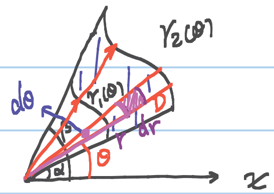
§7.2. 极坐标系下二重积分的计算

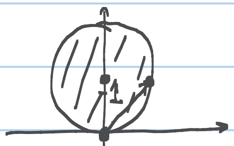


$f(x,y)$ 连续，有界闭区域 D 在极坐标系下的形式为 $D=\{(r,\theta)\mid \alpha\le\theta\le\beta,\ r_1(\theta)\le r\le r_2(\theta)\}$

$$d\sigma = dr\cdot r\,d\theta = r\,dr\,d\theta$$

$$\iint_D f(x,y)\,dxdy = \int_\alpha^\beta d\theta\int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta,\ r\sin\theta)\,r\,dr$$

例1：计算：$I=\iint_D \sqrt{x^2+y^2}\,dxdy \qquad D=\{(x,y)\mid x^2+y^2\le 2y\}$



解：令 $x=r\cos\theta,\ y=r\sin\theta$ 则 $D=\{(r,\theta)\mid 0\le\theta\le\pi,\ 0\le r\le 2\sin\theta\}$

$x^2+y^2=2y$

$r^2=2r\sin\theta$

$\Rightarrow r=2\sin\theta$

$$I=\int_0^\pi d\theta\int_0^{2\sin\theta} r\cdot r\,dr = \int_0^\pi d\theta\int_0^{2\sin\theta} r^2\,dr$$

$$=\int_0^\pi \frac{8}{3}\sin^3\theta\,d\theta = \frac{8}{3}\int_0^\pi \sin^2\theta\,d(-\cos\theta)$$

$$=-\frac{8}{3}\int_0^\pi(1-\cos^2\theta)\,d(\cos\theta) = -\frac{8}{3}\left(\cos\theta-\frac{1}{3}\cos^3\theta\right)\Big|_{\theta=0}^{\theta=\pi}$$

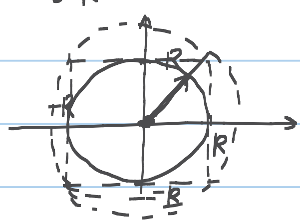
$$=\frac{8}{3}\left(1-\frac{1}{3}\right)\times 2 = \frac{8}{3}\times\frac{2}{3}\times 2 = \frac{32}{9}.$$

例2：利用二重积分计算 $\int_{-\infty}^{+\infty} e^{-x^2}dx$.

解：$I_R=\int_{-R}^{R} e^{-x^2}dx$ 从而 $\int_{-\infty}^{+\infty} e^{-x^2}dx=\lim_{R\to+\infty}\int_{-R}^{R}e^{-x^2}dx$.

$$I_R^2=I_R\cdot I_R=\int_{-R}^{R}e^{-x^2}dx\cdot\int_{-R}^{R}e^{-y^2}dy=\int_{-R}^{R}dx\int_{-R}^{R}e^{-x^2}\cdot e^{-y^2}dy$$

$$=\iint_{[-R,R]^2} e^{-(x^2+y^2)}dxdy$$



$$\iint_{x^2+y^2\le R^2} e^{-(x^2+y^2)}dxdy \le \boxed{I_R^2} \le \iint_{x^2+y^2\le 2R^2} e^{-(x^2+y^2)}dxdy$$

左边 $=\int_0^{2\pi}d\theta\int_0^R e^{-r^2}\cdot r\,dr = 2\pi\cdot\frac{-1}{2}\int_0^R e^{-r^2}d(-r^2) = -\pi\left(e^{-r^2}\right)\Big|_0^R=\pi(1-e^{-R^2})$

右边 $=\pi(1-e^{-2R^2})$. $\quad\lim_{R\to+\infty}\pi(1-e^{-R^2})=\lim_{R\to+\infty}(\pi-\pi e^{-2R^2})=\pi.$

$\lim_{R\to+\infty}I_R^2=\pi \Rightarrow \lim_{R\to+\infty}I_R=\sqrt{\pi}$，即 $\int_{-\infty}^{+\infty}e^{-x^2}dx=\sqrt{\pi}.$