Unit I

1 Set Theory and Logic

1.1 : Sets

Important Points to Remember

1. **Set :** A collection of well defined objects is called set.
2. **Empty set :** A set which does not contain any element is called null set or empty set. It is denoted by ϕ.
3. **The cardinality of a set** is the number of elements in the set.
4. **Power set :** The power set of given set A is the set of all subsets of set A. It is denoted by P (A). If n has n elements then P (A) has 2^n elements.
5. **Finite set :** A set having finite number of elements is called finite set. Otherwise infiite set.
6. If A and B are any two sets then
   (a) $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$,
   $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.
   (b) $A - B = \{x \mid x \in A \text{ but } x \notin B\}$
   (c) $A^c = U - A \ \{x \mid x \notin A \text{ and } x \in U\}$
   (d) $A \oplus B = (A \cup B) - (A \cap B)$ ... [Symmetric difference]
7. **Multiset :** A set in which elements are present more than once.
   The multiplicity of an element in a multiset is defined as the number of times the element appear in the set. It is denoted as μ (a).
   If A and B are multisets then,
   (a) $x \in A \cup B$ and $\mu(x) = \max\{\mu_A(x), \mu_B(x)\}$
   (b) $x \in A \cap B$ and $\mu(x) = \min\{\mu_A(x), \mu_B(x)\}$

Unit I

1 Set Theory and Logic

1.1 : Sets

Important Points to Remember

1. **Set :** A collection of well defined objects is called set.
2. **Empty set :** A set which does not contain any element is called null set or empty set. It is denoted by ϕ.
3. **The cardinality of a set** is the number of elements in the set.
4. **Power set :** The power set of given set A is the set of all subsets of set A. It is denoted by P (A). If n has n elements then P (A) has 2^n elements.
5. **Finite set :** A set having finite number of elements is called finite set. Otherwise infiite set.
6. If A and B are any two sets then
   (a) $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$,
   $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.
   (b) $A - B = \{x \mid x \in A \text{ but } x \notin B\}$
   (c) $A^c = U - A \quad \{x \mid x \notin A \text{ and } x \in U\}$
   (d) $A \oplus B = (A \cup B) - (A \cap B)$ ... [Symmetric difference]
7. **Multiset :** A set in which elements are present more than once.
   The multiplicity of an element in a multiset is defined as the number of times the element appear in the set. It is denoted as μ (a).
   If A and B are multisets then,
   (a) $x \in A \cup B$ and $\mu(x) = \max\{\mu_A(x), \mu_B(x)\}$
   (b) $x \in A \cap B$ and $\mu(x) = \min\{\mu_A(x), \mu_B(x)\}$

(c) $x \in A - B$ and $\mu(x) = \mu_A(x) - \mu_B(x)$ if $\mu(x) \geq 0$

If difference is negative then $\mu(x) = 0$.

(d) $x \in A + B$ and $\mu(x) = \mu_A(x) + \mu_B(x)$

8. Inclusion exclusion theorem :

(a) For two sets : $|A \cup B| = |A| + |B| - |A \quad B|$

(b) For three sets : $|A \cup B \cup C| = |A| + |B| + |C|$

$- A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

9. List of basic identities of set theory :

(a) $A \cap \phi = \phi$, $A \cap A = A$, $A \cap U = A$, $A \cap A'$

$= A' \cap A = \phi$

(b) $A \cup \phi = A$, $A \cup A = A$, $A \cup U = \cup$, $A \cup A'$

$= A' \cup A = U$

(c) $(\overline{\overline{A}}) \doteq A$, $A - B = A \cap \overline{B}$

(d) $A \cup B = B \cup A$, $A \cap B = B \cap A$

(e) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(f) $A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

(g) $(\overline{A \cup B}) = \overline{A} \cap \overline{B}$

$(\overline{A \cap B}) = \overline{A} \cup \overline{B}$

(h) $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$

Q.1 Which of the following are examples of set :

(a) Collection of all clever students of second year B.Tech.

(b) Collection of non negative integers.

(c) Collection of students who are studying in SPPU.

(d) Collection of all good teachers in SPPU.

Ans. : The words clever, beautiful, good, honest, handsome, big, small are not well defined. So examples (a) and (d) are not sets.

But (b) and (c) are properly defined. So (b) and (c) are examples of sets.

These examples explain the difference between the collection and set. Every set is a collection but converse is not true.

Q.2 Let U = Universal set = $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
$A = \{1, 2, 3, 4, 5\}$, $B = \{0, 2, 4, 6, 8\}$, $C = \{0, 3, 6, 9\}$
$D = \{2, 3, 5, 7\}$. Compute $A \cup B$, $B \cup C$, $C \cup D$ $C \cap D$,
$A \cup B - C$, $U - B$, B^C, $A \oplus C$, $B \oplus D$.

Ans. : $A \cup B = \{1, 2, 3, 4, 5\} \cup \{0, 2, 4, 6, 8\} = \{0, 1, 2, 3, 4, 5, 6, 8\}$

$B \cup C = \{0, 2, 4, 6, 8\} \cup \{0, 3, 6, 9\} = \{0, 2, 3, 4, 6, 8, 9\}$

$C \cup D = \{0, 3, 6, 9\} \cup \{2, 3, 5, 7\} = \{0, 2, 3, 5, 6, 7, 9\}$

$C \cap D = \{0, 3, 6, 9\} \cap \{2, 3, 5, 7\} = \{3\}$

$(A \cup B) - C = \{0, 1, 2, 3, 4, 5, 6, 8\} - \{0, 3, 6, 9\} = \{1, 2, 4, 5, 8\}$

$U - B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} - \{0, 2, 4, 6, 8\}$

$= \{1, 3, 5, 7, 9\} = B^C$

$A \oplus C = (A \cup C) - (A \cap C) = \{0, 1, 2, 3, 4, 5, 6, 9\} - \{3\}$

$= \{0, 1, 2, 4, 5, 6, 9\}$

$B \oplus D = (B \cup D) - (B \cap D) = \{0, 2, 3, 4, 5, 6, 7, 8\} - \{2\}$

$= \{0, 3, 4, 5, 6, 7, 8\}$

Q.3 If $U = \{x \in Z' \mid -5 < x < 5\}$ and $A = \{x \in Z' \mid -2 < x < 3\}$
Where Z' = Set of integers. State the elements of the sets.
$\overline{A}$, $\overline{A} \cap \overline{A}$, $A \cap U$, $A \cup U$, $\overline{A} \cap U$, $A \cap \overline{A}$

Ans. : $\overline{A} = \{x \in Z' / (-5 < x < -2) \cup (3 < x < 5)\}$

$\overline{A} \cap \overline{A} = \overline{A}$

$A \cap U = A, \quad A \cup U = U$

$\overline{A} \cap U = \overline{A}$

$\overline{A} \cap \overline{A} = \phi$

Q.4 If $A = \{x, y, \{x, z\} \phi\}$. Determine the following sets
i) $A - \{x, z\}$, ii) $\{\{x, z\} - A$, iii) $A - \{\{x, y\}\}$, iv) $\{x, z\} - A$,
v) $A - P(A)$, vi) $\{x\} - A$, vii) $A - \{x\}$, viii) $A - \phi$,
ix) $\phi - A$, x) $\{x, y, \phi\} - A$ **☞[SPPU : Dec.-08, Marks 4]**

Ans. :

i) $A - \{x, z\} = \{x, y, \phi\}$, ii) $\{\{x, z\}\} - A = \phi$

iii) $A - \{\{x, y\}\} = A$, iv) $\{x, z\} - A = \{z\}$

v) $A - p(A) = \{x, y, \{x, z\}\}$ vi) $\{x\} - A = \phi$

vii) $A - \{x\} = \{y, \{x, z\}, \phi\}$ viii) $A - \phi = \{x, y, \{x, z\}\}$

ix) $\phi - A = \phi$ x) $\{x, y, \phi\} - A = \{\{x, z\}\}$

Q.5 If ϕ is an empty set then find $p(\phi)$, $p(p(\phi))$ $p(p(p(\phi)))$

Ans. :

$$p(\phi) = \{\phi\}$$

$$p(p(\phi)) = \{\phi, \{\phi\}\}$$

$$p(p(p(\phi))) = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$$

Q.6 Salad is made with combination of one or more eatables, how many different salads can be prepared from onion, carrot, cabbage and cucumber ?

[SPPU : Dec.-13, Marks 4]

Ans. : The number of different salads can be prepared from onion, carrot, cabbage and cucumber with combination of one or more eatables is $2^4 - 1 = 16 - 1 = 15$

Q.7 Explain the concepts of countably infinite set with example.

[SPPU : Dec.-14, Marks 4]

Ans. : A set is said to be countable if its all elements can be labelled as 1, 2, 3, 4, ... A set is said to be countably infinite

if, i) It is countable

ii) It has infinitely many elements i.e. It's cardinality is ∞.

For example

1) The set of natural numbers {1, 2, 3, ...} is countably infinite.

2) The set of integers is countably infinite.

3) The set of real numbers is infinite but not countable.

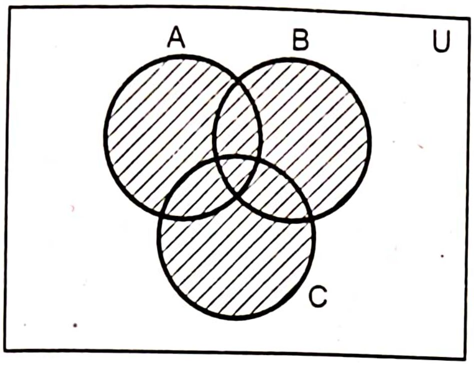
Q.8 Draw Venn diagram and prove the expression. Also write the dual of each of the given statements.

i) $(A \cup B \cup C)^C = (A \cup C)^C \cap (A \cup B)^C$

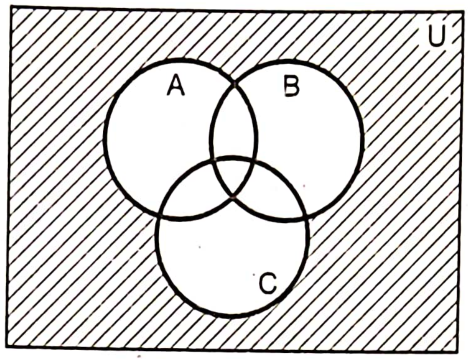
ii) $(U \cap A) \cup (B \cap A) = A$

[SPPU : Dec.-11, Marks 6]

Ans. : i) Consider the following Venn diagrams.

