MODULE · IV

SEQUENCES AND SERIES

Topics

Convergence of Sequence and Series, Convergence of geometric Series, and P- Series (without proof) test of Convergence (Comparison, ratio and root tests without proof) Alternating Series and Leibnitz test, absolute and conditional Convergence.

Sequence

An ordered set of real numbers $a_1, a_2, a_3 \dots a_n$ is called a sequence and is denoted by (a_n). If the number of terms is uncountable then the sequence is said to be an infinite Series. and a_n is its general term.

Eg: $1, 3, 5, 7 \dots (2n-1) \dots$
$1, 1/2, 1/3 \dots 1/n$

Limit

A Sequence is said to tend to a limit l, if for every $\varepsilon > 0$, a value N of n can be found such that $|a_n - l| < \varepsilon$ for $n \geq N$.

ie $\lim_{n \to \infty} a_n = l$.

Convergence

If a sequence has finite limit it is called a convergente sequence. otherwise it is called divergent.

Problems

1. Determine the converge or diverge of the sequence $\left\{\frac{n}{2n+1}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(2+1/n)}$$

$$= \frac{1}{2+\frac{1}{\infty}} = \underline{\underline{1/2}} \qquad \left[\tfrac{1}{\infty}=0\right]$$

Sequence converges to $1/2$.

2. Determine the convergence or divergence of the sequence $\left\{\frac{(n-1)(n+2)}{2n^2}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(n-1)(n+2)}{2n^2}$$

$$= \lim_{n\to\infty} \frac{n(1-1/n)\,n\,(1+2/n)}{2n^2}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}(1-1/n)(1+2/n)}{2\cancel{n^2}}$$

$$= \frac{\left(1-\frac{1}{\infty}\right)\left(1+\frac{2}{\infty}\right)}{2} = \frac{1}{2} \qquad \left[\tfrac{1}{\infty}=0\right]$$

3. Determine the convergence or divergence of the sequence $\left\{\left(\frac{n+3}{n+1}\right)^n\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[\frac{n+3}{n+1}\right]^n$$

$$= \lim_{n\to\infty}\left[\frac{\cancel{n}(1+3/n)}{\cancel{n}(1+1/n)}\right]^n$$

$$= \lim_{n\to\infty} \frac{e^3}{e}$$

$$\boxed{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x}$$

$$= \underline{\underline{e^2}}$$

$\therefore$ The sequence converges to e^2.

4. Discuss the nature of the sequence

$2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4} \dots$

$2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4} \dots \frac{n+1}{n} \dots$

$a_n = \frac{n+1}{n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(1+1/n)}{\cancel{n}}$$

$$= \underline{\underline{1}}$$

Infinite Series

An infinite series is an ordered formal sum of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

of real numbers having infinite number of terms.

The numbers $a_1, a_2 \cdots$ are called terms of the series and 'a_n' is called nth term.

Sequence of partial sum

Consider the series $\sum_{n=1}^{\infty} a_n$.

Put $S_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$

The number S_n is called the nth partial sum of the infinite series and the sequence $\{S_n\}$ is called the sequence of partial sums of infinite series.

Convergent Series

Let $\{S_n\}$ be the sequence of

Partial sum of infinite series $\sum_{n=1}^{\infty} a_n$.
If $\{S_n\}$ converges to a limit S, then infinite series is said to converge to S and S is called the sum of infinite series and write $S = \sum_{n=1}^{\infty} a_n$.
If $\{S_n\}$ diverges, then the infinite series is said to diverge. A divergent series has no sum.

Standard limits

1. $\lim_{n \to \infty} \frac{\log n}{n} = 0$

2. $\lim_{n \to \infty} \left(1+\frac{x}{n}\right)^n = e^x$

3. $\lim_{n \to \infty} n^{1/n} = 1$

4. $\lim_{n \to \infty} \frac{x^n}{n} = 0$

Geometric Series

A series of the form

$a + ar + ar^2 + \cdots + ar^n + \cdots$, where a is a non zero real number is called geometric

Series with first term 'a' and common ratio 'r'.

Convergence of Geometric Series

A geometric series $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots$ converges if $|r| < 1$ and diverges if $|r| \geq 1$.

If the series converges then the sum is $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

Problems

1. Determine whether the series converges, and if so find its sum $\sum_{k=0}^{\infty} \frac{5}{4^k}$.

$$\sum_{k=0}^{\infty} \frac{5}{4^k} = 5 + \frac{5}{4} + \frac{5}{4^2} + \frac{5}{4^3} + \cdots$$

$a = 5 \quad |r| = \frac{1}{4} < 1 \Rightarrow$ the series converges.

$$\text{Sum } S_n = \frac{a}{1-r} = \frac{5}{1-\frac{1}{4}} = \frac{5}{\frac{4-1}{4}} = \frac{5}{\frac{3}{4}}$$

$$= \frac{20}{3}$$

2. Show that the series $\sum_{n=1}^{\infty} (1/2)^n$ converges

$$\sum_{n=1}^{\infty} (1/2)^n = 1/2 + 1/2^2 + 1/2^3 + \cdots$$

$a = 1/2 \quad |r| = 1/2 < 1 \implies$ Series converges

3. Determine whether the series $\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^{k+2}$ converges and if so, find its sum.

$$\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^{k+2} = \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + \left(\frac{3}{4}\right)^6 + \cdots$$

$$= \left(\frac{3}{4}\right)^3 \left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots\right]$$

Geometric series with $a = \left(\frac{3}{4}\right)^3$ and

$|r| = \frac{3}{4} < 1 \implies$ Series converges

Sum $\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^{k+2} = \frac{a}{1-r}$

$$= \frac{(3/4)^3}{1 - \frac{3}{4}} = \frac{3^3}{4^{\cancel{3}2} \times \frac{1}{\cancel{4}}}$$

$$= \frac{27}{16}$$

4. Find the sum of the series

$0.1 + 0.01 + 0.001 + 0.0001 + \cdots$

$0.1 + 0.01 + 0.001 + 0.0001 + \cdots = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots$

