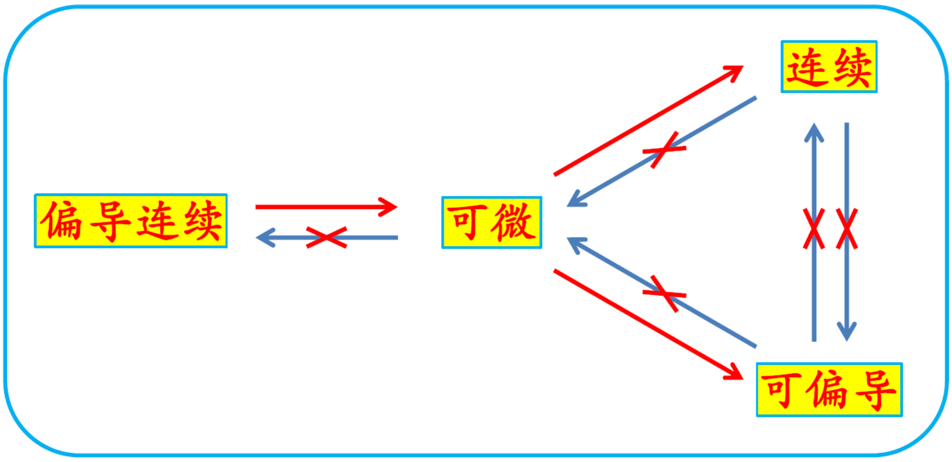
$y=0$

$$\Rightarrow \lim_{\substack{x\to 0\\ y\to 0}} f_x(x,y) \neq f_x(0,0) \Rightarrow \text{偏导不连续}$$

同理. $\lim_{\substack{x\to 0\\ y\to 0}} f_y(x,y) \neq f_y(0,0)$



例5: 讨论函数 $f(x,y)=|x-y|\varphi(x,y)$ 在 $(0,0)$ 点的可微性,

其中函数 $\varphi(x,y)$ 在 $(0,0)$ 点连续.

解: 由 $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{\substack{x\to 0\\ y\to 0}} |x-y|\varphi(x,y) = 0 = f(0,0)$

$\Rightarrow$ 连续.

由 $f_x(0,0) = \lim_{x\to 0} \dfrac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0} \dfrac{|x|}{x}\varphi(x,0)$

$\left(\lim_{x\to 0^-} \dfrac{|x|}{x}\varphi(x,0) = -\varphi(0,0). \quad \lim_{x\to 0^+} \dfrac{|x|}{x}\varphi(x,0) = \varphi(0,0) \right)$

$\Rightarrow$ 当 $\varphi(0,0)\neq 0$. $\Rightarrow$ 不可偏导. $\Rightarrow$ 不可微.

当 $\varphi(0,0)=0$ $\Rightarrow$ 可偏导. 且 $f_x(0,0)=0$. $f_y(0,0)=0$.

由 $\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \dfrac{\Delta z - f_x(0,0)\Delta x - f_y(0,0)\Delta y}{\rho} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \dfrac{|\Delta x-\Delta y|\varphi(\Delta x,\Delta y)}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$

$= 0 \quad \left(\dfrac{|\Delta x-\Delta y|}{\sqrt{(\Delta x)^2+(\Delta y)^2}} \leq \sqrt{2} \right) \Rightarrow$ 可微.

三、全微分的几何意义

函数 $z=f(x,y)$ 在点 (x_0,y_0) 处可微，即

$$\Delta z = f(\underbrace{x_0+\Delta x}_{x},\underbrace{y_0+\Delta y}_{y}) - f(x_0,y_0)$$
$$= f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + o(\rho)$$

$$f(x,y)-f(x_0,y_0) = f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)+o(\rho)$$

