_3 = 26 \longrightarrow (1)$$
$$8x_2 - 4x_3 = -10 \longrightarrow (2)$$
$$6x_3 = 3 \longrightarrow (3)$$

From (3) $x_3 = \frac{3}{6} = \frac{1}{2}$

Substitute in (2)

$8x_2 - 4 \times \frac{1}{2} = -10 \Rightarrow 8x_2 = -10 + 2 = -8$

$\Rightarrow x_2 = -1$

Substitute x_3 & x_2 in (1)

$6x_1 + 2 \times -1 + 8 \times \frac{1}{2} = 26$

$6x_1 = 26 + 2 - 4 = 24$

$x_1 = \frac{24}{6} = 4$

$\therefore$ Solution is $x_1 = 4$, $x_2 = -1$, $x_3 = \frac{1}{2}$

2. Solve the system

$$x_1 - x_2 + x_3 = 0$$
$$-x_1 + x_2 - x_3 = 0$$
$$10x_2 + 25x_3 = 90$$
$$2x_1 + 10x_2 = 80$$

by Gauss elimination method.

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Soln:

$$\tilde{A} = \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_4 \to R_4 - 20R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_4$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 95 & 190 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to 3R_3 - R_2$$

$x_1 - x_2 + x_3 = 0 \longrightarrow (1)$

$30x_2 - 20x_3 = 80 \longrightarrow (2)$

$95x_3 = 190 \longrightarrow (3)$

From (3) $x_3 = \dfrac{190}{95} = 2$

Substitute x_3 in ②

$30x_2 - 20\times 2 = 80$

$30x_2 = 80+40=120 \Rightarrow x_2 = \frac{120}{30} = 4$

Substitute x_2 & x_3 in ①

$x_1 - 4 + 2 = 0 \Rightarrow x_1 - 2 = 0 \Rightarrow x_1 = 2$

$\therefore$ Soln is $x_1 = 2$, $x_2 = 4$, $x_3 = 2$

Note: In Gauss elimination method, the following row operations are used

1) Interchange of any 2 rows
2) Addition of constant multiple of one row to another row
3) Multiplication of a row by a non-zero constant c

to get another system of equations called Row-equivalent system.

Theorem-1:

Row-equivalent linear systems have the same set of solutions.

* These operations are allowed only for rows. No column operations on ~~on~~ augmented matrix are permitted.
* A linear system is called overdetermined if it has more equations than unknowns.
* If m=n, then it is called determined.

* System is called underdetermined if it has fewer equations than unknowns (ie $m<n$)

Three possible ~~Cases of systems:~~

* In each step in Gauss elimination, the coefficient of first unknown in the first equation is called pivotal coefficient. If any of the pivotal coefficient becomes zero, we rewrite the equation in a different order to avoid zero pivotal coefficient. Changing order of equations is called pivoting.

Three possible cases of systems

Gauss elimination can take care of linear systems with a unique solution, with infinitely many solutions and without any solution.

Qn 3. Solve $y+z-2w=0$

$$2x-3y-3z+6w=2$$

$$4x+y+z-2w=4$$

Soln:

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 1 & -2 & : & 0 \\ 2 & -3 & -3 & 6 & : & 2 \\ 4 & 1 & 1 & -2 & : & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & -3 & 6 & : & 2 \\ 0 & 1 & 1 & -2 & : & 0 \\ 4 & 1 & 1 & -2 & : & 4 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\sim \left[\begin{array}{cccc:c} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 7 & 7 & 14 & 0 \end{array}\right] \quad R_3 \to R_3 - 2R_1$$

$$\sim \left[\begin{array}{cccc:c} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$2x - 3y - 3z + 6w = 2 \longrightarrow (1)$

$y + z - 2w = 0 \longrightarrow (2)$

So here we have 2 equations and 4 unknowns.

$\therefore$ Assign arbitrary values to 2 variables

Let $z = t_1$ and $w = t_2$

$\therefore (2) \Rightarrow y + t_1 - 2t_2 = 0$

$\Rightarrow y = -t_1 + 2t_2$

Substitute these in (1)

$2x - 3(-t_1 + 2t_2) - 3t_1 + 6t_2 = 2$

$2x + 3t_1 - 6t_2 - 3t_1 + 6t_2 = 2$

$\Rightarrow 2x = 2 \Rightarrow x = 1$

$\therefore$ Solution is $x = 1$, $y = -t_1 + 2t_2$, $z = t_1$, $w = t_2$ where t_1 and t_2 are any values

$\therefore$ This system has infinite solutions.

Qn 4. Solve $3x_1 + 2x_2 + x_3 = 3$

$2x_1 + x_2 + x_3 = 0$

$6x_1 + 2x_2 + 4x_3 = 6$

$$\tilde{A} = \left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

$$\begin{array}{l} R_2 \to 3R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \end{array} \left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 0 & -1 & 1 & -6 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$$R_3 \to R_3 - 2R_2 \left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 0 & -1 & 1 & -6 \\ 0 & 0 & 0 & 12 \end{array}\right]$$

$$3x_1 + 2x_2 + x_3 = 3 \longrightarrow (1)$$

$$-x_2 + x_3 = -6 \longrightarrow (2)$$

$$0 = 12 \longrightarrow (3)$$

The false statement (3) $0 = 12$ shows that the system has no solutions.