This is a Geometric Series

first term $a = \frac{1}{10}$ and $|r| = \frac{1}{10} < 1$

$$\text{Sum} = \frac{a}{1-r} = \frac{\frac{1}{10}}{1-\frac{1}{10}} = \frac{1}{\cancel{10}\left(\frac{9}{\cancel{10}}\right)} = \frac{1}{9}$$

5. Determine the rational number representing the decimal number 0.764764764 ....

$$0.764764764\ldots = \frac{764}{10^3} + \frac{764}{10^6} + \frac{764}{10^9} + \cdots$$

$$= \frac{764}{10^3}\left[1 + \frac{1}{10^3} + \frac{1}{10^6} + \frac{1}{10^9} + \cdots\right]$$

$$= \frac{\frac{764}{10^3}}{1-\frac{1}{10^3}} = \frac{\frac{764}{\cancel{10^3}}}{\frac{10^3-1}{\cancel{10^3}}}$$

$a = \frac{764}{10^3}$

$r = \frac{1}{10^3}$

$$= \frac{764}{999}$$

6. Find the rational representation of the recurring decimal number 0.412412412 ... using series.

$$0.412412412\ldots = \frac{412}{10^3} + \frac{412}{10^6} + \frac{412}{10^9} + \cdots$$

$$= \frac{412}{10^3}\left[1 + \frac{1}{10^3} + \frac{1}{10^6} + \cdots\right]$$

$$= \frac{\frac{412}{10^3}}{1 - 1/10^3} = \frac{\frac{412}{\cancel{10^3}}}{\frac{999}{\cancel{10^3}}} = \frac{412}{999}$$

P- Series or Hyper Harmonic Series

A P- Series is an infinite series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} + \dots$$

where $p > 0$.

Convergence of P- Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} + \dots$$ ~~Converge~~

Converges if $p > 1$ and diverges if $0 < p \leq 1$.

Problems

1. Examine the convergence of the series

$$1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \dots + \frac{1}{\sqrt[3]{k}} + \dots$$

$$1 + \frac{1}{(2)^{1/3}} + \frac{1}{(3)^{1/3}} + \dots + \frac{1}{(k)^{1/3}} + \dots$$

This is a P- Series with $p = 1/3 < 1$

$\therefore$ the given series is diverge.

2. Test the convergence of the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$$

This is a p series with $p = 1/2 < 1$.

$\Longrightarrow$ the given series diverge

3. Test the convergence of the series $\sum_{k=1}^{\infty} k^{-2/3}$.

$$\sum_{k=1}^{\infty} k^{-2/3} = \sum_{k=1}^{\infty} \frac{1}{k^{2/3}}$$

$\Longrightarrow$ p- series with $p = 2/3 < 1$

$\Longrightarrow$ diverges

4 Test the convergence of the series $\sum_{k=1}^{\infty} \frac{1}{k^4}$

$\sum_{k=1}^{\infty} \frac{1}{k^4} \Longrightarrow$ p-series with $p = 4 > 1$

$\Longrightarrow$ the series converges

Convergence Test

I Comparison Test.

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with non negative terms and suppose that $a_n \le b_n$ for $n = 1, 2, 3 \dots$

1) If the bigger series $\sum b_n$ converges then the smaller series $\sum a_n$ also converge

2) If the smaller series $\sum a_n$ diverges then the bigger series $\sum b_n$ also diverges.

Problems

1. Examine the convergence $\sum_{k=1}^{\infty} \frac{1}{2k^2+k}$

Consider $\sum_{k=1}^{\infty} \frac{1}{2k^2} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2}$

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent p-series $[p = 2 > 1]$

$\therefore \frac{1}{2} \sum \frac{1}{k^2}$ also convergent.

$\sum \frac{1}{2k^2+k} < \sum \frac{1}{2k^2}$

$\Rightarrow$ Bigger series convergent $\Rightarrow$ smaller serie convergent.

$\therefore \sum \frac{1}{2k^2+k}$ is convergent

2. Use Comparison test to determine whether the Series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}-\frac{1}{2}}$ Convergent or divergent.

$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ is divergent p Series $[P = 1/2]$

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}-\frac{1}{2}} > \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

$\Rightarrow$ The Smaller Series is divergent.

$\therefore$ By comparison test the bigger Series also ~~convergent~~ divergent

$\therefore \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}-1/2}$ ~~con~~ divergent.

0.08
0.2

3. Examine the Convergence of $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$.

Consider $\sum_{n=1}^{\infty} \frac{1}{n^2} = \cancel{\sum_{n=1}^{\infty} \frac{1}{n^2}}$

$\sum \frac{1}{n^2}$ is a P Series with $P=2$

$\Rightarrow \sum \frac{1}{n^2}$ Convergent

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} > \sum \frac{\cos n}{n^2}$

Bigger Series converges $\Rightarrow$ Smaller Series converges

4. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{5n-1}$ converges or not.

Consider $\sum_{n=1}^{\infty} \frac{1}{5n} = \frac{1}{5} \sum_{n=1}^{\infty} \frac{1}{n}$

$\Rightarrow$ p - series with $p = 1$

$\Rightarrow$ divergent

$\sum \frac{1}{5n} < \sum \frac{1}{5n-1}$

Smaller series divergent

$\Rightarrow$ the bigger series also divergent

The limit Comparison test

Let $\sum a_n$ and $\sum b_n$ be Series with positive terms and suppose that

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \text{finite quantity} > 0,$$

then the Series both converge or both diverge.

