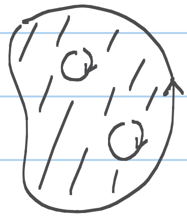
§8.2 格林公式.

1. 单连通: $D\subseteq\mathbb{R}^2$ 为区域. 若 D 中任意的闭曲线围成的区域均包含在 D 中. 则称 D 为单连通区域



复连通区域: 不是单连通的区域称为复连通区域.

2. Green公式: $D\subseteq\mathbb{R}^2$ 有界闭区域, ∂D 逐段光滑

$\vec{F}(x,y)=\{P(x,y), Q(x,y)\}$ 在 D 内具有一阶连续偏导数.

则 $$\oint_{\partial D^+} P\,dx+Q\,dy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy.$$

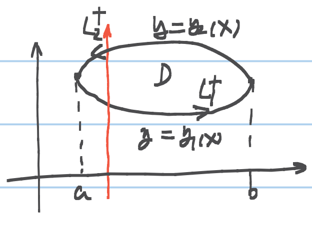
Note 1: ∂D^+ 是指 D 正向边界. 且 ∂D 闭合.

2. $P.Q$ 在 D 内部一阶偏导连续.

3. D 可是单连通域 也可是复连通域.

4. N-L 公式之推广. $\int_a^b f(x)dx=F(b)-F(a)$

简证: 情形1: $D=\{(x,y)\mid a\le x\le b,\ y_1(x)\le y\le y_2(x)\}$. $y_1(a)=y_2(a)$, $y_1(b)=y_2(b)$.



$$\oint_{\partial D^+} P(x,y)\,dx=\int_{L_1^+}+\int_{L_2^+}$$

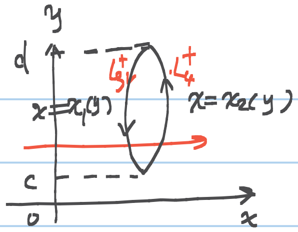
$$=\int_a^b P(x,y_1(x))\,dx+\int_b^a P(x,y_2(x))\,dx$$

$$=\int_a^b \left(P(x,y_1(x))-P(x,y_2(x))\right)dx.$$

$$-\iint_D\frac{\partial P}{\partial y}dx\,dy=-\int_a^b dx\int_{y_1(x)}^{y_2(x)}\frac{\partial P}{\partial y}dy=-\int_a^b\left[P(x,y_2(x))-P(x,y_1(x))\right]dx$$

故 $$\oint_{\partial D^+} P\,dx=-\iint_D\frac{\partial P}{\partial y}dx\,dy$$

情形2: $D=\{(x,y)\mid c\le y\le d,\ x_1(y)\le x\le x_2(y)\}$.　　$x_1(c)=x_2(c)$, $x_1(d)=x_2(d)$.



$$\oint_{\partial D^+} Q(x,y)\,dy = \int_{L_3^+}+\int_{L_4^+}$$

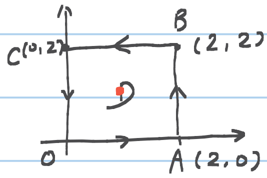
$$= \int_d^c Q(x_1(y),y)\,dy + \int_c^d Q(x_2(y),y)\,dy$$

$$\iint_D \frac{\partial Q}{\partial x}\,dx\,dy = \int_c^d dy\int_{x_1(y)}^{x_2(y)} \frac{\partial Q}{\partial x}\,dx = \int_c^d \big(Q(x_2(y),y)-Q(x_1(y),y)\big)\,dy$$

故 $\oint_{\partial D^+} Q(x,y)\,dy = \iint_D \frac{\partial Q}{\partial x}\,dx\,dy$

$$\oint_{\partial D^+} P\,dx+Q\,dy = \iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

例1: $I=\oint_{L^+}\underline{(x^2-y^2)}\,dx+\boxed{\underline{2xy}}\,dy$　　$L^+: O\to A\to B\to C\to O$

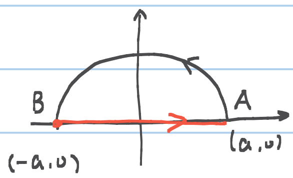


解: 由Green公式

$$I=\iint_D[2y-(-2y)]\,dx\,dy = 4\iint_D y\,dx\,dy$$

$$= 4\,\bar{y}\,S_D = 4\times1\times4=16.$$

例2: 计算 $I=\int_{L^+}(x^2+2x)\,dy$　　$L^+: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(y\ge0)$ 从$(a,0)$到$(-a,0)$为正.



法1: $x=a\cos t$, $y=b\sin t$　　$t:0\to\pi$.

$$I=\int_0^\pi(a^2\cos^2t+2a\cos t)\,b\cos t\,dt$$

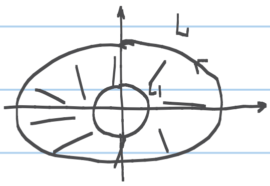
$$= a^2b\underline{\int_0^\pi\cos^3t\,dt}+2ab\int_0^\pi\cos^2t\,dt$$

$$= 2ab\int_0^\pi\frac{1+\cos2t}{2}\,dt = ab\pi.$$

法2: (Green公式). $I+\int_{\overrightarrow{BA}} = \iint_D(2x+2)\,dx\,dy = ab\pi.$

$$I = ab\pi + \int_{\overrightarrow{AB}} = ab\pi + \int_{a}^{-a} (x^2+2x)\, d0 = ab\pi.$$

例3：计算 $I = \oint_{L^+} \frac{y}{x^2+y^2} dx - \frac{x}{x^2+y^2} dy$. $L: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 逆时针为正 $(a.b>0)$.



取 L_1: $x^2+y^2=\varepsilon^2$ $0<\varepsilon<\min\{a,b\}$. 逆时针为正

L_1 与 L 围成区域 D

$$I + \int_{L_1^-} = \iint_D \left[\frac{\partial}{\partial x}\left(-\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right] dxdy = 0$$

$$-\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \qquad \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$I = -\int_{L_1^-} = \int_{L_1^+} = \int_{L_1^+} \frac{y}{\varepsilon^2} dx - \frac{x}{\varepsilon^2} dy$$

$$= \frac{1}{\varepsilon^2} \int_{L_1^+} y\, dx - x\, dy = \frac{1}{\varepsilon^2} \iint_D (-2)\, dxdy$$

$$= \frac{1}{\varepsilon^2} \cdot (-2) \cdot \pi \varepsilon^2 = -2\pi.$$