5. Using Gauss elimination, solve

$$y + 3z = 9$$

$$2x + 2y - z = 8$$

$$-x + 5z = 8$$

Soln:

$$\tilde{A} = \left[\begin{array}{ccc:c} 0 & 1 & 3 & 9 \\ 2 & 2 & -1 & 8 \\ -1 & 0 & 5 & 8 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc:c} 2 & 2 & -1 & 8 \\ 0 & 1 & 3 & 9 \\ -1 & 0 & 5 & 8 \end{array}\right]$$

$$R_3 \to 2R_3 + R_1 \left[\begin{array}{ccc:c} 2 & 2 & -1 & 8 \\ 0 & 1 & 3 & 9 \\ 0 & 2 & 9 & 24 \end{array}\right]$$

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$R_3 \to R_3 - 2R_2$ $\left[\begin{array}{ccc:c} 2 & 2 & -1 & 8 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 3 & 6 \end{array}\right]$

$\therefore 2x + 2y - z = 8 \longrightarrow (1)$

$y + 3z = 9 \longrightarrow (2)$

$3z = 6 \longrightarrow (3)$

$(3) \Rightarrow z = \frac{6}{3} = 2$

$(2) \Rightarrow y + 3 \times 2 = 9 \Rightarrow y = 9 - 6 = 3$

$(1) \Rightarrow 2x + 2 \times 3 - 2 = 8$

$\Rightarrow 2x = 4 \Rightarrow x = 2$

$\therefore$ Solution is $x = 2, y = 3, z = 2$, a unique solution.

H.W. Solve using Gauss elimination

(a) $2x_1 + x_2 + 2x_3 + x_4 = 6$

$6x_1 - 6x_2 + 6x_3 + 12x_4 = 36$

$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1$

$2x_1 + 2x_2 - x_3 + x_4 = 10$

(b) $x_1 + x_2 + x_3 = 4$

$2x_1 + 5x_2 - 2x_3 = 3$

(c) $x + 2y = 3$

$2x + 4y = 7$

(d) $4x + y = 4$

$5x - 3y + z = 2$

$-9x + 2y - z = 5$

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Row-echelon form

At the end of Gauss elimination, the form of the coefficient matrix, the augmented matrix and the system itself are called row echelon form. ie if (i) all non zero rows are above all rows in which every element is zero (if any)

(II) the leading non-zero element of a non-zero row is always strictly to the right of the leading non-zero element of row above it

(III) All entries in a column below a ~~no~~ leading non-zero entry are zeros.

Eg:

$$\begin{bmatrix} 1 & 2 & 5 & 0 \\ 0 & 3 & 6 & 8 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 5 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 2 & -5 & 6 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Qn 1. Reduce $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{bmatrix}$ to row echelon form.

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 14 & 12 \end{bmatrix}$$

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$$\xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank of a matrix :

The number of non-zero rows in the row echelon form of a matrix A is called rank of A denoted by $\rho(A)$ or rank(A).

Eg: In the above question rank(A) = 2.

Fundamental theorem of linear system

Consider the system of equations $AX=B$

(i) If rank([A:B]) $\neq$ rank(A), then the system is inconsistent. (ie no solution)

(ii) If rank [A : B] = rank (A) = no. of unknowns, then system is consistent and has a unique solution.

(iii) If rank [A : B] = $r(A)$ < no. of unknowns then the system is consistent and has infinitely many solution.

* Consider the homogeneous system $AX=0$. $X=0$ is always a solution, called trivial solution or null solution.

$\therefore$ For a homogeneous system $AX=0$,

(i) If rank(A) = no. of unknowns, then system has only trivial solution, $X=0$.

(ii) If rank(A) < no. of unknowns, then system has infinite no. of solutions.

Qn: Find the value of a and b for which the system: $x+y+2z=2$; $2x-y+3z=10$; $5x-y+az=b$, has (i) no solution (ii) unique solution (iii) Infinite no. of solutions.

Ans:
$$[A:B] = \begin{bmatrix} 1 & 1 & 2 & : & 2 \\ 2 & -1 & 3 & : & 10 \\ 5 & -1 & a & : & b \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 5R_1$
$$\begin{bmatrix} 1 & 1 & 2 & : & 2 \\ 0 & -3 & -1 & : & 6 \\ 0 & -6 & a-10 & : & b-10 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$
$$\begin{bmatrix} 1 & 1 & 2 & : & 2 \\ 0 & -3 & -1 & : & 6 \\ 0 & 0 & a-8 & : & b-22 \end{bmatrix}$$

(i) When $a-8=0$ and $b-22 \neq 0$, then rank$[A:B]=3$ and rank$[A]=2$
$\therefore$ no solution if $a=8$ and $b \neq 22$.

