$$\frac{\partial^2 f}{\partial x_n \partial x_1},\ \frac{\partial^2 f}{\partial x_n \partial x_2},\ \cdots\ \frac{\partial^2 f}{\partial x_n^2}$$

例 设 $y=f(x)$ 是 n元函数，Δx 是 x^0 的增量. 求一元函数 $\varphi(t)=f(x^0+t\Delta x)$ 的一阶导数 $\varphi'(t)$，二阶导 $\varphi''(t)$.

解: $$\varphi'(t)=\frac{d\varphi(t)}{dt}=\frac{\partial f}{\partial x_1}\cdot\Delta x_1+\frac{\partial f}{\partial x_2}\cdot\Delta x_2+\cdots+\frac{\partial f}{\partial x_n}\cdot\Delta x_n \qquad \frac{\partial f}{\partial x_1} < \begin{matrix} x_1 \cdots t \\ x_2 \cdots t \\ \vdots \\ x_n \cdots t \end{matrix}$$

$$\cancel{\nabla f(x^0+t\Delta x)^T\cdot\Delta x}$$

$$\vec{x}=x^0+t\Delta x$$

$$\varphi''(t)=\frac{d}{dt}\Big(\frac{\partial f}{\partial x_1}\cdot\underbrace{\Delta x_1}_{\text{不含}t}\Big)=\Big(\frac{\partial^2 f}{\partial x_1^2}\cdot\Delta x_1+\frac{\partial^2 f}{\partial x_1\partial x_2}\Delta x_2+\cdots\frac{\partial^2 f}{\partial x_1\partial x_n}\cdot\Delta x_n\Big)\Delta x_1$$

$$\| $$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 f}{\partial x_i\partial x_j}\Delta x_i\Delta x_j=\Delta x^T\nabla^2 f(x^0+t\Delta x)\Delta x$$

泰勒公式：（多元函数）　微分学的顶峰

定理：n元 $f(x)$ 在 x^0 的某 $U(x^0)$ 内二阶可微，则至少存在一个 $\theta\in(0,1)$

s.t. $f(x)=f(x^0)+\nabla f(x^0)^T(x-x^0)+R_1$,

$$R_1=\frac{1}{2!}(x-x^0)^T\nabla^2 f(x^0+\theta(x-x^0))(x-x^0) \qquad \text{“Lagrange余项”}$$

矩阵

若 $f(x)$ 在 $U(x^0)$ 二阶偏导连续，则

$$f(x)=f(x^0)+\nabla f(x^0)^T(x-x^0)+\frac{1}{2}(x-x^0)^T\nabla^2 f(x^0)(x-x^0)+R_2$$

$$R_2=o(\|x-x^0\|^2)\quad (x\to x^0) \qquad \text{Peano余项}$$

证: 设 $\varphi(t)=f(x^0+t\Delta x)$, $\Delta x=x-x^0$　　$x^0+t\Delta x=x$

上一个例子 $\Rightarrow$ $\varphi(0)=f(x^0)$

$\varphi'(0)=\nabla f(x^0)(x-x_0)$

$\varphi''(0)=(x-x^0)^T\nabla^2 f(x^0)(x-x^0)$

$\varphi''(\theta t)=(x-x^0)^T\nabla^2 f(x^0+\theta t\Delta x)(x-x^0)$　$\forall\theta$.

一元函数泰勒公式

$$\varphi(t)=\varphi(0)+\varphi'(0)t+\frac{1}{2!}\varphi''(\theta t)t^2 \qquad (\theta\in(0,1))$$

$$f(x)=\varphi(1)=\varphi(0)+\varphi'(0)+\frac{1}{2!}\varphi''(\theta t) \qquad \#$$

二元函数 $$f(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$
$$+\frac{1}{\ }(x-x_0,y-y_0)\begin{bmatrix} f_{xx} & f_{xy} \end{bmatrix}\begin{bmatrix} x-x_0 \end{bmatrix}+o(\rho^2)$$

$$+\frac{1}{2!}(x-x_0,\ y-y_0)\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}+o(\rho^2)$$

求泰勒公式，也就是求各偏导数值