Unit II

2 Relations and Functions

2.1 : Relations

Important Points to Remember

1. **Cartesian Product :** Let A and B be two non empty sets. The Cartesian product of A and B is denoted by $A \times B$ and defined as

$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$

(x, y) is known as order pair.

2. **Relation :** Let A and B be two non-empty sets. A relation from A to B is any subset of $A \times B$. It is denoted by $R : A \to B$. Set A is called domain set and B is called co-domain set.

The range set of relation R, is the set of elements of B that are second elements of pairs in R.

3. A relation $R : A \to A$ is called the relation on set A.

4. A relation can be represented in two forms.

(a) **Matrix form :** Let $R : A \to B$ be a relation.

$A = \{a_1, a_2, a_3, \ldots\ldots, a_n\}$, $B = \{b_1, b_2, b_3, \ldots\ldots, b_m\}$.

Then R can be represented by $n \times m$ matrix such that

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, a_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

(b) **Digraph form :** Let R be a relation on set A. Draw small circles or dark dots for each element of A. These circles are called vertices of digraph.

Draw an edge with arrow from a_i to a_j iff $a_i R a_j$ such representation is called digraph or directed graph of R.

5.Types of Relation :

Sr. No.	Name of relation	Condition
1.	Reflexive Relation	If $(a, a) \in R$ for all $a \in A$
2.	Irreflexive	If $(a, a) \notin R$ for all $a \in A$
3.	Symmetric	Whenever $(a, b) \in R$ then $(b, a) \in R$
4.	Asymmetric	Whenever $(a, b) \in R$ then $(b, a) \notin R$
5.	Antisymmetric	Whenever $(a, b) \in R$ and $(b, a) \in R$ then $a = b$
6.	Transitive relation	If $(a, b) \in R$, $(b, c) \in R$ then $(a, c) \in R$
7.	Equivalence relation	If R is reflexive, symmetric and transitive.
8.	Partial ordering relation	If r is reflexive antisymmetric and transitive.

6. Important Properties :

i) *Complement of a relation :* Let R be a relation from A to B. The complement of relation R is denoted by $\overline{R}$ and defined as relation from A to B such that

$$\overline{R} = \{(a, b) \mid (a, b) \notin R\}$$

ii) *Inverse of relation :* Let $R : A \to B$ be a relation then the inverse relation $R^{-1} : B \to A$ is a relation such that xR^{-1} iff y Rx.

iii) *Composite relation :* Let $R_1 : A \to B$ and $R_2 : B \to C$ be two relations. Then the composition of relations R_1 and R_2 is denoted by $R_1 \cdot R_2$ or $R_1 R_2$ and defined as

$$R_1 \cdot R_2 = \{(x, z) \mid xR_1y \text{ and } yR_2z ; x \in A, y \in B, z \in C\}$$

iv) *Equivalence class* : Let $R : A \rightarrow B$ be an equivalence relation. The equivalence class of an element $a \in A$ is the set of all elements of A which are related to a. It is denoted by [a]

$\therefore \quad [a] = \{x \mid xRa \; ; x \in A\}$

7. Closure of Relations :

i) *Reflexive closure* : Let R be a relation on a set A and R is not reflexive relation. A relation $R_1 = R \cup \Delta$ is the reflexive closure of R if $R \cup \Delta$ is the smallest reflexive relation containing R. If $A = \{x_1, x_2, x_3, x_4\}$ then take

$$\Delta = \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_4, x_4)\}$$

ii) *Symmetric closure* : A relation R^* is the symmetric closure of relation R if $R^* = R \cup R^{-1}$.

iii) *Transitive closure* : Let $R : A \rightarrow A$ be a relation which is not transitive. The transitive closure of a relation R is the smallest transitive relation containing R.

Note : *Let A be a set with n elements and R be a relation on set A. Then* $R^* = R \cup R^1 \cup R^2 \cup \ldots\ldots \cup R^n$.

Q.1 Which of the following are relations from A to B where A = {1, 2, 3, 4} ; B = {x, y, z}

(a) $R_1 = \{(1, x), (1, y), (1, z), (4, x)\}$

(b) $R_2 = \{(x, 1), (y, 1), (z, 1), (x, 4)\}$

(c) $R_3 = \{(1, x), (2, y), (3, z), (4, w)\}$

(d) $R_4 = \{(1, 1), (2, 2)\}$

Ans. : Given that $R : A \rightarrow B$

Where A = Domain set

and B = Co-domain set

$$A \times B = \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z), (3, x), (3, y), (3, z), (4, x), (4, y), (4, z)\}$$

(a) R_1 is a relation from A to B because $R_1 \subset A \times B$

(b) R_2 is not a relation from A to B as $R_2 \not\subset A \times B$

(c) R_3 is not a relation from A to B as $(4, w) \in R$ but $w \in B$

(d) R_4 is not a relation from A to B as $R_4 \not\subset A \times B$

Q.2 If R is a relation on set A = {1, 2, 3, 4, 5}, R = {(1, 1) (1, 2) (2, 1) (3, 1) (4, 1) (5, 2)}. Find domain set, codomain set, range. Draw its digraph and find relation matrix.

Ans. : Given that $A = \{1, 2, 3, 4, 5\}$

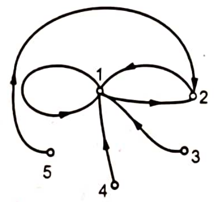
and R is a relation on set A.

$\therefore$ Domain set = A = Co - domain set

$R = \{(1, 1) (1, 2) (2, 1) (3, 1) (4, 1) (5, 2)\}$

$\therefore$ Range set $= \{1, 2\} \subset A$

Its digraph is as follows :



The matrix form of R is

$$M(R) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}_{5 \times 5} \end{array}$$

Q.3 Let A = {1, 2, 3, 4, 5, 6} aRb or (a, b) ∈ R iff a is a multiple of b. Find : (i) Range set and R (6), R (3). (ii) Find relation matrix. (iii) Draw digraph (iv) Find in and out degree of each vertex.

Ans. : By considering given condition.

$R = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 3), (4, 1), (4, 2), (4, 4), (5, 1), (5, 5), (6, 1), (6, 2), (6, 3), (6, 6)\}$

(i) The range set of R is {1, 2, 3, 4, 5, 6}

R (3) = {1, 3} as (3, 1) and (3, 3) $\in$ R

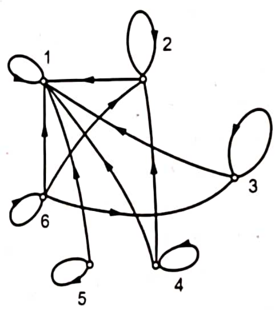
and R (6) = {1, 2, 3, 6} because (6, 1), (6, 2), (6, 3), (6, 6) $\in$ R

(ii) The relational matrix is

$$M(R) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(iii) The digraph of given relation is

vertex set = {1, 2, 3, 4, 5, 6}



(iv) The indegree of a vertex in digraph is the number of edges coming towards that vertex. The outdegree of a vertex is defined as the number of edges going out from a vertex. Consider the following table.

Vertex	Indegee	Outdegree	Degree
1	6	1	7
2	3	2	5
3	2	2	4
4	3	1	4
5	2	1	3
6	4	1	5

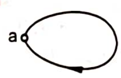
Q.4 Define diagraphs.

Ans. : A relation can be represented pictorially by drawing its graph. Let A be any non empty set and R be a relation on A. R can be represented by graphically by the following procedure.

i) The elements of a set A are represented by small circles or point i.e. (∘) or ... These elements are called as vertices or nodes.

ii) If (a, b) $\in$ R or a R b then vertices a and b are joined by a continuous arc with an arrow from a to b. These arcs are known as edges of the graph i.e. if aRb then

iii) If aRa then the vertex a is joined to itself by a loop around a. e.g. if aRa then



This graphical representation of the relation is called as diagraph or directed graph of R. Let us consider the following examples.

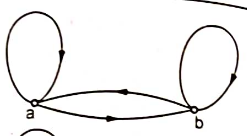
1) aRa

2) aRb

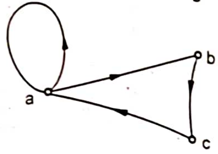
3) bRa

4) aRb $\wedge$ bRa

5) $aRa \wedge aRb \wedge bRa \wedge bRb$

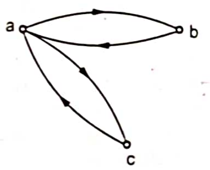


6) $aRb \wedge bRc \wedge CRa \wedge aRa$



7) $aRa \wedge bRb \wedge cRc$

8) $aRb \wedge bRa \wedge aRc \wedge cRa$



Q.5 Let A = R, set of all straight lines in a plane. R_1 = {x R_1 y | x is parallel to y}, R_2 = {x R_2 y | x is perpendicular to y } check whether R_1 and R_2 are equivalence relations or not.

Ans. : i) For relation R_1 : we know that if x, y, z are straight lines in a plane, then x is parallel to x. $\therefore$ R_1 is reflexive.

ii) If $x R_1 y \Rightarrow x$ is $||$ to $y \Rightarrow y$ is parallel to x

$\Rightarrow y R_1 x \Rightarrow R_1$ is symmetric.

iii) If $x R_1 y, y R_1 z \Rightarrow x \| y, y \| z \Rightarrow x \| z \Rightarrow x R_1 z.$

$\Rightarrow R_1$ is transitive relation.

$\therefore$ R_1 is an equivalence relation

iv) For relation R_2 : Any line is not perpendicular to itself.

$\therefore x R_2 x, \forall x \in A$

$\therefore$ R_2 is not reflexive.

$\therefore$ R_2 is not equivalence relation.

Q.6 If A = {x, y, z } and R = {(x, x) (y, y) (z, z)}. It is an equivalence relation ?

Ans. : The given relation R is reflexive, symmetric and transitive.

$\therefore$ R is an equivalence relation.

Q.7 If A = z, R = {(x, y) | x + y = even, x, y $\in$ z }. Is R equivalence relation ? if yes, find equivalence class of 1 and 2.

Ans. : $x + x$ = even number for any $x \in z$.

$\therefore x Rx \Rightarrow R$ is reflexive relation.

Suppose x Ry $\Rightarrow x + y$ = even

$\Rightarrow y + x$ = even

$\Rightarrow$ yRx

$\therefore$R is symmetric relation.

Suppose xRy and yRz $\Rightarrow x + y$ = even, $y + z$ = even

$\Rightarrow x + y + y + z$ = even

$\Rightarrow x + z$ = even $- 2y$ = even

$\Rightarrow x + z$ = even

$\Rightarrow$ xRz

$\therefore$R is a transitive relation.