(ii) When $a \neq 8$ & b any value, then rank$[A:B] = 3 = $ rank$(A) =$ no. of unknowns

$\therefore$ a unique solution.

(iii) If $a = 8$ and $b = 22$,

rank $[A:B] = 2 =$ rank $[A] \leq$ no. of unknowns

$\therefore$ Infinitely many solution.

H.W. (1) For what values of λ for which the following set of equations have infinite no. of solutions

$3x_1 + x_2 - \lambda x_3 = 0$; $4x_1 - 2x_2 - 3x_3 = 0$;

$2\lambda x_1 + 4x_2 + \lambda x_3 = 0$.

(2) For what values of λ and μ, do the system of equations $x + y + z = 6$; $x + 2y + 3z = 10$; $x + 2y + \lambda z = \mu$ have
(i) no solution (ii) unique solution
(iii) more than one solution.

(3) Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & -4 & 2 \\ 3 & 6 & -3 \end{bmatrix}$$

(4) Solve $x + y + z = 0$; $x + 2y + 3z = 0$;
$x + 4y + 9z = 0$.

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Eigen values and eigen vectors

Let $A=[a_{ij}]$ be a given non-zero $n\times n$ square matrix and λ be any scalar such that $AX=\lambda X$. The problem of finding non-zero vector X and a scalar λ satisfying $AX=\lambda X$ is called eigen value ~~of~~ problem.

The value of λ for which $AX=\lambda X$ has a non-zero solution is called an eigen value or characteristic value of the matrix A.

The corresponding non zero solutions X of $AX=\lambda X$ is called eigen vector of A corresponding to λ.

The set of all eigen values of A is called spectrum of A.

The largest of the absolute values of the eigen values of A is called spectral radius of A.

Eigen values of A will be the roots of the characteristic equation $|A-\lambda I|=0$

Corresponding to each eigen value λ, a non-zero solution of $(A-\lambda I)X=0$ will be an eigen vector.

Theorem: The transpose A^T of a square matrix A has same eigen values as A.

1. Find the eigen values and the corresponding eigen vector of the matrix $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

Soln: The characteristic equation is

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$

$\Rightarrow (-5-\lambda)(-2-\lambda) - 2\times 2 = 0$

$\Rightarrow 10 + 5\lambda + 2\lambda + \lambda^2 - 4 = 0$

$\Rightarrow \lambda^2 + 7\lambda + 6 = 0$

$\Rightarrow (\lambda+6)(\lambda+1) = 0 \Rightarrow \lambda = -6, -1$

$\therefore$ eigen values are $\lambda = -6, -1$

Now to find eigen vectors

For $\lambda = -6$, $(A - \lambda I)X = 0$

$\Rightarrow (A + 6I)X = 0$

$\Rightarrow \begin{bmatrix} -5+6 & 2 \\ 2 & -2+6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow x_1 + 2x_2 = 0$

Here rank, $r = 1$ & no. of unknowns, $n = 2$

$\therefore r < n \Rightarrow$ This system has infinitely many solutions

Assign arbitrary value ~~of~~ to $n-r = 2-1 = 1$ variable.

Let $x_2 = a$

$\therefore x_1 + 2a = 0 \Rightarrow x_1 = -2a$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2a \\ a \end{bmatrix} = a\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\therefore$ Eigen vector corresponding to $\lambda = -6$ is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Now for $\lambda = -1$

$(A - \lambda I)X = 0 \Rightarrow (A+I)X = 0$

$$\Rightarrow \begin{bmatrix} -5+1 & 2 \\ 2 & -2+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \xrightarrow{R_2 \to 2R_2 + R_1} \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow -4x_1 + 2x_2 = 0$$

Assign arbitrary value to any one variable.

Let $x_1 = a$

$-4a + 2x_2 = 0 \Rightarrow 4a = 2x_2 \Rightarrow x_2 = 2a$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ 2a \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\therefore$ eigen vector for $\lambda = -1$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

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Multiple Eigen values.

Qn. Find the eigen values and eigen vectors of $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Soln:

$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

$(-2-\lambda)[(1-\lambda)(-\lambda) - 12] - 2[-2\lambda - 6]$
$-3[-4 + (1-\lambda)] = 0$

$\Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$\lambda = 5$ is a root

$\lambda = -3, -3$ are the other roots.

