§9.5 场论简介

一、向量场的散度

二、向量场的旋度

三、几类特殊的场

一、向量场的散度

1. 通量

- 向量场$\vec{A}$通过曲面S指定侧的通量:

$$\Phi=\iint_S \vec{A}\cdot\overrightarrow{dS}$$

$$\begin{cases}\text{流速场}\vec{v}\text{：流量}\\ \text{电场}\vec{E}\text{：电通量}\\ \text{磁场}\vec{B}\text{：磁通量}\end{cases}$$

例1：将点电荷q放于坐标原点，产生的静电场的电场强度为

$$\vec{E}=\frac{q}{4\pi r^2}\vec{r}^0,\text{ 其中}\vec{r}=(x,y,z)$$

$$\vec{E}=\frac{q}{4\pi(x^2+y^2+z^2)^{\frac{3}{2}}}(x\vec{i}+y\vec{j}+z\vec{k})$$

求通过球面$S:x^2+y^2+z^2=R^2$向外的电通量.

解：

$$\Phi=\oiint_S \vec{E}\cdot\overrightarrow{dS}$$

$$=\frac{q}{4\pi R^3}\oiint_S x\,dy\,dz+y\,dz\,dx+z\,dx\,dy$$

$$=\frac{q}{4\pi R^3}\iiint_V 3\,dv$$

$$=\frac{q}{4\pi R^3}\cdot 3\cdot\frac{4}{3}\pi R^3=q$$

2. 散度

- 向量场$\vec{A}$通过闭曲面S外侧的通量：$\Phi=\oiint\limits_{S}\vec{A}\cdot\overrightarrow{\mathrm{d}S}$

$$有源场:\begin{cases}若\Phi>0:\ 称S内有正源\\ 若\Phi<0:\ 称S内有负源\end{cases}$$

- 散度$\mathrm{div}\vec{A}(M)$：向量场$\vec{A}$在点M处的散度

$$\mathrm{div}\vec{A}(M)=\lim_{\Delta V\to M}\frac{\Delta\Phi}{\Delta V}=\lim_{\Delta V\to M}\frac{1}{\Delta V}\oiint\limits_{\Delta S}\vec{A}\cdot\overrightarrow{\mathrm{d}S}$$

$$=\lim_{\Delta V\to M}\frac{1}{\Delta V}\iiint\limits_{\Delta V}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\mathrm{d}V=\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\Big|_M$$

- 若$\mathrm{div}\vec{A}(M)>0$ $(<0,\ =0)$，则点M处有正源(负源，无源).

3. 散度的计算公式

- 向量场$\vec{A}=(P,Q,R)$，其中P,Q,R在G上具有一阶连续偏导，

$$\mathrm{div}\vec{A}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$$

- 若向量场内处处有$\mathrm{div}\vec{A}=0$，则称向量场$\vec{A}$为无源场.

- Gauss公式：$\oiint\limits_{S}\vec{A}\cdot\overrightarrow{\mathrm{d}S}=\iiint\limits_{V}\mathrm{div}\vec{A}\,\mathrm{d}V$

$\mathrm{div}\vec{A}=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot(P,Q,R)$

- 记$\nabla=\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}$ (Nabla算子)

$\boxed{散度\ \mathrm{div}\vec{A}=\nabla\cdot\vec{A}}$　　$\boxed{梯度\ \overrightarrow{\mathrm{grad}}\,u=\nabla u}$ $=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)$

例2：求向量场$A(x,y,z)=(xy,ye^z,xz)$在点$(0,1,0)$处的散度.

解：由 $\mathrm{div}\vec{A}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$

$=y+e^z+x$

$\Rightarrow \mathrm{div}\vec{A}\big|_{(0,1,0)}=2.$

例3: 置于原点的点电荷q产生的静电场的电场强度为

$$\vec{E}=\frac{q}{4\pi r^2}\vec{r}^{\,0},\ \text{其中}\vec{r}=(x,y,z)$$

求静电场中点M处的散度$\operatorname{div}\vec{E}$.