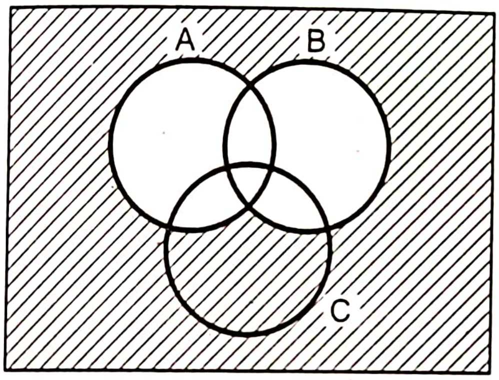


① $A \cup B \cup C =$ ▨

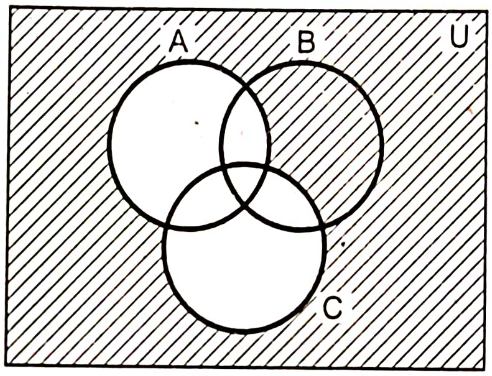


② $(A \cup B \cup C)^c =$ ▨

Fig. Q.8.1

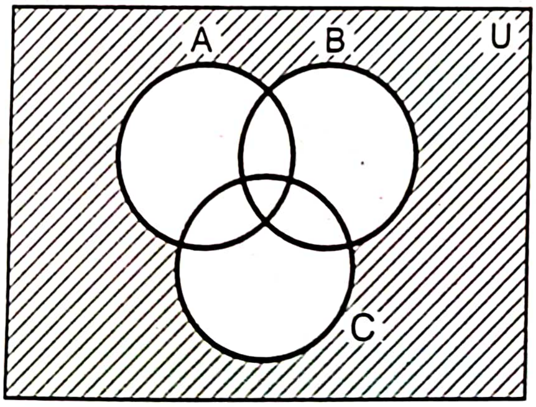


③ $(A \cup B)^c =$ ▨



④ $(A \cup C)^c =$ ▨

Fig. Q.8.2



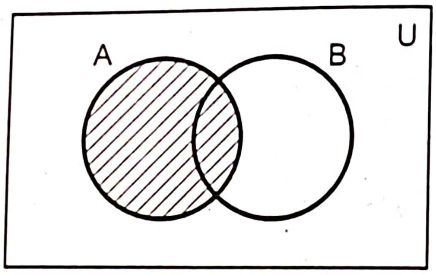
⑤ $(A \cup B)^c \cap (A \cup C)^c$

Fig. Q.8.3

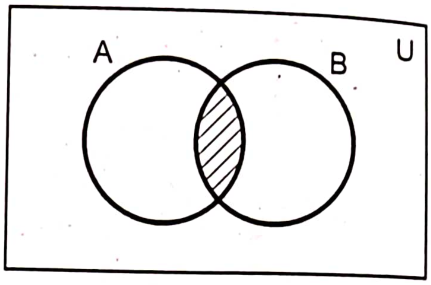
From (2) and (5)

$(A \cup B \cup C)^c = (A \cup C)^c \cap (A \cup B)^c$

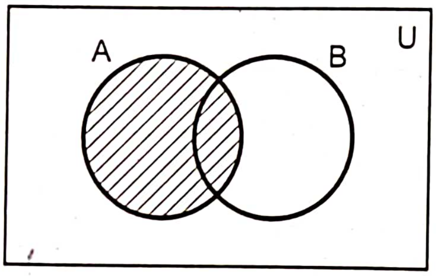
ii) Consider the following Venn diagrams.



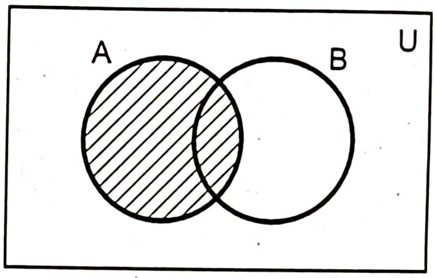
(1) $U \cap A =$



(2) $B \cap A =$



(3) $(U \cap A) \cup (B \cap A) =$



(4) $A =$

Fig. Q.8.4

From Venn diagrams (3) and (4)

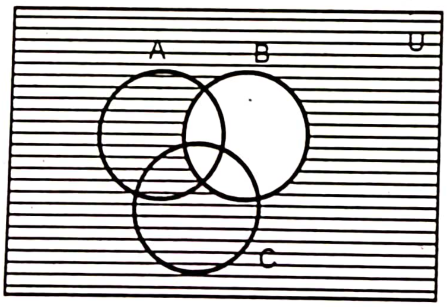
$$(U \cap A) \cup (B \cap A) = A$$

Q.9 Using Venn diagram show that :

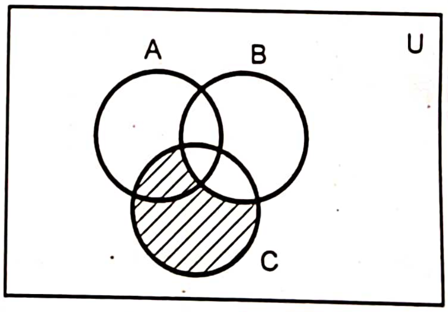
$A \cup (\overline{B} \cap C) = (A \cup \overline{B}) \cap (A \cup C)$ **☞[SPPU : May-05, Marks 4]**

Ans. : Consider the following venn diagrams

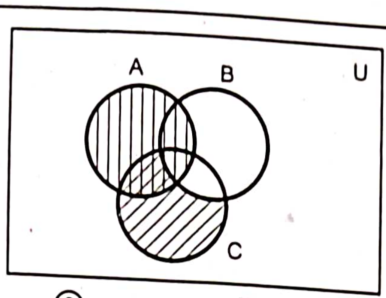
$$\overline{B} = \{x \,/\, x \notin B\}$$



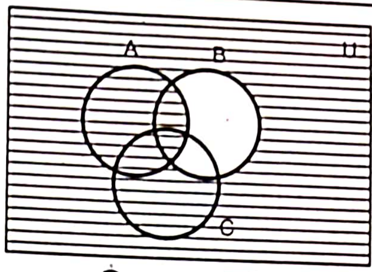
(1) $\overline{B} =$



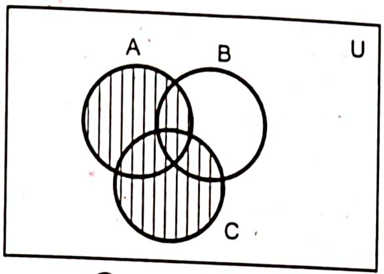
(2) $\overline{B} \cap C =$



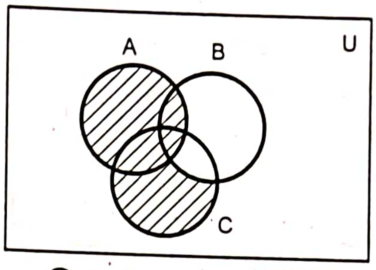
③ $A \cup (\overline{B} \cap C) =$ and



④ $A \cup \overline{B} =$



⑤ $A \cup C =$



⑥ $(A \cup \overline{B}) \cap (A \cup C) =$

From ③ and ⑥, $A \cup (\overline{B} \cap C) = (A \cup \overline{B}) \cap (A \cup C)$

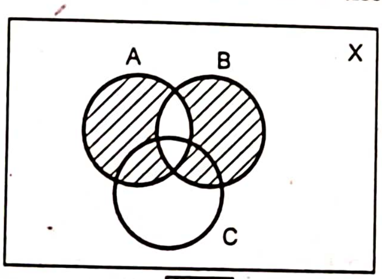
Q.10 Using Venn diagram, prove or disprove.

i) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$

ii) $A \cap B \cap C = A - [(A - B) \cup (A - C)]$ ☞**[SPPU : May-06, Marks 4]**

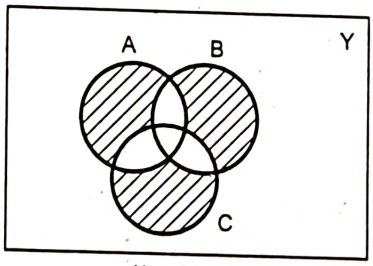
Ans. : i) $A \oplus B = (A \cup B) - (A \cap B)$

$A \oplus B$: elements which are either in A or in B but not in both A and B.



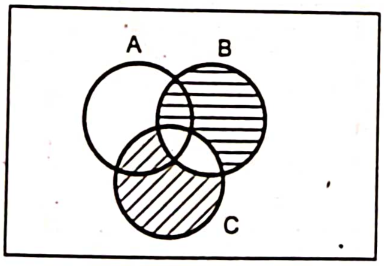
$A \oplus B =$

①



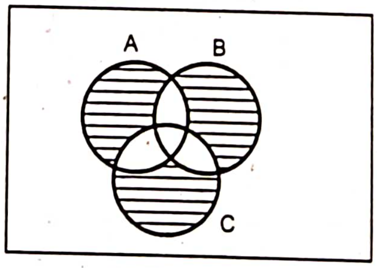
$(A \oplus B) \oplus C$

②



$B \oplus C =$

③ and

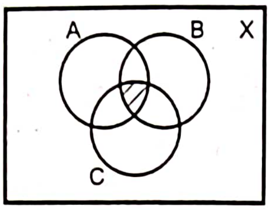


$A \oplus (B \oplus C)$

④

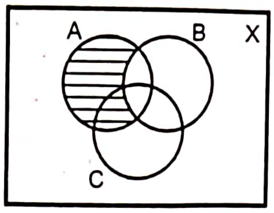
From ② and ④, $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

ii) $A \cap B \cap C = A - [(A - B) \cup (A - C)]$



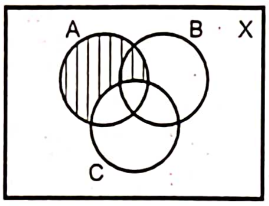
$A \cap B \cap C =$

①



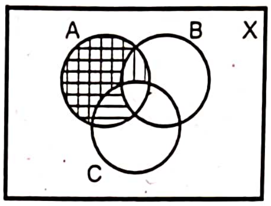
$A - B =$

②



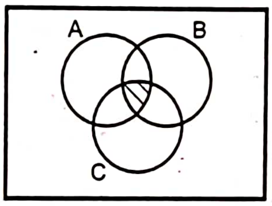
$A - C =$

③



$(A - B) \cup (A - C) =$ and

④



$A - [(A-B) \cup (A-C)] =$

⑤

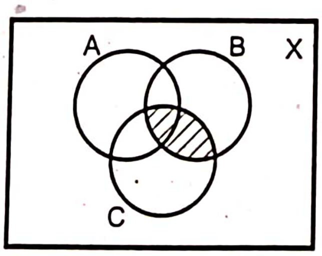
From ① and ⑤

$A \cap B \cap C = A - [(A - B) \cup (A - C)]$

Q.11 Using Venn diagrams show that

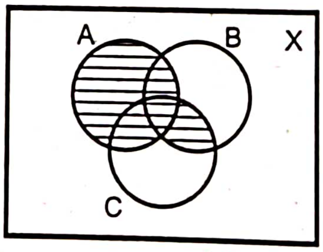
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ **[SPPU : May-08, Dec.-12, Marks 3]**