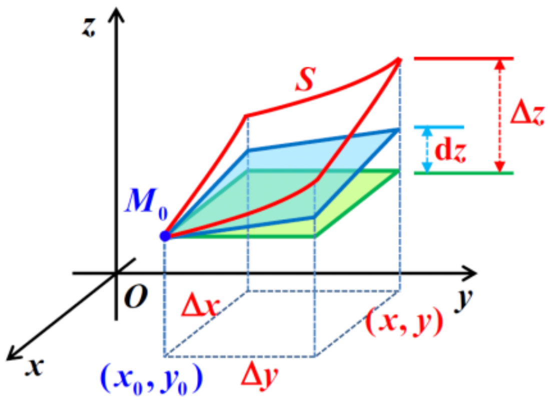
则当 $|\Delta x|$ 和 $|\Delta y|$ 很小时，即在点 (x_0,y_0) 附近，有

$$f(x,y)\approx f(x_0,y_0)+\underbrace{f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)}_{dz}$$

- 令 $z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$

  即 $f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)-(z-f(x_0,y_0))=0$

过点 $M_0(x_0,y_0,f(x_0,y_0))$
法向量 $\vec{n}=(f_x(x_0,y_0),f_y(x_0,y_0),-1)$
切平面



切平面 $f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)=z-f(x_0,y_0)$

- 则当 $|\Delta x|$ 和 $|\Delta y|$ 很小时，即在点 (x_0,y_0) 附近，有 $\Delta z\approx dz$，即

$$f(x,y)\approx f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

四、全微分的计算与应用

1. 全微分的计算

- 全微分计算公式： $\mathrm{d}z=\dfrac{\partial z}{\partial x}\mathrm{d}x+\dfrac{\partial z}{\partial y}\mathrm{d}y$

例6：计算函数 $z=\dfrac{\ln(xy)}{y}$ 在点(2,1)处的全微分.

解：由 $\dfrac{\partial z}{\partial x}=\dfrac{1}{y}\cdot\dfrac{1}{xy}\cdot y=\dfrac{1}{xy}$. $\Rightarrow \left.\dfrac{\partial z}{\partial x}\right|_{(2,1)}=\dfrac{1}{2}$

$\dfrac{\partial z}{\partial y}=-\dfrac{1}{y^2}\ln(xy)+\dfrac{1}{y}\dfrac{1}{xy}\cdot x=\dfrac{1}{y^2}(1-\ln(xy))$

$\Rightarrow \left.\dfrac{\partial z}{\partial y}\right|_{(2,1)}=1-\ln 2 \quad \Rightarrow \left.\mathrm{d}z\right|_{(2,1)}=\dfrac{1}{2}\mathrm{d}x+(1-\ln 2)\mathrm{d}y.$

例7：计算函数 $u=x+\sin\dfrac{y}{2}+\mathrm{e}^{yz}$ 的全微分.

解：由 $\dfrac{\partial u}{\partial x}=1.\quad \dfrac{\partial u}{\partial y}=\dfrac{1}{2}\cos\dfrac{y}{2}+z\,e^{yz}.\quad \dfrac{\partial u}{\partial z}=y\,e^{yz}.$

$\Rightarrow \mathrm{d}u=\dfrac{\partial u}{\partial x}\mathrm{d}x+\dfrac{\partial u}{\partial y}\mathrm{d}y+\dfrac{\partial u}{\partial z}\mathrm{d}z$

$=\mathrm{d}x+\left(\dfrac{1}{2}\cos\dfrac{y}{2}+z\,e^{yz}\right)\mathrm{d}y+y\,e^{yz}\mathrm{d}z.$

- 全微分的四则运算法则

设u,v为多元可微函数，则有：

① $\mathrm{d}(u\pm v)=\mathrm{d}u\pm\mathrm{d}v$

② $\mathrm{d}(uv)=u\,\mathrm{d}v+v\,\mathrm{d}u$

③ $\mathrm{d}\left(\dfrac{u}{v}\right)=\dfrac{v\,\mathrm{d}u-u\,\mathrm{d}v}{v^2}$

例8：求下列函数的全微分.

(1) $u=x^3y+xz^2$;　　(2) $z=\dfrac{x^2-y^2}{x^2+y^2}$.