$\therefore$ eigen values are $5, -3, -3$

$$\begin{array}{r|rrrr} 5 & 1 & 1 & -21 & -45 \\ & & 5 & 30 & 45 \\ \hline & 1 & 6 & 9 & 0 \end{array}$$

$\lambda^2 + 6\lambda + 9 = 0$

$(\lambda+3)^2 = 0 \Rightarrow \lambda = -3$

When $\lambda = 5$

$[A-\lambda I]X = [A-5I]X = \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \xrightarrow[R_3 \to 7R_3 - R_1]{R_2 \to 7R_2 + 2R_1} \begin{bmatrix} -7 & 2 & -3 \\ 0 & -24 & -48 \\ 0 & -16 & -32 \end{bmatrix}$

$R_2 \to R_2/-24$, $R_3 \to R_3/-16$ $\begin{bmatrix} -7 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} -7 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow -7x_1 + 2x_2 - 3x_3 = 0$

$x_2 + 2x_3 = 0$

Let $x_3 = t$

$\Rightarrow x_2 + 2t = 0 \Rightarrow x_2 = -2t$

$-7x_1 + 2(-2t) - 3t = 0$

$\Rightarrow -7x_1 - 7t = 0 \Rightarrow x_1 = -t$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

$\therefore$ Eigen vector corresponding to $\lambda = 5$ is

$\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

Consider $\lambda = -3$

$[A - \lambda I]X = [A + 3I]X = 0 \Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow x_1 + 2x_2 - 3x_3 = 0$

Let $x_3 = t_1$ and $x_2 = t_2$

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$\Rightarrow x_1 + 2t_2 - 3t_1 = 0$

$\Rightarrow x_1 = 3t_1 - 2t_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3t_1 - 2t_2 \\ t_2 \\ t_1 \end{bmatrix} = \begin{bmatrix} 3t_1 - 2t_2 \\ 0t_1 + t_2 \\ t_1 + 0t_2 \end{bmatrix}$$

$$= t_1 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

So we get 2 linearly independent eigen vectors corresponding to $\lambda = -3$

They are $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Note:

① Since real polynomials may have complex roots, a real matrix may have complex eigen values and eigen vectors. For example consider

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0$

$\therefore$ eigen values are $\lambda_1 = i$ and $\lambda_2 = -i$

For $\lambda = i$

$[A - \lambda I] = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \xrightarrow{R_2 \to iR_2 - R_1} \begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix}$

$-ix_1 + x_2 = 0 \Rightarrow x_2 = ix_1$

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Let $x_1 = t \Rightarrow x_2 = it$

$\therefore$ eigen vector $\Rightarrow \begin{bmatrix} t \\ it \end{bmatrix} = t \begin{bmatrix} 1 \\ i \end{bmatrix}$

For $\lambda = -i$

$[A - \lambda I] = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \xrightarrow{R_2 \to iR_2 + R_1} \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow ix_1 + x_2 = 0 \Rightarrow x_2 = -ix_1$

Let $x_1 = t$, then $x_2 = -it$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ -it \end{bmatrix} = t \begin{bmatrix} 1 \\ -i \end{bmatrix}$

(2) If λ is an eigen value of A, then λ^2 is an eigen value of A^2 and $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

(3) Sum of the eigen values of A = sum of diagonal elements of A (ie trace(A))

Product of eigen values of A = det (A)

Qn: Find the eigen values of $A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ without using characteristic equation.

Ans: Sum of eigen values = trace(A)

ie, $\lambda_1 + \lambda_2 = 2 + 2 = 4 \longrightarrow (1)$

Product of eigen values = $|A|$

ie $\lambda_1 \times \lambda_2 = 4 - 16 = -12 \longrightarrow (2)$

We know, $(\lambda_1 + \lambda_2)^2 = (\lambda_1 - \lambda_2)^2 + 4\lambda_1\lambda_2$

$\Rightarrow 4^2 = (\lambda_1 - \lambda_2)^2 + 4 \times -12$

$\Rightarrow (\lambda_1 - \lambda_2)^2 = 16 + 48 = 64$

$\Rightarrow (\lambda_1 - \lambda_2) = 8 \longrightarrow (3)$

$(1) + (3) \Rightarrow 2\lambda_1 = 12 \Rightarrow \lambda_1 = 6$

$(1) \Rightarrow 6 + \lambda_2 = 4 \Rightarrow \lambda_2 = -2$

$\therefore$ eigen values are 6 and -2

2. If two eigen values of $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ are 2 and 8, find the determinant of the matrix.

Ans: $\lambda_1 + \lambda_2 + \lambda_3 =$ trace of matrix

$\cancel{\lambda_1 \Rightarrow}\ 2 + 8 + \lambda_3 = 6 + 3 + 3$

$10 + \lambda_3 = 12 \Rightarrow \lambda_3 = 2$

$\therefore$ eigen values are 2, 2, 8

$\therefore$ determinant = Product of eigen values

$\Rightarrow |A| = 2 \times 2 \times 8 = 32$

H.W Find eigen values and eigen vectors

(a) $\begin{bmatrix} -3 & 0 \\ 5 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 5 & 1 & -1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Symmetric, Skew Symmetric and Orthogonal matrices:

* A real square matrix $A = [a_{jk}]$ is called (1) symmetric if $A^T = A \Rightarrow a_{jk} = a_{kj}$

eg: $\begin{bmatrix} 2 & 3 & 8 \\ 3 & -1 & -7 \\ 8 & -7 & 4 \end{bmatrix}$

(2) Skew symmetric if $A^T = -A \Rightarrow a_{jk} = -a_{kj}$

eg: $\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$

(3) Orthogonal if $A^T = A^{-1}$ (ie $AA^T = I$)

eg: $A = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

$$AA^T = \begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Any real square matrix A may be written as the sum of a symmetric matrix R and a skew symmetric matrix S where $R = \frac{1}{2}(A + A^T)$ and $S = \frac{1}{2}(A - A^T)$

Orthogonal transformation

Orthogonal transformations are transformation $Y = AX$ where A is an orthogonal matrix.

