UNIT-I:

Matrices

Matrix:-

An ordered set of 'mn' numbers real or complex arranged in a rectangular array of 'm' rows & n colums written as.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$ is called as $m \times m$ matrix.

These mn numbers are also called as the elements of the matrix. The symbol 'a_{ij}' denotes the elements in the i^{th} row and j^{th} column. The elements of a matrix may be constants or functions.

Real Matrix:- A matrix whose elements are real numbers is called a real matrix.

Ex:- $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$

Complex matrix:- A matrix in which atleast one element imaginary is called a complex matrix.

Ex:- $\begin{bmatrix} 7i & 4 \\ 9 & 1+4i \end{bmatrix}$

Row Matrix:- A matrix which consists of a single row is called a row matrix or row vector.

(or)

A matrix of order $1 \times n$ is called a row matrix.

Ex:- $[1\ -4\ 3]_{1\times 3}$, $[a\ \ b\ \ c]_{1\times 3}$ $[-1\ 2\ 3\ 5]_{1\times 4}$

Column Matrix:- A matrix which consists of a single column is called a column matrix or column vetor.

(OR)

A matrix of order $m \times 1$ is called a column matrix.

Ex:- $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}_{3\times1}$, $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}_{4\times1}$.

Null matrix :- (or) Zero matrix :-

A matrix in which all the elements are zero is called a zero (or) null matrix. It is denoted by 'O'.

Ex:- $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times3}$.

Square matrix :- A matrix in which the no. of rows is equal to the no. of columns is called a square matrix.

Ex: $\begin{bmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{3\times3}$ $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2}$

The sum of the diagonal elements of a square matrix is called the "trace of that matrix"

Determinant of a matrix :-

A determinant which has same elements as the square matrix 'A' is known as determinant of matrix 'A' and it is denoted by $|A|$ (or) det A.

Ex:- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $|A| = ad - bc$.

If $|A| = 0$ then the matrix 'A' is called singular matrix

If $|A| \neq 0$ then the matrix 'A' is called non-singular matrix

Difference b/w a matrix and determinant :-

* In a determinant the rows & column's must be equal. where as in a matrix the no. of rows and columns may or may not be equal.

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* on interchaning the rows and columns a different matrix is formed where as in determinant an interchange of rows and columns does not change the value of determinant. (2)
* A determinant can be reduced to a number where as a matrix cannot be reduced to a number.

Diagonal elements of a Square matrix & principal diagonal:-

In a matrix $A=[a_{ij}]_{m\times n}$ the elements a_{ij} of 'A' for which $i=j$ [ie., $a_{11}, a_{22}, a_{33}, \ldots a_{nn}$] are called the diagonal elements of A.

The line along which the diagonal elements lie is called the principal diagonal of A.

Diagonal matrix :-

A S. matrix in which all the elements except those along the diagonal are zero is called diagonal matrix.

Ex:- $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Scalar matrix:- A diagonal matrix in which all the diagonal elements are equal is called a scalar matrix.

Ex:- $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

unit matrix (or) identity matrix:- A diagonal matrix in which all the diagonal elements are equal to unity is called a unit matrix.

A unit matrix of order 'n' is denoted by I_n.

Ex:- $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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Triangular Matrix :- A Square matrix in which every element either above (or) below principal diagonal is zero is called a triangular matrix.

(i) **upper triangular matrix :-**

If every element below the principal diagonal is zero is called upper triangular matrix.

Eg: $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 7 \\ 0 & 0 & 2 \end{bmatrix}$

(ii) **lower Triangular matrix :-**

If every element above the Principal diagonal is zero then the matrix is called lower triangular matrix

Symmetric matrix :-

A square matrix $A = [a_{ij}]$ is said to be Symmetric if $a_{ij} = a_{ji}$, ie A' is a Symmetric matrix $\Leftrightarrow A = A^T$.

Ex: $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ $A^T = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ $\therefore A = A^T$.

Skew Symmetric matrix :-

A square matrix $A = [a_{ij}]$ is said to be a skew-Symmetric if $a_{ij} = -a_{ji}$, ie A' is skew symmetric matrix $\Leftrightarrow A = -A^T$.

Ex: $\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 8 \\ 3 & -8 & 0 \end{bmatrix}$

Transpose of A Matrix :-

Matrix obtained by interchanging rows and columns is known as transpose of a given matrix. The transpose of

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a matrix 'A' is denoted as A^T (or) A'.

Properties:-

If A^T and B^T be the transpose of A and B respectively, then (i) $(A^T)^T = A$ (ii) $(A+B)^T = A^T + B^T$ (iii) $(kA)^T = kA^T$ (iv) $(AB)^T = B^T A^T$.

Idempotent Matrix:-

If 'A' is a square matrix such that $A^2 = A$, then A is called idempotent matrix.

Involutary Matrix:-

If 'A' is a square matrix such that $A^2 = I$, then 'A' is called Involutary matrix.

Nilpotent Matrix:-

A square matrix 'A' is called a nilpotent matrix if there exists a positive integer such that $A^m = 0$.

If 'm' is the least positive integer such that $A^m = 0$ then 'm' is called the index of the nilpotent matrix A.

Ex:- If $A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ then $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Periodic Matrix:-

A square matrix 'A' is called a periodic matrix if $A^{k+1} = A$ where 'k' is a positive integer. The least positive integer for which $A^{k+1} = K$ is called the period of A.

Adjoint of a matrix:-

Let $A = [a_{ij}]$ be a square matrix and let A_{ij} denote the matrix formed by the cofactors a_{ij} then the transpose of the matrix A_{ij} is called the adjoint matrix of 'A' and is denoted by Adj A.

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Cofactor of A matrix :-

The cofactor A_{ij} of a_{ij} of 'A' is a signed minor.

ie., $A_{ij} = (-1)^{i+j} |M_{ij}|$ where $|M_{ij}|$ is the minor of a_{ij} of 'A'.

Note :-

Minor of an element a_{ij} of $n \times n$ square matrix is the determinant of the (n-1) square matrix of 'A' obtained by deleting the i^{th} row and j^{th} column from 'A'

Inverse of a Matrix :-

If 'A' is a square matrix & if there exists another matrix 'B' such that $AB = BA = I$, then 'B' is called the Inverse of matrix 'A' and is denoted by A^{-1}. Here both A and B are square matrices of same order. If 'A' is non-singular matrix then $A^{-1} = \frac{adj A}{|A|}$

* Inverse of a matrix is unique.
* only non-singular matrix posses inverses.
* Singular matrix cannot have inverse
* If a square matrix 'A' possess an inverse, then 'A' is said to be inverse
* $(A')^{-1} = (A^{-1})'$.

Theorem :-

(1) If A and B are non-singular matrices of same order then prove that $(AB)^{-1} = B^{-1}A^{-1}$

(2) If 'A' is non singular matrices then prove that the transpose of inverse is equal to the inverse of transpose.

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(1) Find the Inverse of A matrix $A = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

Sol:- $|A| = 2(1-4) - 5(3-2) + 3(6-1) = 2(-3) - 5(1) + 3(5)$

$|A| = -6 - 5 + 15 = -11 + 15 = 4 \neq 0$

$\therefore |A|$ is non-singular matrix $\Rightarrow A^{-1}$ exists.

$\boxed{A^{-1} = \dfrac{\text{adj} A}{|A|}}$ f - First, c - central, L - Last

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$$\begin{matrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 1 & 2 \\ 5 & 3 & 2 & 5 \\ 1 & 2 & 3 & 1 \end{matrix}$$

$$\text{adj} A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

where A_{ij}'s are the cofactors of elements a_{ij}. $\begin{bmatrix} -3 & -1 & 5 \\ 1 & -1 & 1 \\ 7 & 5 & -13 \end{bmatrix}$

Thus the minor of a_{ij} are

cofactors of $A_{ij} = (-1)^{i+j} |M_{ij}|$

$$\text{cofactors matrix } A = \begin{bmatrix} +(-3) & -(1) & +5 \\ -(-1) & +(-1) & -(-1) \\ +(7) & -(-5) & +(-13) \end{bmatrix} = \begin{bmatrix} -3 & -1 & 5 \\ 1 & -1 & 1 \\ 7 & 5 & -13 \end{bmatrix}$$

$$\text{adj} A = [\text{cofactor } A]^T$$

$$= \begin{bmatrix} -3 & -1 & 5 \\ 1 & -1 & 1 \\ 7 & 5 & -13 \end{bmatrix}^T = \begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj} A}{\det A} = \frac{1}{4} \begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix}$$

(2) Find A^{-1} by using adjoint of A, where $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

$|A| = 1(-12-12) - 1(-4-6) + 3(-4+6) = -24 + 10 + 6 = -8 \neq 0$

A is non-singular matrix

Hence A^{-1} exists. $A^{-1} = \dfrac{\text{adj} A}{|A|}$

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cofactor of matrix $A = \begin{bmatrix} +(-24) & -(-10) & +(2) \\ -(8) & +(2) & -(-2) \\ +(-12) & -(-6) & +(2) \end{bmatrix} = \begin{bmatrix} -24 & 10 & 2 \\ -8 & 2 & 2 \\ -12 & 6 & 2 \end{bmatrix}$

$$\operatorname{adj} A = (c.f\ A)^T$$

$$= \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-8}\begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix} = \frac{-2}{8}\begin{bmatrix} -12 & -4 & -6 \\ 5 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}$$

(3) If $A = \begin{bmatrix} 4 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ find A^{-1}?

Sol:- $|A| = 1 \neq 0$ 'A' is non-singular matrix $\Rightarrow A^{-1}$ exists.

$$A^{-1} = \frac{\operatorname{adj} A}{|A|} = \begin{bmatrix} -1 & 2 & 1 \\ -7 & 11 & 6 \\ -2 & 3 & 2 \end{bmatrix}$$

4) $A = \begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -(1+x) \end{bmatrix}$ find 'x' value such that 'A' is singular.

Sol:- G.T. 'A' is singular matrix. $\Rightarrow |A| = 0$

$|A| = (3-x)[-(4-x)(1-x)+4] - 2[-2(1+x)+2] + 2[-8+2(4-x)] = 0$

$= (3-x)[-4-4x+x+x^2+4] - 2[-2-2x+2] + 2[-8+8-2x] = 0$

$= (3-x)[x^2+x-4x] - 2(-2x) + 2(-2x) = 0$

$= 3x^2+3x-12x-x^3-x^2+4x^2+\cancel{4x}-\cancel{4x} = 0$

$= 7x^2-x^2-x^3-9x = 0$

$= -x(x-3)^2 = 0$

$-x = 0$, $(x-3)^2 = 0$

$x = 0$, $x-3 = 0 \Rightarrow x = 3$.

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Equivalent Matrix :-

Two matrix A & B are said to be equivalent if one matrix 'A' can be obtained from B by a squence of elementary transformations. It is denoted by $\boxed{A \sim B.}$

Note:- Two equivalent matrices have the same order & rank.

Minor of a matrix :-

Let 'A' be an $m \times n$ matrix, the determinant of a square sub matrix called a minor of the matrix, if the order of a square submatrix is t, then its determinant is called a minor of order t.

Sub Matrix of a matrix :- 'A' is any matrix obtained by deleting some rows and columns in A.

Ex:- $A = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \end{bmatrix}_{3\times5}$

Here maximum minor of 'A' is of order 3

Minor of order 3 $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$, $\begin{vmatrix} a_1 & b_1 & e_1 \\ a_2 & b_2 & e_2 \\ a_3 & b_3 & e_3 \end{vmatrix}$ ....

Minor of order 2 are $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$, $\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$ ...

Every element of 'A' can be considered as the minor of order 1.

Ex:- Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}_{4\times3}$

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let $B = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 5 & 6 & 7 \end{bmatrix}$

$|B| = 2(7-12) - 1(21-10) + 1(18-5) = -8$ is a minor of order 3.

2. Rank of a matrix :—

A matrix is said to be of rank r, if

(i) Every minor of order $r+1$ (or) more is zero

(ii) atleast one minor of order r is non-zero.

Rank of a matrix 'A' is denoted by $\rho(A)$.

Note :—

* Rank of a matrix is unique
* Rank of a null matrix is zero
* Rank of identity (or) unit matrix, I_n i.e., $\rho(I_n) = n$
* Rank of non-zero row & column matrix is one.
* Rank of non-zero non-singular matrix of order 'n' is n.
* Rank of every non-singular matrix of order 'n' is "$<n$".
* Rank of singular matrix of order 'n' is "$<n$".
* If 'A' is a matrix of order $m \times n$, then $\rho(A) \le \min(m,n)$
* $\rho(A) = \rho(A^T)$
* To determine the rank of A, if m, n are both greater than 4, this definition will not be much of use.

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problems :-

* Find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$

$= 4(-5)+2(1)+2(14)$
$= -20+2+28$
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Sol:- Given $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$

2014 A (1) Find rank $\begin{bmatrix} 4 & -2 & 2 \\ 5 & 3 & 2 \\ 2 & 4 & 1 \end{bmatrix}$

$|A| = 1(0-2)-2(3-4)+1(-1-0)$
$= 1(-2)-2(-1)+1(-1)$
$= -2+2-1$
$= -1$
$|A| \neq 0$

2015 A (2) $\begin{bmatrix} 0 & 3 & 1 \\ 2 & 3 & 5 \\ 2 & 1 & 2 \end{bmatrix}$ $\rho(A)=3$

$0-3(-6)+1(-4)$
$18-4=14$

$\therefore$ A is a non singular matrix and 'A' is 3^{rd} order matrix

$\therefore \rho(A) = 3.$

* Find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}_{3\times4}$

Sol:- Given $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}_{3\times4}$

given matrix is of order 3×4, $\rho(A) \leq \min$ of $(3,4)=$

$\Rightarrow \rho(A) \leq 3$

Consider any minor of order 3,

$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{bmatrix} = 1(0-49)-2(0-56)+3(35-48)$
$= 24$

$\therefore \rho(A) = 3.$

3) Find the rank of matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$

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Given $|A| = 2(72-72) - 3(48-48) + 4(36-36)$
$= 0$

$\therefore$ A is a singular matrix.

Hence $\rho(A) < 3$

consider any minor of order 2,

$$\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} \quad \begin{vmatrix} 2 & 4 \\ 4 & 8 \end{vmatrix} \quad \begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix} \quad \begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} \quad \begin{vmatrix} 3 & 4 \\ 9 & 12 \end{vmatrix} \quad \begin{vmatrix} 2 & 4 \\ 6 & 12 \end{vmatrix}$$

$= 12-12$ $\quad = 0 \quad = 0 \quad = 0 \quad = 0 \quad = 0$
$= 0$

$$\begin{vmatrix} 4 & 6 \\ 6 & 9 \end{vmatrix} \quad \begin{vmatrix} 4 & 8 \\ 6 & 12 \end{vmatrix} \quad \begin{vmatrix} 6 & 8 \\ 9 & 12 \end{vmatrix}$$

$= 0 \quad = 0 \quad = 0$

All second order minors are zero.

$\therefore \rho(A) = 1.$

4) find the value of 'k' such that $\rho(A) = 2$,

where $A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & k & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}_{3\times 4}.$

Sol:- $\therefore \rho(A) \leq \min(3,4) \leq 3$

Given $\rho(A) = 2$

$\therefore$ order of minors of order 3 is equal to zero

$$\therefore \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & k \\ 3 & 1 & 0 \end{vmatrix} = 0 \Rightarrow$$

$= 1(0-k) - 1(0-3k) - 1(1+3) = 0$

$-k + 3k - 4 = 0$

$2k - 4 = 0$

$2k = 4$

$\boxed{k = 2}$

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* Find the value of 'k', the matrix $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & 2 \\ 9 & 9 & k & 3 \end{bmatrix}_{4\times4}$ has rank 3. (7)

Sol:- Let $A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & 2 \\ 9 & 9 & k & 3 \end{bmatrix}$

The given matrix is of order 4x4, and rank is 3.

then we must have $|A| = 0$

$$|A| = \begin{array}{c} \begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} \end{matrix} \\ \begin{vmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & 2 \\ 9 & 9 & k & 3 \end{vmatrix} \end{array}$$

$$= 4\begin{vmatrix} 1 & -1 & 0 \\ 2 & 2 & 2 \\ 9 & k & 3 \end{vmatrix} - 4\begin{vmatrix} 1 & -1 & 0 \\ k & 2 & 2 \\ 9 & k & 3 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 & 0 \\ k & 2 & 2 \\ 9 & 9 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & -1 \\ k & 2 & 2 \\ 9 & 9 & k \end{vmatrix}$$

$$= 4\left[1(6-2k) + 1(6-18)\right] - 4\left[1(6-2k) - 1(3k-18)\right]$$

$$-1\left[1(2k-18) - 1(k^2-18) - 1(9k-18)\right]$$

$|A| = k^2 + 4k - 12$

$\therefore |A| = 0$

$\Rightarrow k^2 + 4k - 12 = 0$

$k^2 + 6k - 2k - 12 = 0$

$k(k+6) - 2(k+6) = 0$

$(k-2)(k+6) = 0$

$k = 2, -6$

* $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ $R_4 \to R_4 - R_1$, $R_3 \to R_3 - R_1$

$C_3 \to C_3 - C_1$, $C_4 \to C_4 - C_1$

$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & -3 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix}$

$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & -3 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix}$ $R_4 \to R_4 - R_2$, $R_3 \to R_3 - 3R_2$

$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\rho(A) = 2$

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Different Methods to find the rank of a matrix:

(I) Echelon form of a matrix:-

A matrix is said to be in echelon form if

(1) All non zero rows, if any precede the zero rows

i.e., zero rows if any (account) occurs, they should be below non zero rows.

(2) The number of zeros preceding the first non-zero element in a row is less than the number of such zero's in the succeeding row.

(3) The first non-zero element in each non zero row is unity (it is optional).

In E.F we must do Row operations

Note:- The rank of a matrix in echelon form is the number of non-zero rows of the matrix.

* Zero rows if any occurs they should be below non-zero rows.
* The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.
* The first non zero element in each non-zero row is equal to 1.

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1) Find the rank of matrix by reducing it into echelon form (8)

$$\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$$

Sol:- let $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 4R_1$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix}$$

$R_3 \to R_3 - R_2$, $R_4 \to R_4 - 3R_2$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is in echelon form, & number of non-zero rows = 2

$\therefore \rho(A) = 2$

2. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 6R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

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$R_2 \leftrightarrow R_4$ (ie., interchanging)

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AB

71, 91, 94, 99
A6, A7, B0, B9,

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_4 \to R_3 + R_4$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ which is in echelon form of no. of non-zero rows = 3

$\therefore \rho(A) = 3$

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3) $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

Sol :- $R_2 \to R_1 + R_2$, $R_3 \to R_3 + 2R_1$

$R_4 \to R_4 - R_1$

$$A \sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix}$$

$R_3 \to 2R_3 - 11R_2$, $R_4 \to R_4 + 2R_2$

$$\sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_4 \to 6R_4 + R_3$

$$\sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$\therefore$ no. of non-zero rows = 4

$\therefore \rho(A) = 4$

4) $\begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -3 \end{bmatrix}$$

$R_2 \to R_2/3$

$R_3 \to R_3/2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & \\ 0 & 1 & 1 & - \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank =

4) Find the rank of $\begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$ (9)

Sol:- $R_4 \to \frac{R_4}{3}$

$$A \sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -1 & 1 & 2 & 2 & 4 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 + R_1$

$R_4 \to R_4 + R_1$

$$\sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 5 & 5 & 0 & 5 \\ 0 & 10 & 10 & 0 & 10 \\ 0 & 5 & 5 & 0 & 5 \end{bmatrix}$$

$R_4 \to R_4 - R_2$, $R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 5 & 5 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2$.

5. Find the l & m. S.T. $\begin{bmatrix} 1 & -1 & 2 & 4 \\ 2 & 1 & -1 & 3 \\ 7 & -1 & l & m \end{bmatrix}$ has rank 2.

I. Method

$$|A| = \begin{vmatrix} 1 & -1 & 2 & 4 \\ 2 & 1 & -1 & 3 \\ 7 & -1 & l & m \end{vmatrix}$$

all minors of order 3 must be zero, since rank is 2.

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$$\Rightarrow \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 7 & -1 & l \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} 1 & -1 & 4 \\ 2 & 1 & 3 \\ 7 & -1 & m \end{vmatrix} = 0$$

$\Rightarrow 1(l-1)+1(2l+7)+2(-2-7)=0$

$l-1+2l+7-18=0$

$3l-12=0$

$\boxed{l=4}$

$1(m+3)+1(2m-21)+4(-2-7)=0$

$m+3+2m-21+36=0$

$3m=54$

$\boxed{m=18}$

II method

$$A = \begin{bmatrix} 1 & -1 & 2 & 4 \\ 2 & 1 & -1 & 3 \\ 7 & -1 & l & m \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 7R_1$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 3 & -5 & -5 \\ 0 & 6 & l-14 & m-28 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 3 & -5 & -5 \\ 0 & 0 & l-4 & m-18 \end{bmatrix}$$

which is in echelon form & its rank is 2. so in 3rd row, all elements are zero.

$\therefore l-4=0$, $m-18=0$

$l=4$, $m=18$

6. find rank $\begin{bmatrix} 3 & 1 & 4 & 6 \\ 2 & 1 & 2 & 4 \\ 4 & 2 & 5 & 8 \\ 1 & 1 & 2 & 2 \end{bmatrix}$ $\rho(A)=3$

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7. $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ $\rho(A)=2$

8. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ $\rho(A)=3$

9. Find rank $\begin{bmatrix} 8 & 0 & 0 & 1 \\ 1 & 0 & 8 & 1 \\ 0 & 0 & 1 & 8 \\ 0 & 8 & 1 & 8 \end{bmatrix}$

$\rho(A)=4$

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Normal form (or) Canonical form:-

Let every non-zero matrix can be reduced by a finite number of elementary row and column transformations to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ (OR) $[I_r]$ (or) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (or) $[I_r \ \ 0]$

where I_r is the unit matrix of order r, then the rank of given matrix is 'r'.

Problems :-

1. Reduce the matrix $\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & 5 \\ 1 & 3 & 2 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ to normal form and find its rank ?

Sol:- Let $A = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & 5 \\ 1 & 3 & 2 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$

$$A \sim \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & 5 \\ 0 & 3 & 5 & -2 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$, $R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & -7 & -17 \\ 0 & 0 & -3 & -7 \end{bmatrix}$$

$C_3 \to C_3 + 3C_1$, $C_4 \to C_4 - 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & -7 & -17 \\ 0 & 0 & -3 & -7 \end{bmatrix}$$

$C_3 \to C_3 - 4C_2$, $C_4 \to C_4 - 5C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -7 & -17 \\ 0 & 0 & -3 & -7 \end{bmatrix}$$

$R_4 \to 7R_4 - 3R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -7 & -17 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$C_4 \to 7C_4 - 17C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 14 \end{bmatrix}$$

$R_3 \to \frac{R_3}{-7}, R_4 \to \frac{R_4}{14}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\sim [I_4]$

$\therefore$ rank $= \rho(A) = 4$

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2. $\begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$

Sol:- $R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \to C_2 - 3C_1$, $C_3 \to C_3 - 6C_1$

$C_4 \to C_4 + C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 + C_2$, $C_4 \to C_4 - 2C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ rank $= 2$.

3) find rank using normal form $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 5 & 6 \end{bmatrix}$

Sol:- $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$, $C_3 \to C_3 - C_1$, $C_4 \to C_4 - 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ rank $= 3$

4) ✓ Apply elementary transformation to find the rank of

$$A = \begin{bmatrix} -1 & -7 & 3 & -3 \\ 7 & 20 & -2 & 25 \\ 5 & -2 & 4 & 7 \end{bmatrix} \quad \rho(A) = 3$$

Ans:- $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ $\therefore \rho(A) = 2$

5) find the rank of matrix by reducing it into canonical form $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$ Ans:- $[I_3 \ 0]$ (11)

6) find rank using normal form $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ (A) $\rho(A)=2$

7) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -3 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ (A) $\rho(A)=4$ (8) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 3 & -1 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & -2 & 1 & 0 \end{bmatrix}$ $\rho(A)=3$

(9) $\begin{bmatrix} 1 & -2 & 1 & 2 \\ 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \end{bmatrix}$

System of linear algebraic equations:-

An equation of the form $a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = b$. where $x_1, x_2, x_3, \dots, x_n$ are unknown and $a_1, a_2, a_3 \dots a_n$, b are constants is called a linear equation in 'n' unknowns.

System of linear non homogeneous equations:-

Consider the system of 'm' linear equations in an 'n' unknowns. $x_1, x_2, \dots, x_n$ are given below.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + \dots + a_{3n}x_n &= b_3 \\ \vdots \qquad\qquad & \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}\right\} \rightarrow (1)$$

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where a_{ij}'s & b_i's $i = 1, 2, 3, \dots m$, $j = 1, 2, 3, \dots, n$ are constants and ordered n-triple $[x_1, x_2, x_3, \dots x_n]$ satisfying all equations in (1) simultaneously is called a solution of system (1). The system of eq (1) can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$$

$$\Rightarrow A \; X = B$$

Here 'A' is called the coefficient matrix and 'B' is constant vector / column matrix and 'X' is set of unknowns / solution vector. when B_i's are not zero i.e, one B_i is atleast non zero then the system is said to be non-homogeneous.

Homogeneous:-

If $B_i = 0$, $i = 1, 2, 3, \dots, m$ all R.H.S constants are zero, then the system is said to be homogeneous.

4 Trivial Solution / zero Solution:-

Trivial solution is a solution where all x_j's are zero, $x_1 = x_2 = x_3 = \dots x_n = 0$

Augmented matrix:-

A matrix which is obtained by attaching the elements of 'B' in the last column of co-efficients matrix is called augmented matrix. It is denoted by $[AB]$ (or) $[A|B]$ or $\tilde{A}$.

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Now Augmented matrix $[AB]$ of system ① is obtained (12)
by augmenting A by column B. ie,

$$[AB] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ - & - & - & - & - & - \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{bmatrix}$$

Consistent :-
The system of equations is said to be consistent if the system has atleast one solution.

In consistent :-
The system of eq is said to be inconsistent if the system has no solutions.

Note :-

* If $\rho(A) = \rho(AB) = n$ (no. of unknowns), then the system has unique solution.
* If $\rho(A) = \rho(AB) < n$ then the system has infinite number of solutions. in terms of $(n-r)$ arbitrary constants, where $r = \rho(A) = \rho([AB])$.
* If $\rho(A) \neq \rho(AB)$ then the system is inconsistent and it has no solution.

2013
1) Solve the system $2x - y + 4z = 12$, $3x + 2y + z = 10$, $x + y + z = 6$ if it is consistent.

Sol :- Given system of equations can be written in matrix form as

$$\underset{A}{\begin{bmatrix} 2 & -1 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 12 \\ 10 \\ 6 \end{bmatrix}}$$

Consider $[AB] = \begin{bmatrix} 2 & -1 & 4 & 12 \\ 3 & 2 & 1 & 10 \\ 1 & 1 & 1 & 6 \end{bmatrix}$

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Now Reducing it into Echelon form.

$R_1 \leftrightarrow R_3$

$$[A\ B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & 2 & 1 & 10 \\ 2 & -1 & 4 & 12 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & -2 & -8 \\ 0 & -3 & 2 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & -2 & -8 \\ 0 & 0 & 8 & 24 \end{bmatrix}$$

$R_2 \to \frac{R_2}{-1}$, $R_3 \to \frac{R_3}{8}$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

which is in echelon form

$\rho(AB) = 3$, $\rho(A) = 3$

$\therefore \rho(A) = \rho([AB])$, Hence the system is Consistent and it has a solution.

Here n = no. of unknowns $= 3$

$\therefore \rho(A) = \rho(AB) = 3 = n$.

$\therefore$ The given System has unique solution.

Now the system of eq is equivalent to

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 3 \end{bmatrix}$$

$x + y + z = 6$

$y + 2z = 8$

$z = 3$

$\therefore y = 8 - 6$ $\quad x = 6 - 2 - 3$

$y = 2$ $\quad x = 1$

$\therefore x = 1, y = 2, z = 3$ is a unique solution

$$\therefore X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

2) Test for Consistency and if Consistent, Solve the system

$5x + 3y + 7t = 4$, $3x + 26y + 2t = 9$, $7x + 2y + 10t = 5$.

Sol:- Consider

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

Now

$$[A\ B] = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix}$$

$R_2 \to 5R_2 - 3R_1$, $R_3 \to 5R_3 - 7R_1$

$$\sim \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & -11 & 1 & -3 \end{bmatrix}$$

$R_2 \to \frac{R_2}{11}$

$$\sim \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 11 & -1 & 3 \\ 0 & -11 & 1 & -3 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 11 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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which is in Echole form $\rho(AB)=2$, $\rho(A)=2$, the given system is consistent & has a solution. (13)

Here $n=3$

$\rho(A)=\rho(AB)=2<n$

hence the given System has infinite number of solution interms of $n-r=1$ (arbitary constant).

Now the system of equations is equivalent to $AX=B$

6/8/19 II CSE-B(AB)

65, 71, 87, 91, 94, 99, B0, B9

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$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

$5x+3y+7t=4$

$11y-t=3$

Let $t=k$ Then $11y=3+k$

$y=\frac{1}{11}(3+k)$

$5x=4-3y-7t$

$=4-\frac{3}{11}(3+k)-7k$

$5x=\frac{44-9-3k-77k}{11}=\frac{35-80k}{11}$

$x=\frac{7-16k}{11}$

$\therefore x=\frac{7-16k}{11}$, $y=\frac{3+k}{11}$, $t=k$.

2015 1) Test the system of Eq $2x+y+5z=4$, $3x-2y+2z=2$, $5x-8y-4z=1$ Consistency It consistent solve them. (A) $\rho(A)\neq\rho(AB)$ It is inconsistent.

2013 2) $u+2v+2w=1$, $2u+v+w=2$, $3u+2v+2w=3$, $v+w=0$ (A) $u=1$, $v=-k$, $w=k$

3) If Consistent, Solve the system of equations $x+y+z+t=4$, $x-y+z+2t=2$, $y+z-3t=-1$, $x+2y-z+t=3$.

Sol:- $\rho(A)=4$, $\rho(AB)=4$ It has unique Solution.

$\rho(A)=\rho(AB)=4=n$

$x+y+z+t=4$, $y+2z-t=2$, $z+2t=3$, $t=1$

$\therefore x=1$, $y=1$, $z=1$, $t=1$

$x+y+z+t=4$

$\cancel{y+2z-t=}$

$x-z=2$

$y-2z=-$

$z-t=0$

$-3t=-4$

$t=4/3$

$x=-1/3$, $y=5/3$, $z=4/3$.

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2008

4) Show that the equations $x-4y+7z=14$, $3x+8y-2z=13$, $7x-8y+26z=5$ are not consistent.

$$[AB]=\begin{bmatrix}1 & -4 & 7 & 14\\ 3 & 8 & -2 & 13\\ 7 & -8 & 26 & 5\end{bmatrix}$$

$R_2 \to R_2-3R_1$, $R_3\to R_3-7R_1$

$$\sim\begin{bmatrix}1 & -4 & 7 & 14\\ 0 & 20 & -23 & -29\\ 0 & 20 & -23 & -93\end{bmatrix}$$

$R_3 \to R_3-R_2$

$$\sim\left[\begin{array}{ccc|c}1 & -4 & 7 & 14\\ 0 & 20 & -23 & -29\\ 0 & 0 & 0 & -64\end{array}\right]$$

which is in echelon form

$\rho([AB])=3$, $\rho(A)=2$

$\therefore \rho(A)\neq\rho(AB)$

Hence given system is inconsistent & has no solutions.

5. Discuss for what values of λ,μ the solution of eq

$x+y+z=6$ (i) no solution

$x+2y+3z=10$ (ii) unique

$x+2y+\lambda z=\mu$ (iii) an infinite no. of solutions.

Ans:— The system of eq's can be written in the form of

$AX=B$.
$$\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 2 & \lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}6\\ 10\\ \mu\end{bmatrix}$$

Consider
$$[AB]=\begin{bmatrix}1 & 1 & 1 & 6\\ 1 & 2 & 3 & 10\\ 1 & 2 & \lambda & \mu\end{bmatrix}$$

$R_2\to R_2-R_1$, $R_3\to R_3-R_1$

$$\sim\begin{bmatrix}1 & 1 & 1 & 6\\ 0 & 1 & 2 & 4\\ 0 & 1 & \lambda-1 & \mu-6\end{bmatrix}$$

$R_3\to R_3-R_2$

$$\sim\begin{bmatrix}1 & 1 & 1 & 6\\ 0 & 1 & 2 & 4\\ 0 & 0 & \lambda-3 & \mu-10\end{bmatrix}$$

which is in echelon form.

i) Let $\lambda=3$, $\mu\neq10$ then

$\rho(A)=2$, $\rho([AB])=3$

$\therefore \rho(A)\neq\rho(AB)$ system is inconsistent & has no solution exists.

In this case the system is inconsistent & has no solution exists.

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(ii) Let $\lambda \neq 3$, μ has any value then $\rho(A) = \rho(AB) = 3 = n$. In this case the system is consistent and has unique solution.

(iii) Let $\mu = 10, \lambda = 3$ then $\rho(A) = \rho(AB) = 2 < (n=3)$. Then the system has infinite number of solutions.

6) Find for what values of λ, the $3x - y + 4z = 3$, $x + 2y - 3z = -2$, $6x + 5y + \lambda z = -3$ have infinite no. of $(r<n)$ solutions and solve them with that λ value?

Sol:-

$$[AB] = \begin{bmatrix} 3 & -1 & 4 & 3 \\ 1 & 2 & -3 & -2 \\ 6 & 5 & \lambda & -3 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$

$$\sim \begin{bmatrix} 1 & 2 & -3 & -2 \\ 3 & -1 & 4 & 3 \\ 6 & 5 & \lambda & -3 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 6R_1$

$$\sim \begin{bmatrix} 1 & 2 & -3 & -2 \\ 0 & -7 & 13 & 9 \\ 0 & -7 & \lambda+18 & 9 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & -3 & -2 \\ 0 & -7 & 13 & 9 \\ 0 & 0 & \lambda+5 & 0 \end{bmatrix}$$

which is in echelon form.

If $\lambda = -5$, Rank of $A = 2$, Rank of $[AB] = 2$

$\therefore \rho(A) = \rho(AB) < n$ The given system is consistent & it has infinite no. of solutions

If $\lambda = -5$ the system becomes $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -7 & 13 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \\ 0 \end{bmatrix}$

$x + 2y - 3z = -2$, $-7y + 13z = 9$

Let $z = k$ $\qquad -7y = 9 - 13z$

$x = -2 - 2y + 3z \qquad y = \frac{1}{7}(13z - 9)$

$x = -2 - \frac{2}{7}(13k - 9) + 3k$

$= \frac{4}{7} - \frac{5}{7}k$

$x = \frac{1}{7}(-5k + 4)$

$\therefore x = \frac{1}{7}(-5k + 4)$

$y = \frac{1}{7}(13k - 9)$

$z = k$.

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7) for what values of 'k', $x+y+z=1$, $4x+y+10z=k^2$, $2x+y+4z=k$ has solution & solve?

Sol:-

$$[AB] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 1 & 10 & k^2 \\ 2 & 1 & 4 & k \end{bmatrix}$$

$R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 6 & k^2-4 \\ 0 & -1 & 2 & k-2 \end{bmatrix}$$

$R_3 \to 3R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 6 & k^2-4 \\ 0 & 0 & 0 & \underbrace{3k-k^2-2}_{(k-2)(k-1)} \end{bmatrix}$$

Here $\rho(A)=2$, $\rho([AB])$ must be 2 for consistency. The system is consistent if $\rho(A) = \rho(AB)$

$\Rightarrow$ if $\rho(AB)=2$ then $(k-2)(k-1)=0$

$k=1,2.$

Case(i):-

when $k=1$ then $[AB] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Now $\rho(A)=\rho(AB)=2<3(=n)$, the system has infinite no. of solutions in terms of $n-r=1$ arbitary value.

The given system

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$$

$x+y+z=1 \Rightarrow x = 1-y-z$
$= 1-1-2k_1-k_1$
$= -3k_1$

$-3y+6z=-3$

Let $z=k_1$

$-3y=-3-6z$

$y=1+2k_1$

$$\therefore X = \begin{bmatrix} -3k_1 \\ 1+2k_1 \\ k_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

for different values of k_1, we get infinite no. of solutions.

Case(ii):- when $k=2$ then

$$[AB] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A)=\rho(AB)=2<n$, the system has infinitely many solutions.

$\therefore$ The given system

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

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$x+y+z=1 \qquad -3y+6z=0 \qquad z=k_2$

$x=1-2k_2-k_2 \qquad -3y=-6k_2$

$x=1-3k_2 \qquad y=2k_2$

$$\therefore X=\begin{bmatrix}1-3k_2\\2k_2\\k_2\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}+k_2\begin{bmatrix}-3\\2\\1\end{bmatrix}$$

HW:- Are the following eq consistent if so solve $x_1-x_2+x_3-x_4+x_5=1$, $2x_1-x_2+3x_3+4x_5=2$, $3x_1-2x_2+2x_3+x_4+x_5=1$, $x_1+x_3+2x_4+x_5=0$.

(A) $\rho(A)=\rho(AB)=3<5(=n)$ Here $n-r=5-3=2$.

2008

8) Test for Consistency $x+y+z=1$, $x-y+2z=1$, $x-y+2z=5$, $2x-2y+3z=1$, $3x+y+z=2$.

Ans.- The system is inconsistent, it has no solution.

$\rho(A)\neq\rho(AB)$

$3\neq 4$.

2009

9) Solve $x+2y+3z=1$, $2x+3y+8z=2$, $x+y+z=3$.

Ans.- The system is consistent, it has unique solution

$$X=\begin{bmatrix}9/2\\-1\\-1/2\end{bmatrix}$$

2010

10) HW. Find whether the following system of eq are consistent. If so solve $x+2y+2z=2$, $3x-2y-z=5$, $2x-5y+3z=-4$, $x+4y+6z=0$

(A) $\rho(A)=\rho(AB)=3=n \quad (x=2, y=1, z=-1)$.

2011

(11) Show that $3x+3y+2z=1$, $x+2y=4$, $10y+3z=-2$, $2x-3y-z=5$ is consistent & hence solve it.

(A) $\rho(A)=\rho(AB)=3$, it has unique solution $X=\begin{bmatrix}2\\1\\-4\end{bmatrix}$

12) If $a+b+c\neq 0$, S.T. system $\begin{matrix}-2x+y+z=a\\x-2y+z=b\\x+y-2z=c\end{matrix}$ has no solution

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Consistency of system of homogeneous linear equation:-

Consider the system of 'm' homogeneous linear equations in 'n' unknowns as $x_1, x_2, x_3, \ldots, x_n$ as given below.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = 0 \\ \text{- - - - - - - - - - -} \\ \text{- - - - - - - - - - -} \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = 0 \end{array}\right\} \rightarrow (1)$$

The system of eq (1) can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ - & - & - & - & - \\ - & - & - & - & - \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \rightarrow (2)$$

$$AX = 0$$

Here 'A' is called coefficient matrix. It is clear that $x_1 = x_2 = x_3 = \cdots = x_n = 0$ is a solution of (1)

$$X = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ is a solution of } (2).$$

This is called trivial solution or zero solution of $AX=0$

This system $AX=0$ is always consistent (ie) it has a solution.

* A zero solution is not linearly independent (ie) it is linearly dependent.

Working Rule:-

Case(i):- If $r=n$, then the system of equations have only trivial solution.

Case(ii):- If $r<n$ then the system of eq's have infinite no. of solutions / non trivial solutions, we shall have $n-r$ linearly independent solutions.

[If $r<n$, to obtain infinite solutions, set $n-r$ variable in a arbitrary values and solve for remaining unknowns].

Case(iii):- If $m<n$, then $r\leq m<n$, in this case also a system of eq's have infinite no. of solutions.

Problems:-

1. Solve the system of equations $x+y+w=0$, $y+z=0$, $x+y+z+w=0$, $x+y+2z=0$.

Sol:- The given system of eq's can be written in form $AX=0$.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now reducing into echelon form.

$R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}$$

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$R_4 \to R_4 - 2R_3$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

which is in echelon form & no. of non zero rows $= 4$

Hence $\rho(A) = 4$, $n = 4$

$\therefore r = n$.

The system has trivial solution (or) zero solution only.

ie, $x = y = z = w = 0$.

2. Solve the system $2x - y + 3z = 0$, $3x + 2y + z = 0$, $x - 4y + 5z = 0$

Sol:- Consider $AX = 0$

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix}$

$R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & -4 & 5 \\ 3 & 2 & 1 \\ 2 & -1 & 3 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & -4 & 5 \\ 0 & 14 & -14 \\ 0 & 7 & -7 \end{bmatrix}$$

$R_3 \to 2R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -4 & 5 \\ 0 & 14 & -14 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 / 14$

$$\sim \begin{bmatrix} 1 & -4 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

which is in echelon form

$\rho(A) = 2$, $n = 3$

$\therefore r < n$, the system has infinite no. of non-zero solutions interms of $n - r = 3 - 2 = 1$ arbitrary value.

The given system of eq's is

$$\sim \begin{bmatrix} 1 & -4 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x - 4y + 5z = 0$, $y - z = 0$, let $z = k$

$x - 4k + 5k = 0$ $\quad y = z$

$x = -k$ $\quad y = k$

$$X = \begin{bmatrix} -k \\ k \\ k \end{bmatrix} = k\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

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$\therefore$ k is a non zero scalar and for different values of k, we get an infinite no of solutions.

