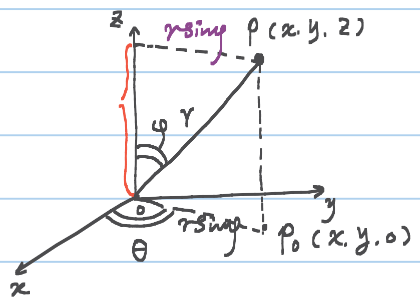
§7.3 球坐标系下三重积分的计算



$$\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$$

(r,θ,φ) 球坐标

$r\in[0,+\infty)$ $\theta\in[0,2\pi)$ $\varphi\in[0,\pi]$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} x_r & x_\theta & x_\varphi \\ y_r & y_\theta & y_\varphi \\ z_r & z_\theta & z_\varphi \end{vmatrix} = \begin{vmatrix} \sin\varphi\cos\theta & -r\sin\varphi\sin\theta & r\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & r\sin\varphi\cos\theta & r\cos\varphi\sin\theta \\ \cos\varphi & 0 & -r\sin\varphi \end{vmatrix}$$

$$= \cos\varphi\left(-r^2\sin\varphi\cos\varphi\sin^2\theta - r^2\sin\varphi\cos\varphi\cos^2\theta\right)$$

$$-r\sin\varphi\cdot\left(r\sin^2\varphi\cos^2\theta + r\sin^2\varphi\sin^2\theta\right)$$

$$= -r^2\sin\varphi\cos^2\varphi - r^2\sin^3\varphi = -r^2\sin\varphi$$

$$\iiint_\Omega f(x,y,z)\,dxdydz = \iiint_{\Omega'} f(r\sin\varphi\cos\theta,\ r\sin\varphi\sin\theta,\ r\cos\varphi)\cdot r^2\sin\varphi\,drd\theta d\varphi$$

例1: 计算 $I = \iiint\limits_{a^2\le x^2+y^2+z^2\le b^2} \frac{1}{(x^2+y^2+z^2)^{3/2}}\,dxdydz$ $(a.b>0)$

解: 用球坐标计算:

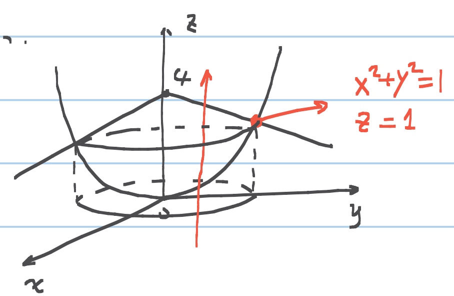
$$I = \int_0^{2\pi}d\theta\int_0^{\pi}d\varphi\int_a^b\frac{1}{r^3}\cdot r^2\sin\varphi\,dr$$

$$= 2\pi\int_0^{\pi}\sin\varphi\,d\varphi\cdot\ln\frac{b}{a}$$

$$= 4\pi\ln\frac{b}{a}.$$

例2: $z=x^2+y^2$. 与 $z=4-3\sqrt{x^2+y^2}$ 围成的空间有界闭区域为 Ω

用不同的方法计算区域 Ω 的体积.



法1: (先一后二. 柱坐标).

$$V = \iiint_\Omega 1\,dxdydz$$

$$= \iint_{x^2+y^2\le 1} dxdy \int_{x^2+y^2}^{4-3\sqrt{x^2+y^2}} 1\, dz$$

$$x^2+y^2 = 4-3\sqrt{x^2+y^2}$$

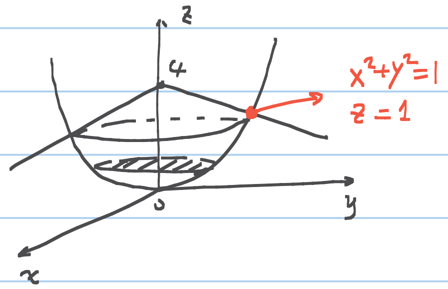
$$x^2+y^2+3\sqrt{x^2+y^2}-4=0$$

$$\sqrt{x^2+y^2}=1$$

$$= \iint_{x^2+y^2\le 1} \left[4-3\sqrt{x^2+y^2}-(x^2+y^2)\right] dxdy$$

$$= \int_0^{2\pi} d\theta \int_0^1 (4-3r-r^2)\, r\, dr = 2\pi\left(2-1-\frac{1}{4}\right) = \frac{3}{2}\pi.$$

法2：(先二后一法).