$\therefore$R is an equivalence relation.

$\therefore$The equivalence class of $1 \in z$ is

$[1]_R = \{xR_1 / x+1 = \text{even}\}$

$= \{xR_1 / x = \text{even} - 1 = \text{odd}\}$

$= \{xR_1 / x \text{ is odd}\}$

$[1]_R$ = Set of all odd integers = $\{\ldots -3, -1, 1, 3, 5, \ldots\}$

$[2]_R = \{xR_2 | x+2=\text{even}\}$

$= \{xR_2 | x=\text{even}\}$

= Set of all even integers

$[2]_R = \{\ldots - 4, - 2, 0, 2, 4 \ldots]$

Theorem 1 : Any two equivalence classes are either identical or disjoint.

Theorem 2 : For any $x \in A$, $x \in A$, $x \in [x]_R$

Theorem 3 : $A = \bigcup_{a \in A} [a]$

Q.8 Let R be a relation on set of natural numbers such that $R = \{(x, y) \mid 2x + 3y$ and $x, y \in N\}$. Find
(i) The domain and codomain of R. ; (ii) R^{-1}

Ans. : i) Given that $x, y \in N$

$\therefore$ The smallest values of x and y are 1, 1 respectively.

$\therefore \quad 2x + 3y = 2(1) + 3(1) = 5$

which is the smallest value of co-domain

$$x = 1, \quad y = 2 \Rightarrow 2x + 3y = 8$$
$$x = 2, \quad y = 1 \Rightarrow 2x + 3y = 7$$
$$x = 2, \quad y = 2 \Rightarrow 2x + 3y = 10$$
$$x = 2, \quad y = 3 \Rightarrow 2x + 3y = 13$$
$$x = 3, \quad y = 2 \Rightarrow 2x + 3y = 12$$

$\therefore$ The domain of R is $= \{x \mid x \in N\}$

and the codomain of R is $= \{y \mid y \in N\}$

(ii) The inverse relation is

$$R^{-1} = \{(y, x) / x, y \in N\}$$

Q.9 Let A = Set of all students in SPPU
xRy iff x and y belongs to same class of SPPU ∀, y ∈ A.
Is R an equivalence relation ? Find the equivalence class of Atharva in A.

Ans. : Given that A = {Set of all students in SPPU}
and xRy iff x and y belong to same class of SPPU.

(i) Any student x and x belong to same class

$\therefore xRx$, $\forall x \in A \Rightarrow$ R is reflexive relation.

(ii) Let $xRy \Rightarrow$ x and y belong to same class of SPPU.

$\Rightarrow$ y and x belong to same class of SPPU.

$\Rightarrow yRx$

i.e. $xRy \Rightarrow yRx$, $\forall x, y \in R$.

Thus R is symmetric relation.

(iii) Let xRy and yRz

$\Rightarrow$ x and y belong to same class and y and z belong to same class of SPPU

$\Rightarrow$ x, y, z belong to same class of SPPU

$\Rightarrow x R z$

$\Rightarrow$ R is transitive relation.

Thus R is an equivalence relation.

Suppose there is one student in SPPU whose name is Atharva.

$\therefore$ The equivalence class of Atharva is the set of all students who are studying in the class of Atharva.

Q.10 Let A = {a, b, c} and M (R) = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.
Determine whether R is an equivalence relation ? Find the equivalence class of b in A.

Ans. : From relation matrix, the relation R is

$$R = \{(a, a)\ (b, b)\ (c, c)\ (b, c)\ (c, b)\}$$

(i) Here $a, b, c \in A$ and (a, a) (b, b) and $(c, c) \in R$

$\therefore$ R is reflexive relation.

(ii) As $(b, c) \in R \Rightarrow (c, b) \in R$

R is symmetric relation.

(iii) In relation R, we have,

$$(b, b), (b, c) \in R \Rightarrow (b, c) \in R$$
$$(b, c), (c, b) \in R \Rightarrow (b, b) \in R$$
$$(b, c), (c, c) \in R \Rightarrow (b, c) \in R$$
$$(c, c), (c, b) \in R \Rightarrow (c, b) \in R$$
$$(c, b), (b, b) \in R \Rightarrow (c, b) \in R$$
$$(c, b), (b, c) \in R \Rightarrow (c, c) \in R$$

So R is transitive relation. Thus R is an equivalence relation.

Now, the equivalence class of b in A is,

$$[b] = \{x \mid (x, b) \text{ or } (b, x) \in R\}$$
$$[b] = \{b, c\}$$

Q.11 Let A = {1, 2, 3, 4, 5, 6, 7, 8, 9} and let N be the relation on A × A defined by (a, b) ~ (c, d) iff a + d = b + c.
(i) Prove that ~ is an equivalence relation.
(ii) Find equivalence class of (2, 5) ☞ **[SPPU : Dec.-11]**

Ans. : Given that (a, b) ~ (c, d) iff a + d = b + c, $\forall$ a, b, c, d $\in$ A

(i) (a) We have a + b = b + a $\Rightarrow$ (a, b) ~ (a, b)

$\Rightarrow$ '~' is reflexive relation.

(b) If (a, b) ~ (c, d) then a + d = b + c.

$\Rightarrow$ b + c = a + d $\Rightarrow$ c + b = d + a

$\Rightarrow$ (c, d) ~ (a, b) by definition

$\Rightarrow$ '~' is symmetric relation.

(c) Suppose (a, b) ~ (c, d) and (c, d) ~ (e, f)

then $a + d = b + c$

and $c + f = d + e$

$\Rightarrow$ $a + \cancel{d} + \cancel{c} + f = b + \cancel{c} + \cancel{d} + e$

$a + f = b + e$

$\Rightarrow$ (a, b) ~ (e, f)

$\therefore$ ~ is a transitive relation.

Thus '~' is reflexive, symmetric and transitive.

$\Rightarrow$ '~' is an equivalence relation.

(ii) We have (2, 5) $\in$ A × A

The equivalence class of (2, 5) is the set of elements of A × A which are equivalent to (2, 5)

$\therefore$ [(2, 5)] = {(x, y) | (x, y) ~ (2, 5) , x, y $\in$ A}

= {(x, y) | x + 5 = y + 2}

= {(x, y) | x - y = -3 or y = x + 3 ; x , y $\in$ A}

Hence, [(2, 5)] = {(1, 3) (2, 5) (3, 6) (4, 7) (5, 8) (6, 9)}

... [$\because$ (7, 10) $\notin$ A × A]

Q.12 Let A = R × R (R is set of real numbers) and define the following relation on A. (a, b) R (c, d) iff $a^2 + b^2 = c^2 + d^2$
(i) Show that (A, R) is an equivalence relation.
(ii) Find equivalence class of (3, 2)

Ans. : Given that (a, b) R (c, d) iff $a^2 + b^2 = c^2 + d^2$

(i) (a) We have, $a^2 + b^2 = a^2 + b^2$

$\Rightarrow$ (a, b) R (a, b)

R is a reflexive relation.

(b) If (a, b) R (c, d) then $a^2 + b^2 = c^2 + d^2$

$\Rightarrow$ $c^2 + d^2 = a^2 + b^2$

$\Rightarrow$ (c, d) R (a, b)

R is symmetric relation.

(c) Suppose (a, b) R (c, d) and (c, d) R (e, f)

$\Rightarrow$ $a^2 + b^2 = c^2 + d^2$ and $c^2 + d^2 = e^2 + f^2$

$\Rightarrow$ $a^2 + b^2 = e^2 + f^2$

$\Rightarrow$ (a, b) R (e, f)

$\therefore$ R is transitive relation.

Thus, R is an equivalence relation.

(ii) An equivalence class of (3, 2) is the set of elements of A which are equivalent to (3, 2)

$\therefore$ [(3, 2)] = {(x, y) | (x, y) R (3, 2) ; x, y $\in$ R}

= {(x, y) | $x^2 + y^2 = 9 + 4 = 13$; x, y $\in \Re$}

[(3, 2)] = The set of points on circle $x^2 + y^2 = 13$.

Q.13 For each of these relations on set A = {1, 2, 3, 4} decide whether it is reflexive, symmetric, transitive or antisymmetric.
R_1 = {(1, 1) (2, 2) (3, 3) (4, 4)}
R_2 = {(1,1)(1, 2) (2, 2) (2,1) (3, 3) (4, 4)},
R_3 = {(1, 3) (1, 4) (2, 3) (2, 4) (3, 1) (3,4,)}. ☞ **[SPPU : Dec.-10]**

Ans. : i) For R_1 :

As $\forall a \in A$, $(a,a) \in R_1$

$\therefore R_1$ is Reflexive, symmetric relation.

$\exists$ (a, b) and $(b,c) \in R_1$ $\therefore R_1$ is transitive relation.

$\exists$ (a, b) and (b, a) $\in R_1$ $\therefore R_1$ is antisymmetric relation.

$\therefore R_1$ reflexive, symmetric, transitive and anti-symmetric relation.

ii) For R_2 :

As $\forall a \in A$ (a, a) $\in R_2$ and $aR_2b \Rightarrow bR_1a$, for a, b $\in$ A

$\therefore R_2$ reflexive and symmetric relation.

For any aR_2b and $bR_2C \Rightarrow aR_2C$

$\therefore R_2$ is transitive realtion

$\therefore R_2$ is an equivalence relation

But (1, 2) and (2, 1) $\in R_2$ and $1 \neq 2$

$\therefore R_2$ is not antisymmetric relation.

iii) For R_3 :

As $2 \in R_3$ but (2, 2) $\notin R_3$

$\therefore R_3$ is not reflexive relation.

As (1, 4) $\in R_3$ but (4, 1) $\notin R_3$

$\therefore R_3$ is not symmetric relation.

As (2, 3) and (3, 1) $\in R_3$ but (2, 1) $\notin R_3$.

$\therefore R_3$ is not transitive relation.

As (1, 3) and (3, 1) $\notin R_3$ but $1 \neq 3$.

$\therefore R_3$ is not transitive relation

As (1, 3) and (3, 1) $\in R_3$ but $1 \neq 3$

$\therefore R_3$ is not antisymmetric relation.

Q.14 Consider the relation on A = { 1, 2, 3, 4, 5, 6} . R = { (i, j) | |i – j| = 2}. Is R reflexive ? Is R symmetric ? Is R transitive ?

[SPPU : Dec.-14]

Ans. : Given that A = {1, 2, 3, 4, 5, 6}

$$R = \{ (1, 3)\ (3, 1)\ (2, 4)\ (4, 2)\ (3, 5)\ (5, 3)\ (4, 6)\ (6, 4)\}$$

As $2 \in A$ but (2, 2) $\notin R$ $\therefore$ R is not reflexive.