Choose $b_n = \frac{1}{n^P}$ where

P = degree of Denominator of a_n − degree of Numerator of a_n

Problems

1. Examine the convergence of $\sum_{k=1}^{\infty} \frac{3k^3 - 2k^2 + 4}{k^7 - k^3 + 2}$

$$a_k = \frac{3k^3 - 2k^2 + 4}{k^7 - k^3 + 2}$$

$$b_k = \frac{1}{k^P}$$ [P = degree Dr − degree Nr = 7 − 3 = 4]

$$= \frac{1}{k^4} \implies \sum b_k \text{ is a convergent p-series } [P = 4 > 1]$$

$$\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{\frac{3k^3 - 2k^2 + 4}{k^7 - k^3 + 2}}{\frac{1}{k^4}}$$

$$= \lim_{k\to\infty} \frac{3k^3-2k^2+4}{k^7-k^3+2} \times \frac{k^4}{1}$$

$$= \lim_{k\to\infty} \frac{\cancel{k^3}\left(3-\frac{2}{k}+\frac{4}{k^3}\right) k^4}{\cancel{k^7}\left(1-\frac{1}{k^4}+\frac{2}{k^7}\right)}$$

$$= \frac{3-\frac{2}{\infty}+\frac{4}{\infty}}{1-\frac{1}{\infty}+\frac{2}{\infty}} = 3 > 0 \text{ , finite quantity}$$

$\sum b_k$ is convergent $\Longrightarrow$ $\sum a_k$ also convergent

2. Test the Convergence of the Series

$$\sum_{k=1}^{\infty} \frac{k(k+3)}{(k+1)(k+2)(k+5)}$$

Let $a_k = \dfrac{k(k+3)}{(k+1)(k+2)(k+5)}$

Choose $b_k = \dfrac{1}{k^p}$ $\quad$ [$p =$ ~~Nr~~ degree Dr $-$ degree Nr $= 3-2 = 1$]

$= \dfrac{1}{k} \Rightarrow \sum b_k$ is a divergent p-series.

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k(k+3)}{(k+1)(k+2)(k+5)} \times \frac{k}{1}$$

$$= \lim_{k\to\infty} \frac{\cancel{k^2}\, k\,(1+3/k)}{\cancel{k}(1+1/k)\cancel{k}(1+2/k)\cancel{k}(1+5/k)}$$

$$= \frac{1+3/\infty}{(1+1/\infty)(1+2/\infty)(1+5/\infty)} = 1 > 0$$

$\sum b_k$ is a divergent p-series

$\Rightarrow \sum a_k$ also divergent

3. Test the convergence of the series
$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{2k-1}}$$

$$a_k = \frac{1}{\sqrt[3]{2k-1}} = \frac{1}{(2k-1)^{1/3}}$$

Choose $b_k = \frac{1}{k^p}$ $\left[\begin{array}{l} p = \text{degree Dr} - \text{degree Nr} \\ = \frac{1}{3} - 0 = 1/3 < 1 \end{array}\right]$

$$= \frac{1}{k^{1/3}}$$

$\sum b_k = \sum \frac{1}{k^{1/3}}$ p series with $p = 1/3 < 1$

$\Rightarrow$ divergent.

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1}{(2k-1)^{1/3}} \times \frac{k^{1/3}}{1}$$

$$= \lim_{k\to\infty} \frac{\cancel{k^{1/3}}}{\cancel{k^{1/3}}\left(2-\frac{1}{k}\right)^{1/3}}$$

$$= \frac{1}{\left(2-\frac{1}{\infty}\right)^{1/3}} = \frac{1}{2^{1/3}} > 0 \text{ and finite.}$$

$\sum b_k$ divergent $\Rightarrow \sum a_k$ also divergent

4. Test for the Convergence of the Series

$\sum \frac{1}{\sqrt{n^3+1}}$

Let $a_n = \frac{1}{\sqrt{n^3+1}} = \frac{1}{(n^3+1)^{1/2}}$

Choose $b_n = \frac{1}{n^p}$ [P= Degree Dr - Degree Nr $= \frac{3}{2} - 0 = 3/2$]

$= \frac{1}{n^{3/2}}$

$\sum b_n = \sum \frac{1}{n^{3/2}}$ p Series $\Longrightarrow$ convergent.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{(n^3+1)^{1/2}} \times \frac{n^{3/2}}{1}$

$= \lim_{n\to\infty} \frac{1}{\cancel{n^{3/2}}\left(1+\frac{1}{n^3}\right)^{1/2}} \frac{\cancel{n^{3/2}}}{1}$

$= \underline{1 > 0}$, finite quantity.

$\therefore \sum a_n$ also convergent

5. Test for convergence the Series

$\frac{1}{1\cdot 2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \frac{5}{3\cdot 4\cdot 5} + \cdots$

$a_n = \frac{2n-1}{n\cdot(n+1)(n+2)}$

even $\rightarrow 2n$
odd $\rightarrow 2n-1$ or $2n+1$

$b_n = \frac{1}{n^p} = \frac{1}{n^2}$ [P= degree Dr - degree Nr $= 3-1 = 2 > 1$]

$\sum b_n \Rightarrow$ P. Series with $p = 2 > 1 \Longrightarrow \sum b_n$ convergent

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n+1}{n(n+1)(n+2)} \times \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(2+1/n)\, n^2}{n \times \cancel{n}(1+1/n)\cancel{n}(1+2/n)}$$

$$= \frac{2+1/\infty}{(1+1/\infty)(1+2/\infty)} = \frac{2}{1} = 2 > 0, \text{ finite}$$

$\therefore \sum a_n$ convergent

6 Test the convergence of the series

$$\sum \sqrt{n^2+1} - n$$

$$a_n = \sqrt{n^2+1} - n = \sqrt{n^2+1} - n \times \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}$$

$$= \frac{\left(\sqrt{n^2+1}\right)^2 - n^2}{\sqrt{n^2+1}+n} = \frac{\cancel{n^2}+1-\cancel{n^2}}{\sqrt{n^2+1}+n} \quad \left[(a-b)(a+b) = a^2-b^2\right]$$

$$= \frac{1}{\sqrt{n^2+1}+n}$$

$$b_n = \frac{1}{n^p} = \frac{1}{n} \qquad [p = 1-0 = 1]$$

$\sum b_n$ divergent p series

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}+n} \times \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n^2(1+1/n^2)}+n} \times \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{1}{n\sqrt{1+\frac{1}{n^2}}+n} \times n$$

$$= \lim_{n\to\infty} \frac{1}{\cancel{n}\left[\sqrt{1+\frac{1}{n^2}}+1\right]} \cancel{n} = \frac{1}{\sqrt{1+\frac{1}{\infty}}+1}$$

$$= 1/2 > 0 \text{, finite}$$

$\therefore \sum a_n$ divergent

7 Test the convergence of the series

$$\sum_{n=1}^{\infty} \sqrt{n^6+1} - \sqrt{n^6-1}$$

$$a_n = \sqrt{n^6+1} - \sqrt{n^6-1} = \sqrt{n^6+1} - \sqrt{n^6-1} \times \frac{\sqrt{n^6+1}+\sqrt{n^6-1}}{\sqrt{n^6+1}+\sqrt{n^6-1}}$$

