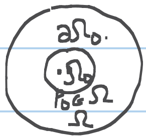
§8.5 场论简介.

$\vec{F}(x,y,z)=\{P(x,y,z),\ Q(x,y,z),\ R(x,y,z)\}$.

$\vec{F}$ 具有一阶连续偏导数.

1. 散度：Ω 空间有界闭区域. $\vec{F}(x,y,z)$ 在 Ω 上一阶连续偏导存在.

$P_0\in\Omega$. 找曲面包含 P_0. $P_0\in\Omega_0$.



(1) 定义：$\displaystyle\oiint_{\partial\Omega_0}\vec{F}\cdot\vec{n}\,dS$ 为通过 $\partial\Omega_0$ 的通量.

称 $\dfrac{\displaystyle\oiint_{\partial\Omega_0}\vec{F}\cdot\vec{n}\,dS}{V(\Omega_0)}$ 为通量体密度.

(2) 定义：若 $\displaystyle\lim_{\Omega_0\to P}\frac{\oiint_{\partial\Omega_0}\vec{F}\cdot\vec{n}\,dS}{V(\Omega_0)}$ 存在. 称其为 $\vec{F}$ 在 P_0 点处的散度 $\operatorname{div}\vec{F}(P_0)$.

(3) 计算：$\vec{F}=\{P,Q,R\}$.

$$\operatorname{div}\vec{F}(P_0)=\lim_{\Omega_0\to P_0}\frac{\oiint_{\partial\Omega_0}\vec{F}\cdot\vec{n}\,dS}{V(\Omega_0)}=\lim_{\Omega_0\to P_0}\frac{\iiint_{\Omega_0}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dxdydz}{V(\Omega_0)}$$

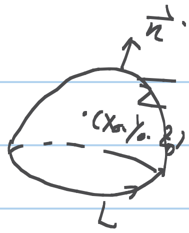
$$=\lim_{\Omega_0\to P_0}\frac{\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)(\xi,\eta,\theta)\,V(\Omega_0)}{V(\Omega_0)}$$

$$=\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\Bigg|_{P_0}$$

$$\operatorname{div}\vec{F}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$$

(4) 若 $\forall(x_0,y_0,z_0)\in\Omega_0$ $\operatorname{div}\vec{F}(x_0,y_0,z_0)=0$ 则称 $\vec{F}$ 为 Ω_0 上的无源场.

2. 旋度：L为封闭正向曲线，Σ以L为边界，L正方向与Σ外法向成右手系



(1) $\oint_{L^+} \vec{F}\cdot d\vec{l}$ 称为通过曲面Σ的环量.

称 $\dfrac{\oint_{L^+} \vec{F}\cdot d\vec{l}}{A(\Sigma)}$ 为通过Σ的环量面密度.

(2) 若 $\lim\limits_{\Sigma\to P_0} \dfrac{\oint_{L^+} \vec{F}\cdot d\vec{l}}{A(\Sigma)}$ 存在 ($P_0\in\Sigma$). 下面定义旋度.

$$\lim_{\Sigma\to P_0} \frac{\oint_{L^+} \vec{F}\cdot d\vec{l}}{A(\Sigma)} = \lim_{\Sigma\to P_0} \frac{\iint_{\Sigma} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy}{A(\Sigma)}$$

$$= \lim_{\Sigma\to P_0} \frac{\iint_{\Sigma} \left\{\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\}\cdot \vec{n}\, dS}{A(\Sigma)}.$$

$$= \left\{\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\}\Bigg|_{P_0} \cdot \vec{n}\Big|_{P_0}$$

定义： $\mathrm{rot}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

(3) 若 $\mathrm{rot}\,\vec{F} \equiv 0$ $P(x,y,z)\in\Omega$. 称 $\vec{F}$ 为 Ω 中的无旋场.

三度：梯度，散度，旋度

f可微 $\mathrm{grad}\,f = \left\{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\} = \nabla f$

$\vec{F}=\{P,Q,R\}$ 可微. $\mathrm{div}\,\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla\cdot\vec{F}$

$\mathrm{rot}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \nabla\times\vec{F}.$

$\nabla = \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}$

Green: $\oint_{\partial D^+} \vec{F}\cdot\vec{dl} = \iint_D \mathrm{rot}\vec{F}\cdot\vec{k}\,d\sigma$

Gauss: $\oiint_{\partial\Sigma^+} \vec{F}\cdot\vec{dS} = \iiint_\Omega \mathrm{div}\vec{F}\,dv$

Stokes: $\oint_{L^+} \vec{F}\cdot\vec{dl} = \iint_\Sigma \mathrm{rot}\vec{F}\cdot\vec{dS}$.

3. 特殊向量场.

保守场: 积分与路径无关 $\int_L \vec{F}\cdot\vec{dl}$

有势场: $\vec{F} = \left\{\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right\} = \nabla\varphi$.

无旋场: $\mathrm{rot}\,\vec{F} = 0$.

保守场 $\Leftrightarrow$ 有势场 $\underset{\Omega\text{单连通}}{\overset{\vec{F}\in C'}{\rightleftarrows}}$ 无旋场