解: $\vec{E}=\dfrac{q}{4\pi(x^2+y^2+z^2)^{\frac{3}{2}}}(x\vec{i}+y\vec{j}+z\vec{k})$

则: $P=\dfrac{qx}{4\pi(x^2+y^2+z^2)^{\frac{3}{2}}}$. $Q=\dfrac{qy}{\cdots}$. $R=\dfrac{qz}{\cdots}$

$\hookrightarrow \dfrac{q}{4\pi}x(x^2+y^2+z^2)^{-\frac{3}{2}}$

$$\Rightarrow \frac{\partial P}{\partial x}=\frac{q}{4\pi}(x^2+y^2+z^2)^{-\frac{3}{2}}-\frac{q}{4\pi}x\cdot\frac{3}{2}(x^2+y^2+z^2)^{-\frac{5}{2}}\cdot 2x$$

$$=\frac{q}{4\pi}\frac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}=\frac{q}{4\pi}\frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$\frac{\partial Q}{\partial y}=\frac{q}{4\pi}\frac{x^2+z^2-2y^2}{(\cdots)^{\frac{5}{2}}}.\quad \frac{\partial R}{\partial z}=\cdots.\quad \Rightarrow \operatorname{div}\vec{E}=0$$

4. 散度的运算法则

- 对任意常数C, 有$\operatorname{div}(C\vec{A})=C(\operatorname{div}\vec{A})$, 即: $\nabla\cdot(C\vec{A})=C(\nabla\cdot\vec{A})$
- $\operatorname{div}(\vec{A}\pm\vec{B})=\operatorname{div}\vec{A}+\operatorname{div}\vec{B}$, 即: $\nabla\cdot(\vec{A}\pm\vec{B})=\nabla\cdot\vec{A}\pm\nabla\cdot\vec{B}$
- 对数量值函数u, 有$\operatorname{div}(u\vec{A})=u(\operatorname{div}\vec{A})+\nabla u\cdot\vec{A}$,

  即: $\nabla\cdot(u\vec{A})=u(\nabla\cdot\vec{A})+\nabla u\cdot\vec{A}$ ✓

例4: 已知$u=\sin(xyz)$, $\vec{A}=x\vec{i}+y\vec{j}+2z\vec{k}$, 求$\operatorname{div}(u\vec{A})$.

解: $u\vec{A}=\underbrace{x\sin(xyz)}_{P}\vec{i}+\underbrace{y\sin(xyz)}_{Q}\vec{j}+\underbrace{2z\sin(xyz)}_{R}\vec{k}$

$$\Rightarrow \operatorname{div}(u\vec{A})=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=\cdots$$

(法二). $\operatorname{div}(u\vec{A})=u\operatorname{div}\vec{A}+\nabla u\cdot\vec{A}$.

二、向量场的旋度

1. 环量

- 向量场$\vec{A}$沿闭曲线L给定方向的环量:

$$\Gamma=\oint_L \vec{A}\cdot\vec{ds}=\oint_L P\,dx+Q\,dy+R\,dz$$

2. 旋度

- 向量场$\vec{A}$沿闭曲线L给定方向的环量:

$$\Gamma=\oint_L \vec{A}\cdot\vec{ds}=\oint_L P\,dx+Q\,dy+R\,dz=\iint_S \begin{vmatrix} dydz & dzdx & dxdy \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$=\iint_S \left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)dydz+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)dzdx+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$$

- 旋度$\overrightarrow{\mathrm{rot}}\vec{A}(M)$: 向量场$\vec{A}$在点$M$处的旋度

$$\overrightarrow{\mathrm{rot}}\vec{A}(M)=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z},\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x},\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\Bigg|_M$$

3. 旋度的计算公式

- 向量场$\vec{A}=(P,Q,R)$, 其中P,Q,R在G上具有一阶连续偏导,

$$\overrightarrow{\mathrm{rot}}\vec{A}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z},\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x},\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

- 若向量场内处处有$\overrightarrow{\mathrm{rot}}\vec{A}=\vec{0}$, 则称向量场$\vec{A}$为无旋场.

