$$\iint_D (x+y)\,dxdy = \iint_{\substack{1\le u\le 2\\1\le v\le 2}} u\cdot\frac{u}{(1+v)^2}\,dudv = \int_1^2 u^2 du \int_1^2 \frac{1}{(1+v)^2}dv$$

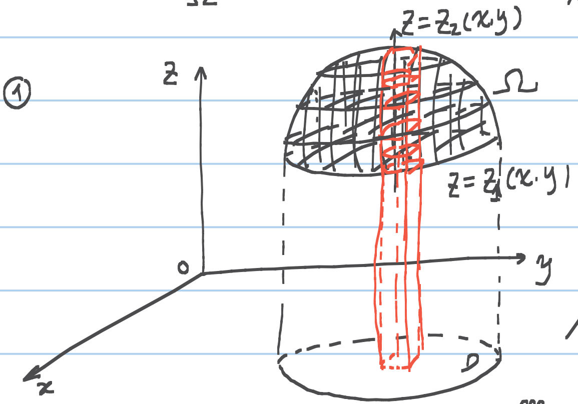
$$= \frac{7}{3}\times\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{7}{18}.$$

§7.3 直角坐标系下三重积分的计算

$f(x,y,z)\in C(\Omega)$ Ω为有界闭区域.

$$\iiint_\Omega f(x,y,z)\,dxdydz = \lim_{\lambda(T)\to 0}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} f(\xi_i,\eta_j,\theta_k)\,\Delta x_i\Delta y_j\Delta z_k$$

①



$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} f(\xi_i,\eta_j,\theta_k)\,\Delta x_i\Delta y_j\Delta z_k$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{m}\Delta x_i\Delta y_j\sum_{k=1}^{l} f(\xi_i,\eta_j,\theta_k)\Delta z_k.$$

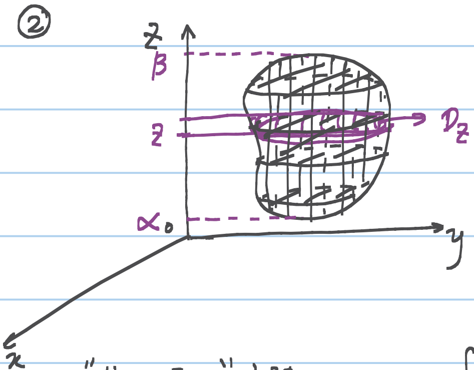
令 $\lambda(T)\to 0$ 有

"先一后二"法.

$$\iiint_\Omega f(x,y,z)\,dxdydz = \iint_D dxdy\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz$$

面密度.

②



$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} f(\xi_i,\eta_j,\theta_k)\,\Delta x_i\Delta y_j\Delta z_k$$

$$=\sum_{k=1}^{l}\Delta z_k\sum_{i=1}^{n}\sum_{j=1}^{m} f(\xi_i,\eta_j,\theta_k)\,\Delta x_i\Delta y_j$$

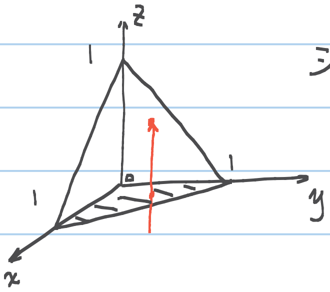
令 $\lambda(T)\to 0$

"先二后一"方法:

$$\iiint_\Omega f(x,y,z)\,dxdydz = \int_\alpha^\beta dz\iint_{D_z} f(x,y,z)\,dxdy$$

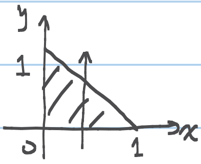
线密度.

例: Ω 由平面 $x+y+z=1$ 与三个坐标面围成. 计算 $\iiint\limits_{\Omega} x\,dxdydz$



法一: (先一后二法)

$$\iiint\limits_{\Omega} x\,dxdydz = \iint\limits_{\substack{x,y\geq 0\\ x+y\leq 1}} dxdy \int_0^{1-x-y} x\,dz$$

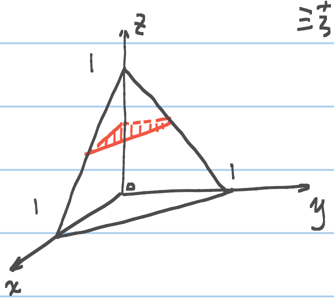


$$= \iint\limits_{\substack{x.y\geq 0\\ x+y\leq 1}} x(1-x-y)\,dxdy = \int_0^1 dx \int_0^{1-x} (x-x^2-xy)\,dy$$

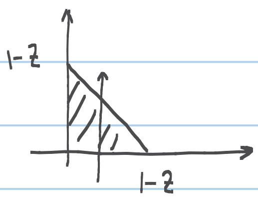
$$= \int_0^1 \left[(x-x^2)(1-x) - \frac{1}{2}x(1-x)^2\right] dx$$

$$= \int_0^1 \left[x-2x^2+x^3-\frac{1}{2}x-\frac{1}{2}x^3+x^2\right] dx = \int_0^1 \left(\frac{1}{2}x^3-x^2+\frac{1}{2}x\right) dx = \frac{1}{8}-\frac{1}{3}+\frac{1}{4} = \frac{1}{24}$$

法2: (先二后一法).



$$I = \iiint\limits_{\Omega} x\,dxdydz = \int_0^1 dz \iint\limits_{\substack{x.y\geq 0\\ x+y\leq 1-z}} x\,dxdy$$



$$= \int_0^1 dz \left[\int_0^{1-z} dx \int_0^{1-z-x} x\,dy\right]$$

$$= \int_0^1 dz \int_0^{1-z} (x-x^2-xz)\,dx$$

$$= \int_0^1 \left[\underline{\frac{1}{2}(1-z)^2} - \frac{1}{3}(1-z)^3 - z\cdot\underline{\frac{1}{2}(1-z)^2}\right] dz$$

$$= \int_0^1 \frac{1}{6}(1-z)^3\,dz = -\int_0^1 \frac{1}{6}(1-z)^3\,d(1-z)$$

$$= -\frac{1}{24}(1-z)^4\Big|_{z=0}^{z=1} = \frac{1}{24}.$$

法3: (先二后一).

$$\iiint\limits_{\Omega} x\,dxdydz = \int_0^1 dx \iint\limits_{\substack{y.z\geq 0\\ y+z\leq 1-x}} x\,dydz$$

$$= \int_0^1 x\,\frac{1}{2}(1-x)^2\,dx = \frac{1}{24}.$$