$$= \frac{(\sqrt{n^6+1})^2 - (\sqrt{n^6-1})^2}{\sqrt{n^6+1}+\sqrt{n^6-1}} = \frac{\cancel{n^6}+1-\cancel{n^6}+1}{\sqrt{n^6+1}+\sqrt{n^6-1}} = \frac{2}{\sqrt{n^6+1}+\sqrt{n^6-1}}$$

$$b_n = \frac{1}{n^p} \quad [p = 3-0 = 3 > 1]$$

$$= \frac{1}{n^3} \implies \sum b_n \text{ convergent p-series}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2}{\sqrt{n^6+1}+\sqrt{n^6-1}} \times \frac{n^3}{1}$$

$$= \lim_{n\to\infty} \frac{2}{\sqrt{n^6(1+1/n^6)}+\sqrt{n^6(1-1/n^6)}} \times n^3$$

$$= \lim_{n\to\infty} \frac{2}{\cancel{n^3}\left[\sqrt{1+1/n^6}+\sqrt{1-1/n^6}\right]} \cancel{n^3}$$

$$= \frac{2}{\sqrt{1+1/\infty}+\sqrt{1-1/\infty}} = \frac{2}{2} = 1 > 0 \text{, finite}$$

$\therefore \sum a_n$ also convergent

8 Determine whether $\sum_{k=1}^{\infty} \frac{1}{(3k-1)(3k+2)}$ is converg

$$a_k = \frac{1}{(3k-1)(3k+2)}$$

$$b_k = \frac{1}{k^p} = \frac{1}{k^2} \quad [p = 2-0 = 2]$$

$\sum b_k$ convergent p-series

$$\therefore \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1}{(3k-1)(3k+2)} \times \frac{k^2}{1}$$

$$= \lim_{k\to\infty} \frac{\cancel{k^2}}{\cancel{k}(3-1/k)\,\cancel{k}(3+2/k)}$$

$$= \frac{1}{(3-1/\infty)(3+2/\infty)} = \frac{1}{3\times 3} = \frac{1}{9} > 0, \text{ finite}$$

$\Longrightarrow$ By limit comparison test

$\sum a_k$ is convergent

The Ratio Test

Let $\sum a_n$ be a series with positive term and $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = l$

then the series $\sum a_n$ is

1) Convergent if $l < 1$
2) divergent if $l > 1$
3) If $l = 1$, the series may converge or diverge [If $l=1$, the test fail]

Problems

1. Test the convergence of $\sum\limits_{k=1}^{\infty} \frac{99^k}{k!}$

Let $a_k = \frac{99^k}{k!}$

$$a_{k+1} = \frac{99^{k+1}}{(k+1)!}$$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{99^{k+1}}{(k+1)!}}{\frac{99^k}{k!}}$$

$$= \lim_{k\to\infty} \frac{99^{k+1}}{(k+1)!} \times \frac{k!}{99^k} = \lim_{k\to\infty} \frac{\cancel{99^k}\,99}{(k+1)\cancel{k!}} \frac{\cancel{k!}}{\cancel{99^k}}$$

$$= \lim_{k\to\infty} \frac{99}{k+1} = \frac{99}{\infty} = 0 < 1$$

$\therefore$ By ratio test the Series Converges

2. Test the convergence of $\sum_{k=1}^{\infty} \frac{(k+4)!}{4!\, k!\, 4^k}$

$$a_k = \frac{(k+4)!}{4!\, k!\, 4^k} \qquad a_{k+1} = \frac{(k+1+4)!}{4!\,(k+1)!\, 4^{k+1}}$$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(k+5)!}{4!\,(k+1)!\,4^{k+1}} \times \frac{4!\, k!\, 4^k}{(k+4)!}$$

$$= \lim_{k\to\infty} \frac{(k+5)\cancel{(k+4)!}}{\cancel{4!}\,(k+1)\cancel{k!}\,\cancel{4^k}\cdot 4^1} \times \frac{\cancel{4!}\ \cancel{k!}\ \cancel{4^k}}{\cancel{(k+4)!}}$$

$$= \lim_{k\to\infty} \frac{k+5}{4(k+1)} = \lim_{k\to\infty} \frac{\cancel{k}(1+5/k)}{4\cancel{k}(1+1/k)}$$

$$= \frac{1+\frac{5}{\infty}}{4.(1+1/\infty)} = \frac{1}{4} < 1$$

$\therefore$ By ratio test the Series Converges

3. Examine the Convergence of the Series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$a_n = \frac{n^n}{n!} \qquad a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n \cancel{(n+1)}}{\cancel{(n+1)}\, \cancel{n!}} \times \frac{\cancel{n!}}{n^n}$$

$$= \lim_{n\to\infty} \left[\frac{n+1}{n}\right]^n = \lim_{n\to\infty} \left[\frac{\cancel{n}(1+1/n)}{\cancel{n}}\right]^n$$

$$= \lim_{n\to\infty} \left[1+\frac{1}{n}\right]^n \qquad \boxed{\lim_{n\to\infty} \left(1+\frac{x}{n}\right) = e^x}$$

$$= e > 1$$

$\Rightarrow$ the Series diverges

4. Determine whether the Series $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ converge or diverge.

$$a_n = \frac{(2n)!}{(n!)^2} \qquad a_{n+1} = \frac{[2(n+1)]!}{[(n+1)!]^2} = \frac{(2n+2)!}{[(n+1)!]^2}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+2)!}{[(n+1)!]^2} \times \frac{(n!)^2}{(2n)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)\, \cancel{2n!}}{(n+1)^2 \cancel{(n!)^2}} \quad \frac{\cancel{(n!)^2}}{\cancel{(2n)!}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(2+2/n)\, \cancel{n}(2+1/n)}{\cancel{n^2}\cdot(1+1/n)^2}$$

$$= \frac{(2+2/\infty)(2+1/\infty)}{(1+1/\infty)^2} = \frac{2\times 2}{1} = 4 > 1$$