解：(1). $\mathrm{d}u=\mathrm{d}(x^3y)+\mathrm{d}(xz^2)$

$=x^3\mathrm{d}y+y\,\mathrm{d}(x^3)+z^2\mathrm{d}x+x\,\mathrm{d}(z^2)$

$=(3x^2y+z^2)\mathrm{d}x+x^3\mathrm{d}y+2xz\,\mathrm{d}z$

$$= (3x^2y + z^2)dx + x^3dy + 2xz\,dz$$

$$\left(\frac{\partial u}{\partial x} = 3x^2y + z^2,\quad \frac{\partial u}{\partial y} = x^3,\quad \frac{\partial u}{\partial z} = 2xz\right)$$

(2). $$dz = \frac{(x^2+y^2)\,d(x^2-y^2) - (x^2-y^2)\,d(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)(2x\,dx - 2y\,dy) - (x^2-y^2)(2x\,dx + 2y\,dy)}{(x^2+y^2)^2}$$

$$= \frac{4xy^2\,dx - 4x^2y\,dy}{(x^2+y^2)^2}$$

2. 全微分的应用

- 近似计算

中值公式： 若函数 $f(x,y)$ 在点 (x,y) 附近可偏导，则有

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) = f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) + \cdots$$

$$= f_x(x+\theta_1\Delta x, y+\Delta y)\Delta x + f_y(x, y+\theta_2\Delta y)\Delta y \quad (0<\theta_1,\theta_2<1)$$

增量公式： 若函数 $f(x,y)$ 在点 (x,y) 处可微，则有

$$\Delta z = f_x(x,y)\Delta x + f_y(x,y)\Delta y + o(\rho)$$

$$= \mathrm{d}z + o(\rho) \qquad (\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2})$$

当 $|\Delta x|$ 和 $|\Delta y|$ 很小时，即在点 (x,y) 附近，有 $\Delta z \approx \mathrm{d}z$，即

$$f(x+\Delta x, y+\Delta y) \approx f(x,y) + f_x(x,y)\Delta x + f_y(x,y)\Delta y$$

例9：计算 $1.04^{2.02}$ 的近似值.

解：令 $z = f(x,y) = x^y$.

取 $x=1,\ y=2,\ \Delta x = 0.04,\ \Delta y = 0.02$.

$f_x(x,y) = yx^{y-1}$, $\quad f_y(x,y) = x^y \ln x$.

$$\Rightarrow 1.04^{2.02} = f(x+\Delta x, y+\Delta y)$$

$$\approx f(x,y) + f_x(x,y)\Delta x + f_y(x,y)\Delta y$$

$$\Rightarrow 1.04 \quad = f(x+\Delta x, y+\Delta y)$$

$$\approx f(x,y) + f_x(x,y)\Delta x + f_y(x,y)\Delta y$$

$$= f(1,2) + f_x(1,2)\times 0.04 + f_y(1,2)\times 0.02$$

$$= 1^2 + 2\times 0.04 + 0$$

$$= 1.08$$

例10: 一圆柱体受压后发生形变, 半径由20cm增大到20.05cm, 高度由100cm减少到99cm, 求此圆柱体体积的近似改变量.

解: 由 $V=\pi r^2 h \qquad \Delta r = 0.05. \qquad \Delta h = -1.$

$$\Delta V \approx dV = 2\pi r h\Delta r + \pi r^2 \Delta h \qquad r = 20. \quad h = 100.$$

$$= 2\pi\times 20\times 100\times 0.05 - \pi\times 20^2$$

$$= -200\pi \ (cm^3)$$

$$\left|\frac{\Delta V}{V}\right| = 0.5\%$$

- 误差估计

$$\Delta z \approx \mathrm{d}z = f_x(x,y)\Delta x + f_y(x,y)\Delta y$$

设测量x产生的绝对误差(限)为δ_x, 测量y的绝对误差(限)为δ_y

即: $|\Delta x| \le \delta_x$, $|\Delta y| \le \delta_y$

则由公式$z = f(x,y)$计算z所产生的绝对误差(限)约为

$$\delta_z = |f_x(x,y)|\delta_x + |f_y(x,y)|\delta_y$$

相对误差(限)约为

$$\frac{\delta_z}{|z|} = \left|\frac{f_x(x,y)}{f(x,y)}\right|\delta_x + \left|\frac{f_y(x,y)}{f(x,y)}\right|\delta_y$$

注:

- 当$z = xy$时: $\dfrac{\delta_z}{|z|} = \left|\dfrac{y}{xy}\right|\delta_x + \left|\dfrac{x}{xy}\right|\delta_y = \dfrac{\delta_x}{|x|} + \dfrac{\delta_y}{|y|}$

- 当$z = \dfrac{y}{x}$时: $\dfrac{\delta_z}{|z|} = \left|\dfrac{x}{y}\cdot\left(-\dfrac{y}{x^2}\right)\right|\delta_x + \left|\dfrac{x}{y}\cdot\dfrac{1}{x}\right|\delta_y = \dfrac{\delta_x}{|x|} + \dfrac{\delta_y}{|y|}$

乘除后的运算结果, 相对误差变大.

