Report for assignment 1

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Matrix manipulation using analytical inversion, LU, and singular value decomposition methods

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Matrix manipulation is a critical tool in physics for modelling multidimensional systems. This paper explores three methods of matrix inversion employed to solve simultaneous equations describing a physical system. The analytical method is the first method considered whereby a matrix of co-factors and a determinant is generated from the original matrix to produce the inverted matrix. The other two methods considered are the LU decomposition method and the singular valued decomposition, which rely on the factorization of the original matrix instead of the inversion technique.

INTRODUCTION

A system of simultaneous equation may be represented in one entity: a matrix. In this system, the linear algebra equations may have multiple unknown variables, which are solvable using matrix multiplication rules:

$$Ax = B \tag{1}$$

where A is a matrix of coefficients, x is the vector of the unknown variables of the system and B is the vector of known solutions.

By multiplying each side of the equation by the inverse of matrix A it is possible to extract the unknown variables:

$$x = A^{-1}B \tag{2}$$

The analytical method enables the calculation of the inverse of the matrix using the standard inversion formula:

$$A^{-1} = \frac{1}{det A}C^T \tag{3}$$

where C is the matrix of cofactors and T is the transpose of the matrix.

The matrix A must be iterated over the ith and jth columns to find the matrix of minors, in order to compute the elements of the co-factors and the determinant. The LU decomposition method, introduced by Alan Turing [1] allows the steps of Gaussian elimination to be recorded. The factorization of the original matrix A produces two triangular matrices, an upper (U) and a lower (L) matrix. Matrix U is calculated using the Gaussian inversion method, whereas matrix L holds the ones of the diagonal and records the elements of the lower triangle. A pivot matrix, P, is introduced, which is the identity matrix with the rows switched in accordance with the pivot of A [2]. To solve the EQ. 1 the original matrix is multiplied by the pivoting matrix. LU is then substituted for PA which leads to the easily solvable equations:

$$L\mathbf{y} = \mathbf{d} \tag{4}$$

and

$$U\mathbf{x} = \mathbf{y} \tag{5}$$

The third method considered in this paper is the singular value decomposition method (SVD). The original matrix A is factorized in the form:

$$A = UDV^T \tag{6}$$

where U and V are othonormal and the matrix D is diagonal.

Other techniques of spectral decomposition in linear algebra produce non-invertible systems of eigenvectors [3]. The applicability of SVD extends to all defined matrices of mxn dimensions provided m $\geq$ n.

COMPARISON OF METHODS

To determine if the analytical method using the iterative approach produced the required result, the relationship:

$$A^{-1}A = I \tag{7}$$

was utilised. It was possible to check this relation up until N = 8 after which the analytical method became increasingly inefficient. It was also observed that when increasing N, the non-diagonal matrix elements became non-zero.

To analyze the efficiency of the analytical method, the time taken for the matrix to solve a random set of simultaneous equations was recorded.

The time taken for the analytical matrix to be calculated over the range of N = 2-8 was taken over 5 repeats. It was found that the error bars significantly increased around N = 7. The standard deviation of the error for N = 1 was ± 0.00034 compared to N = 8 with ± 6.6.

Using the analytical method to produce the inverse of matrix A became inefficient after N = 9 due to the calculation of the determinant. This is due to the fact that the determinant must iterate over the ith and jth columns. Hence, for an N = 9 matrix, the function using the iteration begins to break down.

The time taken to run the analytical method is N factorial due to the iterative method used to find the matrix of minors [4]. This due to the fact that for an N dimensional matrix, the minors of the minors must be calculated until the minor has dimensions of N = 2. Further error arises in the evaluation

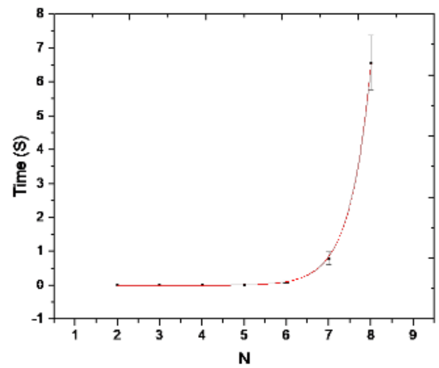

FIG. 1. Time of take for the analytical inversion method versus to the dimensions of the matrix

of the determinant in the analytical method due to rounding errors.

There are alternative ways to extend the analytical method including Chio's condensation method, matrix mirroring and extended condensation, which improves the time complexity.

In comparison, the decomposition methods are able to run up to the order of N = 10^4. This is due to the fact that these methods rely on factorisation rather than iterations.