For any (a, b) $\in R \Rightarrow$ (b, a) $\in R$ $\therefore$ R is symmetric.

As (1, 3) (3, 1) $\in R$ but (1, 1) $\notin R$ $\therefore$ R is not transitive.

Q.15 Let A = { 1, 2, 3, 4, 5, 6, 7 } and R = {(x, y) | x – y is divisible by 3}. Show that R is an equivalence relation ? Draw graph of R.

[SPPU : May-14, Dec.-12]

Ans. : We have R = {x, y) | x – y is divisible by 3}

We know that x – x = 0 is divisible by 3

x R x, $\forall x \in A \Rightarrow$ R is reflexive relation.

As x R y $\Rightarrow$ x – y is divisible by 3

$\Rightarrow$ y – x is also divisible by 3

$\Rightarrow$ y – z is divisible by 3

$\Rightarrow$ y R x for x, y $\in$ A

$\therefore$ R is a symmetric relation.

As,

x R y and y R $\Rightarrow$ x – y and y – z are divisible by 3

$\Rightarrow$ (x – y) + (y – z) is also divisible by 3

$\Rightarrow$ x – z is divisible by 3

$\Rightarrow$ x R z

$\therefore$ R is a transitive relation.

$\therefore$ R is an equivalence relation.

and R = {(1, 1), (2, 2) (3, 3) (4, 4) (5, 5) (6, 6) (7, 7) (1, 4) (4, 1) (1, 7) (7, 1) (2, 5), (5, 2) (3, 6) (6, 3)}

Its graph is as follows

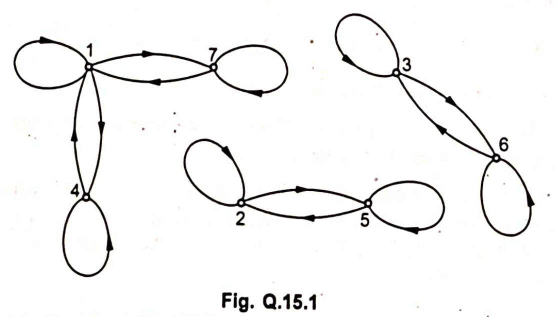


Fig. Q.15.1

Q.16 Let $A = \{x, y, z, u, v\}$, $\pi = \{\{x, y\}, \{z\}, \{u, v\}\}$. Find the equivalence relation induced by π.

Ans. :

We have $A = \{x, y, z, u, v\}$ and Define

x Ry iff x and y belongs to the same block of the partition of A.

$$\pi = \{\{x, y\}, \{z\}, \{u, v\}\} \text{ has 3 blocks.}$$

The first block $\{x, y\} \rightarrow x, y \in$ same block

$\therefore$ x Rx, xRy, yRx, yRy

The second block, $\{z\} \Rightarrow$ zRz

The third block, $\{u, v\} \Rightarrow$ uRu, uRv, vRu, vRv

$\therefore$ The required relation is

$$R = \{(x,x)\ (x,y),\ (y,x),\ (y,y),\ (z,z),\ (u,u),\ (u,v),\ (v,u),\ (v,v)\}$$

The relation R is reflexive, symmetric and transitive

$\therefore$ R is an equivalence relation.

Q.17 Define partition of a set. Let $x = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Determine whether or not each of the following is a partition of x :
$A = \{\{2, 4, 5, 8\}, \{1, 9\}, \{3, 6, 7\}\}$, $B = \{\{1, 3, 6\}, \{2, 8\}, \{5, 7, 9\}\}$

[SPPU : Dec.-11]

Ans. :

Definition :

Let A be any non empty set. A set

$P = \{A_1, A_2, A_3, ... A_n\}$ of non empty subsets of A is called a partition of set A if

i) $A_1 \cup A_2 \cup A_3 \cup .... \cup A_n = A = \bigcup_{i=1}^{n} A_i$

i.e. Set A is the union of the sets $A_1, A_2, ... A_n$.

ii) $A_i \cap A_j = \phi$ for $i \neq j$ i.e. All sets A_i are mutually disjoints.

The partition of a set A is denoted by π.

An element of a partition set is called a block. The rank of a partition is called as the number of blocks of that partitions. It is denoted by $r(\pi)$. For any non empty set, its partitions are not unique. There are different partitions of the same set.

e.g.

1) Let A = Z = Set of integers

A_1 = Set of even integers

A_2 = Set of odd integers

$A_3 = \{1, 2, 3, 4, ...\}$,

$A_4 = \{0\}$

$A_5 = \{-1, -2, -3, -4, ...\}$

$P_1 = \{A_1, A_2\}$ and $P_2 = \{A_3, A_4, A_5\}$

are different partitions of set A.

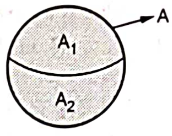


Fig. Q.17.1

2) If $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and it's subsets

$A_1 = \{1, 4, 9\}$ $A_2 = \{2, 6, 8, 10\}$, $A_3 = \{3, 5, 7\}$,

The set $P = \{A_1, A_2, A_3\}$ is such that

i) A_1, A_2, A_3 are non empty sets

ii) $A = A_1 \cup A_2 \cup A_3$

iii) $A_1 \cap A_2 = \phi$ $A_1 \cap A_3 = \phi$ $A_2 \cap A_3 = \phi$

Hence $\{A_1, A_2, A_3\}$ form a partition for set A.

i) For set A :

The set A has 3 blocks,

$A_1 = \{2, 4, 5, 8\}$, $A_2 = \{1, 9\}$ $A_3 = \{3, 6, 7\}$

- The union of all these blocks is a set X

$A_1 \cup A_2 \cup A_3 = X$

- These blocks are mutually disjoints.

$\therefore$ The set A forms a partition for the set X

ii) For set B :

$\therefore$ The set B has 3 blocks.

$B_1 = \{1, 3, 6\}$, $B_2 = \{2, 8\}$ $B_3 = \{5, 7, 9\}$ As $B_1 \cup B_2 \cup B_3 \neq X$, B is not a partition of set X.

Q.18 If S = {1, 2, 3, 4, 5, 6, 7, 8, 9}. Determine whether or not each of the following is a partition of S.

i) A = {{1, 3, 5}, {2, 6}, {4, 8, 9}}

ii) B = {{1, 3, 5}, {2, 4, 6, 8}, {7, 9}}

iii) C = {{1, 3, 5}}, {2, 4, 6, 8}, {5, 7, 9}}

iv) D = {{5}}

☞ **[SPPU : Dec.-12]**

Ans. : Given that

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

i) A is not a partition of S because S is not the union of all blocks of A.

i.e. $S \neq A_1 \cup A_2 \cup A_3$

ii) B is the partition of S as $B_1 \cup B_2 \cup B_3 = S$ and B_1, B_2, B_3 are mutually disjoints.

iii) As blocks of set C are not disjoints. ∴ The set C is not a partition of S.

iv) D = {{s}} is a partition of S, called as trivial partition.

Q.19 Find the symmetric closure of the following relations.

On A = {1, 2, 3}. R_1 = {(1, 1) (2, 1)},

R_2 = {(1, 2) (2, 1) (3, 2) (2, 2)},

R_3 = {(1, 1) (2, 2) (3, 3)}.

☞ **[SPPU : Dec.-12]**

Ans. : Given that

We have $A = \{1, 2, 3\}$

i) $R_1^{-1} = \{(1, 1) (1, 2)\}$

∴ $R = R_1 \cup R_1^{-1} = \{(1, 1) (1, 2) (2, 1)\}$ is the symmetric closure of R_1

ii) $R_2^{-1} = \{(2, 1) (1, 2) (2, 3) (2, 2)\}$

∴ $R = R_2 \cup R_2^{-1} = \{(1, 2) (2, 1) (3, 2) (2, 3) (2, 2)\}$ is the symmetric closure of R_2

iii) R_3 is the symmetric relation.

∴ R_3 itself is the symmetric closure.

2.2 : Warshall's Algorithm to Find Transitive Closure

Important Points to Remember

To find the transitive closure of a relation by computing various powers of R or product of the relation matrix is quite impractical for large relations. Warshall's algorithm gives an alternate method for finding transitive closure of R. Warshall's algorithm is practical and efficient method.

Consider the following steps to find transitive closure of the relation R on a set A.

Step 1 : We have |A| = n

∴ We require $W_o, W_1, W_2, \ldots W_n$. Warshall sets

$$W_o = \text{Relation Matrix of } R = M_R.$$

Step 2 : To find the transitive closure of relation R on set A, with |A| = n

Procedure to compute W_k from W_{k-1} is as follows

i) Copy 1 to all entries in W_k from W_{k-1}, where there is a 1 in W_{k-1}.

ii) Find the row numbers $p_1, p_2, p_3 \ldots$ for which there is 1 in column k in W_{k-1} and the column numbers $q_1, q_2, q_3 \ldots$ for which there is 1 in row k of W_{k-1}.

iii) Mark entries in W_k as 1 for (p_i, q_i). If there are not already 1.

Step 3 : Stop the procedure when W_n is obtained and it is the required transitive closure of R.

Q.20 Find the transitive closure of R by Warshall's algorithm. Where A = {1, 2, 3, 4, 5, 6} and R = {(x, y)/(x – y) = 2}

☞ **[SPPU : Dec.-05, 12, 13, 14]**

Ans. :

Step 1 : We have |A| = 6,

$$R = \{(1, 3), (3, 1), (2, 4), (4, 2), (4, 6), (6, 4), (3, 5), (5, 3)\}$$

Thus we have to find Warshall's sets,

$W_o, W_1, W_2, W_3, W_4, W_5$ and W_6.

The first set W_o is same as M_R. Which is shown below

$$W_0 = M_R = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 0 & 0 & 1 \\ 5 & 0 & 0 & 1 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}$$

Step 2 : To find W_1 :

To find W_1 from W_o, we consider the first column and first row.

In a column C_1, 1 is present at R_3

In a row R_1, 1 is present at C_3

Thus add new entry in W_1, at (R_3, C_3) which is given below

$$W_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 3 : To find W_2 :

To find W_2 from W_1, we consider the second column C_2 and second row R_2.

In C_2, 1 is present at R_4

In R_2, 1 is present at C_4

Thus add new entry in W_2 at (R_4, C_4), which is given below

$$W_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 4 : To find W_3 :

To find W_3 from W_2, we consider the third column and third row.

