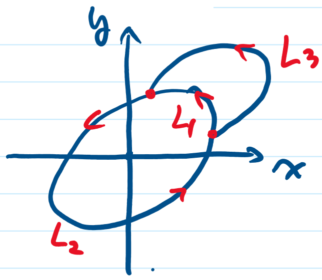
第九章习题课

例1： 已知曲线积分 $\oint_L \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{\varphi(x)+y^2} = A$ (常数), 其中 $\varphi(x)$ 有连续导数且 $\varphi(1)=1$, L 是绕 $(0,0)$ 一周的任一分段光滑正向闭曲线, 试求 $\varphi(x)$ 及 A.



解: 设 $\oint_{L_1+L_2} = A$ $\oint_{L_3+L_2} = A$

$\Rightarrow \oint_{L_3+L_1^-} = 0$ (不包含 $(0,0)$)

令 $P = \frac{-y}{\varphi(x)+y^2}$. $Q = \frac{x}{\varphi(x)+y^2}$

$\Rightarrow \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \Rightarrow x\varphi'(x) = 2\varphi(x)$.

解之. $\varphi(x) = Cx^2$ 又 $\varphi(1)=1$. $\Rightarrow \varphi(x) = x^2$.

$\Rightarrow A = \oint_L \frac{x\,dy - y\,dx}{x^2+y^2} = 2\pi$.

例2： 计算 $I = \oint_L xy\,\mathrm{d}x + z^2\mathrm{d}y + xz\,\mathrm{d}z$, 其中 L 为锥面 $z=\sqrt{x^2+y^2}$ 与柱面 $x^2+y^2=2ax\ (a>0)$ 的交线, 从 z 轴正向看 L 为逆时针方向.

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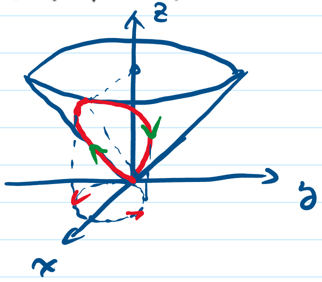
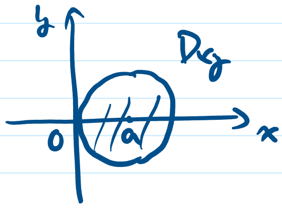
解: $L:\begin{cases} z=\sqrt{x^2+y^2} \\ x^2+y^2=2ax \end{cases} \Leftrightarrow \begin{cases} z=\sqrt{2ax} \\ (x-a)^2+y^2=a^2 \end{cases}$



$\Rightarrow L: \begin{cases} x=a(1+\cos t) \\ y=a\sin t \\ z=2a\cos\frac{t}{2} \end{cases} \quad (t: 0\to 2\pi)$

$\Rightarrow I=\int_0^{2\pi}[\cdots]dt=\pi a^3.$



(法二). 设 S: $z=\sqrt{x^2+y^2}$. 上侧.

上侧: $\vec{n}=\left(-\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1\right)$

$$\Rightarrow I=\iint_S \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & z^2 & xz \end{vmatrix}$$

$$=\iint_S \underbrace{(-2z)}_{P}dydz \underbrace{-z}_{Q}dzdx \underbrace{-x}_{R}dxdy$$

$$=\iint_{D_{xy}}(x+y)dxdy=\pi a^3.$$

(法三). 由 $L\in S$: $z=\sqrt{x^2+y^2}$. 即: $z^2=x^2+y^2$.

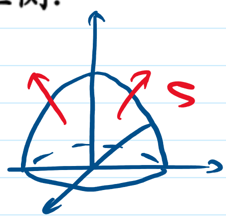
$\Rightarrow zdz=xdx+ydy \qquad \Rightarrow L': x^2+y^2=2ax.$

$$\Rightarrow I=\oint_{L'} xydx+(x^2+y^2)dy+x(xdx+ydy)=\oint_{L'}(xy+x^2)dx+(x^2+y^2+xy)dy$$

$$=\iint_{D_{xy}}(x+y)dxdy=\pi a^3.$$

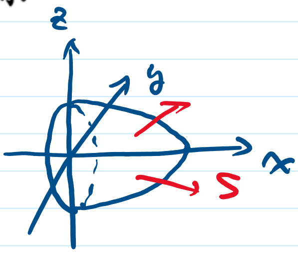
例3: 计算$I=\iint\limits_S x\,dydz+(y^3+2)dzdx+z^3dxdy$,

其中S是曲面$z=1-x^2-y^2\ (z\geq 0)$, 取上侧.