Taking the range matrices between N = 2 to N = 1000, the time taken for the SVD method compared to the LUD method produce a graph which shows:

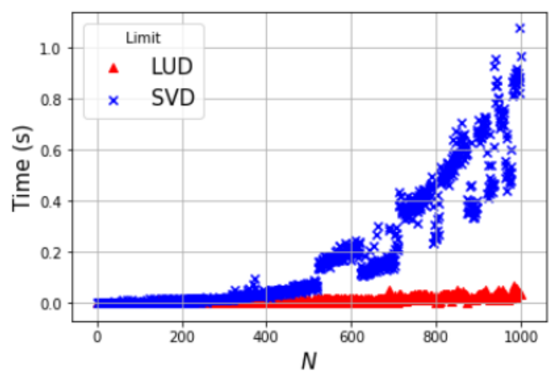

FIG. 2. Time taken for the SVD and LUD inversion method versus to the dimensions of the matrix

From this graph, it is difficult to calculate the trend of time with N dimensions. However, in a paper by Dezprez et al. [5] the time for the LUD method is polynomial of O(n^3). The time taken for the SVD has two fundamental steps due to the use of the U V matrices. It is found that the SVD also scales with a O(n^3) polynomial.

Time is one important factor when comparing methods of solving simultaneous equations. However, the ability for these equations to work close to a singularity is paramount [6]. Consider the matrix multiplication for the following set of simultaneous equations:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 3 & k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix} \tag{8}$$

The matrix becomes close to singular as k tends to zero. Solving this set of equations using all three methods gave the unanimous results that x = 0, y = 5, and z = 0. However, as $k \to 0$ the methods gave varied results. The LUD method showed constant solutions with no variation as k tends to zero. However, the analytical and SVD method do not arrive at constant solutions:

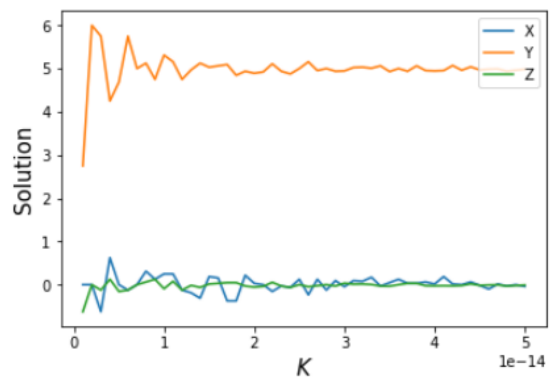

FIG. 3. Computation of a matrix close to a singularity using the analytical method

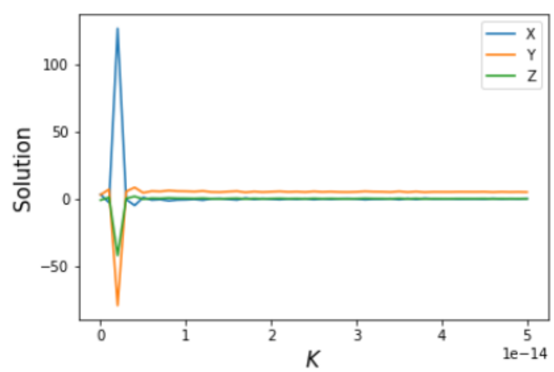

FIG. 4. Computation of a matrix close to a singularity using the SVD method

The analytical method begins to fluctuate significantly around k = 1e-14 as shown in FIG 3. However, the SVD method breaks down close to zero as shown in FIG 4. as the x peaks past 100 with the true value as x = 5.

In summary, the analytical method has the capability of solving a matrix of N = 10 dimensions after which the efficiency exponentially decreases and the non-diagonal elements of the matrix are no longer zero. The SVD and LUD methods are much more efficient and have the capability of solving matrix systems in the N = 10^4 order. However, around the point of a singularity the LUD proves to be the best method.

PHYSICS PROBLEM: THE TRAPEZE ARTIST

Consider a trapeze artist suspended in the air by three wires in the setup as shown below, where the trapeze artist has maximum height 7m:

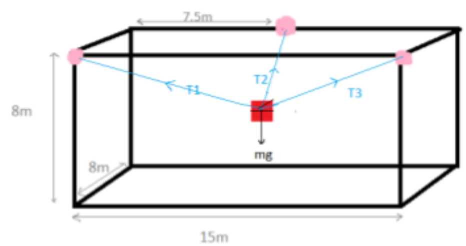

FIG. 5. Time of take for the analytical inversion method versus to the dimensions of the matrix

Following Newtons second law, F=ma, it is possible to derive three simultaneous equations in the x,y and z direction of the components of the force. Using the method of directional cosines, it is then possible to express a the tensor of the components:

$$\begin{bmatrix} cos\theta_{x1} & cos\theta_{x2} & cos\theta_{x3} \\ cos\theta_{y1} & cos\theta_{y2} & cos\theta_{y3} \\ cos\theta_{z1} & cos\theta_{z2} & cos\theta_{z3} \end{bmatrix} \begin{bmatrix} T1 \\ T2 \\ T3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} \quad (9)$$

Consider first a simplification of the trapeze artist problem whereby the front two wires are utilised. Using the inversion of the matrix, the tension in each cable is extracted as function of the artists position:

As expected the maximum tension, T1 = 2589 N occurs at the position (7,7) for wire 1. This is due to the fact that the wire must support both the mass of the trapeze artist in the z component yet also counterbalance the second wire in the x component.