$\therefore$ By ratio test, the Series diverges

5. Test the Convergence of $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$

$a_n = \frac{1}{3^n+1} \qquad a_{n+1} = \frac{1}{3^{n+1}+1}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{3^{n+1}+1} \times \frac{3^n+1}{1}$$

$$= \lim_{n\to\infty} \frac{1}{3^{n+1}\left[1+\frac{1}{3^{n+1}}\right]} \, 3^n\left[1+\frac{1}{3^n}\right]$$

$$= \lim_{n\to\infty} \frac{1}{3} \frac{\left[1+1/3^n\right]}{\left[1+1/3^{n+1}\right]}$$

$$= \frac{1}{3}\left[\frac{1+1/\infty}{1+1/\infty}\right] = 1/3 < 1$$

Hence by ratio test, the Series Convergent

6. Discuss the Convergence of $\sum_{k=1}^{\infty} \frac{(2k)!}{4^k}$

$a_k = \frac{(2k)!}{4^k} \qquad a_{k+1} = \frac{[2(k+1)]!}{4^{k+1}} = \frac{(2k+2)!}{4^{k+1}}$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(2k+2)!}{4^{k+1}} \times \frac{4^k}{(2k)!}$$

$$= \lim_{k\to\infty} \frac{(2k+2)(2k+1)\cancel{(2k)!}}{\cancel{4^k}.4^1} \times \frac{\cancel{4^k}}{\cancel{(2k)!}}$$

$$= \lim_{k\to\infty} \frac{k(2+2/k)\,k(2+1/k)}{4} = \infty > 1$$

using ratio test Series diverges

7. Test the Convergence of the Series

$$\frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \frac{x^4}{4\cdot 5} + \cdots$$

$$a_n = \frac{x^n}{n(n+1)} \qquad a_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(\cancel{n+1})(n+2)} \times \frac{n(\cancel{n+1})}{x^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{x^n}\cdot x\, \cancel{n}}{\cancel{n}(1+2/n)\, \cancel{x^n}} = \frac{x}{1+2/\infty} = x.$$

By ratio test the Series Converges if $x<1$ and diverges if $x>1$.

If $x=1$ [Ratio test fail]

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$$

$$a_n = \frac{1}{n(n+1)} \qquad \cancel{a_{n+1} = \frac{1}{(n+1)(n+2)}}$$

$$b_n = \frac{1}{n^p} = \frac{1}{n^2} \qquad [\, p = 2-0 = 2\,]$$

$$\sum b_n = \sum \frac{1}{n^2} \text{ Convergent}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim \frac{1}{\cancel{n}(n+1)} \times \frac{n^2}{1} = \lim \frac{1}{\cancel{n}(1+1/n)}$$

$$= \frac{1}{1+1/\infty} = 1 > 0 \text{ finite quantity.}$$

$\therefore$ $\sum a_n$ Convergent

$\therefore$ The Series is convergent if $x \le 1$ and divergent if $x>1$

8. Test the convergence of

$$\frac{3}{4}+\frac{3\cdot4}{4\cdot6}+\frac{3\cdot4\cdot5}{4\cdot6\cdot8}+\frac{3\cdot4\cdot5\cdot6}{4\cdot6\cdot8\cdot10}+\cdots$$

$$a_n = \frac{3\cdot 4\cdot 5\cdots(n+2)}{4\cdot 6\cdot 8\cdots(2n+2)}$$

$$a_{n+1} = \frac{3\cdot 4\cdot 5\cdots(n+2)(n+3)}{4\cdot 6\cdot 8\cdots(2n+2)(2n+4)}$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\cancel{3\cdot4\cdot5}\cdots\cancel{(n+2)}(n+3)}{\cancel{4\cdot6\cdot8}\cdots\cancel{(2n+2)}(2n+4)}\times\frac{\cancel{4\cdot6\cdot8}\cdots\cancel{(2n+2)}}{\cancel{3\cdot4\cdot5}\cdots\cancel{(n+2)}}$$

$$= \lim_{n\to\infty}\frac{n+3}{2n+4} = \lim_{n\to\infty}\frac{\cancel{n}(1+3/n)}{\cancel{n}(2+4/n)}$$

$$= \tfrac{1}{2} < 1$$

$\therefore$ By Ratio test, the series converge

9. Examine the convergence of the series

$$1+\frac{1\cdot2}{1\cdot3}+\frac{1\cdot2\cdot3}{1\cdot3\cdot5}+\frac{1\cdot2\cdot3\cdot4}{1\cdot3\cdot5\cdot7}+\cdots$$

$$a_n = \frac{1\cdot2\cdot3\cdots n}{1\cdot3\cdot5\cdots 2n-1}$$

$$a_{n+1} = \frac{1\cdot2\cdot3\cdots n\,(n+1)}{1\cdot3\cdot5\cdots(2n-1)(2n+1)}$$

$$\boxed{\begin{aligned}&2(n+1)-1\\&2n+2-1\\&=2n+1\end{aligned}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \cancel{\lim_{n\to\infty}}$$

$$= \lim_{n\to\infty} \frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdots\cancel{n}(n+1)}{\cancel{1}\cdot\cancel{3}\cdot\cancel{5}\cdots\cancel{(2n-1)}(2n+1)} \times \frac{\cancel{1}\cdot\cancel{3}\cdot\cancel{5}\cdots\cancel{(2n-1)}}{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdots\cancel{n}}$$

$$= \lim_{n\to\infty} \frac{(n+1)}{2n+1} = \lim_{n\to\infty} \frac{\cancel{n}(1+1/n)}{\cancel{n}(2+1/n)}$$

$$= 1/2 < 1$$

$\therefore$ By Ratio test Series converges

10. Examine the convergence of the series

$$1 + \frac{1\cdot 3}{3!} + \frac{1\cdot 3\cdot 5}{5!} + \frac{1\cdot 3\cdot 5\cdot 7}{7!} + \cdots$$

$$a_n = \frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{(2n-1)!}$$

$$a_{n+1} = \frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)(2n+1)}{(2n+1)!}.$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} =$$

$$\lim_{n\to\infty} \frac{\cancel{1}\cdot\cancel{3}\cdot\cancel{5}\cdot\cancel{7}\cdots\cancel{(2n-1)}(2n+1)}{(2n+1)!} \times \frac{(2n-1)!}{\cancel{1}\cdot\cancel{3}\cdot\cancel{5}\cdot\cancel{7}\cdots\cancel{2n-1}}$$