Ans. :



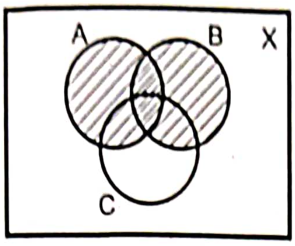
$B \cap C =$

①



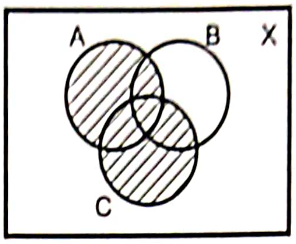
$A \cup (B \cap C) =$

②



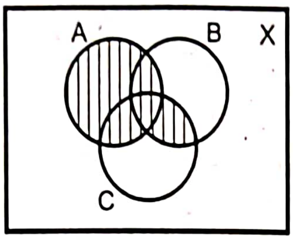
$A \cup B =$ ▨

③



$A \cup C =$ ▨

④



$(A \cup B) \cap (A \cup C) =$ ▥

⑤

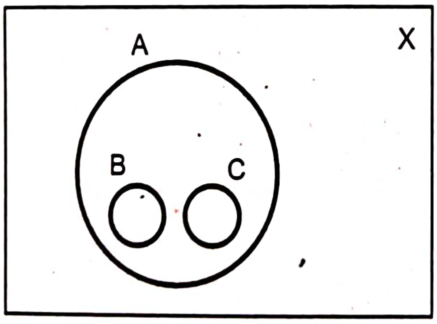
From ② and ⑤, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Q.12 Let A, B, C be sets. Under what conditions the following statements are true ?

i) $(A - B) \cup (A - C) = A$ ii) $(A - B) \cup (A - C) = \varnothing$

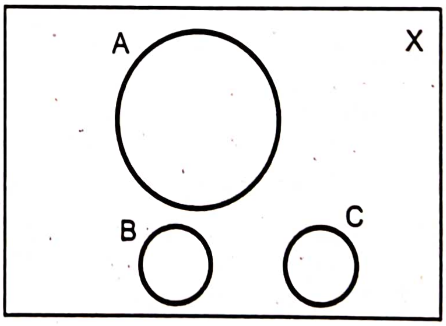
☞[SPPU : Dec.-06, 15, Marks 3]

Ans. : i) $(A - B) \cup (A - C) = A$



①

OR



②

If $B \subset A$ and $C \subset A$

then $(A - B) \cup (A - C) = A$

Or A, B, C are disjoint sets.

i.e. $A \cap B \cap C = \varnothing$

Then $(A - B) \cup (A - C) = A$

ii) $(A - B) \cup (A - C) = \varnothing$ is true.

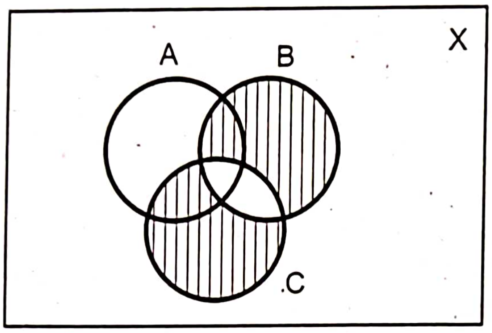
If A is empty set i.e. $A = \varnothing$

Q.13 Prove the following using Venn diagram.

$$\mathbf{A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)}$$

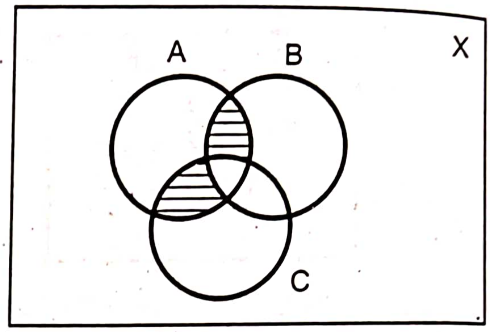
[SPPU : May-08, 14, Dec.-12, Marks 3]

Ans. :



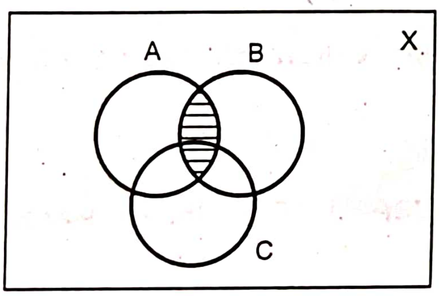
$B \oplus C =$

(1)



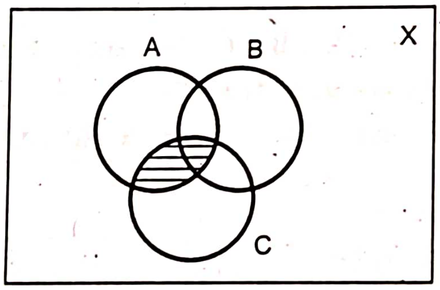
$A \cap (B \oplus C) =$

(2)



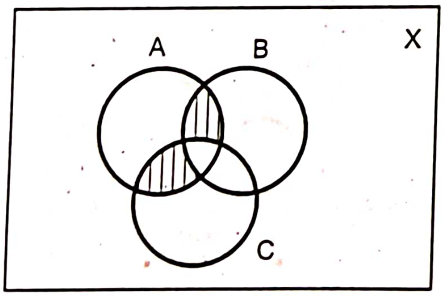
$A \cap B =$

(3)



$A \cap C =$

(4)



$(A \cap B) \oplus (A \cap C) =$

(5)

From (2) and (5)

$$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$$

Q.14 Among the integers 1 to 300 find how many are not divisible by 3, not by 5. Find also how many are divisible by 3 but not by 7.

☞[SPPU : Dec.-08, Marks 6]

Ans. : Let A denotes the set of integers 1 to 300 divisible by 3, B denotes the set of integers 1 to 300 divisible by 5, C denotes the set of integers 1 to 300 divisible by 7.

$$|A| = \left[\frac{300}{3}\right] = 100,\ |B| = \left[\frac{300}{5}\right] = 60,\ |C| = \left[\frac{300}{7}\right] = 42,$$

$$|A \cap B| = \left[\frac{300}{3 \times 5}\right] = 20$$

Find $|\overline{A} \cap B|$ and $|A - C|$

We have $\overline{A} \cap \overline{B} = \overline{A \cup B} = U - (A \cup B)$

$$\therefore \quad |A \cup B| = |A| + |B| - |A \cap B|$$

$$= 100 + 60 - 20 = 140$$

$$\therefore \quad |\overline{A} \cap \overline{B}| = |U| - |A \cup B| = 300 - 140 = \mathbf{160}$$

Hence 160 integers between 1 – 300 are not divisible by 3, not by 5.

$$|A - C| = |A| - |A \cap C|, \qquad |A \cap C| = \left[\frac{300}{3 \times 7}\right] = 14$$

$$|A - C| = 100 - 14 = 86$$

Hence, 86 integers between 1 – 300 are not divisible by 3 but not by 7.

Q.15 It is known that at the university 60 percent of the professors play tennis, 50 percent of them play bridge. 70 percent jog, 20 percent play tennis and bridge, 30 percent play tennis and jog, and 40 percent play bridge and jog. If some one claimed that 20 percent of the professors jog and play bridge and tennis, would you believe this claim ? Why ?

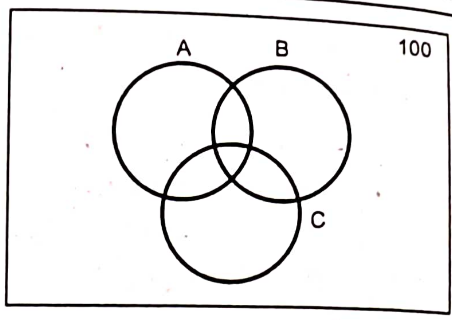
☞[SPPU : Dec.-13, Marks 6]

Ans. : Let A, B, C, denotes the number of professors play tennis, bridge and jog respectively.

$$|A| = 60$$

$$|B| = 50$$

$$|C| = 70$$
$$|A \cap B| = 20$$
$$|A \cap C| = 30$$
$$|B \cap C| = 40$$
$$|A \cap B \cap C| = 20$$



$|A \cup B \cup C| =$
$|A|+|B|+|C|-[|A \cap B|+|A \cap C|+|B \cap C|]+|A \cap B \cap C|$

$$= 60 + 50 + 70 - [20 + 30 + 40] + 20 = \mathbf{110}$$

which is not possible as $|A \cup B \cup C| \subset X$ and the number of elements in $|A \cup B \cup C|$ cannot exceed number of elements in the universal set X.

Q.16 Consider a set of integers 1 to 500. Find

i) How many of these numbers are divisible by 3 or 5 or by 11 ?

☞ [SPPU : Dec.-14, Marks 6]

ii) Also indicate how many are divisible by 3 or by 11 but not by all 3, 5 and 11.

iii) How many are divisible by 3 or 11 but not by 5 ?

☞[SPPU : May-05, Marks 6]

Ans. : Let A denote numbers divisible by 3.

B denote numbers divisible by 5. C denote numbers divisible by 11.

$|A|$ denotes cardinality of A similarly $|B|$ and $|C|$ denotes cardinality of B and C.

$$|A| = \left[\frac{500}{3}\right] = 166 \qquad |B| = \left[\frac{500}{5}\right] = 100$$

$$|C| = \left[\frac{500}{11}\right] = 45 \qquad |A \cap B| = \left[\frac{500}{3 \times 5}\right] = 33$$

$$|A \cap C| = \left[\frac{500}{3 \times 11}\right] = 15 \qquad |B \cap C| = \left[\frac{500}{5 \times 11}\right] = 9$$

$$|A \cap B \cap C| = \left[\frac{500}{3 \times 5 \times 11}\right] = 3$$

i) $|A \cup B \cup C| = |A|+|B|+|C|-[|A \cap B|+|A \cap C|+|B \cap C|]+|A \cap B \cap C|$

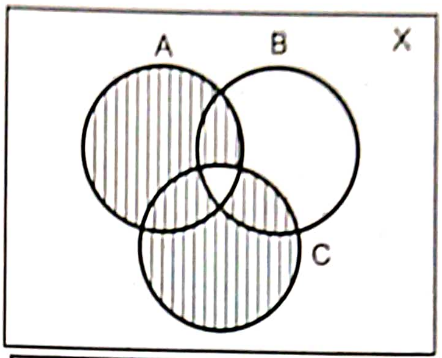
$$= 166 + 100 + 45 - [13 + 15 + 9] + 3 = \mathbf{257}$$

ii) Number of integers divisible by 3 or by 11 but not by all 3, 5 and 11.