Eg: $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Here $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\therefore AA^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\therefore Y = AX$ is an orthogonal transformation.

Similar matrices

An $n \times n$ matrix $\hat{A}$ is called similar to an $n \times n$ matrix A, if $\hat{A} = P^{-1}AP$ for some $n \times n$ matrix P.

If $\hat{A}$ is similar to A, then $\hat{A}$ has the same eigen values as A.

If x is an eigen vector of A, then $Y = P^{-1}x$ is an eigen vector of $\hat{A}$ corresponding to same eigen value.

Qn 1. For $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ and $P = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$, verify that A and $\hat{A} = P^{-1}AP$ have equal eigen values. Also if Y is an eigen vector of $\hat{A}$, show that $X = PY$ is an eigen vector of A.

Soln: $|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0$

$\Rightarrow (3-\lambda)(-3-\lambda) - 16 = 0$

$\Rightarrow -(9-\lambda^2) - 16 = 0 \Rightarrow \lambda^2 - 25 = 0$

$\Rightarrow \lambda = \pm 5$

$\hat{A} = P^{-1}AP$

$P^{-1} = \begin{bmatrix} \frac{1}{2} & 1 \\ 3/2 & 2 \end{bmatrix}$

$P^{-1} = \frac{\text{adj} P}{|P|}$

$|P| = 4 - 6 = -2$

$\text{adj } P = \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}$

$\hat{A} = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$

$= \begin{bmatrix} -25 & 12 \\ -50 & 25 \end{bmatrix}$

$|\hat{A} - \lambda I| = 0 \Rightarrow \begin{vmatrix} -25-\lambda & 12 \\ -50 & 25-\lambda \end{vmatrix} = 0$

$\Rightarrow (-25-\lambda)(25-\lambda) + 600 = 0$

$\Rightarrow \lambda^2 - 625 + 600 = 0 \Rightarrow \lambda^2 - 25 = 0$

$\Rightarrow \lambda = \pm 5$

$\therefore$ Eigen values of A and $\hat{A}$ are same.

Consider $\lambda = 5$

$[A - \lambda I] = \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \quad R_2 \to R_2 + 2R_1 \begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow -2x_1+4x_2=0$

Let $x_2=t \Rightarrow -2x_1+4t=0$

$2x_1=4t \Rightarrow x_1=2t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\therefore$ Eigen vector, $X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$[\hat{A}-\lambda I] = \begin{bmatrix} -30 & 12 \\ -50 & 20 \end{bmatrix} \xrightarrow{R_2 \to 3R_2 - 5R_1} \begin{bmatrix} -30 & 12 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow -30x_1+12x_2=0$

Let $x_2=t \Rightarrow -30x_1=-12t$

$\Rightarrow x_1=\frac{12t}{30}=\frac{2}{5}t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{5}t \\ t \end{bmatrix} = \frac{t}{5}\begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$\therefore$ eigen vector, $Y = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$$PY = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = X$$

For $\lambda=-5$

$$[A-\lambda I] = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \xrightarrow{R_2 \to 2R_2 - R_1} \begin{bmatrix} 8 & 4 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow 8x_1+4x_2=0$

Let $x_2=t \Rightarrow 8x_1=-4t \Rightarrow x_1=-t/2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t/2 \\ t \end{bmatrix} = -\frac{t}{2}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$\therefore X = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$[\hat{A} - 5I] = \begin{bmatrix} -20 & 12 \\ -50 & 30 \end{bmatrix} \xrightarrow{R_2 \to 2R_2 - 5R_1} \begin{bmatrix} -20 & 12 \\ 0 & 0 \end{bmatrix}$$

$-20x_1 + 12x_2 = 0$

Let $x_2 = t \Rightarrow 20x_1 = 12t \Rightarrow x_1 = \frac{12t}{20} = \frac{3t}{5}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3t}{5} \\ t \end{bmatrix} = \frac{t}{5}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$\therefore$ Eigen vector of $\hat{A}$, $Y = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$$PY = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} = -2\underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{X}$$

$\therefore PY = X$

H.W Verify the same result for $A = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$

Diagonalisation of a matrix

If an $n \times n$ matrix A has a basis of eigen ~~val~~ vectors, then $D = X^{-1}AX$ is a diagonal matrix with eigen values of A as the entries on the main diagonal. Here X is the matrix with these eigen vectors as column vectors.