In C_3, 1 is present at R_1, R_3, R_5

In R_3, 1 is present at C_1, C_3, C_5

Thus add new entries in W_3 at (R_1, C_1), (R_1, C_3), (R_1, C_5) (R_2, C_1), (R_2, C_3), (R_2, C_5) (R_3, C_1), (R_3, C_3), (R_3, C_5) which is given below

$$W_3 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 5 : To find W_4 :

To find W_4 from W_3, we consider the fourth column and fourth row.

In C_4, 1 is present at R_2, R_4, R_6

In R_4, 1 is present at C_2, C_4, C_6

Thus add new entries in W_4 at (R_2, C_2), (R_2, C_4), (R_2, C_6), (R_4, C_2), (R_4, C_4), (R_4, C_6), (R_6, C_2), (R_6, C_4), (R_6, C_6) which is given below

$$W_4 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Step 6 : To find W_5 :

To find W_5 from W_4, we consider the 5^{th} column and 5^{th} row.

In C_5, 1 is present at R_1, R_3, R_5

In R_5, 1 is present at C_1, C_3, C_5

Thus add new entries in W_5 at (R_1, C_1), (R_1, C_3), (R_1, C_5), (R_3, C_1), (R_3, C_3), (R_3, C_5), (R_5, C_1), (R_5, C_3), (R_5, C_5) which is given below

$$W_5 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} = W_4$$

Step 7 : To find W_6 :

To find W_6 from W_5 we consider the 6^{th} column and 6^{th} row.

In C_6, 1 is present at R_2, R_4, R_6

In R_6, 1 is present at C_2, C_4, C_6

Thus add new entries in W_6 at (R_2, C_2), (R_2, C_4), (R_2, C_6), (R_4, C_2), (R_4, C_4), (R_4, C_6), (R_6, C_2), (R_6, C_4), (R_6, C_6) which is given below

$$W_6 = W_5 = W_4$$

Hence W_6 is the relation matrix of R^*

$$\therefore \quad R^* = \left\{ \begin{matrix} (1, 1), (1, 3)\ (1, 5)\ (2, 2)\ (2, 4)\ (2, 6)\ (3, 1)\ (3, 3)\ (3, 5) \\ (4, 2)\ (4, 4)\ (4, 6)\ (5, 1)\ (5, 3)\ (5, 5)\ (6, 2)\ (6, 4)\ (6, 6) \end{matrix} \right\}$$

Q.21 Let R = {(a, d), (b, a), (b, d), (c, b), (c, d), (d, c)}
Use Warshall's algorithm to find the matrix of transitive closure where A = {a, b, c, d} ☞ **[SPPU : Dec.-15]**

Ans. : Step 1 : We have

$$A = \{a, b, c, d\} \qquad \therefore \quad |A| = 4$$

$$R = \{(a, d)\ (b, a), (b, d)\ (c, b), (c, d), (d, c)\}$$

Thus we have to find Warshall's sets W_o, W_1, W_2, W_3, W_4

The first set $\quad W_o = M_R$

$$\therefore W_0 = M_R = \begin{array}{c c} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Step 2 : To find W_1 :

To find W_1 from W_o, we consider the first column and first row.

In C_1, 1 is present at R_2

In R_1, 1 is present at C_4

Thus add new entry in W_1 at (R_2, C_4)

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Step 3 : To find W_2

To find W_2, from W_1, we consider the second column and second row.

In C_2, 1 is present at R_3

In R_2, 1 is present at C_1 and C_4

Thus add new entries in W_2 at (R_3, C_1), (R_3, C_4) which is given below.

$$\therefore W_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Step 4 : To find W_3 :

To find W_3 from W_2, we consider the 3^{rd} column and 3^{rd} row.

In C_3, 1 is present at R_4

In R_3, 1 is present at C_1, C_2, C_4

Thus add news entries in W_3 at (R_4, C_1), (R_4, C_2), (R_4, C_4)

Which is given below

$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Step 5 : To find W_4 :

To find W_4 from W_3, we consider the 4^{th} column and 4^{th} row.

In C_4, 1 is present at R_1, R_2, R_3, R_4

In R_4, 1 is present at C_1, C_2, C_3, C_4

Thus we add new entries in W_4 at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_1, C_4), (R_2, C_1), (R_2, C_2), (R_2, C_3), (R_2, C_4), (R_4, C_3), (R_3, C_1), (R_3, C_2), (R_3, C_4), (R_4, C_1), (R_4, C_2), (R_4, C_3), (R_4, C_4)

Which is given below

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Hence W_4 is the relation matrix of R^*.

$$\text{and } R^* = \left\{ \begin{matrix} (a, a), & (a, b) & (a, c) & (a, d) & (b, a) & (b, b) & (b, c) & (b, d) \\ (c, a), & (c, b) & (c, c) & (c, d) & (d, a) & (d, b) & (d, c) & (d, d) \end{matrix} \right\}$$

Q.22 Find the transitive closure of the relation R on $A = \{1, 2, 3, 4\}$ defined by $R = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (3, 4), (3, 2), (4, 2), (4, 3)\}$

☞ [SPPU : Dec.-07, May-15]

Ans. : Step 1 :

We have $|A| = 4$, Thus we have to find

Warshall's sets, W_0, W_1, W_2, W_3, W_4

The first set $W_0 = M_R$

$$\therefore W_0 = M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Step 2 : To find W_1 :

To find W_1 from W_0, we consider the first column and first row.

In C_1, 1 is present at R_2

In R_1, 1 is present at C_2, C_3, C_4

Thus add new entries in W_1 at (R_2, C_2), (R_2, C_3), (R_2, C_4) which is given below

$$W_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Step 3 : To find W_2 :

To find W_2 from W_1, we consider the 2^{nd} column and 2^{nd} row.

In C_2, 1 is present at R_1, R_2, R_3, R_4

In R_2, 1 is present at C_1, C_2, C_3, C_4

Thus we add new entries in W_2 at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_1, C_4), (R_2, C_1), (R_2, C_2), (R_2, C_3), (R_2, C_4), (R_3, C_1), (R_3, C_2), (R_3, C_2), (R_3, C_4), (R_4, C_1), (R_4, C_2), (R_4, C_3), (R_4, C_4)

Which is given below

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

All entries in W_2 are 1

Hence W_2 is the relation matrix of transitive closure of R.

$$\text{and } R^* = \left\{ \begin{matrix} (1, 1), & (1, 2) & (1, 3) & (1, 4) & (2, 1) & (2, 2) & (2, 3) & (2, 4) \\ (3, 1), & (3, 2) & (3, 3) & (3, 4) & (4, 1) & (4, 2) & (4, 3) & (4, 4) \end{matrix} \right\}$$

Q.23 Find the transitive closure of R by Warshall's algorithm. A = {Set of positive integers $\leq$ 10}. R = {(a, b) / a divides b}

☞ [SPPU : May-07]

Ans. :

Step 1 : We have $|A| = 10$. Thus we have to find

Warshall's sets $W_0, W_1, W_2, \ldots W_{10}$

$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8),$
$(1, 9), (1, 10),$
$(2, 2), (2, 4), (2, 6), (2, 8), (2, 10), (3, 3), (3, 6), (3, 9),$
$(4, 4), (4, 8),$
$(5, 5), (5, 10), (6, 6), (7, 7), (8, 8), (9, 9), (10, 10)\}$

The first set $W_0 = M_R$

$$\therefore \quad W_0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The relation R itself is a transitive relation on the set of positive integers. Hence $R = R^*$ and

$W_0 = M_R$ is the relation matrix of R^*

Q.24 Use Warshall's Algorithm to find transitive closure of R where

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } A = \{1, 2, 3\}$$

☞ [SPPU : May-06, May-08]

Ans. :

Step 1 : We have $|A| = 3$. Thus we have to find Warshall's sets W_0, W_1, W_2, W_3

The first set is

$$W_0 = M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Step 2 : To find W_1 :

To find W_1 from W_0, we consider the first column and the first row.

In C_1, 1 is present at R_1, R_3

In R_1, 1 is present at C_1 and C_3

Thus add new entries in W_1 at (R_1, C_1), (R_1, C_3), (R_3, C_1), (R_3, C_3)

Which is given below

$$W_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Step 3 : To find W_2 :

To find W_2 from W_1, we consider the 2^{nd} column and 2^{nd} row.

In C_2, 1 is present at R_2, R_3

In R_2, 1 is present at C_2

Thus add new entries in W_2 at (R_2, C_2), (R_3, C_2) which is given below

$$W_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = W_1$$

Step 4 : To find W_3

To find W_3 from W_2, we consider the 3^{rd} column and 3^{rd} row.

In C_3, 1 is present at R_1, R_3

In R_3, 1 is present at C_1, C_2, C_3

2.3 : Partially Ordered Set

Important Points to Remember

A relation R on a set A is called a partially ordered relation iff R is reflexive, anti-symmetric and transitive relation.

The set A together with partially ordered relation is called a partially ordered set or POSET.

It is denoted by (A, R) or (A, ⪯) where ≤ is a partially ordered relation.

Examples :

1) ($\Re$, ⪯) (N, ⪯) are Posets.

where '≤' is reflexive, antisymmetric and transitive relation.

2) If A = P(S) where S = (a, b, c) and for X, Y ∈ A, Define $X \leq Y$ or XRY iff $X \subseteq Y$.

As $X \leq X \Rightarrow X \leq X$. ∴' ≤ ' is reflexive.

If $X \leq Y$, $Y \leq Z \Rightarrow X \subseteq Y$ and $Y \subseteq X \Rightarrow X = Y$

∴'≤' is antisymmetric relation.

If $X \leq Y, Y \leq Z \Rightarrow X \subseteq Y$ and $Y \subseteq Z \Rightarrow X \subseteq Z \Rightarrow X \leq Z$

⇒ '≤' is transitive relation.

∴ (P(S), ⊆) or (P(S), ⪯) is a poset.

I) Comparable elements : Let (A, ⪯) be a poset. Two elements a, b in A are said to be comparable elements if a ≤ b or b ≤ a. Two elements a and b of a set A are said to be non-comparable if neither a ≤ b nor b ≤ a.

In above example (2),

The comparable elements are

{a} = {a, b}, {b} = {a, b, c} {b,c}⊆ {a,b,c}

Non comparable elements are

{a} ⊄ {b} {a} ⊄ {a,c}

II) Totally ordered set : Let A be any nonempty set. The set A is called linearly ordered set or totally ordered set if every pair of elements in A are comparable.

i.e. for any a, b ∈ A either a ≤ b or b ≤ a.

2.4 : Hasse Diagram

Important Points to Remember

It is useful tool, which completely describes the associated partially ordered relation. It is also known as ordering diagram.