$= |A \cup C| - |A \cap B \cap C|$

$= [|A| + |C| - |A \cap C|] - |A \cap B \cap C|$

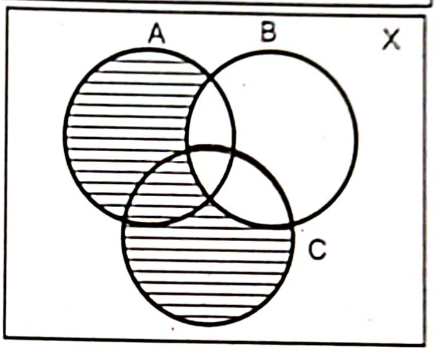
$= 166 + 45 - 15 - 3 = \mathbf{193}$



iii) Number of integers divisible by 3 or 11 but not by 5.

$= |A \cup B \cup C| - |B|$

$= 257 - 100 = \mathbf{157}$



Q.17 Out of the integers 1 to 1000.

i) How many of them are not divisible by 3, nor by 5, nor by 7 ?

ii) How many are not divisible by 5 and 7 but divisible by 3 ?

☞[SPPU : May-06, May-08, May-14, Marks 6]

Ans. : i) Let A denote numbers divisible by 3.

B denote numbers divisible by 5. and C denote numbers divisible by 7.

$$|A| = \left[\frac{1000}{3}\right] = 333 \qquad |B| = \left[\frac{1000}{5}\right] = 200$$

$$|C| = \left[\frac{1000}{7}\right] = 142 \qquad |A \cap B| = \left[\frac{1000}{3\times5}\right] = 66$$

$$|A \cap C| = \left[\frac{1000}{3\times7}\right] = 47 \qquad |B \cap C| = \left[\frac{1000}{5\times7}\right] = 28$$

$$|A \cap B \cap C| = \left[\frac{1000}{3\times5\times7}\right] = 9$$

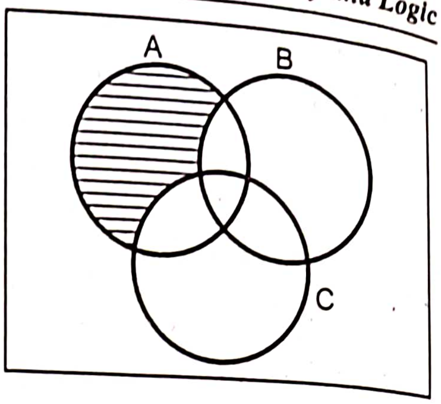
$$|A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C|$$

$$= 333 + 200 + 142 - [66 + 47 + 28] + 9 = 543$$

This show 543 numbers are divisible by 3 or 5 or 7. Hence numbers which are not divisible by 3, nor by 5, nor by 7.

$$= |A' \cap B' \cap C'| = |X| - |A \cup B \cup C| = 1000 - 543 = \mathbf{457}$$

ii) Number of integers divisible by 3 but not by 5 and not by 7 is $|A \cap B' \cap C'|$.



$$|A \cap B' \cap C'| = |A \cap (B \cup C)'|$$

$$= |A| - [|A \cap B| + |A \cap C|]$$

$$+ |A \cap B \cap C|$$

$$= 333 - [66 + 47] + 9 = 229$$

Q.18 It was found that in the first year computer science class of 80 students, 50 knew COBOL, 55 'C' and 46 PASCAL. It was also know that 37 knew 'C' and COBOL, 28 'C' and PASCAL and 25 PASCAL and COBOL. 7 students however knew none of the languages. Find

i) How many knew all the three languages ?

ii) How many knew exactly two languages ?

iii) How many knew exactly one language ?

☞[SPPU : May-15, Dec.-15, Marks 4]

Ans. : Let A denote the set of students who know COBOL.

B denote the set of students who know 'C'. and C denote the set of students who know PASCAL. and X denote universal set.

Then $|X| = 80, |A| = 50, |B| = 55, |C| = 46$

$$|A \cap B| = 37, |B \cap C| = 28, |A \cap C| = 25$$

$$|A' \cap B' \cap C'| = 7$$

Hence $|(A \cup B \cup C)'| = 7$

Also $|A \cup B \cup C| = |X| - |(A \cup B \cup C)'|$

Hence $|A \cup B \cup C| = 80 - 7 = 73$

i) $|A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C|$

$$73 = 50 + 55 + 46 - [37 + 28 + 25] + |A \cap B \cap C|$$

$\Rightarrow |A \cap B \cap C| = 12$

ii) Number of students who know only COBOL and 'C' but not PASCAL.

$$= |A \cap B| - |A \cap B \cap C| = 37 - 12 = \mathbf{25} \quad \text{... (Q.18.1)}$$

Number of students who know only COBOL and PASCAL but not 'C'.

$$= |A \cap C| - |A \cap B \cap C|$$

$$= 25 - 12 = \mathbf{13} \quad \text{... (Q.18.2)}$$

Number of students who know only 'C' and PASCAL but not COBOL.

$$= |B \cap C| - |A \cap B \cap C|$$

$$= 28 - 12 = \mathbf{16} \quad \text{... (Q.18.3)}$$

Hence number of students who know exactly two languages.

$$= 25 + 13 + 16 = \mathbf{54}$$

iii) Number of students who know only COBOL but not 'C' and not PASCAL.

$$= |A| - [|A \cap B| + |A \cap C|] + |A \cap B \cap C|$$

$$= 50 - [37 + 25] + 12 = \mathbf{0} \quad \text{... (Q.18.4)}$$

Number of students who know only 'C' but not COBOL and not PASCAL.

$$= |B| - [|A \cap B| + |B \cap C|] + |A \cap B \cap C|$$

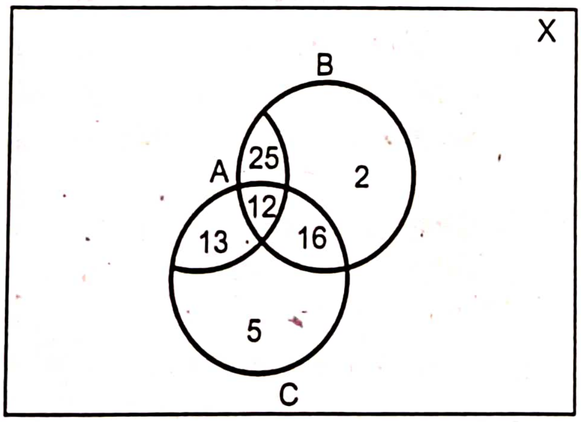
$$= 55 - [37 + 28] + 12 = \mathbf{2} \quad \text{... (Q.18.5)}$$

Number of students who know only PASCAL but not COBOL and not 'C'.

$$= |C| - [|A \cap C| + |B \cap C|] + |A \cap B \cap C|$$

$$= 46 - [25 + 28] + 12 = \mathbf{5} \quad \text{... (Q.18.6)}$$

Hence number of students who know only one language = 0 + 2 + 5 = **7**



Q.19 Find the multiset to solve equation

$A \cap [1, 2, 2, 3, 4] = [1, 2, 3, 4]$

$A \cup [1, 2, 2, 3] = [1, 1, 2, 2, 3, 3, 4]$

Ans. : Maximum multiplicity of each element is as follows

$$\mu(1) = 2, \quad \mu(2) = 2, \quad \mu(3) = 2, \quad \mu(4) = 1$$

and minimum multiplicity of each element in A is as follows

$$\mu(1) = 1, \quad \mu(2) = 1, \quad \mu(3) = 1, \quad \mu(4) = 1$$

Therefore, $A = [1, 2, 3, 4]$ or $A = [1, 1, 2, 2, 3, 3, 4]$

Q.20 Explain examples of multisets with its significance.

[SPPU : Dec.-13, Marks 4]

Ans. : 1) Multisets are used to denote roots of the polynomial.

$\therefore \quad x^3 + 3x^2 + 3x + 1 = 0$

Roots are $-1, -1, -1$

$$\therefore \quad A = [-1, -1, -1]$$

2) Multisets are used to denote prime factors of every non-negative integer.

e.g. prime factors of 72 are $2 \times 2 \times 2 \times 3 \times 3$

$$\therefore \quad A = [2, 2, 2, 3, 3]$$

3) Multisets are used to denote zeros or poles of analytic functions.

4) In Computer Science, multisets are applied in a variety of search and sort procedure.

1.2 : Multiset

Important Points to Remember

1) **Multiset :** A collection of objects that are not necessarily distinct is called a multiset (or Msets). To distinguish set and multiset we denote multiset by enclosing elements in a square brackets
e.g. [a, a, b]

2) **Multiplicity of an Element :** Let S be a multiset and $x \in S$. The multiplicity of x is defined as the number of times the element x appears in the multiset S. It is denoted by $\mu(x)$.

For example i) $S = [a, b, c, d, d, d, e, e]$

$$\mu(a) = 1, \mu(b) = 1, \mu(c) = 1, \mu(d) = 3, \mu(e) = 2$$

3) **Equality of Multisets :** Let A and B be two multisets. A and B are said to be equal multisets iff
$\mu_A(x) = \mu_B(x), \forall x \in A \text{ or } B$

e.g. $[a, b, a, b] = [a, a, b, b]$ but $[a, a] \neq [a]$

- **Subset of Multisets :** A multiset 'A' is said to be the multiset of B if multiplicity of each element in A is less or equal to it's multiplicity in B.

e.g. $[a] \subseteq [a, a]$, $[a, b, a, b, a] \subseteq [a, a, a, b, b, b]$.

4) **Union and Intersection of Multisets :** If A and B are multisets then $A \cup B$ and $A \cap B$ are also multisets.

- The multiplicity of an element $x \in A \cup B$ is equal to the maximum of the multiplicity of x in A and in B.
- The multiplicity of an element $x \in A \cap B$ is equal to the minimum of the multiplicity of x in A and in B.
- Example :

$$A = [a, b, c, a, b, c, a, a, a], \quad B = [a, a, b, b, b, b]$$
$$A \cup B = [a, a, a, a, a, b, b, b, b, c, c]$$
$$A \cap B = [a, a, b]$$

5) **Difference of Multisets :** Let A and B be two multisets. The difference A – B is a multiset such that $x \in A - B$
if $(\mu_A(x) - \mu_B(x)) \geq 1$

e.g. 1) $A = [a, a, b, a]$, $B = [a, b]$

$\therefore$ $A - B = [a, a]$

2) $A = [1, 2, 3, 4, 2, 2, 3, 3]$,

$B = [1, 1, 1, 2, 2, 2, 3, 3, 3, 3, 4]$

$A - B = [\ ]$ or ϕ

6) **Sum of Multisets :** Let A and B be two multisets. The sum of A and B is denoted by A + B and defined as for each

$$x \in A + B, \ \mu(x) = \mu_A(x) + \mu_B(x)$$

e.g. $A = [a, b, c, c]$, $B = [a, a, b, b, c, c]$

$$A + B = [a, a, a, b, b, b, c, c, c, c]$$

Q.21 Find the multiset to solve equation

$A \cap [1, 2, 2, 3, 4] = [1, 2, 3, 4]$

$A \cup [1, 2, 2, 3] = [1, 1, 2, 2, 3, 3, 4]$

Ans. : Maximum multiplicity of each element is as follows

$$\mu(1) = 2, \quad \mu(2) = 2, \quad \mu(3) = 2, \quad \mu(4) = 1$$

and minimum multiplicity of each element in A is as follows

$$\mu(1) = 1, \quad \mu(2) = 1, \quad \mu(3) = 1, \quad \mu(4) = 1$$

Therefore, $A = [1, 2, 3, 4]$ or $A = [1, 1, 2, 2, 3, 3, 4]$

Q.22 Explain examples of multisets with its significance.