3. Solve the system of eq's $4x+2y+z+3w=0$, $6x+3y+4z+7w=0$, $2x+y+w=0$.

Sol:— Consider $AX=0$

$$A=\begin{bmatrix}4&2&1&3\\6&3&4&7\\2&1&0&1\end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\sim\begin{bmatrix}2&1&0&1\\6&3&4&7\\4&2&1&3\end{bmatrix}$$

$R_2 \to R_2-3R_1$, $R_3 \to R_3-2R_1$

$$\sim\begin{bmatrix}2&1&0&1\\0&0&4&4\\0&0&1&1\end{bmatrix}$$

$R_3 \to 4R_3-R_4$

$$\sim\begin{bmatrix}2&1&0&1\\0&0&4&4\\0&0&0&0\end{bmatrix}$$

$R_2 \to R_2/4$

$$\sim\begin{bmatrix}2&1&0&1\\0&0&1&1\\0&0&0&0\end{bmatrix}$$

rank $r=2$, $n=4$

$\therefore r<n$, the system has infinite no. of non zero solutions and $n-r=4-2=2$ arbitary values.

$$\begin{bmatrix}2&1&0&1\\0&0&1&1\\0&0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$2x+y+w=0$

$z+w=0$

Let $w=k_1$, $y=k_2$

$z+k_1=0$

$z=-k_1$

$2x=-w-y$

$=-k_1-k_2$

$x=-\frac{1}{2}(k_1+k_2)$

$$\therefore X=\begin{bmatrix}-\frac{1}{2}(k_1+k_2)\\k_2\\-k_1\\k_1\end{bmatrix}$$

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4) Show that the only real number 'λ' for which the system $x+2y+3z=\lambda x$, $3x+y+2z=\lambda y$, $2x+3y+z=\lambda z$ has non-zero solution is 6 & solve them, when $\lambda=6$. (8)

Sol :-

$$x+2y+3z=\lambda x$$
$$x-\lambda x+2y+3z=0$$
$$(1-\lambda)x+2y+3z=0$$
$$3x+(1-\lambda)y+2z=0$$
$$2x+3y+(1-\lambda)z=0$$

$Ax=0$

$$\begin{bmatrix}1-\lambda & 2 & 3\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Here number of variables $n=3$

The given system of eq's possess a non-zero solution if $r<n$.

W.K.T. if the system has non trivial solution then coefficient matrix 'A' must be singular.

$$|A|=0 \Rightarrow \begin{vmatrix}1-\lambda & 2 & 3\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{vmatrix}=0$$

$R_1 \to R_1+R_2+R_3$

$$\sim\begin{vmatrix}6-\lambda & 6-\lambda & 6-\lambda\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{vmatrix}=0$$

$$\sim(6-\lambda)\begin{vmatrix}1 & 1 & 1\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{vmatrix}=0$$

$C_2 \to C_2-C_1$
$C_3 \to C_3-C_1$

$$\sim(6-\lambda)\begin{vmatrix}1 & 0 & 0\\ 3 & -\lambda-2 & -1\\ 2 & 1 & -\lambda-1\end{vmatrix}=0$$

$$(6-\lambda)\left[(\lambda+2)(\lambda+1)+1\right]=0$$

$$(6-\lambda)\left[\lambda^2+3\lambda+3\right]=0$$

$6-\lambda=0$ & $\lambda^2+3\lambda+3=0$

$\lambda=6$

$$\lambda=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-3\pm\sqrt{9-4\cdot 3}}{2}$$
$$=\frac{-3\pm\sqrt{9-12}}{2}$$
$$=\frac{-3\pm\sqrt{-3}}{2}$$
$$=\frac{-3\pm\sqrt{3}\,i}{2}$$
$(\sqrt{-1}=i)$

$\therefore$ $\lambda=6$ is the only real value & other values are complex.

If $\lambda=6$ Then

$$A=\begin{bmatrix}-5 & 2 & 3\\ 3 & -5 & 2\\ 2 & 3 & -5\end{bmatrix}$$

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$R_2 \to 5R_2 + 3R_1$, $R_3 \to 5R_3 + 2R_1$

$$\sim \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 19 & -19 \end{bmatrix}$$

$R_3 \to R_3 + R_2$, $R_2 \to R_2/19$

$$\sim \begin{bmatrix} -5 & 2 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$ & $n = 3$

$n - r = 3 - 2 = 1$ arbitary values

$-5x + 2y + 3z = 0$, $-y + z = 0$, let $z = k$

$x = k$. $y = z$

$y = k$

$$\therefore X = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

5. Solve the system of equations

$$x + 3y - 2z = 0$$
$$2x - y + 4z = 0$$
$$x - 11y + 14z = 0$$

$r = 2, n = 3$

A) $X = \begin{bmatrix} -\frac{10}{7}k \\ \frac{8}{7}k \\ k \end{bmatrix}$

6. find all solutions of S.O.E.

$$x + 2y - z = 0$$
$$2x + y + z = 0$$
$$x - 4y + 5z = 0$$

$r = 2, n = 3$

$X = \begin{bmatrix} -k \\ k \\ k \end{bmatrix}$

7) Determine whether the following eq's have non-trivial Solutions, if so solve the

$$3x + 4y - z - 6w = 0$$
$$2x + 3y + 2z - 3w = 0$$
$$2x + y - 14z - 9w = 0$$
$$x + 3y + 13z + 3w = 0$$

$r = 2, n = 4$

$X = \begin{bmatrix} 11k_1 + 6k_2 \\ -(8k_1 + 3k_2) \\ k_1 \\ k_2 \end{bmatrix}$

Note:—

* Homogeneous linear eq's $AX = 0$ always consistent.
* $\rho(A)$ = no. of unknowns ie $r = n$, then the system has trivial solution (or) zero solutions.

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* $\rho(A) <$ no. of unknowns i.e., $r < n$ then the system has infinite no. of Solutions.
* If $m = n$ the coefficient matrix 'A' will be order $n \times n$ matrix, then for the system the system of eq's $AX = 0$

The system has

(i) 'A' is singular matrix $\Leftrightarrow$ X has non trivial solution

(ii) A is non-singular matrix $\Leftrightarrow$ X has trivial solution.

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(5) Eigen values and Eigen vectors:—

Let $A=[a_{ij}]$ be an $n \times n$ matrix. A non-zero vector 'X' is said to be a characteristic vector of 'A' if there exist a scalar 'λ' such that $AX=\lambda X$.

Let 'X' be eigen vector of 'A' corresponding to the eigen value 'λ' then by definition $AX=\lambda X$

$$AX=\lambda I X$$

$(A-\lambda I)X=0$ This is homogeneous system of 'n' equations in 'n' unknowns. This will have a non-zero solution iff $|A-\lambda I|=0$.

Here $(A-\lambda I)$ is called the characteristic matrix of 'A' and $|A-\lambda I|=0$ is called the characteristic eq of A. There will be a polynominal eq in 'λ' of degree 'n' ie, $|A-\lambda I|=0$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & a_{3n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & & a_{nn}-\lambda \end{vmatrix}=0$$

Solving this eq, we get roots $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ of the characteristic equation. The roots of C.E. of 'A' are called characteristic roots (or) the eigen values (or) latent values (or) proper values. Corresponding to each one of these 'n' eigen values, we can find characteristic vector X.

Consider homogeneous system.

$$[A-\lambda_i I]X=0 \quad \text{for } i=1,2,3,\ldots n.$$

The non-zero solution X_i of this system is the eigen vector of 'A' corresponding to eigen value λ_i.

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Note:- An eigen value of a square matrix 'A' can be zero but a zero vector cannot be an eigen vector of 'A'.

* 2006,08

1. Determine the characteristic roots and the corresponding characteristic vector of matrix $A=\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic equation of 'A' is $|A-\lambda I|=0$

$$|A-\lambda I|=0 \Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

(or $\lambda^3 - \lambda^2(\text{Trace}) + \lambda(\text{Sum of minors of diagonals}) - |A|$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)[(7-\lambda)(3-\lambda)-16]+6[-6(3-\lambda)+8]+2[24-2(7-\lambda)]=0$$

$$\Rightarrow (8-\lambda)[21-10\lambda+\lambda^2-16]+6[-18+6\lambda+8]+2[24-14+2\lambda]=0$$

$$\Rightarrow (8-\lambda)[\lambda^2-10\lambda+5]+6[6\lambda-10]+2[2\lambda+10]=0$$

$$\Rightarrow 8\lambda^2-80\lambda+40-\lambda^3+10\lambda^2-5\lambda+36\lambda-60+4\lambda+20=0$$

$$\Rightarrow -\lambda^3+18\lambda^2-45\lambda=0$$

$$\lambda^3-18\lambda^2+45\lambda=0$$

$$\lambda(\lambda^2-18\lambda+45)=0$$

$$\lambda=0, \quad \lambda^2-18\lambda+45=0$$

$$\lambda^2-15\lambda-3\lambda+45=0$$

$$\lambda(\lambda-15)-3(\lambda-15)=0$$

$$(\lambda-3)(\lambda-15)=0$$

$$\lambda=3,15$$

Sum of E.V = Trace

Product of E.V = determine |A|

$\therefore \lambda=0,3,15$ are eigen values

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The eigen vector x of A corresponding to eigen value λ is given by $[A-\lambda I]X=0$.

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

The characteristic roots are $0, 3, 15$

Case(i):- If $\lambda=0$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$8x_1-6x_2+2x_3=0 \rightarrow (1)$

$-6x_1+7x_2-4x_3=0 \rightarrow (2)$

$2x_1-4x_2+3x_3=0 \rightarrow (3)$

from (1) & (2) b c a b

$$\begin{matrix} -6 & 2 & 8 & -6 \\ 7 & -4 & -6 & 7 \end{matrix}$$

$$\frac{x_1}{24-14}=\frac{x_2}{-12+32}=\frac{x_3}{56-36}=k_1$$

$$\frac{x_1}{10}=\frac{x_2}{20}=\frac{x_3}{20}=k_1 \Rightarrow \frac{x_1}{1}=\frac{x_2}{2}=\frac{x_3}{2}=k_1$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}k_1$$

Case(ii):- If $\lambda=3$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 5x_1-6x_2+2x_3=0$

$-6x_1+4x_2-4x_3=0, \; 2x_1-4x_2=0$

$$\begin{matrix} -6 & 2 & 5 & -6 \\ 4 & -4 & -6 & 4 \end{matrix}$$

$$\frac{x_1}{24-8}=\frac{x_2}{-12+20}=\frac{x_3}{20-36}=k_2$$

$$\frac{x_1}{16}=\frac{x_2}{8}=\frac{x_3}{-16}=k_2$$

$$\frac{x_1}{2}=\frac{x_2}{1}=\frac{x_3}{-2}=k_2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}k_2$$

Case(iii):- If $\lambda=15$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x_1-6x_2+2x_3=0$

$-6x_1-8x_2-4x_3=0$

$2x_1-4x_2-12x_3=0$

$$\begin{matrix} -6 & 2 & -7 & -6 \\ -4 & -12 & 2 & -4 \end{matrix}$$

$$\frac{x_1}{72+8}=\frac{x_2}{4-84}=\frac{x_3}{28+12}=k_3$$

$$\frac{x_1}{80}=\frac{x_2}{-80}=\frac{x_3}{40}=k_3$$

$$\frac{x_1}{2}=\frac{x_2}{-2}=\frac{x_3}{1}=k_3$$

$$X=k_3\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore$ The eigen vectors are

$$X_1=\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix},\; X_2=\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix},\; X_3=\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

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2013 2) find Eigen values and eigen vectors of matrix. (41)

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$14 + (8) + 14$ $\quad 3(14)+1(-2)+1(-4)$

$(8) \quad 42-2-4$

$42-6$

36

Sol:- The characteristic eq of A is $|A-\lambda I|=0$

$$A = \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$8-44+72-36$

$80-80$

$$(3-\lambda)\left[(5-\lambda)(3-\lambda)-1\right]+1\left[-(3-\lambda)+1\right]+1\left[1-(5-\lambda)\right]=0$$

$1-11$

$$(3-\lambda)\left[15-8\lambda+\lambda^2-1\right]+1\left[-3+\lambda+1\right]+1\left[1-5+\lambda\right]=0$$

$$(3-\lambda)(14-8\lambda+\lambda^2)+1[\lambda-2]+1[\lambda-4]=0$$

$$42-24\lambda+3\lambda^2-14\lambda+8\lambda^2-\lambda^3+\lambda-2+\lambda-4=0$$

$$-\lambda^3+11\lambda^2-36\lambda+36=0$$

$$\lambda^3-11\lambda^2+36\lambda-36=0$$

$$\begin{array}{r|rrrr} \lambda=2 & 1 & -11 & 36 & -36 \\ & 0 & 2 & -18 & 36 \\ \hline & 1 & -9 & 18 & 0 \end{array}$$

$x^2-9x+18=0$

$|A| \Rightarrow 36 = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$

$$\therefore \lambda(\lambda^2-11\lambda+36)-36=0$$

$\frac{+9 \pm \sqrt{81-4(18)}}{2}$

$\lambda_1 = 6, 3, 2.$ $\quad \lambda^3-\lambda^2(\text{Trace})+\lambda(\text{sum of minors of diagonal})-|A|=0$

$\frac{+9 \pm \sqrt{81-72}}{2}$

$\frac{+9 \pm \sqrt{9}}{2}$

Case (i):- $\lambda=2$ $\quad \lambda^3-\lambda^2(11)+36-36=0$

$\frac{+9 \pm 3}{2} =$

$\frac{+9+3}{2} = \frac{6}{2}$ 3

$\frac{+9-3}{2} = \frac{12}{2}$ 6

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 - x_2 + x_3 = 0$

$-x_1 + 3x_2 - x_3 = 0$

$x_1 - x_2 + x_3 = 0$

$$\begin{array}{cccc} -1 & 1 & 1 & -1 \\ 3 & -1 & -1 & 3 \end{array} \quad \frac{x_1}{1-3} = \frac{x_2}{-1+1} = \frac{x_3}{3-1} = k_1$$

$$\frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2} = k_1$$

$$X = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

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Case(ii) $\lambda = 3$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_2 + x_3 = 0$

$-x_1 + 2x_2 - x_3 = 0$

$x_1 - x_2 = 0$

$$\begin{matrix} -1 & 1 & 0 & -1 \\ 2 & -1 & -1 & 2 \end{matrix}$$

$$\frac{x_1}{1-2} = \frac{x_2}{-1-0} = \frac{x_3}{0-1} = k_2$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1} = k_2$$

$$X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case(iii):- $\lambda = 6$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x_1 - x_2 + x_3 = 0$

$-x_1 - x_2 - x_3 = 0$

$x_1 - x_2 - 3x_3 = 0$

$$\begin{matrix} -1 & 1 & -3 & -1 \\ 1 & 1 & 1 & 1 \end{matrix}$$

$$\frac{x_1}{-1-1} = \frac{x_2}{1+3} = \frac{x_3}{-3+1} = k_3$$

$$X = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

3. $$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

37

$-1+12-36-32$

$\frac{1}{8} - 6 + 18 - 32$

$1 - 48 + 144 - 32$

$1 - 12 + 36 - 32$

$$\begin{matrix} 1 & -12 & 36 & -32 \end{matrix}$$

Sol:- $$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$8 - 24 + 72 - 32$

80 56

$\Rightarrow (6-\lambda)\left[(3-\lambda)^2 - 1\right] + 2\left[-2(3-\lambda)+2\right] + 2\left[2 - 2(3-\lambda)\right] = 0$

$\Rightarrow (6-\lambda)\left[9 + \lambda^2 - 6\lambda - 1\right] + 2\left[-6 + 2\lambda + 2\right] + 2\left[2 - 6 + 2\lambda\right] = 0$

$\Rightarrow (6-\lambda)\left(8 + \lambda^2 - 6\lambda\right] + 2\left[-4 + 2\lambda\right] + 2\left[2\lambda - 4\right] = 0$

$\Rightarrow 48 + 6\lambda^2 - 36\lambda - \cancel{8\lambda} - \lambda^3 + 6\lambda^2 - 8 + 4\lambda + 4\lambda - 8 = 0$

$\Rightarrow -\lambda^3 + 12\lambda - 36\lambda + 32 = 0$

$\Rightarrow \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$\lambda = 2, 2, 8.$

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The eigen vector X of A corresponding to eigen value 'λ' is given by (22)

$$[A-\lambda I][X]=0$$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case(i) when $\lambda=2$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$4x_1-2x_2+2x_3=0$, $\quad -2x_1+x_2-x_3=0$, $\quad 2x_1-x_2+x_3=0$

$2x_1-x_2+x_3=0 \qquad 2x_1-x_2+x_3=0 \qquad 2x_1-x_2+x_3=0$

Let $x_1=k_1$, $\qquad x_3=x_2-2x_1$

$x_2=k_2 \qquad = k_2-2k_1$

$$X=\begin{bmatrix} k_1 \\ k_2 \\ k_2-2k_1 \end{bmatrix}=k\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}+k_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii) :- when $\lambda=8$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_1-2x_2+2x_3=0$, $\quad -2x_1-5x_2-x_3=0$, $\quad 2x_1-x_2-5x_3=0$

$+x_1+x_2-x_3=0 \qquad 2x_1+5x_2+x_3=0, \qquad 2x_1-x_2-5x_3=0$

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$$\begin{array}{cccccc} x_1 & & x_2 & & x_3 & \\ 1 & -1 & & 1 & & 1 \\ 5 & & 1 & & 2 & 5 \end{array}$$

$$\frac{x_1}{1+5} = \frac{x_2}{-2-1} = \frac{x_3}{5-2}$$

$$\Rightarrow \frac{x_1}{6} = \frac{x_2}{-3} = \frac{x_3}{3} = k_4$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1} = k_4$$

$\therefore x_1 = 2k_4, \; x_2 = -k_4, \; x_3 = k_4$

$$\therefore X = \begin{bmatrix} 2k_4 \\ -k_4 \\ k_4 \end{bmatrix} = k_4 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

2008

4) Find the eigen values and eigen vectors of matrix

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Sol:- The C.E. of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda = 5, -3, -3$ are the eigen value

Case (i) when $\lambda = 5$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x_1 + 2x_2 - 3x_3 = 0, \quad 2x_1 - 4x_2 - 6x_3 = 0$

$-x_1 - 2x_2 - 5x_3 = 0$

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$2 \quad -3 \quad -7 \quad 2$

$-2 \quad -5 \quad -1 \quad -2$

$$\frac{x_1}{-10-6} = \frac{x_2}{3-35} = \frac{x_3}{14+2}$$

$$\frac{x_1}{-16} = \frac{x_2}{-32} = \frac{x_3}{16}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1} = k_1$$

$$\therefore X = \begin{bmatrix} k_1 \\ 2k_2 \\ -k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Case ii) $\lambda = -3$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & +3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 - 3x_3 = 0, \quad 2x_1 + 4x_2 - 6x_3 = 0.$

$-x_1 - 2x_2 + 3x_3 = 0 \quad x_1 + 2x_2 - 3x_3 = 0.$

$x_1 + 2x_2 + 3x_3 = 0$

All three eq's are same. So the system has only on independent eq.

$x_1 + 2x_2 - 3x_3 = 0$

Let $x_3 = k_2, \; x_2 = k_3, \; x_1 = 3k_2 - 2k_3$

$$x = \begin{bmatrix} 3k_2 - 2k_3 \\ k_3 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + k_3 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

where k_2, k_3 are non-zero scalars

$\therefore$ Eigen vectors are $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

2006

4) Find the eigen value & eigen vectors of $\begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$

eigen value $\lambda = 3, 9, 6$

eigen vector $\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$

5) $\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$

$\lambda^3 - 8\lambda^2 + 17\lambda - 10 = 0$

$\lambda = 1, 2, 5$

$\lambda = 5 \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\lambda = 2 \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$\lambda = 1 \Rightarrow \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$

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Diagonalization of a matrix:-

[A matrix 'A' is diagonalizable if there exists an invertible matrix 'P' such that $P^{-1}AP = D$ where 'D' is a diagonal matrix. Also the matrix 'P' is then said to diagonalize A or transform A to diagonal form.]

Let 'A' be a square matrix if there exists a non singular matrix 'P' and a diagonal matrix D such that $P^{-1}AP = D$, then 'A' is said to be diagonalisable and 'D' is said to be a diagonal form (or) Canonical form of matrix 'A' whose diagonal elements are the eigen values of 'A'.

First alternat [illegible]

Similar matrix:—

Two matrix A and B are of same order, then 'A' is said to be similar to B, if there exists a non singular matrix 'P' such that $B = P^{-1}AP$. This transformation of A to B is known as similarity transformation.

Spectrum:—

The set of eigen values of matrix 'A' is known as Spectrum.

Modal matrix:—

The matrix 'P' which diagonalises 'A' is called the modal matrix of A & is obtained by grouping the vectors of 'A' into square matrix.

Spectral matrix:-

The diagonal matrix 'D' is known as spectral matrix.

Note:-

If $x_1, x_2, x_3, \ldots, x_n$ are not linearly independent then the given 'A' is not diagonalisable.

Working Rule:-

1. Let 'A' be the square matrix which is to be diagonalised.
2. Find the eigen values of matrix.
3. Find the eigen ~~val~~ vector of matrix.
4. Check whether the eigen vectors are linearly independent (or) not. If the eigen vectors are linearly independent; $|P| \neq 0$ then the matrix is diagonalisable, otherwise not.
5. Form modal matrix $P = [X_1 \; X_2 \; X_3]$, where X_1, X_2, X_3 are eigen vectors of A.
6. Find the inverse of P.
7. Find diagonal matrix $D = P^{-1}AP$, whose diagonal elements are the eigen values of A.

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Calculation of Powers of a matrix:—

we can obtain the powers of a matrix by using diagonalisation. Let 'A' be the Square matrix, then non singular matrix 'P' can be found such that

$$D = P^{-1}AP$$

$$D^2 = (P^{-1}AP)(P^{-1}AP)$$

$$= P^{-1}APP^{-1}AP$$

$$= P^{-1}AIAP$$

$$= P^{-1}A^2P$$

Similarly $D^3 = P^{-1}A^3P \ldots D^n = P^{-1}A^nP \rightarrow (1)$

To obtain A^n, premultiply (1) with 'P' and post multiply with P^{-1} then

$$PD^nP^{-1} = PP^{-1}A^nPP^{-1}$$

$$= IA^nI$$

$$\boxed{PD^nP^{-1} = A^n}$$

$$\therefore \quad A^n = PD^nP^{-1}$$

$$= P\begin{bmatrix} \lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^n \end{bmatrix}P^{-1}$$

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Problems :-

1) Diagonalize the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ and also find A^4.

Sol :- $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0$$
$\Rightarrow \lambda = -2, 3, 6$ are the eigen values.

Case(i) :- when $\lambda = -2$

$$(A + 2I)\, X = 0$$

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3x_1 + x_2 + 3x_3 = 0 \; -(1)$, $\quad x_1 + 7x_2 + x_3 = 0 \; -(2)$, $\quad 3x_1 + x_2 + 3x_3 = 0$

From (1) & (2)

x_1		x_2	x_3	
1	3	3	1	
7	1	1	7	

$$\frac{x_1}{1-21} = \frac{x_2}{3-3} = \frac{x_3}{21-1} = k_1$$

$$\frac{x_1}{-\cancel{20}\;1} = \frac{x_2}{0} = \frac{x_3}{\cancel{20}\;1} = k_1$$

$\therefore X = k_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ where k_1 is non-zero scalar.

$\therefore X = k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

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case(ii): — when $\lambda = 3$

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X = k_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

case(iii) :- when $\lambda = 6$ $\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix}$

$$x_3 = k_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

at $x_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$\therefore$ modal matrix $P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ & $|P| = 6 \neq 0$

$\therefore$ x_1, x_2, x_3 are linearly independent.

$\therefore$ A is diagonalizable.

To prove $P^{-1}AP = D$

Since 'P' is non-singular $\Rightarrow$ P^{-1} exists.

$$\therefore P^{-1} = \frac{1}{|P|} \operatorname{adj} P$$

$$\operatorname{adj} P = \begin{pmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{pmatrix} \quad \therefore P^{-1} = \frac{1}{6}\begin{pmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

Now, $P^{-1}AP = \frac{1}{6}\begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

and $A^4 = PD^4P^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2^4 & 0 & 0 \\ 0 & 3^4 & 0 \\ 0 & 0 & 6^4 \end{bmatrix}\begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

$$\therefore A^4 = \begin{pmatrix} 251 & 405 & 235 \\ 405 & 891 & 405 \\ 235 & 405 & 251 \end{pmatrix}$$

2) Diagonalise the matrix $\begin{bmatrix} -1 & 2 & 2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

Sol:-

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 2 & 2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 1, \quad \lambda = \pm\sqrt{5} \pm 1$$

Case (i) when $\lambda = 1$

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Case (ii) :- when $\lambda = \sqrt{5}$.

$$\begin{bmatrix} -1-\sqrt{5} & 2 & -2 \\ 1 & 2-\sqrt{5} & 1 \\ -1 & -1 & -\sqrt{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-(1+\sqrt{5})x_1 + 2x_2 - 2x_3 = 0$ &

$x_1 + x_2 + \sqrt{5}x_3 = 0$

$\Rightarrow x_1 + (2-\sqrt{5})x_2 + x_3 = 0,$

$2-\sqrt{5} \quad 1 \quad 1 \quad 2-\sqrt{5}$

$1 \quad \sqrt{5} \quad 1 \quad 1$

$2 \quad -2 \quad -(1+\sqrt{5}) \quad 2$

$2-\sqrt{5} \quad 1 \quad 1 \quad 2-\sqrt{5}$

$$\frac{x_1}{2+2(2-\sqrt{5})} = \frac{x_2}{-2+1+\sqrt{5}} = \frac{x_3}{-(1+\sqrt{5})(2-\sqrt{5})-2} = k_1$$

$$\frac{x_1}{4-2\sqrt{5}} = \frac{x_2}{-1+\sqrt{5}} = \frac{x_3}{-2+\sqrt{5}-2\sqrt{5}+5-2} = k_1$$

$$\frac{x_1}{2(2-\sqrt{5})} = \frac{x_1}{\sqrt{5}-1} = \frac{x_3}{1-\sqrt{5}} = k_1$$

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$$\frac{x_1}{2(2-\sqrt{5})} = \frac{x_2}{-(1-\sqrt{5})} = \frac{x_3}{(1-\sqrt{5})} = k_1$$

$$X_1 = k_1 \begin{bmatrix} 2(2-\sqrt{5}) \\ -1 \\ 1 \end{bmatrix}$$

<u>Case (iii)</u> <u>$\lambda = -\sqrt{5}$</u>

$$X_3 = \begin{bmatrix} -(\sqrt{5}+1) \\ 1 \\ -1 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} -1 & 2(2-\sqrt{5}) & -(\sqrt{5}+1) \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \quad \& \quad |P| = 2\sqrt{5} \neq 0$$

$\therefore X_1, X_2, X_3$ are linearly independent.

$\therefore$ A is diagonalisable.

$$\text{adj } P = \begin{bmatrix} 0 & 1 & -1 \\ 2\sqrt{5} & 2+\sqrt{5} & \sqrt{5}-2 \\ 2\sqrt{5} & 1 & -1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{2\sqrt{5}} \begin{bmatrix} 0 & 1 & -1 \\ 2\sqrt{5} & 2+\sqrt{5} & \sqrt{5}-2 \\ 2\sqrt{5} & 1 & -1 \end{bmatrix}$$

$$\therefore P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$$

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3) $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -2 & -1 \end{bmatrix}$ & A^4 & A^8 (1, 2, 1)

4) $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ (1, 2, 3) $\underline{\lambda=3}$ $X_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\underline{\lambda=1}$ $X_2 = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$ $\underline{\lambda=2}$ $X_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

5) Determine the eigenvalue ~~Evalues~~ of A^{-1}, where $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

$\lambda = 1, 2, 3$ $P = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$ $A^4 = \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$

6) Diagonalize the matrix $A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$ Hence find A^3

6 Cayley - Hamilton Theorem:-

St:- "Every square matrix satisfies its own C.E. (characteristic equation)

Proof:- Let 'A' be a square matrix of order 'n', then the characteristic equation of 'A' is $|A-\lambda I| = 0$

$= (-1)^n (\lambda-\lambda_1)(\lambda-\lambda_2) \cdots (\lambda-\lambda_n)$

Let $|A-\lambda I| = (-1)^n \left[\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0\right]$

where $a_0, a_1, a_2, \ldots a_{n-1}$ are constants. Since all the elements of $(A-\lambda I)$ are atmost of first degree in λ all the elements of $\operatorname{adj}(A-\lambda I)$ are polynomials in λ of degree $(n-1)$ (or) less and hence $\operatorname{adj}(A-\lambda I)$ can be written as a matrix polynomial in λ.

Let $\operatorname{adj}(A-\lambda I) = B_{n-1}\lambda^{n-1} + B_{n-2}\lambda^{n-2} + \cdots + B_1\lambda + B_0$, where $B_0, B_1, \ldots, B_{n-1}$ are matrices of order $n \times n$.

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we know that $|A|I = A(\text{adj}A)$ $\rightarrow A^{-1} = \frac{\text{adj}A}{|A|} \Rightarrow A^{-1}|A| = \text{adj}A$

$A A^{-1}|A| = A\,\text{adj}A$

$I|A| = A \cdot \text{adj}A$

(or)

replace A by $(A - \lambda I)$ we get

$$|A - \lambda I| I = (A - \lambda I)\, \text{adj}(A - \lambda I)$$

$$\Rightarrow (-1)^n \left[\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0\right] I$$

$$= (A - \lambda I)\left[B_{n-1}\lambda^{n-1} + B_{n-2}\lambda^{n-2} + \cdots + B_1\lambda + B_0\right]$$

Equating the co-eff of same ~~pollers~~ powers of 'λ' o.b.s. we get

$$(-1)^n I = -B_{n-1}$$

$$(-1)^n a_{n-1} I = AB_{n-1} - B_{n-2}$$

$$\cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots$$

$$(-1)^n a_1 I = AB_1 - B_0$$

$$(-1)^n a_0 I = AB_0$$

pre-multiply the above eq successively by $A^n, A^{n-1} \ldots A, I$ and adding, we get

$$(-1)^n \left[A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0\right] I = 0$$

$-\cancel{A^n B_{n-1}} + \cancel{A^n B_{n-1}}$
$\cancel{A^{n-1} B_{n-2}} + \cdots$
$+ \cancel{A^2 B_1} - \cancel{AB_0} +$
$\cancel{AB_0}$

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0 \rightarrow ①$$

Hence A satisfies its C.E.

To find A^{-1} :- If A satisfies its C.E then eq ① exists. If 'A' is non singular then A^{-1} exists.

$$A^{-1}\left[A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1 A + a_0\right] = 0 \Rightarrow A^{-1} = \frac{-1}{a_0}\left[A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I\right]$$

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7 Problems:-

1) verify C.H.T. for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

hence find A^{-1}

Sol:- The C.E of A is $\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 5 \\ 3 & 5 & 6-\lambda \end{vmatrix} = 0$

$$\Rightarrow \lambda^3 - 11\lambda^2 - 4\lambda + 1 = 0.$$

C.H.T. states that every square matrix satisfies its own C.E.. To verify C.H.T. we have to show that.

$$A^3 - 11A^2 - 4A + 1 = 0 \rightarrow (1)$$

Now $A^2 = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$, $A^3 = \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix}$

$\therefore A^3 - 11A^2 - 4A + I =$

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2) If $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ verify C.H.T. & find A^4 & A^{-1} using C.H.T.

$\begin{bmatrix} 1-\lambda & 2 & -1 \\ 2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{bmatrix} = 0$

Sol:- $|A-\lambda I| = 0 \Rightarrow \lambda^3 - 3\lambda^2 - 3\lambda + 9 = 0$

To verify C.H.T. we have to P.T. $A^3 - 3A^2 - 3A + 9I = 0$ → ①

$A^2 = \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix}$, $A^3 = \begin{bmatrix} 3 & 24 & -21 \\ 6 & 21 & -24 \\ 6 & -6 & 3 \end{bmatrix}$

To find A^4 multiply ① with A, we get

$A^4 - 3A^3 + 3A^2 - 9A = \begin{bmatrix} 9 & 72 & -72 \\ 0 & 81 & -72 \\ 0 & 0 & 9 \end{bmatrix}$

$A^{-1}(A^3 - 3A^2 - 3A + 9I) = 0$

$A^2 - 3A - 3 + 9A^{-1} = 0$

$9A^{-1} = -A^2 + 3A + 3I$

$A^{-1} = \frac{1}{9}[-A^2 + 3A + 3I]$

To find A^{-1} ① $\times A^{-1}$, we get

$A^{-1} = \frac{1}{9}\left[-A^2 + 3A + 3I\right] = \frac{1}{9}\begin{bmatrix} 3 & 0 & 3 \\ 6 & -3 & 0 \\ 6 & -6 & 3 \end{bmatrix}$

$\frac{1}{9}\left(\begin{bmatrix} -3 & -6 & +6 \\ 0 & -9 & 6 \\ 0 & 0 & -3 \end{bmatrix} + \begin{bmatrix} 3 & 6 & -3 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right)$

$= \frac{1}{9}\begin{bmatrix} 3 & 0 & 3 \\ 6 & -3 & 0 \\ 6 & -6 & 3 \end{bmatrix}$

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29A

$$\therefore A^3 - 11A^2 - 4A + A = \begin{bmatrix} 157 & 283 & 352 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix} - 11\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$$

$$-4\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore$ C.H.T is verified.

multiplying eq ① with A^{-1} on both sides, we get

$$A^{-1}(A^3 - 11A^2 - 4A + I) = 0$$

$$A^2 - 11A - 4I + A^{-1} = 0$$

$$A^{-1} = -A^2 + 11A + 4I$$

$$= -\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} + 11\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

2) verify caley - Hamilton Theorem for the matrix

$$A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

ie., $A^3 - 6A^2 - A + 22 = 0$.

Sol: $C.E = |A - \lambda I| = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 - \lambda + 22 = 0$

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3) verify C.H.T. for the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

find A^4 & A^{-1} using C.H.T.

Sol:— C.E. is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & -1 \\ 2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 3\lambda^2 - 3\lambda + 9 = 0$$

To verify C.H.T, we have to prove $A^3 - 3A^2 - 3A + 9 = 0$ → ①

$$A^2 = \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 3 & 24 & -21 \\ 6 & 21 & -24 \\ 6 & -6 & 3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 3 & 24 & -21 \\ 6 & 21 & -24 \\ 6 & -6 & 3 \end{bmatrix} - 3\begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

To find A^4 multiply eq ①, with A.

$$A\left[A^3 - 3A^2 - 3A + 9I\right] = 0$$

$$A^4 - 3A^3 - 3A^2 + 9A = 0$$

$$A^4 = 3A^3 + 3A^2 - 9A$$

$$= 3\begin{bmatrix} 3 & 24 & -21 \\ 6 & 21 & -24 \\ 6 & -6 & 3 \end{bmatrix} + 3\begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix} - 9\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 72 & -72 \\ 0 & 81 & -72 \\ 0 & 0 & 9 \end{bmatrix}$$

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Complex matrix :-

Real matrix :- A matrix $A = a_{ij}$ is said to be a real matrix if every element a_{ij} of 'A' is real.

Complex matrix :- If the elements of a matrix are complex, then the matrix is called Complex matrix.

Symmetric matrix :- A real square matrix $A = [a_{ij}]$ is said to be Symmetric if $A = A^T$ i.e., $a_{ij} = a_{ji}$ $\forall$ i & j

Skew Symmetric matrix :- A real square matrix $A = [a_{ij}]$ is said to be Skew symmetric if $A = -A^T$ i.e., $a_{ij} = -a_{ji}$ $\forall$ i & j.

Orthogonal matrix :- A real square matrix $A = a_{ij}$ is said to be orthogonal if $AA^T = A^TA = I$ (or) $A^{-1} = A^T$.

Note :- The determinant of an orthogonal matrix is only one.

Conjugate of matrix :-

The matrix obtained by replacing each element of a matrix by its Complex conjugate is called the conjugate of a matrix. The Conjugate of matrix is denoted by $\bar{A}$.

Transposed Conjugate of a matrix / Conjugate Transpose of a matrix :-

The transpose of the conjugate of a square matrix is called the transposed conjugate of a matrix

ie., If 'A' is a square matrix and its conjugate is $\bar{A}$ then the transpose of $\bar{A}$ is $(\bar{A})^T$ we observe that $(\bar{A})^T = \overline{(A^T)}$. The transpose conjugate of 'A' is denoted by A^* (or) A^θ (or) A^{H}.

$$\therefore A^* = (\bar{A})^T = \overline{(A^T)}$$

Hermitian matrix :-

A square matrix $A = [a_{ij}]$ is said to be Hermitian if $A = A^*$ ie., $a_{ij} = \bar{a}_{ij}$ $\forall$ i & j.

Eg:- If $A = \begin{bmatrix} 1 & 2+i & 3-2i \\ 2-i & 0 & 2i \\ 3+2i & -2i & -4 \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 1 & 2-i & 3+2i \\ 2+i & 0 & -2i \\ 3-2i & 2i & -4 \end{bmatrix}$$

$$A^* = (\bar{A})^T = \begin{bmatrix} 1 & 2+i & 3-2i \\ 2-i & 0 & 2i \\ 3+2i & -2i & -4 \end{bmatrix} = A$$

$\therefore$ 'A' is Hermitian.

Note :-

1. The elements of the principal diagonal of a Hermitian matrix must be real.
2. If A^* and B^* be the transpose conjugate of A and B respectively then

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$(A^*)^* = A$

$(A \pm B)^* = A^* \pm B^*$

$(KA)^* = \bar{K} A^*$, where 'K' is a complex

$(AB)^* = B^* A^*$

Skew Hermitian matrix:-

A square matrix $A = a_{ij}$ is said to be skew-Hermitian if $A = -A^*$ ie., $a_{ij} = -a_{ji}$ $\forall$ i & j.

Eg:- $A = \begin{bmatrix} -3i & 2+i \\ -2+i & -i \end{bmatrix}$, $\bar{A} = \begin{bmatrix} 3i & 2-i \\ -2-i & i \end{bmatrix}$

$A^* = (\bar{A})^T = \begin{bmatrix} 3i & -2-i \\ 2-i & i \end{bmatrix} = -A.$

Note :-

* The principal diagonal elements of a skew hermitian matrix are purely imaginary ~~&~~ or zero.

Unitary matrix:-

A square matrix 'A' is said to be unitary if $AA^* = A^*A = I$. (or) $A^{-1} = A^*$

Theorem:- Every real square matrix can be expressed as unique as a sum of symmetric and skew symmetric matrix.

Proof:- If 'A' is real square matrix, Then 'A' can be written as.

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$= \frac{1}{2}[R+S+R^T+S^T]$ $\quad Q = S.$

$= \frac{1}{2}(R+S+R-S]$

$= \underline{R}.$

Hence every real matrix can be expressed as unique as sum of symmetric & skew symmetric matrix.

Note:-

1. If 'A' is any real square matrix, then $A+A^T$ is symmetric & $A-A^T$ is skew symmetric.

2. The inverse of a non singular symmetric matrix is symmetric.

Let 'A' be a non-singular symmetric matrix then $A^T = A$ and A^{-1} exists.

Now to prove A^{-1} is symmetric ie, $(A^{-1})^T = A^{-1}$.

Consider $(A^{-1})^T = (A^T)^{-1}$

$= A^{-1}$

$\Rightarrow A^{-1}$ is symmetric.

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Problems

Express the matrix $A = \begin{bmatrix} 3 & -2 & -6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$ as a sum of symmetric and skew-symmetric matrix.

Sol:- Given $A = \begin{bmatrix} 3 & -2 & -6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$

$$A^T = \begin{bmatrix} 3 & 2 & 5 \\ -2 & 7 & 4 \\ -6 & -1 & 0 \end{bmatrix}$$

sum of symmetric and skew symmetric = P+Q

where $P = \frac{1}{2}(A+A^T)$ and $Q = \frac{1}{2}(A-A^T)$

Consider $A+A^T = \begin{bmatrix} 3 & -2 & -6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 5 \\ -2 & 7 & 4 \\ -6 & -1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 6 & 0 & -1 \\ 0 & 14 & 3 \\ -1 & 3 & 0 \end{bmatrix}$$

let $P = \frac{1}{2}(A+A^T) = \frac{1}{2}\begin{bmatrix} 6 & 0 & -1 \\ 0 & 14 & 3 \\ -1 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -1/2 \\ 0 & 7 & 3/2 \\ -1/2 & 3/2 & 0 \end{bmatrix}$

Consider $A - A^T = \begin{bmatrix} 3 & -2 & -6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 5 \\ -2 & 7 & 4 \\ -6 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 & -11 \\ 4 & 0 & -5 \\ 11 & 5 & 0 \end{bmatrix}$

let $Q = \frac{1}{2}(A - A^T) = \frac{1}{2}\begin{bmatrix} 0 & -4 & -11 \\ 4 & 0 & -5 \\ 11 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -11/2 \\ 2 & 0 & -5/2 \\ 11/2 & 5/2 & 0 \end{bmatrix}$

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$$\text{NOW } P+Q = \begin{bmatrix} 3 & 0 & -1/2 \\ 0 & 7 & 3/2 \\ -1/2 & 3/2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -2 & -11/2 \\ 2 & 0 & -5/2 \\ 11/2 & 5/2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 & -6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix} = A$$

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Note:-

1) The Symmetric matrix 'A' is called matrix of Q.F.
2) The determinant of a matrix 'A' ie., $|A|$ is called discriminant of Q.F.
3) If $|A|=0$, the Q.F is singular
4) If $|A|\neq 0$, The Q.F is non-singular
5) Given a symmetric matrix, a Q.F. can be written and given Q.F. a symmetric matrix can be written, ie.,

Symmetric matrix $\Leftrightarrow$ Q.F.

