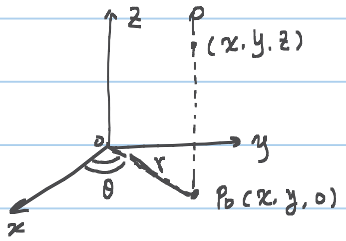
§7.3 柱坐标系下三重积分的计算



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$ (r,θ,z) 柱坐标.

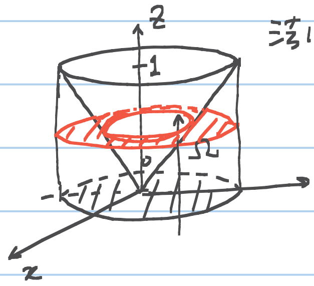
$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$$ 可微. 且 $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} \neq 0$ 则

$$\iiint_{\Omega} f(x,y,z)\,dxdydz = \iiint_{\Omega'} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

柱坐标 $\iiint_{\Omega} f(x,y,z)\,dxdydz = \iiint_{\Omega'} f(r\cos\theta, r\sin\theta, z)\, r\,drd\theta dz$

例：计算三重积分 $I=\iiint\limits_{\Omega}(x^2+y^2)z\,dxdydz$，其中 Ω 由柱面 $x^2+y^2=1$，锥面 $z=\sqrt{x^2+y^2}$ 和平面 $z=0$ 围成



法1：$I=\int_0^{2\pi}d\theta\int_0^1 r\,dr\int_0^r r^2\cdot z\,dz$ （柱坐标）

$= 2\pi\int_0^1 r^3\,\frac{1}{2}r^2\,dr$

$= 2\pi\cdot\frac{1}{2}\cdot\frac{1}{6}=\frac{\pi}{6}.$

法2：$I=\int_0^1 dz\iint\limits_{z^2\le x^2+y^2\le 1}(x^2+y^2)z\,dxdy$

$= \int_0^1 z\,dz\int_0^{2\pi}d\theta\int_z^1 r^2\cdot r\,dr$

$= 2\pi\int_0^1 z\cdot\frac{1}{4}(1-z^4)\,dz$

$= \frac{\pi}{2}\left(\frac{1}{2}-\frac{1}{6}\right)=\frac{\pi}{6}.$