A diagram of graph which is drawn by considering comparable and non-comparable elements is called Hasse diagram of that relation. Therefore while drawing Hasse diagram following points must be followed.

1) The elements of a relation R are called vertices and denoted by points.
2) All loops are omitted as relation is reflexive on poset.
3) If aRb or a ≤ b then join a to b by a straight line called an edge the vertex b appears above the level of vertex a. Therefore the arrows may be omitted from the edges in Hasse diagram.
4) If a ≰ b and b ≰ a i.e. a and b are non - comparable elements, then they lie on same level and there is no edge between a and b.
5) If a ≤ b and b ≤ c then a ≤ c. So there is a path a → b → c. Therefore do not join a to c directly i.e. delete all edges that are implied by transitive relation.

Q.25 Draw Hasse diagram of a poset (P(s), ⊆) where S = {a, b, c}.

Ans. : P (s) = { ϕ, {a}, {b}, {c}, {a, b} {a, c} {b, c} {a, b, c}}

Now find the comparable and non comparable elements.

ϕ ⊆ {a}, ϕ ⊆ {b}, ϕ ⊆ {c}, ∴ {a}, {b}, {c} lie above the level of ϕ

{a}⊆ {a,b},{b} ⊆ {a,b},{c}⊆ {a,c} ∴ {a, b}, {b, c} {a,c} lies above the level of {a}, {b}, {c}.

{a, b} ⊆ S, {b, c} ⊆ S, {a, c} ⊆ S ∴S lies above the level of {a, b}, {a, c}, {b, c}

But {a}, {b} {c} are non comparable ∴ {a}, {b}, {c} lie on same level.

{a, b}, {a, c}, {b, c} are non comparable ∴ lie on same level.

By considering the above observations, the Hasse diagram is as follows :

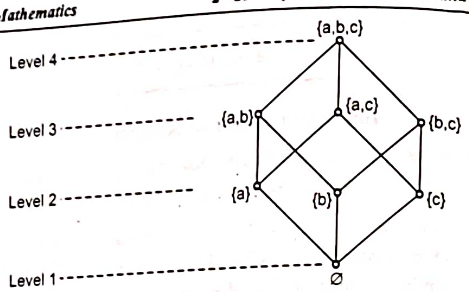


Fig. Q.25.1

2.5 : Chains, Antichains and Elements of Poset

Important Points to Remember

Let (A, ≤) be a poset. A subset of A is called a chain if every pair of elements in the subset are related.

A subset of A is called antichain if no two distinct elements in a subset are related. e.g. In above (Q.25.1)

1) The chains are
$\{\phi, \{a\}, \{a,b\}, \{a,b,c\}\}$, $\{\{a\}, \{a,c\}, \{a,b,c\}\}$, $\{\{b,c\}, \{a,b,c\}\}$

2) Antichains are $\{\{a\}, \{b\}, \{c\}\}$

> **Note :** 1) The number of elements in the chain is called the length of chain.
> 2) If the length of chain is n in a poset (A, ≤) then the elements in A can be partitioned into n disjoint antichains.

3) Let (A,≤) be a poset. An element a ∈ A is called a **maximal element** of A if there is no element c∈ A such that a ≤c.

4) An element b ∈ A is called a **minimal element** of A if there is no element c ∈ A such that c ≤ b

5) Greatest element : An element x∈ A is called a **greatest element** of A if for all a ∈ A, a ∈ x. It is denoted by 1 and is called the unit element.

6) Least element : An element y ∈ A is called a **least element** of A if for all a ∈ A, y ≤a.

It is denoted by 0 and is called as **zero element.**

7) Least upper bound (*l*ub) : Let (A,≤) be a poset. For a, b, c ∈ A, an element C is called **upper bound** of a and b if a ≤ c and b ≤ c C is called as least upper bound of a and b in A if C is an upper bound a and b there is no upper bound d of a and b such that d ≤ c. It is also known as **supremum.**

8) Greatest lower bound (glb) : Let (A,≤) be a poset. for a, b, *l* ∈ A, an element *l* is called the **lower bound** of a and b if *l* ≤ a and *l* ≤ b.

An element *l* called the **greatest lower bound** of a and b if *l* is the lower bound of a and b and there is no lower bound f of a and b such that

• *l* ≤ f.

glb is also called as **infimum.**

Q.26 Determine the greatest and least elements of the poset whose Hasse diagrams are shown below.

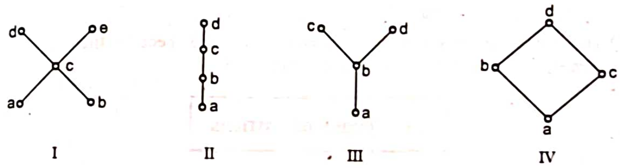


Fig. Q.26.1

Ans. : The Poset shown in Fig. Q.26.1 (I) has neither greatest not least element.

The Poset shown in Fig. Q.26.1 (II), has greatest and a as least element.

The Poset shown in Fig. Q.26.1 (III), has no greatest element but a is the least element.

The Poset shown in Fig. Q.26.1 (IV), has greatest and a as least element.

Q.27 Find glb, lwb, ub, lb, maximal, minimal, of the poset (A, R), Here aRb if a / b where. A = { 2, 3, 5, 6, 10, 15, 30, 45}

Ans. : We have A = {2, 3, 5, 6, 10, 15, 30, 45} and aRb iff a|b.

Hasse diagram is as follows :

1) Here 10 and 30 are upper bounds of 2 and 5, But 10 is the least upper bound of 2 and 5.

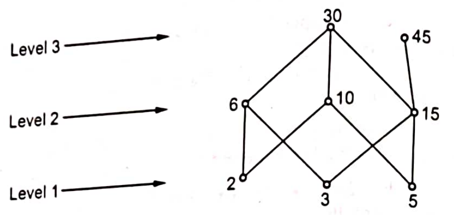


Fig. Q.27.1

2) 5, 15, 3 are lower bounds of 30 and 45. But 15 is the greatest lower bound of 30 to 45.

3) This Poset has neither greatest element nor least element.

4) This poset has two maximal elements 45 and 30 as there is no element c such that $45 \leq C$ and $30 \leq C$.

5) This poset has three minimal elements 2, 3 and 5. because there is no element $x \in A$ such that $x \leq 2$, $x \leq 3$ and $x \leq 5$

2.6 : Types of Lattices

Important Points to Remember

A lattice is a poset in which every pair of elements has a least upper bound (lub) and a greatest lower (glb).

Let $(A, \leq)$ be a poset and $a, b \in A$ then lub of a and b is denoted by $a \vee b$. It is called the join of a and b.

i.e. $a \vee b$ = lub (a, b)

The greatest lower bound of a and b is called the meet of a and b and it is denoted by $a \wedge b$

$\therefore \quad a \wedge b = \text{glb}(a, b)$

From the above discussion, it follows that a lattic is a mathematical structure with two binary operations $\vee$ (join) and $\wedge$ (meet). It is denoted by $\{L, \vee, \wedge\}$.

Properties of a Lattice

Let $(L, \wedge, \vee)$ be a lattice and $a, b, c \notin L$. Then L satisfies the following properties.

1) Commutative property

$a \wedge b = b \wedge a$ and $a \vee b = b \vee a$

2) Associative law

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

3) Absorption law

$a \wedge (a \vee b) = a$ and $a\vee (a \wedge b) = a$

$a \wedge a = a$, $a \vee a = a$

$a \wedge b = a$ iff $a \vee b = b$

Types of Lattices

I) Bounded lattice : A lattice L is called a bounded lattice if it has a greatest element 1 and least element 0.

II) Sublattice : Let, $(L, \vee, \wedge)$ be a lattice. A non empty subset L_1 of L is called a sublattice of L if L_1 itself is a lattice w.r.t. the operations of L.

III) Distributive lattice : A lattice $(L, \vee, \wedge)$ is called a distributive lattice if for any elements $a, b, c \in L$, it satisfies the following properties,

i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

If the lattice does not satisfy the above properties then it is called a non distributive lattice.

Theorem : Let, $(L, \wedge, \vee)$ be a lattice with universal bounds 0 and 1 then for any $a \in L$, $a \vee 1 = 1$, $a \wedge 1 = a$, $0 \vee a = a$, $0 \wedge a = 0$

IV) Complement lattice : Let $(L, \wedge, \vee)$ be a lattice with universal bounds 0 and 1 for any $a \in L$, $b \in L$ is said to be complement of a if $a \vee b = 1$ and $a \wedge b = 0$.

A Lattice in which every element has a complement in that lattice, is called the complemented lattice.

e.g. 1) The Hasse diagram is here $0 \approx 1$ and $1 \approx 30$.

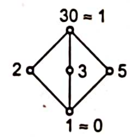


Fig. 2.1

i) $2 \wedge 3 = 0$ and $2 \vee 3 = 1$, $2 \wedge 5 = 0$ and $2 \vee 5 = 1$

$\therefore$ 2 has two compliments 3 and 5

Hence the complement is not unique.

Principle of Duality

Any statement about lattice involving $\wedge, \vee, \leq, \geq$ remains true if '$\wedge$' is replaced by '$\vee$', $\vee$ by $\wedge$', '$\leq$ by $\geq$', '$\geq$ by $\leq$', '0 by 1' and '1 by 0'.

e.g. 1) $a \vee (b \wedge c) = a \wedge (b \vee c)$

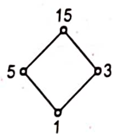
2) $a \wedge (b \vee 1) = a \vee (b \wedge 0)$

Q.28 Let A be the set of positive factors of 15 and R be a relation on A s.t. R = {xRy| x divides y, x y ∈ A}. Draw Hasse diagram and give and $\wedge$ and $\vee$ for lattice. **[SPPU : Dec.-06]**

Ans. : We have A = {1, 3, 5, 15}

R = {(1,1) (1,3) (1,5) (1, 15) (3,15) (5,15) (15,15)}

Hasse diagram of R is : Table for $\wedge$ and $\vee$



$\vee$	1	3	5	15
1	1	3	5	15
3	3	3	15	15
5	5	15	5	15
15	15	15	15	15

$\wedge$	1	3	5	15
1	1	1	1	1
3	1	3	1	3
5	1	1	5	5
15	1	3	5	15

Fig. Q.28.1

Every pair of elements has lub and glb. $\therefore$ It is a lattice.