例11: 利用公式$S = \dfrac{1}{2}ab\sin C$计算三角形的面积. 现测得

$$a = 12.5 \pm 0.01,\ \ b = 8.3 \pm 0.01,\ \ C = 30^\circ \pm 0.1^\circ$$

求计算三角形的面积时的绝对误差与相对误差.

解: $\delta_S = \left|\frac{\partial S}{\partial a}\right|\delta_a + \left|\frac{\partial S}{\partial b}\right|\delta_b + \left|\frac{\partial S}{\partial C}\right|\delta_C$

$= \left|\frac{1}{2}b\sin C\right|\delta_a + \left|\frac{1}{2}a\sin C\right|\delta_b + \left|\frac{1}{2}ab\cos C\right|\delta_C$

$a = 12.5$. $\delta_a = 0.01$. $b = 8.3$. $\delta_b = 0.01$. $C = \frac{\pi}{6}$. $\delta_C = \frac{\pi}{1800}$

$\Rightarrow \delta_S = 0.13$. (绝对误差).

相对误差. $\left|\frac{\delta_S}{S}\right|$. $S = \frac{1}{2}ab\sin C \approx 25.94$

$\Rightarrow \left|\frac{\delta_S}{S}\right| \approx 0.5\%$

五、高阶全微分

$d^2z = (df_x(x,y))\cdot dx + (df_y(x,y))\cdot dy$
$= (f_{xx}(x,y)dx + f_{xy}(x,y)dy)\cdot dx$
$+ (\cdots)dy.$

定义： 设函数 $z=f(x,y)$ 在点 (x,y) 处可微，则

$$dz = f_x(x,y)dx + f_y(x,y)dy$$

dz 仍是 x,y 的函数，若它在点 (x,y) 处也可微，则称函数 $f(x,y)$ **二阶可微**，并且称 dz 的全微分为函数 $z=f(x,y)$ 在点 (x,y) 处的**二阶全微分**，记作 d^2z.

$dx\cdot dx$ $\quad$ $d(x^2)=2xdx$

$$d^2z = f_{xx}(x,y)dx^2 + (f_{xy}(x,y)+f_{yx}(x,y))dxdy + f_{yy}(x,y)dy^2$$

- 若 $f_{xy}(x,y)$ 和 $f_{yx}(x,y)$ 在点 (x,y) 处连续，则

$$d^2z = f_{xx}(x,y)dx^2 + 2f_{xy}(x,y)dxdy + f_{yy}(x,y)dy^2$$

- n阶全微分： $d^nz = d(d^{n-1}z)$

例12： 设 $z=\ln(x^2+y^2)$，求 dz 和 d^2z.

解：1由 $\frac{\partial z}{\partial x} = \frac{2x}{x^2+y^2}$. $\quad \frac{\partial z}{\partial y} = \frac{2y}{x^2+y^2}$

$\Rightarrow dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{2xdx+2ydy}{x^2+y^2}$.

2由 $\frac{\partial^2 z}{\partial x^2} = \frac{2(x^2+y^2)-2x\cdot 2x}{(x^2+y^2)^2} = 2\frac{y^2-x^2}{(x^2+y^2)^2}$.

$\frac{\partial^2 z}{\partial x\partial y} = -\frac{4xy}{(x^2+y^2)^2} = \frac{\partial^2 z}{\partial y\partial x}$.

$\frac{\partial^2 z}{\partial y^2} = 2\frac{x^2-y^2}{(x^2+y^2)^2}$

$\Rightarrow d^2z = \frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x\partial y}dxdy + \frac{\partial^2 z}{\partial y^2}dy^2$

$= \cdots$