§7.2 偏导数与高阶偏导数

一、偏导数

二元函数 $z=f(x,y)$ 在 (x_0,y_0) 的某邻域内有定义，给定自变量 x 和 y 在 (x_0,y_0) 处的增量 Δx 和 Δy，相应有函数值的增量：

- 全增量：$\Delta z=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)$
- 偏增量：$\Delta_x z=f(x_0+\Delta x,y_0)-f(x_0,y_0)$

  $\Delta_y z=f(x_0,y_0+\Delta y)-f(x_0,y_0)$

定义：若极限 $\lim\limits_{\Delta x\to0}\dfrac{\Delta_x z}{\Delta x}=\lim\limits_{\Delta x\to0}\dfrac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$ 存在，

则称该极限为函数 $f(x,y)$ 在 (x_0,y_0) 处对 x 的偏导数. 记作：

$\left.\dfrac{\partial z}{\partial x}\right|_{(x_0,y_0)}$，$\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)}$，$z_x(x_0,y_0)$，$f_x(x_0,y_0)$ 或 $z'_x(x_0,y_0)$，$f'_x(x_0,y_0)$.

$f_1'(x_0,y_0)$

若极限 $\lim\limits_{\Delta y\to0}\dfrac{\Delta_y z}{\Delta y}=\lim\limits_{\Delta y\to0}\dfrac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}$ 存在，

则称该极限为函数 $f(x,y)$ 在 (x_0,y_0) 处对 y 的偏导数. 记作：

$\left.\dfrac{\partial z}{\partial y}\right|_{(x_0,y_0)}$，$\left.\dfrac{\partial f}{\partial y}\right|_{(x_0,y_0)}$，$z_y(x_0,y_0)$，$f_y(x_0,y_0)$ 或 $z'_y(x_0,y_0)$，$f'_y(x_0,y_0)$.

$f_2'(x_0,y_0)$

注：
$$f_x(x_0,y_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$$
$$=\lim_{x\to x_0}\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}=\frac{\mathrm{d}}{\mathrm{d}x}f(x,y_0)\Big|_{x=x_0}$$

$$f_y(x_0,y_0)=\lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}$$
$$=\lim_{y\to y_0}\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}=\frac{\mathrm{d}}{\mathrm{d}y}f(x_0,y)\Big|_{y=y_0}$$

- 若$z=f(x,y)$在点(x_0,y_0)处对x和对y的偏导数都存在，则称$f(x,y)$在点(x_0,y_0)处可偏导.
- 若$z=f(x,y)$在区域D内每一点(x,y)处都存在偏导数，它们构成(x,y)的函数，称之为$z=f(x,y)$的偏导函数，简称$z=f(x,y)$的偏导数，记作：

$$\frac{\partial z}{\partial x},\ \frac{\partial f}{\partial x},\ z_x,\ z_x(x,y),\ f_x(x,y)\text{或}z'_x(x,y),\ f'_x(x,y).$$
$$\frac{\partial z}{\partial y},\ \frac{\partial f}{\partial y},\ z_y,\ z_y(x,y),\ f_y(x,y)\text{或}z'_y(x,y),\ f'_y(x,y).$$

- 二元函数$z=f(x,y)$在(x,y)处的**偏导数**

$$f_x(x,y)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}=\frac{\mathrm{d}}{\mathrm{d}x}f(x,y)$$

$$f_y(x,y)=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}=\frac{\mathrm{d}}{\mathrm{d}y}f(x,y)$$

- 三元函数$u=f(x,y,z)$在(x,y,z)处的**偏导数**

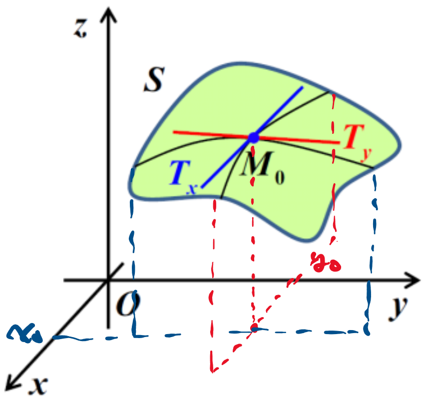
$$f_x(x,y,z)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y,z)-f(x,y,z)}{\Delta x}=\frac{\mathrm{d}}{\mathrm{d}x}f(x,y,z)$$

$f_y(x,y,z)$和$f_z(x,y,z)$的计算公式类似.