6) To write the symmetric matrix of Q.F. the coefficient of x_i^2 is placed in a_{ij} position & $\frac{1}{2}$(coeff of $x_i x_j$) is placed in each of the position a_{ij} & a_{ji}.

Transformation of a Q.F. to Canonical form (C.F):-

Let X^TAX be Q.F. in 'n' variables. Let $X=\hat{P}Y$ be the linear transformation which transforms $x_1, x_2, \dots x_n$ to $y_1, y_2, \dots y_n$.

Apply transformation $X=\hat{P}Y$ to the Q.F. X^TAX then

$$X^TAX = (\hat{P}Y)^T A(\hat{P}Y)$$
$$= Y^T(\hat{P})^T A(\hat{P})(Y)$$
$$= Y^T(\hat{P}^T A\hat{P})Y \qquad \because D=(\hat{P})^T A\hat{P}$$
$$= Y^T DY$$

Here $\hat{P}$ is the normalised modal matrix of A, then $\hat{P}$ is an orthogonal matrix $[(\hat{P})^T=(\hat{P})^{-1}]$

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$X^TAX = Y^TDY$

$\therefore Y^TDY$, is canonical form in variables $y_1, y_2, \dots, y_n$.

Canonical form (or) normal form (or) sum of the square (or) principal axis form

The process of transforming Q.F. from one variable to another variable which is a C.F. without cross product terms.

C.F. of real Q.F., $Q = X^TAX$ is Y^TDY

$$\therefore Y^TDY = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}_{1\times n} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \lambda_3 \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots & \lambda_n \end{bmatrix}_{n\times n} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

$$= \sum_{i=1}^{n} \lambda_i y_i^2$$ where $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigen values of A.

Rank:- The no. of non zeros terms in canonical form of Q.F is called the rank of Q.F. It is denoted by r.

Index:- The no. of positive terms in C.F. is ... of Q.F. it is denoted

Signature:- The difference b/w the no. of positive and negative terms in C.F. is signature of Q.F

ie., $S-(r-S) = 2S-r$

Nature of Q.F :-

The Q.F X^TAX in 'n' variables is said to be

(i) **positive definite:-**

If all eigen values of 'A' are positive ie., $\lambda_i > 0$

(or) rank and index are equal ie $r=n$, $S=n$.

ii) **Negative definite:-**

If all eigen value of 'A' are negative ie $\lambda_i < 0$

If index $S=0$. ie $r=n$, $S=0$ (zero)

iii) **positive semi definite:-**

If atleast one eigen value of 'A' is zero & all other positive.

iv) **Negative semi definite:-**

If atleast one eigen value of 'A' is zero & all others negative ie $\lambda_i \le 0$.

v) **Indefinite:-**

If eigen value of A are both positive & -ve

Eg:- ① write the matrix of the Q.F. $x_1^2 - 2x_1x_2 + 2x_2^2$

Let 'A' be the matrix of Q.F $A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

② $x_1^2 + 6x_1x_2 + 5x_2^2$ $\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$

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3) $x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3 + 5x_2x_3$

$$\begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & 5/2 \\ 4 & 5/2 & -7 \end{bmatrix} \text{(or)} \quad \begin{array}{c|ccc} & x & y & z \\ \hline x & x^2 & xy/2 & xz/2 \\ y & \frac{xy}{2} & y^2 & yz/2 \\ z & \frac{zx}{2} & \frac{zy}{2} & z^2 \end{array} \quad \begin{array}{c|ccc} & x & y & z \\ \hline x & 1 & -4/2 & 8/2 \\ y & -4/2 & 2 & 5/2 \\ z & 8/2 & 5/2 & -7 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & 5/2 \\ 4 & 5/2 & -7 \end{bmatrix}$$

4) $x^2 + 2y^2 + 3z^2 + 4xy + 5yz + 6zx$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix}$$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix}$

5) To verify $x^T A x = x^2 + 2y^2 + 3z^2 + 4xy + 5yz + 6zx$

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad x^T = [x \quad y \quad z] \qquad [x\,y\,z]\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\cancel{x+2y+3z} \quad \cancel{2x+2y+\frac{5}{2}z}$

6) $x_1^2 + 2x_2^2 + 4x_2x_3 + x_3x_4$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1/2 & 0 \end{bmatrix}$$

7) write the quadratic form of the matrix $\begin{bmatrix} 1 & -9 \\ -9 & 5 \end{bmatrix}$,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & -2 \\ 5 & -2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & -9 \\ -9 & 5 \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then $X^T = [x_1 \quad x_2]$

$Q.F = Q = X^T A X$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & -9 \\ -9 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - 9x_2 & -9x_1 + 5x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= x_1^2 - 9x_1x_2 + x_2(-9x_1 + 5x_2)$$

$$= x_1^2 - 9x_1x_2 - 9x_1x_2 + 5x_2^2$$

$$= x_1^2 - 18x_1x_2 + 5x_2^2.$$

2) Find the Q.F Relating to the symmetric matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$

A) $x_1^2 + y^2 + z^2 + 4xy + 6yz + 6zx$

3) Find Nature, index & signature of Q.F. $2x_1x_2 + 2x_2x_3 + 2x_1x_3$. A) $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

The characteristic Eq of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 3\lambda - 2 = 0$$

$\lambda = 2, -1, -1$ are eigen value.

Nature of Q.F is indefinite

Index of Q.F. is one

Signature $= -1$

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Note:- Norm of a vector 'x' denoted by $\|x\|$ length of vector

If $x = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a vector, then norm $\|x\| = \sqrt{a^2+b^2+c^2}$

8) A set of vector $x_1, x_2 \ldots x_n$ are said to form an orthogonal system if $x_i^T x_j = \delta_{ij} = 0$ for $i \neq j$.

1) Reduce the Q.F. $3x^2 - 2y^2 - z^2 + 12yz + 8zx - 4xy$ to C.F. by orthogonal reduction & state the nature of Q.F., rank, index, signature, Also find the corresponding transformation.

Let 'A' be the matrix of given Q.F. then

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & -2 & 6 \\ 4 & 6 & -1 \end{bmatrix} \qquad \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}$$

The C.E. of 'A' is $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -2 & 4 \\ -2 & -2-\lambda & 6 \\ 4 & 6 & -1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(-2-\lambda)(-1-\lambda) - 36\right] + 2\left[-2(-1-\lambda) - 24\right] + 4\left[-12 - 4(-2-\lambda)\right] = 0$$

$$-\lambda^3 + 63\lambda - 162 = 0$$

$$\lambda^3 - 63\lambda + 162 = 0 \implies \lambda = -9, 6, 3.$$

Case (i) $\lambda = 3$

$$\Rightarrow (A - 3I)(X) = 0$$

$$\begin{bmatrix} 0 & -2 & 4 \\ -2 & -5 & 6 \\ 4 & 6 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$-2y+4z=0$, $-2x-5y+6z=0$, $4x+6y-4z=0$

$y-2z=0 \to ①$ $2x+5y-6z=0 \to ②$, $2x+3y-2z=0 \to ③$

② & ③

5 -6 2 5

3 -2 2 3

$\frac{x}{-10+18} = \frac{y}{-12+4} = \frac{z}{6-10}$

$\frac{x}{8} = \frac{y}{-8} = \frac{z}{-4} \Rightarrow \frac{x}{2} = \frac{y}{-2} = \frac{z}{-1} = k_1$

$\therefore X = k_1 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ where k_1 is non zero scalar.

Case (ii):— when $\lambda = 6$

$$\begin{bmatrix} -3 & -2 & 4 \\ -2 & -8 & 6 \\ 4 & 6 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x-2y+4z=0$, $-2x-8y+6z=0$, $4x+6y-7z=0$

$3x+2y-4z=0 \to ①$ $x+4y-3z=0 \to ②$ $4x+6y-7z=0 \to ③$

① & ②

2 -4 3 2

4 -3 1 4

$\Rightarrow \frac{x}{10} = \frac{y}{5} = \frac{z}{10}$

$\Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{2} = k_2$

$\therefore X = k_2 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

Case (iii):— when $\lambda = 9$

$$\begin{bmatrix} 12 & -2 & 4 \\ -2 & 7 & 6 \\ 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$12x-2y+4z=0$, $-2x+7y+6z=0$, $4x+6y+8z=0$

$6x-y+2z=0 \to ①$ $2x-7y-6z=0 \to ②$ $2x+3y+4z=0 \to ③$

① & ③

-1 2 6 -1

3 4 2 3

$\frac{x}{-10} = \frac{y}{-20} = \frac{z}{20} \Rightarrow \frac{x}{1} = \frac{y}{2} = \frac{z}{-2} = k_3$

$X = k_3 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

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The eigen vectors $\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ are pairwise orthogonal

[Note:- If the eigen values are distinct then the corresponding eigen vectors are pairwise orthogonal]

at $x_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $x_3 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

$\|x_1\| = \sqrt{4+4+1} = \sqrt{9} = 3$, $\|x_2\| = \sqrt{4+1+4} = \sqrt{9} = 3$

$\|x_3\| = \sqrt{1+4+4} = \sqrt{9} = 3.$

$e_1 = \dfrac{x_1}{\|x_1\|} = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$, $e_2 = \dfrac{x_2}{\|x_2\|} = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$, $e_3 = \dfrac{x_3}{\|x_3\|} = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$

Normalized modal matrix $\hat{P} = [e_1 \ e_2 \ e_3]$

$$= \begin{bmatrix} -2/3 & 2/3 & 1/3 \\ 2/3 & 1/3 & 2/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$$

$$\hat{P}^T = \begin{bmatrix} -2/3 & 2/3 & 1/3 \\ 2/3 & 1/3 & 2/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$$

$\hat{P}\hat{P}^T = I$

lly $\hat{P}^T \hat{P} = I$

$\therefore \hat{P}$ is orthogonal matrix, so $\hat{P}^T = \hat{P}^{-1}$

$$\therefore \hat{P}^T A \hat{P} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -9 \end{bmatrix} = D$$

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Let $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ Then $Y^T = [y_1\ y_2\ y_3]$

$\therefore$ C.F. is $Y^T D Y = [y_1\ y_2\ y_3] \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -9 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$= \cancel{3y_1\ 6y_2}\ [3y_1\ \ 6y_2\ \ -9y_3] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$= [3y_1^2 + 6y_2^2 - 9y_3^2]$

The nature of given Q.F. is indefinite.

Rank of given Q.F. is three, Index = two, Signature = one

The corresponding transformation is $X = PY$. $x = -\frac{2}{3}y_1 + \frac{2}{3}y_2 + \frac{1}{3}y_3$

$y = \frac{2}{3}y_1 + \frac{2}{3}y_2 + \frac{2}{3}y_3$

$z = \frac{1}{3}y_1 + \frac{2}{3}y_2 - \frac{2}{3}y_3$.

2) Reduce the Q.F. $3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$ into sum of square form by an orthogonal transformation & give the matrix of transformation also find the linear matrix.

Sol:- Let the matrix of the Q.F $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

The ~~Cova~~ C.E of A is $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda - 1)(\lambda^2 - 8\lambda + 16) = 0$

$\lambda^3 - 8\lambda^2 + 16\lambda - \lambda^2 + 8\lambda - 16 = 0$

$\lambda = 1, 4, 4$.

$\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$

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Case (i) when $\lambda = 1$

Consider $(A - I)(x) = 0$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} 2x_1 + x_2 + x_3 = 0 \rightarrow (1) \\ x_1 + 2x_2 - x_3 = 0 \rightarrow (2) \\ x_1 + x_2 + 2x_3 = 0 \rightarrow (3) \end{array}$$

Solving (1) & (2)

$$\begin{array}{cccc} 1 & 1 & 2 & 1 \\ 2 & -1 & 1 & 2 \end{array} \quad \frac{x_1}{-1-2} = \frac{x_2}{1+2} = \frac{x_3}{4-1} = k_1$$

$$\therefore X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii):- when $\lambda = 4$

Consider $[A - 4I][x] = 0$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} -x_1 + x_2 + x_3 = 0 \Rightarrow x_1 - x_2 - x_3 = 0 \rightarrow (1) \\ x_1 - x_2 - x_3 = 0 \rightarrow (2) \\ x_1 - x_2 - x_3 = 0 \rightarrow (3) \end{array}$$

All the eq are same. Let $x_3 = k_2$, $x_2 = k_3$ then

$x_1 = k_3 + k_2$

$$\therefore X = \begin{bmatrix} k_3 + k_2 \\ k_3 \\ k_2 \end{bmatrix} = k_2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + k_3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$ where k_2, k_3 are non zero scalars.

Thus, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ are the eigen vectors for $\lambda = 4$.

$\therefore$ The eigen vectors $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ are pair wise othogonal

Let $X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

To find X_3, using orthogonal property, Let $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

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$\because X_1^T X_3 = 0$ & $X_2^T X_3 = 0$

$$\begin{bmatrix} -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \& \quad \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$-x_1 + x_2 + x_3 = 0 \rightarrow (4)$ $\quad$ $x_1 + x_3 = 0 \rightarrow (5)$

solving (4) & (5) By cross multi

$$\begin{matrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{matrix} \qquad \frac{x_1}{1-0} = \frac{x_2}{1+1} = \frac{x_3}{0-1}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1} = k$$

$$\therefore X = k\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \Rightarrow X_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$\therefore$ we observe that X_1, X_2, X_3 are pairwise orthogonal

$\|X_1\| = \sqrt{3}$

Normalized eigen vectors are $e_1 = \dfrac{X_1}{\|X_1\|} = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

$\|X_2\| = \sqrt{2}$ $\quad$ $\|X_3\| = \sqrt{6}$

$$e_2 = \frac{X_2}{\|X_2\|} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad e_3 = \frac{X_3}{\|X_3\|} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

2)

Normalized modal matrix $\hat{P} = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$

$$\hat{P}^T = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix}$$

$$\hat{P} \cdot \hat{P}^T = \hat{P}^T \hat{P} = I.$$

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Consider $\hat{P}^T A \hat{P} = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = D.$$

The O.T is $X = \hat{P} Y$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$x_1 = \frac{-1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 + 1/\sqrt{6}\, y_3$$

$$x_2 = \frac{1}{\sqrt{2}} y_1 + 2/\sqrt{6}\, y_3$$

$$x_3 = \frac{1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 - \frac{1}{\sqrt{6}} y_3.$$

3) $2x^2 + 2y^2 + 2z^2 - 2xy + 2xz - 2yz$

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

C.E of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)\left((2-\lambda)^2 - 1\right) + 1\left(-1(2-\lambda)+1\right)$

$+ 1\left(1-(2-\lambda)\right) = 0$

$\Rightarrow (2-\lambda)(4+\lambda^2-4\lambda-1) + 1(-2+\lambda+1)$

$+ 1 - 2 + \lambda = 0$

$\Rightarrow 6 + 2\lambda^2 - 8\lambda - 3\lambda - \lambda^3 + 4\lambda^2 - 2$

$+\lambda + 1 + 1 - 2 + \lambda = 0$

$\Rightarrow -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$

$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \quad \lambda = 1, 1, 4$

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Case (i) $\lambda = 4$

$(A-4I)x = 0$

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x - y + z = 0 \Rightarrow 2x + y - z = 0 \rightarrow (1)$

$-x - 2y - z = 0 \Rightarrow x + 2y + z = 0 \rightarrow (2)$

$x - y + 2z = 0 \rightarrow (3)$

Solving (1) & (2)

$$\begin{matrix} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 2 \end{matrix} \qquad \frac{x}{1+2} = \frac{y}{-1-2} = \frac{z}{4-1}$$

$$\frac{x}{3} = \frac{y}{-3} = \frac{z}{3} = k_1$$

$$x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Case (ii) $\lambda = 1$

$(A-I)X = 0$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x - y + z = 0 \rightarrow (1)$

$-x + y - z = 0 \Rightarrow x - y + z = 0 \rightarrow (2)$

$x - y + z = 0 \rightarrow (3)$

All eq are same equal

Let $z = k_2$, $y = k_3$ Then

$x_1 = y - z = k_3 - k_2$

$$\therefore X = \begin{bmatrix} k_3 - k_2 \\ k_3 \\ k_2 \end{bmatrix} = k_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + k_3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Thus, $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ are eigen vectors for $\lambda = 1$.

$\therefore$ The eigen vectors are $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ are pair wise not orthogonal.

Let $X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

To find X_3 using orthogonal property, let $X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\therefore X_1^T X_3 = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \qquad X_2^T X_3 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$= x - y + z = 0 \rightarrow (1)$

$-x + z = 0$

$x - z = 0 \rightarrow (2)$

solving (1) & (2)

$$\begin{matrix} -1 & 1 & 1 & -1 \\ 0 & -1 & 1 & 0 \end{matrix} \qquad \frac{x}{1-0} = \frac{y}{1+1} = \frac{z}{0+1} = k_4$$

$$X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

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$$x_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\|x_1\| = \sqrt{1+1+1} = \sqrt{3} \quad \|x_2\| = \sqrt{1+1} = \sqrt{2} \quad \|X_3\| = \sqrt{1+4+1} = \sqrt{6}$$

$$e_1 = \frac{x_1}{\|x_1\|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \quad e_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad e_2 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

Normalize eigen vectors are $\hat{P} = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$

$$\hat{P}^T = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

$\therefore \hat{P}\hat{P}^T = \hat{P}^T\hat{P} = I.$

Consider $\hat{P}^T A \hat{P} = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$$

The O.T. is $X = \hat{P} Y$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$x_1 = \frac{1}{\sqrt{3}} y_1 - \frac{1}{\sqrt{2}} y_2 + \frac{1}{\sqrt{6}} y_3, \quad x_2 = \frac{-1}{\sqrt{3}} y_1 + \frac{2}{\sqrt{6}} y_2$

$x_3 = \frac{1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 + \frac{1}{\sqrt{6}} y_3.$

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Pb:- Discuss the nature of the quadratic from

$x^2+4xy+6xz-y^2+2yz+4z^2$

Sol:- Given Q.F. is $x^2+4xy+6xz-y^2+2yz+4z^2$

Matrix form is $\begin{bmatrix}1&2&3\\2&-1&1\\3&1&4\end{bmatrix}$

C.E. of A is $|A-\lambda I|=0$

ie., $\left|\begin{bmatrix}1&2&3\\2&-1&1\\3&1&4\end{bmatrix}-\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right|=0$

$$\begin{vmatrix}1-\lambda&2&3\\2&-1-\lambda&1\\3&1&4-\lambda\end{vmatrix}=0$$

$\lambda^3-4\lambda^2-15\lambda=0$

$\lambda=0$ (or) $\lambda^2-4\lambda-15=0$

$\lambda=0,\ 2+\sqrt{19},\ 2-\sqrt{19}.$

$\therefore$ The give Q.F. is indefinite.

2) Find the nature of the Q.F. $2x^2+2y^2+2z^2+2yz$.

Sol:- Q.F. is $2x^2+2y^2+2z^2+2yz$

Matrix of Q.F. is $\begin{bmatrix}2&0&0\\0&2&1\\0&1&2\end{bmatrix}$

The C.E. is $A-\lambda I=0$

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Properties of Eigen values :-

1) The sum of the eigen values of a Square matrix is equal to its trace and product of the eigen values is equal to its determinant.

ie., if 'A' is an $n \times n$ matrices and $\lambda_1, \lambda_2 \cdots \lambda_n$ are its n eigen values, Then

$$\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n = Tr(A) \quad \& \quad \lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n = det(A)$$

Proof :- The C.E. of A is $|A - \lambda I| = 0$

ie., $$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

$\Rightarrow [(a_{11}-\lambda)(a_{22}-\lambda)(a_{23}-\lambda)\cdots(a_{nn}-\lambda)] - a_{12}$[a polynomial of degree of $(n-2)$] $+ a_{13}$[a polynomial of degree $(n-2)$] $+ \cdots = 0$

ie., $(-1)^n(\lambda - a_{11})(\lambda - a_{22})\cdots(\lambda - a_{nn})$ + a polynomial of degree $(n-2) = 0$

ie., $(-1)^n[\lambda^n - (a_{11}+a_{22}+\cdots+a_{nn})\lambda^{n-1}$ + a polynomial of degree $(n-2)$ + a polynomial of degree $(n-2)$ in $\lambda = 0$

$\therefore (-1)^n \lambda^n + (-1)^n (\text{Trace } A)\lambda^{n-1}$ + a polynomial of degree $(n-2)$ in $\lambda = 0$

If $\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_n$ are the roots of this equations.

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Sum of the roots $= -\dfrac{(-1)^{n+1}\,Tr(A)}{(-1)^n} = Tr(A)$

Further $|A-\lambda I| = (-1)^n \lambda^n + \cdots + a_0$

put $\lambda=0$, Then $|A| = a_0$

$(-1)^n \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_0 = 0$

$\Rightarrow$ Product of the roots $= \dfrac{(-1)^n a_0}{(-1)^n} = a_0 = |A| = \det A$

Hence the result.

* 2) If 'λ' is an eigen value of 'A' corresponding to the eigen vector X, Then λ^n is eigen value of A^n corresponding to the eigen vector X.

(OR)

P.T. if $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$ are the eigenvalues of A Then $\lambda_1^k, \lambda_2^k, \lambda_3^k, \ldots \lambda_n^k$ are the eigen values of A^k.

<u>Proof</u>:- Since 'λ' is an eigen value of 'A' corresponding to the eigen vector X, we have $AX = \lambda X \to ①$

Premultiply ① by A, $A(AX) = A(\lambda X)$

ie., $A^2X = \lambda(AX) = \lambda \cdot \lambda X = \lambda^2 X$. ($\because$ from ①)

Hence λ^2 is eigen value of A^2 with 'X' itself as the corresponding eigen vector. Thus the Theorem is true for $n=2$.

Let the result be true for $n=k$.

Then $A^k X = \lambda^k X$.

Premultiply this by A, we get

$A(A^k X) = A(\lambda^k X)$

$A^{k+1}X = \lambda^k (AX)$ (from ①)

$= \lambda^k(\lambda X) = \lambda^{k+1} X$

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$\Rightarrow \lambda^{k+1}$ is eigen value of A^{k+1} with x itself as the corresponding eigen vector. ②

$\therefore$ The Theorem is true for all +ve integers n.

3) show that if $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$ are the latent roots of A, then A^3 has latent roots $\lambda_1^3, \lambda_2^3, \lambda_3^3 \ldots \lambda_n^3$.

Sol :- put $n=3$

we know that $A^{k+1} x = \lambda^{k+1} x$

put $n=3$

$$A^3 x = \lambda^3 x$$

So $\therefore \lambda_1, \lambda_2 \lambda_3 \ldots \lambda_n$ are the latent roots of A.

Then A^3 has latent roots $\lambda_1^3, \lambda_2^3, \ldots \lambda_n^3$.

4) A Square matrix A & its transpose A^T have the same eigen values.

Sol:- we have $(A - \lambda I)^T = A^T - \lambda I^T = A^T - \lambda I$.

$$\therefore |(A-\lambda I)^T| = |A^T - \lambda I| \text{ (or) } |A - \lambda I| = |A^T - \lambda I|$$

$$(\because |A^T| = |A|)$$

$$\therefore |A - \lambda I| = 0 \Leftrightarrow |A^T - \lambda I| = 0.$$

i.e., λ is an eigen value of A iff λ is eigen value of A^T.

Thus the eigen values of A & A^T are same.

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5) If A and B are n rowed square matrices and if A is invertible show that $A^{-1}B$ & BA^{-1} have same eigen values

Sol:- Given 'A' is invertible ⇒ A^{-1} exists.

W.K.T. if A & P are square matrices of order n such that 'P' is non-singular, Then A & $P^{-1}AP$ have the same eigen values

Taking $A = BA^{-1}$ & $P = A$, we have

BA^{-1} and $A^{-1}(BA^{-1})A$ have the same eigenvalues

i.e., BA^{-1} and $(A^{-1}B)(A^{-1}A)$ " " " "

BA^{-1} & $(A^{-1}B)I$ " " " "

(or) BA^{-1} & $A^{-1}B$ " " " ".

6) If $\lambda_1, \lambda_2, \lambda_3, \dots \lambda_n$ are the eigen values of a matrix A, Then $k\lambda_1, k\lambda_2, k\lambda_3, \dots k\lambda_n$ are the eigen values of kA, where k is a non-zero scalar.

Sol:- Let 'A' be a square matrix of order n.

Then $|kA - \lambda kI| = |k(A - \lambda I)| = k^n|A - \lambda I|$ $[\because |kA| = k^n|A|]$

Since $k \neq 0$,

$\therefore |kA - \lambda kI| = 0 \iff |A - \lambda I| = 0$

i.e., $k\lambda$ is an eigen value of kA iff λ is an eigen value of A

$\therefore k\lambda_1, k\lambda_2, \dots k\lambda_n$ are the eigenvalues of the matrix kA if $\lambda_1, \lambda_2, \dots \lambda_n$ are the eigenvalues of the matrix A.

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7) If λ is an eigen value of the matrix A Then $\lambda+k$ is an eigen value of the matrix $A+kI$.

sol :- let 'λ' be an eigen value of A and X the corresponding eigen vector.

Then by definition $AX=\lambda X \rightarrow (1)$

(from (1))

Now, $(A+kI)X = AX+kIX = \lambda X+kX=(\lambda+k)X$

we from (2) $\lambda+k$ is an eigen value of $(A+kI)$ and X is eigen vector.

** 8) If $\lambda_1, \lambda_2, \cdots \lambda_n$ are the eigen values of A, Then $\lambda_1-k, \lambda_2-k, \lambda_3-k, \cdots \lambda_n-k$ are the eigen values of the matrix $(A-kI)$, where 'k' is a non-zero scalar.

sol :- $\because \lambda_1, \lambda_2, \cdots \lambda_n$ are eigen values of A.

$\therefore$ The C.E. of A is The characteristic polynomial of A is

$$|A-\lambda I| = (\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda)\cdots(\lambda_n-\lambda) \rightarrow (1)$$

Thus The c. polynomial of $[A-kI]$ is

$$[A-kI-\lambda I]x = |A-I(k+\lambda)|$$

$$=[\lambda_1-(\lambda+k)][\lambda_2-(\lambda+k)][\lambda_3-(\lambda+k)]\cdots[\lambda_n-(\lambda+k)] \text{ from (1)}$$

$$=[(\lambda_1-k)-\lambda][(\lambda_2-k)-\lambda][(\lambda_3-k)-\lambda]\cdots[(\lambda_n-k)-\lambda]$$

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This shows that eigen values of $A-kI$ is

$\lambda_1-k, \lambda_2-k, \lambda_3-k, \dots \lambda_n-k$.

9) If $\lambda_1, \lambda_2, \lambda_3, \dots \lambda_n$ are the eigen values of A, find the eigen values of the matrix $(A-\lambda I)^2$

Sol:- first we will find the eigen values of the matrix $A-\lambda I$.

$\because \lambda_1, \lambda_2, \lambda_3, \dots \lambda_n$ are the eigen values of A.

$\therefore$ The C. polynomial of A is $|A-kI| = (\lambda_1-k)(\lambda_2-k)\dots(\lambda_n-k) \rightarrow ①$

where k is a scalar.

The characteristic polynomial of the matrix $(A-\lambda I)$ is

$$(A-\lambda I-k) = |A-(\lambda+k)I| \quad \text{(from ①)}$$

$$= [\lambda_1-(\lambda+k)][\lambda_2-(\lambda+k)]\dots(\lambda_n-(\lambda+k))$$

$$= [(\lambda_1-\lambda)-k][(\lambda_2-\lambda)-k] \dots [(\lambda_n-\lambda)+k]$$

which shows that the eigen values of $A-\lambda I$ are $\lambda_1-\lambda, \lambda_2-\lambda$

$\lambda_3-\lambda, \dots \lambda_n-\lambda$.

w.k.T. if the eigen values of A are $\lambda_1, \lambda_2, \dots \lambda_n$ then the eigen values of A^2 are $\lambda_1^2, \lambda_2^2, \dots \lambda_n^2$.

$\therefore$ The eigen values of $(A-\lambda I)^2$ are $(\lambda_1-\lambda)^2, (\lambda_2-\lambda)^2, \dots (\lambda_n-\lambda)^2$

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10) If λ is an eigen values of a non-singular matrix A (4)
* corresponding to the eigen vector X, Then λ^{-1} is an eigen value of A^{-1} and corresponding eigen vector X itself.

(OR)

P.T. the eigen values of A^{-1} are the reciprocals of the eigen values of A.

Sol:- $\because$ 'A' is a non-singular & product of the eigen values is equal to |A|, it follows that none of the eigen values of 'A' is 0.

$\therefore$ If λ is an eigen value of the non-singular matrix A & X is the corresponding eigen vector, $\lambda \neq 0$ and $AX = \lambda X$. Premultiplying this with A^{-1}, we get

$$A^{-1}(AX) = A^{-1}(\lambda X)$$

$$(A^{-1}A)X = A^{-1}\lambda X$$

$$I X = \lambda(A^{-1}X)$$

$$\therefore X = \lambda A^{-1} X$$

$$\Rightarrow A^{-1}X = \lambda^{-1}X \quad (\lambda \neq 0)$$

Hence by Def it follows that λ^{-1} is an eigen value of A^{-1} and X is the corresponding eigen vector.

11) If 'λ' is an eigen value of a non-singular matrix A,
* Then $\frac{|A|}{\lambda}$ is an eigen value of the matrix adj A.

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Sol:- $\because \lambda$ is an eigen value of a non-singular matrix,

$\therefore \lambda \neq 0$.

Also λ is an eigen value of A implies that there exists a non-zero vector X such that

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$(\mathrm{adj}A)A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= \begin{bmatrix} ad-bc & bd-bd \\ -ac+ac & ad-bc \end{bmatrix}$

$\begin{bmatrix} ad-bc \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$|A|I$.

$$AX = \lambda X \to ①$$

$$(\mathrm{adj}A)AX = (\mathrm{adj}A)\lambda X$$

$$[(\mathrm{adj}A)A]X = \lambda(\mathrm{adj}A)X$$

$$|A|IX = \lambda(\mathrm{adj}A)X \quad (\because (\mathrm{adj}A)A = |A|I)$$

$$\frac{|A|}{\lambda}X = (\mathrm{adj}A)X \quad \text{(or)} \quad (\mathrm{adj}A)X = \frac{|A|}{\lambda}X.$$

$\because X$ is a non-zero vector

$\therefore$ from ① it is clear that $\frac{|A|}{\lambda}$ is an eigen value of the matrix adj A.

Problems:-

1) Find the eigen values of the following matrices.

1) $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$

Sol:- C.E. is $|A-\lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 1-3i \\ 1+3i & 7-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(7-\lambda)-(1-9i^2) = 0$$

$$28-11\lambda+\lambda^2-10 = 0$$

$$\lambda^2-11\lambda+18 = 0, \quad \lambda = 9, 2.$$

Also $\bar{A} = \begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix} = A^T$

$\therefore$ A is Hermitian.

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2) $A = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

$\bar{A} = \begin{bmatrix} -3i & 2-i \\ -2-i & i \end{bmatrix}$

$A^T = \begin{bmatrix} 3i & -2+i \\ 2+i & -i \end{bmatrix} = -\begin{bmatrix} -3i & 2-i \\ -2-i & i \end{bmatrix} = -\bar{A}$

$\therefore A^T = -\bar{A}$ Thus B is a Skew - Hermitian.

The C.E. of A is $|A-\lambda I| = 0$

$\begin{vmatrix} 3i-\lambda & +2+i \\ -2+i & -i-\lambda \end{vmatrix} = 0$

$\lambda^2 - 2i\lambda + 8 = 0$ roots are $\lambda = 4i, -2i$

$\therefore$ eigen values are imaginary & roots of skew ~~symm~~ Hermitian is imaginary.

() $\begin{bmatrix} \frac{1}{2}i & \sqrt{3}/2 \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i \end{bmatrix}$ $\lambda = \frac{\sqrt{3}}{2} + \frac{1}{2}i, \frac{-\sqrt{3}}{2} + \frac{1}{2}i.$

$\left|\pm\frac{\sqrt{3}}{2} + \frac{1}{2}i\right| = 1$

Thus the C. Roots of unitary matrix have absolute value 1.

4) P.T. $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary & determine the eigen value & eigen vector.

Ans: Let $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$, $A^T = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -i \\ 1+i & -1 \end{bmatrix}$

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$(A^{\theta}) = (\bar{A})^{T} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$$AA^{\theta} = \frac{1}{3}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1+2 & (1+i)-(1+i) \\ (1-i)-(1-i) & 2+1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$\therefore$ A is unitary matrix.

Pb:- P.T. the determinant of a unitary matrix is of unit modulus.

Sol:- Let 'A' be an unitary matrix.

Thus $AA^{\theta} = I \Rightarrow |AA^{\theta}| = |I|$

$|A||A^{\theta}| = 1 \Rightarrow |A||\bar{A}^{T}| = 1$

$|A||\bar{A}| = 1 \quad (\bar{A}^{T} = \bar{A})$

$|A|^{2} = 1 \quad (A = \bar{A})$

$|A|$ is unit modulus.

Hence A is unitary Then $|A|$ is of unit modulus.

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Gauss Seidel iteration method:-

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \rightarrow (1)$$

where the diagonal coefficients are not zero and large compared to other coefficients. Such a system is called a Diagonally Dominant System.

The system of eq's (1) may be written as

$$x_1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3\right] \rightarrow (2)$$

$$x_2 = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3\right] \rightarrow (3)$$

$$x_3 = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1 - a_{32}x_2\right] \rightarrow (4)$$

Let the initial approximate solution be $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}$

Sub $x_2^{(0)}, x_3^{(0)}$ in the first eq. of (2), we get

$$x_1^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\right] \rightarrow (5)$$

This is taken as the first approximation

Sub $x_1^{(1)}$ for x_1 and $x_3^{(0)}$ for $x_3^{(0)}$ in the second eq of (2), we get

$$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}\right] \rightarrow (6)$$

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This is taken as the first approximation of x_2

Next, Sub $x_1^{(1)}$ for x_1 & $x_2^{(1)}$ for x_2 in the last eq of

(4), we get

$$x_3^{(1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}\right] \rightarrow (7)$$

This is taken as the first approximation of x_3.

This is taken as the first approximation

we have repeated this by same approximation

of x_1 x_2 x_3.

Problems:-

1) Use Gauss-Seidel iteration method to solve the system.

$10x+y+z=12$, $2x+10y+z=13$, $2x+2y+10z=14$.

Sol:- The given system is diagonally dominant & we written as

$$x = \frac{1}{10}\left[12-y-z\right] \rightarrow (1), \quad y = \frac{1}{10}\left[13-2x-z\right] \rightarrow (2)$$

$$z = \frac{1}{10}\left[14-2x-2y\right] \rightarrow (3)$$

we start iteration by taking $y=0$, $z=0$ in (1)

$$x^{(1)} = \frac{1}{10}(12) = 1.2$$

putting $x = x^{(1)} = 1.2$, $z=0$ in (2)

$$y^{(1)} = 1.06$$

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putting $x = 1.2$, $y = 1.06$ in (3)

$$z^{(1)} = 0.95$$

Now taking $y^{(1)}, z^{(1)}$ as the initial values in (1)

$$x^{(2)} = \frac{1}{10}[12 - 1.06 - 0.95] = 0.999$$

Taking $x = x^{(2)}$ & $z = z^{(1)}$ in (2), we get

$$y^{(2)} = \frac{1}{10}[13 - 1.998 - 0.95] = 1.005$$

Next, Taking $x = x^{(2)}$ & $y = y^{(2)}$ in (3), we get

$$z^{(2)} = \frac{1}{10}[14 - 1.998 - 2.010] = 0.999.$$

again taking $x^{(2)}, y^{(2)}, z^{(2)}$ as the intial values.

we get

$$x^{(3)} = \frac{1}{10}(12 - 1.005 - 0.999) = 0.9996 = 1.00$$

$$y^{(3)} = \frac{1}{10}(13 - 2.0 - 0.999) = 1.0001 = 1.00$$

$$z^{(3)} = \frac{1}{10}(14 - 2.0 - 2.0) = 1.00$$

lly, we find the fouth approximations of x, y, z & get them as $x^{(4)} = 1.00$, $y^{(4)} = 1.00$, $z^{(4)} = 1.00$.

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The tabulate the results

variable	Ist approximation	II	III	IV
x	1.20	0.999	1.00	1.00
y	1.06	1.005	1.00	1.00
z	0.95	0.999	1.00	1.00.

Thus the solution of the given system of eq is $x = 1$, $y = 1$, $z = 1$.

2) solve the following system of equations by Gauss- Seidel method. $8x_1 - 3x_2 + 2x_3 = 20$
$4x_1 + 11x_2 - x_3 = 33$, $6x_1 + 3x_2 + 12x_3 = 36$

IV approx:- $x = 2.9997$, $y = 1.9998$, $z = 1.0002$

V approx:- $x = 2.9998$, $y = 2.0000$, $z = 1.0000$

3) $x_1 + 10x_2 + x_3 = 6$, $10x_1 + x_2 + x_3 = 6$, $x_1 + x_2 + 10x_3 = 6$.

II. $x = 0.4974$, $y = 0.5017$, $z = 0.4992$

III $x = 0.4999$, $y = 0.5000$, $z = 0.5000$.

4) $20x + 2y + 6z = 28$, $x + 20y + 9z = -23$, $2x - 7y - 20z = -57$

A $x = 0.5149$, $y = -2.9451$, $z = 3.9323$

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UNIT-II:
Mean Value Theorems

Rolle's Theorem:

UNIT - II

Statement: Let $f(x)$ be a function such that $x \in [a,b]$

i) $f(x)$ is continuous in closed interval $[a,b]$

ii) $f(x)$ is differentiable in open interval (a,b) and

iii) $f(a) = f(b)$.

Then there exists atleast one point c in open (a,b) such that $f'(c) = 0$.

PROBLEMS:

1) Verify Rolle's Theorem for $f(x) = 2x^3 + x^2 - 4x - 2$ in $[-\sqrt{3}, \sqrt{3}]$

Soln: Given

$f(x) = 2x^3 + x^2 - 4x - 2$ is a single valued function

And interval $[a,b]$ is $[-\sqrt{3}, \sqrt{3}]$

Verifing Rolle's Theorem we have to check the following conditions.

i) Continuity:

$f(x)$ is a polynomial fn of 'x'.

Every polynomial fn is continuous on $\mathbb{R}$.

$\therefore f(x)$ is continuous for every value of x.

In perticuler, $f(x)$ is continuous in $[-\sqrt{3}, \sqrt{3}]$.

ii) Derivability:

$f(x) = 2x^3 + x^2 - 4x - 2$

$f'(x) = 6x^2 + 2x - 4$

$f'(x)$ exists forevery $x \in (-\sqrt{3}, \sqrt{3})$

$\Rightarrow f(x)$ is derivable in $(-\sqrt{3}, \sqrt{3})$.

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iii) For $f(a)=f(b)$.

$[a,b]=[-\sqrt{3},\sqrt{3}]$

$f(-\sqrt{3})=2(-\sqrt{3})^3+(-\sqrt{3})^2-4(-\sqrt{3})-2$

$f(-\sqrt{3})=-6\sqrt{3}+3+4\sqrt{3}-2$

$f(-\sqrt{3})=1-2\sqrt{3}$

$f(\sqrt{3})=2(\sqrt{3})^3+(\sqrt{3})^2-4(\sqrt{3})-2$

$f(\sqrt{3})=6\sqrt{3}+3-4\sqrt{3}-2$

$f(\sqrt{3})=1+2\sqrt{3}$

Here $f(-\sqrt{3})\neq f(\sqrt{3})$

$f(x)$ is not satisfying the condition $f(a)=f(b)$.

$\therefore$ Rolle's theorem is not applicable for $f(x)$ in $[-\sqrt{3},\sqrt{3}]$.

2) Verify Rolle's Theorem for $f(x)=(x+2)^3(x-3)^4$ in $[-2,3]$.

Soln: Given

$f(x)=(x+2)^3(x-3)^4$ is a single valued function.

And interval $[a,b]$ is $[-2,3]$.

i) Continuity:

$f(x)$ is a polynomial fn of 'x'.

Every polynomial fn is continuous on $\mathbb{R}$.

Hence $f(x)$ is continuous in closed interval $[-2,3]$.

ii) Derivability:

$f(x)=(x+2)^3(x-3)^4$

$f'(x)=3(x+2)^2(x-3)^4$
$+(x+2)^3\cdot 4(x-3)^3$

$=(x+2)^2(x-3)^3$
$[3(x-3)+4(x+2)]$

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$f'(x) = (x+2)^2(x-3)^3$
$(3x-9+4x+8)$

$f'(x) = (x+2)^2(x-3)^3(7x-1)$.

$f'(x)$ exists for every $(-2,3)$

$\Rightarrow f(x)$ is derivable in $(-2,3)$.

iii> For $f(a) = f(b)$.

$[a,b] = [-2,3]$

$f(-2) = (-2+2)^3(-2-3)^{+4} = 0$

$f(3) = (3+2)^3 \cdot (3-3)^4 = 0$

Further $f(-2) = f(3)$

Thus all the three conditions of Rolle's theorem are satisfied.

Hence there exists a point c in $(-2,3)$ such that $f'(c) = 0$.

Verification:

Consider $f'(c) = 0$

$\Rightarrow (c+2)^2(c-3)^3(7c-1) = 0$

$\Rightarrow c = -2$ or 3 or $1/7$

clearly $c = \frac{1}{7} \in (-2,3)$

i.e., $-2 < \frac{1}{7} < 3$

Hence Rolle's theorem is verified.

3> Verify Rolle's Theorem for $f(x) = \tan x$ in $[0,\pi]$.

Soln: Given $f(x) = \tan x$, is single valued function.

And interval is $[0,\pi]$

i> Continuity:

$f(x)$ is not exist at $x = \pi/2$ in $[0,\pi]$

$f(x)$ is discontinuous in $[0,\pi]$.

