某些变系数线性微分方程

1. 欧拉方程　　$x^k y^{(k)}$

$$x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_n y = f(x)$$

令 $x = e^t$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\cdot\frac{1}{\frac{dx}{dt}} = \boxed{\frac{1}{x}}\boxed{\frac{dy}{dt}}$$　　$xy' = Dy$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right),$$　　$x^2y'' = (D^2-D)y$

$$\frac{d^3y}{dx^3} = \frac{1}{x^3}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right),$$　　$x^3y''' = D(D-1)(D-2)y$

⋮

微分算子 $D = \frac{d}{dt} \to y$　　$\frac{dy}{dt}$　　$x^k y^{(k)}$.

$D^2 = \frac{d^2}{dt^2} \to y$

例 $x^2y'' + xy' - y = x^2$.

令 $x = e^t$

$$D(D-1)y + Dy - y = e^{2t}$$

$$D^2y - y = e^{2t}.$$

$$\frac{d^2y}{dt^2} - y = e^{2t}$$　二阶常.非齐次线.

$$y = C_1e^t + C_2e^{-t} + \tfrac{1}{3}e^{2t}.$$

降阶法　二阶 $y''+p_1(x)y'+p_2(x)y=0$

已知 $y_1(x)$ 是解 $(\neq 0)$ 只需再找一个与 $y_1(x)$ 线性无关

设 $y(x)=y_1(x)\,z(x)$

$y'=y_1'z+y_1z'$　　$y''=y_1''z+y_1'z'+y_1'z'+y_1z''$

整理 $\underbrace{\underset{\text{已知}}{y_1}z''+[2\underset{\text{已知}}{y_1'}+\underset{\text{已知}}{p_1(x)}\underset{\text{已知}}{y_1}]}_{\text{关于}x\text{的函数}}z'+\underbrace{[y_1''+p_1(x)y_1'+p_2(x)y_1]}_{=0}z=0$

$F(z'',z',x)=0$

令 $z'=u$，则 $u'+\underbrace{\dfrac{2y_1'+p_1(x)y_1}{y_1}}_{\text{已知的}x\text{的函数}}u=0$　分离变量法

例 $(2x-1)y''-(2x+1)y'+2y=0$.　猜 e^x

令 $y=e^xz$　　$y'=e^xz+e^xz'$　$y''=e^xz+2e^xz'+e^xz''$

$\Rightarrow (2x-1)z''+(2x-3)z'=0$

$(2x-1)u'+(2x-3)u=0$

$y=C_1(2x+1)+C_2e^x$.