- 斯托克斯公式: $\oint_L \vec{A}\cdot\vec{ds}=\iint_S \overrightarrow{\mathrm{rot}}\vec{A}\cdot\vec{dS}$

旋度$\overrightarrow{\mathrm{rot}}\vec{A}=\nabla\times\vec{A}$　　散度$\mathrm{div}\vec{A}=\nabla\cdot\vec{A}$　　梯度$\overrightarrow{\mathrm{grad}}\,u=\nabla u$

4. 旋度的运算法则

- 对任意常数C，有$\overrightarrow{\mathrm{rot}}(C\vec{A})=C(\overrightarrow{\mathrm{rot}}\vec{A})$，即：$\nabla\times(C\vec{A})=C(\nabla\times\vec{A})$
- $\overrightarrow{\mathrm{rot}}(\vec{A}\pm\vec{B})=\overrightarrow{\mathrm{rot}}\vec{A}+\overrightarrow{\mathrm{rot}}\vec{B}$，即：$\nabla\times(\vec{A}\pm\vec{B})=\nabla\times\vec{A}\pm\nabla\times\vec{B}$
- 对数量值函数u，有$\overrightarrow{\mathrm{rot}}(u\vec{A})=u(\overrightarrow{\mathrm{rot}}\vec{A})+\nabla u\times\vec{A}$，

  即：$\nabla\times(u\vec{A})=u(\nabla\times\vec{A})+\nabla u\times\vec{A}$
- $\overrightarrow{\mathrm{rot}}(\nabla u)=\vec{0}$，即：$\nabla\times(\nabla u)=\vec{0}$

  $\vec{A}=\nabla u=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)$

  $\Rightarrow$ $\vec{A}$为无旋场.
- $\mathrm{div}(\overrightarrow{\mathrm{rot}}\vec{A})=0$，即：$\nabla\cdot(\nabla\times\vec{A})=0$

5. 环量密度

- 向量场$\vec{A}=(P,Q,R)$在点M处沿给定方向$\vec{n}$的环量密度：

$$\mu_n=\lim_{\Delta S\to M}\frac{1}{\Delta S}\oint_{\Delta L}\vec{A}\cdot\vec{ds}=\lim_{\Delta S\to M}\frac{1}{\Delta S}\iint_S\overrightarrow{\mathrm{rot}}\vec{A}\cdot\vec{dS}$$

$$=\lim_{\Delta S\to M}\frac{1}{\Delta S}\iint_S\overrightarrow{\mathrm{rot}}\vec{A}\cdot\vec{n}_0\mathrm{d}S=\overrightarrow{\mathrm{rot}}\vec{A}(M)\cdot\vec{n}_0$$

- 环量密度μ_n不仅与点M的位置有关，还与$\vec{n}_0$的方向有关
- 当$\vec{n}_0$与$\overrightarrow{\mathrm{rot}}A(M)$方向一致时，环量密度$\mu_n$最大，为$|\overrightarrow{\mathrm{rot}}A(M)|$.

  (类似于方向导数与梯度的关系)

例5：求向量场$\vec{A}=x^2yz\,\vec{i}+xy^2z\,\vec{j}+xyz^2\,\vec{k}$在点$M(1,2,-3)$处的旋度和沿方向$\vec{n}=2\vec{i}-2\vec{j}+\vec{k}$的环量密度.

解：$\overrightarrow{\mathrm{rot}}\vec{A}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ x^2yz&xy^2z&xyz^2\end{vmatrix}=(xz^2-xy^2)\vec{i}+(x^2y-yz^2)\vec{j}+(y^2z-x^2z)\vec{k}$

$\Rightarrow \overrightarrow{\mathrm{rot}}\vec{A}\big|_M=5\vec{i}-16\vec{j}-9\vec{k}$

由 $\vec{n}_0=\left(\frac{2}{3},-\frac{2}{3},\frac{1}{3}\right)$.

$\Rightarrow \mu_n=\overrightarrow{\mathrm{rot}}\vec{A}\big|_M\cdot\vec{n}_0=(5,-16,-9)\cdot\left(\frac{2}{3},-\frac{2}{3},\frac{1}{3}\right)=11.$

$$\Rightarrow \mu_n = \overrightarrow{\mathrm{rot} A}\Big|_M \cdot \vec{n}_0 = (5, -16, -9)\cdot\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) = 11.$$

例6：置于原点的点电荷q产生的静电场的电场强度为

$$\vec{E} = \frac{q}{4\pi r^2}\vec{r}^{\,0},\ 其中\vec{r} = (x, y, z)$$

求静电场中点M处的散度$\overrightarrow{\mathrm{rot}\,E}$.

