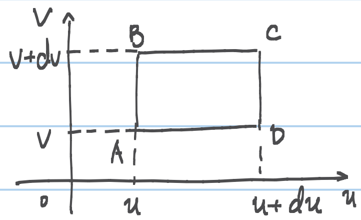
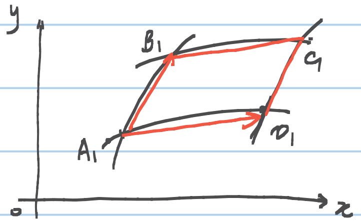
§7.2 二重积分的变量代换

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$ 且 $x(u,v)$, $y(u,v)$ 可微 $\dfrac{\partial(x,y)}{\partial(u,v)} \neq 0$.





$A_1(x(u,v),\ y(u,v))$ $B_1(x(u,v+dv),\ y(u,v+dv))$

$C_1(x(u+du,v+dv),\ y(u+du,v+dv))$ $D_1(x(u+du,v),\ y(u+du,v))$

$$\overrightarrow{A_1B_1} = \left(x(u,v+dv)-x(u,v),\ y(u,v+dv)-y(u,v)\right) \approx \left(\frac{\partial x}{\partial v}dv,\ \frac{\partial y}{\partial v}dv\right)$$

同理 $\overrightarrow{A_1D_1} \approx \left(\dfrac{\partial x}{\partial u}du,\ \dfrac{\partial y}{\partial u}du\right)$

$$\left|\overrightarrow{A_1B_1} \times \overrightarrow{A_1D_1}\right| = \left|\det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial v}dv & \frac{\partial y}{\partial v}dv & 0 \\ \frac{\partial x}{\partial u}du & \frac{\partial y}{\partial u}du & 0 \end{bmatrix}\right| + o(\rho) \qquad \rho = \sqrt{(du)^2+(dv)^2}$$

$$= \left|\det\begin{bmatrix} \frac{\partial x}{\partial v}dv & \frac{\partial y}{\partial v}dv \\ \frac{\partial x}{\partial u}du & \frac{\partial y}{\partial u}du \end{bmatrix}\vec{k}\right| = \left|\det\begin{bmatrix} \frac{\partial x}{\partial v}dv & \frac{\partial y}{\partial v}dv \\ \frac{\partial x}{\partial u}du & \frac{\partial y}{\partial u}du \end{bmatrix}\right|$$

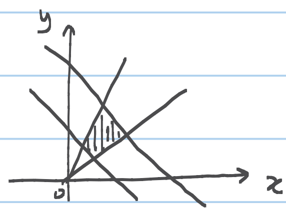
$$= \frac{\partial(x,y)}{\partial(u,v)}\,dudv$$ （u, v 从小参数积分到大参数）

$$d\sigma = \frac{\partial(x,y)}{\partial(u,v)}\,dudv$$

$$\iint_{D_{xy}} f(x,y)\,dxdy = \iint_{D_{uv}} f(x(u,v),\ y(u,v))\,\frac{\partial(x,y)}{\partial(u,v)}\,dudv.$$

例：D 由 $x+y=1$　$x+y=2$.　$y=x$.　$y=2x$ 围成的有界闭区域

计算 $\iint_D (x+y)\,dxdy$



解 $$\begin{cases} x+y = u \\ \dfrac{y}{x} = v \end{cases}$$ $D_{uv} = \{(u,v) \mid 1 \le u \le 2,\ 1 \le v \le 2\}$.

$$\begin{cases} x = \dfrac{u}{1+v} \\ y = \dfrac{uv}{1+v} \end{cases}$$ 则 $$\frac{\partial(x,y)}{\partial(u,v)} = \left|\det\begin{bmatrix} \frac{1}{1+v} & \frac{v}{1+v} \\ \frac{-u}{(1+v)^2} & \frac{u}{(1+v)^2} \end{bmatrix}\right|$$

$$= \frac{u}{(1+v)^2}$$

$$\iint_D (x+y)\,dxdy = \iint_{\substack{1\le u\le 2\\ 1\le v\le 2}} u\cdot\frac{u}{(1+v)^2}\,dudv = \int_1^2 u^2\,du \int_1^2 \frac{1}{(1+v)^2}\,dv$$

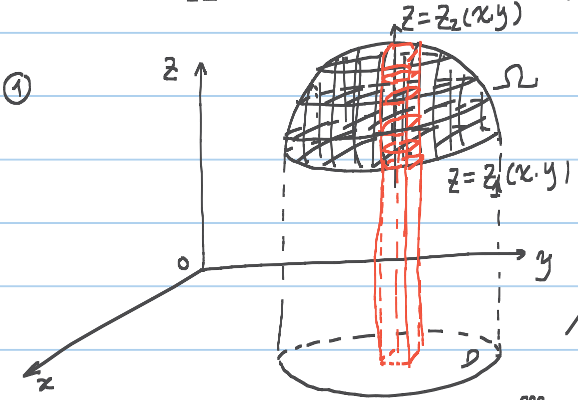
$$= \frac{7}{3}\times\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{7}{18}.$$

§7.3 直角坐标系下三重积分的计算

$f(x,y,z)\in C(\Omega)$ Ω为有界闭区域.

$$\iiint_\Omega f(x,y,z)\,dxdydz = \lim_{\lambda(T)\to 0} \underline{\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} f(\xi_i,\eta_j,\theta_k)\,\Delta x_i\Delta y_j\Delta z_k}$$

①



$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} f(\xi_i,\eta_j,\theta_k)\,\Delta x_i\Delta y_j\Delta z_k$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{m}\Delta x_i\Delta y_j \sum_{k=1}^{l} f(\xi_i,\eta_j,\theta_k)\,\Delta z_k.$$

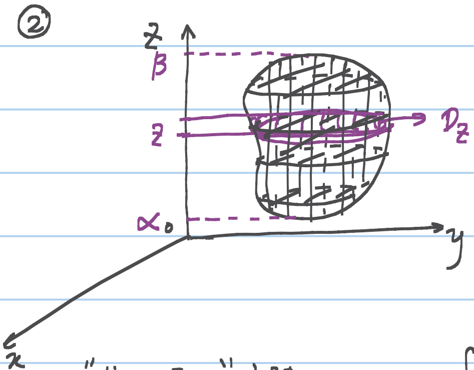
令 $\lambda(T)\to 0$ 有

"先一后二"法.

$$\iiint_\Omega f(x,y,z)\,dxdydz = \iint_D dxdy \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz$$

面密度.

②



$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} f(\xi_i,\eta_j,\theta_k)\,\Delta x_i\Delta y_j\Delta z_k$$

$$=\sum_{k=1}^{l}\Delta z_k \sum_{i=1}^{n}\sum_{j=1}^{m} f(\xi_i,\eta_j,\theta_k)\,\Delta x_i\Delta y_j$$

令 $\lambda(T)\to 0$

"先二后一"方法:

$$\iiint_\Omega f(x,y,z)\,dxdydz = \int_\alpha^\beta dz \iint_{D_z} f(x,y,z)\,dxdy$$

线密度.