$f(x)$ is not satisfying the condition of continuity in Rolle's Theorem

$\therefore$ Rolle's Theorem is not applicable.

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4> Using Rolle's theorem, show that $g(x) = 8x^3 - 6x^2 - 2x + 1$ between zero and one. i.e., $x \in [0, 1]$.

Soln: Given

$$g(x) = 8x^3 - 6x^2 - 2x + 1$$

is single valued function. And interval $[0,1]$

i> **Continuity:**

$g(x)$ is a polynomial fn of 'x'.

Every polynomial fn is continuous on $\mathbb{R}$.

Hence $g(x)$ is continuous, in closed interval $[0,1]$

ii> **Derivability:**

$$g(x) = 8x^3 - 6x^2 - 2x + 1$$

$$g'(x) = 24x^2 - 12x - 2$$

$g'(x)$ exists for every $(0,1)$.

iii> $g(a) = g(b)$

$$[a,b] = [0,1]$$

$$g(0) = 8(0) - 6(0) - 2(0) + 1$$

$$g(0) = 1$$

$$g(1) = 8(1) - 6(1) - 2(1) + 1$$

$$g(1) = 1$$

$$\therefore g(0) = g(1).$$

Hence all the conditions of Rolle's theorem are satisfied on $[0,1]$.

Therefore there exists a number $c \in (0,1)$ such that $g'(c) = 0$.

Verification:

Consider $g'(c) = 0$

$$\Rightarrow 24c^2 - 12c - 2 = 0$$

$$12c^2 - 6c - 1 = 0$$

$$c = \frac{6 \pm \sqrt{36+48}}{24}$$

$$c = \frac{6 \pm \sqrt{84}}{24} = \frac{3 \pm \sqrt{21}}{12}$$

$$c = 0.6319 \text{ or } -0.1319$$

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only the value $c = 0.6319$ lies in $(0,1)$.

Thus there exists atleast one root b/n 0 and 1.

5> Verify Rolle's Theorem for $f(x) = x(x+3)e^{-x/2}$ in the interval $[-3, 0]$. **

Soln: Given $f(x) = x(x+3)\cdot e^{-x/2}$ is a single valued function. And interval $[-3,0]$

i> Continuity:

$x(x+3)$ being a polynomial is continuous for all values of x and $e^{-x/2}$ is also continuous for all x,

their product $x(x+3)\cdot e^{-x/2} = f(x)$ is also continuous for every value of x. And in particular $f(x)$ is continuous in the closed interval $[-3, 0]$.

ii> Derivability:

$$f(x) = (x^2+3x)e^{-x/2}$$

$$f'(x) = (x^2+3x)\cdot -\frac{1}{2}\cdot e^{-x/2} + e^{-x/2}(2x+3)$$

$$f'(x) = e^{-x/2}\left[2x+3 - \frac{x^2+3x}{2}\right]$$

$$f'(x) = e^{-x/2}\left[\frac{4x+6-x^2-3x}{2}\right]$$

$$f'(x) = e^{-x/2}\left[\frac{6+x-x^2}{2}\right]$$

Since $f'(x)$ doesnot become infinite or indeterminate at any point of the interval $(-3,0)$.

$\therefore f(x)$ is derivable in the open interval $(-3,0)$.

iii> $f(a) = f(b)$

$[a,b] = [-3,0]$

$$f(-3) = -3(-3+3)e^{3/2} = 0$$

$$f(0) = 0(0+3)e^{0/2} = 0$$

$$\therefore f(-3) = f(0).$$

Thus $f(x)$ satisfies all the three conditions of Rolle's Theorem in the interval $[-3,0]$.

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Hence there exist atleast one value of c of x in the interval $(-3,0)$ such that $f'(c)=0$

i.e., $e^{-c/2}\left(\frac{6+c-c^2}{2}\right)=0$

$$c^2-c-6=0$$

$$\Rightarrow (c-3)(c+2)=0$$

$$\therefore c=3,-2$$

clearly, the value $c=-2$ lies within the open interval $(-3,0)$ which verifies Rolle's Theorem.

6> Verify Rolle's Theorem for $f(x)=|x|$ in $-1\leq x\leq 1$.

Soln: Given $f(x)=|x|$ is a single valued function.

And interval is $[-1,1]$

i.e, $f(x)=-x,\ x<0$

$\qquad = x,\ x\geq 0$

$$f(x)=|x|=\begin{cases}-x, & -1\leq x<0\\ x, & 0\leq x\leq 1\end{cases}$$

i> Continuity:

$f(x)$ is continuous for every value of x.

$\therefore f(x)$ is continuous in the closed $[-1,1]$.

ii> Derivability:

L.H.D:

$$Lf'(0)=\underset{x\to 0^-}{Lt}\frac{f(x)-f(a)}{x-a}$$

$$=\underset{x\to 0^-}{Lt}\frac{-x-0}{x-0}$$

$$=\underset{h\to 0^-}{Lt}\frac{h-0}{-h-0}$$

$$=\underset{h\to 0^-}{Lt}(-1)$$

$$=-1$$

put $x=a-h$

$x=0-h$

$x=-h$

$h\to 0$

R.H.D:

$$Rf'(0)=\underset{x\to 0^+}{Lt}\frac{f(x)-f(a)}{x-a}$$

$$=\underset{x\to 0^+}{Lt}\frac{x-0}{x-0}$$

$$=\underset{h\to 0}{Lt}\frac{0+h-0}{0+h-0}=\underset{h\to 0}{Lt}(1)$$

$$=1$$

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Since $Lf'(0) \neq Rf'(0)$,

therefore $f(x)$ is not derivable at $x=0$.

$\therefore$ $f(x)$ is not derivable in the open interval $(-1,1)$.

Hence Rolle's theorem is not applicable to $f(x)=|x|$ in $[-1,1]$.

---

7) Verify Rolle's theorem for the function of $\log\left[\frac{x^2+ab}{x(a+b)}\right]$ in $[a,b]$, $a>0,\ b>0$. **

Soln: Given

$$f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$$

is a single valued fn.

And interval is $[a,b]$.

The given eqn can be written as,

$$f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right] \quad 23$$

$$f(x) = \log(x^2+ab) - \log(x(a+b))$$

$$f(x) = \log(x^2+ab) - \log x - \log(a+b)$$

i) Continuity:

Since $f(x)$ is a composite fn of continuous fns in $[a,b]$.

It is continuous in $[a,b]$.

ii) Derivability:

$$f'(x) = \frac{1}{x^2+ab}(2x) - \frac{1}{x} - 0$$

$$f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x}$$

$$f'(x) = \frac{2x^2 - x^2 - ab}{x(x^2+ab)}$$

$$f'(x) = \frac{x^2 - ab}{x(x^2+ab)}$$

$\therefore$ $f'(x)$ exists $\forall\ x \in (a,b)$, $a>0, b>0$.

$\therefore$ $f(x)$ is derivable in the open interval (a,b).

iii) for $f(a) = f(b)$.

$$f(a) = \log\left[\frac{a^2+ab}{a(a+b)}\right]$$

$$= \log\left[\frac{a(a+b)}{a(a+b)}\right]$$

$$f(a) = \log 1 = 0$$

$$f(b) = \log\left[\frac{b^2+ab}{b(a+b)}\right]$$

$$= \log\left[\frac{b(b+a)}{b(b+a)}\right]$$

$$f(b) = \log 1 = 0.$$

$$\therefore f(a) = f(b)$$

Thus $f(x)$ satisfies all the three conditions of Rolle's Theorem.

$\therefore$ There exists $c \in (a, b)$ such that $f'(c) = 0$.

$$\Rightarrow \frac{c^2 - ab}{c(c^2+ab)} = 0$$

$$\Rightarrow c^2 - ab = 0$$

$$\Rightarrow c = \pm\sqrt{ab}$$

$$\therefore c = \sqrt{ab} \in (a, b)$$

Hence Rolle's Theorem is verified.

8) Verify Rolle's theorem for the function $f(x) = (x-a)^m (x-b)^n$ where m, n are positive integers in $[a, b]$.

Soln:

Repeated Prob. (2) with variables.

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8> Verify Rolle's theorem for the function

$f(x) = \frac{\sin x}{e^x}$ (or) $e^{-x}\sin x$ in $[0, \pi]$

Soln: Given $f(x) = \frac{\sin x}{e^x}$

is a single valued fn

And interval $[a,b]$ is $[0,\pi]$

i) Continuity:

Since $\sin x$ and e^x are both continuous functions in $[0,\pi]$, therefore $\frac{\sin x}{e^x}$ is also continuous in $[0,\pi]$

ii) Derivability:

$$f(x) = \frac{\sin x}{e^x}$$

$$f'(x) = \frac{e^x \cos x - \sin x \cdot e^x}{(e^x)^2}$$

$$f'(x) = \frac{\cos x - \sin x}{e^x}$$

$f'(x)$ exists $\forall\ x \in (0,\pi)$

$\therefore f(x)$ is derivable in $(0,\pi)$

iii) for $f(a) = f(b)$

$[a,b] = [0,\pi]$

$$f(0) = \frac{\sin 0}{e^0} = \frac{0}{1} = 0$$

$$f(\pi) = \frac{\sin \pi}{e^\pi} = \frac{0}{e^\pi} = 0$$

$$f(0) = f(\pi)$$

Thus all the three conditions of Rolle's theorem are satisfied.

Hence there exists a point c in $(0,\pi)$ such that $f'(c) = 0$

Verification:

$$f'(c) = 0$$

$$f'(c) = \frac{\cos c - \sin c}{e^c} = 0$$

$$\cos c - \sin c = 0$$

$$\Rightarrow \tan c = 1 \Rightarrow c = \pi/4$$

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$c = \pi/4 \in (0, \pi)$

Hence Rolle's theorem is verified

9) Verify Rolle's theorem for $f(x) = e^x(\sin x - \cos x)$ in $[\pi/4, 5\pi/4]$

<u>Soln:</u> Given

$f(x) = e^x(\sin x - \cos x)$ is a single valued function.

And interval $[a,b]$ is $[\pi/4, 5\pi/4]$

i) Continuity:

Since e^x, $\cos x$, $\sin x$ are continuous functions in $[\pi/4, 5\pi/4]$.

$\therefore f(x) = e^x(\sin x - \cos x)$ is exists $\forall x \in [\pi/4, 5\pi/4]$.

$\therefore f(x)$ is continuous in $[\pi/4, 5\pi/4]$.

ii) Derivability:

for the function

$f(x) = e^x(\sin x - \cos x)$

$f'(x) = e^x(\cos x + \sin x) + e^x(\sin x - \cos x)$

$f'(x) = e^x \cos x + e^x \sin x + e^x \sin x - e^x \cos x$

$f'(x) = 2e^x \sin x$

$f'(x) = 2e^x \sin x$ exists $\forall x \in (\pi/4, 5\pi/4)$

$\therefore f(x)$ is derivable in $(\pi/4, 5\pi/4)$

iii) For $f(a) = f(b)$

$[a,b] = [\pi/4, 5\pi/4]$

$f(\pi/4) = e^{\pi/4}(\sin \pi/4 - \cos \pi/4) = 0$

$f(5\pi/4) = e^{5\pi/4}(\sin(\frac{5\pi}{4}) - \cos(\frac{5\pi}{4})) = 0$

$f(\pi/4) = f(5\pi/4)$

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Thus $f(x)$ satisfies all the three conditions of Rolle's theorem.

Hence there exists a point c in $(\pi/4, 5\pi/4)$ such that $f'(c)=0$.

Verification:

$$f'(c)=0.$$

$$f'(c) = 2e^{x}\sin c = 0$$

$$\sin c = 0$$

$$c = \pi$$

$$c = \pi \in (\pi/4, 5\pi/4)$$

Hence Rolle's theorem is verified.

(*) $|x| = \begin{cases} -x, & -1 \le x < 0 \\ x, & 0 \le x \le 1 \end{cases}$

$f(x) = |x||\cos x|$

$= x\cos x$, $[|x| = x,\ x>0]$

10) State whether Rolle's theorem can be applied for $|x\cos x|$ where $0 \le x \le 3\pi/2$ 27

(*)

Soln: Given $f(x) = |x\cos x|$
$= |x|\cos x = x\cos x$.
is a single valued function

And interval $[a, b]$ is $[0, 3\pi/2]$

We've to check the following conditions for $f(x)$ whether applicable to Rolle's theorem.

i) Continuity:

$f(x) = x\cos x$ exists $\forall$ $x \in [0, 3\pi/2]$.

$\therefore$ $f(x)$ is continuous in $[0, 3\pi/2]$.

ii) Derivability:

$$f(x) = x\cos x$$

$$f'(x) = x(-\sin x) + \cos x$$

$$f'(x) = \cos x - x\sin x.$$

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$f'(x)$ is exists $\forall\ x \in (0, \frac{3\pi}{2})$

$\therefore f(x)$ is derivable in $(0, 3\pi/2)$.

iii) for $f(a) = f(b)$.

$f(x) = x\cos x$

$[a, b] = [0, 3\pi/2]$

$f(0) = 0 \cdot \cos(0) = 0$

$f(3\pi/2) = \frac{3\pi}{2} \cdot \cos\frac{3\pi}{2} = 0$

$f(0) = f(3\pi/2)$

Thus $f(x)$ satisfies all the three conditions of Rolle's theorem.

Hence Rolle's theorem is applicable.

11) Verify Rolle's theorem for $f(x) = x^3$ in $[1, 3]$.

Soln: Given $f(x) = x^3$ in $[1, 3]$

i) Clearly $f(x)$ is continuous in $[1, 3]$

ii) $f(x) = x^3$

$f'(x) = 3x^2$ which exists.

$\therefore f(x)$ is derivable in $(1, 3)$.

iii) $f(x) = x^3$

$[1, 3] = [a, b]$

$f(a) = f(1) = 1$

$f(b) = f(3) = 27$

$f(a) \neq f(b)$.

$f(x)$ doesnot satisfy all the conditions of Rolle's Th.

Hence Rolle's Theorem can't be applied for $f(x)$.

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LAGRANGE'S MEAN VALUE THEOREM:

STATEMENT: Let $f(x)$ be a function, $x \in [a,b]$ such that,

i) $f(x)$ is continuous in closed interval $[a,b]$ and

ii) $f(x)$ is differentiable in open interval (a,b)

Then there exists atleast one point c in open interval (a,b) such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Another Form Of The Lagrange's Mean Value Theorem:

In Lagrange's mean value theorem, put $b = a+h$ so that, $h = b-a$.

Any point $x = c$ in open (a,b) i.e., in $(a, a+h)$ will be of the form $c = a+\theta h$ for some θ lying between 0 and 1.

Further, $\dfrac{f(b)-f(a)}{b-a} = \dfrac{f(a+h)-f(a)}{h}$

Now $f'(c) = \dfrac{f(b)-f(a)}{h} \Rightarrow f'(a+\theta h) = \dfrac{f(a+h)-f(a)}{h}$

Lagrange's mean value theorem can be stated alternately as,

Let $f(x)$ be a function, $x \in [a, a+h]$ such that,

i) $f(x)$ is continuous in closed interval $[a, a+h]$ and

ii) $f(x)$ is differentiable in open $(a, a+h)$

Then $\exists$ a positive real number θ, $0 < \theta < 1$ such that

$$f(a+h) = f(a) + h f'(a+\theta h).$$

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NOTE: Rolle's theorem can be deduced from Lagrange's theorem by substituting $f(a)=f(b)$.

$$\therefore f'(c) = \frac{f(b)-f(a)}{f(a)} = \frac{f(a)-f(a)}{f(a)} = 0.$$

PROBLEMS:

Verify Lagrange's Theorem and
1) Find 'c' of the Lagrange's theorem for $f(x) = (x-1)(x-2)(x-3)$ on $[0,4]$.

Soln: Given $f(x) = (x-1)(x-2)(x-3)$

$$f(x) = (x-1)(x^2-5x+6)$$

$$f(x) = x^3-5x^2+6x-x^2+5x-6$$

$$f(x) = x^3-6x^2+11x-6$$

is a single valued function

And interval is $[0,4]$

$$a=0,\ b=4$$

Verifying Lagrange's theorem we've to check the following conditions.

i) Continuity:

Given $f(x)$ is a polynomial function.

Every polynomial function is continuous

$\therefore f(x)$ is continuous $\forall\ [0,4]$

ii) Differentiability:

$$f(x) = x^3-6x^2+11x-6$$

$$f'(x) = 3x^2-12x+11$$

$f'(x)$ exists $\forall\ x \in (0,4)$

$\therefore f(x)$ is differentiable in $(0,4)$

$f(x)$ satisfies all conditions of Lagrange's mean value theorem.

Hence by Lagrange's mean value theorem, there exists a point c in open interval $(0,4)$ such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$f(b) = f(4) = (4)^3-6(4)^2+11(4)-6$$

$$f(4) = 64-96+44-6$$

$$f(4) = 6$$

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$f(a) = f(0) = (0)^3 - 6(0)^2 + 11(0) - 6$

$f(a) = f(0) = -6$

$\Rightarrow f'(c) = \frac{6-(-6)}{4-0} = \frac{12}{4} = 3$

$\Rightarrow 3c^2 - 12c + 11 = 3$

$\Rightarrow 3c^2 - 12c + 8 = 0$

$c = \frac{12 \pm \sqrt{144-96}}{6}$

$c = \frac{12 \pm \sqrt{48}}{6} = \frac{12 \pm 2\sqrt{12}}{6}$

$c = \frac{12 \pm 4\sqrt{3}}{6} = \frac{6 \pm 2\sqrt{3}}{3}$

$c = \frac{6 \pm 3.4641}{3}$

$c = 0.8453, 3.1547$

Both values lie in the open interval $(0,4)$.

$\therefore$ Lagrange's mean value theorem is Verified.

2> Verify Lagrange's mean value theorem for $f(x) = 2x^2 - 7x + 10$; $a = 2$; $b = 5$.

Soln: Given $f(x) = 2x^2 - 7x + 10$ is a single valued function and interval is $[2, 5]$

$a = 2$; $b = 5$

Verifying Lagrange's theorem we've to check the following conditions:

i> Continuity:

Given $f(x)$ is a polynomial fn.

Every polynomial fn is continuous

$\therefore f(x)$ is continuous $\forall$ $[2, 5]$.

ii> Differentiability:

$f(x) = 2x^2 - 7x + 10$

$f'(x) = 4x - 7$

$f'(x)$ exists $\forall$ $x \in (2, 5)$

$\therefore f(x)$ is differentiable in $(2,5)$.

$\therefore f(x)$ satisfies all the conditions of Lagrange's mean Value theorem

Hence by Lagrange's mean value theorem, there exists a point c in open interval $(2, 5)$ such that,

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$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$f(a) = f(2) = 2(2)^2 - 7(2) + 10 = 4$$

$$f(b) = f(5) = 2(5)^2 - 7(5) + 10 = 25$$

$$\Rightarrow 4c - 7 = \frac{25-4}{5-2}$$

$$\Rightarrow 4c - 7 = \frac{21}{3} = 7$$

$$\Rightarrow 4c = 7 + 7 = 14$$

$$\Rightarrow c = \frac{14}{4} = \frac{7}{2}$$

$$c = \frac{7}{2} \in (2,5)$$

Lagrange's mean value theorem is verified.

Then $\exists$ a point c in $(2,3)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{c}{\sqrt{c^2-4}} = \frac{\sqrt{5}-0}{1}$$

$$\frac{c^2}{c^2-4} = 5$$

$$5c^2 - 20 - c^2 = 0$$

$$4c^2 = 20$$

$$c^2 = 5$$

$$c = \sqrt{5} = 2.2361$$

3> Verify Lagrange's Mean Value Theorem for $f(x) = \sqrt{x^2-4}$ in $[2,3]$.

Soln: Given $f(x) = \sqrt{x^2-4}$ is a single valued function and interval is $[2,3]$

$$a = 2;\ b = 3.$$

Verifying Lagrange's theorem we've to check the following conditions.

i> Continuity:

Given $f(x) = \sqrt{x^2-4}$ is exists $\forall\ x \in [2,3]$

$\therefore f(x)$ is continuous $\forall\ x \in [2,3]$.

ii> Derivability:

$$f(x) = \sqrt{x^2-4}$$

$$f'(x) = \frac{1}{2\sqrt{x^2-4}}(2x) = \frac{x}{\sqrt{x^2-4}}$$

$f'(x)$ is not exists in $(2,3)$

$\therefore f(x)$ is not derivable in $(2,3)$.

Hence Lagrange's Mean Value theorem is not applicable.

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4) Verify Lagrange's Mean Value theorem for $f(x) = \log x$ in $[1, e]$

Soln: Given $f(x) = \log x$

is a single valued function

and interval is $[1, e]$

$a = 1;\ b = e$

Verifying Lagrange's theorem we've to check the following conditions

i) Continuity:

Given $f(x) = \log x$

is exists $\forall\ x \in [1, e]$

$\therefore f(x)$ is continuous $\forall\ [1, e]$

ii) Differentiability:

$f(x) = \log x$

$f'(x) = \frac{1}{x}$

$f'(x)$ is exists $\forall\ x \in (1, e)$

$\therefore f(x)$ is derivable in $(1, e)$

Hence $f(x)$ satisfies all the conditions of Lagrange's mean value theorem.

By Lagrange's mean value theorem, there exists a point c in open interval $(1, e)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$f'(c) = 1/x$

$f(b) = \log_e e = 1$

$f(a) = \log_e 1 = 0$

$$\Rightarrow \frac{1}{c} = \frac{1 - 0}{e - 1}$$

$$\Rightarrow c = e - 1 \in (1, e)$$

$\therefore$ Lagrange's mean value theorem is verified.

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5) State whether Lagrange's mean Value theorem can be applied for $f(x) = x^{4/3}$ in $[-1, 1]$

Soln: Given $f(x) = x^{4/3}$ is a single valued function and interval $[a,b]$ is $[-1,1]$

$a = -1, b = 1$

We've to check the following conditions for $f(x)$ whether applicable to Lagrange's mean value Theorem.

i) Continuity:

$f(x) = x^{4/3}$ is exists $\forall$ $x \in [-1, 1]$

$\therefore f(x)$ is continuous $\forall$ $[-1,1]$

ii) Differentiability:

$f(x) = x^{4/3}$

$f'(x) = \frac{4}{3} x^{1/3}$

$f'(x)$ exists $\forall$ $x \in (-1,1)$

$\therefore f(x)$ is Differentiable in $(-1,1)$

$f(x)$ satisfies all the conditions of Lagrange's mean Value theorem Lagrange's mean value Th., is applicable.

VERIFICATION:

Hence by Lagrange's mean Value theorem, there exists a point c in open interval $(-1,1)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{4}{3} \cdot c^{1/3} = \frac{f(1)-f(-1)}{1+1}$$

$$\frac{4}{3} \cdot c^{1/3} = \frac{1+1}{1+1} = 1$$

$$c^{1/3} = 1 \cdot \frac{3}{4} \Rightarrow c = 0.4219 \in (-1,1)$$

$\therefore$ Lagrange's mean Value theorem is verified.

6) For $f(x) = \frac{1}{x}$ in $[-1,1]$

Soln: Given $f(x) = 1/x$ is a single valued function. And interval $[a,b]$ is $[-1,1]$.

$a = -1, b = 1$

We've to check the following conditions for $f(x)$ whether applicable to LMVT.

i) Continuity: $f(x) = 1/x$ doesnot exists at $x = 0$ in $[-1,1]$

$\therefore f(x)$ is not continuous in $[-1,1]$

$f(x)$ is not satisfying the cond., of LMVT continuity in

$\therefore$ LMVT is not applicable.

7) F

Soln

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7) For $f(x) = x(x-1)(x-2)$ in $[0, \frac{1}{2}]$

Soln: Given $f(x) = x(x-1)(x-2)$ is a single valued function and interval $[a, b]$ is $[0, \frac{1}{2}]$

$a = 0, b = \frac{1}{2}$

We've to check the following conditions for $f(x)$ whether applicable to Lagrange's mean value theorem

i) Continuity:

$$f(x) = x(x-1)(x-2)$$

$$f(x) = x^3 - x^2 - 2x^2 + 2x$$

$$f(x) = x^3 - 3x^2 + 2x$$

$f(x)$ is a polynomial function. Every polynomial function is continuous.

$f(x)$ is continuous $\forall\, x \in [0, \frac{1}{2}]$

ii) Differentiability:

$$f(x) = x^3 - 3x^2 + 2x$$

$$f'(x) = 3x^2 - 6x + 2$$

$f'(x)$ exists $\forall\, x \in (0, \frac{1}{2})$

$\therefore f(x)$ is differentiable in $(0, \frac{1}{2})$.

$f(x)$ satisfies all the conditions of Lagrange's mean value theorem.

Lagrange's Mean Value Theorem is applicable.

Verification:

Hence by Lagrange's mean value theorem, there exists a point c in open interval $(0, \frac{1}{2})$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$3c^2 - 6c + 2 = \frac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0}$$

$$3c^2 - 6c + 2 = \frac{3/8}{1/2} = 3/4$$

$$3c^2 - 6c + 5/4 = 0$$

$$\Rightarrow 12c^2 - 24c + 5 = 0$$

$$c = \frac{6 \pm \sqrt{21}}{6} \in (0, \frac{1}{2})$$

Lagrange's Mean Value theorem is verified.

8) Find the region in which $f(x) = 1-4x-x^2$ is increasing and the region in which it is decreasing using Mean Value Theorem.

Soln: Given $f(x) = 1-4x-x^2$

$f(x)$ being a polynomial fun is Continuous on $[a,b]$ and differentiable on (a,b) for $a,b \in \mathbb{R}$.

$\therefore$ f Satisfies the Conditions of Lagrange mean value theorem on every interval

$$f'(x) = -4-2x = -2(2+x) \quad \forall x \in \mathbb{R}$$

$$f'(x) = 0 \text{ if } x = -2.$$

For $x < -2$, $f'(x) > 0$ and

For $x > -2$, $f'(x) < 0$

Hence $f(x)$ is strictly increasing on $(-\infty, -2)$ and strictly decreasing on $(-2, \infty)$.

9) Verify LMVT for
$$f(x) = \begin{cases} x \cdot \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases} \text{ in } [-1,1]$$

Soln: Given
$$f(x) = \begin{cases} x \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is a step function

And interval $[a,b]$ is $[-1,1]$.

Verifying LMVT we've to check the following conditions.

i) Continuity:

Continuity of $f(x)$ at $x=0$.

$$f(0) = 0.$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x \sin\tfrac{1}{x} = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x \cdot \sin 1/x = 0$$

$$f(0) = 0$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$\therefore$ $f(x)$ is continuous in $[-1,1]$

ii) Derivability of $f(x)$ at $x=0$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{x \sin 1/x}{x}$$

$$\lim_{x\to 0^-} \sin 1/x = \infty$$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x\sin 1/x}{x}$$

$$= \lim_{x\to 0^+} \sin 1/x = \infty$$

$\therefore f(x)$ is not derivable in $(-1, 1)$

Hence Lagrange's theorem is not applicable.

10) Verify LMVT for $f(x) = lx^2+mx+n$ in $[a, b]$.

Soln: Given $f(x) = lx^2+mx+n$ is a single valued function and interval $[a,b]$ is $[a,b]$.

Verifying LMVT we've to check the following conditions.

i) Continuity:

$f(x)$ is a polynomial fun,

Every polynomial function is continuous.

$\therefore f(x)$ is continuous function.

ii) Derivability:

$$f(x) = lx^2+mx+n$$

$$f'(x) = 2lx+m$$

$f'(x)$ exists $\forall$ (a,b)

$\therefore f(x)$ is Derivable in (a,b).

$\therefore f(x)$ satisfies all conditions of LMVT.

Hence by LMVT, there exist a point c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$2lc+m = \frac{lb^2+mb+n-(la^2+ma+n)}{b-a}$$

$$2lc+m = \frac{l(b^2-a^2)+m(b-a)}{b-a}$$

$$2lc+m = l(b+a)+m$$

$$2lc = l(b+a)$$

$$c = \frac{b+a}{2} \in (a,b).$$

$\therefore$ Lagrange's mean value theorem is verified.

11) Verify LMVT for $f(x) = x^3-3x-1$ in $\left[-\frac{11}{7}, \frac{13}{7}\right]$

Soln: Given $f(x) = x^3-3x-1$ is a single valued function.

And interval $[a,b]$ is $\left[-\frac{11}{7}, \frac{13}{7}\right]$.

Verifying LMVT we've to check the following conditions.

i) Continuity:

$f(x)$ is a polynomial function.

Every polynomial function is Continuous.

$\therefore f(x)$ is Continuous function.

ii) Differentiability:

$$f(x) = x^3 - 3x - 1$$

$$f'(x) = 3x^2 - 3$$

$f'(x)$ exists $\forall \left(-\frac{11}{7}, \frac{13}{7}\right)$.

$\therefore f(x)$ is Differentiable in $\left(-\frac{11}{7}, \frac{13}{7}\right)$.

$\therefore f(x)$ satisfies all the conditions of LMVT.

Hence by LMVT, there exist a point c in (a,b) i.e., $\left(-\frac{11}{7}, \frac{13}{7}\right)$ Such that,

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$3c^2 - 3 = \frac{f\left(\frac{13}{7}\right) - f\left(-\frac{11}{7}\right)}{\frac{13}{7} + \frac{11}{7}}$$

$$3c^2 - 3 = 0$$

$$3c^2 = 3 \Rightarrow c^2 = 1$$

$$c = \pm 1$$

$$c = \pm 1 \in \left(-\frac{11}{7}, \frac{13}{7}\right)$$

LMVT is verified for $f(x)$.

* **Deductions Of LMVT:** →

✓ If $f'(x) = 0 \ \forall \ x \in (a,b)$ then $f(x)$ is said to be a constant function in $[a,b]$.

✓ If $f'(x) > 0, \ \forall \ x \in (a,b)$ then $f(x)$ is said to be an increasing function in $[a,b]$.

✓ If $f'(x) < 0, \ \forall \ x \in (a,b)$ then $f(x)$ is said to be decreasing function in $[a,b]$.

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12) Calculate an approximate value of $\sqrt{85}$ using Lagrange's MVT.

<u>Soln:</u> $\sqrt{85} = \sqrt{9^2+4}$

By LMVT, there exists $\theta \in (0,1)$ such that,

$$f'(a+\theta h) = \frac{f(a+h)-f(a)}{h}$$

$$a = 9^2 ; h = 4$$

$$f(x) = \sqrt{x} ; \quad f'(x) = \frac{1}{2\sqrt{x}}$$

$$f'(81+4\theta) = \frac{f(81+4)-f(81)}{4}$$

$$\frac{1}{2\sqrt{81+4\theta}} = \frac{\sqrt{85}-\sqrt{81}}{4}$$

$$\frac{1}{\sqrt{81+4\theta}} = \frac{\sqrt{85}-9}{2}$$

$$\sqrt{81+4\theta} = \frac{2}{\sqrt{85}-9}$$

$$\sqrt{85}-9 = \frac{2}{\sqrt{81+4\theta}}$$

$$\sqrt{85}-9 = \frac{2}{\sqrt{81+4^2(\frac{1}{x})}} \quad \begin{cases} \theta = \frac{1}{2} \\ \in (0,1) \end{cases}$$

$$\sqrt{85}-9 = \frac{2}{\sqrt{83}} = \frac{2}{9.1104}$$

$$9.21 \quad \sqrt{85} = 9 + \frac{2}{9.1104}$$

$$\sqrt{85} = 9 + 0.2193$$

$$\sqrt{85} = 9.2193$$

13) $\sqrt[6]{65} =$

<u>Soln:</u> $\sqrt[6]{65} = (65)^{1/6}$

$$= (2^6+1)^{1/6}$$

By LMVT, there exists $\theta \in (0,1)$ such that,

$$f'(a+\theta h) = \frac{f(a+h)-f(a)}{h}$$

$$a = 2^6 ; h = 1$$

$$f(x) = x^{1/6} ; \quad f'(x) = \frac{1}{6x^{5/6}}$$

$$f'(64+\theta) = \frac{f(64+1)-f(64)}{1}$$

$$\frac{1}{6(64+\theta)^{5/6}} = \sqrt[6]{65} - \sqrt[6]{64}$$

$$\frac{1}{6(64+\theta)^{5/6}} = \sqrt[6]{65} - (64)^{1/6}$$

$$\frac{1}{6(64+\theta)^{5/6}} = \sqrt[6]{65} - (2^6)^{1/6}$$

$$\sqrt[6]{65} = \frac{1}{6(64+\theta)^{5/6}} + 2$$

$$\sqrt[6]{65} = \frac{1}{6(64+\frac{1}{2})^{5/6}} + 2$$

$$\sqrt[6]{65} = \frac{1}{6(129)^{5/6}} + 2$$

$$\sqrt[6]{65} = \frac{1}{344.331} + 2 = 0.0029 + 2$$

$$\sqrt[6]{65} = 2.0029$$

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Inequality Problems:

1) Apply MVT to prove that

$x > \sin x > x - \frac{1}{6}x^3, \forall\ 0 \le x \le \pi/2$

Soln: Let $f(x) = \sin x$ on $[0, \pi/2]$

$f(0) = \sin 0 = 0$

$f(\pi/2) = \sin \pi/2 = 1$.

Case (i): To prove that,

$x > \sin x$

Let $f(x) = x - \sin x > 0$

$f'(x) = 1 - \cos x > 0$

for $x \in (0, \pi/2)$.

$\therefore f(x)$ is an increasing fn of x in $[0, \pi/2]$.

Hence for $x > 0$

$\Rightarrow f(x) > f(0)$

$\Rightarrow x - \sin x > 0$

$\Rightarrow x > \sin x \longrightarrow (1)$

Case (ii): To prove that,

$\sin x > x - \frac{1}{6}x^3$

Let $g(x) = \sin x - x + \frac{1}{6}x^3$

$g'(x) = \cos x - 1 + \frac{x^2}{2} > 0$

$\forall\ x \in (0, \pi/2)$

$\therefore g(x)$ is an increasing fn of x in $[0, \pi/2]$.

Hence for $x > 0$

$\Rightarrow g(x) > g(0)$

$\Rightarrow \sin x - x + \frac{1}{6}x^3 > 0$

$\Rightarrow \sin x > x - \frac{1}{6}x^3 \longrightarrow (2)$

From (1) and (2), we have

$x > \sin x > x - \frac{1}{6}x^3$

Hence proved

2) Prove that by MVT

$\frac{\pi}{6} + \frac{1}{2\sqrt{3}} < \sin^{-1}\left(\frac{3}{4}\right) < \frac{\pi}{6} + \frac{1}{\sqrt{7}}$

Soln: Let $f(x) = \sin^{-1} x$ and an interval is $[a, b]$.

$f'(x) = \frac{1}{\sqrt{1-x^2}}$

By LMVT there exists a point c in (a, b) such that $a < c < b$

$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\sin^{-1} b - \sin^{-1} a}{b - a}$

$f'(c) = \frac{1}{\sqrt{1-c^2}}$

Now, $a < c < b$

$\Rightarrow a^2 < c^2 < b^2$

$\Rightarrow 1 - a^2 > 1 - c^2 > 1 - b^2$

$\Rightarrow \sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$

$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$

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$$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{f(b)-f(a)}{b-a} < \frac{1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1}(b)-\sin^{-1}(a)}{b-a} < \frac{1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{b-a}{\sqrt{1-a^2}} < \sin^{-1}(b)-\sin^{-1}(a) < \frac{b-a}{\sqrt{1-b^2}}$$

Let $a = 1/2$; $b = 3/4$

$$\frac{\frac{3}{4}-\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} < \sin^{-1}\left(\frac{3}{4}\right)-\sin^{-1}\left(\frac{1}{2}\right) < \frac{\frac{3}{4}-\frac{1}{2}}{\sqrt{1-\frac{9}{16}}}$$

$$\frac{1}{2\sqrt{3}} < \sin^{-1}\left(\frac{3}{4}\right)-\frac{\pi}{6} < \frac{1}{\sqrt{7}}$$

$$\frac{1}{2\sqrt{3}}+\frac{\pi}{6} < \sin^{-1}\left(\frac{3}{4}\right) < \frac{1}{\sqrt{7}}+\frac{\pi}{6}$$

$$\frac{\pi}{6}+\frac{1}{2\sqrt{3}} < \sin^{-1}\left(\frac{3}{4}\right) < \frac{\pi}{6}+\frac{1}{\sqrt{7}}$$

Hence Proved.

3> For $0<a<b<1$, Prove that, $1-\frac{a}{b} < \log\frac{b}{a} < \frac{b}{a}-1$. Hence show that $\frac{1}{6} < \log\frac{6}{5} < \frac{1}{5}$.

Soln: Let $f(x) = \log x$ is a single valued function. And interval is $[a,b]$.

i> $f(x)$ exists $\forall\ x \in [a,b]$

$\therefore f(x)$ is continuous in $[a,b]$

ii> $f(x) = \log x$

$f'(x) = \frac{1}{x}$ exists $\forall\ x \in (a,b)$

$\therefore f(x)$ is derivable in (a,b)

The fn $f(x)$ satisfies all the conditions of LMVT then $\exists$ $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{1}{c} = \frac{\log b - \log a}{b-a}$$

$$\frac{1}{c} = \frac{\log(b/a)}{b-a}$$

Now, $a < c < b$

$$\Rightarrow \frac{1}{a} > \frac{1}{c} > \frac{1}{b}$$

$$\Rightarrow \frac{1}{a} > \frac{\log(b/a)}{b-a} > \frac{1}{b}$$

$$\Rightarrow \frac{b-a}{a} > \log(b/a) > \frac{b-a}{b}$$

$$\Rightarrow \frac{b}{a}-1 > \log(b/a) > 1-\frac{a}{b}$$

$$\Rightarrow 1-\frac{a}{b} < \log(b/a) < \frac{b}{a}-1$$

put $b=6$; $a=5$

$$\Rightarrow 1-\frac{5}{6} < \log(6/5) < \frac{6}{5}-1$$

$$\frac{1}{6} < \log(6/5) < \frac{1}{5}$$

Hence Proved.

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4> Show that $\frac{h}{1+h} < \log(1+h) < h$ where $h > 0$. $x > \log(1+x) > \frac{x}{1+x}, x > 0$

Soln: Let $f(x) = \log(1+x)$ is a single valued function

And interval is $[0, h]$.

i> $f(x)$ exists $\forall\ x \in [0, h]$

$\therefore f(x)$ is continuous in $[0, h]$

ii> $f(x) = \log(1+x)$

$f'(x) = \frac{1}{1+x}$ exists $\forall\ (0, h)$

$\therefore f(x)$ is derivable in $(0, h)$

The fn $f(x)$ satisfies all the conditions of LMVT then $\exists$ $c \in (0, h)$ such that

$$f'(c) = \frac{f(h) - f(0)}{h - 0}$$

$$\frac{1}{1+c} = \frac{\log(1+h) - 0}{h}$$

$$\frac{1}{1+c} = \frac{\log(1+h)}{h}$$

Now, $0 < c < h$

$$1 < c+1 < h+1$$

$$1 > \frac{1}{1+c} > \frac{1}{1+h}$$

$$1 > \frac{\log(1+h)}{h} > \frac{1}{1+h}$$

$$h > \log(1+h) > \frac{h}{1+h}$$

$$\Rightarrow \frac{h}{1+h} < \log(1+h) < h$$

Hence Proved.

5> Prove that, $\frac{b-a}{1+a^2} > \tan^{-1} b - \tan^{-1} a > \frac{b-a}{1+b^2}$ if $0 < a < b < 1$. Hence Deduce

i> $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}$

ii> $\frac{5\pi + 4}{20} < \tan^{-1}(2) < \frac{\pi + 2}{4}$

Soln: Consider $f(x) = \tan^{-1} x$ in $[a, b]$ for $0 < a < b < 1$.

Since $f(x)$ is continuous in closed interval $[a, b]$ and $f(x)$ is derivable in open (a, b)

The fn $f(x)$ satisfies all the conditions of LMVT then $\exists$ $c \in (a, b)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f'(x) = \frac{1}{1+x^2} \; ; \; f'(c) = \frac{1}{1+c^2}$$

$$\Rightarrow \frac{1}{1+c^2} = \frac{\tan^{-1} b - \tan^{-1} a}{b - a}$$

Now, $a < c < b$

$$\Rightarrow a^2 < c^2 < b^2$$

$$\Rightarrow 1 + a^2 < 1 + c^2 < 1 + b^2$$

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$$\Rightarrow \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

$$\Rightarrow \frac{1}{1+a^2} > \frac{\tan^{-1}b - \tan^{-1}a}{b-a} > \frac{1}{1+b^2}$$

$$\Rightarrow \frac{b-a}{1+a^2} > \tan^{-1}b - \tan^{-1}a > \frac{b-a}{1+b^2}$$

$$\Rightarrow \frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$$

HENCE PROVED.

DEDUCTIONS:

i) Let $a = 1$; $b = \frac{4}{3}$

We've,

$$\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$$

$$\frac{\frac{4}{3}-1}{1+\frac{16}{9}} < \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}(1) < \frac{\frac{4}{3}-1}{1+1}$$

$$\frac{\frac{1}{3}}{\frac{25}{9}} < \tan^{-1}\left(\frac{4}{3}\right) - \frac{\pi}{4} < \frac{\frac{1}{3}}{2}$$

$$\frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) - \frac{\pi}{4} < \frac{1}{6}$$

$$\frac{3}{25} + \frac{\pi}{4} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{1}{6} + \frac{\pi}{4}$$

$$\Rightarrow \frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}$$

Hence Proved

ii) Let $a = 1$; $b = 2$

We've

$$\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$$

$$\frac{2-1}{1+4} < \tan^{-1}(2) - \tan^{-1}(1) < \frac{2-1}{1+1}$$

$$\Rightarrow \frac{1}{5} < \tan^{-1}(2) - \frac{\pi}{4} < \frac{1}{2}$$

$$\Rightarrow \frac{1}{5} + \frac{\pi}{4} < \tan^{-1}(2) < \frac{1}{2} + \frac{\pi}{4}$$

$$\Rightarrow \frac{4+5\pi}{20} < \tan^{-1}(2) < \frac{4+2\pi}{8}$$

$$\Rightarrow \frac{5\pi+4}{20} < \tan^{-1}(2) < \frac{\pi+2}{4}$$

Hence Proved.