Q.29 Let A = {1, 2, 3, 4, 6, 9, 12} Let a relation R on a set A is R = {(a,b)/ a divides b ∀ a, b ∈ A}. Give list of R. Prove that it is a partial ordering relation. Draw Hasse diagram of the same. Prove or disprove it is a lattice. **[SPPU : Dec.-11]**

Ans. : We have A = {1, 2, 3, 4, 6, 9, 12}

and R =

$$\left\{\begin{array}{l}(1,1),(1,2),(1,3),(1,4)(1,6),(1,9),(1,12),(2,2),(2,4),(2,6)(2,12)(3,3),\\(3,6),(3,9)\ (3,12),\ (4,4),\ (4,12),\ (6,6),(6,12),(9,9),(12,12)\end{array}\right\}.$$

We know that for any $a \in A$, $a \mid a$ $\therefore$ aRa

$\therefore$ R is a reflexive relation.

As $a \mid b$ and $b \mid a \Rightarrow a = b$ $\therefore$ R is antisymmetric relation.

As $a \mid b$ and $b \mid c \Rightarrow a \mid c \Rightarrow$ R is a transitive relation.

$\therefore$ R is reflexive antisymmetric and transitive

$\therefore$ (A, R) is a poset and R is a partial ordering relation.

Hasse diagram is as follows :

In above diagram $6 \vee 9$ does not exist. $\therefore$ It is not a lattice.

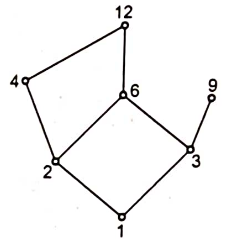
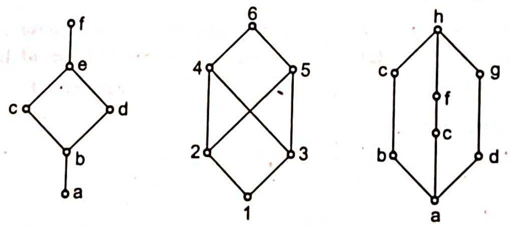


Fig. Q.29.1

Q.30 Determine whether the poset represented by each of the Hasse diagram are lattices. Justify your answer. **[SPPU : Dec.-10]**



I II III

Fig. Q.30.1

Ans. :

I) In Q.30.1 (I), every pair of element has glb and lub. $\therefore$ It is a lattice.

II) In Fig. Q.30.1 (II), every pair of elements has lub and glb. $\therefore$ It is a lattice.

III) In Fig. Q.30.1 (III), every pair of elements gas lub and glb. $\therefore$ It is a lattice.

Q.31 Show that the set of all divisors of 36 forms a lattice. **[SPPU : Dec.-14]**

Ans. : Let A = {1, 2, 3, 4, 6, 9,12, 18, 36 } and Let '≤' is a divisor of. It's Hasse diagram is as follows.

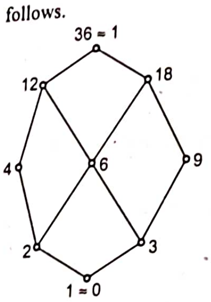


Fig. Q.31.1

The universal upper bound 1 is 36 and lower bound 0 is 1. Every pairs of elements of this poset has lub and glb.

∴ It is a lattice.

Q.32 Let n be a positive integer, S_n be the set of all divisors of n, Let D denote the relation of divisor. Draw the diagram of lattices for n = 24, 30, 6. ☞ **[SPPU : May-15]**

Ans. : Given that

i) We have S_6 = {1,2,3,6}, D is the relation of divisor.

ii) S_{24} = {1, 2, 3, 4, 6, 8, 12, 24}

iii) S_{30} = {1, 2, 3, 5, 6, 10, 15, 30}

Diagrams of Lattices are as follows.

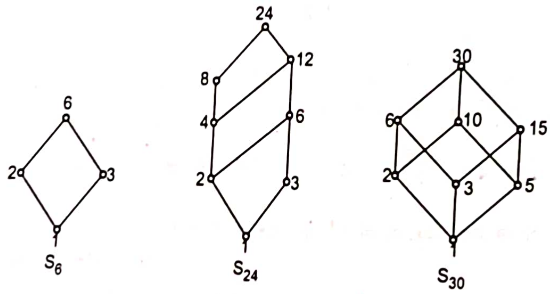


Fig. Q.32.1

Q.33 Show that the set of all divisors of 70 forms a lattice. ☞ **[SPPU : Dec.-13]**

Ans. :

Let A = {1, 2, 5, 7, 10, 14, 35, 70}

and Let '≤' is "a divisor of".

The universal upper bound 1 is 70 and the lower bound 0 is 1.

It's Hasse diagram is as follows :

Every pair of elements of A has ∧ and ∨.

∴ It is a lattice [write table of ∧ and ∨].

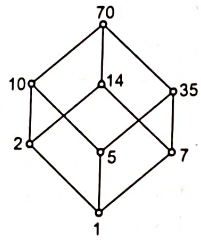


Fig. Q.33.1

Q.34 Let A = {1, 2, 3, 4 , 5, 6, 7, 8 ,9, 12, 18, 24} be ordered by the relation x divides y. Show that the relation is a partial ordering and draw Hasse diagram. ☞ **[SPPU : Dec-15]**

Ans. : We have A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 18, 24}

R = {(x,y) | x divides y, for x, y ∈ A}

R = {(1, 2) (1, 3), (1, 4), (1, 5) (1, 6) (1, 7) (1, 8) (1, 9) (1, 12) (1, 18) (1, 24)
(2, 2) (2, 4) (2, 6) (2, 8) (2, 12) (2, 18) (2, 24) (5, 5) (6, 6) (6, 12) (6, 18)
(6, 24), (7, 7) (8, 8) (8, 24) (9, 9) (9, 18) (12, 12) (12, 24) (18, 18) (4, 24)}

We have for any x ∈ A, x|x ⇒ R is a reflexive for x|y and y|x ⇒ x = 0 ⇒ R is antisymmetric. If x|y and y|z ⇒ x|z ∴ R is a transitive relation.

∴ R is a partial ordering relation, It's Hasse diagram is as follows.

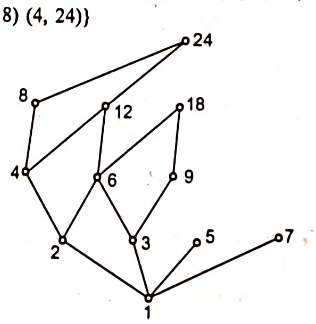


Fig. Q.34.1

Q.35 Let x = {2, 3, 6, 12, 24, 36} and x ≤ y iff x divides y find i) Maximal element ii) Minimal element iii) Chain iv) Antichain v) Is Poset lattice ☞ **[SPPU : May-14]**

Ans. : We have x = {2, 3, 6, 12, 24, 36}

The relation 'R' = '≤'

∴ R = {(2, 2) (2, 6) (2, 12) (2, 24) (2, 36) (3, 3) (3, 6) (3, 12) (3, 24) (3, 36), (6, 6) (6, 12) (6, 24) (6, 36) (12, 12) (12, 24) (12, 36) (24, 24) (36, 36)}.

It's Hasse diagram is as follows .

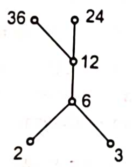


Fig. Q.35.1

i) Maximal elements are 24, 36

ii) Minimal elements are 2, 3

iii) Chain { 2, 6, 12, 24}, { 2, 6, 12, 36}, {3, 6, 12, 24}, {3, 6, 12, 36}

iv) Antichain : {2, 3} {24, 36}

v) The given poset is not a lattice as $2 \wedge 3$ does not exist.

2.7 : Functions

Important Points to Remember

1. **Function :** Let A and B be two non empty sets. A function from A to B, denoted as $f : A \rightarrow B$, is a relation from A to B such that for every element $a \in A$, there exists a unique element $b \in B$ such that (a, b) f or f (a) = b.

where A = Domain set,

B = Codomain set

Range set = $\{b \in B \mid \exists\, a \in A \text{ s.t. } f(a) = b\}$

= {Set of all image points}

2. **Equality of functions :** two functions f and g from A to B are said to be equal iff $f(a) = g(a)\ \forall\, a \in A$. we write as $f \equiv g$.

3. **Types of functions :**

(a) Identity function : Let A be any non empty set. A function f from $A \rightarrow A$ is said to be the Identity function iff f (a) = a, $\forall\, a \in A$.

(b) One to one function : (Injective function) : A function $f : A \rightarrow B$ is said to be one to one function if for all elements x, $y \in A$ such that $f(x) = f(y) \Rightarrow x = y$.

(c) Into function : A function $f : A \rightarrow B$ is said to be into function if range set of f is a proper subset of B i.e. ∃ at least one element in B which has no pre–image in A.

(d) Onto function (Surjective function) : A function $f : A \rightarrow B$ is said to be onto function if range set of f is equal to set B i.e. every element in B has pre–image in A.

(e) Bijective function : A function which is one to one (injective) as well as onto (surjective) is called bijective function.

(f) Inverse function : Let A, B any non–empty sets and $f : A \rightarrow B$ is a bijection function. The inverse function $f^{-1} : B \rightarrow A$ is a function defined as

$f^{-1}(b) = a$ iff $f(a) = b$ for $a \in A$ and $b \in B$

(g) Composite function : Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two function. The composition of f and g is denoted by gof and gof : $A \rightarrow C$ such that

$(gof)(x) = g[f(x)],\ \forall\, x \in A$.

Q.36 Explain classification of functions with example.

Ans. : Depending upon the nature of function, there are mainly two functions.

1. **Algebraic function :** A function which consists of a finite number of terms involving powers and roots of the independent variable and four fundamental operations addition, subtraction, multiplication and division, is called an algebraic function.

There are three types of algebraic functions.

a) Polynomial function : A function of the form

$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \; + a_n$, where n is positive integer, a_0 , a_1, a_2, , a_n are real numbers and a_0 0, is called polynomial function of x in degree n.

e.g. $f(x) = x^3 - 3x^2 + 2x - 5$

b) Rational function : A function of the form $\frac{f(x)}{g(x)}$ where f (x) and g (x) are polynomial functions and $g(x) \neq 0$ is called rational function e.g. $\frac{x^2-3x+5}{x^2+1}$

c) Irrational function : A function involving radicals is called irrational function. e.g. $f(x) = x^{2/3} + 5x^2 + 1, \sqrt[2]{x+1}$.

2. Transcendental function : A function which is not algebraic is called transcendental function.

e.g. $f(x) = \sin x + x^3 + 5x$

a) Trigonometric function : The six functions sin x, cos x, tan x, sec x, cosec x, cot x, where x is in radians are called trigonometric functions.

e.g. $f(x) = \sin x + \tan x$.

b) Inverse trigonometric function : The six functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\text{cosec}^{-1} x$, $\sec^{-1} x$ and $\cot^{-1} x$ are called inverse trigonometric functions.

e.g. $f(x) = \cos^{-1} x + 5 \tan^{-1} x$

c) Exponential function : A function of the form $f(x) = a^x$ $(a > 0)$ satisfying $a^x \cdot a^y = a^{x+y}$ and $a' = a$ is called exponential function. e.g. $f(x) = 5^x$.

d) Logarithmic function : The inverse function of the exponential function is called logarithmic function. e.g. $f(x) = \log x$.