For the second wire, the graph is the mirror image of FIG. 6. The maximum tension occurs at T2 = 2589N, which is an expected result due to the symmetry of the system. However, the position produces an unexpected result. The maximum tension occurs at the position (8,7), despite the symmetry of

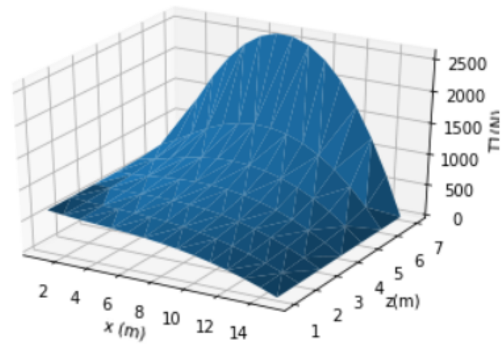

FIG. 6. Plot of the tension in wire 1 with varying tension

the system. This result may be due to a fault in the tensor of the components of the tension.

Next, consider the full system using all three wires to suspend the trapeze artist modelled by EQ. 9. The results for the maximum tension are as follows: T1 = 669N, T2 = 679N, and T3 = 683N. This result is surprising due to the fact that the tension in T1 and T2 are expected to be the same due to the symmetry of the system. In terms of error checking, the main point of error must be in the tensor of the tensions.

However, the position where the maximum tension occurs for wire 1 and wire 2 are symmetrical. They both occur in the corner below the drums which hold the wire at (1,1,7) and (15,1,7) respectively. This result is expected as this is the point where the individual wires will have the largest tensions in the system:

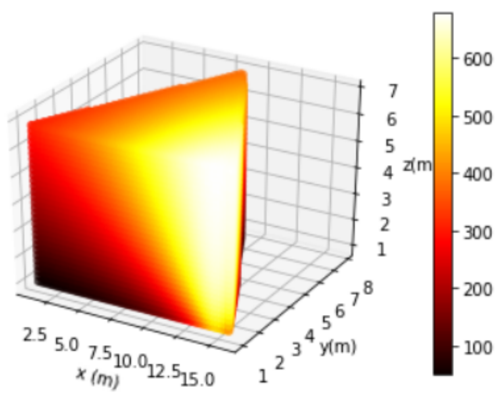

FIG. 7. 3d position plot with tension colour overlay for wire 2

The plot for wire 1 is the same as FIG 7. but in the mirror image. These plots are in agreement to the theory since the tension should have the same but mirrored distribution.

The maximum tension in wire 3 was found to occur at T3 = 682 N at the position (8,7,6). From intuition, the plot for the third wire describes the physical system well. As the angle

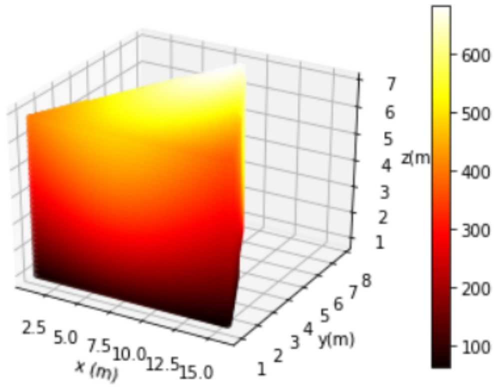

FIG. 8. 3d position plot with tension colour overlay for wire 3

to the horizontal is increased, more force is exerted by the string in the horizontal direction. There will be a decrease in the force exerted on the trapeze artist in the vertical direction. Hence in compensation for this decrease the string must exert a larger force on the trapeze artist.

As the angle between the wires and the horizontal plane tends to zero, the tension in the string will become infinite. This predicts that the greatest tension will occur when the wires become parallel to the horizontal, which is shown to be the case in both FIG 7 and FIG 8.

CONCLUSION

In conclusion, the analytical method produces more errors due to the rounding in the determinant. It becomes inefficient past N = 10 whereas the LUD and SVD methods are able to solve matrix equations for N in the order of 10^4. The SVD breaks down close to a singularity whereas the LUD and analytical methods stay closer to the true values of the system.

In terms of efficiency, the analytical method was inefficient due to its factorial progression whereas the LVD and SUD methods had a time complexity polynomial in the $O(n^3)$.

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