6) Show that

$$\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right) > \frac{\pi}{3} - \frac{1}{8}$$

Soln: Let $f(x) = \cos^{-1}x$ and an interval $[a, b]$.

Given $f(x)$ is continuous in $[a, b]$.

$$f'(x) = \frac{-1}{\sqrt{1-x^2}}$$

$f'(x)$ exists $\forall\ x \in (a, b)$.

$\therefore f(x)$ is derivable at (a, b).

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$f(x)$ satisfies all the conditions of LMVT then $\exists\ c \in (a,b)$ such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$f'(c) = \frac{\cos^{-1}b - \cos^{-1}a}{b-a} = \frac{-1}{\sqrt{1-c^2}}$$

where $a < c < b$

$$\Rightarrow a^2 < c^2 < b^2$$

$$\Rightarrow 1-a^2 > 1-c^2 > 1-b^2$$

$$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{-1}{\sqrt{1-a^2}} > \frac{-1}{\sqrt{1-c^2}} > \frac{-1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{-1}{\sqrt{1-a^2}} > \frac{-1}{\sqrt{1-c^2}} > \frac{-1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{-1}{\sqrt{1-a^2}} > \frac{\cos^{-1}b - \cos^{-1}a}{b-a} > \frac{-1}{\sqrt{1-b^2}}$$

$$\Rightarrow -\frac{(b-a)}{\sqrt{1-a^2}} > \cos^{-1}b - \cos^{-1}a > -\frac{(b-a)}{\sqrt{1-b^2}}$$

Let $b = 3/5$, $a = 1/2$. Then

$$\Rightarrow \frac{-\left(\frac{3}{5}-\frac{1}{2}\right)}{\sqrt{1-\frac{1}{4}}} > \cos^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(\frac{1}{2}\right) > \frac{-\left(\frac{3}{5}-\frac{1}{2}\right)}{\sqrt{1-\frac{9}{25}}}$$

$$\Rightarrow \frac{-\frac{1}{10}}{\frac{\sqrt{3}}{2}} > \cos^{-1}\left(\frac{3}{5}\right) - \frac{\pi}{3} > \frac{-\frac{1}{10}}{\frac{4}{5}}$$

$$\Rightarrow -\frac{1}{5\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right) - \frac{\pi}{3} > -\frac{1}{8}$$

$$\Rightarrow \frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right) > \frac{\pi}{3} - \frac{1}{8}$$

Hence Proved.

7) show that $\frac{1}{b} < \frac{\log b}{b-1} < 1$ for $b > 1$

Soln: Let $f(x) = \log x$ is a single valued function and interval $[a,b]$ is $[1,b]$

$\log x$ exists $\forall\ x \in [1,b]$

$\therefore f(x) = \log x$ continuous in $[1,b]$.

$f'(x) = \frac{1}{x}$ exists $\forall\ x \in (1,b)$.

$\therefore f(x)$ is derivable in $(1,b)$.

$\therefore f(x)$ satisfies all the conditions of LMVT then $\exists\ c \in (a,b)$ i.e., $c \in (1,b)$ such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(b)-f(1)}{b-1}$$

$$f'(c) = \frac{\log b - \log 1}{b-1} = \frac{\log b}{b-1} = \frac{1}{c}$$

Now $1 < c < b$

$$1 > \frac{1}{c} > \frac{1}{b}$$

$$1 > \frac{\log b}{b-1} > \frac{1}{b} \Rightarrow \frac{1}{b} < \frac{\log b}{b-1} < 1$$

Hence Proved.

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8) Show that for any $x>0$,

$1+x < e^x < 1+xe^x$

Soln: Given $f(x) = e^x$ in $[0,x]$

i) $f(x)$ exists $\forall\ x \in [0,x]$

$f(x)$ is continuous in $[0,x]$

ii) $f(x) = e^x \Rightarrow f'(x) = e^x$

$\therefore f'(x)$ exists $\forall\ x \in (0,x)$

$f(x)$ is derivable in $(0,x)$

$\therefore f(x)$ satisfies all the conditions of LMVT.

Then $\exists$, $c \in (0,x)$ such that

$$f'(c) = \frac{f(x)-f(0)}{x-0}$$

$$e^c = \frac{e^x - e^0}{x-0}$$

$$e^c = \frac{e^x - 1}{x}$$

Now, $0 < c < x$

$$e^0 < e^c < e^x$$

$$1 < \frac{e^x-1}{x} < e^x$$

$$x < e^x - 1 < xe^x$$

$$1+x < e^x < 1+xe^x$$

Hence Proved.

9) show that

$h < \sin^{-1}h < \frac{h}{\sqrt{1-h^2}}$ in $[0 \le h \le 1]$

Soln: Given $f(x) = \sin^{-1}x$ and in interval $[0,h]$.

Where $0 \le h \le 1$.

i) $f(x)$ exists $\forall\ x \in [0,h]$

$f(x)$ is continuous in $[0,h]$

ii) $f'(x) = \frac{1}{\sqrt{1-x^2}}$

$f'(x)$ exists $\forall\ x \in (0,h)$

$f(x)$ is derivable in $(0,h)$

$\therefore f(x)$ satisfies all the conditions of LMVT.

$\therefore$ LMVT is applicable.

Then $\exists$ $c \in (0,h)$ such that,

$$f'(c) = \frac{f(h)-f(0)}{h-0}$$

$$\frac{1}{\sqrt{1-c^2}} = \frac{\sin^{-1}h - \sin^{-1}0}{h-0}$$

$$\frac{1}{\sqrt{1-c^2}} = \frac{\sin^{-1}h}{h} \longrightarrow (1)$$

Now, $0 < c < h$

$\Rightarrow 0^2 < c^2 < h^2$

$\Rightarrow 0 > -c^2 > -h^2$

$\Rightarrow 1 > 1-c^2 > 1-h^2$

$\Rightarrow 1 > \sqrt{1-c^2} > \sqrt{1-h^2}$

$\Rightarrow 1 < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-h^2}}$

$\Rightarrow 1 < \frac{\sin^{-1}h}{h} < \frac{1}{\sqrt{1-h^2}}$ [$\because$ from(1)]

$\Rightarrow h < \sin^{-1}h < \frac{h}{\sqrt{1-h^2}}$

Hence the result follows.

10> Prove that

$\frac{\pi}{6} + \frac{1}{5\sqrt{3}} < \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{6} + \frac{1}{8}$ (or)

$\frac{\pi}{6} + \frac{\sqrt{3}}{15} < \sin^{-1}(0.6) < \frac{\pi}{6} + \frac{1}{8}$.

Soln: Let $f(x) = \sin^{-1}x$,

$\sin^{-1}x$ is continuous and differentiable in $[0,\pi]$ & $(0,\pi)$

$f'(x) = \frac{1}{\sqrt{1-x^2}}$

By LMVT, $\exists$ a point c in $(0,\pi)$ such that $0 < c < \pi$ and

$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{1}{\sqrt{1-c^2}}$

$\frac{1}{\sqrt{1-c^2}} = \frac{\sin^{-1}b - \sin^{-1}a}{b-a}$

Now $a < c < b$

$a^2 < c^2 < b^2$

$\Rightarrow 1-a^2 > 1-c^2 > 1-b^2$

$\Rightarrow \sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$

$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$

$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1}b - \sin^{-1}a}{b-a} < \frac{1}{\sqrt{1-b^2}}$

$\Rightarrow \frac{b-a}{\sqrt{1-a^2}} < \sin^{-1}b - \sin^{-1}a < \frac{b-a}{\sqrt{1-b^2}}$

put $a = 1/2$; $b = 3/5$

Then,

$\frac{\frac{3}{5}-\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} < \sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\frac{1}{2}\right) < \frac{\frac{3}{5}-\frac{1}{2}}{\sqrt{1-\frac{9}{25}}}$

$\Rightarrow \frac{2}{10\sqrt{3}} < \sin^{-1}\left(\frac{3}{5}\right) - \frac{\pi}{6} < \frac{1}{10 \cdot \frac{4}{5}}$

$\Rightarrow \frac{\pi}{6} + \frac{1}{5\sqrt{3}} < \sin^{-1}\left(\frac{3}{5}\right) < \frac{1}{8} + \frac{\pi}{6}$

$\Rightarrow \frac{\pi}{6} + \frac{\sqrt{3}}{15} < \sin^{-1}(0.6) < \frac{\pi}{6} + \frac{1}{8}$.

Hence Proved.

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GENERALISED MEAN VALUE THEOREMS:

TAYLOR's THEOREM:

Let $f(x)$ be a function and if $f:[a,b] \to \mathbb{R}$ is such that

a) $f^{(n-1)}(x)$ is continuous on $[a,b]$

b) $f^{(n-1)}(x)$ is derivable on (a,b) (or) $f^{(n)}(x)$ exists on (a,b)

and $p \in z^{+}$ then there exist a point $c \in (a,b)$ such that

$$f(b) = f(a) + \frac{b-a}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n$$

where $R_n = \dfrac{(b-a)^p (b-c)^{n-p} f^{(n)}(c)}{(n-1)!\,p}$

NOTE: Various Taylor's Remainders are as follows:

1) Schlomilch – Roche's form of Remainder

$$R_n = \frac{(b-a)^p \cdot (b-c)^{n-p} \cdot f^n(c)}{(n-1)!\,p}$$

2) Lagrange's form of Remainder

Put $p=n$ in S-R form of Remainder, we get

$$R_n = \frac{(b-a)^n \cdot f^n(c)}{n!}$$

3) Cauchy's form of Remainder

Put $p=1$, in S-R form of Remainder, we get

$$R_n = \frac{(b-a)(b-c)^{n-1} \cdot f^n(c)}{(n-1)!}$$

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Another form of Taylor's Theorem :

Let $f(x)$ be a function and if $f:[a,a+h] \longrightarrow \mathbb{R}$ is such that,

a) $f^{(n-1)}(x)$ is continuous on $[a,a+h]$

b) $f^{(n-1)}(x)$ is derivable on $(a,a+h)$

and $p \in z^+$ then there exists a real number $0<\theta<1$ such that

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n$$

Where $R_n = \dfrac{h^n (1-\theta)^{n-p} f^{(n)}(a+\theta h)}{p(n-1)!}$

(It is obtained from Taylor's Theorem $b = a+h$ and $c = a+\theta h$)

NOTE :

1) Schlomilch - Roche's form of Remainder

$$R_n = \frac{h^n (1-\theta)^{n-p} f^{(n)}(a+\theta h)}{p(n-1)!}$$

2) Lagrange's form of Remainder

put $p=n$ in S-R form of Remainder, we get

$$R_n = \frac{h^n \cdot f^n(a+\theta h)}{n!}$$

3) Cauchy's form of Remainder

put $p=1$ in S-R form of Remainder, we get

$$R_n = \frac{h^n (1-\theta)^{n-1} f^{(n)}(a+\theta h)}{(n-1)!}$$

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Taylor's Series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^n(a) + \cdots$$

is called Taylor's series expansion of $f(x)$ about $x = a$.
or powers of '$x-a$'. **

PROBLEMS:

1) Obtain the Taylor's series expansion of $\sin x$ in powers of $x - \pi/2$. (1)

Soln: We know that, The Taylor's series expansion of $f(x)$ in powers of $x-a$ is given by,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots \quad (2)$$

Given that,
Here $f(x) = \sin x$
$x - a = x - \pi/2$
$\Rightarrow a = \pi/2$ (1)

(4)

$f(x) = \sin x \Rightarrow f(\pi/2) = 1$

$f'(x) = \cos x \Rightarrow f'(\pi/2) = 0$

$f''(x) = -\sin x \Rightarrow f''(\pi/2) = -1$

$f'''(x) = -\cos x \Rightarrow f'''(\pi/2) = 0$

$f^4(x) = \sin x \Rightarrow f^4(\pi/2) = 1$

$f^5(x) = \cos x \Rightarrow f^5(\pi/2) = 0$

$\vdots$

Hence

$$\sin x = f\left(\tfrac{\pi}{2}\right) + \left(x-\tfrac{\pi}{2}\right)f'\left(\tfrac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^2}{2!}f''\left(\tfrac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^3}{3!}f'''\left(\tfrac{\pi}{2}\right) + \cdots \quad (3)$$

(5) (4) ⇒

$$\sin x = 1 + \left(x-\tfrac{\pi}{2}\right)(0) + \frac{\left(x-\frac{\pi}{2}\right)^2}{2!}(-1) + \frac{\left(x-\frac{\pi}{2}\right)^3}{3!}(0) + \frac{\left(x-\frac{\pi}{2}\right)^4}{4!}(1) + \frac{\left(x-\frac{\pi}{2}\right)^5}{5!}(0) + \cdots$$

$$\sin x = \left[1 - \frac{\left(x-\frac{\pi}{2}\right)^2}{2!} + \frac{\left(x-\frac{\pi}{2}\right)^4}{4!} - \cdots\right.$$

2) Obtain the Taylor's Series expansion of $\cos x$ in powers of $x + \pi$. (2)

Soln: The Taylor's series expansion of $f(x)$ about $x = a$ is,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^n(a) + \cdots$$

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Here $f(x) = \cos x$ $\quad \cos n\pi = (-1)^n$

$a = -\pi$

$f(x) = \cos x \Rightarrow f(-\pi) = -1$

$f'(x) = -\sin x \Rightarrow f'(-\pi) = 0$

$f''(x) = -\cos x \Rightarrow f''(-\pi) = 1$

$f'''(x) = +\sin x \Rightarrow f'''(-\pi) = 0$

$f^4(x) = \cos x \Rightarrow f^4(-\pi) = -1$

$f^5(x) = -\sin x \Rightarrow f^5(-\pi) = 0$

$f^6(x) = -\cos x \Rightarrow f^6(-\pi) = 1$

$\vdots$

$$\cos x = f(-\pi) + (x+\pi) f'(-\pi) + \frac{(x+\pi)^2}{2!} f''(-\pi) + \frac{(x+\pi)^3}{3!} f'''(-\pi) + \cdots$$

$$\cos x = -1 + 0 + \frac{(x+\pi)^2}{2!}(1) + 0 + \frac{(x+\pi)^4}{4!}(-1) + 0 + \cdots$$

$$\cos x = -1 + \frac{(x+\pi)^2}{2!} - \frac{(x+\pi)^4}{4!} + \cdots$$

3> Obtain the Taylor's series expansion of $2x^3 + 7x^2 + x - 6$ in powers of $x - 2$. (3)

Soln: We know that,

By Taylor's series expansion of $f(x)$ in powers of $x - a$

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

Given $f(x) = 2x^3 + 7x^2 + x - 6$

$\Rightarrow f(2) = 2(8) + 7(4) + 2 - 6$

$\Rightarrow f(2) = 40$

$f'(x) = 6x^2 + 14x + 1$

$f'(2) = 6(4) + 28 + 1 = 53$

$f''(x) = 12x + 14$

$\Rightarrow f''(2) = 12(2) + 14 = 38$

$f'''(x) = 12 \Rightarrow f'''(2) = 12$

$$2x^3 + 7x^2 + x - 6 = 40 + (x-2)(53) + \frac{(x-2)^2}{2!}(38) + \frac{(x-2)^3}{3!}(12) + \cdots$$

$$= 40 + (x-2)53 + (x-2)^2(19) + (x-2)^3(2) + \cdots$$

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4> Verify Taylor's theorem for $f(x) = (1-x)^{5/2}$ with Lagrange's form of remainder upto 2 terms in the interval $[0, 1]$ (6)

Soln: Consider

$$f(x) = (1-x)^{5/2} \text{ in } [0,1]$$

i> $f(x)$, $f'(x)$ exists $\forall\, x \in [0,1]$

$f'(x)$ is continuous in $[0,1]$

ii> $f''(x)$ is differentiable in $(0,1)$

Thus $f(x)$ satisfies the conditions of Taylor's Theorem.

We consider Taylor's Theorem with Lagrange's form of Remainder, upto n terms is,

$$f(b) = f(a) + \frac{b-a}{1!} f'(a) + \dots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(b-a)^n f^n(c)}{n!} \quad **$$

Here $n = 2$; $a = 0$; $b = 1$

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(c)$$

$$\Rightarrow f(1) = f(0) + (1-0) f'(0) + \frac{1}{2} f''(c)$$

$$\Rightarrow f(1) = f(0) + f'(0) + \frac{1}{2} f''(c). \quad (1)$$

$$f(x) = (1-x)^{5/2}$$

$$f(1) = (1-1)^{5/2} = 0$$

$$f(0) = (1-0)^{5/2} = 1$$

$$f'(x) = \frac{-5}{2}(1-x)^{3/2}$$

$$f'(0) = -\frac{5}{2}(1-0)^{3/2} = -\frac{5}{2}$$

$$f''(x) = +\frac{5}{2} \cdot \frac{3}{2}(1-x)^{\frac{1}{2}} = \frac{15}{4}(1-x)^{1/2}$$

$(1) \Longrightarrow$

$$0 = 1 + \left(-\frac{5}{2}\right) + \frac{1}{2} \cdot \frac{15}{4}(1-c)^{1/2}$$

$$0 = 1 - \frac{5}{2} + \frac{15}{8}(1-c)^{1/2}$$

$$0 = -\frac{3}{2} + \frac{15}{8}(1-c)^{1/2}$$

$$\frac{15}{8}(1-c)^{1/2} = \frac{3}{2}$$

$$(1-c)^{1/2} = \frac{\cancel{3}}{\cancel{2}} \times \frac{\cancel{8}^{4}}{\cancel{15}_{5}} = \frac{4}{5}$$

$$\Rightarrow (1-c) = \frac{16}{25}$$

$$\Rightarrow c = 1 - \frac{16}{25} = \frac{9}{25}$$

$$\Rightarrow c = 0.36 \in (0, 1).$$

c lies between 0 and 1.

Thus Taylor's Theorem is Verified.

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5> Obtain Taylor's Series expansion of $\log_e x$ in powers of $x-1$ and hence evaluate $\log_e 1.1$ correct to 4 decimals. (A)

Soln: Given $f(x) = \log_e x$

$$x - a = x - 1$$

$$\Rightarrow a = 1$$

Taylor's Series expansion of $f(x)$ in powers of $(x-a)$ is

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^4}{4!} f^4(a) + \cdots$$

$$\log_e x = f(1) + \frac{(x-1)}{1!} f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \frac{(x-1)^4}{4!} f^4(1) + \cdots \longrightarrow (1)$$

$$f(x) = \log_e x \Rightarrow f(1) = \log_e 1 = 0$$

$$f'(x) = \frac{1}{x} \Rightarrow f'(1) = \frac{1}{1} = 1$$

$$f''(x) = -\frac{1}{x^2} \Rightarrow f''(1) = -\frac{1}{1} = -1$$

$$f'''(x) = +2 \cdot \frac{1}{x^3} \Rightarrow f'''(1) = \frac{2}{1} = 2$$

$$f^4(x) = \frac{-6}{x^4} \Rightarrow f^4(1) = \frac{-6}{1} = -6$$

$\vdots$

Substituting above values in (1),

$$\Rightarrow \log_e x = 0 + (x-1)(1) + \frac{(x-1)^2}{2}(-1) + \frac{(x-1)^3}{6}(2) + \frac{(x-1)^4}{24}(-6) + \cdots$$

$$\Rightarrow \log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

$$\Rightarrow \log_e 1.1 = (1.1-1) - \frac{(1.1-1)^2}{2} + \frac{(1.1-1)}{3} - \frac{(1.1-1)^4}{4} + \cdots$$

$$\Rightarrow \log_e 1.1 = 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \cdots$$

$$\Rightarrow \log_e 1.1 = 0.1003 - 0.0050$$

$$\Rightarrow \log_e 1.1 = 0.0953$$

** Taylor's Series expansion of $f(x+h)$ in powers of h is,

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$

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6) Find the first 4 terms in the expansion of $\log[\sin(x+h)]$ in ascending powers of h. (7)

Soln: Given

$$f(x+h) = \log[\sin(x+h)]$$

$$\Rightarrow f(x) = \log(\sin x)$$

Taylor's series expansion of $f(x+h)$ in powers of h

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$

$$\Rightarrow \log[\sin(x+h)] = \log(\sin x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots \quad (1)$$

$$f(x) = \log(\sin x) \Rightarrow f'(x) = \frac{1}{\sin x}\cdot\cos x$$

$$\Rightarrow f'(x) = \cot x$$

$$\Rightarrow f''(x) = -\csc^2 x$$

$$\Rightarrow f'''(x) = -2\csc x(-\csc x\cdot\cot x)$$

$$f'''(x) = 2\csc^2 x\cdot\cot x.$$

$\vdots$

Substituting above values in eqn (1),

$$\log[\sin(x+h)] = \log(\sin x) + \frac{h}{1!}\cot x + \frac{h^2}{2!}(-\csc^2 x) + \frac{h^3}{3\times 2}2\csc^2 x\cot x$$

$$\log[\sin(x+h)] = \log(\sin x) + h\cot x - \frac{h^2}{2}\csc^2 x + \frac{h^3}{3}\csc^2 x\cot x$$

7) Calculate the approximate value of $\sqrt{10}$ to 4 decimal places using Taylor's Series. (8)

Soln: $\sqrt{10} = \sqrt{9+1}$

Let $f(x+h) = \sqrt{x+h} = \sqrt{10} = \sqrt{9+1}$

$$\Rightarrow x = 9;\ h = 1$$

$$f(x) = \sqrt{x}$$

Taylor's series expansion of $f(x+h)$ in powers h is,

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$

$$\sqrt{10} = f(9) + \frac{1}{1!}f'(9) + \frac{1}{2!}f''(9) + \frac{1}{3!}f'''(9) + \cdots \quad (1)$$

$$f(x) = \sqrt{x} = x^{1/2} \Rightarrow f(9) = (9)^{1/2} = 3$$

$$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

$$f''(x) = \frac{1}{2}\cdot\left(-\frac{1}{2}\right)x^{-3/2} \Rightarrow f''(9) = \frac{-1}{4}\left(\frac{1}{27}\right) = \frac{-1}{108}$$

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$$f'''(x) = \frac{3}{8} x^{-\frac{5}{2}} \Rightarrow f'''(9) = \frac{3}{8} \cdot \frac{1}{243}$$

$$f'''(9) = \frac{1}{648}$$

Substituting the above values in eqn (1),

$$\sqrt{10} = 3 + \frac{1}{6} + \frac{1}{2}\left(-\frac{1}{108}\right) + \frac{1}{6}\left(\frac{1}{648}\right) + \dots$$

$$\sqrt{10} = 3 + \frac{1}{6} - \frac{1}{216} + \frac{1}{3888}$$

$$\sqrt{10} = 3 + 0.1667 - 0.0046 + 0.0002$$

$$\sqrt{10} = 3.1623$$

8) Express $\tan\left(\frac{\pi}{4}+x\right)$ in ascending powers of x. (a)

Soln: Let $f(x) = \tan x$

Given $f(a+h) = \tan\left(\frac{\pi}{4}+x\right)$

By Taylor's series expansion in $[a, a+h]$

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \frac{h^4}{4!} f^4(a) + \dots$$

$\tan x \longrightarrow \sec^2 x$

$\sec x \longrightarrow \sec x \cdot \tan x$

$$\Rightarrow \tan\left(\frac{\pi}{4}+x\right) = f(\pi/4) + x \cdot f'(\pi/4) + \frac{x^2}{2!} f''(\pi/4) + \frac{x^3}{3!} f'''(\pi/4) + \frac{x^4}{4!} f^4(\pi/4) + \dots \longrightarrow (1)$$

$f(x) = \tan x \Rightarrow f(\pi/4) = \tan(\pi/4) = 1$

$f'(x) = \sec^2 x \Rightarrow f'(\pi/4) = (\sqrt{2})^2 = 2$

$f''(x) = 2\sec x \sec x \tan x$

$f''(x) = 2\sec^2 x \cdot \tan x$

$\Rightarrow f''\left(\frac{\pi}{4}\right) = 2 \cdot 2 \cdot 1 = 4$

$f'''(x) = 2\sec^2 x \cdot \sec^2 x + 2\tan x (2\sec x)(\sec x \tan x)$

$f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x$

$\Rightarrow f'''(\pi/4) = 2(4) + 4(2) * 1 = 1$

$f^4(x) = (8)\sec^3 x(\sec x \tan x) + 4\sec^2 x \, 2\tan x \cdot \sec^2 x + 4\tan^2 x \, 2\sec x(\sec x \tan x)$

$f^4(x) = 8\sec^4 x \tan x + 8\sec^4 x \tan x + 8\sec^2 x \tan^3 x$

$\Rightarrow f^4(\pi/4) = 8(4)(1) + 8(4)(1) + 8(2)(1)$

$f^4(\pi/4) = 80 \dots$

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Substitute the above values in eqn (1).

$$\tan\left(\frac{\pi}{4}+x\right) = 1 + x \cdot 2 + \frac{x^2}{2}(4) + \frac{x^3}{6}(16) + \frac{x^4}{24}(80) + \cdots$$

$$\tan\left(\frac{\pi}{4}+x\right) = 1 + 2x + 2x^2 + \frac{8x^3}{3} + \frac{10x^4}{3} + \cdots$$

9) Show that

$$\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \cdots$$

for $0 < x < 2$ (5)

Soln: Given $f(x) = \sqrt{x}$

$$x - a = x - 1$$

$$a = 1$$

Taylor's series expansion of $f(x)$ in powers of $x-a$ is,

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

$$\sqrt{x} = f(1) + \frac{x-1}{1!} f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \cdots \longrightarrow (1)$$

$$f(x) = \sqrt{x} \Rightarrow f(1) = \sqrt{1} = 1$$

$$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4} \cdot x^{-3/2} \Rightarrow f''(1) = -\frac{1}{4}$$

$$f'''(x) = \frac{3}{8} x^{-5/2} \Rightarrow f'''(1) = \frac{3}{8}$$

sub, the above values in eqn (1)

$$(1) \Rightarrow \sqrt{x} = 1 + (x-1) \cdot \frac{1}{2} + \frac{(x-1)^2}{2}\left(-\frac{1}{4}\right) + \frac{(x-1)^3}{\cancel{6}_2}\left(\frac{3}{8}\right) + \cdots$$

$$\Rightarrow \sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 + \cdots$$

Hence Proved

MACLAURIN'S THEOREM:

Let $f(x)$ be a function and it is defined on $[0,x]$ such that,

i) $f^{n-1}(x)$ is continuous in $[0,x]$

ii) $f^{n-1}(x)$ is derivable in $(0,x)$.

and $p \in z^{+}$ then there exists atleast one positive value θ, $0<\theta<1$ such that,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \dots\dots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + R_n$$

$$\longrightarrow R_n = \frac{(x-0)^p (x-\theta x)^{n-p} f^n(\theta x)}{(n-1)!\,p}$$

where $R_n = \dfrac{x^n (1-\theta)^{n-p} f^n(\theta x)}{(n-1)!\,p} \Leftarrow \dfrac{x^p \cdot x^{n-p} (1-\theta)^{n-p} f^n(\theta x)}{(n-1)!\,p}$

$\star \rightarrow$

NOTE: Various Maclaurin's remainder are as follows:

1) Schlomilch – Roche's form of Remainder

$$R_n = \frac{x^n (1-\theta)^{n-p} f^n(\theta x)}{(n-1)!\,p}$$

2) Lagrange's form of Remainder

put $p=n$ in S-R form of remainder, we get

$$R_n = \frac{x^n f^n(\theta x)}{n!}$$

3) Cauchy's form of Remainder

put $p=1$ in S-R form of remainder, we get

$$R_n = \frac{x^n (1-\theta)^{n-1} f^n(\theta x)}{(n-1)!}$$

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4> $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots\cdots + \frac{x^n}{n!} f^n(0) + \cdots\cdots$

is called Maclaurin's series expansion of $f(x)$ about $x=0$ or in powers of 'x'. (Maclaurin's Infinite Series)

*→ Maclaurin's theorem obtained from Taylor's Theorem by substituting $a=0, b=x$ and $c=\theta x$ in Taylor's Theorem.

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PROBLEMS:

1) Obtain the Maclaurin's Series expansion of e^x.

Soln: Given $f(x) = e^x$

By Maclaurin's series of expansion of $f(x)$ in $[0, x]$

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \longrightarrow (1)$$

$f(x) = e^x \Rightarrow f(0) = e^0 = 1$

$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1$

$f''(x) = e^x \Rightarrow f''(0) = e^0 = 1$

$f'''(x) = e^x \Rightarrow f'''(0) = e^0 = 1$

$\vdots$

Substituting the values in eqn (1)

$$e^x = 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

2) Obtain the Maclaurin's Series expansion of $\sin^2 x$ in powers of x.

Soln: Given $f(x) = \sin^2 x$

By Maclaurin's series of expansion of $f(x)$ in $[0, x]$

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \longrightarrow (1)$$

$f(x) = \sin^2 x \Rightarrow f(0) = 0$

$f'(x) = 2\sin x \cos x = \sin 2x$

$\Rightarrow f'(0) = 0$

$f''(x) = 2\cos 2x \Rightarrow f''(0) = 2(1) = 2$

$f'''(x) = -4\sin 2x \Rightarrow f'''(0) = 0$

$f^4(x) = -8\cos 2x \Rightarrow f^4(0) = -8$

$f^5(x) = 16\sin 2x \Rightarrow f^5(0) = 0$

$f^6(x) = 32\cos 2x \Rightarrow f^6(0) = 32$

Substituting the values in eqn (1),

$$\sin^2 x = 0 + \frac{x}{1!}(0) + \frac{x^2}{2}(2) + 0 + \frac{x^4}{4!}(-8) + \cdots$$

$$\sin^2 x = x^2 - \frac{x^4}{4 \times 3 \times 2}(8) + \frac{x^6}{720}(32) + \cdots$$

$$\sin^2 x = x^2 - \frac{x^4}{3} + \frac{4x^6}{9} + \cdots$$

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3) Obtain Maclaurin's series expansion of $(1+x)^n$.

Soln: Given $f(x) = (1+x)^n$

By Maclaurin's series of expansion of $f(x)$ in $[0, x]$

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \longrightarrow (1)$$

$f(x) = (1+x)^n \Rightarrow f(0) = 1$

$f'(x) = n(1+x)^{n-1} \Rightarrow f'(0) = n$

$f''(x) = n(n-1)(1+x)^{n-2}$

$\Rightarrow f''(0) = n(n-1)$

$f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$

$\Rightarrow f'''(0) = n(n-1)(n-2)$

$f^4(x) = n(n-1)(n-2)(n-3)(1+x)^{n-4}$

$\Rightarrow f^4(0) = n(n-1)(n-2)(n-3)$

⋮

$$(1) \Rightarrow (1+x)^n = 1 + \frac{x}{1!}(n) + \frac{x^2}{2!} n(n-1) + \frac{x^3}{3!} n(n-1)(n-2) + \frac{x^4}{4!} n(n-1)(n-2)(n-3) + \cdots$$

4) Show that

$$\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots$$

and hence deduce that

$$\frac{e^x}{e^x+1} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$$

Soln: $f(x) = \log(1+e^x)$

$$f'(x) = \frac{1}{1+e^x} \cdot e^x = \frac{e^x}{1+e^x}$$

$$f''(x) = \frac{(1+e^x)e^x - e^x \cdot e^x}{(1+e^x)^2}$$

$$f''(x) = \frac{e^x + e^{2x} - e^{2x}}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$$

$$f'''(x) = \frac{(1+e^x)^2 e^x - e^x \cdot 2(1+e^x) \cdot e^x}{(1+e^x)^4}$$

$$f'''(x) = \frac{(1+e^x)\left[(1+e^x)e^x - 2e^{2x}\right]}{(1+e^x)^4}$$

$$f'''(x) = \frac{e^x + e^{2x} - 2e^{2x}}{(1+e^x)^3} = \frac{e^x - e^{2x}}{(1+e^x)^3}$$

$$f^4(x) = \frac{(1+e^x)^3(e^x - 2e^{2x}) - (e^x - e^{2x})3(1+e^x)^2 e^x}{(1+e^x)^6}$$

$$f^4(x) = \frac{(1+e^x)^2\left[(1+e^x)(e^x - 2e^{2x}) - 3e^x(e^x - e^{2x})\right]}{(1+e^x)^6}$$

$$f^4(x) = \frac{e^x + e^{2x} - 2e^{2x} - 2e^{3x} - 3e^{2x} \ldots}{(1+e^x)^4}$$

$$f^4(x) = \frac{e^{3x} - 4e^{2x} + e^x}{(1+e^x)^4}$$

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At $x=0$

$$\Rightarrow f(0) = \log(1+e^0) = \log 2$$

$$\Rightarrow f'(0) = \frac{e^0}{1+e^0} = \frac{1}{1+1} = \frac{1}{2}$$

$$\Rightarrow f''(0) = \frac{e^0}{(1+e^0)^2} = \frac{1}{(1+1)^2} = \frac{1}{4}$$

$$\Rightarrow f'''(0) = \frac{e^0 - e^{2(0)}}{(1+e^0)^3} = \frac{1-1}{(1+1)^3} = 0$$

$$\Rightarrow f^4(0) = \frac{e^{3(0)} - 4e^{2(0)} + e^0}{(1+e^0)^4} = \frac{1-4+1}{(1+1)^4}$$

$$\Rightarrow f^4(0) = \frac{-2}{16} = -\frac{1}{8}.$$

$\vdots$

By Maclaurin's Series,

$$f(x) = f(0) + x\cdot f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

$$\log(1+e^x) = \log 2 + x\cdot\frac{1}{2} + \frac{x^2}{2}\cdot\frac{1}{4} + \frac{x^3}{6}\cdot(0) + \frac{x^4}{24}\left(-\frac{1}{8}\right) + \cdots$$

$$\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots \longrightarrow (1)$$

Diff., eqn (1) w.r.t. 'x',

$$\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{2x}{8} - \frac{4x^3}{192} + \cdots$$

$$\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$$

5> Find Maclaurin's Theorem with Lagrange's form of remainder for $f(x) = \cos x$. (OR)

Show that,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n \cdot x^{2n}}{(2n)!}\cos n\theta x + (-1)^{n+1}\cdot\frac{x^{2n+1}}{(2n+1)!}\sin\theta x.$$

Soln: We know that,

Lagrange's form of Maclaurin's series

i.e., $R_n = \dfrac{x^n f^n(\theta x)}{n!}$

Then Maclaurin's Theorem Expansion is;

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + R_n$$

$$\text{i.e., } f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^n(\theta x)$$

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{2n}}{(2n)!}f^{2n}(\theta x) + \frac{x^{2n+1}}{(2n+1)!}f^{2n+1}(\theta x) \longrightarrow (1)$$

Given $f(x) = \cos x$

$$\Rightarrow f(0) = \cos 0 = 1$$

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$f'(x) = -\sin x \Rightarrow f'(0) = 0$

$f''(x) = -\cos x \Rightarrow f''(0) = -1$

$f'''(x) = \sin x \Rightarrow f'''(0) = 0$

$f^{4}(x) = \cos x \Rightarrow f^{4}(0) = 1$

$f^{2n}(x) = (-1)^n \cos x \Rightarrow f^{2n}(\theta x) = (-1)^n \cos\theta x$

$f^{2n+1}(x) = (-1)^{n+1} \sin x \Rightarrow f^{2n+1}(\theta x) = (-1)^{n+1} \sin\theta x$

Substitute above values in eqn (1),

$$(1) \Rightarrow \cos x = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \cdots + \frac{x^{2n}}{(2n)!}(-1)^n \cos\theta x + \frac{x^{2n+1}}{(2n+1)!}(-1)^{n+1} \sin\theta x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + (\cdots\cdots + (-1)^n \cdot \frac{x^{2n}}{(2n)!} \cos\theta x + (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!} \sin\theta x$$

Hence the result follows.

6) Show that

$$\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + 4\frac{x^3}{3!} + \cdots\cdots$$

Soln: Let $f(x) = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$

$\Rightarrow f(0) = 0$

From Given,

$\sqrt{1-x^2} \cdot f(x) = \sin^{-1}x \longrightarrow (1)$

Diff., (1) w.r.t. 'x', we get

$\sqrt{1-x^2} \cdot f'(x) + f(x) \dfrac{1}{2\sqrt{1-x^2}}(-2x) = \dfrac{1}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2) f'(x) - x \cdot f(x) = 1 \longrightarrow (2)$

Now $f'(0) = 1$

Diff., (2) w.r.t. 'x', we get

$(1-x^2) f''(x) + f'(x)(-2x) - [x \cdot f'(x) + f(x)] = 0$

$(1-x^2) f''(x) - 2x f'(x) - x f'(x) - f(x) = 0$

$\Rightarrow (1-x^2) f''(x) - 3x f'(x) - f(x) = 0 \longrightarrow (3)$

$\Rightarrow f''(0) = 0$

Diff., (3), w.r.t. 'x', we get

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$$(1-x^2) f'''(x) + f''(x)(-2x) - 3[x f''(x) + f'(x)] - f'(x) = 0$$

$$(1-x^2) f'''(x) - 2x f''(x) - 3x f''(x) - 3f'(x) - f'(x) = 0$$

$$(1-x^2) f'''(x) - 5x f''(x) - 4f'(x) = 0$$

$$\Rightarrow f'''(0) - 5(0)(0) - 4(1) = 0$$

$$\Rightarrow f'''(0) = 4.$$

$\vdots$

We have by Maclaurin's Series

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$\Rightarrow \frac{\sin^{-1}x}{\sqrt{1-x^2}} = 0 + \frac{x}{1}(1) + \frac{x^2}{2}(0) + \frac{x^3}{3!}(4) + \cdots$$

$$\Rightarrow \frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + 4\frac{x^3}{3!} + \cdots$$

Hence the result follows.

7) Using Maclaurin's Series expand $\tan x$ upto the 5^{th} power of x and hence find series for $\log \sec x$.

Soln: From the given,
consider $f(x) = \tan x$.

WKT, the M.S expansion of $f(x)$ is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots \longrightarrow (1)$$

$$f(x) = \tan x \Rightarrow f(0) = \tan 0 = 0$$

$$f'(x) = \sec^2 x \Rightarrow f'(0) = \sec^2 0 = 1$$

$$f''(x) = 2\sec x \cdot \sec x \tan x$$

$$= 2\sec^2 x \tan x$$

$$\Rightarrow f''(0) = 2\sec^2 0 \cdot \tan 0 = 0$$

$$f'''(x) = 2[\sec^2 x \sec^2 x + 2\sec x \sec x \cdot \tan^2 x]$$

$$= 2[\sec^4 x + 2\sec^2 x \tan^2 x]$$

$$\Rightarrow f'''(0) = 2[1+0] = 2$$

$$f^{iv}(x) = 2[4\sec^3 x \cdot \sec x \cdot \tan x] + 4[\sec^2 x \cdot 2\tan x \cdot \sec^2 x + \tan^2 x \cdot 2\sec x \tan x]$$

$$= 8[\sec^4 x \tan x + \sec^4 x \tan x + \sec x \tan^3 x]$$

$$\Rightarrow f^{iv}(0) = 0$$

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$$f^{v}(x) = 8\left[2\sec^4 x \sec^2 x + (\tan x\, 4\sec^3 x \sec x \tan x)2 + \tan^3 x \sec x \tan x\right]$$

$$+ \sec x\, 3\tan^2 x \sec^2 x + \tan^3 x \sec x + 3\tan^2 x \sec^3 x + \sec x \tan^4 x.$$

$$f^{v}(x) = 16\sec^6 x + 8\sec^4 x \tan^2 x + 3\tan^2 x \sec^3 x + \sec x \tan^4 x.$$

$$\Rightarrow f^{v}(0) = 16(1) = 16$$

$$(1) \Rightarrow \tan x = 0 + x\cdot 1 + \frac{x^2}{2!}(0) + \frac{x^3}{3!}2 + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(16) + \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{x^5}{15}(2) + \cdots \longrightarrow (2)$$

Integrating $\underline{eqn}$ (2) on both sides,

$$\log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + 2\cdot\frac{x^6}{90} + \cdots$$

$$\Rightarrow \log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \cdots$$

UNIT-III

Multivariable Calculus

UNIT – III

Multivariable Calculus

Partial differentiation :-

Let $z = f(x,y)$ be a function of two variables x and y. Then $\lim_{x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$, if it exists, is said to be partial derivative. It is denoted by the symbol $\frac{\partial z}{\partial x}$ (or) $\frac{\partial f}{\partial x}$ (or) f_x.

Note:-

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \quad ; \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \quad ; \quad f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{xy} = f_{yx}$$

1. Find first and second order partial derivatives of $x^3 + y^3 - 3axy$ and verify $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

Sol:-

Let $f(x,y) = x^3 + y^3 - 3axy$

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay \quad , \quad \frac{\partial f}{\partial y} = 3y^2 - 3ax$$

$$\frac{\partial^2 f}{\partial x^2} = 6x \quad , \quad \frac{\partial^2 f}{\partial y^2} = 6y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(3y^2 - 3ax\right) = -3a$$

$$\frac{\partial^2 f}{\partial y \partial x} = -3a$$

Thus $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ is verified.