例4：计算$I=\iint\limits_S 2x^3\mathrm{d}y\mathrm{d}z+2y^3\mathrm{d}z\mathrm{d}x+3(z^2-1)\mathrm{d}x\mathrm{d}y$，

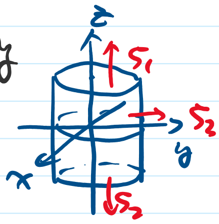
其中S是曲面$x=\sqrt{1-3y^2-3z^2}$，取前侧.



例5：计算$I=\oiint\limits_S \frac{x\,\mathrm{d}y\mathrm{d}z+z^2\,\mathrm{d}x\mathrm{d}y}{x^2+y^2+z^2}$，其中$S$是由柱面$x^2+y^2=R^2$

及平面$z=R$和$z=-R\,(R>0)$所围成立体表面，取外侧.

解: $I = \underbrace{\iint\limits_S \frac{x}{x^2+y^2+z^2}dydz}_{I_1} + \underbrace{\iint\limits_S \frac{z^2}{x^2+y^2+z^2}dxdy}_{I_2}$



对于 I_1

$\Rightarrow I_1 = \left(\underset{=}{\iint\limits_{S_1}} + \underset{=0}{\iint\limits_{S_2}} + \iint\limits_{S_3}\right) \frac{x}{x^2+y^2+z^2}dydz$

对于 I_2.

$\Rightarrow I_2 = \left(\iint\limits_{S_1} + \iint\limits_{S_2} + \underset{=0}{\iint\limits_{S_3}}\right) \frac{z^2}{x^2+y^2+z^2}dxdy$

例6: 计算 $I = \iint\limits_S \frac{x\,dydz + y\,dzdx + z\,dxdy}{(x^2+y^2+z^2)^{3/2}}$, 其中

(1) S为上半球面 $z = \sqrt{a^2-x^2-y^2}$ $(a>0)$的上侧.

(2) S为上半椭球面 $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{25}=1$ $(z\geq 0)$的上侧.

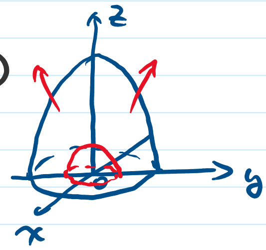
解: (1) $I = \frac{1}{a^3}\iint\limits_S x\,dydz + y\,dzdx + z\,dxdy$

补 S': $z=0$. 下侧. Gauss.

$\uparrow z$

补 S': $z=0$. 下侧. Gauss.

(2). 补 S_1: $x^2+y^2+z^2=1$ $(z\geq 0)$ 下侧.



补 S_2: $z=0$. $(x,y)\in D_{xy}$ 下侧.

但 $\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0$.

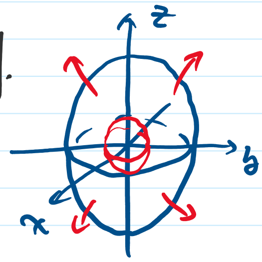
$\Rightarrow I = \oiint_{S+S_1+S_2} - \iint_{S_1} - \iint_{S_2}$

$= 0 - (1) - 0$

例7: 计算 $I=\oiint_S \frac{x\,dydz+y\,dzdx+z\,dxdy}{(x^2+y^2+z^2)^{3/2}}$,

其中 S 为椭球面 $2x^2+2y^2+z^2=4$, 取外侧.

解: 补 S': $x^2+y^2+z^2=1$. 内侧.



$\Rightarrow I = \oiint_{S+S'} - \oiint_{S'}$

$= 0 + \oiint_{S'^-}$

例8：设对半空间$x>0$内任意光滑有向闭曲面S都有

$$\oint\!\!\!\oint_S xf(x)\,\mathrm{d}y\mathrm{d}z - xyf(x)\,\mathrm{d}z\mathrm{d}x - \mathrm{e}^{2x}z\,\mathrm{d}x\mathrm{d}y = 0$$

其中$f(x)$在$(0,+\infty)$内一阶导数连续，且$\lim\limits_{x\to 0^+} f(x)=1$，求$f(x)$.