☞ **[SPPU : Dec.-13, Marks 4]**

Ans. : 1) Multisets are used to denote roots of the polynomial.

$\therefore \quad x^3 + 3x^2 + 3x + 1 = 0$

Roots are –1, –1, –1

$\therefore \quad A = [-1, -1, -1]$

2) Multisets are used to denote prime factors of every non-negative integer.

e.g. prime factors of 72 are $2 \times 2 \times 2 \times 3 \times 3$

$\therefore \quad A = [2, 2, 2, 3, 3]$

3) Multisets are used to denote zeros or poles of analytic functions.

4) In Computer Science, multisets are applied in a variety of search and sort procedure.

Q.23 If P = {a, a, a, c, d, d}, Q = {a, a, b, c, c}. Find union, intersection and difference of P and Q. ☞ **[SPPU : May-19, Marks 3]**

Ans. : We have

$$P \cup Q = \{a, a, a, b, c, c, d, d\}$$

$$P \cap Q = \{a, a, c\}$$

$$P - Q = \{a, d, d\}$$

1.3 : Logic

Important Points to Remember

1. **Statement (Propositions)** : A statement or propositions is a declarative sentence which is either true or false but not both. These two values 'True' and 'False' are denoted by 'T' (or 1) and 'F' (or 0) respectively.

2. **Logical connectives** : The words or symbols which are used to form compound statements are called connectives. There are five logical connectively i.e. negation, conjunction, disjunction, conditional and biconditional.

(i)Negation (NOT) : If p is any statement then the negation of p is denoted by ~p or $\neg P$ and it is a statement read as 'not p'. It's truth table is

p	~p
T	F
F	T

(ii) Conjunction (AND) : If p and q are two statements then the conjunction of p and q is the compound statement 'p and q' and It is denoted by $p \wedge q$. The compound statement is true if both p and q are true, otherwise false.

(iii) Disjunction (OR) : If p and q are two statements then the disjunction of p and q is the compound statement 'p OR q' and it is denoted by $P \vee q$. The compound statement $p \vee q$ is false if both p and q are false otherwise true.

(iv) Conditional statement *(If ......then)* : If p and q are two statements then the conditional statement of p and q is the compound statement. 'If p then q' and it is denoted by $p \Rightarrow q$. The compound statement $P \Rightarrow q$ is false if p is true and q is false otherwise true.

(v) Biconditional statement : *(If and only if)* : If p and q are two statements then the biconditional statement of p and q is the compound 'p if and only if q' and it is denoted by $p \Leftrightarrow q$. The statement $p \Leftrightarrow$ is true if both p and q are true or both are false, otherwise false.

Truth table for all above types is

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

3. If $p \rightarrow q$ is the conditional statement then
   (i) $q \rightarrow p$ is called its converse statement
   (ii) $\sim p \rightarrow \sim q$ is called inverse statement.
   (iii) $\sim q \rightarrow \sim p$ is called its contrapositive statement.
4. **Tautology :** A statement which is true for all possible values of its propositional variables is called *tautology.*
5. **Contradiction :** A statement which is false for all possible values of its propositional variables is called *contradiction.* It is true negation of tautology.
6. **Contingency :** A statement which is neither tautology nor contradiction is called *contingency.*
7. **Predicates :** An assertion that contains one or more variables is called predicates. It's truth value is predicated after assigning truth values to its variables.
8. **Quantifiers :**

(i) The existential quantifiers tell us that for some objects, given statement is true. It is denoted by $\exists$. This symbol ($\exists$) read as *'there exist'* or 'for some'.

(ii) The universal quantifiers tell us that all objects possess the property. It is denoted by $\forall$ and read as 'for all' or for each or for every.

(iii) Negation :

Statement	Negation
1. $\forall x \ [p(x)]$	$\exists x \exists n [\sim p(n)]$
2. $\exists x (\sim p(x))$	$\forall n (p(n))$
3. $\forall x (\sim p(x))$	$\exists n (p(n))$
4. $\exists x \ p(x)$	$\forall n [\sim p(n)]$

Q.24 Let p denote the statement, "The material is interesting'. q denote the statement, "The exercises are challenging", and r denote the statement, "The course is enjoyable".

Write the following statements in symbolic form :

i)The material is interesting and exercises are challenging.

ii) The material is interesting means the exercises are challenging and conversely.

iii)Either the material is interesting or the exercises are not challenging but not both.

iv) If the material is not interesting and exercises are not challenging, then the course is not enjoyable.

v) The material is uninteresting, the exercises are not challenging and the course is not enjoyable.

☞[SPPU : Dec.-06, May-08, Marks 6]

Ans. : i) $p \wedge q$ ii) $(p \rightarrow q) \wedge (q \rightarrow p)$

iii) $p \oplus \sim q$ iv) $(\sim p \wedge \sim q) \rightarrow \sim r$ v) $\sim p \wedge \sim q \wedge \sim r$

Q.25 Express following statement in propositional form :

i) There are many clouds in the sky but it did not rain.

ii) I will get first class if and only if I study well and score above 80 in mathematics.

iii) Computers are cheap but softwares are costly.

iv) It is very hot and humid or Ramesh is having heart problem.

v) In small restaurants the food is good and service is poor.

vi)If I finish my submission before 5.00 in the evening and it is not very hot I will go and play a game of hockey.

☞[SPPU : May-05, Marks 6]

Ans. :

i) p : There are many clouds in the sky

q : It rain

∴ $p \wedge \sim q$

ii) p : I will get first class

q : I study well

r : Score above 80 in mathematics

∴ $p \leftrightarrow (q \wedge r)$

iii) p : Computers are cheap

q : Softwares are costly

$\therefore$ $p \wedge q$

iv) p : It is very hot

q : It is very humid

r : Ramesh is having heart problem

$\therefore$ $(p \wedge q) \vee r$

v) p : In small restaurant food in good

q : Service is poor

$\therefore$ $p \wedge q$

vi) p : I finish my submission before 5:00 p.m.

q : It is very hot

r : I will go

s : I will play a game of hockey

$\therefore$ $(p \wedge \sim q) \rightarrow (r \wedge s)$

Q.26 Express the contrapositive, converse and inverse forms of the following statement if 3 < b and 1 + 1 = 2, then $\sin\frac{\pi}{3} = \frac{1}{2}$.

[SPPU : May-07, Marks 6]

Ans. : p : $3 < b$

q : $1 + 1 = 2$

r : $\sin\frac{\pi}{3} = \frac{1}{2}$

Symbolic form : $p \wedge q \rightarrow r$

Contrapositive : $(\sim r \rightarrow \sim(p \wedge q))$

i.e. $\sim r \rightarrow (\sim p \vee \sim q)$

i.e. if $\sin\frac{\pi}{3} \neq \frac{1}{2}$ then $3 \geq b$ or $1+1 \neq 2$

Converse : $r \rightarrow (p \wedge q)$

i.e. if $\sin\frac{\pi}{3} = \frac{1}{2}$ then $3 < b$ and $1 + 1 = 2$

Inverse : $\sim(p \wedge q) \rightarrow \sim r$

i.e. $(\sim p \vee \sim q) \rightarrow \sim r$ i.e. if $3 \geq b$ or $1+1 \neq 2$ then $\sin\frac{\pi}{3} \neq \frac{1}{2}$

Q.27 Show that $p \wedge \sim p$ is a contradiction and $\sim (p \wedge \sim p)$ is tautology.

Ans. : We construct truth table for $\sim (p \wedge \sim p)$

p	$\sim p$	$p \wedge \sim p$	$\sim (p \wedge \sim p)$
T	F	F	T
F	T	F	T

As $p \wedge \sim p$ is always false. Hence $p \wedge \sim p$ is a contradiction. As $\sim (p \wedge \sim p)$ is always true, Hence $\sim (p \wedge \sim p)$ is a tautology.

Q.28 Determine whether each of the following statement formula is a tautology, contradiction or contingency.

i) $(p \wedge q) \wedge \sim (p \vee q)$

ii) $(p \rightarrow q) \leftrightarrow (q \vee \sim p)$

iii) $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$

iv) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

☞[SPPU : Dec.-12, Marks 6]

Ans. : i) Consider the truth table

p	q	$p \wedge q$	$p \vee q$	$\sim (p \vee q)$	$(p \wedge q) \wedge \sim (p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Hence $(p \wedge q) \wedge \sim (p \vee q)$ is a contradiction

ii) Consider the truthtable

p	q	$\sim p$	$p \rightarrow q$	$q \vee \sim p$	$(p \rightarrow q) \leftrightarrow (q \vee \sim p)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

As $(p \rightarrow q) \leftrightarrow (q \vee \sim p)$ is always true. Hence it is a tautology.

iii) Consider the truth table

p	q	r	$p \to q$	$p \to r$	$q \to r$	$p \to (q \to r)$	$(p \to q) \to (p \to r)$	$[p \to (q \to r)] \to [(p \to q) \to (p \to r)]$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	T	T	T	T
T	F	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

Hence $[(p \to (q \to r)] \to [(p \to q) \to (p \to r)]$ is a tautology.

iv) Consider the truth table

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Hence given statement formula is a tautology.

Q.29 Show that $(p \to q) \wedge \sim q \to \sim p$ is a tautology without using truth table.

Ans. : We know that $p \rightarrow q$ is true if p is true and q is also true.

$\therefore$ We need only to show that $p \rightarrow q$ and $\sim q$ both are true imply $\sim p$ is true.

As the truth value of $\sim q$ is T, the truth value of q is F. And as $p \rightarrow q$ is true, this means that p is false ($\because F \rightarrow F$ is true)

$\therefore$ The truth value of p is T. Hence the proof.

Q.30 Prove that $[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r)] \rightarrow (q \vee s)$ is a tautology.