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2. If $U = \frac{1}{\sqrt{x^2+y^2+z^2}}$, $x^2+y^2+z^2 \neq 0$ then prove that $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$

2

Sol:

Given $U = (x^2+y^2+z^2)^{-1/2}$

$$\frac{\partial U}{\partial x} = \frac{-1}{2}(x^2+y^2+z^2)^{-3/2}(2x) = -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 U}{\partial x^2} = -\left[(x^2+y^2+z^2)^{-3/2} - x\frac{3}{2}(x^2+y^2+z^2)^{-5/2}(2x)\right]$$

$$\frac{\partial^2 U}{\partial x^2} = -(x^2+y^2+z^2)^{-5/2}\left[x^2+y^2+z^2-3x^2\right]$$

$$= (x^2+y^2+z^2)^{-5/2}\left[2x^2-y^2-z^2\right] \longrightarrow (1)$$

Similarly, $\frac{\partial^2 U}{\partial y^2} = (x^2+y^2+z^2)^{-5/2}\left[-x^2+2y^2-z^2\right] \longrightarrow (2)$

$$\frac{\partial^2 U}{\partial z^2} = (x^2+y^2+z^2)^{-5/2}\left[-x^2-y^2+2z^2\right] \longrightarrow (3)$$

$(1)+(2)+(3) \Rightarrow \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$

3. verify $\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x}$ for the function $u = \tan^{-1}\left(\frac{x}{y}\right)$

Sol:

let $u = \tan^{-1}\left(\frac{x}{y}\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{1}{y}\right) = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2+y^2-y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \longrightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{-x}{y^2}\right) = \frac{-x}{y^2+x^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{(y^2+x^2)(-1)+x(2x)}{(y^2+x^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \longrightarrow (2)$$

from (1) & (2), we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

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(P) If $u = \tan^{-1}\left[\frac{2xy}{x^2-y^2}\right]$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (2) (3)

Sol:-
$$\frac{\partial u}{\partial x} = \frac{1}{1+\frac{4x^2y^2}{(x^2-y^2)^2}}\left[\frac{(x^2-y^2)(2y) - 2xy \cdot 2x}{(x^2-y^2)^2}\right]$$

$$= \frac{2yx^2 - 2y^3 - 4x^2y}{(x^2-y^2)^2 + 4x^2y^2} = \frac{-2y^3 - 2x^2y}{(x^2+y^2)^2} = \frac{-2y(x^2+y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial x} = \frac{-2y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{4xy}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{2x}{x^2+y^2}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{-4xy}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(P) If $z = \log(e^x + e^y)$ show that $rt - s^2 = 0$, where $r = \frac{\partial^2 z}{\partial x^2}$, $t = \frac{\partial^2 z}{\partial y^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$

Sol:-
$$z = \log(e^x + e^y)$$

$$\frac{\partial z}{\partial x} = \frac{e^x}{e^x+e^y}, \quad \frac{\partial z}{\partial y} = \frac{e^y}{e^x+e^y}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{(e^x+e^y)e^x - e^x \cdot e^x}{(e^x+e^y)^2} = \frac{e^x e^y}{(e^x+e^y)^2} = \frac{e^{x+y}}{(e^x+e^y)^2}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{e^{x+y}}{(e^x+e^y)^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y} = -\frac{e^{x+y}}{(e^x+e^y)^2}$$

$$\therefore rt - s^2 = 0$$

?) If $u = e^{xyz}$, show that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$

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Homogeneous function :-

A function $f(x,y)$ is said to be homogeneous function of degree n if the degree of each term in $f(x,y)$ is n, where n is a real number. (OR)

A function $f(x,y)$ is said to be a homogeneous function of degree or order n in variables x, y if $f(kx, ky) = k^n f(x,y)$ (n is a real number)

Ex :- 1. $f(x,y) = x^2+y^2$

$f(kx, ky) = k^2(x^2+y^2) = k^2 f(x,y)$

$\therefore$ $f(x,y)$ is a homogeneous function of order 2.

2. $f(x,y) = \sin(xy)$ (not Homo) $\Rightarrow f(kx,ky) = \sin(k^2xy) \neq k^n f(x,y)$

Euler's Theorem on Homogeneous functions :-

If $z = f(x,y)$ is a homogeneous function of degree n, then $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = nz$, $\forall\ x, y$ in the domain of the function.

Ex :- If $f(x,y) = \dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} \Rightarrow f(kx, ky) = k^0 f(x,y)$

$\Rightarrow$ $f(x,y)$ is a homogeneous function of degree 0

By Euler's thm, $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = 0 \Rightarrow x f_x + y f_y = 0$

(P) Verify Euler's thm for the function $xy+yz+zx$

Sol :-

let $f(x,y,z) = xy+yz+zx$

$f(kx, ky, kz) = k^2 f(x,y,z) \longrightarrow (1)$

$\therefore$ This is a homogeneous function of degree 2.

we have $\dfrac{\partial f}{\partial x} = y+z$, $\dfrac{\partial f}{\partial y} = x+z$, $\dfrac{\partial f}{\partial z} = x+y$

$x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} + z\dfrac{\partial f}{\partial z} = 2f(x,y,z)$

Hence the Euler's thm is verified.

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Ⓟ If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$ ⑤

Sol:- Given $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{1}{y}\right) + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{-x}{y^2}\right) + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Ⓟ If $u(x,y) = \log\left(\frac{x^4+y^4}{x+y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$

Sol:- $u = \log\left(\frac{x^4+y^4}{x+y}\right) \Rightarrow e^u = \frac{x^4+y^4}{x+y} = f(x,y)$

$f(x,y)$ is a homogeneous function of degree 3.

By Euler's thm, i.e; e^u is a homogeneous function in x, y of degree 3

$$x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = 3e^u$$

$$x\, e^u \frac{\partial u}{\partial x} + y\, e^u \frac{\partial u}{\partial y} = 3e^u$$

$$\therefore\; x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$$

The chain rule of partial Differentiation :-

(i) If $z = f(x,y)$ where $x = \phi(t)$, $y = \psi(t)$ then z is called a Composite function of a variable t.

(ii) If $z = f(x,y)$ where $x = \phi(u,v)$, $y = \psi(u,v)$ then z is called a Composite function of two Variables u and v.

Note:-

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \quad \text{and}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$z = f(u,v)$
$u = \phi(x,y)$
$v = g(x,y)$

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Total Differential Coefficient :-

let $z = f(x, y)$; $x = \phi(t)$, $y = g(t)$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

(P) If $u = f(y-z, z-x, x-y)$ show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Sol:-

let $r = y-z$, $s = z-x$, $t = x-y$

Then $u = f(r, s, t)$

$$\frac{\partial r}{\partial x} = 0, \quad \frac{\partial r}{\partial y} = 1, \quad \frac{\partial r}{\partial z} = -1$$

$$\frac{\partial s}{\partial x} = -1, \quad \frac{\partial s}{\partial y} = 0, \quad \frac{\partial s}{\partial z} = 1$$

$$\frac{\partial t}{\partial x} = 1, \quad \frac{\partial t}{\partial y} = -1, \quad \frac{\partial t}{\partial z} = 0$$

$\therefore$ By Chain rule of partial differentiation, we get

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \longrightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \longrightarrow (2)$$

and $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial z} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial z}$$

$$\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \longrightarrow (3)$$

(1) + (2) + (3) $\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ Hence the result.

(P) If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ then Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

(P) If $u = x^2 + y^2$, $x = at^2$, $y = 2at$ then find $\frac{du}{dt}$

Sol:-

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= 4a^2t(t^2+2)$$

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$

$\frac{dx}{dt} = 2at$, $\frac{dy}{dt} = 2a$

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(4) (7)

(P) If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$ prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

Sol:- $u = f(r) \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = f'(r)\frac{\partial r}{\partial x}$

$$\frac{\partial^2 u}{\partial x^2} = f''(r).\frac{\partial r}{\partial x}.\frac{\partial r}{\partial x} + f'(r)\frac{\partial^2 r}{\partial x^2} = f''(r)\left(\frac{\partial r}{\partial x}\right)^2 + f'(r)\frac{\partial^2 r}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial y^2} = f''(r)\left(\frac{\partial r}{\partial y}\right)^2 + f'(r)\frac{\partial^2 r}{\partial y^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] + f'(r)\left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right] \longrightarrow (1)$$

We write $x^2 + y^2 = r^2$

$$2x = 2r\frac{\partial r}{\partial x} \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r.1 - x\frac{\partial r}{\partial x}}{r^2} = \frac{r - x.\frac{x}{r}}{r^2} = \frac{r^2 - x^2}{r^3} = \frac{y^2}{r^3}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{y^2}{r^3}$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}$$

$$\therefore (1) \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left[\frac{x^2}{r^2} + \frac{y^2}{r^2}\right] + f'(r)\left[\frac{y^2}{r^3} + \frac{x^2}{r^3}\right]$$

$$= f''(r)[1] + f'(r)\left[\frac{1}{r}\right]$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

(P) If $x = r\cos\theta$, $y = r\sin\theta$ then prove that $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$ and $\frac{1}{r}\frac{\partial x}{\partial \theta} = r\frac{\partial \theta}{\partial x}$

Sol:- Given $x = r\cos\theta$, $y = r\sin\theta \Rightarrow \theta = \tan^{-1}(y/x)$

$$\frac{\partial x}{\partial r} = \cos\theta$$

$$r^2 = x^2 + y^2 \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \Rightarrow \frac{1}{r}\frac{\partial x}{\partial \theta} = -\sin\theta \ \& \ \frac{\partial \theta}{\partial x} = -y/r^2 = \frac{-\sin\theta}{r}$$

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Jacobian :-

1. $$\frac{\partial(u,v)}{\partial(x,y)} = J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

2. $$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Properties :-

1. $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$ (or) If $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$ then $JJ' = 1$

2. Chain rule for Jacobians

If u, v are functions of r, s and r, s are functions of x, y,

then $$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$$

(P) If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$ show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$

Sol:- $\frac{\partial u}{\partial x} = \frac{-yz}{x^2}$, $\frac{\partial u}{\partial y} = \frac{z}{x}$, $\frac{\partial u}{\partial z} = \frac{y}{x}$

$\frac{\partial v}{\partial x} = \frac{z}{y}$, $\frac{\partial v}{\partial y} = \frac{-zx}{y^2}$, $\frac{\partial v}{\partial z} = \frac{x}{y}$

$\frac{\partial w}{\partial x} = \frac{y}{z}$, $\frac{\partial w}{\partial y} = \frac{x}{z}$, $\frac{\partial w}{\partial z} = \frac{-xy}{z^2}$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{xyz} \begin{vmatrix} -\frac{xyz}{x^2} & \frac{zy}{x} & \frac{yz}{x} \\ \frac{xz}{y} & -\frac{xyz}{y^2} & \frac{xz}{y} \\ \frac{xy}{z} & \frac{xy}{z} & -\frac{xyz}{z^2} \end{vmatrix}$$ ($\because$ Multiply C_1 by x, C_2 by y, C_3 by z)

$$= \frac{1}{xyz} \cdot \frac{yz}{x} \cdot \frac{xz}{y} \cdot \frac{xy}{z} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4$$

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(9)

(P) If $z = f(x+ay) + \phi(x-ay)$, prove that $\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$

Sol:- $\frac{\partial z}{\partial x} = f'(x+ay) + \phi'(x-ay)$; $\frac{\partial z}{\partial y} = af'(x+ay) - a\phi'(x-ay)$

$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + \phi''(x-ay)$

$\frac{\partial^2 z}{\partial y^2} = a^2 [f''(x+ay) + \phi''(x-ay)]$

$= a^2 \frac{\partial^2 z}{\partial x^2}$

(P) If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, $x+y \neq 0$, prove that

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$

Sol:- $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ which is not homogeneous function

let $z = \sin u = \frac{x^2+y^2}{x+y}$

z is a homogeneous function of x, y of degree 1

By Euler's th^{m}, we have

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1 \cdot z$

$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \sin u$

$x\cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = \sin u$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$

(P) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$; prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

Sol:- $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ is not Homo. function.

let $f = \tan u = \frac{x^3+y^3}{x+y}$

$\therefore$ f is a homogo. function of degree 2

By Euler's th^{m}, $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2 \cdot f$

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$$x \frac{\partial}{\partial x} \tan f(t) + y \frac{\partial}{\partial y} \tan f(t) = 2 \tan f(t)$$

$$x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2 \tan u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2 \tan u}{\sec^2 u} = \sin 2u$$

(P) If $u = \sec^{-1}\left(\frac{x^3 - y^3}{x + y}\right)$ show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \cot u$

then evaluate $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$

Sol:n

$$f = \sec u = \frac{x^3 - y^3}{x + y} \quad \text{degree} = 2$$

By Euler's thm,
$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 2f$$

$$x \frac{\partial}{\partial x} \sec u + y \frac{\partial}{\partial y} \sec u = 2 \sec u$$

$$x \sec u \tan u \frac{\partial u}{\partial x} + y \sec u \tan u \frac{\partial u}{\partial y} = 2 \sec u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \cot u = g(u) \text{ (say)}$$

By Euler's thm of second order, we get

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)\left[g'(u) - 1\right]$$

$$= 2 \cot u\left[-2 \operatorname{cosec}^2 u - 1\right]$$

$$= -2 \cot u\left[2 \operatorname{cosec}^2 u + 1\right]$$

(P) If $x = r\cos\theta$, $y = r\sin\theta$,

Show that $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$

(P) If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$ then

prove that $\frac{1}{2} \frac{\partial u}{\partial x} + \frac{1}{3} \frac{\partial u}{\partial y} + \frac{1}{4} \frac{\partial u}{\partial z} = 0$

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Ⓟ (i) If $x = u(1+v)$, $y = v(1+u)$ then prove that $\dfrac{\partial(x,y)}{\partial(u,v)} = 1+u+v$

(ii) If $u = x^2 - 2y$, $v = x+y+z$, $w = x-2y+3z$ find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$

(iii) If $x+y+z = u$, $y+z = uv$, $z = uvw$ then evaluate

(a) $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ (b) $J\left(\dfrac{u,v,w}{x,y,z}\right)$

Sol:

(i) $x = u(1+v)$, $y = v(1+u)$

$$\frac{\partial x}{\partial u} = 1+v, \quad \frac{\partial x}{\partial v} = u, \quad \frac{\partial y}{\partial u} = v, \quad \frac{\partial y}{\partial v} = 1+u$$

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$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = (1+v)(1+u) - uv$$

$$= 1+u+v$$

(ii) We have $\dfrac{\partial u}{\partial x} = 2x$, $\dfrac{\partial u}{\partial y} = -2$, $\dfrac{\partial u}{\partial z} = 0$

$$\frac{\partial v}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = 1, \quad \frac{\partial v}{\partial z} = 1$$

$$\frac{\partial w}{\partial x} = 1, \quad \frac{\partial w}{\partial y} = -2, \quad \frac{\partial w}{\partial z} = 3$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 2x & -2 & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= 2x(3+2) + 2(3-1) = \cancel{6x+4}\; 10x+4$$

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(iii)

(a) $x+y+z=u \longrightarrow (1)$, $\quad y+z=uv \longrightarrow (2)$, $\quad z=uvw \longrightarrow (3)$

$\cancel{\frac{\partial x}{\partial u}}$ $(1) \Rightarrow \quad x = u-(y+z) \Rightarrow x = u-uv$

$(2) \Rightarrow \quad y = uv - z \Rightarrow y = uv - uvw$

and $\quad z = uvw$

$$\frac{\partial x}{\partial u} = 1-v, \quad \frac{\partial x}{\partial v} = -u, \quad \frac{\partial x}{\partial w} = 0$$

$$\frac{\partial y}{\partial u} = v - vw, \quad \frac{\partial y}{\partial v} = u - uw, \quad \frac{\partial y}{\partial w} = -uv$$

$$\frac{\partial z}{\partial u} = vw, \quad \frac{\partial z}{\partial v} = uw; \quad \frac{\partial z}{\partial w} = uv$$

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1-v)\left[(u-uw)uv + uv(uw)\right] + u\left[(v-vw)uv + uv(vw)\right]$$

$$= (1-v)\left[uv\{u-uw+uw\}\right] + u\left[uv\{v-vw+vw\}\right]$$

$$= (1-v)u^2v + u^2v^2 = u^2v$$

$\cancel{(b)}$ $\quad \cancel{\frac{\partial u}{\partial x} = 1}, \quad \cancel{\frac{\partial u}{\partial y} = 1}, \quad \cancel{\frac{\partial u}{\partial z} = 1}$

Note:- <u>Jacobian of Implicit functions :-</u>

If u, v, w and x, y, z are implicitly connected by the equations such as $f_1(u,v,w,x,y,z) = 0$, $f_2(u,v,w,x,y,z) = 0$, $f_3(u,v,w,x,y,z) = 0$ then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \, \frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} \div \frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}$$

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(b) Let $f_1 = u - x - y - z$, $f_2 = uv - y - z$, $f_3 = uvw - z$

$$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} \div \frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} \longrightarrow (1)$$

Now,
$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix} = -1$$

and
$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix}$$

$$= u^2 v$$

$$(1) \Rightarrow J\left(\frac{u,v,w}{x,y,z}\right) = (-1)\frac{-1}{u^2 v} = \frac{1}{u^2 v}$$

Note:-
$$J\left(\frac{x,y,z}{u,v,w}\right) = u^2 v$$

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(P) (i) If $x = \frac{u^2}{v}$, $y = \frac{v^2}{u}$ find $\frac{\partial(u,v)}{\partial(x,y)}$ Sol: $\frac{1}{3}$ (14)

(ii) If $x = uv$, $y = \frac{u}{v}$ then find $\frac{\partial(x,y)}{\partial(u,v)}$ Sol: $-\frac{2u}{v}$

(iii) If $x = uv$, $y = \frac{u}{v}$ verify that $\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1$

(P) If $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(x,y)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(x,y)}$.

Also show that $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$

Sol: Given that $x = r\cos\theta$, $y = r\sin\theta$

$$r^2 = x^2+y^2 \quad ; \quad \theta = \tan^{-1}(y/x)$$

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-y/x^2\right) = \frac{-y}{x^2+y^2} = \frac{-r\sin\theta}{r^2} = \frac{-1}{r}\sin\theta$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+y^2/x^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{1}{r}\cos\theta$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

Also, $$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ \frac{-1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{vmatrix} = \frac{1}{r}$$

$$\cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \Rightarrow \frac{\partial r}{\partial x} = \cos\theta$$

$$2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r} \Rightarrow \frac{\partial r}{\partial y} = \sin\theta$$

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(P) If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, Show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$ also find $\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)}$

(OR)

Find the Jacobian transformation from 3D - Cartesian Coordinates to Spherical polar Coordinates.

Sol:-
$\frac{\partial x}{\partial r} = \sin\theta\cos\phi$, $\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$, $\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$

$\frac{\partial y}{\partial r} = \sin\theta\sin\phi$, $\frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi$, $\frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$

$\frac{\partial z}{\partial r} = \cos\theta$, $\frac{\partial z}{\partial \theta} = -r\sin\theta$, $\frac{\partial z}{\partial \phi} = 0$

$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} x_r & x_\theta & x_\phi \\ y_r & y_\theta & y_\phi \\ z_r & z_\theta & z_\phi \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta\left[r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi\right]$$
$$+ r\sin\theta\left[r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi\right]$$

$$= \cos\theta\left[r^2\sin\theta\cos\theta\right] + r\sin\theta\left[r\sin^2\theta\right]$$

$$= r^2\sin\theta\left[\cos^2\theta + \sin^2\theta\right] = r^2\sin\theta$$

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since $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} \cdot \frac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = 1$ $(\because J \cdot J' = 1)$

$\Rightarrow \frac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = \frac{1}{r^2 \sin\theta}$

(P) find $J\left(\frac{u,v}{x,y}\right)$ if

(i) $u = e^x, \quad v = e^y \longrightarrow e^{x+y}$

(ii) $u = e^{x+y}, \quad v = e^{-x+y} \longrightarrow 2e^{2y}$

(iii) $u = \frac{y^2}{x}, \quad v = \frac{x^2}{y} \longrightarrow 2e^{2y}$

(iv) $u = x\cos y, \quad v = y\sin x \longrightarrow 2(x+1)$

(v) $u = \frac{2x-y}{2}, \quad v = \frac{y}{2}$

(P) If $u = x^2+y^2+z^2, \quad v = xyz$ find $J\left(\frac{x,y}{u,v}\right)$

(P) find $\frac{\partial(u,v)}{\partial(r,\theta)}$ if

(i) $u = 2axy$ and $v = a(x^2-y^2) \longrightarrow -4a^2r^3$

(ii) $u = 2xy, \quad v = x^2-y^2 \longrightarrow 4r^3$

Where $x = r\cos\theta, \quad y = r\sin\theta$

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(P) If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ and $u = r\sin\theta\cos\phi$, (8)

$v = r\sin\theta\sin\phi$, $w = r\cos\theta$ find $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$ (17)

Sol:- Given $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ and

$u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$

Since x, y, z are functions of u, v, w and u, v, w are functions of r, θ, ϕ. Then

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} \rightarrow ① \text{ (By chain rule)}$$

$$\frac{\partial x}{\partial u} = 0, \quad \frac{\partial x}{\partial v} = \sqrt{w} \cdot \frac{1}{2\sqrt{v}} = \frac{1}{2}\sqrt{\frac{w}{v}}, \quad \frac{\partial x}{\partial w} = \frac{1}{2}\sqrt{\frac{v}{w}}$$

$$\frac{\partial y}{\partial u} = \frac{1}{2}\sqrt{\frac{w}{u}}, \quad \frac{\partial y}{\partial v} = 0, \quad \frac{\partial y}{\partial w} = \frac{1}{2}\sqrt{\frac{u}{w}}$$

$$\frac{\partial z}{\partial u} = \frac{1}{2}\sqrt{\frac{v}{u}}, \quad \frac{\partial z}{\partial v} = \frac{1}{2}\sqrt{\frac{u}{v}}, \quad \frac{\partial z}{\partial w} = 0.$$

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \frac{1}{2}\sqrt{\frac{w}{v}} & \frac{1}{2}\sqrt{\frac{v}{w}} \\ \frac{1}{2}\sqrt{\frac{w}{u}} & 0 & \frac{1}{2}\sqrt{\frac{u}{w}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} & 0 \end{vmatrix}$$

$$= -\frac{1}{2}\sqrt{\frac{w}{v}}\left[-\frac{1}{4}\sqrt{\frac{v}{w}}\right] + \frac{1}{2}\sqrt{\frac{v}{w}}\left[\frac{1}{4}\sqrt{\frac{w}{v}}\right]$$

$$= \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$$

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$$\frac{\partial u}{\partial r} = \sin\theta \cos\phi, \quad \frac{\partial u}{\partial \theta} = r\cos\theta\cos\varphi, \quad \frac{\partial u}{\partial \varphi} = -r\sin\theta\sin\varphi$$

$$\frac{\partial v}{\partial r} = \sin\theta \sin\varphi, \quad \frac{\partial v}{\partial \theta} = r\cos\theta\sin\varphi, \quad \frac{\partial v}{\partial \varphi} = r\sin\theta\cos\varphi$$

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$$\frac{\partial w}{\partial r} = \cos\theta, \quad \frac{\partial w}{\partial \theta} = -r\sin\theta, \quad \frac{\partial w}{\partial \varphi} = 0$$

$$\therefore \quad \frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = \begin{vmatrix} u_r & u_\theta & u_\varphi \\ v_r & v_\theta & v_\varphi \\ w_r & w_\theta & w_\varphi \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta\left[r^2\cos^2\varphi\sin\theta\cos\theta + r^2\sin^2\varphi\sin\theta\cos\theta\right]$$
$$+ r\sin\theta\left[r\sin^2\theta\cos^2\varphi + r\sin^2\theta\sin^2\varphi\right]$$

$$= \cos\theta \,.\, r^2\sin\theta\cos\theta\left[\sin^2\varphi + \cos^2\varphi\right]$$
$$+ r^2\sin^3\theta\left[\sin^2\varphi + \cos^2\varphi\right]$$

$$= r^2\sin\theta\cos^2\theta + r^2\sin^3\theta = r^2\sin\theta\left[\sin^2\theta + \cos^2\theta\right]$$
$$= r^2\sin\theta$$

Hence $$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(r,\theta,\phi)}$$

$$= \frac{1}{4} \cdot r^2\sin\theta$$

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(P) If $u = 2xy$, $v = x^2 - y^2$, $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial(u,v)}{\partial(r,\theta)}$ (9) (19)

Sol:

$u = 2(r\cos\theta)(r\sin\theta) \Rightarrow u = r^2\sin 2\theta$

$v = r^2\cos^2\theta - r^2\sin^2\theta \Rightarrow v = r^2\cos 2\theta$

$u_r = 2r\sin 2\theta$, $u_\theta = 2r^2\cos 2\theta$

$v_r = 2r\cos 2\theta$, $v_\theta = -2r^2\sin 2\theta$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} 2r\sin 2\theta & 2r^2\cos 2\theta \\ 2r\cos 2\theta & -2r^2\sin 2\theta \end{vmatrix}$$

$$= -4r^3\sin^2 2\theta - 4r^3\cdot\cos^2 2\theta$$

$$= -4r^3$$

(P) If $x = e^r\sec\theta$ —(1), $y = e^r\tan\theta$ —(2) prove that $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$

Sol:

$\frac{\partial x}{\partial r} = e^r\sec\theta$, $\frac{\partial x}{\partial\theta} = e^r\sec\theta\tan\theta$

$\frac{\partial y}{\partial r} = e^r\tan\theta$, $\frac{\partial y}{\partial\theta} = e^r\sec^2\theta$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = e^{2r}\sec^3\theta - e^{2r}\sec\theta\tan^2\theta$$

$$= e^{2r}\sec\theta(\sec^2\theta - \tan^2\theta) \longrightarrow 1$$

$$= e^{2r}\sec\theta$$

$\frac{(2)}{(1)} \Rightarrow \frac{y}{x} = \frac{\tan\theta}{\sec\theta} = \sin\theta$ and

$x^2 - y^2 = e^{2r}(\sec^2\theta - \tan^2\theta) = e^{2r}$

$x^2 - y^2 = e^{2r} \Rightarrow 2r = \log(x^2 - y^2)$

$r = \frac{1}{2}\log(x^2 - y^2) \longrightarrow (3)$

$\theta = \sin^{-1}(y/x) \longrightarrow (4)$

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$$\frac{\partial r}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2-y^2}(2x) = \frac{x}{x^2-y^2}$$

$$\frac{\partial r}{\partial y} = \frac{1}{2} \cdot \frac{1}{x^2-y^2}(-2y) = \frac{-y}{x^2-y^2}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{\sqrt{1-y^2/x^2}}\left(-y/x^2\right) = \frac{-y}{x\sqrt{x^2-y^2}}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{\sqrt{1-y^2/x^2}}\left(\frac{1}{x}\right) = \frac{1}{\sqrt{x^2-y^2}}$$

$$\therefore \quad \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix} = \begin{vmatrix} \dfrac{x}{x^2-y^2} & \dfrac{-y}{x^2-y^2} \\ \dfrac{-y}{x\sqrt{x^2-y^2}} & \dfrac{1}{\sqrt{x^2-y^2}} \end{vmatrix}$$

$$= \frac{x}{(x^2-y^2)^{3/2}} - \frac{y^2}{x\left(\sqrt{x^2-y^2}\right)^{3/2}}$$

$$= \frac{1}{(x^2-y^2)^{3/2}}\left[x - \frac{y^2}{x}\right] = \frac{1}{x\sqrt{x^2-y^2}}$$

$$= \frac{1}{e^r \sec\theta} \cdot \frac{1}{e^r} = \frac{1}{e^{2r}\sec\theta}$$

Hence $\dfrac{\partial(x,y)}{\partial(r,\theta)} \cdot \dfrac{\partial(r,\theta)}{\partial(x,y)} = e^{2r}\sec\theta \cdot \dfrac{1}{e^{2r}\sec\theta}$

$$= 1$$

(P) (i) If $x = \frac{u^2}{v}$, $y = \frac{v^2}{u}$ find $\frac{\partial(u,v)}{\partial(x,y)}$

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Sol:- $\frac{\partial x}{\partial u} = \frac{2u}{v}$, $\frac{\partial x}{\partial v} = -\frac{u^2}{v^2}$, $\frac{\partial y}{\partial u} = -\frac{v^2}{u^2}$, $\frac{\partial y}{\partial v} = \frac{2v}{u}$

$$\frac{\partial(x,y)}{\partial(u,v)} = 3 \Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{3}$$

(ii) If $x = uv$, $y = \frac{u}{v}$ then find $\frac{\partial(x,y)}{\partial(u,v)}$

Sol:- $\frac{\partial x}{\partial u} = v$, $\frac{\partial x}{\partial v} = u$, $\frac{\partial y}{\partial u} = \frac{1}{v}$, $\frac{\partial y}{\partial v} = -\frac{u}{v^2}$

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{2u}{v}$$

(iii) If $x = uv$, $y = \frac{u}{v}$ verify that $\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1$

Sol:- $u = \frac{x}{v} \Rightarrow u = x\left(\frac{y}{u}\right)$

$$\Rightarrow u^2 = xy \Rightarrow u = \sqrt{xy}$$

$$v = \frac{u}{y} \Rightarrow v = \frac{\sqrt{xy}}{y} = \sqrt{\frac{x}{y}}$$

(iv) If $x = u(1-v)$, $y = uv$ prove that $J \cdot J' = 1$

Sol:- $J = \frac{\partial(x,y)}{\partial(u,v)} = u$

$$x = u - uv = u - y \Rightarrow x + y = u$$

$$y = uv \Rightarrow v = \frac{y}{u} \Rightarrow v = \frac{y}{x+y}$$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{x+y} = \frac{1}{u}$$

$$\therefore J \cdot J' = 1$$

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(22)

Functional Dependence :-

Let $u = f(x, y)$ and $v = g(x, y)$ are two given differentiable functions in the independent variables x, y. Suppose these functions are connected by a relation $F(u, v) = 0$, where F is differentiable. We say that u and v functionally dependent on one another, if the partial derivatives u_x, u_y, v_x, v_y are all not zero at a time.

→ If the Jacobian $J\left(\frac{u, v}{x, y}\right) \neq 0$ then u and v are said to be functionally independent.

→ If the Jacobian $J\left(\frac{u, v}{x, y}\right) = 0$ then u and v are said to be functionally Dependent.

(P) verify whether the following functions are functionally dependent or not, if so, find the relation b/w. them.

(i) $u = \dfrac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$

(ii) $u = e^x \sin y$, $v = e^x \cos y$

(iii) $u = \dfrac{x}{y}$, $v = \dfrac{x+y}{x-y}$

$\boxed{\text{(iv)}\quad u = \dfrac{x}{y} \quad \text{and} \quad v = \dfrac{x+y}{x-y}}$

(iv) $u = x+y+z$, $v = xy+yz+zx$, $w = x^2+y^2+z^2$

(vi) $u = \dfrac{x^2-y^2}{x^2+y^2}$, $v = \dfrac{2xy}{x^2+y^2}$

(vii) $u = \dfrac{x+y}{x-y}$, $v = \dfrac{xy}{(x-y)^2}$

(viii) $u = \dfrac{x-y}{x+y}$, $v = \dfrac{xy}{(x-y)^2}$

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Sol:- (i) $u = \dfrac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$

$$\frac{\partial u}{\partial x} = \frac{(1-xy)-(x+y)(-y)}{(1-xy)^2} = \frac{1-xy+xy+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial x} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1+x^2}{(1-xy)^2}, \quad \frac{\partial v}{\partial x} = \frac{1}{1+x^2}, \quad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

Now, $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \\ \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

Hence u and v are functionally dependent

Now $v = \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) = \tan^{-1}u$

$\therefore$ $v = \tan^{-1}u$ is the functional relation b/w u an.

(ii) $$\frac{\partial(u,v)}{\partial(x,y)} = -e^{2x} \neq 0$$

$\therefore$ u and v are functionally independent

(iii) $$\frac{\partial(u,v)}{\partial(x,y)} = 0$$

Hence u and v are functionally dependent

Now $v = \dfrac{x+y}{x-y} \Rightarrow v = \dfrac{u+1}{u-1}$

$v = \dfrac{u+1}{u-1}$ is the functional relation b/w u and v

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(iv) $u = x+y+z$, $v = xy+yz+zx$, $w = x^2+y^2+z^2$

$u_x = 1$, $u_y = 1$, $u_z = 1$

$v_x = y+z$, $v_y = x+z$, $v_z = y+x$

$w_x = 2x$, $w_y = 2y$, $w_z = 2z$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ y+z & x+z & y+x \\ 2x & 2y & 2z \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ y+z & x+z & y+x \\ x & y & z \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ x & y & z \end{vmatrix} \quad R_2 \to R_2+R_3.$$

$$= 2(x+y+z)(0) \quad (\because R_1 \,\&\, R_2 \text{ are identical})$$

$$= 0$$

Hence u, v and w are functionally dependent.

i.e; The functional relationship exists b/w u, v and w.

Now, $u^2 = (x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx)$

$u^2 = w+2v$, is the functional relationship b/w u, v and w

(v) $u = \dfrac{x^2-y^2}{x^2+y^2}$, $v = \dfrac{2xy}{x^2+y^2}$

$$u_x = \frac{(x^2+y^2)(2x)-(x^2-y^2)(2x)}{(x^2+y^2)^2} = \frac{2x(\cancel{x^2}+y^2-\cancel{x^2}+y^2)}{(x^2+y^2)^2} = \frac{4xy^2}{(x^2+y^2)^2}$$

$$u_y = \frac{-4x^2y}{(x^2+y^2)^2}, \quad v_x = \frac{2y(y^2-x^2)}{(x^2+y^2)^2}, \quad v_y = \frac{2x(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = 0 \Rightarrow u, v \text{ are functionally dependent.}$$

$$u^2+v^2 = \frac{(x^2+y^2)^2}{(x^2+y^2)^2} = 1 \Rightarrow u^2+v^2=1 \text{ is the functional relation b/w } u \text{ and } v.$$

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(vi) $u = \dfrac{x+y}{x-y}$, $v = \dfrac{xy}{(x-y)^2}$

$$\frac{\partial u}{\partial x} = \frac{-2y}{(x-y)^2}, \quad \frac{\partial u}{\partial y} = \frac{2x}{(x-y)^2}$$

$$\frac{\partial v}{\partial x} = \frac{-y(x+y)}{(x-y)^3}, \quad \frac{\partial v}{\partial y} = \frac{x(x+y)}{(x-y)^3}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = 0$$

$\therefore$ u and v are functionally dependent.

We have

$$u^2 - 4v = \left(\frac{x+y}{x-y}\right)^2 - 4\left(\frac{xy}{(x-y)^2}\right) = \frac{(x-y)^2}{(x-y)^2} = 1$$

$u^2 - 4v = 1 \Rightarrow u^2 = 1+4v$ is the functional relationship b/w u and v.

(vii) $\dfrac{\partial(u,v)}{\partial(x,y)} = 0$ & $u^2+4v = 1$

(P) If $x = u\sqrt{1-v^2} + v\sqrt{1-u^2}$ and $y = \sin^{-1}u + \sin^{-1}v$ then showthat x and y are functionally related. Also find the relationship.

Sol:- $J\left(\dfrac{x,y}{u,v}\right) = 0$

$$y = \sin^{-1}u + \sin^{-1}v = \sin^{-1}\left(u\sqrt{1-v^2} + v\sqrt{1-u^2}\right)$$

$$\sin y = u\sqrt{1-v^2} + v\sqrt{1-u^2}$$

$\sin y = x$ is the required functional relation b/w x & y

(P) $u = \sin^{-1}x + \sin^{-1}y$, $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

Sol:- $v = \sin u$

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Ⓟ Show that the functions $u = x+y+z$, $v = x^2+y^2+z^2-2xy-2yz-2zx$ and $w = x^3+y^3+z^3-3xyz$ are functionally related. 26

Sol:-
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ 2(x-y-z) & 2(y-x-z) & 2(z-y-x) \\ 3(x^2-yz) & 3(y^2-xz) & 3(z^2-xy) \end{vmatrix}$$

$$= 6\begin{vmatrix} 0 & 0 & 1 \\ 2(x-y) & 2(y-z) & z-y-x \\ (x-y)(x+y+z) & (y-z)(x+y+z) & z^2-xy \end{vmatrix} \quad \begin{matrix} C_1 \to C_1-C_2 \\ C_2 \to C_2-C_3 \end{matrix}$$

$$= 12(x-y)(y-z)\begin{vmatrix} 1 & 1 \\ x+y+z & x+y+z \end{vmatrix}$$

$$= 0$$

Hence the functional relationship exists b/w u, v and w.

Ⓟ If $f(x)+f(y) = f\left(\frac{x+y}{1-xy}\right)$ and $f'(x) = \frac{1}{1+x^2}$ then using Jacobians show that $f(x)+f(y) = f\left(\frac{x+y}{1-xy}\right)$

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Maximum and Minimum of functions of Two variables :-

Def:- let $f(x,y)$ be a function of two variables x and y. At $x=a$, $y=b$, $f(x,y)$ is said to be have maximum or minimum value, if $f(a,b) > f(a+h, b+k)$ or

$f(a,b) < f(a+h, b+k)$ respectively

where h and k are small values.

Extreme Value : $f(a,b)$ is said to be an extreme value of f, if it is a maximum or minimum value.

Stationary value : $f(a,b)$ is said to be a stationary value of $f(x,y)$ if $f_x(a,b)=0$; $f_y(a,b)=0$.

Thus every extreme value is a stationary value but the converse may not be true.

Note:- let $l = \frac{\partial^2 f}{\partial x^2}$, $m = \frac{\partial^2 f}{\partial x \partial y}$, $n = \frac{\partial^2 f}{\partial y^2}$

(i) $f(a,b)$ is a maximum value if $ln-m^2>0$ and $l<0$

(ii) $f(a,b)$ is a minimum value if $ln-m^2>0$ and $l>0$

(iii) $f(a,b)$ is not an extreme value (ie; neither maximum nor minimum) if $ln-m^2<0$. In this case (a,b) is a saddle point.

(iv) If $ln-m^2=0$, then $f(x,y)$ fails to have maximum or minimum value and it needs further investigation.

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1. If $f(x,y) = xy + (x-y)$ then find the stationary points. (or) critical points

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Sol:- $\frac{\partial f}{\partial x} = \cancel{xy+x}\ y+1$, $\frac{\partial f}{\partial y} = x-1$

The stationary points are given by $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$y+1 = 0 \Rightarrow y = -1$; $x-1=0 \Rightarrow x=1$

Hence the stationary points are $(1,-1)$

2. Find the maximum and minimum values of

(i) x^3+y^3-3axy (ii) x^3+y^3-3xy

(iii) $x^3+3xy^2-15x^2-15y^2+72x$

(iv) $xy + \frac{a^3}{x} + \frac{a^3}{y}$ (v) $3x^4-2x^3-6x^2+6x+1$

(vi) $x^3y^2(1-x-y)$ (vii) $x^4+y^4-x^2-y^2+1$

Sol:- (i) let $f(x,y) = x^3+y^3-3axy$

We have $\frac{\partial f}{\partial x} = 3x^2-3ay$, $\frac{\partial f}{\partial y} = 3y^2-3ax$

Equating $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ to zero, we get

$3x^2-3ay = 0 \Rightarrow y = \frac{3x^2}{3a}$; $3y^2-3ax=0$

$y = \frac{x^2}{a}$ $\Rightarrow 3\left(\frac{x^4}{a^2}\right)-3ax=0$

$x^4 - a^3x = 0$

If $x=0$ then $y=0$ $x(x^3-a^3)=0$

$x=a$ then $y=a$ $x=0,\ x=a$

The points are $(0,a)$ and (a,a)

Now, $l = \frac{\partial^2 f}{\partial x^2} = 6x$, $m = \frac{\partial^2 f}{\partial x \partial y} = -3a$, $n = \frac{\partial^2 f}{\partial y^2} = 6y$

At $(0,0)$, $ln-m^2 = 36xy-9a^2 < 0$

At (a,a), $ln-m^2 = 27a^2 > 0$ and $l = 6a > 0$ if $a>0$

$l = 6a < 0$ if $a<0$

Thus, if $a>0$, $f=-a^3$ is the maximum value and

if $a<0$, $f=-a^3$ is the minimum value. At $(0,0)$, f does not have any extreme value

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(vi) Let $f(x,y) = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = x^2y^2(3-4x-3y)$$

$$\frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2 = x^3y(2-2x-3y)$$

$$l = \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3 = 6xy^2(1-2x-y)$$

$$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[2x^3y - 2x^4y - 3x^3y^2\right]$$

$$= 6x^2y - 8x^3y - 9x^2y^2 = x^2y(6-8x-9y)$$

$$n = \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y = 2x^3(1-x-3y)$$

$$\therefore \; ln - m^2 = 6xy^2(1-2x-y) \cdot 2x^3(1-x-3y) - (x^2y)^2(6-8x-9y)$$

$$= (x^2y)^2\left[12(1-2x-y)(1-x-3y) - (6-8x-9y)^2\right]$$

for maxima and minima, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\Rightarrow$ $x^2y^2(3-4x-3y) = 0$ and $x^3y(2-2x-3y) = 0$

$\Rightarrow$ $x=0,\ y=0$ (or) $3-4x-3y = 0 \rightarrow$ ① and

$x=0,\ y=0$ (or) $2-2x-3y = 0 \rightarrow$ ②

$3-4x-2+2x = 0$

$1-2x = 0$

$x = 1/2$

$\Rightarrow$ The possible extremum of $f(x,y)$ are

If $x=0$ ~~then~~ ① $\Rightarrow y = 1$ $(0,1)$

$y=0$ then ① $\Rightarrow x = 4/3$ $(4/3, 0)$ and

If $x=0$ ② $\Rightarrow y = 2/3$ $(0, 2/3)$

$y=0$ ② $\Rightarrow x = 1$ $(1,0)$

$(0,0)$, $(\frac{1}{2}, 1/3)$

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At all these points except $(\frac{1}{2}, \frac{1}{3})$,

$ln - m^2 = 0 \Rightarrow$ There is no extremum value

At $(\frac{1}{2}, \frac{1}{3})$, $ln - m^2 = \frac{1}{9.64} > 0$

and $l = -\frac{1}{9} < 0$

$\therefore (\frac{1}{2}, \frac{1}{3})$ is a point of maximum value

$\therefore$ Maximum value of $f = (\frac{1}{8} \cdot \frac{1}{9})(1 - \frac{1}{2} - \frac{1}{3})$

$= \frac{1}{72}(\frac{1}{2} - \frac{1}{3}) = \frac{1}{432}$

(iv) let $f(x,y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$

$\frac{\partial f}{\partial x} = y - \frac{a^3}{x^2}$, $\frac{\partial f}{\partial y} = x - \frac{a^3}{y^2}$

$\frac{\partial^2 f}{\partial x^2} = \frac{2a^3}{x^3}$, $\frac{\partial^2 f}{\partial y^2} = \frac{2a^3}{y^3}$, $\frac{\partial^2 f}{\partial x \partial y} = 1$

$\frac{\partial f}{\partial x} = 0 \Rightarrow y - \frac{a^3}{x^2} = 0 \Rightarrow y = \frac{a^3}{x^2}$

$\frac{\partial f}{\partial y} = 0 \Rightarrow x - \frac{a^3}{y^2} = 0 \Rightarrow x = \frac{a^3}{y^2}$

$y = a^3 \left[\frac{y^4}{a^6}\right] \Rightarrow y = \frac{y^4}{a^3} \Rightarrow ya^3 - y^4 = 0$

$\Rightarrow y(a^3 - y^3) = 0 \Rightarrow y = 0$ (or) $y = a$

If $y = 0 \Rightarrow x = \infty \Rightarrow$ It is not possible

$y = a \Rightarrow x = a$

$\therefore$ The extremum point is (a, a)

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At $(a,a) \Rightarrow \quad ln - m^2 = 3 > 0, \quad l = 2 > 0$

$\therefore$ $f(x,y)$ has minimum ~~maximum~~ at (a,a).

