第四章习题课

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例1： 求以 $y=c_1x^2+c_2e^x$ (c_1,c_2 为任意常数)为通解的线性微分方程.

解：由 $y = c_1x^2 + c_2e^x$

$\Rightarrow y' = 2c_1x + c_2e^x$

$y'' = 2c_1 + c_2e^x$

联立，消去 c_1、c_2

$\Rightarrow (x^2-2x)y'' - (x^2-2)y' + 2(x-1)y = 0$

例2： 求方程 $xy\,dx+(y^4-x^2)\,dy=0$ 的通解.

$\frac{dy}{dx} = \frac{xy}{x^2-y^4}$

解：$\frac{dx}{dy} = \frac{x^2-y^4}{xy} = \frac{x}{y} - \frac{1}{x}y^3$

即：$\frac{dx}{dy} + \frac{1}{x}y^3 = \frac{x}{y}$. $\Leftrightarrow$ $\frac{dx}{dy} - \frac{x}{y} = -\frac{1}{x}y^3$ ✓

$\left(\frac{dy}{dx} + \frac{1}{y}x^3 = \frac{y}{x} \quad \Leftrightarrow \quad \frac{dy}{dx} - \frac{y}{x} = -\frac{1}{y}x^3 \right)$

Bernoulli

同乘 x. 令 $x^2 = z(y)$ $\Rightarrow 2x\frac{dx}{dy} = \frac{dz}{dy}$.

$\Rightarrow \frac{dz}{dy} - \frac{2}{y}z = -2y^3$. $\left(y = e^{-\int p(x)dx}\left(C + \int q(x)e^{\int p(x)dx}dx\right)\right)$

$\left(p(y) = -\frac{2}{y}.\quad q(y) = -2y^3\right)$

$\Rightarrow$ 通解：$z = y^2(c-y^2)$. 即：$x^2 = y^2(c-y^2)$

(法二). $\frac{dx}{dy} = \frac{x^2-y^4}{xy} = \frac{x}{y} - \frac{y^3}{x} = y\left(\frac{x}{y^2} - \frac{y^2}{x}\right)$

即：$\frac{dy}{dx} = \frac{1}{y} \cdot \frac{1}{\frac{x}{y^2} - \frac{y^2}{x}}$

即: $\dfrac{dy}{dx} = \dfrac{1}{y} \cdot \dfrac{1}{\dfrac{x}{y^2} - \dfrac{y^2}{x}}$

令 $\dfrac{y^2}{x} = u(x)$. $\Rightarrow$ $y^2 = x \cdot u(x)$ $\Rightarrow$ $2y\dfrac{dy}{dx} = u + x\dfrac{du}{dx}$

$\Rightarrow$ $u + x\dfrac{du}{dx} = \dfrac{2}{\dfrac{1}{u} - u}$ (可分离)

(法二). $\dfrac{dx}{dy} = \dfrac{x}{y} - \dfrac{y^3}{x} = \dfrac{x}{y} - \dfrac{y}{x} \cdot y^2$

令 $\dfrac{x}{y} = u(y)$ $\Rightarrow$ $x = y \cdot u(y)$ $\Rightarrow$ $\dfrac{dx}{dy} = u + y \cdot \dfrac{du}{dy}$

$\Rightarrow$ $u + y \cdot \dfrac{du}{dy} = u - \dfrac{1}{u} \cdot y^2$ (可分离)

例3: 求方程 $2y'y'' = 1$ 的通解.

解: (不显含 y)

令 $y' = z(x)$ $\Rightarrow$ $y'' = z'(x)$

$\Rightarrow$ $2z \cdot \dfrac{dz}{dx} = 1$. 即: $2z\,dz = dx$

积分: $z^2 = x + C_1$ $\Rightarrow$ $z = \pm\sqrt{x + C_1}$ $(= y')$

积分. $\Rightarrow$ $y = \pm\dfrac{2}{3}(x + C_1)^{\frac{3}{2}} + C_2$

即: $9(y - C_2)^2 = 4(x + C_1)^3$.

法二. (不显含 x)

令 $y' = z(y)$ $\Rightarrow$ $y'' = z\,\dfrac{dz}{}$

法二.(不显含 x)

令 $y' = z(y)$. $\Rightarrow y'' = z\dfrac{dz}{dy}$

$\Rightarrow 2z^2\dfrac{dz}{dy} = 1$. 即: $2z^2 dz = dy$

积分: $z^3 = \frac{3}{2}y + C_1 \Rightarrow z = \left(\frac{3}{2}y + C_1\right)^{\frac{1}{3}}$

即: $\dfrac{dy}{dx} = \left(\frac{3}{2}y + C_1\right)^{\frac{1}{3}}$

分离: $\left(\frac{3}{2}y + C_1\right)^{-\frac{1}{3}} dy = dx$

积分: $\left(\frac{3}{2}y + C_1\right)^{\frac{2}{3}} = x + C_2$

即: $(3y + C_1)^2 = 4(x + C_2)^3$

(法三).(不显含 x)

令 $y' = z(y)$. $\Rightarrow y'' = z\dfrac{dz}{dy}$

$dy = z\,dx$

代入. $\Rightarrow dy = 2z^2 dz$ ①

$\Rightarrow dx = 2z\,dz$ ②.

积分. $\Rightarrow \begin{cases} ① \to y = \frac{2}{3}z^3 + C_1 \\ ② \to x = z^2 + C_2 \end{cases}$

消去 z. $\Rightarrow$ 通解.