解：$\vec{E} = \frac{q}{4\pi r^3}(x\vec{i} + y\vec{j} + z\vec{k})$

令 $u = \frac{q}{4\pi r^3} = \frac{q}{4\pi (x^2+y^2+z^2)^{\frac{3}{2}}}$

$\vec{A} = x\vec{i} + y\vec{j} + z\vec{k} = (x, y, z)$.

$\Rightarrow \overrightarrow{\mathrm{rot}\,E} = \overrightarrow{\mathrm{rot}\,(u\vec{A})} = u\,\overrightarrow{\mathrm{rot}\,A} + \nabla u \times \vec{A}$

$= 0$

三、几类特殊的场

1. 场论三度

- 记$\nabla=\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})$ (Nabla算子)

  数量场$u=u(x,y,z)$, 向量场$\vec{A}=P\vec{i}+Q\vec{j}+R\vec{k}=(P,Q,R)$

- 梯度 $\overrightarrow{\mathrm{grad}}\,u=(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z})=\nabla u$
- 散度 $\mathrm{div}\vec{A}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=\nabla\cdot\vec{A}$
- 旋度 $\overrightarrow{\mathrm{rot}}\vec{A}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}=\nabla\times\vec{A}$

2. 几类特殊的场

- 无源场 $\mathrm{div}\vec{A}(M)=0\quad(\forall M\in G)$
- 无旋场 $\overrightarrow{\mathrm{rot}}\vec{A}(M)=\vec{0}\quad(\forall M\in G)$
- 有势场(梯度场) $\boxed{v=-u\text{为向量场}\vec{A}\text{的势函数}}$

  $\exists u=u(x,y,z)$, 有$\vec{A}=(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z})=\overrightarrow{\mathrm{grad}}\,u$ $\Rightarrow du=Pdx+Qdy+Rdz$

  $\boxed{\vec{A}\text{为无旋场}\Leftrightarrow\vec{A}\text{为有势场}}$

- 调和场(无源、无旋场) $\mathrm{div}\vec{A}(M)=0,\ \overrightarrow{\mathrm{rot}}\vec{A}(M)=\vec{0}\quad(\forall M\in G)$

  势函数$v=-u$满足Laplace方程: $\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$

由 $\overrightarrow{\mathrm{rot}}\vec{A}=\vec{0}$. $\Rightarrow \vec{A}=(\underset{P}{\frac{\partial u}{\partial x}}, \underset{Q}{\frac{\partial u}{\partial y}}, \underset{R}{\frac{\partial u}{\partial z}})$

又由 $\mathrm{div}\vec{A}=0$. $\Rightarrow \frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$

例7：求 a,b,c 使向量场 $\vec{A}=(\underbrace{ax+b^2yz}_{P},\underbrace{by+axz}_{Q},\underbrace{cxy-2z}_{R})$ 是一个调和场，并求势函数.

解：由 $\operatorname{div}\vec{A}=0$. 即：$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0$

$\Rightarrow a+b-2=0$ ①

又由 $\operatorname{rot}\vec{A}=\vec{0}$. 即：
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax+b^2yz & by+axz & cxy-2z \end{vmatrix}=\vec{0}$$

即：$\begin{cases} cx-ax=0 \\ b^2y-cy=0 \\ az-b^2z=0. \end{cases}$ 即：$a=c=b^2$ ②

由 ①. ②. 解之：$a=b=c=1$. $a=c=4$. $b=-2$

当 $a=b=c=1$.

$$\Rightarrow du=(x+yz)dx+(y+xz)dy+(xy-2z)dz$$
$$=(xdx+ydy-2zdz)+(yzdx+xzdy+xydz)$$
$$=d\left(\frac{x^2}{2}+\frac{y^2}{2}-z^2+xyz\right)$$

$\Rightarrow u=\frac{x^2}{2}+\frac{y^2}{2}-z^2+xyz+C$ $\Rightarrow$ 势函数 $v=-u$

当 $a=c=4$. $b=-2$.

$\Rightarrow u=2x^2-y^2-z^2+4xyz+C$