$\therefore$ The minimum value $= 3a^2$

(iii) let $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x \longrightarrow ①$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72 \quad ; \quad \frac{\partial f}{\partial y} = 6xy - 30y = 6y(x-5)$$

$$l = \frac{\partial^2 f}{\partial x^2} = 6x - 30, \quad n = \frac{\partial^2 f}{\partial y^2} = 6x - 30$$

$$m = \frac{\partial^2 f}{\partial x \partial y} = 6y$$

The critical points of f are $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

$$3x^2 + 3y^2 - 30x + 72 = 0, \quad 6y(x-5) = 0$$

$$x^2 + y^2 - 10x + 24 = 0 \longrightarrow ② \quad , \quad y = 0 \text{ (or) } x = 5$$

Case (i) :-

If $y = 0$ then $② \Rightarrow x^2 - 10x + 24 = 0$

$$x^2 - 6x - 4x + 24 = 0$$

$$(x-6)(x-4) = 0 \Rightarrow x = 4 \text{ or } 6$$

The points are $(4,0)$, $(6,0)$

Case (ii) :-

If $x = 5$ then $② \Rightarrow 5^2 + y^2 - 50 + 24 = 0$

$$y^2 - 1 = 0 \Rightarrow y = \pm 1$$

The points are $(1,5)$, $(-1,5)$

$\therefore$ The critical points are $A(4,0)$, $B(6,0)$, $C(1,5)$, $D(-1,5)$

~~At (4,0)~~ $ln - m^2 = (6x-30)^2 - 36y^2$

$$= 36[(x-5)^2 - y^2]$$

At $(4,0)$, $ln - m^2 = 36 > 0$, $l = 24 - 30 = -6 < 0$

$(6,0)$, $ln - m^2 = 36 > 0$, $l = 36 - 30 = 6 > 0$

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At $(5,1)$, $ln - m^2 = -36 < 0$

At $(5,-1)$, $ln - m^2 = -36 < 0$

C and D are Saddle points

A and B are extremum points.

$\Rightarrow$ A is the point of Maximum for f and B is the point of Minimum for f.

$\therefore$ Maximum value $= 112$

Minimum value $= 108$

(v) let $f(x,y) = 3x^4 - 2x^3 - 6x^2 + 6x + 1 \longrightarrow (1)$

$$\frac{\partial f}{\partial x} = 6(2x^3 - x^2 - 2x + 1) \quad ; \quad \frac{\partial f}{\partial y} = 0$$

for maxima or minima, $\frac{\partial f}{\partial x} = 0$

$$2x^3 - x^2 - 2x + 1 = 0 \Rightarrow (x-1)(x+1)(2x-1) = 0$$

$$x = 1, -1, \tfrac{1}{2}$$

These are the possible extreme points

$$\frac{\partial^2 f}{\partial x^2} = 12(3x^2 - x - 1)$$

When $x = 1 \Rightarrow \frac{\partial^2 f}{\partial x^2} = 12 > 0 \Rightarrow f(x)$ has minimum at $x = 1$

When $x = -1 \Rightarrow \frac{\partial^2 f}{\partial x^2} = 36 > 0 \Rightarrow f(x)$ has minimum at $x = -1$

When $x = \frac{1}{2} \Rightarrow \frac{\partial^2 f}{\partial x^2} = -9 < 0 \Rightarrow f(x)$ has maximum at $x = \frac{1}{2}$

$\therefore$ Maximum value of $f = \frac{39}{16}$

Minimum values of $f = -6, 2$

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(vii) let $f(x,y) = x^4 + y^4 - x^2 - y^2 + 1$

$$\frac{\partial f}{\partial x} = 4x^3 - 2x \qquad \frac{\partial f}{\partial y} = 4y^3 - 2y$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 4x^3 - 2x = 0 \quad ; \quad \frac{\partial f}{\partial y} = 0 \Rightarrow 2y(2y^2-1) = 0$$

$$2x(2x^2-1) = 0$$

$x = 0$ (or) $x^2 = 1/2$ $\qquad$ $y = 0$ (or) $y = \pm\frac{1}{\sqrt{2}}$

$x = \pm\frac{1}{\sqrt{2}}$

The possible points are $(0,0)$, $(0, \pm\frac{1}{\sqrt{2}})$, $(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}})$, $(\pm\frac{1}{\sqrt{2}}, 0)$

$$l = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 2 \quad , \quad n = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 2$$

$$m = \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$ln - m^2 = (12x^2-2)(12y^2-2)$$

Max. at $(0,0)$, Maximum value $= 1$

Min. at $(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$, Minimum value $= \frac{1}{2}$

(P) Show that the function $f(x,y) = x^3 + y^3 - 63(x+y) + 12xy$ is maximum at $(-7,-7)$ and minimum at $(3,3)$.

(P) find the extreme values of $u = x^2y^2 - 5x^2 - 8xy - 5y^2$

Soln: $(-1,1)$, $(1,-1)$, $(3,3)$, $(-3,-3)$, $(0,0)$

$(1,-1)$, $(3,3)$, $(-3,-3)$ are Saddle points

$(1,-1) \Rightarrow ln - m^2 < 0$

$(0,0) \Rightarrow$ Maximum & Max. value $= 0$

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(P) A rectangular box open at the top is to have volume 32 cubic ft. Find the dimensions of the box requiring least material for its Construction. (34)

Sol:-

let x ft, y ft, z ft be the dimensions of the box

let S be the surface of the box.

Then $S = xy + 2yz + 2zx$ ($\because$ open at the top) $\longrightarrow (1)$

Its volume, $xyz = 32 \longrightarrow (2)$

$$z = \frac{32}{xy}$$

$$(1) \Rightarrow S = xy + 2y\left(\frac{32}{xy}\right) + 2\left(\frac{32}{xy}\right)x$$

$$S = xy + \frac{64}{x} + \frac{64}{y}$$

$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2}, \quad \frac{\partial S}{\partial y} = x - \frac{64}{y^2}$$

$$\frac{\partial S}{\partial x} = 0, \quad \frac{\partial S}{\partial y} = 0 \Rightarrow y = \frac{64}{x^2}, \quad x = \frac{64}{y^2}$$

$$y = \frac{64}{\left(\frac{64}{y^2}\right)^2} \Rightarrow y = \frac{64y^4}{(64)^2}$$

$$64y = y^4$$

$$64y - y^4 = 0 \Rightarrow y = 0 \text{ (or) } y = 4$$

If $y = 0 \Rightarrow x = \infty$, It is not possible

$y = 4 \Rightarrow x = 4$

$$(2) \Rightarrow z = \frac{32}{(4)(4)} = 2$$

$\therefore$ The dimensions of the box for least material of its Construction are 4, 4, 2.

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$$\frac{216}{216}$$

(8) A rectangular box open at the top is to have a given capacity. Find the dimensions of the box requiring least material for its construction.

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Sol:-

let x, y, z be the dimensions of the box.

let S be the surface of the box.

then $S = xy + 2yz + 2zx \rightarrow (1)$ (open at the top)

let $V = xyz \Rightarrow z = \frac{V}{xy} \rightarrow (2)$

$(1) \Rightarrow S = xy + 2y\left(\frac{V}{xy}\right) + 2x\left(\frac{V}{xy}\right)$

$$S = xy + \frac{2V}{x} + \frac{2V}{y} \rightarrow (3)$$

$\frac{\partial S}{\partial x} = 0 \Rightarrow y - \frac{2V}{x^2} = 0 \Rightarrow y = \frac{2V}{x^2} \longrightarrow (4)$

$\frac{\partial S}{\partial y} = 0 \Rightarrow x - \frac{2V}{y^2} = 0 \Rightarrow x = \frac{2V}{y^2} \longrightarrow (5)$

$y = 2V \cdot \frac{1}{\left(\frac{2V}{y^2}\right)^2} \Rightarrow y = \cancel{2V} \cdot \frac{y^4}{(2V)^{\cancel{2}}}$

$$y = \frac{y^4}{2V}$$

$$y^4 = 2Vy$$

$y[y^3 - 2V] = 0 \Rightarrow y = 0$ (or) $y^3 = 2V$

$y = (2V)^{1/3}$

If $y = 0 \Rightarrow x = \infty$, it is not possible

If $y = (2V)^{1/3} \Rightarrow (5) \rightarrow x = \frac{2V}{(2V)^{2/3}} \Rightarrow x = (2V)^{1/3}$

$(2) \Rightarrow z = V \cdot \frac{1}{(2V)^{2/3}} \Rightarrow z = \frac{1}{2^{2/3}} \cdot V^{1-\frac{2}{3}}$

$$z = V^{1/3}/2^{2/3}$$

The dimensions of the box requiring least material = $x = y = (2V)^{1/3}$, $z = V^{1/3}/2^{2/3}$

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(36)

(P) Examine for minimum and maximum values of $\sin x + \sin y + \sin(x+y)$

Sol:- let $f(x,y) = \sin x + \sin y + \sin(x+y) \longrightarrow (1)$

$$\frac{\partial f}{\partial x} = \cos x + \cos(x+y) \quad ; \quad \frac{\partial f}{\partial y} = \cos y + \cos(x+y)$$

$$l = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y) \quad ; \quad n = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$$

$$m = \frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow \cos x + \cos(x+y) = 0 \longrightarrow (2)$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow \cos y + \cos(x+y) = 0 \longrightarrow (3)$$

$$(2) - (3) \Rightarrow \cos x - \cos y = 0 \Rightarrow \cos x = \cos y$$

$$x = y$$

$$(2) \Rightarrow \cos x + \cos 2x = 0$$

$$2\cos\frac{3x}{2}\cos\frac{x}{2} = 0$$

$$\Rightarrow \cos\frac{3x}{2} = 0 \quad (\text{or}) \quad \cos\frac{x}{2} = 0$$

$$\frac{3x}{2} = \pm\frac{\pi}{2} \quad (\text{or}) \quad \frac{x}{2} = \pm\frac{\pi}{2}$$

$$x = \pm\frac{\pi}{3} \quad (\text{or}) \quad x = \pm\pi$$

$$\therefore y = \pm\frac{\pi}{3} \quad (\text{or}) \quad y = \pm\pi$$

$\therefore$ The critical points are $\left(\pm\frac{\pi}{3}, \pm\frac{\pi}{3}\right), (\pm\pi, \pm\pi)$

$$ln - m^2 = [\sin x + \sin(x+y)][\sin y + \sin(x+y)] - [\sin(x+y)]^2 \longrightarrow (4)$$

At $(\pm\pi, \pm\pi)$, $ln - m^2 = 0$. There is a need for further investigation

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At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $ln - m^2 = \left(\sin\frac{\pi}{3} + \sin\frac{2\pi}{3}\right)^2 - \left(\sin\frac{2\pi}{3}\right)^2$

$= \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2$

$= 3 - \frac{3}{4} = \frac{9}{4} > 0$

and $l = -\sqrt{3} < 0$

$\Rightarrow$ f has maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$\therefore$ Maximum value of $f = \frac{3\sqrt{3}}{2}$

At $\left(-\frac{\pi}{3}, -\frac{\pi}{3}\right)$, $ln - m^2$ = positive > 0 and l = positive > 0

$\Rightarrow$ f has minimum at $\left(-\frac{\pi}{3}, -\frac{\pi}{3}\right)$

$\therefore$ Minimum value of $f = -\frac{3\sqrt{3}}{2}$

(P) Find the stationary points of $u(x,y) = \sin x \sin y \sin(x+$ where $0 < x < \pi$, $0 < y < \pi$ and find the maximum u.

Sol:-

$x = \frac{\pi}{3}, \frac{\pi}{3}$

$ln - m^2 = > 0$, $l < 0$

Maximum $= \frac{3\sqrt{3}}{8}$

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(P) If the sum of three numbers is a constant, then prove that their product is maximum when they are equal.

(38)

Sol:- let x, y, z be the required three numbers

Then $x+y+z = k$ →(1) and $f(x,y,z) = xyz$ →(2)

$$f(x,y) = xy(k-x-y)$$

$f_x = 0 \Rightarrow y(k-2x-y) = 0 \Rightarrow y=0$ (or) $2x+y=k$

$f_y = 0 \Rightarrow x(k-x-2y) = 0 \Rightarrow x=0$ (or) $x+2y=k$

Solving, $x = y = \frac{k}{3}$

$l = \frac{\partial^2 f}{\partial x^2} = -2y$, $m = \frac{\partial^2 f}{\partial x \partial y} = k-2x-2y$

$n = \frac{\partial^2 f}{\partial y^2} = -2x$

$$ln - m^2 = 4xy - (k-2x-2y)^2$$

At $\left(\frac{k}{3}, \frac{k}{3}\right)$, $ln - m^2 = \frac{k^2}{3} > 0$, $l = \frac{-2k}{3} < 0$

$f(x,y)$ has a maximum at $\left(\frac{k}{3}, \frac{k}{3}\right)$

(1) $\Rightarrow z = k - \frac{2k}{3} = \frac{k}{3}$

$\therefore x = y = z = \frac{k}{3}$

The product is maximum when all the three numbers are equal.

(P) Find three positive numbers whose sum is 100 and whose product is maximum.

(P) Divide 24 into three parts such that the continued product of the first, square of the second and cube of the third is maximum.

Sol:- $x+y+z = 24$, $f(x,y,z) = x^3y^2z$, $x = 12, y = 8, z = 4$

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P Find the minimum value of $x^2+y^2+z^2$ given that $xyz = a^3$

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Sol:- let $f(x,y,z) = x^2+y^2+z^2$, $xyz = a^3$

$\Rightarrow z = \frac{a^3}{xy}$

$f(x,y) = x^2+y^2+\frac{a^6}{x^2y^2}$

$\frac{\partial f}{\partial x}=0 \Rightarrow 2x - \frac{2a^6}{x^3y^2} = 0$, $\frac{\partial f}{\partial y} = 0 \Rightarrow 2y - \frac{2a^6}{x^2y^3} = 0$

Solving $x = a, y = a$

At (a,a), $ln-m^2 = 48>0$, $l>0$

f has minimum at (a,a)

minimum value $= 3a^2$

P Find the shortest distance from origin to the surface $xyz^2 = 2$

Sol:- Let $P(x,y,z)$ be any point on the surface $xyz^2 = 2$ $\rightarrow$ ①

Then $OP = d = \sqrt{x^2+y^2+z^2}$

let $f(x,y,z) = d^2 = x^2+y^2+z^2$

$f(x,y) = x^2+y^2+\frac{2}{xy}$

$f_x = 2x - \frac{2}{x^2y}$, $f_y = 2y - \frac{2}{xy^2}$

$f_x = 0 \Rightarrow x^3y = 1$, $xy^3 = 1$

$\Rightarrow x^3y = xy^3 \Rightarrow xy(x^2-y^2) = 0$ $(\because x\neq 0, y\neq 0)$

$x^2-y^2=0 \Rightarrow x = \pm y$

The critical points are $(1,1)$, $(1,-1)$

$f_{xx} = 2+\frac{4}{x^3y}$, $f_{yy} = 2+\frac{4}{xy^3}$, $f_{xy} = \frac{2}{x^2y^2}$

At $(1,1) \Rightarrow ln-m^2 = 32>0$, $l = 6>0$

f has minimum at $(1,1)$; $z = \pm\sqrt{2}$

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So, minimum occurs at $(1, 1, \sqrt{2})$

Hence the shortest distance from the origin is

$$= \sqrt{1+1+2} = 2$$

(P) find the points on the surface $z^2 = xy+1$ that are nearest to the origin.

Sol:- let $P(x,y,z)$ be any point on the surface

let ~~A(x,y,z)~~ $z^2 = xy+1$

let $OP = d = \sqrt{x^2+y^2+z^2}$

let $f(x,y,z) = d^2 = x^2+y^2+z^2$

$f(x,y) = x^2+y^2+xy+1$

$f_x = 2x+y$, $f_y = 2y+x$

$l = f_{xx} = 2$, $n = f_{yy} = 2$, $m = f_{xy} = 1$

$ln - m^2 = 4-1 = 3 > 0$, $l = 2 > 0$

$f_x = 0$, $f_y = 0$ $\Rightarrow$ $2x+y=0$, $2y+x=0$

$2(-2y)+y=0$

$z^2 = 1$ $\qquad -4y+y=0 \Rightarrow y=0, \quad x=0$

$z = \pm 1$

$\therefore$ The points are $A(0,0,1)$, $B(0,0,-1)$

At A, B $\Rightarrow$ f has minimum

$\therefore$ Required points are $(0,0,1)$, $(0,0,-1)$

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Lagrange's Method of undetermined Multipliers :-

To find the maxima or minima for a function $f(x,y,z) = 0$ subject to the Conditions $\phi(x,y,z) = 0$, $\psi(x,y,z) = 0$, then Lagrangian function as

$$F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z) + \mu\,\psi(x,y,z)$$

where λ and μ are the Lagrange multipliers.

① Find the point on the plane $x+2y+3z = 4$ that is closest to the origin.

Sol:- Let $P(x,y,z)$ be a point on the given plane $(x+2y+3z=4)$

Then $op = \sqrt{x^2+y^2+z^2} \Rightarrow op^2 = x^2+y^2+z^2$

let $f(x,y,z) = x^2+y^2+z^2 \longrightarrow ①$

Now we have to minimize ① subject to the Condition

$$\phi(x,y,z) = x+2y+3z-4 = 0 \longrightarrow ②$$

Consider the Lagrangean function

$$F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)$$

$$F(x,y,z) = (x^2+y^2+z^2) + \lambda(x+2y+3z-4).$$

for F to be minima, $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$

$$2x + \lambda = 0 \Rightarrow x = -\frac{\lambda}{2}$$

$$2y + 2\lambda = 0 \Rightarrow y = -\lambda$$

$$2z + 3\lambda = 0 \Rightarrow z = -3\lambda/2$$

② $\Rightarrow$ $-\frac{\lambda}{2} + 2(-\lambda) + 3\left(-\frac{3\lambda}{2}\right) - 4 = 0$

$$-\lambda - 4\lambda - 9\lambda - 8 = 0 \Rightarrow -14\lambda - 8 = 0$$

$$-14\lambda = 8 \Rightarrow \lambda = \frac{-4}{7}$$

$\therefore$ $x = \frac{2}{7}$, $y = \frac{4}{7}$, $z = 6/7$

Hence $\left(\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right)$ is the point on the given plane which is nearest to the origin

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(P) Find the point on the plane $3x+2y+z-12=0$, which is nearest to the origin. 42

Sol:- $op = \sqrt{x^2+y^2+z^2}$, $op^2 = x^2+y^2+z^2 = f(x,y,z)$ (say).

$\phi(x,y,z) = 3x+2y+z-12=0$

$x = \frac{-3}{2}\lambda$, $y = -\lambda$, $z = -\frac{\lambda}{2}$ $\Rightarrow$ $\lambda = \frac{-12}{7}$

$x = \frac{18}{7}$, $y = \frac{12}{7}$, $z = \frac{6}{7}$

Note:- Minimum value of $op = \sqrt{x^2+y^2+z^2} = \sqrt{\frac{72}{7}}$

(P) Find the minimum value of $x^2+y^2+z^2$, given that $xyz = a^3$

Sol:- $f(x,y,z) = x^2+y^2+z^2$, $\phi(x,y,z) = xyz-a^3=0$ ——— ①

$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x+\lambda yz=0 \Rightarrow \lambda = \frac{-2x}{yz}$

$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y+\lambda xz=0 \Rightarrow \lambda = \frac{-2y}{xz}$

$\frac{\partial F}{\partial z} = 0 \Rightarrow 2z+\lambda yx=0 \Rightarrow \lambda = \frac{-2z}{xy}$

$\therefore \frac{x}{yz} = \frac{y}{xz} = \frac{z}{xy} = \frac{-\lambda}{2}$

Take, $\frac{x}{yz} = \frac{y}{xz} \Rightarrow x^2 = y^2$

Take, $\frac{y}{xz} = \frac{z}{xy} \Rightarrow y^2 = z^2$

$\Rightarrow x^2 = y^2 = z^2 \Rightarrow x = y = z$

① $\Rightarrow x^3 - a^3 = 0 \Rightarrow x = a,\ y = a,\ z = a$

$\therefore$ Minimum value of $f = 3a^2$

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(P) Find the minimum value of $x^2+y^2+z^2$ given $x+y+z=3a$

Sol:- $x = y = z = a$, Minimum value of $f = 3a^2$ 43

(P) Find the maximum and minimum values of $x+y+z$ subject to $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$

Sol:- $x = y = z = \pm\sqrt{\lambda}$ $\Rightarrow$ $\frac{1}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}=1$ $\Rightarrow$ $\frac{3}{\sqrt{\lambda}}=1$ $\rightarrow$ $\sqrt{\lambda}=3$

$\therefore$ $x = y = z = \pm 3$

$\therefore$ Maximum value of $f = 9$

Minimum value of $f = -9$

(P) ~~Find the~~ Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

(OR)

Find the rectangular parallelopiped of maximum volume that can be inscribed in a sphere.

Sol:- Let 'a' be the radius of the given sphere

Let x, y, z be the length, breadth and height of a rectangular parallelopiped inscribed in the given sphere.

The equation of the sphere is $x^2+y^2+z^2=a^2$ —①

Volume of the rectangular parallelopiped is $V = xyz$ —②

Take $F(x,y,z) = f(x,y,z) + \lambda(x^2+y^2+z^2-a^2)$

$F(x,y,z) = xyz + \lambda(x^2+y^2+z^2-a^2)$

for maxima & minima,

$\frac{\partial F}{\partial x} = 0 \Rightarrow yz + \lambda(2x) = 0$ ③ $\Rightarrow$ $x = \frac{-yz}{2\lambda}$, $y = \frac{-xz}{2\lambda}$, $z = \frac{-xy}{2\lambda}$

① $\Rightarrow$ $\frac{1}{4\lambda^2}+\frac{1}{4\lambda^2}+\frac{1}{4\lambda^2}=a^2$ $\Rightarrow$ $\frac{3}{4\lambda^2}=a^2$ $\rightarrow$ $\lambda^2=\frac{3}{4a^2}$

$\lambda = \pm\frac{\sqrt{3}}{2a}$

$x = \frac{1}{2\left(\frac{\sqrt{3}}{2a}\right)} = \frac{a}{\sqrt{3}}$, $y = \frac{a}{\sqrt{3}}$, $z = a/\sqrt{3}$

$\therefore$ Maximum value $= xyz = a^3/3\sqrt{3}$ cu. units

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$\frac{\partial F}{\partial y} = 0 \Rightarrow xz + \lambda(2y) = 0 \longrightarrow (4) \Rightarrow y = \frac{-xz}{2\lambda}$

$\frac{\partial F}{\partial z} = 0 \Rightarrow z = \frac{-xy}{2\lambda} \Rightarrow xy + \lambda(2z) = 0 \longrightarrow (5)$

$(3) \Rightarrow xyz = -2\lambda x^2$

$(4) \Rightarrow xyz = -2\lambda y^2$

$(5) \Rightarrow xyz = -2\lambda z^2$

from the above $x^2 = y^2 = z^2 \Rightarrow x = y = z$

$(3) \Rightarrow x^3 = -2\lambda x^2 \Rightarrow x = -2\lambda \longrightarrow (6)$

$(1) \Rightarrow (-2\lambda)^2 + (-2\lambda)^2 + (-2\lambda)^2 = a^2$

$4\lambda^2 + 4\lambda^2 + 4\lambda^2 = a^2 \Rightarrow 12\lambda^2 = a^2 \Rightarrow \lambda^2 = \frac{a^2}{12}$

$\lambda = \pm \frac{a}{2\sqrt{3}}$

$x = -2\left(-\frac{a}{2\sqrt{3}}\right) \Rightarrow x = \frac{a}{\sqrt{3}}$

$x = y = z = \frac{a}{\sqrt{3}}$

$\therefore$ Maximum value $= (v) = xyz$

$= \left(\frac{a}{\sqrt{3}}\right)^3 = \frac{a^3}{3\sqrt{3}}$ cu. units

(P) Find the volume of the largest rectangular Parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Sol:- let $2x, 2y, 2z$ be the length, breadth and height of the rectangular parallelopiped that can be inscribed in the ellipsoid. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \longrightarrow (1)$

let V be the volume of the parallelopiped then

$V = 8xyz \longrightarrow (2)$

$= f(x,y,z)$

Consider the Lagrangean function:

$F(x,y,z) = f(x,y,z) + \lambda\, \phi(x,y,z)$

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$$F(x,y,z) = (8xyz) + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$ 45

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0$$

$$8yz + \lambda\left(\frac{2x}{a^2}\right) = 0 \Rightarrow \lambda = -8yz\left(\frac{a^2}{2x}\right)$$

$$\frac{\lambda}{-4} = \frac{a^2yz}{x} \longrightarrow (3)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\lambda}{-4} = \frac{b^2xz}{y} \quad \text{—} (4)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\lambda}{-4} = \frac{c^2xy}{z} \quad \text{—} (5)$$

from (3), (4), (5) $\Rightarrow$ $$\frac{a^2yz}{x} = \frac{b^2xz}{y} = \frac{c^2xy}{z} = -\frac{\lambda}{4}$$

$$\frac{a^2y}{x} = \frac{b^2x}{y} \Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2} \quad \text{—} (6)$$

$$\frac{b^2x}{y} = \frac{c^2y}{z} \Rightarrow \frac{y^2}{b^2} = \frac{z^2}{c^2} \quad \text{—} (7)$$

$$\therefore \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

$$(1) \Rightarrow \frac{3x^2}{a^2} = 1 \Rightarrow x^2 = \frac{a^2}{3} \Rightarrow x = \frac{a}{\sqrt{3}}$$

$$\therefore y = \frac{b}{\sqrt{3}}, \quad z = \frac{c}{\sqrt{3}}$$

$\therefore$ volume $(V) = 8xyz$

$$= \frac{8abc}{3\sqrt{3}} \text{ cu. units}$$

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① If $x = a \cosh u \cos v$, $y = a \sinh u \sin v$

then P.T $\frac{\partial(x,y)}{\partial(u,v)} = \frac{a^2}{2}[\cosh 2u - \cos 2v]$

Sol:-

$$\left[\because \begin{aligned} &\cosh^2 u - \sinh^2 u = 1 \\ &\cosh^2 u + \sinh^2 u = \cosh 2u \\ &\sin^2 u + \cos^2 u = 1, \quad \cos^2 u - \sin^2 u = \cos 2u \end{aligned}\right]$$

$\frac{\partial x}{\partial u} = a \sinh u \cos v$; $\frac{\partial x}{\partial v} = -a \cosh u \sin v$

$\frac{\partial y}{\partial u} = a \cosh u \sin v$; $\frac{\partial y}{\partial v} = a \sinh u \cos v$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} a \sinh u \cos v & -a \cosh u \sin v \\ a \cosh u \sin v & a \sinh u \cos v \end{vmatrix}$$

$$= a^2 \sinh^2 u \cos^2 v + a^2 \cosh^2 u \sin^2 v$$

$$= a^2 \sinh^2 u \cos^2 v + a^2 (1 + \sinh^2 u) \sin^2 v$$

$$= a^2 \sinh^2 u \cos^2 v + a^2 \sinh^2 u \sin^2 v + a^2 \sin^2 v$$

$$= a^2 \sinh^2 u + a^2 \sin^2 v$$

$$= \frac{a^2}{2}[2 \sinh^2 u + 2 \sin^2 v]$$

$$= \frac{a^2}{2}[\sinh^2 u + \sinh^2 u + \sin^2 v + \sin^2 v]$$

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$$= \frac{a^2}{2}\left[\sinh^2 u + \cosh^2 u - \cancel{1} + \sin^2 v + \cancel{1} - \cos^2 v\right]$$

$$= \frac{a^2}{2}\left[\cosh 2u - (\cos^2 v - \sin^2 v)\right]$$ 47

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{a^2}{2}\left[\cosh 2u - \cos 2v\right]$$

UNIT-IV

Multiple Integrals

UNIT-IV

MULTIPLE INTEGRALS

Let $y = f(x)$ be a function of single independent var., defined and bounded on $[a, b]$.

Let $[a, b]$ be divided into sub intervals by points $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ such that $a = x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n = b$

The limit is defined to be the definite integral $\int_a^b f(x)\,dx$ between the limits $x = a$ and $x = b$.

The generalisation of this definition to two dimensions $f(x, y)$ is called Double Integral and to three dimensions $f(x, y, z)$ is called Triple Integral.

Double Integrals:

Let $f(x, y)$ be a continuous function of the independent variables x, y defined on the region $\mathbb{R}$ of the xy-plane. If the region R is bounded by the curves $x = x_1, x = x_2, y = y_1$ and $y = y_2$. Then $\iint_R f(x, y)\,dx\,dy$ is called Double integral.

$$\iint_R f(x, y)\,dx\,dy = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y)\,dx\,dy.$$

Eg: 1) $\int_0^2\int_0^x y\,dy\,dx$ 2) $\int_{x=0}^{a}\int_{y=0}^{b} (x^2 + y^2)\,dy\,dx$

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Properties of Double Integrals:

Let f and g be functions of x and y, defined and continuous in a region R. Then

i) $\iint_R (f+g)\,dx\,dy = \iint_R f\,dx\,dy + \iint_R g\,dx\,dy$

ii) $\iint_R k f\,dx\,dy = k\iint_R f\,dx\,dy$, where k is a constant

iii) $\iint_R f\,dx\,dy = \iint_{R_1} f\,dx\,dy + \iint_{R_2} f\,dx\,dy$

when R_1 and R_2 are two disjoint regions whose union is R.

iv) There exists atleast one point (x_0, y_0) in R such that we have

$$\iint_R f(x,y)\,dx\,dy = f(x_0,y_0)\cdot A \quad (\text{given that } f \text{ is continuous in } R)$$

where A is the area of R and this is called the mean value theorem for double integrals.

NOTE: Iterated Integral: An expression of the form

$$\int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx \quad (\text{OR}) \quad \int_a^b \int_{x_1(y)}^{x_2(y)} f(x,y)\,dx\,dy$$

is called an iterated integral.

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Evaluation Of Double Integrals :

Double Integrals over a region R may be evaluated by two successive integrations as follows:

i) Suppose that R can be described by inequalities of the form $a \le x \le b$, $y_1(x) \le y \le y_2(x)$ represent the boundary of R.

Then $$\int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx = \int_{x=a}^{b}\left[\int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy\right]dx$$

✓ The integral within the [ ] brackets is evaluated first by integrating the integrand partially w.r.t. 'y' treating x as constant. And then the resulting fn of x from a to b.

ii) Similarly, if R can be described by inequalities of the form $a \le y \le b$, $x_1(y) \le x \le x_2(y)$ then we obtain,

$$\iint_R f(x,y)\,dx\,dy = \int_a^b\left[\int_{x_1(y)}^{x_2(y)} f(x,y)\,dx\right]dy$$

We now integrate first over x (treating y as a constant) and then the resulting fn of y from a to b.

iii) Suppose the limits of x are x_1 and x_2 i.e., $x = x_1$, $x = x_2$ and the limits of y are y_1 and y_2 i.e., $y = y_1$, $y = y_2$. And if x_1, x_2, y_1, y_2 all are constant limits, then the order of integration is not considered.

$$\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2}\left[\int_{x_1}^{x_2} f(x,y)\,dx\right]dy = \int_{x_1}^{x_2}\left[\int_{y_1}^{y_2} f(x,y)\,dy\right]dx$$

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iv) If the limits of y i.e., $y_1 = \phi(x_1)$; $y_2 = \phi(x_2)$ and the limits of x i.e., x_1, x_2 are constants then the order of integration is

$$\iint_R f(x,y)\,dy\,dx = \int_{x_1}^{x_2}\left[\int_{y=\phi(x_1)}^{y=\phi(x_2)} f(x,y)\,dy\right]dx$$

v) If the limits of x i.e., $x_1 = \psi(y_1)$; $x_2 = \psi(y_2)$ and the limits of y i.e., y_1, y_2 are constants then the order of integration is

$$\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2}\left[\int_{x_1=\psi(y_1)}^{x_2=\psi(y_2)} f(x,y)\,dx\right]dy.$$

vi) If $f(x,y) = 1$ then $\iint_R f(x,y)\,dx\,dy = \iint_R dx\,dy = R$ represents the area of the region 'R'.

Double Integrals In Polar Co-ordinates :

✓ To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$ over the region bounded by the lines $\theta = \theta_1$, $\theta = \theta_2$ and the curves $r = r_1$, $r = r_2$ We first integrate w.r.t. 'r' between limits $r = r_1$ and $r = r_2$ keeping θ fixed. The resulting expression is integrated w.r.t. θ from θ_1 to θ_2.

✓ The region R is defined by the lines r is function of θ i.e., $r_1 = \phi(\theta_1)$ and $r_2 = \phi(\theta_2)$ and $\theta = \alpha$; $\theta = \beta$ then to evaluate

$$\iint_R f(r,\theta)\,dr\,d\theta = \int_{\alpha}^{\beta}\left[\int_{r_1=\phi(\theta_1)}^{r_2=\phi(\theta_2)} f(r,\theta)\,dr\right]d\theta$$

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PROBLEMS:

1) Evaluate $\int_0^2 \int_0^x e^{x+y}\, dy\, dx$ (2000)

Soln: Given Integral is,

$$\int_0^2 \int_0^x e^{x+y}\, dy\, dx$$

$$\Rightarrow \int_0^2 \int_0^x e^x \cdot e^y\, dy\, dx$$

$$\Rightarrow \int_0^2 e^x \left[\int_0^x e^y\, dy\right] dx$$

$$\Rightarrow \int_0^2 e^x \left[e^y\right]_0^x dx$$

$$\Rightarrow \int_0^2 e^x (e^x - e^0)\, dx$$

$$\Rightarrow \int_0^2 (e^{2x} - e^x)\, dx$$

$$= \left(\frac{e^{2x}}{2} - e^x\right)_0^2$$

$$= \frac{e^4}{2} - \frac{e^0}{2} - e^2 + e^0$$

$$= \frac{e^4}{2} - e^2 - \frac{1}{2} + 1$$

$$= \frac{e^4}{2} - e^2 + \frac{1}{2} = \frac{1}{2}(e^4 - 2e^2 + 1)$$

$$= \frac{1}{2}(e^2 - 1)^2.$$

2) Evaluate $\int_0^{\pi/2} \int_{-1}^1 x^2 y^2\, dx\, dy$ (2001)

Soln: Given Integral is,

$$\int_0^{\pi/2} \int_{-1}^1 x^2 y^2\, dx\, dy$$

$$= \int_0^{\pi/2} y^2 \left[\int_{-1}^1 x^2\, dx\right] dy$$

$$= \int_0^{\pi/2} y^2 \left[\frac{x^3}{3}\right]_{-1}^1 dy$$

$$= \int_0^{\pi/2} y^2 \left(\frac{1}{3} + \frac{1}{3}\right) dy$$

$$= \frac{2}{3}\int_0^{\pi/2} y^2\, dy = \frac{2}{3} \cdot \left[\frac{y^3}{3}\right]_0^{\pi/2}$$

$$= \frac{2}{3} \cdot \left[\frac{1}{3}\left(\frac{\pi^3}{8} - 0\right)\right]$$

$$= \frac{2}{9} \cdot \left(\frac{\pi^3}{8}\right) = \frac{\pi^3}{36}.$$

3) Evaluate $\int_0^5 \int_0^{x^2} x(x^2+y^2)\, dx\, dy$ 04, 06, 07, 2011

Soln: Given integral is,

$$\int_0^5 \int_0^{x^2} x(x^2+y^2)\, dx\, dy$$

$$= \int_{x=0}^5 \int_{y=0}^{x^2} (x^3 + xy^2)\, dx\, dy$$

$$= \int_{x=0}^5 \left[\int_{y=0}^{x^2} (x^3 + xy^2)\, dy\right] dx$$

$$= \int_0^5 \left(x^3 y + x\frac{y^3}{3}\right)_0^{x^2} dx$$

$$= \int_0^5 \left[x^3 \cdot x^2 - x^3(0) + x \cdot \frac{x^6}{3} - x \cdot \frac{0}{3}\right] dx$$

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(2001) $= \int_0^5 \left(x^5 + \frac{x^7}{3}\right) dx$

$= \left[\frac{x^6}{6} + \frac{1}{3}\cdot\frac{x^8}{8}\right]_0^5$

$= \frac{5^6}{6} + \frac{5^8}{24} = 5^6\left(\frac{1}{6} + \frac{5^2}{24}\right)$

$= 5^6\left(\frac{4+25}{24}\right)$

$= \frac{5^6 \cdot (29)}{24}$

4> Evaluate $\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{1-x^2}\sqrt{1-y^2}}$ (3)

Soln: The given integral

$\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{1-x^2}\sqrt{1-y^2}}$

$= \int_0^1 \frac{1}{\sqrt{1-y^2}}\left[\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx\right]dy$

$= \int_0^1 \frac{1}{\sqrt{1-y^2}}\left(\sin^{-1}x\right)_0^1 dy$

$= \int_0^1 \frac{1}{\sqrt{1-y^2}}\left[\sin^{-1}(1) - \sin^{-1}(0)\right] dy$

$= \frac{\pi}{2}\int_0^1 \frac{1}{\sqrt{1-y^2}}\,dy$

$= \frac{\pi}{2}\left(\sin^{-1}y\right)_0^1 = \frac{\pi}{2}\cdot\frac{\pi}{2}$

$= \frac{\pi^2}{4}$

5> Evaluate $\int_0^3\int_1^2 xy(x+y)\,dx\,dy$

2008

Soln: The given integral is,

(1) $\int_0^3\int_1^2 xy(x+y)\,dx\,dy$

$= \int_0^3\int_1^2 (x^2y + xy^2)\,dx\,dy$

$= \int_0^3 \left[\int_1^2 (x^2y + xy^2)\,dx\right]dy$

$= \int_0^3 \left[\frac{x^3y}{3} + \frac{x^2y^2}{2}\right]_1^2 dy$

$= \int_0^3 \left[\frac{y}{3}(8-1) + \frac{y^2}{2}(4-1)\right]dy$

$= \int_0^3 \left(\frac{7}{3}y + \frac{3}{2}y^2\right)dy$

$= \left(\frac{7}{3}\cdot\frac{y^2}{2} + \frac{3}{2}\cdot\frac{y^3}{3}\right)_0^3$

$= \left(\frac{7}{\cancel{3}}\cdot\frac{\cancel{9}\,3}{2} + \frac{3}{2}\cdot\frac{27}{\cancel{3}}\right)$

$= \left(\frac{21}{2} + \frac{27}{2}\right) = \frac{48}{2} = 24.$

6> Evaluate $\int_0^2\int_0^{x^2} e^{y/x}\,dy\,dx$ (4)

Soln: The given integral is,

$\int_0^2\int_0^{x^2} e^{y/x}\,dy\,dx$

$= \int_0^2 \left[\int_0^{x^2} e^{y/x}\,dy\right]dx$

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$$= \int_0^2 \left[\frac{e^{y/x}}{1/x} \right]_0^{x^2} dx$$

$$= \int_0^2 \left(\frac{x e^{y/x}}{1} \right)_0^{x^2} dx$$

$$= \int_0^2 (x \cdot e^{x^2/x} - x \cdot e^0)\, dx$$

$$= \int_0^2 (x \cdot e^x - x)\, dx \quad \int x e^x - e^x = e^x(x-1)$$

$$= ((x-1)e^x)_0^2 - \left(\frac{x^2}{2}\right)_0^2$$

$$= (2-1)e^2 - (0-1)e^0 - \frac{4}{2}$$

$$= e^2 + 1 - 2 = e^2 - 1$$

7) Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\, dx\, dy$ H.W

2011

Soln: From the given problem we noticed that the limits of the interior integration are functions of x.

Hence we must understand that these are limits of y.

The limits 0, 1 of the outermost integration correspond to x.