$$= \lim_{n\to\infty} \cancel{(2n+1)} \times \frac{\cancel{(2n-1)!}}{\cancel{(2n+1)}(2n)\cancel{(2n-1)!}}$$

$$= \lim_{n\to\infty} \frac{1}{2n} = \frac{1}{\infty} = 0 < 1$$

$\therefore$ By Ratio test Series converges

11. Examine the Convergence of

$$\frac{1}{3} + \frac{1\cdot 2}{3\cdot 5} + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7} + \cdots$$

$$a_n = \frac{1\cdot 2\cdot 3 \cdots n}{3\cdot 5\cdot 7 \cdots (2n+1)}$$

$$a_{n+1} = \frac{1\cdot 2\cdot 3 \cdots n(n+1)}{3\cdot 5\cdot 7 \cdots (2n+1)(2n+3)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\cancel{1}\cdot \cancel{2}\cdot \cancel{3} \cdots \cancel{n}(n+1)}{\cancel{3}\cdot \cancel{5}\cdot \cancel{7} \cdots \cancel{(2n+1)}(2n+3)} \times \frac{\cancel{3}\cdot\cancel{5}\cdot\cancel{7}\cdots\cancel{(2n+1)}}{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdots\cancel{n}}$$

$$= \lim_{n\to\infty} \frac{n+1}{2n+3}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(1+1/n)}{\cancel{n}(2+3/n)}$$

$$= 1/2 < 1$$

Hence by ratio test, the given Series Converges

The Root Test

Let $\sum a_n$ be a Series with positive terms and $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} (a_n)^{1/n} = l$

1) If $l<1$, the Series Converges
2) If $l>1$, the Series diverges
3) If $l=1$, the Series may converge or diverge.
   [If $l=1$ test failed]

Problems

1. Examine the Convergence of $\sum_{n=2}^{\infty} \left(\frac{4n-5}{2n+1}\right)^n$

$a_n = \left(\frac{4n-5}{2n+1}\right)^n$

$l = \lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left[\left(\frac{4n-5}{2n+1}\right)^n\right]^{1/n}$

$= \lim_{n\to\infty} \frac{\cancel{n}(4-5/n)}{\cancel{n}(2+1/n)} = \frac{4}{2} = 2 > 1$

$l>1 \Rightarrow$ Series diverges

2. Examine the Convergence of $\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{k^2}$

Imp

$a_k = \left[\frac{k}{k+1}\right]^{k^2}$

$$l = \lim_{k\to\infty} (a_k)^{1/k} = \lim_{k\to\infty} \left\{\left[\frac{k}{k+1}\right]^{k^2}\right\}^{1/k}$$

$$= \lim_{k\to\infty} \left[\frac{k}{k[1+1/k]}\right]^k$$

$$= \lim_{k\to\infty} \frac{1}{\left[1+\frac{1}{k}\right]^k} = \frac{1}{e} < 1$$

$l < 1 \Rightarrow$ the series converges

3 Test the convergence of the series

$$\sum_{k=1}^{\infty} \frac{k}{2^k}.$$

$$a_k = \frac{k}{2^k}$$

$$l = \lim_{k\to\infty} \left(\frac{k}{2^k}\right)^{1/k} = \lim_{k\to\infty} \frac{k^{1/k}}{2}$$

$$\boxed{\lim_{n\to\infty} n^{1/n} = 1}$$

$$= \frac{1}{2} \lim_{k\to\infty} k^{1/k} = \frac{1}{2}$$

4 Examine the convergence of $\sum_{n=1}^{\infty} \frac{1}{[\ln(n+1)]^n}$

$$a_n = \frac{1}{[\ln(n+1)]^n}$$

$$l = \lim_{n\to\infty} \left[\frac{1}{[\ln(n+1)]^n}\right]^{1/n} = \lim_{n\to\infty} \frac{1}{\ln(n+1)} = \frac{1}{\infty} = 0$$

< 1

5. ## Alternating Series

A Series in which the terms are alternately positive and negative, is called an alternating series.

An alternating series has one of the following two forms.

1. $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$

2. $\sum_{n=1}^{\infty} (-1)^{n} a_n = -a_1 + a_2 - a_3 + a_4 - \cdots$

where each a_n is assumed to be positive for all n.

eg: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots$

Leibnitz Test [The alternating Series test]

The alternating Series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges if the following two conditions are satisfied.

(i) $a_n \geq a_{n+1}$ for all n.

(ii) $\lim_{n\to\infty} a_n = 0$.

Problems

1. Prove that the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$ is convergent.

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}$$

$$a_n = \frac{1}{n} \qquad a_{n+1} = \frac{1}{n+1}$$

$\Longrightarrow$ $a_n > a_{n+1}$ for all n.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n+1} = \frac{1}{\infty} = 0$$

The alternating series $\sum (-1)^{n+1}\frac{1}{n}$ satisfies all the conditions of Leibnitz test.

Hence the given series converges

2. Use the alternating series test to show that the series $\sum_{k=1}^{\infty} (-1)^{k+1}\frac{k+3}{k(k+1)}$ converges.

Ans: Given series $\sum_{k=1}^{\infty} (-1)^{k+1}\frac{k+3}{k(k+1)}$ is an alternating series

$$a_k = \frac{k+3}{k(k+1)} \qquad a_{k+1} = \frac{k+4}{(k+1)(k+2)}$$

$$\frac{a_{k+1}}{a_k} = \frac{k+4}{(\cancel{k+1})(k+2)} \times \frac{k(\cancel{k+1})}{k+3}$$

$$= \frac{k(k+4)}{(k+2)(k+3)} = \frac{k^2+4k}{k^2+5k+6} = \frac{k^2+4k}{(k^2+4k)+k+6} < 1$$

$$\Longrightarrow \frac{a_{k+1}}{a_k} < 1$$

$\Longrightarrow a_k > a_{k+1}$ for all k

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k+3}{k(k+1)}$$

$$= \lim_{k\to\infty} \frac{\cancel{k}(1+3/k)}{\cancel{k}(k+1)} = \frac{1+3/\infty}{\infty} = \underline{0}$$

$\Longrightarrow$ The alternating Series Satisfies all the conditions of Leibnitz test.