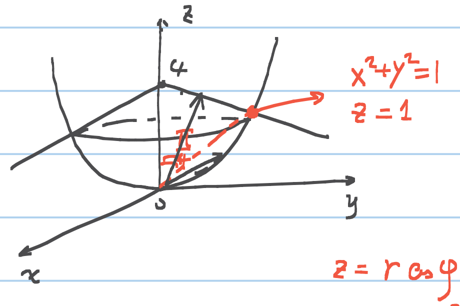
$$V = \iiint_\Omega 1\, dxdydz$$

$$= \int_0^1 dz \iint_{x^2+y^2\le z} 1\, dxdy + \int_1^4 dz \iint_{x^2+y^2\le\left(\frac{4-z}{3}\right)^2} 1\, dxdy$$

$$= \int_0^1 \pi z\, dz + \int_1^4 \pi\left(\frac{4-z}{3}\right)^2 dz$$

$$= \frac{\pi}{2} + \frac{\pi}{9}\cdot\frac{1}{3}(z-4)^3\Big|_{z=1}^{z=4} = \frac{\pi}{2} + \frac{\pi}{27}\cdot 27 = \frac{3}{2}\pi.$$

法3：(球坐标法).



$$V = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} d\varphi \int_0^{\frac{4}{\cos\varphi+3\sin\varphi}} r^2\sin\varphi\, dr$$

$$+ \int_0^{2\pi} d\theta \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\cos\varphi}{\sin^2\varphi}} r^2\sin\varphi\, dr$$

$$z = r\cos\varphi = 1$$

$$x^2+y^2 = r^2\sin^2\varphi = 1$$

$$\Rightarrow \tan\varphi = 1$$

$$\varphi = \frac{\pi}{4}$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \sin\varphi\, \frac{1}{3}\left(\frac{4}{\cos\varphi+3\sin\varphi}\right)^3 d\varphi$$

$$+ 2\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\varphi\, \frac{1}{3}\, \frac{\cos^3\varphi}{\sin^6\varphi}\, d\varphi$$

$$z = 4-3\sqrt{x^2+y^2}$$

$$r\cos\varphi = 4-3r\sin\varphi$$

$$r = \frac{4}{\cos\varphi+3\sin\varphi}$$

$$z = x^2+y^2 \qquad r\cos\varphi = r^2\sin^2\varphi \Rightarrow r = \frac{\cos\varphi}{\sin^2\varphi}$$

$$= \frac{128\pi}{3} \int_0^{\frac{\pi}{4}} \boxed{\frac{1}{\sin^2\varphi}} \cdot \left(\frac{1}{\cot\varphi+3}\right)^3 d\varphi$$

$$+ \frac{2\pi}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1-\sin^2\varphi}{\sin^5\varphi}\, d(\sin\varphi)$$

$$= \frac{-128\pi}{3} \int_0^{\frac{\pi}{4}} \frac{1}{(\cot\varphi+3)^3}\, d(\cot\varphi) + \frac{2\pi}{3}\left(-\frac{1}{4}\,\frac{1}{\sin^4\varphi} + \frac{1}{2}\,\frac{1}{\sin^2\varphi}\right)\Bigg|_{\varphi=\frac{\pi}{4}}^{\varphi=\frac{\pi}{2}}$$

$$= \frac{128\pi}{3} \times \frac{1}{2}\, \frac{1}{(\cot\varphi+3)^2}\Bigg|_0^{\frac{\pi}{4}} + \frac{2\pi}{3}\left[\frac{1}{4} - \left(-\frac{1}{4}\times 4 + \frac{1}{2}\times 2\right)\right] = \frac{\pi}{6} + \frac{4\pi}{3} = \frac{3\pi}{2}.$$