Q.37 Let $f(x) = x+2$, $g(x) = x - 2$, $h(x) = 3x$, for $x \in \Re$ Where $\Re$ is the set of real numbers, Find i) gof ii) fog iii) fof iv) hog v) gog vi) foh vii) hof viii) fohog ix) gofoh. ☞[SPPU : May-08, 15, Dec.-12, Marks 4]

Ans. : Let $x \in \Re$ be any real number.

i) $gof(x) = g[f(x)] = g[x + 2] = x + 2 - 2 = x$

ii) $fog(x) = f[g(x)] = f[x - 2] = x - 2 + 2 = x$

iii) $fof(x) = f[f(x)] = f[x + 2] = x + 2 + 2 = x + 4$

iv) $hog(x) = h[g(x)] = h[x - 2] = 3(x - 2) = 3x - 6$

v) $gog(x) = g[g(x)] = g[x - 2] = x - 2 - 2 = x - 4$

vi) $foh(x) = f[h(x)] = f[3x] = 3x + 2$

vii) $hof(x) = h[f(x)] = h[x + 2] = 3(x + 2) = 3x + 6$

viii) $fohog(x) = f[h(g(x))] = f[h(x - 2)] = f[3(x - 2)] = f(3x - 6)$
$= 3x - 6 + 2 = 3x - 4$

ix) $gofoh(x) = g[f(h(x))] = g[f(3x)] = g[3x + 2] = 3x + 2 - 2 = 3x$

Q.38 Let functions f and g be defined by $f(x) = 2x + 1$, $g(x) = x^2 - 2$, Find a) gof(4) and fog (4), b) gof(a + 2) and fog(a + 2) c) fog(5) d) gof(a + 3)

☞[SPPU : May-07, Dec.-07, Marks 4]

Ans. :

a) $gof(4) = g[f(4)] = g[2(4) + 1] = g[9] = 9^2 - 2 = 79$

$fog(4) = f[g(4)] = f(4^2 - 2) = f(14) = 2(14) + 1 = 29$

b) $gof(a + 2) = g[f(a + 2)] = g[2(a + 2) + 1] = g[2a + 5]$
$= (2a+5)^2 - 2 = 4a^2 + 20a + 23$

$fog(a + 2) = f[g(a + 2)] = f[(a+2)^2 - 2] = f[a^2 + 4a + 2]$
$= 2[(a^2 + 4a + 2] + 1 = 2a^2 + 8a + 5$

c) $fog(5) = f[g(5)] = f[25 - 2] = f(23) = 2(23) + 1 = 47$

d) $gof(a + 3) = g[f(a + 3)] = g[2(a + 3) + 1] = g[2a + 7]$
$= [2a + 7]^2 - 2 = 4a^2 + 28a + 47$

Q.39 Give examples of functions of the following types by diagrams. a) Injective function but not surjective, b) Surjective but not injective, c) Neither injective nor surjective, d) Injective as well as surjective, e) Into function, f) Inverse function

Ans. :

a) Injective but not surjective

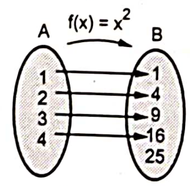


Fig. Q.39.1

b) Surjective but not injective

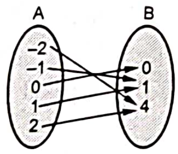


Fig. Q.39.2

c) Neither injective nor surjective

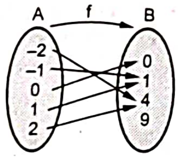


Fig. Q.39.3

d) Injective as well as surjective i.e. bijective

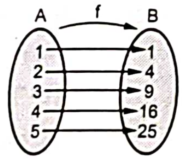


Fig. Q.39.4

e) Into function

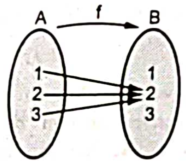


Fig. Q.39.5

f) Inverse function

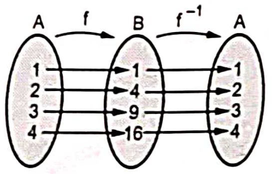


Fig. Q.39.6

Q.40 Determine if each is a function. If yes is it injective, surjective, bijective ?

a) Each person in the earth is assigned a number which corresponds to his age.

b) Each student is assigned a teacher.

c) Each country has assigned it's capital. ☞**[SPPU : Dec.-11, Marks 4]**

Ans. : a) Every person has unique age

$\therefore$ It is a function. Two person's may have same age. $\therefore$ It is not injective. There is no person whose age is 300 years. $\therefore$ It is not surjective. $\therefore$ Function is not bijective.

b) It is a function. It is not injective. It is not surjective. $\therefore$ It is not bijective.

c) It is a function. It is injective as well as surjective. $\therefore$ It is bijective.

Q.41 Let A = B be the set of real numbers.

$f : A \to B$ given by $f(x) = 2x^3 - 1$

$g : B \to A$ given by $g(y) = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}}$

Show that f is a bijective function and g is also bijective function :

☞**SPPU : Dec.-10, Marks 4]**

Ans. :

1) Suppose $f(x_1) = f(x_2)$

$$\Rightarrow \quad 2x_1^3 - 1 = 2x_2^3 - 1$$

$$\Rightarrow \quad x_1^3 = x_2^3$$

$$\Rightarrow \quad x_1 = x_2$$

$\therefore$ f is injective mapping

Let $y \in B$ and $f(x) = y \Rightarrow 2x^3 - 1 = y$

$$\therefore \quad 2x^3 = 1 + y \Rightarrow x^3 = \frac{1+y}{2}$$

$$\Rightarrow \quad x = \sqrt[3]{\frac{1+y}{2}} = \sqrt[3]{\frac{y}{2} + \frac{1}{2}} \in A = \Re \text{ for any } y \in B$$

$\therefore$ f is a surjective mapping.

Thus f is injective as well as surjective function.

Hence f is a bijective function.

We have $f(x) = y \Rightarrow x = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}} \Rightarrow f^{-1}(y) = x = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}} = g(y)$

Thus $f^{-1} = g$. We know that if f is bijective function then f^{-1} is also bijective. Hence g is bijective function.

Q.42 If $f : A \to B$ is bijective function then f^{-1} is unique.

Ans. : Let $f\ A \to B$ is a bijective function.

Claim : Show that f^{-1} is unique.

Suppose f^{-1} is not unique, so there are two inverse functions say, g and h.

Let $x_1, x_2 \in A, \exists$ y in B such that

$f^{-1}(y) = x_1 \Rightarrow g(y) = x_1 \Rightarrow f(x_1) = y$

and $f^{-1}(y) = x_1 \Rightarrow h(y) = x_2 \Rightarrow f(x_2) = y$

This implies $f(x_1) = f(x_2)$, but f is 1 – 1 function.

$\therefore \quad x_1 = x_2$

$\Rightarrow \quad g(y) = h(y)\ \forall\ y$

$\Rightarrow \quad g \equiv h$ i.e. g and h are equal function.

Hence universe of f is unique.

Q.43 If $f : A \to B$ and $g : B \to C$ are bijective functions the gof is also bijective.

Ans. : Let $f : A \to B$ and $g : B \to C$ be two bijective functions then gof $A \to C$ is a function.

(i) Let $x_1, x_2 \in A$ and suppose $gof(x_1) = gof(x_2)$

$g[f(x_1)] = g[f(x_2)]$

$\Rightarrow \quad f(x_1) = f(x_2)$ ... ($\because$ g is 1 – 1 function)

$\Rightarrow \quad x_1 = x_2$ ... ($\because$ f is 1 – 1 function)

$\Rightarrow$ gof is 1 – 1 function.

(ii) Let $z \in C$.

$\therefore\ \exists$ y in B such that g (y) = z and $\exists$ x in A such that f(x) = y.

$\therefore\ g(y) = z \quad \Rightarrow \quad z = g(f(x)) = gof(x)$ where $x \in A$

$\Rightarrow$ gof is onto function.

Hence gof is bijective function.

Q.44 Determine whether the function is bijective.

$f : I \to I$ such that $f(i) = \frac{i}{2}$ if i is even

$= \frac{(i-1)}{2}$ if i is odd

Ans. : Let a and b be any integers such that $a \neq b$.

There are three possibilities

Case 1 : If a and b are odd

$\therefore \quad f(a) = \frac{a-1}{2}, \quad f(b) = \frac{b-1}{2}$

$f(a) = f(b) \Rightarrow \frac{a-1}{2} = \frac{b-1}{2}$

$\Rightarrow a = b$

Case 2 : If a and b are even

then $f(a) = \frac{a}{2}, \quad f(b) = \frac{a}{2}$

$f(a) = f(b) \Rightarrow \frac{a}{2} = \frac{b}{2} \Rightarrow a = b$

Case 3 : If a is odd and b is even

then $f(a) = \frac{a-1}{2}$ and $f(b) = \frac{b}{2}$

Now, $f(a) = f(b) \Rightarrow \frac{a-1}{2} = \frac{b}{2} \Rightarrow \frac{a}{2} - \frac{b}{2} = \frac{1}{2}$

$\Rightarrow \quad a - b = 1 \quad \therefore \quad a \neq b$

In particular a = 7, b = 6

$f(7) = \frac{7-1}{2} = 3 \quad f(6) = \frac{6}{2} = 3$

$f(7) = f(6)$ but $6 \neq 7$

Thus f is not one one function. Hence f is not bijective function.

Q.45 Let $f(x) = ax^2 + b$ and $g(x) = cx^2 + d$, where a, b, c, d are constants. Determine for which values of constants gof (x) = fog (x).

Ans. : Given that, $f(x) = ax^2 + b$ and $g(x) = cx^2 + d$

Suppose $\quad fog(x) = gof(x)$

$f[g(x)] = g[f(x)]$

$$f[cx^2 + d] = g[ax^2 + b]$$

$\Rightarrow$ $a[cx^2 + d]^2 + b = c[ax^2 + b]^2 + d$

$\Rightarrow$ $a[c^2x^4 + 2cdx^2 + d^2] + b = c(a^2x^4 + 2abx^2 + b^2) + d$

$\Rightarrow$ $ac^2x^4 + 2acdx^2 + ad^2 + b = ca^2x^4 + 2cabx^2 + cb^2 + d$

$\Rightarrow$ Coefficient of x^4 = coefficient of x^4

and Coefficient of x^2 = coefficient of x^2

$\Rightarrow$ $ac^2 = ca^2 \Rightarrow a = c$

and $2acd = 2acb \Rightarrow b = d$

Thus $a = c$ and $b = d$

Q.46 Show that f, g : $N \times N \rightarrow N$ as f (x, y) = x + y. g(x, y) = xy are onto but not one-one.

