第七章习题课

例1: 讨论 $f(x,y)=e^{\sqrt{x^2+y^4}}$ 在$(0,0)$点的偏导数.

解: $f_x(0,0)=\lim\limits_{x\to 0}\dfrac{f(x,0)-f(0,0)}{x-0}$

$=\lim\limits_{x\to 0}\dfrac{e^{|x|}-1}{x}=\lim\limits_{x\to 0}\dfrac{|x|}{x}$ 不存在.

$f_y(0,0)=\lim\limits_{y\to 0}\dfrac{f(0,y)-f(0,0)}{y-0}$

$=\lim\limits_{y\to 0}\dfrac{e^{y^2}-1}{y}=\lim\limits_{y\to 0}\dfrac{y^2}{y}=0.$

$\left(f_x(x,y)=e^{\sqrt{x^2+y^4}}\cdot\dfrac{x}{\sqrt{x^2+y^4}}.\quad (x,y)\neq(0,0)\right)$

例2: 设函数$z=z(x,y)$由方程$F(\dfrac{y}{x},\dfrac{z}{x})=0$确定, 其中$F$为可微函数, 且$F_2'\neq 0$, 则$x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=(\quad)$

A. x B. z C. $-x$ D. $-z$

解: 直接求.

关于 x 求偏导. $F_1'\cdot(-\dfrac{y}{x^2})+F_2'\cdot\dfrac{\frac{\partial z}{\partial x}\cdot x-z}{x^2}=0$

-- y -- --. $F_1'\cdot\dfrac{1}{x}+F_2'\cdot\dfrac{1}{x}\dfrac{\partial z}{\partial y}=0$

$\Rightarrow \dfrac{\partial z}{\partial x}=\cdots.\quad \dfrac{\partial z}{\partial y}=\cdots$

(法二) 全微分.

$F_1'\cdot\dfrac{x\,dy-y\,dx}{x^2}+F_2'\cdot\dfrac{x\,dz-z\,dx}{x^2}=0$

即: $xF_2'\,dz=(zF_2'+yF_1')\,dx-xF_1'\,dy$

$\Rightarrow \dfrac{\partial z}{\partial x}=\dfrac{zF_2'+yF_1'}{xF_2'}.\quad \dfrac{\partial z}{\partial y}=-\dfrac{F_1'}{F_2'}$

$\Rightarrow x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=z.$

例3：设 $z=\left(\dfrac{y}{x}\right)^{\frac{x}{y}}$，求 $\left.\dfrac{\partial z}{\partial x}\right|_{(1,2)}$.

解：由 $x=1.\ y=2.\ \Rightarrow z=\sqrt{2}$.

由 $\ln z=\frac{x}{y}\cdot(\ln y-\ln x)$

即：$y\ln z=x(\ln y-\ln x)$

关于 x 求偏导. $\Rightarrow \frac{y}{z}\cdot\frac{\partial z}{\partial x}=(\ln y-\ln x)-1$

代入. $x=1.\ y=2.\ z=\sqrt{2}$.

$\Rightarrow \left.\frac{\partial z}{\partial x}\right|_{(1,2)}=\frac{\ln 2-1}{\sqrt{2}}$

例4：$f(x,y)$在$(0,0)$点可微的充分条件是(　　)

$\begin{cases}\lim\limits_{(x,y)\to(0,0)} f_x(x,y)=f_x(0,0)\\ \lim\limits_{(x,y)\to(0,0)} f_y(x,y)=f_y(0,0)\end{cases}$

偏导连续 $\underset{\not\Leftarrow}{\Rightarrow}$ 可微 $\underset{\not\Leftarrow}{\Rightarrow}$ 连续、$\not\Leftrightarrow$ 可偏导.

A. $\lim\limits_{(x,y)\to(0,0)}[f(x,y)-f(0,0)]=0$ ✗

$f(x,y)$连续

B. $\lim\limits_{x\to 0}\dfrac{f(x,0)-f(0,0)}{x}=\lim\limits_{y\to 0}\dfrac{f(0,y)-f(0,0)}{y}=0$ ✗

$f_x(0,0)=0$. $f_y(0,0)=0$. 可偏导.

C. $\lim\limits_{(x,y)\to(0,0)}\dfrac{f(x,y)-f(0,0)}{\sqrt{x^2+y^2}}=0$ ✓

D. $\lim\limits_{x\to 0}[f_x(x,0)-f_x(0,0)]=0,\ \lim\limits_{y\to 0}[f_y(0,y)-f_y(0,0)]=0$ ✗

$\lim\limits_{x\to 0} f_x(x,0)=f_x(0,0)$　　$\lim\limits_{y\to 0} f_y(0,y)=f_y(0,0)$

C. 由 $\lim\limits_{(x,y)\to(0,0)}\dfrac{f(x,y)-f(0,0)}{\sqrt{x^2+y^2}}=0$

① $\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)=f(0,0)$. 即：$f(x,y)$在$(0,0)$处连续.

② $\Rightarrow$ 设 $y=0$. $\Rightarrow \lim\limits_{x\to 0}\dfrac{f(x,0)-f(0,0)}{|x|}=0$

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$$\Rightarrow f_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{|x|}\cdot\frac{|x|}{x}=0.$$

设 $x=0$. $\Rightarrow \lim\limits_{y\to 0}\dfrac{f(0,y)-f(0,0)}{|y|}=0.$

$$\Rightarrow f_y(0,0)=0$$

即：$f(x,y)$ 在 $(0,0)$ 处可偏导.