Hence we shall rewrite the given integral as,

$$\int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} (x^2+y^2)\, dy\, dx$$

$$\int_{x=0}^{1} \left[\int_{y=x}^{\sqrt{x}} (x^2+y^2)\, dy \right] dx$$

$$= \int_{x=0}^{1} \left[x^2 y + \frac{y^3}{3} \right]_x^{\sqrt{x}} dx$$

$$= \int_{x=0}^{1} \left[x^2\sqrt{x} + \frac{(\sqrt{x})^3}{3} - \left(x^2(x) + \frac{x^3}{3} \right) \right] dx$$

$$= \int_{x=0}^{1} \left[x^{5/2} + \frac{x^{3/2}}{3} - \frac{4x^3}{3} \right] dx$$

$$= \left[\frac{x^{7/2}}{\frac{7}{2}} + \frac{x^{5/2}}{\frac{5}{2}\times 3} - \frac{4 \cdot x^4}{4\times 3} \right]_0^1$$

$$= \left[\frac{2}{7} \cdot x^{7/2} + \frac{2}{15} \cdot x^{5/2} - \frac{1}{3} \cdot x^4 \right]_0^1$$

$$= \frac{2}{7} + \frac{2}{15} - \frac{1}{3} = \frac{30+14-35}{105}$$

$$= \frac{9}{105}$$

8) Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx$ 5

Soln: Given integral as

$$\int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy$$

$$= \int_{y=0}^{a} \left[\int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{a^2-y^2-x^2}\, dx \right] dy$$

$$= \int_{y=0}^{a} \left[\int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{\left\{\sqrt{a^2-y^2}\right\}^2 - x^2}\, dx \right] dy$$

$$\left[\because \int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right]$$

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$$= \int_0^a \left[\frac{x}{2}\sqrt{(\sqrt{a^2-y^2})^2 - x^2} + \frac{(\sqrt{a^2-y^2})^2}{2}\sin^{-1}\frac{x}{\sqrt{a^2-y^2}}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \left[\frac{\sqrt{a^2-y^2}}{2}\sqrt{a^2-y^2-a^2+y^2} + \frac{a^2-y^2}{2}\cdot\sin^{-1}\left(\frac{\sqrt{a^2-y^2}}{\sqrt{a^2-y^2}}\right)\right] dy$$

$$= \int_0^a \left[0 + \frac{a^2-y^2}{2}\sin^{-1}(1)\right] dy$$

$$= \int_0^a \left[\frac{\pi}{2}\cdot\frac{a^2-y^2}{2}\right] dy$$

$$= \frac{\pi}{2}\int_0^a \frac{a^2-y^2}{2}\, dy$$

$$= \frac{\pi}{2}\left[\frac{a^2y}{2} - \frac{y^3}{2\cdot 3}\right]_0^a$$

$$= \frac{\pi}{2}\left[\frac{a^2\cdot a}{2} - \frac{a^3}{6}\right]$$

$$= \frac{\pi}{2}\left[\frac{3a^3-a^3}{6}\right]$$

$$= \frac{\pi}{2}\left(\frac{\cancel{2}a^3}{\cancel{6}_3}\right) = \frac{\pi a^3}{6}$$

9> Evaluate $\int_0^2\int_0^{\sqrt{2x-x^2}} xy\, dy\, dx$. H.W

Soln: Given integral is,

$$\int_0^2\int_0^{\sqrt{2x-x^2}} xy\, dy\, dx$$

$$= \int_0^2 x\left[\int_0^{\sqrt{2x-x^2}} y\, dy\right] dx$$

$$= \int_0^2 x\left[\frac{y^2}{2}\right]_0^{\sqrt{2x-x^2}} dx$$

$$= \int_0^2 x\left[\frac{(\sqrt{2x-x^2})^2}{2} - 0\right] dx$$

$$= \int_0^2 x\cdot\frac{2x-x^2}{2}\, dx$$

$$= \frac{1}{2}\int_0^2 (2x^2-x^3)\, dx$$

$$= \frac{1}{2}\left[2\frac{x^3}{3} - \frac{x^4}{4}\right]_0^2$$

$$= \frac{1}{2}\left[2\cdot\frac{8}{3} - \frac{16}{4}\right]$$

$$= \frac{1}{2}\left[\frac{16}{3} - 4\right]$$

$$= \frac{1}{2}\left[\frac{16-12}{3}\right]$$

$$= \frac{4}{6} = \frac{2}{3}$$

10> Evaluate $\int_{-1}^2\int_{x^2}^{x+2} dy\, dx$ H.W

Soln: Given integral is

$$\int_{-1}^2\int_{x^2}^{x+2} dy\, dx$$

$$= \int_{-1}^2\left[\int_{x^2}^{x+2} dy\right] dx$$

$$= \int_{-1}^2 [y]_{x^2}^{x+2}\, dx$$

$$= \int_{-1}^2 (x+2-x^2)\, dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^2$$

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$$=\left[\frac{4}{2}+4-\frac{8}{3}-\left(\frac{1}{2}-2+\frac{1}{3}\right)\right]$$

$$=\left(2+4-\frac{8}{3}-\frac{1}{2}+2-\frac{1}{3}\right)$$

$$= 6-3+\frac{3}{2} = 3+\frac{3}{2}=\frac{9}{2}$$

11> Evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$

2004

Soln: Given integral is, 6

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

$$=\int_0^\infty\int_0^\infty (e^{-x^2}\cdot e^{-y^2})\,dx\,dy$$

$$=\int_0^\infty e^{-y^2}\left[\int_0^\infty e^{-x^2}dx\right]dy$$

$$\left[\because \int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}\right]$$

$$=\int_0^\infty e^{-y^2}\cdot\frac{\sqrt{\pi}}{2}\,dy$$

$$=\frac{\sqrt{\pi}}{2}\int_0^\infty e^{-y^2}dy=\frac{\sqrt{\pi}}{2}\cdot\frac{\sqrt{\pi}}{2}$$

$$=\frac{\pi}{4}$$

12> Evaluate $\int_0^\pi\int_0^{a\sin\theta} r\,dr\,d\theta$

Soln: Given integral is,

$$\int_0^\pi\int_0^{a\sin\theta} r\,dr\,d\theta$$

$$=\int_0^\pi\left[\int_0^{a\sin\theta} r\,dr\right]d\theta$$

$$=\int_0^\pi\left(\frac{r^2}{2}\right)_0^{a\sin\theta}d\theta$$

$$=\int_0^\pi \frac{a^2\sin^2\theta}{2}\,d\theta$$

$$=\frac{a^2}{2}\int_0^\pi \sin^2\theta\,d\theta$$

$$=\frac{a^2}{2}\int_0^\pi \left(\frac{1-\cos2\theta}{2}\right)d\theta$$

$$=\frac{a^2}{2\cdot2}\left[\theta-\frac{\sin2\theta}{2}\right]_0^\pi$$

$$=\frac{a^2}{4}\left[\pi-\frac{\sin2\pi}{2}\right]$$

$$=\frac{a^2}{4}(\pi-0)=\frac{\pi a^2}{4}$$

13> Evaluate $\int_0^{\pi/2}\int_0^{2a\cos\theta} r^2\sin\theta\,d\theta\,dr$

Soln: Given integral as,

$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r^2\sin\theta\,dr\,d\theta$$

$$=\int_0^{\pi/2}\sin\theta\left[\int_0^{2a\cos\theta} r^2\,dr\right]d\theta$$

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$$= \int_0^{\pi/2} \sin\theta \left(\frac{r^3}{3}\right)_0^{2a\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \sin\theta \left[\frac{8a^3\cos^3\theta}{3}\right] d\theta$$

$$= \frac{8a^3}{3}\int_0^{\pi/2} (\sin\theta)(\cos^3\theta)\, d\theta$$

$\int [f(x)]^n \cdot f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1}$

$$= -\frac{8a^3}{3}\int_0^{\pi/2} \underbrace{(-\sin\theta)}_{f'(x)}\underbrace{(\cos^3\theta)}_{f(x)}\, d\theta$$

$$= -\frac{8a^3}{3}\left[\frac{\cos^4\theta}{4}\right]_0^{\pi/2}$$

$$= -\frac{8a^3}{12}\left[(\cos \pi/2)^4 - (\cos 0)^4\right]$$

$$= -\frac{8a^3}{12}(-1) = \frac{2a^3}{3}$$

14> $\int_0^{\pi/2}\int_0^{a\cos\theta} r\cdot\sqrt{a^2-r^2}\, dr\, d\theta \rightarrow$ Evaluate

15 a)

Soln: Given integral is,

$$\int_0^{\pi/2}\int_0^{a\cos\theta} r\cdot\sqrt{a^2-r^2}\, dr\, d\theta$$

$$= \int_0^{\pi/2}\left[\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\, dr\right] d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left[\int_0^{a\cos\theta}(-2r)(a^2-r^2)^{1/2}\, dr\right] d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left(\frac{(a^2-r^2)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right)_0^{a\cos\theta} d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\frac{2}{3}\left[(a^2-r^2)^{3/2}\right]_0^{a\cos\theta} d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}\left[(a^2-a^2\cos^2\theta)^{3/2} - (a^2 - \right.$$

$$= -\frac{1}{3}\int_0^{\pi/2}\left[(a^2\sin^2\theta)^{3/2} - a^3\right] d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}(a^3\sin^3\theta - a^3)\, d\theta$$

$$= -\frac{a^3}{3}\int_0^{\pi/2}(\sin^3\theta - 1)\, d\theta$$

$$= -\frac{a^3}{3}\left(\int_0^{\pi/2}\sin^3\theta\, d\theta - \int_0^{\pi/2} 1\cdot d\theta\right)$$

$\left(\because \int_0^{\pi/2}\sin^n\theta\, d\theta = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3}\right.$ if n is odd $\Big)$

$$= -\frac{a^3}{3}\left[\frac{3-1}{3} - (\theta)_0^{\pi/2}\right]$$

$$= -\frac{a^3}{3}\left(\frac{2}{3} - \frac{\pi}{2}\right)$$

$$= \frac{\pi a^3}{6} - \frac{2a^3}{9}$$

15> Evaluate $\int_0^{\pi}\int_0^{a(1-\cos\theta)} 2\pi r^2\sin\theta\, d\theta\, dr$

14 b

Soln: Given integral as

$$\int_{\theta=0}^{\pi}\int_{r=0}^{a(1-\cos\theta)} 2\pi r^2\sin\theta\, dr\, d\theta$$

$$= \int_0^{\pi}\left[\int_0^{a(1-\cos\theta)} 2\pi r^2\sin\theta\, dr\right] d\theta$$

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$$= \int_0^{\pi} 2\pi \sin\theta \left[\int_0^{a(1-\cos\theta)} r^2 \, dr \right] d\theta$$

$$= \int_0^{\pi} 2\pi \sin\theta \left(\frac{r^3}{3} \right)_0^{a(1-\cos\theta)} d\theta$$

$$= \frac{2\pi}{3} \int_0^{\pi} (a^3 (1-\cos\theta)^3 \sin\theta) \, d\theta$$

$$= \frac{2\pi}{3} a^3 \int_0^{\pi} \frac{(1-\cos\theta)^4}{4} \, d\theta$$

$$= \frac{2\pi a^3}{3} \cdot \frac{1}{4} \left[(1-\cos\theta)^4 \right]_0^{\pi}$$

$$= \frac{\pi a^3}{6} \left[(1-\cos\pi)^4 - (1-\cos 0)^4 \right]$$

$$= \frac{\pi a^3}{6} (1+1)^4 = \frac{\pi a^3}{\cancel{6}_3} \cdot \cancel{16}^{8}$$

$$= \frac{8\pi a^3}{3}$$

17> Evaluate $\int_0^{\pi} \int_0^{a(1-\cos\theta)} r^2 \sin\theta \, dr \, d\theta$

14 a)

Soln: Given integral is,

$$\int_0^{\pi} \int_0^{a(1-\cos\theta)} r^2 \sin\theta \, dr \, d\theta$$

$$= \int_0^{\pi} \left[\int_0^{a(1-\cos\theta)} r^2 \sin\theta \, dr \right] d\theta$$

$$= \int_0^{\pi} \sin\theta \left[\int_0^{a(1-\cos\theta)} r^2 \, dr \right] d\theta$$

$$= \int_0^{\pi} \sin\theta \left(\frac{r^3}{3} \right)_0^{(1-\cos\theta)a} d\theta$$

$$= \int_0^{\pi} \sin\theta \left[\frac{1}{3} a^3 (1-\cos\theta)^3 \right] d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi} [\sin\theta \, (1-\cos\theta)^3] \, d\theta$$

$$= \frac{a^3}{3} \left[\frac{(1-\cos\theta)^4}{4} \right]_0^{\pi}$$

$$= \frac{a^3}{3} \left[\frac{(1-\cos\pi)^4}{4} - \frac{(1-\cos 0)^4}{4} \right]$$

$$= \frac{a^3}{3} \left[\frac{(1+1)^4}{4} - \frac{(1-1)^4}{4} \right]$$

$$= \frac{a^3}{3} \left(\frac{16}{4} \right) = \frac{4a^3}{3}$$

16> Evaluate $\int_0^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2} \, dr \, d\theta$

Soln: Given integral is,

$$\int_0^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2} \, dr \, d\theta$$

$$= \int_0^{\pi/4} \left[\int_0^{\sqrt{\cos 2\theta}} r \cdot (1+r^2)^{-2} \, dr \right] d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2} \left[\int_0^{\sqrt{\cos 2\theta}} 2r \cdot (1+r^2)^{-2} \, dr \right] d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2} \left[\frac{(1+r^2)^{-2+1}}{-2+1} \right]_0^{\sqrt{\cos 2\theta}} d\theta$$

$$= -\frac{1}{2} \int_0^{\pi/4} \left(\frac{1}{1+r^2} \right)_0^{\sqrt{\cos 2\theta}} d\theta$$

$$= -\frac{1}{2} \int_0^{\pi/4} \left(\frac{1}{1+\cos 2\theta} - \frac{1}{1+0} \right) d\theta$$

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$$= -\frac{1}{2}\int_0^{\pi/4}\left(\frac{1}{2\cos^2\theta}-1\right)d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/4}\frac{1}{2}[(\sec^2\theta-2)]\,d\theta$$

$$= -\frac{1}{4}\int_0^{\pi/4}(\sec^2\theta-2)\,d\theta$$

$$= -\frac{1}{4}(\tan\theta-2\theta)_0^{\pi/4}$$

$$= -\frac{1}{4}\left(\tan\frac{\pi}{4}-2\cdot\frac{\pi}{4}\right)$$

$$= -\frac{1}{4}\left(1-\frac{\pi}{2}\right)$$

$$= -\frac{1}{4}\left(\frac{2-\pi}{2}\right)$$

$$= \frac{\pi-2}{8}$$

NOTE:

> $\cos n\pi = (-1)^n$

> $\sin n\pi = \begin{cases}(-1)^{\frac{n-1}{2}}, & \text{if } n \text{ is odd}\\ 0, & \text{if } n \text{ is even}\end{cases}$

> $\int e^{f(x)}\cdot f'(x)\,dx = e^{f(x)}+c$

> $e^{-\infty}=0$ & $e^{\infty}=\infty$

18> Evaluate $\int_0^{\infty}\int_0^{\pi/2} e^{-r^2}\, r\,d\theta\,dr$

Soln: $\int_0^{\infty}\int_0^{\pi/2} e^{-r^2}\, r\,d\theta\,dr$

$$= \int_{r=0}^{\infty}\left[\int_{\theta=0}^{\pi/2} 1\,d\theta\right]e^{-r^2}\, r\,dr$$

$$= \int_{r=0}^{\infty}(\theta)_0^{\pi/2}\, r e^{-r^2}\,dr$$

$$= \frac{-1}{2}\cdot\frac{\pi}{2}\int_{r=0}^{\infty}(-2r)e^{-r^2}\,dr$$

$$= -\frac{\pi}{4}\int_{r=0}^{\infty}(-2r)e^{-r^2}\,dr$$

$$= -\frac{\pi}{4}\left(e^{-r^2}\right)_0^{\infty}$$

$$= -\frac{\pi}{4}\left(e^{-\infty}-e^{0}\right)$$

$$= -\frac{\pi}{4}(0-1) = \frac{\pi}{4}.$$

15> b> Evaluate

$$\int_0^{\pi/4}\int_0^{a\sin\theta}\frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta$$

Soln: From the given,

$$\int_{\theta=0}^{\pi/4}\frac{-1}{2}\int_{\theta=r}^{a\sin\theta}(-2r)(a^2-r^2)^{-1/2}\,dr\,d\theta$$

$$-\frac{1}{2}\int_{\theta=0}^{\pi/4}\left[\frac{(a^2-r^2)^{1/2}}{1/2}\right]_0^{a\sin\theta} d\theta$$

$$=(-1)\int_0^{\pi/4}\left(\sqrt{a^2-a^2\sin^2\theta}-\sqrt{a^2-0}\right)d\theta$$

$$=(-a)\int_0^{\pi/4}(\cos\theta-1)\,d\theta$$

$$=(-a)\left[\cancel{\cos}(\sin\theta-\theta)\right]_0^{\pi/4}$$

$$=(-a)\left[\sin\pi/4-\pi/4\right]$$

$$=(-a)\left(\frac{1}{\sqrt{2}}-\frac{\pi}{4}\right)$$

$$=a\left(\frac{\pi}{4}-\frac{1}{\sqrt{2}}\right).$$

19) Evaluate $\int_0^{\pi/2}\int_0^{\infty}\frac{r\,dr\,d\theta}{(r^2+a^2)^2}$.

17)

Soln: $\pi/4a^2$.

$$\int_{\theta=0}^{\pi/2}\left[\int_{r=0}^{\infty}\frac{r}{(r^2+a^2)^2}dr\right]d\theta$$

$$\frac{1}{2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty}\frac{2r}{(r^2+a^2)^2}dr\,d\theta$$

$$\frac{1}{2}\int_0^{\pi/2}\left(\frac{(r^2+a^2)^{-2+1}}{-2+1}\right)_0^{\infty}d\theta$$

$$-\frac{1}{2}\int_0^{\pi/2}\left(\frac{1}{(r^2+a^2)}\right)_0^{\infty}d\theta$$

$$-\frac{1}{2}\int_0^{\pi/2}\left[\frac{1}{\infty}-\frac{1}{a^2}\right]d\theta$$

$$\frac{1}{2a^2}\int_0^{\pi/2}1\,d\theta$$

$$\frac{1}{2a^2}(\theta)_0^{\pi/2}$$

$$\frac{1}{2a^2}\left(\frac{\pi}{2}-0\right)$$

$$=\frac{\pi}{4a^2}.$$

$$\int f'(x)\,(f(x))^n\,dx=\frac{(f(x))^{n+1}}{n+1}$$

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Triple Integrals:

Let $f(x,y,z)$ be a function of the independent variables x,y,z defined over the finite 3-dimensional region V.

If V bounded by the curves $x=x_1$, $x=x_2$, $y=y_1$, $y=y_2$, $z=z_1$ and $z=z_2$ then

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \int_{x=x_1}^{x_2}\int_{y=y_1}^{y_2}\int_{z=z_1}^{z_2} f(x,y,z)\,dz\,dy\,dx.$$

Evaluation Of Triple Integrals:

1) If $x_1, x_2, y_1, y_2, z_1, z_2$ are constants, order of evaluation is not considered.

i.e., $$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f\,dz\,dy\,dx = \int_{y_1}^{y_2}\int_{z_1}^{z_2}\int_{x_1}^{x_2} f\,dx\,dz\,dy = \int_{z_1}^{z_2}\int_{x_1}^{x_2}\int_{y_1}^{y_2} f\,dy\,dx\,dz$$

2) If the limits of x are constant limits i.e, $x=x_1$ & x_2

the limits of y are variable limits i.e., fns of x i.e, $y_1=\psi_1(x)$ & $\psi_2(x)=y_2$

the limits of z are also var limits i.e, fns of x,y i.e, $z_1=\phi_1(x,y)$ and $z_2=\phi_2(x,y)$.

Then evaluate the triple integral first w.r.t. z (treating x,y are constants) then w.r.t. y (treating x as constant) and then finally w.r.t 'x'.

i.e., $$\iiint_V f(x,y,z)\,dx\,dy\,dz = \int_{x_1}^{x_2}\left[\int_{y_1=\psi_1(x)}^{y_2=\psi_2(x)}\left[\int_{z_1=\phi_1(x,y)}^{z_2=\phi_2(x,y)} f(x,y,z)\,dz\right]dy\right]dx$$

3) If $f(x,y,z)=1$, then from

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_V dx\,dy\,dz = \iiint_V dv = V.$$

represents the volume of region.

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PROBLEMS:

1) Evaluate $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz$ (1)

Soln: Given Integral is,

$$\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz$$

$$\int_0^1\int_0^1\int_0^1 e^x\cdot e^y\cdot e^z\,dz\,dy\,dx$$

$$=\int_0^1\int_0^1 e^x\cdot e^y\left[\int_0^1 e^z\,dz\right]dy\,dx$$

$$=\int_0^1\int_0^1 e^x\cdot e^y\,[e^z]_0^1\,dy\,dx$$

$$=\int_0^1\int_0^1 e^x\cdot e^y\,(e^1-e^0)\,dy\,dx$$

$$=(e-1)\int_0^1 e^x\left[\int_0^1 e^y\,dy\right]dx$$

$$=(e-1)\int_0^1 e^x\,(e^y)_0^1\,dx$$

$$=(e-1)\int_0^1 e^x\,(e^1-e^0)\,dx$$

$$=(e-1)^2\int_0^1 e^x\,dx$$

$$=(e-1)^2\,(e^x)_0^1$$

$$=(e-1)^2\,(e^1-e^0)$$

$$=(e-1)^3$$

2) Evaluate $\int_0^a\int_0^a\int_0^a (yz+zx+xy)\,dx\,dy\,dz$ (2)

Soln: Given Integral is,

$$\int_0^a\int_0^a\int_0^a (yz+zx+xy)\,dx\,dy\,dz$$

$$=\int_0^a\int_0^a\left[\int_0^a (yz+zx+xy)\,dx\right]dy\,dz$$

$$=\int_0^a\int_0^a\left[yz(x)_0^a+z\left(\frac{x^2}{2}\right)_0^a+y\left(\frac{x^2}{2}\right)_0^a\right]$$

$$=\int_0^a\int_0^a\left[ayz+\frac{a^2}{2}z+\frac{a^2}{2}y\right]dy\,dz$$

$$=\int_0^a\left[\int_0^a\left(ayz+\frac{a^2}{2}z+\frac{a^2}{2}y\right)dy\right]dz$$

$$=\int_0^a\left[az\left(\frac{y^2}{2}\right)_0^a+\frac{a^2}{2}z(y)_0^a+\frac{a^2}{2}\left(\frac{y^2}{2}\right)_0^a\right]$$

$$=\int_0^a\left[\frac{a^3}{2}z+\frac{a^3}{2}z+\frac{a^4}{4}\right]dz$$

$$=\frac{a^3}{2}\left(\frac{z^2}{2}\right)_0^a+\frac{a^3}{2}\left(\frac{z^2}{2}\right)_0^a+\frac{a^4}{4}(z)_0^a$$

$$=\frac{a^5}{4}+\frac{a^5}{4}+\frac{a^5}{4}=\frac{3a^5}{4}.$$

3) Evaluate $\int_0^1\int_1^2\int_2^3 xyz\,dx\,dy\,dz$ (3)

Soln: Given integral is,

$$\int_0^1\int_1^2\int_2^3 xyz\,dx\,dy\,dz$$

$$=\int_{z=0}^{1}\int_{y=1}^{2}\int_{x=2}^{3} xyz\,dx\,dy\,dz$$

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$$= \int_0^1 \int_1^2 \left[\int_2^3 xyz \, dx \right] dy \, dz$$

$$= \int_0^1 \int_1^2 yz \left(\frac{x^2}{2} \right)_2^3 dy \, dz$$

$$= \int_0^1 \int_1^2 yz \left(\frac{9}{2} - \frac{4}{2} \right) dy \, dz$$

$$= \frac{5}{2} \int_0^1 \left[\int_1^2 yz \, dy \right] dz$$

$$= \frac{5}{2} \int_0^1 \left[z \left(\frac{y^2}{2} \right)_1^2 \right] dz$$

$$= \frac{5}{2} \int_0^1 z \left(\frac{4}{2} - \frac{1}{2} \right) dz$$

$$= \frac{5}{2} \cdot \frac{3}{2} \int_0^1 z \, dz$$

$$= \frac{15}{4} \cdot \left(\frac{z^2}{2} \right)_0^1$$

$$= \frac{15}{4} \left(\frac{1}{2} - 0 \right) = \frac{15}{8}$$

4a) Evaluate $\int_0^1 \int_0^2 \int_1^2 x^2 yz \, dx \, dy \, dz$ (4)

Soln: Given integral is,

$$\int_0^1 \int_0^2 \int_1^2 x^2 yz \, dx \, dy \, dz$$

$$\int_0^1 \int_0^2 \left[\int_1^2 x^2 yz \, dx \right] dy \, dz$$

$$\int_0^1 \int_0^2 \left[yz \left(\frac{x^3}{3} \right)_1^2 \right] dy \, dz$$

$$\int_0^1 \int_0^2 yz \left(\frac{8}{3} - \frac{1}{3} \right) dy \, dz$$

$$= \frac{7}{3} \int_0^1 \left[\int_0^2 yz \, dy \right] dz$$

$$= \frac{7}{3} \int_0^1 \left[z \left(\frac{y^2}{2} \right)_0^2 \right] dz$$

$$= \frac{7}{3} \int_0^1 z \left(\frac{4}{2} \right) dz$$

$$= \frac{14}{3} \int_0^1 z \, dz = \frac{14}{3} \left(\frac{z^2}{2} \right)_0^1$$

$$= \frac{14}{3} \left(\frac{1}{2} \right) = \frac{7}{3}$$

4b) Evaluate $\int_0^1 \int_0^2 \int_1^2 x^2 yz \, dz \, dy \, dx$

Solns: Given integral is,

$$\int_0^1 \int_0^2 \int_1^2 x^2 yz \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^2 \left[\int_1^2 x^2 yz \, dz \right] dy \, dx$$

$$= \int_0^1 \int_0^2 \left[x^2 y \left(\frac{z^2}{2} \right)_1^2 \right] dy \, dx$$

$$= \int_0^1 \int_0^2 x^2 y \left(\frac{4}{2} - \frac{1}{2} \right) dy \, dx$$

$$= \int_0^1 \int_0^2 x^2 y \left(\frac{3}{2} \right) dy \, dx$$

$$= \frac{3}{2} \int_0^1 \left[\int_0^2 x^2 y \, dy \right] dx$$

$$= \frac{3}{2} \int_0^1 x^2 \left(\frac{y^2}{2} \right)_0^2 dx$$

$$= \frac{3}{2} \int_0^1 x^2 \left(\frac{4}{2} \right) dx$$

$$= 3 \int_0^1 x^2 \, dx = 3 \left(\frac{x^3}{3} \right)_0^1 = 3 \cdot \frac{1}{3} = 1$$

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5> Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$

Soln: Given integral as,

$$\int_{z=-1}^{1}\int_{x=0}^{z}\int_{y=x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$$

$$\int_{-1}^{1}\int_{0}^{z}\left[\int_{x-z}^{x+z}(x+y+z)\,dy\right]dx\,dz$$

$$=\int_{-1}^{1}\int_{0}^{z}\left[x(y)_{x-z}^{x+z}+\left(\frac{y^2}{2}\right)_{x-z}^{x+z}+z(y)_{x-z}^{x+z}\right]dx\,dz$$

$$=\int_{-1}^{1}\int_{0}^{z}\left[x(x+z-x+z)+\frac{1}{2}\left[(x+z)^2-(x-z)^2\right]+z(x+z-x+z)\right]dx\,dz$$

$$=\int_{-1}^{1}\int_{0}^{z}\left[\left(2zx+\frac{1}{2}(4zx)+2z^2\right)\right]dx\,dz$$

$$\int_{-1}^{1}\left[\int_{0}^{z}(2zx+2zx+2z^2)\,dx\right]dz$$

$$=\int_{-1}^{1}\left[\int_{0}^{z}(4zx+2z^2)\,dx\right]dz$$

$$=\int_{-1}^{1}\left[\left(4z\frac{x^2}{2}+2z^2x\right)_0^z\right]dz$$

$$=\int_{-1}^{1}\left[4z\frac{z^2}{2}+2z^2(z)\right]dz$$

$$=\int_{-1}^{1}(2z^3+2z^3)\,dz$$

$$=\int_{-1}^{1}4z^3\,dz=4\left(\frac{z^4}{4}\right)_{-1}^{1}$$

$$=4\left(\frac{1}{4}-\frac{1}{4}\right)=0.$$

6> Evaluate $\int_{0}^{1}\int_{y}^{1}\int_{0}^{1-x}x\,dz\,dx\,dy$

06.07

Soln: Given integral is

$$\int_{y=0}^{1}\int_{x=y}^{1}\int_{z=0}^{1-x}x\,dz\,dx\,dy$$

$$\int_{0}^{1}\int_{y}^{1}\left[\int_{0}^{1-x}x\,dz\right]dx\,dy$$

$$=\int_{0}^{1}\int_{y}^{1}\left[x(z)_0^{1-x}\right]dx\,dy$$

$$=\int_{0}^{1}\int_{y}^{1}\left[x(1-x)\right]dx\,dy$$

$$=\int_{0}^{1}\int_{y}^{1}(x-x^2)\,dx\,dy$$

$$=\int_{0}^{1}\left[\int_{y}^{1}(x-x^2)\,dx\right]dy$$

$$=\int_{0}^{1}\left(\frac{x^2}{2}-\frac{x^3}{3}\right)_y^1 dy$$

$$=\int_{0}^{1}\left(\frac{1}{2}-\frac{1}{3}-\frac{y^2}{2}+\frac{y^3}{3}\right)dy$$

$$=\int_{0}^{1}\left(\frac{1}{6}-\frac{y^2}{2}+\frac{y^3}{3}\right)dy$$

$$=\frac{1}{6}(y)_0^1-\left(\frac{y^3}{3}\right)_0^1\cdot\frac{1}{2}+\left(\frac{y^4}{4}\right)_0^1\cdot\frac{1}{3}$$

$$=\frac{1}{6}(1)-\frac{1}{2}\left(\frac{1}{3}-0\right)+\left(\frac{1}{4}-0\right)\frac{1}{3}$$

$$=\frac{1}{6}-\frac{1}{6}+\frac{1}{12}=\frac{1}{12}.$$

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7) Evaluate $\iiint (x+y+z)\,dx\,dy\,dz$ taken over the volume bounded by the plane $x=0, x=1, y=0, y=1$ $z=0; z=1$

Soln: Given integral is

$$\iiint (x+y+z)\,dx\,dy\,dz$$

$$\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (x+y+z)\,dz\,dy\,dx$$

$$\int_0^1\int_0^1 \left[\int_0^1 (x+y+z)\,dz\right] dy\,dx$$

$$=\int_0^1\int_0^1 \left[xz+yz+\frac{z^2}{2}\right]_0^1 dy\,dx$$

$$=\int_0^1\int_0^1 \left(x+y+\frac{1}{2}\right) dy\,dx$$

$$=\int_0^1 \left[\int_0^1 \left(x+y+\frac{1}{2}\right) dy\right] dx$$

$$=\int_0^1 \left[xy+\frac{y^2}{2}+\frac{1}{2}y\right]_0^1 dx$$

$$=\int_0^1 \left(x+\frac{1}{2}+\frac{1}{2}\right) dx$$

$$=\int_0^1 (1+x)\,dx = \left(x+\frac{x^2}{2}\right)_0^1$$

$$=1+\frac{1}{2}=\frac{3}{2}$$

8) Evaluate $\int_1^3\int_{1/x}^{1}\int_0^{\sqrt{xy}} xyz\,dz\,dy\,dx$

Soln: Given integral is,

$$\int_1^3\int_{1/x}^{1}\int_0^{\sqrt{xy}} xyz\,dz\,dy\,dx$$

$$\int_{x=1}^{3}\int_{y=1/x}^{1}\int_{z=0}^{\sqrt{xy}} (xyz)\,dz\,dy\,dx$$

$$\int_1^3\int_{1/x}^{1}\left[\int_0^{\sqrt{xy}} xyz\,dz\right] dy\,dx$$

$$=\int_1^3\int_{1/x}^{1}\left[xy\left(\frac{z^2}{2}\right)_0^{\sqrt{xy}}\right] dy\,dx$$

$$=\int_1^3\int_{1/x}^{1}\left(xy\left(\frac{xy}{2}\right)\right) dy\,dx$$

$$=\frac{1}{2}\int_1^3\left[\int_{1/x}^{1} x^2y^2\,dy\right] dx$$

$$=\frac{1}{2}\int_1^3\left[x^2\int_{1/x}^{1} y^2\,dy\right] dx$$

$$=\frac{1}{2}\int_1^3\left[x^2\left(\frac{y^3}{3}\right)_{1/x}^{1}\right] dx$$

$$=\frac{1}{2}\int_1^3\left[x^2\left(\frac{1}{3}\right)\left(-\frac{1}{x^3}+1\right)\right] dx$$

$$=\frac{1}{6}\int_1^3\left[x^2-\frac{1}{x}\right] dx$$

$$=\frac{1}{6}\left[\frac{x^3}{3}-\log x\right]_1^3$$

$$=\frac{1}{6}\left[\frac{27}{3}-\log 3-\frac{1}{3}\right]$$

$$=\frac{1}{6}\left[\frac{26}{3}-\log 3\right]=\frac{1}{6}(8.6667-0.4771)$$

$$=1.3649$$

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9) Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz\,dx\,dy$

Soln: Given integral is,

$$\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz\, dx\, dy$$

$$= \int_1^e \int_1^{\log y} \left[\int_1^{e^x} \log z \, dz \right] dx . dy$$

$$= \int_1^e \int_1^{\log y} \left\{ \left[z(\log z - 1) \right]_1^{e^x} \right\} dx\, dy$$

$$\because \int \log x \cdot 1\, dx$$
$$= \log x (x) - \int \frac{1}{x} \cdot x\, dx$$
$$= x \log x - x$$
$$= x(\log x - 1)$$

$$= \int_1^e \int_1^{\log y} \left[e^x (\log_e e^x - 1) - 1(\log 1 - 1) \right] dx\,dy$$

$$= \int_1^e \int_1^{\log y} \left[e^x (x-1) - 1(-1) \right] dx\,dy$$

$$= \int_1^e \left[\int_1^{\log y} (xe^x - e^x + 1) dx \right] dy$$

$$= \int_1^e \left[e^x (x-1) - e^x + x \right]_1^{\log y} dy$$

$$= \int_1^e \left[e^{\log y} (\log y - 1) - e^{\log y} + \log y - e^1(1-1) + e^1 - 1 \right] dy$$

$$= \int_1^e \left[y(\log y - 1) - y + \log y + e - 1 \right] dy$$

$$= \int_1^e \left[y \log y - y - y + \log y + e - 1 \right] dy$$

$$= \int_1^e \left[y \log y - 2y + \log y + e - 1 \right] dy$$

$$= \left[\log y \cdot \frac{y^2}{2} - \int \frac{1}{y} \cdot \frac{y^2}{2} dy - 2\frac{y^2}{2} + y(\log y - 1) + ey - y \right.$$

$$= \left[\frac{1}{4}(2y^2 \log y - y^2) - y^2 + y \log y - 2y + ey \right]_1^e$$

$$= \frac{1}{4}(2e^2 \log e - e^2) - \cancel{e^2} + e \log e - 2e + \cancel{e^2} - \frac{1}{4}(2\log 1 - 1) + 1 - 1\log 1 + 2.1 - e$$

$$= \frac{1}{4}(2e^2 - e^2) + \cancel{e} - 2e + \frac{1}{4} + 3 - \cancel{e}$$

$$= \frac{e^2}{4} - 2e + \frac{13}{4}$$

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10) Evaluate $\iiint x-2y-z \, dx\,dy\,dz$ bounded by the curves $0 \le x \le 1$; $0 \le y \le x^2$; $0 \le z \le x+y$.

Soln: Given Integral is,

$$\iiint (x-2y-z)\,dx\,dy\,dz$$

$$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{x^2}\left[\int_{z=0}^{x+y}(x-2y-z)\,dz\right]dy\,dx$$

$$\Rightarrow \int_0^1\int_0^{x^2}\left[xz-2yz-\frac{z^2}{2}\right]_0^{x+y} dy\,dx$$

$$\Rightarrow \int_0^1\int_0^{x^2}\left[x(x+y)-2y(x+y)-\frac{(x+y)^2}{2}\right]dy\,dx$$

$$\Rightarrow \int_0^1\int_0^{x^2}\left[x^2+xy-2xy-2y^2-\frac{x^2}{2}-\frac{y^2}{2}-\frac{\cancel{2}xy}{\cancel{2}}\right]dy\,dx$$

$$\Rightarrow \int_0^1\int_0^{x^2}\left[\frac{x^2}{2}-\frac{5y^2}{2}-2xy\right]dy\,dx$$

$$\Rightarrow \int_{x=0}^{1}\left[\int_{y=0}^{x^2}\left(\frac{x^2}{2}-\frac{5y^2}{2}-2xy\right)dy\right]dx$$

$$\Rightarrow \int_{x=0}^{1}\left[y\frac{x^2}{2}-\frac{5}{2}\cdot\frac{y^3}{3}-2x\frac{y^2}{2}\right]_0^{x^2}dx$$

$$\Rightarrow \int_{x=0}^{1}\left[x^2\cdot\frac{x^2}{2}-\frac{5}{6}x^6-\frac{\cancel{2}}{\cancel{2}}x\cdot x^4\right]dx$$

$$\Rightarrow \int_{x=0}^{1}\left[\frac{x^4}{2}-\frac{5}{6}x^6-x^5\right]dx$$

$$\Rightarrow \left[\frac{x^5}{10}-\frac{5}{6}\cdot\frac{x^7}{7}-\frac{x^6}{6}\right]_0^1$$

$$\Rightarrow \frac{1}{10}-\frac{5}{42}-\frac{1}{6}$$

$$= -0.1857$$

11) Evaluate

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\frac{1}{\sqrt{1-x^2-y^2-z^2}}\,dx\,dy\,dz$$

12)

Soln: Given Integral is,

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\frac{1}{\sqrt{1-x^2-y^2-z^2}}\,dx\,dy\,dz$$

$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\left[\int_0^{\sqrt{1-x^2-y^2}}\frac{1}{\sqrt{1-x^2-y^2-z^2}}\,dz\right]dy\,dx$$

$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\left[\int_0^{\sqrt{1-x^2-y^2}}\frac{1}{\sqrt{\left(\sqrt{1-x^2-y^2}\right)^2-z^2}}\,dz\right]dy\,dx$$

$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\left[\sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}}\right)\right]_0^{\sqrt{1-x^2-y^2}}dy\,dx$$

$$\because \int\frac{1}{\sqrt{a^2-x^2}}\,dx=\sin^{-1}\left(\frac{x}{a}\right)$$

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$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\left[\sin^{-1}\frac{\sqrt{1-x^2-y^2}}{\sqrt{1-x^2-y^2}}\right]dy\,dx$$

$$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\left[\pi/2\right]dy\,dx$$

$$\Rightarrow \frac{\pi}{2}\int_{x=0}^{1}\left[\int_{y=0}^{\sqrt{1-x^2}}dy\right]dx$$

$$\Rightarrow \frac{\pi}{2}\int_{x=0}^{1}\left[y\right]_0^{\sqrt{1-x^2}}dx$$

$$\Rightarrow \frac{\pi}{2}\int_{x=0}^{1}\left(\sqrt{1-x^2}\right)dx$$

$$\because \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

$$\Rightarrow \frac{\pi}{2}\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right)\right]_0^1$$

$$\Rightarrow \frac{\pi}{2}\left[\frac{1}{2}\sqrt{1-1}+\frac{1}{2}\sin^{-1}\left(\frac{1}{1}\right)\right]$$

$$= \frac{\pi}{2}\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)$$

$$= \frac{\pi^2}{8}$$

12) Evaluate 137

$$\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{(x+y+z+1)^3}\,dx\,dy\,dz$$

Soln: Given integral is,

$$\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{(x+y+z+1)^3}\,dx\,dy\,dz$$

$$\int_{x=0}^{1}\left[\int_{y=0}^{1-x}\left[\int_{z=0}^{1-x-y}\frac{1}{(x+y+z+1)^3}dz\right]dy\,dx\right.$$

$$\int_{x=0}^{1}\left[\int_{y=0}^{1-x}\left[\frac{(x+y+z+1)^{-3+1}}{(-3+1)(1)}\right]_0^{1-x-y}dy\,dx\right.$$

$$\int_{x=0}^{1}\left[\int_{y=0}^{1-x}\left(-\frac{1}{2}\right)\left[(x+y+z+1)^{-2}\right]_0^{1-x-y}dy\,dx\right.$$

$$\left(-\frac{1}{2}\right)\int_{x=0}^{1}\left[\int_{y=0}^{1-x}\left[(1-x-y+y+x+1)^{-2}-(x+y+1)^{-2}\right]dy\,d\right.$$

$$\left(-\frac{1}{2}\right)\int_{x=0}^{1}\left[\int_{y=0}^{1-x}\left(\frac{1}{4}-\frac{1}{(x+y+1)^2}\right)dy\right.$$

$$\left(-\frac{1}{2}\right)\int_{x=0}^{1}\left[\frac{1}{4}y-\frac{(x+y+1)^{-2+1}}{-2+1}\right]_0^{1-x}d$$

$$\left(-\frac{1}{2}\right)\int_{x=0}^{1}\left[\frac{1}{4}(1-x)+(x+1-x+1\right.$$
$$\left.-\frac{1}{4}(0)-(x+1)^{-1}\right]d$$

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