例4: 求解初值问题 $\begin{cases} yy'' = 2y'(y'-1) & ① \\ y(0) = 1, y'(0) = 2 & ② \end{cases}$.

解: (不显含 x)

令 $y' = z(y)$. $\Rightarrow y'' = z\cdot\dfrac{dz}{dy}$.

$\Rightarrow y\cdot z\dfrac{dz}{dy} = 2z(z-1)$

即 $y\dfrac{dz}{dy} = 2(z-1)$

即：$y\frac{dz}{dy}=2(z-1)$

积分：$\Rightarrow z=C_1y^2+1.$

由②. $x=0.\ y=1.\ y'=z=2.\Rightarrow C_1=1$

即：$z=y^2+1=\frac{dy}{dx}$

分离.积分. $\Rightarrow \arctan y=x+C_2$

即：$y=\tan(x+C_2)$

由②. $x=0.\ y=1\ \Rightarrow C_2=\frac{\pi}{4}$

即：$y=\tan(x+\frac{\pi}{4})$

例5： 设 $y(x)$ 连续，且满足：$y(x)=4xe^x+\int_0^x t\,y(\underline{x-t})\,dt$，求 $y(x)$.

解：由 $\int_0^x t\,y(x-t)dt \xlongequal{u=x-t} \int_x^0 (x-u)\,y(u)\,(-du)$

$=\underline{x\int_0^x y(u)du}-\int_0^x u\,y(u)du.$

$y(0)=0$

对方程求导

$\Rightarrow y'=4(x+1)e^x+\int_0^x y(u)du \longrightarrow y'(0)=4$

再求导. $\Rightarrow y''=4(x+2)e^x+y$

即：$y''-y=4(x+2)e^x.$

特征方程：$\lambda^2-1=0$

$\Rightarrow$ 特征根：$\lambda_1=1.\ \lambda_2=-1$

又由 $f(x)=4(x+2)e^x$. 令特解 $y^*=xe^x(ax+b)$

代入方程. $\Rightarrow a=1.\ b=3$

即：$y^*=xe^x(x+3)$

即: $y^* = xe^x(x+3)$

$\Rightarrow$ 通解: $y = C_1 e^x + C_2 e^{-x} + xe^x(x+3)$

又由 $y(0)=0$. $y'(0)=4$. $\Rightarrow C_1 = \frac{1}{2}$. $C_2 = -\frac{1}{2}$

即: $y = \frac{1}{2}(e^x - e^{-x}) + xe^x(x+3)$

例6: 已知微分方程 $y''+(x+e^y)y'^3=0$, 将其转化为 x 为因变量, y 为自变量的微分方程, 并求通解.

解: 由 $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{y'}$ $\left(= \frac{1}{y'(x)} = \frac{1}{y'(x(y))} \right)$

$$\Rightarrow \frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dy}\left(\frac{1}{y'}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{y'}\right)\cdot\frac{dx}{dy}$$

$$= -\frac{y''}{y'^2}\cdot\frac{1}{y'} = -\frac{y''}{y'^3}$$

代入原方程. $\Rightarrow \frac{d^2x}{dy^2} - (x+e^y) = 0$

即: $\frac{d^2x}{dy^2} - x = e^y$

特征方程. $\lambda^2 - 1 = 0$. $\Rightarrow$ 特征根 $\lambda_1 = 1$. $\lambda_2 = -1$

$f(y) = e^y$. $\Rightarrow$ 令特解 $x^* = Aye^y$

代入. $\Rightarrow A = \frac{1}{2}$

代入. $\Rightarrow A = \frac{1}{2}$

$\Rightarrow$ 通解为: $x = c_1 e^{y} + c_2 e^{-y} + \frac{1}{2} y e^{y}$.

例7: 求方程 $y''' + y'' - 2y = e^{x}(5 - 16x\sin x)$ 的一个特解.

解: $f(x) = e^{x}(5 - 16x\sin x) = 5e^{x} - 16x e^{x}\sin x$

令 $f_1(x) = 5e^{x}$. $f_2(x) = 16x e^{x}\sin x$.

特征方程: $\lambda^3 + \lambda^2 - 2 = 0$ $\quad (\lambda - 1)(\lambda^2 + 2\lambda + 2)$

$(\lambda+1)^2 + 1$

$\Rightarrow$ 特征根: $\lambda_1 = 1$. $\lambda_{2,3} = -1 \pm i$

对于 $y''' + y'' - 2y = 5e^{x}$

特解 $y_1^* = x e^{x} A$ 代入. $\Rightarrow A = 1$

对于 $y''' + y'' - 2y = 16x e^{x}\sin x$ $\quad \begin{pmatrix} \alpha = 1. \ \beta = 1 \\ A_0(x) = 0 \\ B_1(x) = 16x \end{pmatrix}$

特解 $y_2^* = e^{x}[(ax+b)\cos x + (cx+d)\sin x]$

代入. $\Rightarrow a = -2$. $b = -1$. $c = -2$. $d = 4$.

$\Rightarrow$ 特解: $y^* = y_1^* - y_2^*$

例8: 已知 $y_1^* = -e^{x^2}$, $y_2^* = e^{x^2}(e^{x} - 1)$ 是非齐次线性微分方程

$$y'' - 4xy' - (3 - 4x^2)y = e^{x^2}$$

的两个特解, 试求此方程的通解.