二元函数偏导数的几何意义

$$f_x(x_0,y_0)=\frac{\mathrm{d}}{\mathrm{d}x}f(x,y_0)\Big|_{x=x_0},$$

$$f_y(x_0,y_0)=\frac{\mathrm{d}}{\mathrm{d}y}f(x_0,y)\Big|_{y=y_0}$$



注: 一元函数$f(x)$在x_0点可导 $\Longrightarrow$ $f(x)$在x_0点连续

- 二元函数$z=f(x,y)$在点(x_0,y_0)处对x和y**偏导存在**,

$\not\Longrightarrow$ $f(x,y)$在(x_0,y_0)点连续

例如: $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0. & (x,y)=(0,0)\end{cases}$

因 $\lim\limits_{(x,y)\to(0,0)}f(x,y)$ 不存在. $\Rightarrow$ 在$(0,0)$点不连续.

$$\left.\begin{aligned}f_x(0,0)&=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{0-0}{x-0}=0\\ f_y(0,0)&=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0}=\lim_{y\to 0}\frac{0-0}{y-0}=0\end{aligned}\right\}\text{可偏导.}$$

例1：设 $z=\ln(x+\ln y)$，求 $\left.\frac{\partial z}{\partial x}\right|_{(1,e)}$，$\left.\frac{\partial z}{\partial y}\right|_{(1,e)}$.

解：$z|_{y=e}=\ln(x+1)$. $\Rightarrow \left.\frac{\partial z}{\partial x}\right|_{(1,e)}=\left.\frac{1}{x+1}\right|_{x=1}=\frac{1}{2}$

$z|_{x=1}=\ln(1+\ln y)$ $\Rightarrow \left.\frac{\partial z}{\partial y}\right|_{(1,e)}=\left.\frac{1}{1+\ln y}\cdot\frac{1}{y}\right|_{y=e}=\frac{1}{2e}$

(法二). $\frac{\partial z}{\partial x}=\frac{1}{x+\ln y}$ $\Rightarrow \left.\frac{\partial z}{\partial x}\right|_{(1,e)}=\left.\frac{1}{x+\ln y}\right|_{(1,e)}=\frac{1}{2}$

$\frac{\partial z}{\partial y}=\frac{1}{x+\ln y}\cdot\frac{1}{y}$ $\Rightarrow \left.\frac{\partial z}{\partial y}\right|_{(1,e)}=\frac{1}{2e}$.

例2：设 $z=x^y$ $(x>0$, 且 $x\neq 1)$，验证：$\frac{x}{y}\frac{\partial z}{\partial x}+\frac{1}{\ln x}\frac{\partial z}{\partial y}=2z$.

解：$\frac{\partial z}{\partial x}=y\cdot x^{y-1}$.　　$\frac{\partial z}{\partial y}=x^y\cdot\ln x$.

$\Rightarrow \frac{x}{y}\frac{\partial z}{\partial x}+\frac{1}{\ln x}\frac{\partial z}{\partial y}=x^y+x^y=2z$

例3：求 $r=\sqrt{x^2+y^2+z^2}$ 的偏导数.

解：$\frac{\partial r}{\partial x}=\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot 2x=\frac{x}{\sqrt{x^2+y^2+z^2}}=\frac{x}{r}$.

同理：$\frac{\partial r}{\partial y}=\frac{y}{\sqrt{x^2+y^2+z^2}}=\frac{y}{r}$.

$\frac{\partial r}{\partial z}=\frac{z}{\sqrt{x^2+y^2+z^2}}=\frac{z}{r}$

$\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2+\left(\frac{\partial r}{\partial z}\right)^2=1$

$$\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2+\left(\frac{\partial r}{\partial z}\right)^2=1.$$

例4: 已知某定量的理想气体的状态方程为 $pV=RT$ (R为常数),

证明: $\frac{\partial p}{\partial V}\cdot\frac{\partial V}{\partial T}\cdot\frac{\partial T}{\partial p}=-1.$

导数. $\frac{dy}{dx}\cdot\frac{dx}{dy}=1.$

$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$

$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$

$\frac{\partial z}{\partial x}$. $\frac{\partial z}{\partial y}$.

整体记号.

解: 由 $p=\frac{RT}{V}$. $\Rightarrow \frac{\partial p}{\partial V}=-\frac{RT}{V^2}$.