Ans. : (i) Suppose $f(x_1, y_1) = f(x_2, y_2)$

$\Rightarrow$ $x_1 + y_1 = x_2 + y_2$

$\Rightarrow$ $x_1 \neq x_2$ and $y_1 \neq y_2$

e.g. $x_1 = 5$, $x_2 = 2$, $y_1 = 4$, $y_2 = 7$.

$\therefore$ f is not one to one.

Now, $g(x_1, y_1) = g(x_2, y_2)$

$\Rightarrow$ $x_1 y_1 = x_2 y_2$

$\Rightarrow$ x_1 may or may not be equal to x_2

and y_1 may or may not be equal to y_2

$\therefore$ g is not one to one.

(ii) $f(x, y) = x + y$

$\therefore$ Every element of N can be written as the sum of two elements of N. Hence f is onto.

Now, $g(x, y) = xy$

Every element of can be written as the product of two elements of N.

$\therefore$ g is onto.

2.8 Infinite Set and Countable Sets

1. Infinite Set

A set A is said to be an infinite set if there exists an injective mapping (function) $f : A \rightarrow A$ such that f(A) is a proper subset of A.

If no such injective function exists, then set is finite.

Examples :

1) Let $f : \mathbf{N} \rightarrow \mathbf{N}$ such that $f(n) = 2n$, $\forall n \in N$ = Natural number set.

There range set = $f(\mathbf{N})$ = {Set of positive even natural numbers} $\subset \mathbf{N}$

$\therefore$ **N** is an infinite set.

2) Define $f : \Re \rightarrow \Re$ such that $f(x) = \begin{cases} x+2 & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}$

f is an injective mapping and for $x \geq 0$, $f(x) = x + 2$

i.e. $f(x) \geq 2$ and for $x < 0$, $f(x) = x < 0$

$\therefore$ Range set = $f(\Re) = \{y \in \Re / f(x) \geq 2 \text{ or } f(x) < 0\}$

$\therefore 1 \notin f(\Re)$

Thus $f(\Re) \subsetneq \Re$

Hence $\Re$ is an infinite set.

I) Properties of an infinite sets

1) If A is an infinite set the $A \times A$, P (A) are infinite sets.

2) If A and B are non empt sets and either A or B is an infinite set then $A \times B$ is an infinite set.

3) If either A or B is an infinite set then $A \cup B$ is an infinite set.

4) If $A \subseteq B$ and A is an infinite set then B is also an infinite set.

i.e. the superset of an infinite set is an infinite.

2. Countable Sets

We know that the codinality of a set is the number of elements of that set. If set is finite then, we can list elements as 1, 2, 3, ...

Therefore every finite set is countable. As $|\phi| = 0$, the null set is also countable. A question remains same for an infinite sets. Let us define countable infinite sets.

Definition :

An infinite set A is said to be countable if there exists a bijection $f : \mathbf{N} \to A$

$\therefore \quad A = \{ f(1), f(2), f(3), \ldots\}$

A countably infinite set is also known as a denumarable set.

i.e. If A is a denumerable set then we can least elements of A as $a_1, a_2, a_3, \ldots a_n \ldots$ or $f(1), f(x) \ldots f(n) \ldots$.

e.g. 1) ϕ is countable

2) $A = \{1, 2, 3, 4, \ldots 1000\}$ is countable as $|A| = 1000$.

3) As $f : \mathbf{N} \to \mathbf{N}$ defined as $f(x) = n$, $\therefore$ **N** is countable

4) The set of integers is countable

As $f : \mathbf{N} \to \mathbf{Z}$ such that $f(n) = \frac{n+1}{2}$ if $n = 1, 3, 5, \ldots$

$= \frac{-n}{2}$ if $n = 0, 2, 4, 6, \ldots$

is bijective mapping.

5) The set of rational numbers is countable.

6) The set of real numbers is not countable

7) The set of complex numbers is not countable.

8) The set of real numbers in $[a, b]$, $a < b$ is not countable.

9) The countable union of countable sets is countable.

Properties of countable sets :

1) A subset of a countable set is countable.

2) Let A and B be countable sets then $A \cup B$ is countable.

$\Rightarrow$ Define $f : A \cup B \to \mathbf{N}$ as $f(a_i) = 2i - 1$ and $f(b_i) = 2i$

f is bijective $\therefore A \cup B$ is countable

3) Prove that the set of rational numbers is countable.

Proof :

We know that the countable union of countable sets is countable. Therefore it is sufficient to prove that the set of rational numbers in [0, 1] is countable.

We have to prove that $\exists$ atleast one function f.

$f : [0, 1] \to N$ such that f is injective.

We arrange the rational numbers of the interval according to increasing denominators as

$$0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots$$

then the one to one correspondence is as follows

$0 \leftrightarrow 1$

$1 \leftrightarrow 2$

$\frac{1}{2} \leftrightarrow 3$

$\frac{1}{3} \leftrightarrow 4$

$\frac{2}{3} \leftrightarrow 5$

$\frac{1}{4} \leftrightarrow 6$ and so on

Hence set of rationals in [0, 1] is countable. Thus the set of rational numbers is countable.

4) The set of irrational numbers is uncountable.

Proof :

We know that $\Re = Q \cup \overline{Q}$ where Q = set of rational numbers, $\overline{Q}$ = set of irrational numbers.

Suppose $\overline{Q}$ is countable $\Rightarrow Q \cup \overline{Q} = \Re$ is countable which is contradiction.

$\therefore \overline{Q}$ is not countable i.e. uncountable.

5) The set of real numbers in (0, 1) is not countable. Assume that the set is countable. $\therefore A = \{x_1, x_2, x_3, \cdots x_n, \cdots\}$.

Proof :

Any real number in (0, 1) can be written in a unique decimal without an infinite string of 9's at the end. i.e. 0.3459999 will be represented as 0.345000. Let the infinite sequence be given by,

$1 \rightarrow x_1 = 0 \cdot a_{11}\ a_{12}\ a_{13} \cdots$

$2 \rightarrow x_2 = 0 \cdot a_{21}\ a_{22}\ a_{23} \cdots$

$3 \rightarrow x_3 = 0 \cdot a_{31}\ a_{32}\ a_{33} \cdots$

$\vdots$

$n \rightarrow x_n = 0 a_{n1}\ a_{n2}\ a_{n3} \cdots$

$\vdots$

Construct a new number $y = 0 \cdot b_1\ b_2\ b_3\ \ldots$

Where $b_i = 0$ if $a_{ii} \neq 0$

$b_i = 1$ if $a_{ii} = 0$

Hence $b_1 \neq a_{11}, b_2 \neq a_{22}, b_3 \neq a_{33} \ldots b_n \neq a_{nn}$

$\therefore b_i \neq a_{ii}, \forall i$

$\therefore y \neq x_1, y \neq x_i\ \forall i$

Hence y is not in the list of numbers $\{x_1, x_2, \cdots x_n \cdots\}$ Thus $y \in (0, 1)$ and it is different from elements in $\{x_1, x_2, \cdots x_n \cdots\}$ which is contradiction that A is countable.

Hence A is not countable.

Thus the set of real numbers is not countable.

2.9 : Pigeon Hole Principle

Q.47 If 11 shoes are selected from 10 pairs of shoes then there must be a pair and matched shoes among the selection.

Ans. : In the pigeonhole principle, 11 shoes are pigeons and the 10 pairs are the pigeon holes.

Q.48 Show that if seven numbers from 1 to 12 are chosen then two of them will add upto 13.

Ans. : We have A = {1, 2, 3, 4, 5, ... 12}

We form the six different sets each containing 2 numbers that add uptot 13.

$A_1 = \{1, 12\}, A_2 = \{2, 11\}, A_3 = \{3, 10\}, A_4 = \{4, 9\}, A_5 = \{5, 8\}, A_6 = \{6, 7\}$

Each of the seven numbers chosen must belong to one of these sets. As there are only six sets, by pigeonhole principle two of the chosen numbers must belong to the same set and their sum is 13.

II) The extended pigeon hole principle

If n pigeons are assigned to m pigeon holes, then one of the pigeon holes must be occupied by at least $\left[\frac{n-1}{m}\right] + 1$ pigeons. It is also known as generalized pigeon hole principle. Here $\left[\frac{n-1}{m}\right]$ is the integer division of n – 1 by m. e.g. $\left[\frac{9}{2}\right] = 4, \left[\frac{16}{5}\right] = 3, \left[\frac{8}{3}\right] = 2.$

Q.49 Show that 7 colours are used to paint 50 bicycles, then at least 8 bicycles will be of same colour. **[SPPU : Dec.-09, 12, Marks 4]**

Ans. : By the extended pigeonhole principle, at least $\left[\frac{n-1}{m}\right] + 1$ pigeons will occupy one piegeonhole.

Here n = 50, m = 7 and m < n then

$$\left[\frac{50-1}{7}\right] + 1 = 7 + 1 = 8$$

Thus 8 bicycles will be of the same colour.

Q.50 Write generalized pigeonhole principle. Use any form of pigeonhole principle to solve the given problem.

i) Assume that there are 3 mens and 5 womens in a party show that if these people are lined up in a row at least two women will be next to each other.

ii) Find the minimum number of students in the class to be sure that three of them are born in the same month.

Ans. : I) This principle states that if there are n + 1 pigeons and only n pigeon holes then two pigeons will share the same whole.

This principle is stated by using the analogy of the bijective mapping i.e. If A and B are any two sets such that $|A| > |B|$ then there does not exist bijective mapping from A to B.

i) By using analogy of pigeon hole principle, we get

3 men = pigeonholes and 5 women = pigeon

Pigeons are more than pigeon holes.

$\therefore$ At least two pigeons share the same pigeon hole i.e. at least two women in a row will be next to each other.

ii) Let h = Number of pigeons = Number of students

n = Number of pigeon holes = Number of months = 12

Given that three students in the class are born in the same month.

$$\therefore \quad \left\lceil \frac{n-1}{m} \right\rceil + 1 = 3$$

$$\Rightarrow \quad \frac{n-1}{12} = 3 - 1 = 2$$

$$n = 2 \times 12 + 1 = 25$$

Therefore there are 25 minimum number of students in the class.

END...