☞[SPPU : Dec.-10, Marks 4]

Ans. : Consider truth table

p	q	r	s	$p \rightarrow q$	$r \rightarrow s$	$p \vee r$	$(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r)$ (I)	$q \vee s$	$I \rightarrow (q \vee s)$
T	T	T	T	T	T	T	T	T	T
T	T	T	F	T	F	T	F	T	T
T	T	F	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T	T	T
T	F	T	T	F	T	T	F	T	T
T	F	T	F	F	F	T	F	F	T
T	F	F	T	F	T	T	F	T	T
T	F	F	F	F	T	T	F	F	T
F	T	T	T	T	T	T	T	T	T
F	T	T	F	T	F	T	F	T	T
F	T	F	T	T	T	F	F	T	T
F	T	F	F	T	T	F	F	T	T
F	F	T	T	T	T	T	T	T	T
F	F	T	F	T	F	T	F	F	T
F	F	F	T	T	T	F	F	T	T
F	F	F	F	T	T	F	F	F	T

Hence given statement formula is a tautology.

Q.31 Prove by truth table $p \rightarrow (Q \vee R) \equiv (P \rightarrow Q) \vee (P \rightarrow R)$.

☞[SPPU : Dec.-12]

Ans. : Consider the truth table

				I			II
P	Q	R	$Q \vee R$	$P \to (Q \vee R)$	$P \to Q$	$P \to R$	$(P \to Q) \vee (P \to R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

From truth table

$$p \to (Q \vee R) \equiv (P \to Q) \vee (P \to R)$$

Q.32 Prove by constructing the truth table
$p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$. **[SPPU : Dec.-12, Marks 3]**

Ans. : Consider truth table

p	q	r	$q \vee r$	$p \to (q \vee r)$	$p \to q$	$p \to r$	$(p \to q) \vee (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

In the columns of $p \to (q \vee r)$ and $(p \to q) \vee (p \to r)$, truth values are same for all possible choices of truth values of p, q and r. Hence

$$p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$$

Q.33 Prove that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p) \equiv (\sim p \vee q) \wedge (\sim q \vee p)$.

Ans. : Consider the truth table

p	q	$\sim p$	$\sim q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$	$p \leftrightarrow q$	$\sim p \vee q$	$\sim q \vee p$	$(\sim p \vee q) \wedge (\sim q \vee p)$
T	T	F	F	T	T	T	T	T	T	T
T	F	F	T	F	T	F	F	F	T	F
F	T	T	F	T	F	F	F	T	F	F
F	F	T	T	T	T	T	T	T	T	T

From above table

$$(p \leftrightarrow q) \equiv (p \to q) \wedge (q \to p) \equiv (\sim p \vee q) \wedge (\sim q \vee p)$$

1.4 : Logical Identities

Important Points to Remember

Sr. No.	Name of identity	Identity
1.	Idempotence of $\vee$	$p \equiv p \vee p$
2.	Idempotence of $\wedge$	$p \equiv p \wedge p$
3.	Commutativity of $\vee$	$p \vee q \equiv q \vee p$
4.	Commutativity of $\wedge$	$p \wedge q \equiv q \wedge p$
5.	Associativity of $\vee$	$p \vee (q \vee r) \equiv (p \vee q) \vee r$
6.	Associativity of $\wedge$	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
7.	Distributivity of $\wedge$ over $\vee$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
8.	Distributivity of $\vee$ over $\wedge$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
9.	Double negation	$p \equiv \sim (\sim p)$
10.	De Morgan's laws	$\sim (p \vee q) \equiv \sim p \wedge \sim q$
11.	De Morgan's laws	$\sim (p \wedge p) \equiv \sim p \wedge \sim q$
12.	Tautology	$p \vee \sim q \equiv$ Tautology

13.	Contradiction	$p \wedge \sim p \equiv$ Contradiction
14.	Absorption laws	$p \vee (p \wedge q) \equiv p$
15.	Absorption laws	$p \wedge (p \vee q) \equiv p$

Q.34 De Morgan's laws i) $\sim (p \vee q) \equiv \sim p \wedge \sim q$ ii) $\sim (p \wedge q) \equiv \sim p \vee \sim q$

Ans. : i) $\sim (p \vee q) \equiv \sim p \wedge \sim q$

Consider the truth table

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim (p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

From the table, truth values of $\sim (p \vee q)$ and $\sim p \wedge \sim q$ are same for each choice of p and q

Hence $\sim (p \vee q) \equiv \sim p \wedge \sim q$

ii) $\sim (p \vee q) \equiv \sim p \vee \sim q$

Consider the table

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim (p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

From the table, truth values of $\sim (p \wedge q)$ and $(\sim p \vee \sim q)$ are same for each choice of p and q.

Hence $\sim (p \wedge q) \equiv (\sim p \vee \sim q)$

Q.35 Absorption laws i) $p \vee (p \wedge q) \equiv p$ ii) $p \wedge (p \vee q) \equiv p$

Ans. : i) $p \vee (p \wedge q) \equiv p$

Consider the table

p	q	$p \wedge q$	$p \vee (p \wedge q)$	p
T	T	T	T	T
T	F	F	T	T
F	T	F	F	F
F	F	F	F	F

From the table, in last two columns truth tables of $p \vee (p \wedge q)$ and p are same for each choice of p and q.

Hence $p \vee (p \wedge q) \equiv p$

ii) $p \wedge (p \vee q) \equiv p$

Consider the table

p	q	$p \vee q$	$p \wedge (p \vee q)$	p
T	T	T	T	T
T	F	T	T	T
F	T	T	F	F
F	F	F	F	F

From the table, last two columns are identical hence $p \wedge (p \vee q) \equiv p$

Q.36 $p \rightarrow q$ and $\sim p \vee q$ are logically equivalent.

Ans. : Consider the table

p	q	$\sim p$	$\sim p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Hence $\sim p \vee q$ and $p \rightarrow q$ are logically equivalent.

1.5 : Quantifiers

Important Points to Remember

Consider the statement $\forall x p(x)$. It's negation is "It is not the case that for all x, p(x) is true". This means that for some x = a, p(a) is not true or $\exists$ x s.t. ~ p(x) is true.

Hence the negation of $\forall x p(x)$ is logically equivalent to $\exists x [\sim p(x)]$.

Sr. No.	Statement	Negation
1.	$\forall x p(x)$	$\exists x [\sim p(x)]$
2.	$\exists x [\sim p(x)]$	$\forall x p(x)$
3.	$\forall x [\sim p(x)]$	$\exists x\, p(x)$
4.	$\exists x\, p(x)$	$\forall x [\sim p(x)]$

I) Equivalence involving quantifiers

1) Distributivity of $\exists$ over $\vee$

$$\exists x [p(x) \vee Q(x)] \equiv \exists x p(x) \vee \exists x Q(x)$$

$$\exists x [p \vee Q(x)] \equiv p \vee (\exists x Q(x))$$

2) Distributivity of $\forall$ over $\wedge$

$$\forall x [p(x) \wedge Q(x)] \equiv \forall x p(x) \wedge \forall x Q(x)$$

$$\forall x [p \wedge Q(x)] \equiv p \wedge (\forall x Q(x))$$

3) $\exists x [p \wedge Q(x)] \equiv p \wedge [\exists x Q(x)]$

4) $\forall x [p \vee Q(x)] \equiv p \wedge [\forall x Q(x)]$

5) $\sim [\exists x p(x)] \equiv \forall x [\sim p(x)]$

6) $\sim [\forall x p(x)] \equiv \exists x [\sim p(x)]$

7) $\forall x p(x) \Rightarrow \exists x p(x)$

8) $\forall x p(x) \vee \forall x Q(x) \Rightarrow \forall x (p(x) \vee Q(x))$

9) $\exists x (p(x) \wedge Q(x)) \Rightarrow \exists x p(x) \wedge \exists x Q(x)$

II) Rules of inference for addition and deletion of quantifiers

1) Rule 1 : Universal Instantiation

$\dfrac{\forall x p(x)}{\therefore\ p(k)}$, k is some element of the universe

2) Rule 2 : Existential Instantiation

$\dfrac{\exists x p(x)}{\therefore\ p(k)}$, k is some element for which p(k) is true.

3) Rule 3 : Universal Generalization

$\dfrac{px}{\forall x p(x)}$

4) Rule 4 : Existential Generalization

$\dfrac{p(k)}{\therefore\ \exists x p(x)}$

k is some element of the universe

III)

Sr. No.	Quantifiers	Expression
1.	$\exists x \forall y\, p(x,y)$	There exists a value of x such that for all values of y, p(x, y) is true.
2.	$\forall y\, \exists x p(x,y)$	For each value of y, there exists x such that p(x, y) is true.
3.	$\exists x \exists y\, p(x,y)$	There exist value of x and value of y such that p(x, y) is true.
4.	$\forall x\, \forall y\, p(x,y)$	For all values of x and y. p(x, y) is true.

Q.37 Represent the arguments using quantifiers and find its correctness. All students in this class understand logic. Ganesh is a student in this class. Therefore Ganesh understands logic.

[SPPU : Dec.-11, Marks 4]

Ans. : Let C(x) : x is a student in this class

L(x) : x understands logic

In symbolic form

$$\begin{array}{l} \forall x(C(x) \rightarrow L(x)) \\ \quad C(a) \\ \hline \therefore \quad L(a) \end{array}$$

Here a means Ganesh

This is Modus Ponen. Therefore this argument is valid.

Q.38 Let p(x) : x is even, Q(x) : x is a prime number, R(x, y) : x + y is even

a) Using the information given above write the following sentences in symbolic form.

i) Every integer is an odd integer

ii) Every integer is even or prime

iii) The sum of any two integers is an odd integer.

[SPPU : Dec.-10, Marks 4]

Ans. : i) $\forall x[\sim p(x)]$

ii) $\forall x[p(x) \vee Q(x)]$

iii) $\forall x \forall y[\sim R(x, y)]$

Q.39 Using information in Q.38 write an english sentence for each of the symbolic statement given below

i) $\forall x(\sim Q(x))$ ii) $\exists y(\sim p(y))$ iii) $\sim[\exists x (p(x) \wedge Q(x))]$

[SPPU : Dec.-10, Marks 4]

Ans. : i) All integers are not prime numbers

ii) At least one integer is not even.

iii) It is not the case that there exists an integer which is even and prime.

Q.40 Determine the validity of the following argument

s_1 : All my friends are musician, s_2 : John is my friend

s_3 : Name of my neighbours are musician,

s : John is not my neighbour.

Ans. : Let the universe of discourse be the set of people.

Let F(x) : x is my friend

M(x) : x is a musician

N(x) : x is my neighbour

It's symbolic form is

$$\begin{array}{l} s_1 : \forall x[F(x) \rightarrow M(x)] \\ s_2 : F(a)\ (a = \text{John}] \\ s_3 : \forall x[N(x) \rightarrow \sim M(x)] \\ \hline \therefore \quad s : \sim N(a) \end{array}$$

Suppose ~N(a) has value F.

$\therefore$ N(a) is T. Since s_3 is T, we must have ~M(a) is T or M(a) is F. But s_1 is T. Hence we must have F(a) to be false but this is contradiction. Hence if s is false either of s_1 or s_3 should be false. Hence argument is valid.