Hence the given Series is convergent.

3. Examine the convergence of $\sum_{k=1}^{\infty} (-1)^{k+1} e^{-k}$

$a_k = e^{-k}$ $\quad a_{k+1} = e^{-(k+1)}$

$a_k > a_{k+1}$ for all k.

$\therefore \lim_{k\to\infty} a_k = \lim_{k\to\infty} e^{-k} = e^{-\infty} = \underline{0}$

alternating Series Satisfies all the conditions of Leibnitz test.

$\therefore$ Series Convergent

4. Examine the convergence of $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+1}{4k+1}$.

$a_k = \frac{k+1}{4k+1}$ $\qquad a_{k+1} = \frac{k+2}{4k+5}$

$$\frac{a_{k+1}}{a_k} = \frac{k+2}{4k+5} \times \frac{4k+1}{k+1}$$

$$= \frac{4k^2+8k+k+2}{4k^2+4k+5k+5}$$

$$= \frac{4k^2+9k+2}{4k^2+9k+5} < 1$$

$\therefore \frac{a_{k+1}}{a_k} < 1 \implies a_{k+1} < a_k$

or $a_k > a_{k+1}$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k+1}{4k+1}$$

$$= \lim_{k\to\infty} \frac{\cancel{k}(1+1/k)}{\cancel{k}(4+1/k)}$$

$$= \frac{1}{4} \neq 0.$$

$\therefore$ By alternating series test, the series diverges

Absolute Convergence

A Series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$ is said to converge absolutely if the series of absolute values, $\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + \cdots + |a_n| + \cdots$ converge and is said to diverge absolutely if the series of absolute value diverges.

Eg: Examine whether the series

$$1 - \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} - \cdots$$

The series of absolute value is

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots,$$

which is a p-series with $p = 2 > 1$.

$\therefore$ It is convergent

$\therefore$ The given series converges absolutely

Absolute Convergence test

Every absolutely convergent series is convergent. ie if the series $\sum_{n=1}^{\infty} |a_n|$

converges so does $\sum_{n=1}^{\infty} a_n$.

Note

The converse of the above test is false.

eg: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

By Leibnitz test $a_n > a_{n+1}$ and

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = \frac{1}{\infty} = 0$

$\therefore \sum (-1)^{n+1} \frac{1}{n}$ Convergent.

The series of absolute values a_n is

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum \frac{1}{n}$,

is a p series with $p = 1$

$\Rightarrow$ $\Sigma 1$ is a divergent series.

$\sum a_n$ convergent. But $\sum |a_n|$ does not convergent

Conditionally Convergent Series

A series that converges but does not converge absolutely is said to be

Conditionally Convergent.

i.e if $\sum a_n$ converges and $\sum |a_n|$ diverges, then $\sum a_n$ is said to converge conditionally (or to be conditionally convergent).

Eg: The alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges conditionally.

Problems

1. Determine whether the series $1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \dots$ converge absolutely.

The series of absolute value is

$$\sum |a_n| = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$$

This is a geometric series, $|r| = \frac{1}{2} < 1$

$\Rightarrow$ convergent

$\therefore$ The given series converge absolutely

2. Determine whether the series $\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$ converge absolutley.

$$\sum |a_k| = \sum \left|\frac{\cos k}{k^2}\right|$$

$$a_k = \left|\frac{\cos k}{k^2}\right| < \left|\frac{1}{k^2}\right|$$

$\sum \frac{1}{k^2}$ is a convergent p series.

By comparison test the bigger series convergent $\Rightarrow$ the smaller series also convergent.

$\therefore \sum \left|\frac{\cos k}{k^2}\right|$ convergent absolutely.

$\Rightarrow$ the given series converges

3 Prove that the series $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \dots$ converges absolutely.

The given series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n^3}$$

The series of absolute value is

$$\sum_{n=1}^{\infty} |a_n| = \sum \left|\frac{\sin nx}{n^3}\right|$$

$$|a_n| = \left|\frac{\sin n\pi}{n^3}\right| \le \left|\frac{1}{n^3}\right|$$

and $\sum \frac{1}{n^3}$ convergent p series with $p = 3 > 1$.

$\therefore$ By comparison test the bigger series converges $\Longrightarrow$ the smaller series $\sum |a_n|$ also convergent.

$\Longrightarrow$ the given series converges absolutely.

4. Determine whether the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ is absolutely convergent.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

The absolute value of the series

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right| \le \sum_{n=1}^{\infty} \left|\frac{1}{n^2}\right|$$

$\sum \frac{1}{n^2}$ is a convergent p-series

$\therefore$ By comparison test $\sum_{n=1}^{\infty} |a_n|$ convergent

The Ratio test for Absolute Convergence

Let $\sum a_n$ be a Series with non zero terms and suppose that

$$l = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

(1) If $l<1$, then the Series $\sum a_n$ converges absolutely and therefore converges.

(2) If $l>1$ then the Series $\sum a_n$ diverges.

(3) If $l=1$, no conclusion about convergence or absolutely convergence.

Problems

1. Test for Convergence of the Series

$$\sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!}{3^k}$$

Taking the absolute value of the general term

$$|a_k| = \left| (-1)^k \frac{(2k-1)!}{3^k} \right| = \frac{(2k-1)!}{3^k}.$$

$$l = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} \frac{\frac{[2(k+1)-1]!}{3^{k+1}}}{\frac{(2k-1)!}{3^k}}$$

$$= \lim_{k\to\infty} \frac{(2k+1)!}{3^{k+1}} \times \frac{3^k}{(2k-1)!}$$

$$= \lim_{k\to\infty} \frac{(2k+1)\, 2k\, \cancel{(2k-1)!}}{\cancel{3^k}\, 3^1} \times \frac{\cancel{3^k}}{\cancel{(2k-1)!}}$$

$$= \lim_{k\to\infty} \frac{k(2+1/k)\, 2k}{3} = \infty$$

$\Longrightarrow$ Series converges absolutley

$\Longrightarrow$ the given series converges

2. Determine whether the series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3^{2k-1}}{k^2+1}$ is absolutely convergent.