③ 由 $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\dfrac{\Delta z-f_x(0,0)\Delta x-f_y(0,0)\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$

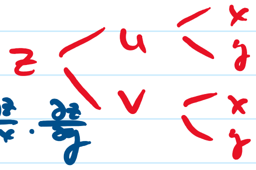
$$=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{f(\Delta x,\Delta y)-f(0,0)}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=0.$$

即：$f(x,y)$ 在 $(0,0)$ 处可微.

例5: 设 $z(x,y)=\varphi(x+y)+\varphi(x-y)+\int_{x-y}^{x+y}\psi(t)\mathrm{d}t$, 其中 φ 有二阶导数, ψ 有一阶导数, 证明: $\dfrac{\partial^2 z}{\partial x^2}=\dfrac{\partial^2 z}{\partial y^2}$.

解：令 $u=x+y$. $v=x-y$.

$$\Rightarrow z=\varphi(u)+\varphi(v)+\int_v^u\psi(t)\,dt.$$



$\dfrac{\partial z}{\partial x}\cdot\dfrac{\partial z}{\partial y}$

$$\Rightarrow \frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=(\varphi'(u)+\psi(u))\cdot 1+(\varphi'(v)-\psi(v))\cdot 1$$

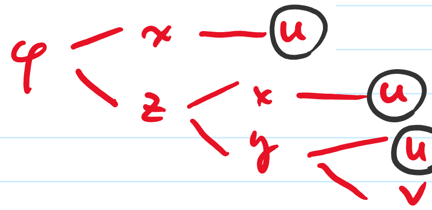
$$\frac{\partial z}{\partial y}=\varphi'(u)+\psi(u)-(\varphi'(v)-\psi(v))$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2}=(\varphi''(u)+\psi'(u))\cdot 1+(\varphi''(v)-\psi'(v))\cdot 1$$

$$\frac{\partial^2 z}{\partial y^2}=\varphi''(u)+\psi'(u)+(\varphi''(v)-\psi'(v))$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial y^2}$$

例6: 设 $z=z(x,y)$ ① 满足 $x^2\dfrac{\partial z}{\partial x}+y^2\dfrac{\partial z}{\partial y}=z^2$, 且 $u=x$ ②, $v=\dfrac{1}{y}-\dfrac{1}{x}$ ③, $\varphi=\dfrac{1}{z}-\dfrac{1}{x}$ ④, 对函数 $\varphi(u,v)$, 证明: $\dfrac{\partial\varphi}{\partial u}=0$.



解: 由 $x=u$. $\quad v=\frac{1}{y}-\frac{1}{x}=\frac{1}{y}-\frac{1}{u}$

$$\Rightarrow y=\frac{u}{1+uv} \Rightarrow \frac{1}{1+uv}=\frac{y}{x}$$

$$\Rightarrow \frac{\partial\varphi}{\partial u}=\frac{\partial\varphi}{\partial x}\cdot\frac{dx}{du}+\frac{\partial\varphi}{\partial z}\left(\frac{\partial z}{\partial x}\cdot\frac{dx}{du}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\right)$$

$$=\frac{1}{x^2}\cdot 1-\frac{1}{z^2}\left(\frac{\partial z}{\partial x}\cdot 1+\frac{\partial z}{\partial y}\cdot\frac{1}{(1+uv)^2}\right)$$

$$=\frac{1}{x^2}-\frac{1}{z^2}\left(\frac{\partial z}{\partial x}+\frac{y^2}{x^2}\cdot\frac{\partial z}{\partial y}\right)$$

$$=\frac{1}{x^2}-\frac{1}{x^2z^2}\left(x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}\right)$$

$$=0$$

例7: 设变换 $\begin{cases}u=x+ay\\ v=x+by\end{cases}$ 把方程 $2\dfrac{\partial^2 z}{\partial x^2}-3\dfrac{\partial^2 z}{\partial x\partial y}+\dfrac{\partial^2 z}{\partial y^2}=0$ 简化为 $\dfrac{\partial^2 z}{\partial u\partial v}=0$, 其中 z 有二阶连续偏导数, 求 a,b.

例8： 设函数 $f(u)$ 具有二阶连续导数, 且 $f(0)=1,\ f'(0)=3$, $z=f(\mathrm{e}^x\sin y)$ 满足方程 $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=\mathrm{e}^{2x}z$, 求 $f(u)$ 的表达式.

例9： 设 $u=f(x,y,xyz),\ z=z(x,y)$ 由方程 $\int_{xy}^{z}g(xy+z-t)\mathrm{d}t=\mathrm{e}^{xyz}$ 确定, 其中 f 可微, g 连续, 求 $x\dfrac{\partial u}{\partial x}-y\dfrac{\partial u}{\partial y}$.

例10： 求函数 $u=\ln(x+\sqrt{y^2+z^2})$ 在点 $A(1,0,1)$ 处沿 A 点指向点 $B(3,-2,2)$ 方向的方向导数，并指出在该点沿什么方向方向导数最大，沿什么方向方向导数最小，最大值和最小值分别为多少？

例11：求函数$u=x^2+y^2-z^2$沿曲面$S:2x^2+2y^2+z^2=5$上点$M(1,1,1)$处外法线方向$\vec{n}$的方向导数.

例12：在曲线$x=t,y=-t^2,z=t^3$的所有切线中，与平面$x+2y+z=4$平行的切线(　　)

A. 只有1条　　B. 只有2条　　C. 至少有3条　　D. 不存在

例13：设椭球面$x^2+2y^2+3z^2=21$上点P处的切平面Π过直线$L:\dfrac{x-6}{2}=\dfrac{y-3}{1}=\dfrac{2z-1}{-2}$，求点$P$的坐标和切平面$\Pi$的方程.