偏导数与高阶偏导数

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定义

定义1. 设函数$z=f(x,y)$在点(x_0,y_0)的某邻域内有定义，若极限

$$\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$$

存在，则称此极限为函数$z=f(x,y)$在点(x_0,y_0)处对x的偏导数，记为

$$\left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)};\ \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)};\ z_x\big|_{(x_0,y_0)};\ f_x(x_0,y_0).$$

定义

类似地，设函数$z = f(x, y)$在点(x_0, y_0)的某邻域内有定义,若极限

$$\lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

存在，则称此极限为函数$z = f(x, y)$在点(x_0, y_0)处对y的偏导数，记为

$$\left.\frac{\partial z}{\partial y}\right|_{(x_0, y_0)};\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)};\left.z_y\right|_{(x_0, y_0)};f_y(x_0, y_0).$$

定义

若函数$z = f(x, y)$在域D内每一点(x, y)处对x或y偏导数存在，则该偏导数称为偏导函数，也简称为偏导数，记为

$$\frac{\partial z}{\partial x};\frac{\partial f}{\partial x};z_x;f_x(x, y)$$

$$\frac{\partial z}{\partial y};\frac{\partial f}{\partial y};z_y;f_y(x, y)$$

推广

偏导数的概念可以推广到二元以上的函数。如

三元函数$u = f(x, y, z)$在点(x, y, z)处

对x的偏导数定义为：$f_x(x, y, z) = \lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$

对y的偏导数定义为：$f_y(x, y, z) = \lim\limits_{\Delta y \to 0} \dfrac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$

对z的偏导数定义为：$f_z(x, y, z) = \lim\limits_{\Delta z \to 0} \dfrac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$

偏导数与导数

$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} = \frac{d}{dx} f(x, y_0)\Big|_{x = x_0}$$

$$f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} = \frac{d}{dx} f(x_0, y)\Big|_{y = y_0}$$

二元函数$z = f(x, y)$对x的偏导数$f_x(x_0, y_0)$

$\rightarrow$ 固定$y = y_0$时，一元函数$f(x, y_0)$在x_0处的导数

二元函数$z = f(x, y)$对y的偏导数$f_y(x_0, y_0)$

$\rightarrow$ 固定$x = x_0$时，一元函数$f(x_0, y)$在y_0处的导数

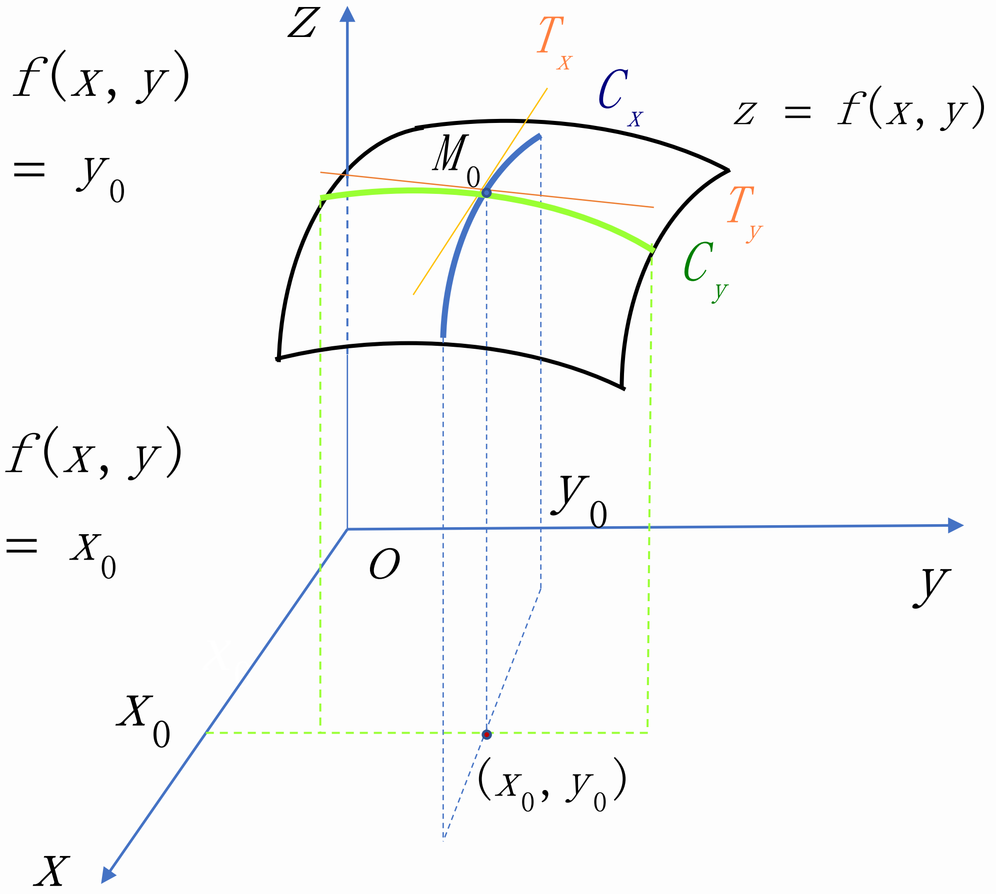
偏导数的几何意义

$\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \left.\dfrac{\mathrm{d}}{\mathrm{d}x} f(x, y_0)\right|_{x=x_0}$ 是曲线 C_x : $\begin{cases} z = f(x, y) \\ y = y_0 \end{cases}$

在点 M_0 处的切线 M_0T_x 对 x 轴的斜率

$\left.\dfrac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \left.\dfrac{\mathrm{d}}{\mathrm{d}y} f(x, y_0)\right|_{y=y_0}$ 是曲线 C_y : $\begin{cases} z = f(x, y) \\ x = x_0 \end{cases}$

在点 M_0 处的切线 M_0T_y 对 y 轴的斜率



高阶偏导数

设函数$z = f(x, y)$在区域D内具有偏导数

$$\frac{\partial z}{\partial x} = f_x(x, y), \frac{\partial z}{\partial y} = f_y(x, y).$$

若这两个函数的偏导数也存在，则称它们是函数$z = f(x, y)$的二阶偏导数.

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y), \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}(x, y),$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}(x, y), \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y).$$

高阶偏导数

类似地，可以定义更高阶的偏导数，如

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x \partial y}\right) = \frac{\partial^3 z}{\partial x \partial y^2} = f_{xy^2}(x, y)$$

谢谢！