Also $D^m = X^{-1}A^mX$

Steps for diagonalisation

1. Determine the eigen values
2. Determine corresponding eigen vectors

③ Write these eigen vectors as columns of X

④ Find X^{-1}

⑤ Calculate $D = X^{-1}AX$, which will be a diagonal matrix with eigen ~~diagonal~~ values ~~entries~~ as the diagonal entries.

Diagonalisation can be used to find the powers of a matrix A.

$$D^m = X^{-1}A^mX$$

$$\Rightarrow XD^m = XX^{-1}A^mX = IA^mX = A^mX$$

$$\Rightarrow XD^mX^{-1} = A^mXX^{-1} = A^mI = A^m$$

$$\therefore A^m = XD^mX^{-1}$$

Qn Diagonalise the matrix $A = \begin{bmatrix} 5 & -3 \\ -6 & 2 \end{bmatrix}$ and hence find A^4.

Soln: $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & -3 \\ -6 & 2-\lambda \end{vmatrix} = 0$

$$\Rightarrow (5-\lambda)(2-\lambda) - 18 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda - 8 = 0 \Rightarrow (\lambda-8)(\lambda+1) = 0$$

$$\Rightarrow \lambda = -1, 8$$

For $\lambda = -1$

$$[A - \lambda I] = \begin{bmatrix} 6 & -3 \\ -6 & 3 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} 6 & -3 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow 6x_1 - 3x_2 = 0$$

Let $x_2 = t \Rightarrow 6x_1 = 3t \Rightarrow x_1 = \frac{t}{2}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t/2 \\ t \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\therefore$ eigen vector for $\lambda = -1$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For $\lambda = 8$

$$[A - \lambda I] = [A - 8I] = \begin{bmatrix} -3 & -3 \\ -6 & -6 \end{bmatrix} \longrightarrow \begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow -3x_1 - 3x_2 = 0$

Let $x_2 = t \Rightarrow 3x_1 = -3t \Rightarrow x_1 = -t$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = -t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\therefore$ eigen vector for $\lambda = 8$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\therefore X = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \qquad |X| = -1 - 2 = -3$$

$$\therefore X^{-1} = \frac{\text{adj } X}{|X|} = \frac{1}{-3} \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$\therefore D = X^{-1} A X = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -6 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 8 \\ -2 & -8 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -3 & 0 \\ 0 & 24 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix}$$

$$A^4 = X D^4 X^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix}^4 \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} (-1)^4 & 0 \\ 0 & 8^4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 4096\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix}1 & 4096\\ 2 & -4096\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix}8193 & -4095\\ -8190 & 4098\end{bmatrix} = \begin{bmatrix}2731 & -1365\\ -2730 & 1366\end{bmatrix}$$

2. Diagonalise the matrix, $A = \begin{bmatrix}0 & 0 & 2\\ 0 & 3 & 2\\ 1 & 0 & 1\end{bmatrix}$

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix}-\lambda & 0 & 2\\ 0 & 3-\lambda & 2\\ 1 & 0 & 1-\lambda\end{vmatrix} = 0$$

$$-\lambda\left[(3-\lambda)(1-\lambda)-0\right] - 0 + 2\left[0-(3-\lambda)\right] = 0$$

$$-\lambda\left[3-4\lambda+\lambda^2\right] - 6 + 2\lambda = 0$$

$$-\lambda^3 + 4\lambda^2 - \lambda - 6 = 0$$

$\lambda = -1$ is a root

$$\begin{array}{r|rrrr} -1 & -1 & 4 & -1 & -6\\ & & 1 & -5 & 6\\ \hline & -1 & 5 & -6 & 0 \end{array}$$

$$-\lambda^2 + 5\lambda - 6 = 0$$

$$-(\lambda-3)(\lambda-2) = 0$$

$$-(\lambda+1)(\lambda-3)(\lambda-2) = 0$$

$$\lambda = -1, 2, 3.$$

Consider $\lambda = -1$

$$[A-\lambda I] = \begin{bmatrix}1 & 0 & 2\\ 0 & 4 & 2\\ 1 & 0 & 2\end{bmatrix} \rightarrow \begin{bmatrix}1 & 0 & 2\\ 0 & 4 & 2\\ 0 & 0 & 0\end{bmatrix} \quad R_3 \rightarrow R_3 - R_1$$

$\Rightarrow x_1 + 2x_3 = 0 \ \& \ 4x_2 + 2x_3 = 0$

Let $x_2 = t \Rightarrow 2x_3 = -4t \Rightarrow x_3 = -2t$

$x_1 = -2x_3 = -2(-2t) = 4t$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4t \\ t \\ -2t \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$

$\therefore$ eigen vector for $\lambda = -1$ is $\begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$

