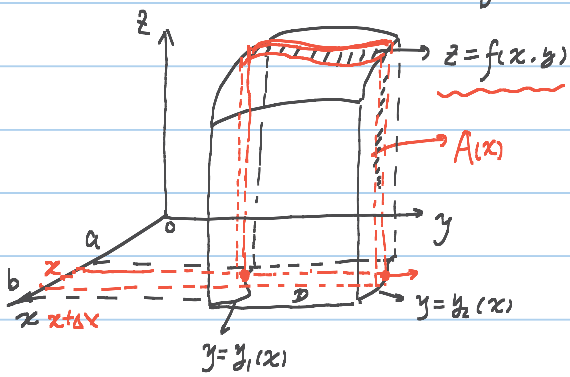
§7.2 直角坐标系下二重积分的计算

例1: $f(x,y)$ 在有界闭区域 D 上连续.

$D=\{(x,y)\mid a\le x\le b,\ y_1(x)\le y(x)\le y_2(x)\}$.

如何计算 $\iint\limits_D f(x,y)\,dxdy$.



$A(x)=\int_{y_1(x)}^{y_2(x)} f(x,y)\,dy$

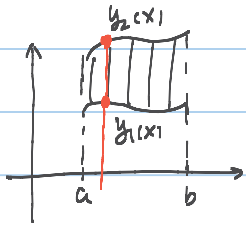
当 x 变到 $x+\Delta x$

$dV=A(x)\,\Delta x=A(x)\,dx$

从而 $V=\int_a^b A(x)\,dx$

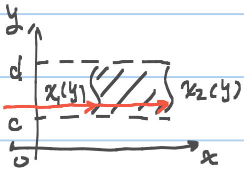
即 $\iint\limits_D f(x,y)\,dxdy=\int_a^b A(x)\,dx=\int_a^b\left(\int_{y_1(x)}^{y_2(x)} f(x,y)\,dy\right)dx$.

约定 $\int_a^b dx\int_{y_1(x)}^{y_2(x)} f(x,y)\,dy$ 累次积分.



"穿线法"

$D=\{(x,y)\mid c\le y\le d,\ x_1(y)\le x\le x_2(y)\}$



$\iint\limits_D f(x,y)\,dxdy=\int_c^d dy\int_{x_1(y)}^{x_2(y)} f(x,y)\,dx$

例1: $f(x,y)$ 连续. 将累次积分 $\int_{-1}^{0} dy\int_{1-y}^{2} f(x,y)\,dx$ 换序.

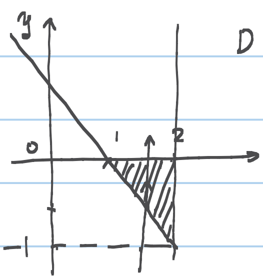
解: 由图得 $D=\{(x,y)\mid -1\le y\le 0,\ 1-y\le x\le 2\}$.

D 也可写成 $D=\{(x,y)\mid 1\le x\le 2,\ 1-x\le y\le 0\}$

$x=1-y$

$x+y=1$

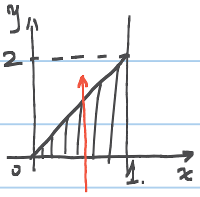
$x=2$



于是 $\int_{-1}^{0} dy\int_{1-y}^{2} f(x,y)\,dx$

$=\int_1^2 dx\int_{1-x}^{0} f(x,y)\,dy$

例2：求 $I=\iint_D \sqrt{4x^2-y^2}\,dxdy$　D 由 $y=0$．$x=1$．$y=2x$ 围成．



$$I=\iint_D \sqrt{4x^2-y^2}\,dxdy=\int_0^1 dx\int_0^{2x}\sqrt{4x^2-y^2}\,dy$$

$$=\int_0^1\left(\frac{1}{4}\pi\cdot(2x)^2\right)dx=\pi\int_0^1 x^2\,dx=\frac{\pi}{3}.$$

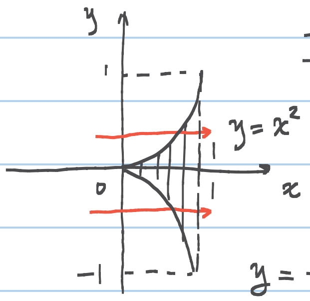
$\int_0^a\sqrt{a^2-x^2}\,dx$ $(a>0)$．

$=\frac{1}{4}\pi a^2$．

① D．
② $f(x,y)$

$\int\sqrt{x^2-a^2}\,dx$

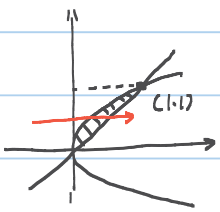
例3：交换积分次序 $I=\int_0^1 dx\int_{-x^2}^{x^2} f(x,y)\,dy$



$\Rightarrow x=\sqrt{-y}$

$$I=\int_{-1}^{0}dy\int_{\sqrt{-y}}^{1}f(x,y)\,dx+\int_0^1 dy\int_{\sqrt{y}}^{1}f(x,y)\,dx.$$

例4：D 由 $y=x$，$x=y^2$ 围成．计算 $\iint_D\frac{\sin y}{y}\,dxdy$



$$I=\iint_D\frac{\sin y}{y}\,dxdy$$

$$=\int_0^1 dy\int_{y^2}^{y}\frac{\sin y}{y}\,dx.$$

$$=\int_0^1\frac{\sin y}{y}\cdot(y-y^2)\,dy$$

$$=\int_0^1(\sin y-y\sin y)\,dy=1-\sin 1.$$