Q.41 For the universe of all integers. Let p(x) : x > 0, Q(x) : x is even, R(x) : x is a perfect square, S(x) : x is divisible by 4, T(x) : is divisible by 7

Write the following statement in symbolic form

i) At least one integer is even

ii) There exists a positive integer that is even

iii) If x is even then x is not divisible by 7

iv) No even integer is divisible by 7

v) There exists an even integer divisible by 7

vi) If x is even and x is perfect square then x is divisible by 4.

Ans. : i) $\exists x Q(x)$

ii) $\exists x[p(x) \wedge Q(x)]$

iii) $\forall x[Q(x) \rightarrow \sim T(x)]$

iv) $\forall x[Q(x) \rightarrow \sim T(x)]$

v) $\exists x[Q(x) \wedge T(x)]$

vi) $\forall x[Q(x) \wedge R(x) \rightarrow s(x)]$

Q.42 Rewrite the following statements using quantifier variables and predicate symbols. i) All birds can fly, ii) Not all birds can fly, ii) Some men are genius, iii) Some numbers are not rational

v) There is a student who likes Maths but not Hindi

vi) Each integer is either even or odd **[SPPU : Dec.-08, Marks 4]**

Ans. : i) Let B(x) : x is a bird

F(x) : x can fly

Then the statement can be written as

$\forall x[B(x) \rightarrow F(x)]$

ii) $\exists x[B(x) \wedge \sim F(x)]$

iii) Let $M(x)$: x is a man $G(x)$: x is a genius

The statement in symbolic form as $\exists x[M(x) \wedge G(x)]$

iv) Let $N(x)$: x is a number $R(x)$: x is rational

The statement in symbolic form as $\exists x[N(x) \wedge \sim R(x)]$ or $\sim[\forall x(N(x) \rightarrow R(x)]$

v) Let $S(x)$: x is a student, $M(x)$: x likes Maths

$H(x)$: x likes Hindi

$\therefore$ The statement in symbolic form as $\exists x[S(x) \wedge M(x) \wedge \sim H(x)]$

vi) Let $I(x)$: x is an integer

$E(x)$: x is even $O(x)$: x is odd

The statement in symbolic form as $\forall x[I(x) \rightarrow E(x) \vee O(x)]$

Q.43 Negate each of the following statements i) $\forall x, |x| = x$ ii) $\exists x, x^2 = x$ ☞ **[SPPU : Dec.-09, 15, May-15, Marks 4]**

Ans. : i) $\exists x, |x| \neq x$ ii) $\forall x, x^2 \neq x$.

Q.44 Negate the following i) If there is a riot, then someone is killed. ii) It is day light and all the people are arisen. ☞ **[SPPU : May-15, Dec.-15, Marks 4]**

Ans. : i) It is not the case that if there is a riot then someone is killed.
ii) It is not the case that it is day light and all the people are arisen.

OR

i) Let p : There is a riot, q : Someone is killed

Given statement is $p \rightarrow q$

Hence $\sim(p \rightarrow q) \equiv \sim(\sim p \vee q) \equiv p \wedge \sim q$

$\equiv$ There is a riot and someone is not killed.

ii) Let p : It is a day light q : All the people are arisen

Given statement is $p \wedge q$

Hence $\sim(p \wedge q) = \sim p \vee \sim q$

Hence either it is not a day light or all the people are not arisen.

1.6 : Mathematical Induction

Important Points of Remember

1. First Principle of Mathematical Induction Statement :

Let P(n) be a statement involving a natural number $n \geq n_0$ such that,

1) If P(n) is true for $n = n_0$ where $n_0 \leq N$ and

2) Assume that P(k) is true for $k = \geq n_0$

- We prove P(k+1) is also true,
- Then P(n) is true for all natural numbers $n = n_0$.
- Step 1 is called as the basis of induction.
- Step 2 is called as the induction step.

2. Second Principle of Mathematical Induction Statement (Strong Mathematical Induction)

- Let P(n) be a statement involving a natural number $n \geq n_0$ such that,

1) If P(n) is true for $n = n_0$ where $n_0 \leq N$ and

2) Assume that P (n) is true for $n_0 < n \leq k$
i.e. $P(n_0 + 1)(n_0 + 2)\ldots\ldots\ldots\ldots P(k)$ are true.
we prove that P(k + 1) is true,

Then P(n) is true for all natural numbers $n \geq n_0$.

Q.45 Prove that :

$$1 + 2 + 3 + 4 \ldots + n = \frac{n(n+1)}{2}$$

☞ **[SPPU : May-17, Dec.-18, Marks 4]**

Ans. : Let, P(n) be the given statement
Consider the following steps,

Step 1 : Basis of induction

For n = 1, L.H.S. = 1

$$\text{R.H.S.} = \frac{1\,(1+1)}{2} = 1$$

$\therefore$ For n = 1, L.H.S. = R.H.S.

$\therefore$ P(1) is true.

Step 2 : Assume that P(k) is true

i.e. $$1+2+3+\ldots\ldots+k=\frac{k(k+1)}{2} \quad \ldots (Q.45.1)$$

Consider, $1+2+3+\ldots\ldots+k+(k+1)$

$$=\frac{k(k+1)}{2}+k+1=\frac{k(k+1)+2(k+1)}{2}=\frac{(k+1)+(k+2)}{2}$$

Hence P(k + 1) is true

$\therefore$By the principle of mathematical induction P(n) is true for all n.

Q.46 Prove by mathematical induction for $n \geq 1$.

$$1\cdot2+2\cdot3+3\cdot4+\ldots+n(n+1)=\frac{n(n+1)(n+2)}{3}$$

☞ **[SPPU : May-05, Marks 6]**

Ans. : Let P(n) the given statement

1. Basis of Induction

For $n_0 = 1$ L.H.S. = 1.2 = 2

R.H.S. $= \frac{1(2)(3)}{3} = 2 \Rightarrow$ L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step

Assume that, P(k) is true

i.e. $$1\cdot2+2\cdot3+3\cdot4+\ldots+k(k+1)=\frac{k(k+1)(k+2)}{3} \quad \ldots (Q.46.1)$$

Then we have

$$[1\cdot2+2\cdot3+3\cdot4+\ldots+k(k+1)]+(k+1)(k+2)$$

$$=\frac{k(k+1)(k+2)}{3}+(k+1)(k+2) \quad \ldots \text{(Using Q.46.1)}$$

$$=(k+1)(k+2)\left[\frac{k}{3}+1\right]=\frac{(k+1)(k+2)(k+3)}{3}$$

Hence assuming P(k) is true, P(k + 1), is also true. Therefore by mathematical induction P(n) is true for all $n \geq 1$.

Q.47 Show by induction that , $n \geq 1$

$$1^2+3^2+5^2+\ldots(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$$

☞ **[SPPU : May-14, Marks 4]**

Ans. : Let P(n) be the given statement,

1. Basis of Induction :

For n = 1, L.H.S. $= 1^2 = 1$,

R.H.S. $= \frac{1(1)(3)}{3} = 1$

$\Rightarrow$ L.H.S. = R.H.S.

Hence P(1) is true

2. Induction step : Assume that P(k) is true.

i.e. $$1^2+3^2+5^2+\ldots+(2k-1)^2=\frac{k(2k-1)(2k+1)}{3} \quad \ldots (Q.47.1)$$

Hence

$$[1^2+3^2+5^2+\ldots+(2k-1)^2]+(2k+1)^2$$

$$=\frac{k(2k-1)(2k+1)}{3}+(2k+1)^2 \quad \ldots \text{(Using Q.47.1)}$$

$$=\frac{(2k+1)}{3}[2k^2-k+3(2k+1)]$$

$$=\frac{(2k+1)}{3}[2k^2+5k+3]$$

$$=\frac{(2k+1)}{3}[2k^2+2k+3k+3]$$

$$=\frac{(2k+1)}{3}[(2k+3)(k+1)]$$

$$=\frac{(k+1)(2k+1)(2k+3)}{3}$$

$$=\frac{(k+1)[2(k+1)-1][2(k+1)+1]}{3}$$

Hence assuming P(k) is true P(k + 1) is also true. Therefore by mathematical induction P(n) in true for all $n \geq 1$.

Q.48 Show that

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2 (n+1)^2}{4} = (1+2+3+\ldots+n)^2$$

☞ [SPPU : Dec.-12, May-18, Marks 4]

Ans. : Let P(n) be the given statement,

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2 (n+1)^2}{4}$$

1. Basis of induction :

For n = 1, L.H.S. = 1,

$$\text{R.H.S.} = \frac{1(1+1)^2}{4} = 1$$

$\Rightarrow$ L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step : Assume that P(k) is true.

i.e. $1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2 (k+1)^2}{4}$ ... (Q.48.1)

Then we have

$$(1^3 + 2^3 + 3^3 + \ldots + k^3) + (k+1)^3$$

$$= (1+2+3+\ldots+k)^2 + (k+1)^3 \quad \text{(Using Q.48.1)}$$

$$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

$$= (k+1)^2 \left[\frac{k^2}{4} + k + 1\right]$$

$$= (k+1)^2 \left[\frac{k^2 + 4k + 4}{4}\right]$$

$$= \frac{(k+1)^2 (k+2)^2}{4}$$

$$= \left(\frac{(k+1)(k+2)}{2}\right)^2 = \frac{(k+1)^2 (k+2)^2}{4}$$

Hence assuming P(k) is true, P(k + 1) is also true. Therefore by mathematical induction P(n) is true for all $n \geq 1$.

Q.49 Show that

a) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

b) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

c) $\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \ldots + \frac{1}{(3n-1)(3n+1)} = \frac{n}{3n+1}$

☞ [SPPU : Dec.-05, Marks 6]

Solution : Let P(n) be the given statement,

a) 1. Basis of induction :

For n = 1 $\quad \text{L.H.S.} = \frac{1}{1.2} = \frac{1}{2}$

$$\text{R.H.S.} = \frac{1}{1+1} = \frac{1}{2}$$

L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step : Assume that P(k) is true.

i.e. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{k(k+1)} = \frac{k}{k+1}$ ... (Q.49.1)

Then we have

$$\left[\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{k(k+1)}\right] + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \quad \text{... (Using Q.49.1)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2 + 2k + 1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2} = \frac{(k+1)}{(k+1)+1}$$

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) in true for all $n \geq 1$.

b) Let P(n) : $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

1. Basis of induction : For n = 1

$$\text{L.H.S.} = \frac{1}{1.3} = \frac{1}{3}, \text{ R.H.S.} = \frac{1}{3}$$

$$\frac{1}{3} = \frac{1}{2.1+1}$$

$\Rightarrow$ L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step : Assume that P(k) is true

i.e. $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$ ... (Q.49.2)

Then we have,

$$\left[\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ldots+\frac{1}{(2k-1)(2k+1)}\right]+\frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)} = \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$

$$= \frac{k+1}{2k+3} = \frac{k+1}{2(k+1)+1}$$

Hence assuming P(k) is true. P(k + 1) is also true. Therefore P(n) is true for all n ≥ 1.

c) Let P(n) : $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\ldots+\frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$

1. Basis of induction

For n = 1, $\text{L.H.S.} = \frac{1}{1.4} = \frac{1}{4}$, R.H.S. $= \frac{1}{4}$

$$\frac{1}{14} = \frac{1}{3.1+1}$$

$\Rightarrow$ L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step : Assume that P(k) is true.

i.e. $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\ldots+\frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1}$ ... (Q.49.3)

Then we have,

$$\left[\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\ldots+\frac{1}{(3k-2)(3k+1)}\right]+\frac{1}{(3k+1)(3k+4)}$$ ... (Using Q.49.3)

$$= \frac{k}{(3k+1)}+\frac{1}{(3k+1)(3k+4)}$$

$$= \frac{3k^2+4k+1}{(3k+1)(3k+4)} = \frac{(3k+1)(k+1)}{(3k+1)(3(k+1)+1)}$$

$$= \frac{k+1}{3(k+1)+1}$$

Hence assuming P(k) is true. P(k + 1) is also true. Therefore P(n) is true for all n ≥ 1.

