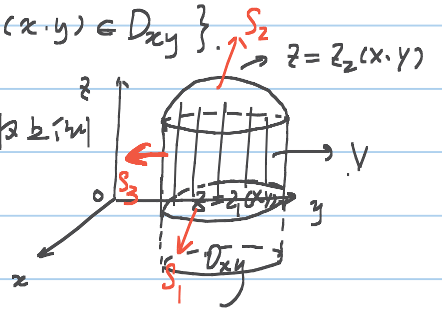
§8.4 高斯公式

定理：(Gauss) 设空间闭区域 V 是由光滑或分片光滑的封闭曲面 S 围成的单连通区域，P、Q、R 在 V 上具有一阶连续偏导数，则

$$\oiint_{S^+} P\,dydz + Q\,dzdx + R\,dxdy = \iiint_V \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dxdydz$$

S 取外侧

简单证明：$V=\{(x,y,z) \mid z_1(x,y)\le z\le z_2(x,y),\ (x,y)\in D_{xy}\}$.



V 底面 S_1，S_1 取下侧　V 顶面 S_2：$z=z_2(x,y)$ 取上侧

V 侧面 S_3：取外侧

$$\iiint_V \frac{\partial R}{\partial z}dxdydz = \iint_{D_{xy}} dxdy\int_{z_1(x,y)}^{z_2(x,y)} \frac{\partial R}{\partial z}dz$$

$$= \iint_{D_{xy}} \left[R(x,y,z_2(x,y)) - R(x,y,z_1(x,y))\right]dxdy.$$

$$\oiint_S R(x,y,z)\,dxdy = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$$

S_1：$z=z_1(x,y)$

$\vec{n} /\!/ \{\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\}$

$$= \iint_{D_{xy}} R(x,y,z_1(x,y))(-1)\,dxdy$$

S_2　$z=z_2(x,y)$

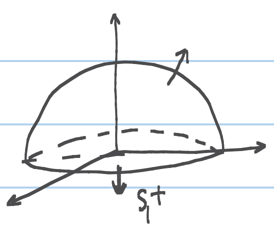
$\vec{n} /\!/ \{\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\}$

$$+ \iint_{D_{xy}} R(x,y,z_2(x,y))\cdot 1\,dxdy + 0$$

$$= \iint_{D_{xy}} \left[R(x,y,z_2(x,y)) - R(x,y,z_1(x,y))\right]dxdy.$$

其余同理可证.

例：$I=\iint_\Sigma xz\,dydz + 2zy\,dzdx + 3xy\,dxdy$　Σ^+：$z=1-x^2-\frac{y^2}{4}$　$0\le z\le 1$. 上侧为正.



S_1：$\begin{cases} z=0 \\ x^2+\frac{y^2}{4}\le 1.\end{cases}$　取下侧为正.

$$I+\iint_{S_1^+} = \iiint_\Omega (z+2z+0)\,dxdydz$$

$$= 3\int_0^1 dz \iint_{x^2+\frac{y^2}{4}\le 1-z} z\,dxdy$$

$$= 3\int_0^1 z\,\pi(1-z)\cdot 2\,dz$$

$$= 6\pi\left(\frac{1}{2}-\frac{1}{3}\right) = \pi.$$

$\vec{n} = \{0, 0, 1\}$

$$I = \pi + \iint_{S_1^-} xz\,dydz + 2zy\,dzdx + 3xy\,dxdy$$

$$= \pi + \iint_{D_{xy}} 3xy\,dxdy = \pi.$$