For $\lambda = 2$

$[A - \lambda I] = \begin{bmatrix} -2 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \ 2R_3 + R_1$

$-2x_1 + 2x_3 = 0$

$x_2 + 2x_3 = 0$

Let $x_1 = t \Rightarrow 2x_3 = 2t \Rightarrow x_3 = t$

$x_2 = -2t$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$\therefore$ eigen vector for $\lambda = 2$ is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

For $\lambda = 3$

$[A - \lambda I] = \begin{bmatrix} -3 & 0 & 2 \\ 0 & 0 & 2 \\ 1 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} -3 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -4 \end{bmatrix}$

$\rightarrow \begin{bmatrix} -3 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$-3x_1 + 2x_3 = 0$

$2x_3 = 0 \Rightarrow x_3 = 0$

$\therefore x_1 = 0$

x_2 can be any non zero number, t

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ eigen vector for $\lambda = 3$ is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$\therefore X = \begin{bmatrix} 0 & 4 & 1 \\ 1 & 1 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

$$|X| = 0 - 4(1-0) + 1(-2-0) = -6$$

$$X^{-1} = \frac{\text{adj } X}{|X|} = \frac{1}{-6} \begin{bmatrix} -3 & -6 & -9 \\ -1 & 0 & 1 \\ -2 & 0 & -4 \end{bmatrix}$$

$$\therefore D = X^{-1} A X = -\frac{1}{6} \begin{bmatrix} -3 & -6 & -9 \\ -1 & 0 & 1 \\ -2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 4 & 1 \\ 1 & 1 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= -\frac{1}{6} \begin{bmatrix} -9 & -18 & -27 \\ 1 & 0 & -1 \\ -4 & 0 & 8 \end{bmatrix} \begin{bmatrix} 0 & 4 & 1 \\ 1 & 1 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= -\frac{1}{6} \begin{bmatrix} -18 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -12 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} //$$

H.W ① Diagonalise $A = \begin{bmatrix} 10 & 3 \\ 4 & 6 \end{bmatrix}$. Also find A^3.

② Diagonalise $A = \begin{bmatrix} -1 & 2 & -2 \\ 2 & 4 & 1 \\ 2 & 1 & 4 \end{bmatrix}$

Quadratic Forms

A quadratic form Q in the components $x_1, x_2, \dots x_n$ of a vector X is a sum of n^2 terms, namely $Q = X^T A X = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} x_j x_k$

For example

i) Two components, $Q = ax_1^2 + 2hx_1x_2 + bx_2^2$

ii) Three components, $Q = ax_1^2 + bx_2^2 + cx_3^2 + 2hx_1x_2 + 2gx_2x_3 + 2fx_1x_3$.

~~Every~~

i) $Q = ax_1^2 + 2hx_1x_2 + bx_2^2$

$$= [x_1, x_2] \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = X^T A X$$

ii) $Q = ax_1^2 + bx_2^2 + cx_3^2 + 2hx_1x_2 + 2gx_2x_3 + 2fx_1x_3$

$$= [x_1, x_2, x_3] \begin{bmatrix} a & h & f \\ h & b & g \\ f & g & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = X^T A X$$

Qn 1. Give the matrix form associated with the quadratic form $6x_1^2 + 15x_2^2 + 3x_3^2 - 4x_1x_2 + 14x_2x_3 - 7x_1x_3$

Soln: $Q = [x_1\ x_2\ x_3]\begin{bmatrix} 6 & -2 & -7/2 \\ -2 & 15 & 7 \\ -7/2 & 7 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Canonical form

We can transform a quadratic form to canonical form or principal axes form $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$, where $\lambda_1, \lambda_2, \dots \lambda_n$ are the eigen values of the symmetric matrix A.

Qn: What kind of conic section (or pair of straight lines) is given by the following quadratic form? Transform it to Principal axes. Express $x^T = [x_1, x_2]$ in terms of new coordinate vector $Y^T = [y_1, y_2]$.

a) $17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$

Soln: $Q = 17x_1^2 - 30x_1x_2 + 17x_2^2$

$= [x_1, x_2]\begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = X^T A X$

$A = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 17-\lambda & -15 \\ -15 & 17-\lambda \end{vmatrix} = 0$

$\Rightarrow (17-\lambda)^2 - 225 = 0$

$$\Rightarrow (17-\lambda)^2 = 225$$

$$\Rightarrow 17-\lambda = \pm 15$$

$$\Rightarrow \lambda = 2 \text{ or } 32.$$

$\therefore \lambda_1 = 2 \quad \lambda_2 = 32.$

$\therefore$ Canonical form $2y_1^2 + 32y_2^2 = 128$

$$\frac{2y_1^2}{128} + \frac{32y_2^2}{128} = 1$$

$$\Rightarrow \frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1,$$ which represents an ellipse.

Now to express X^T in terms of Y^T, we have to find the eigen vector corresponding to each eigen values.