Q.50 Use mathematical induction to show that $n(n^2-1)$ is divisible by 24. Where n is any odd positive number.

☞ [SPPU : Dec.-14, Marks 4]

Ans. : If $n(n^2-1) = n^3-n$ is divisible by 24.

Then $n^3 - n = 24(m)$ where m is any positive integral.

Let P(n) be the given statement,

1. Induction step : For n = 1,

$n(n^2-1) = 0$ which is divisible by 24.

For n = 3, $n\left(n^2-1\right) = 24$ which is divisible by 24.

∴ P(1) and P(3) is true.

2. Induction step :

Assume that P(k) is true.

i.e. $k(k^2-1) = k^3-k$ is divisible by 24.

∴ $k(k^2-1) = k^3-k = 24(m_0),\ m_0 \leftarrow z$ ... (Q.50.1)

Consider

$$(k+1)[(k+1)^2-1] = (k+1)^3-(k+1)$$

$$= k^3+3k^2+3k+1-k-1$$

$$= k^3+3k^2+2k$$

$$= (k^3-k)+3k^2+3k \quad \text{... (Using Q.50.1)}$$

$$= 24\,m_0+3k(k+1)$$

(As k(k + 1) is multiple of 8 for k odd positive integer and $k \geq 3$)

$$= 24\,m_0+3(8m_1)$$

$$= 24\,(m_0+m_1)$$

$$= 24\,m_2\ (\because m_0+m_1=m_2)$$

$\therefore$ P(k + 1) is true.

$\therefore$ By mathematical induction P(n) is true for all n odd positive number.

Q.51 Show that n^4-4n^2 is divisible by 3 for all $n \geq 2$.

☞ **[SPPU : Dec.-15, Marks 4]**

Ans. : Let P(n) be the given statement,

1. Basis of induction

For $n = 2$

$$2^4-4(2^2) = 16-16$$

$= 0$ is divisible by 3 as 0 is divisible by every number

$\therefore$ P(2) is true.

2. Induction step : Assume that P(k) is true

i.e. k^4-4k^2 is divisible by 3

Then we have,

$$(k+1)^4-4(k+1)^2 = k^4+4k^3+6k^2+4k+1-4(k^2+2k+1)$$

$$= (k^4-4k^2)+4(k^3+2k)+6k^2+12k-3$$

k^4-4k^2 is divisible by 3.

k^3+2k is divisible by 3

Also $\quad 6k^2+12k-3 = 3(2k^2+4k-1)$ is divisible by 3.

Hence $(k+1)^4-4(k+1)^2$ is divisible by 3.

Hence assuming P(k) is true.

P(k + 1) in also true. Therefore P(n) is true for $n \geq 2$.

Q.52 Prove that : $7^{2n}+2^{3n-3}\cdot 3^{n-1}$ is divisible by 25 $\forall$ n.

☞ **[SPPU : May-19, Marks 3]**

Ans. : Let P(n) be the given statement.

1) Basis of induction : For n = 1,

$7^2+2^0\cdot 3^0 = 49+1 = 50$, which is divisible by 25.

$\therefore$ P(1) is true.

2) Induction step :

Assume that P(k) is true.

i.e. $7^{2k}+2^{3k-3}\,3^{k-1} = 25\,(m)$

Consider

$$7^{2(k+1)}+2^{3(k+1)-3}\cdot 3^{k+1-1} = 7^{2k+2}+2^{3k}\,3^k$$

$$= 49 7^{2k}+2^{3k}\cdot 3k$$

$$= 49[25m-2^{3k-3}\cdot 3^{k-1}]+2^{3k}\cdot 3^k$$

$$= 25\,(49m)-49\,2^{3k-3}\cdot 3^{k-1}+2^{3k-3}\cdot 3^{k-1}(8\times 3)$$

$$= 25\,(49m)+2^{3k-3}\cdot 3^{k-1}\,(-49+24)$$

$$= 25(49\,m)+2^{3k-3}3^{k-1}(-25)$$

$$= 25\,[49\,m-2^{3k-3}\cdot 3^{k-1}]$$

$$= 25\,(P) : P \in N$$

$\therefore$ P(k+1) is true.

$\therefore$ By mathematical induction, P(n) is true for all n.

Q.53 Prove by mathematical induction that for $n \geq 1$:
$1\cdot 1!+2\cdot 2!+3\cdot 3!+\ldots+n\cdot n! = (n+1)!-1$.

☞ **[SPPU : May-08, 15, Marks 6]**

Ans. : Let P(n) be the given statement,

1. Basis of Induction

For $n = 1$, L.H.S = 1, R.H.S. = 1

$\Rightarrow$ L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step : Assume that, P(k) is true.

i.e. $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + k \cdot k!$

$$= (k+1)! - 1 \qquad \ldots (Q.53.1)$$

Then we have,

$$[1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + k \cdot k!] + (k+1) \cdot (k+1)! \quad \ldots \text{(Using Q.53.1)}$$

$$= [(k+1)! - 1] + (k+1) \cdot (k+1)!$$

$$= (k+1)! + (k+1) \cdot (k+1)! - 1$$

$$= (k+1)!\,[k+1+1] - 1$$

$$= (k+2)\,(k+1)! - 1$$

$$= (k+2)! - 1$$

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) is true for $n \geq 1$.

Q.54 Prove that $8^n - 3^n$ is a multiple of 5 by mathematical induction for $n \geq 1$. **☞ [SPPU : May-06, 07, Dec.-13, Marks 6]**

Ans. : Let P(n) be the given statement,

1. Basis of Induction :

For n = 1 $\quad 8^1 - 3^1 = 5$

$$= 5 \cdot 1$$

Obviously a multiple of 5.

$\therefore$ P(1) is true.

2. Induction step : Assume that, P(k) in true.

i.e. $8^k - 3^k$ is multiple of 5 say 5 r

i.e. $\quad 8^k - 3^k = 5r \qquad \ldots (Q.54.1)$

where r is an integer

Then we have,

$$8^{k+1} - 3^{k+1} = 8^k \cdot 8 - 3^k \cdot 3$$

$$= 8^k \cdot (5+3) - 3^k \cdot 3$$

$$= 8^k \cdot 5 + (8^k \cdot 3 - 3^k \cdot 3)$$

$$= 8^k \cdot 5 + 3(8^k - 3^k)$$

Obviously $8^k \cdot 5$ is multiple of 5 and also $8^k - 3^k$ is multiple of 5.

Therefore, $8^{k+1} - 3^{k+1}$ is multiple of 5.

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) is true for all $n \geq 1$.

Q.55 Show that the sum of the cubes of three consecutive natural number is divisible by 9. **☞ [SPPU : Dec.-06, Marks 6]**

Ans. : Let n, n + 1, n + 2 be three consecutive natural numbers.

We have to show that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

Let P(n) be the above statement,

1. Basis of Induction : For n = 1

$1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36$ which is divisible by 9.

$\therefore$ P(1) is true.

2. Induction step : Assume that P(k) is true.

i.e. $k^3 + (k+1)^3 + (k+2)^3$ in divisible by 9.

Then we have,

$$(k+1)^3 + (k+2)^3 + (k+3)^3$$

$$= [(k+1)^3 + (k+2)^3] + [k^3 + {}^3C_1 k^2 (3) + {}^3C_2 k(3)^2 + {}^3C_3 (3^3)]$$

$$= (k+1)^3 + (k+2)^3 + k^3 + [9k^2 + 27k + 27]$$

$$= [k^3 + (k+1)^3 + (k+2)^3] + 9[k^2 + 3k + 3]$$

$k^3 + (k+1)^3 + (k+2)^3$ is divisible by 9 and $9(k^3 + 3k + 3)$ is divisible by 9.

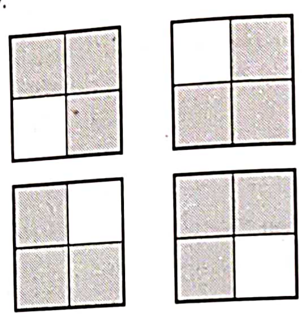
$\Rightarrow (k+1)^3 + (k+2)^3 + (k+3)^3$ is divisible by 9.

Hence assuming P(k) in true, P(k + 1) is also true. Therefore P(n) is true for all $n \geq 1$.

Q.56 Let n be a positive integer. Show that any $2^n \times 2^n$ chessboard with one square removed can be covered by L-shaped pieces, where each piece covers three squares at a time.

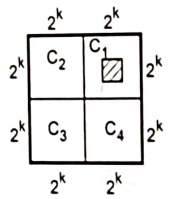
Ans. : Let P(n) be the proposition that any $2^n \times 2^n$ chessboard with one square removed can be covered using L-shaped pieces.

Basis of Induction : For n = 1, P(1) implies that any 2×2 chessboard with one square removed can be covered using L shaped pieces. P(1) is true, as seen below.



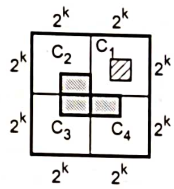
Induction step : Assume that, P(k) is true i.e. any $2^k \times 2^k$ chessboard with one square removed can be covered using L-shaped pieces.

Then, we have to show that P(k + 1) is true. For this consider, a $2^{k+1} \times 2^{k+1}$ chesshoard with one square removed. Divide the chessboard into four equal halves of size $2^k \times 2^k$, as shown below.



The square which has been removed, would have been removed from one of the four chessboards, say C_1. Then by induction hypothesis, C_1 can be covered using L-shaped pieces. Now, from each of the remaining

chessboards, remove that particular piece (or tile), lying at the centre of the large chessboards.



Then by induction hypothesis, each of these $2^k \times 2^k$chessboards with a piece (or tile) removed can be covered by the L-shaped pieces. Also the three tiles removed from the centre can be covered by one L-shaped piece. Hence the chessboard of $2^{k+1} \times 2^{k+1}$ can be covered by L-shaped pieces.

Hence proved.

END...