$V=\frac{RT}{p} \Rightarrow \frac{\partial V}{\partial T}=\frac{R}{p}$

$T=\frac{pV}{R} \Rightarrow \frac{\partial T}{\partial p}=\frac{V}{R}$

$\Rightarrow \frac{\partial p}{\partial V}\cdot\frac{\partial V}{\partial T}\cdot\frac{\partial T}{\partial p}=-\frac{RT}{pV}=-1.$

二、高阶偏导数

定义: 设 $z=f(x,y)$ 在区域 D 内具有偏导数,

$$\frac{\partial z}{\partial x}=f_x(x,y),\qquad \frac{\partial z}{\partial y}=f_y(x,y)$$

它们仍是 x,y 的函数, 若它们的偏导数也存在, 则称它们是 $z=f(x,y)$ 的二阶偏导数.　类似可定义三阶偏导数

- 四个二阶偏导数

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}=f_{xx}(x,y),\quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x\partial y}=f_{xy}(x,y)$$

f''_{11}　　f''_{12}

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y\partial x}=f_{yx}(x,y),\quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}=f_{yy}(x,y)$$

f''_{21}　　f''_{22}

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y\partial x}=f_{yx}(x,y),\quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}=f_{yy}(x,y)$$

f''_{21} $\quad$ f''_{22}

例5：求函数 $z=e^{x+2y}$ 的二阶偏导数及 $\dfrac{\partial^3 z}{\partial y\partial x^2}$.

解：$\dfrac{\partial z}{\partial x}=e^{x+2y}$. $\quad \dfrac{\partial z}{\partial y}=e^{x+2y}\cdot 2=2e^{x+2y}$.

$\dfrac{\partial^2 z}{\partial x^2}=e^{x+2y}$. $\quad \dfrac{\partial^2 z}{\partial y^2}=4e^{x+2y}$

$\dfrac{\partial^2 z}{\partial x\partial y}=2e^{x+2y}$. $\quad \dfrac{\partial^2 z}{\partial y\partial x}=2e^{x+2y}$

$$\frac{\partial^3 z}{\partial y\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y\partial x}\right)=2e^{x+2y}.$$

例6：设 $f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$,

求 $f_{xy}(0,0)$ 和 $f_{yx}(0,0)$.

解：当 $x^2+y^2=0$. $\quad f_x(0.0)=\lim\limits_{x\to 0}\dfrac{f(x.0)-f(0.0)}{x-0}=\lim\limits_{x\to 0}\dfrac{0-0}{x-0}=0$

$f_y(0.0)=\lim\limits_{y\to 0}\dfrac{f(0.y)-f(0.0)}{y-0}=\lim\limits_{y\to 0}\dfrac{0-0}{y-0}=0$.

当 $x^2+y^2\neq 0$. $\quad f_x(x.y)=y\dfrac{x^2-y^2}{x^2+y^2}+xy\dfrac{2x(x^2+y^2)-2x(x^2-y^2)}{(x^2+y^2)^2}$

$$\Rightarrow f_x(x.y)=\begin{cases}\dfrac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2}\cdot y. & x^2+y^2\neq 0. \\ 0. & x^2+y^2=0\end{cases}$$

$$\text{同理}\ f_y(x.y)=\begin{cases}\dfrac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2}\cdot x. & x^2+y^2\neq 0 \\ 0. & x^2+y^2=0\end{cases}$$

$$\Rightarrow f_{xy}(0,0)=\lim_{y\to0}\frac{f_x(0,y)-f_x(0,0)}{y-0}=\lim_{y\to0}\frac{-y}{y}=-1$$

$$f_{yx}(0,0)=\lim_{x\to0}\frac{f_y(x,0)-f_y(0,0)}{x-0}=\lim_{x\to0}\frac{x}{x}=1.$$

(注:). $f_x(0,0)=0$. $f_y(0,0)=0$.

$$f_x(0,y)=\lim_{x\to0}\frac{f(x,y)-f(0,y)}{x-0}=\lim_{x\to0}y\frac{x^2-y^2}{x^2+y^2}=-y$$

$$f_y(x,0)=x$$

定理：若函数$z=f(x,y)$的两个二阶混合偏导数$\dfrac{\partial^2 z}{\partial x\partial y}$和$\dfrac{\partial^2 z}{\partial y\partial x}$在区域$D$内连续，则在该区域内两者相等.

例7：验证函数$u=\dfrac{1}{\sqrt{x^2+y^2+z^2}}$满足方程$\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=0.$

Laplace方程.

解：
$$\frac{\partial u}{\partial x}=-\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot 2x$$
$$=-x(x^2+y^2+z^2)^{-\frac{3}{2}}.$$

$$\frac{\partial^2 u}{\partial x^2}=-(x^2+y^2+z^2)^{-\frac{3}{2}}+\frac{3}{2}x(x^2+y^2+z^2)^{-\frac{5}{2}}\cdot 2x$$
$$=\frac{3x^2-(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}=\frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}.$$

同理：$\dfrac{\partial^2 u}{\partial y^2}=\dfrac{2y^2-x^2-z^2}{(\cdots)^{\frac{5}{2}}}$. $\dfrac{\partial^2 u}{\partial z^2}=\dfrac{2z^2-x^2-y^2}{(\cdots)^{\frac{5}{2}}}$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0.$$