For $\lambda = 2$

$$(A-\lambda I) = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix} \longrightarrow \begin{bmatrix} 15 & -15 \\ 0 & 0 \end{bmatrix} \; R_2 \to R_2 + R_1$$

$$\Rightarrow 15x_1 - 15x_2 = 0$$

$$\Rightarrow x_1 = x_2 = t \text{ (say)}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\therefore$ eigen vector for $\lambda = 2$ is, $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore$ Normalised eigen vector of $X_1 = \frac{X_1}{\|X_1\|}$

$$\|X_1\| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\therefore \frac{X_1}{\|X_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

For $\lambda = 32$

$$[A - 32I] = \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix} \longrightarrow \begin{bmatrix} -15 & -15 \\ 0 & 0 \end{bmatrix} \quad R_2 \mp R_1$$

$\therefore -15x_1 - 15x_2 = 0$

$\Rightarrow x_1 = -x_2$

Let $x_2 = t \Rightarrow x_1 = -t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\therefore$ eigen vector for $\lambda = 32$ is $X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\|X_2\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\therefore$ Normalised eigen vector $\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Now define $X = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$x = Xy = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{y_1}{\sqrt{2}} - \frac{y_2}{\sqrt{2}} \\ \frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{2}} \end{bmatrix}$$

$\therefore x_1 = \frac{y_1 - y_2}{\sqrt{2}}$ and $x_2 = \frac{y_1 + y_2}{\sqrt{2}}$

b $x_1^2 - 12x_1x_2 + x_2^2 = 70$

$$Q = [x_1, x_2] \begin{bmatrix} 1 & -6 \\ -6 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -6 \\ -6 & 1 \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -6 \\ -6 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)^2 - 36 = 0 \Rightarrow 1-\lambda = \pm 6$

$\lambda = 7, -5$

$\therefore$ Canonical form $7y_1^2 - 5y_2^2 = 70$

$$\frac{7y_1^2}{70} - \frac{5y_2^2}{70} = 1$$

$$\Rightarrow \frac{y_1^2}{10} - \frac{y_2^2}{14} = 1,$$ which represents a hyperbola.

For $\lambda = 7$

$$[A-\lambda I] = \begin{bmatrix} -6 & -6 \\ -6 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} -6 & -6 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow -6x_1 = 6x_2$

Let $x_2 = t \Rightarrow x_1 = -t \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\therefore$ normalised eigenvector $= \cancel{\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

For $\lambda = -5$

$$[A-\lambda I] = \begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 6 & -6 \\ 0 & 0 \end{bmatrix}$$

$6x_1 - 6x_2 = 0 \Rightarrow x_1 = x_2 = t \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore$ normalised eigenvector $= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$\therefore X = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\therefore x = XY = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -\frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{2}} \\ \frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{2}} \end{bmatrix}$$

$\therefore\ x_1 = \frac{-y_1+y_2}{\sqrt{2}}, \quad x_2 = \frac{y_1+y_2}{\sqrt{2}}$

c) $9x_1^2 + 6x_1x_2 + x_2^2 = 10$

$Q = [x_1\ x_2]\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$A = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$

$|A - \lambda I| = \begin{vmatrix} 9-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (9-\lambda)(1-\lambda) - 9 = 0$

$\Rightarrow 9 - 10\lambda + \lambda^2 - 9 = 0 \Rightarrow \lambda^2 - 10\lambda = 0$

$\Rightarrow \lambda(\lambda - 10) = 0 \Rightarrow \lambda = 0, 10$

$\therefore$ Canonical form, $0y_1^2 + 10y_2^2 = 10$

$\Rightarrow y_2^2 = 1 \Rightarrow y_2 = \pm 1$, which represent a pair of straight line.

For $\lambda = 0$

$[A - \lambda I] = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 9 & 3 \\ 0 & 0 \end{bmatrix}$

$9x_1 + 3x_2 = 0 \Rightarrow x_2 = -3x_1$

Let $x_1 = t \Rightarrow x_2 = -3t \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$\therefore$ normalised eigen vector $= \begin{bmatrix} \frac{1}{\sqrt{10}} \\ -3/\sqrt{10} \end{bmatrix}$

For $\lambda = 10$

$[A - \lambda I] = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow -x_1 + 3x_2 = 0 \Rightarrow x_1 = 3x_2$$

Let $x_2 = t \Rightarrow x_1 = 3t \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\therefore$ normalised eigen vector $\begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$

$$\therefore X = \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$$

$$\therefore x = Xy = \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{y + 3y_2}{\sqrt{10}} \\ \frac{-3y_1 + y_2}{\sqrt{10}} \end{bmatrix}$$

H.W Transform the Quadratic form to canonical form. What kind of conic section does it represent?

1. $3x_1^2 + 22x_1x_2 + 3x_2^2 = 0$
2. $7x_1^2 + 6x_1x_2 + 7x_2^2 = 200$
3. $4x_1^2 + 6x_1x_2 - 4x_2^2 = 10$.