Taking absolute value of general term

$$a_k = \left| (-1)^{k+1} \frac{3^{2k-1}}{k^2+1} \right| = \frac{3^{2k-1}}{k^2+1}$$

$$l = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{3^{2(k+1)-1}}{(k+1)^2+1} \times \frac{k^2+1}{3^{2k-1}}$$

$$= \lim_{k\to\infty} \frac{\cancel{3^{2k}}\, 3^1}{\cancel{k^2}\left[\left(1+\frac{1}{k}\right)^2+\frac{1}{k^2}\right]} \times \frac{\cancel{k^2}\left[1+1/k^2\right]}{\cancel{3^{2k}}\cdot 3^{-1}}$$

$$= \lim_{k\to\infty} 3\times 3 \frac{[1+1/k^2]}{\left[\left(1+\frac{1}{k}\right)^2+\frac{1}{k^2}\right]}$$

$$= 9 \frac{[1+1/\infty]}{\left[\left(1+\frac{1}{\infty}\right)^2+\frac{1}{\infty}\right]} = 9 > 1$$

$\therefore$ Series diverges

3. Use ratio test for absolute convergence of the series determine the convergence

$$\sum_{k=1}^{\infty} (-1)^k \frac{2^k}{k!}.$$

Taking absolute value of the general term

$$|a_k| = \left|(-1)^k \frac{2^k}{k!}\right| = \frac{2^k}{k!}$$

$$l = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{2^{k+1}}{(k+1)!} \times \frac{k!}{2^k}$$

$$= \lim_{k\to\infty} \frac{2^k\cdot 2^1}{(k+1)\cancel{k!}} \frac{\cancel{k!}}{2^k} = \lim \frac{2}{k+1}$$

$$= \frac{2}{\infty} = 0 < 1$$

$\therefore$ The Series Converges absolutley and therefore Converges

4. Determine whether the alternating Series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+7}{k(k+4)}$ is absolutely converged or not.

$$|a_k| = \left| (-1)^{k+1} \frac{k+7}{k(k+4)} \right| = \frac{k+7}{k(k+4)}$$

$$l = \lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \lim_{k \to \infty} \frac{k+8}{(k+1)(k+5)} \times \frac{k(k+4)}{k+7}$$

$$= \lim_{k \to \infty} \frac{\cancel{k}(1+8/k)}{\cancel{k}(1+1/k)\cancel{k}(1+5/k)} \times \frac{\cancel{k} \times \cancel{k}(1+4/k)}{k(1+7/k)}$$

$$= \underline{\underline{1}}$$

Test fail.

$$a_k = \frac{k+7}{k(k+4)}$$

$$b_k = \frac{1}{k^p} = \frac{1}{k} \qquad [2-1=1]$$

$\sum \frac{1}{k}$ divergent p- Series

$$\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{k+7}{k(k+4)} \times \frac{\cancel{k}}{1}$$

$$= \lim_{k \to \infty} \frac{\cancel{k}(1+7/k)}{\cancel{k}(1+4/k)} = 1 > 0 \text{ finite}$$

$\therefore$ By limit Comparison test $\sum b_k$ diverg

$\Longrightarrow \sum a_k$ also diverges

5. Test the absolute Convergence of

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(2k)!}{(3k-2)!}$$

$$|a_k| = \left| (-1)^{k+1} \frac{(2k)!}{(3k-2)!} \right| = \frac{(2k)!}{(3k-2)!}$$

$$l = \lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \lim \frac{(2k+2)!}{(3k+1)!} \times \frac{(3k-2)!}{(2k)!}$$

$$= \lim_{k \to \infty} \frac{(2k+2)(2k+1)\cancel{(2k)!}}{(3k+1)(3k)(3k-1)\cancel{(3k-2)!}} \times \frac{\cancel{(3k-2)!}}{\cancel{(2k)!}}$$

$$= \lim_{k \to \infty} \frac{\cancel{k}(2+2/k)\,\cancel{k}(2+1/k)}{\cancel{k}(3+1/k)\,3\cancel{k}\,k(3-1/k)}$$

$$= \frac{(2+2/\infty)(2+1/\infty)}{\infty} = 0 < 1$$

$\Longrightarrow$ The Series Converges absolutely

6. Test the absolute Convergence of

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^4}{4^n}$$

$$|a_n| = \left| (-1)^n \frac{n^4}{4^n} \right| = \frac{n^4}{4^n}$$

$$l = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} =$$

$$= \lim_{n \to \infty} \frac{(n+1)^4}{4^{n+1}} \times \frac{4^n}{n^4}$$

$$= \lim_{n\to\infty} \frac{n^4 (1+1/n)^4}{4^n \cdot 4^1} \frac{4^n}{n^4}$$

$$= 1/4 < 1$$

$\Rightarrow$ The Series Converges absolutely

7. Check whether the series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^k}{k!}$ is absolutely convergent or not.

$$|a_k| = \frac{k^k}{k!}$$

$$l = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(k+1)^{k+1}}{(k+1)!} \times \frac{k!}{k^k}$$

$$= \lim_{k\to\infty} \frac{(k+1)^k (k+1)}{(k+1)!} \frac{k!}{k^k}$$

$$= \lim_{k\to\infty} \frac{\cancel{k^k} (1+1/k)^k \cancel{(k+1)}}{\cancel{(k+1)} \cancel{k!}} \frac{\cancel{k!}}{\cancel{k^k}}$$

$$= \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k$$

$$= e > 1$$

$\therefore$ The Series diverges

8. Use ratio test for absolute convergence to find whether the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} 2^k}{k!}$.

$$a_k = \frac{2^k}{k!}$$

$$l = \lim_{k \to \infty} \frac{a_{k+1}}{a_k}$$

$$= \lim_{k \to \infty} \frac{2^{k+1}}{(k+1)!} \times \frac{k!}{2^k}$$

$$= \lim_{k \to \infty} \frac{\cancel{2^k} \cdot 2^1}{(k+1)\,\cancel{k!}} \cdot \frac{\cancel{k!}}{\cancel{2^k}}$$

$$= \lim_{k \to \infty} \frac{2}{k+1} = \frac{2}{\infty} = \underline{0 < 1}$$

∴ The series converges absolutely