# 3 Counting Principles

## 3.1 : Basic Counting Principles

### Important Points to Remember

**1. Sum Rule (Principle of Disjunctive Counting)**

We know that, if $S = A \cup B$ and A and B are disjoint sets i.e. $A \cap B = \phi$ then

$$|S| = |A| + |B|$$

i.e. A and B are disjoint partitions of S.

Now we can extend this logic to state sum rule.

**Sum rule :**

If one experiment $E_1$ has $n_1$ possible outcomes and another experiment $E_2$ has $n_2$ possible outcomes and $E_1$ and $E_2$ are disjoint (exclusive) then there are $n_1 + n_2$ possible outcomes when $E_1$ or $E_2$ take place.

**Product Rule (The Principle of Sequential Counting)**

We have, if A and B are non empty sets and $|A| = n$, $|B| = n$. then the number of elements in the cartesian product of A and B is equal to $n \times m$

i.e. $|A \times B| = n \times m$

Now extend this analogy for the product rule.

**Product Rule :**

If one experiment $E_1$ has $n_1$ possible outcomes and another experiment $E_2$ has $n_2$ possible outcomes then there are $n_1 \cdot n_2$ possible outcomes when the sequence of experiment $E_1$ first followed by $E_2$.

**Permutations**

An arrangement in a sequence of elements of a set is called a permutation of elements.

Depending upon the nature of arrangements, there are three types of permutations.

**Type I)** Permutations when all objects are distinct : A permutation of n objects taken r at a time is an arrangement of r objects out of n objects where $r \le n$.

It is called r - permutations or r - arrangements and denoted by P (n, r) or $^{n}P_{r}$.

e.g. 1) Consider the three letters a, b, c. The arrangements of the letters a, b, c taken two at a time are ab, ba, ac, ca, bc, cb.

∴ The number of 2 - arrangements are 6. i.e. the number of permutations of 3 symbols taken two at a time = $^{3}P_{2} = 6$.

Therefore as discussed above, the first place in the sequence can be filled up in n - ways, the second place in (n − 1) ways, the third place in (n − 2) ways and proceeding in this manner the $r^{th}$ place can be filled up in $n - (r - 1) = n - r + 1$ ways.

Hence
$$^{n}P_{r} = n\cdot(n-1)\cdot(n-2)\ldots(n-(r-1))$$
$$= \frac{n\cdot(n-1)(n-2)\ldots(n-r+1)\times(n-r)\,!}{(n-r)\,!}$$
$$^{n}P_{r} = \frac{n\,!}{(n-r)\,!}\ ;\ 0 \le r \le n$$

**Properties :** 1) $^{n}P_{n} = \frac{n\,!}{(n-n)\,!} = \frac{n\,!}{0\,!} = n!$

2)
$$^{n}P_{1} = \frac{n\,!}{(n-1)\,!} = \frac{n\,(n-1)\,!}{(n-1)\,!} = n$$
$$^{n}P_{2} = \frac{n\,!}{(n-2)\,!} = \frac{n\,(n-1)\,[(n-2)\,!]}{(n-2)\,!} = n\,(n-1)$$
$$^{n}P_{3} = n\,(n-1)\,(n-2) \text{ and so on.}$$

3) $0\,! = 1$

**Q.1 Given that A = {1, 2, 3, 4, 5, 6}, find the number of permutations of A taken**
**i) 2 at a time ii) 3 at a time iii) 4 at a time iv) 5 at a time v) 6 at a time.**





**Ans. :** We have A = {1, 2, 3, 4, 5, 6}

∴ $|A| = 6$

i) The permutation at 6 letters taken 2 at a time is
$$^{6}P_{2} = \frac{6\,!}{(6-2)\,!} = \frac{6\,!}{4\,!} = \frac{6\times 5\times 4\,!}{4\,!} = 30$$

ii) The permutation of 6 letters taken 3 at a time is
$$^{6}P_{3} = \frac{6\,!}{(6-3)\,!} = \frac{6\times 5\times 4\times 3\,!}{3\,!} = 120$$

Similarly,

iii) $^{6}P_{4} = \frac{6\,!}{(6-2)\,!} = \frac{6\,!}{4\,!} = 30$

iv) $^{6}P_{5} = \frac{6\,!}{(6-5)\,!} = \frac{6\,!}{1\,!} = 6\times 5\times 4\times 3\times 2\times 1 = 720$

v) $^{6}P_{6} = \frac{6\,!}{(6-6)\,!} = \frac{6\,!}{0\,!} = 6\,! = 720$

**Q.2 How many four digit numbers can be formed out of digits 1, 2, 3, ...9. if i) No repetition is permitted.**
**ii) How many of these will be greater than 3000.**

**Ans. :**

i) The number of ways of selecting 4 digits out of 9 digits is
$$^{9}P_{4} = \frac{9\,!}{(9-4)\,!} = \frac{9\times 8\times 7\times 6\times 5\,!}{5\,!} = 3024$$

ii) There is a restriction that the 4 digit numbers so formed must be greater than 3000.

Therefore the thousandth position can be filled with numbers 3, 4, 5, 6, 7, 8, 9 i.e. the thousandth place can be selected in 7 different ways.

Now out of remaining 8 digits, hundredth position can be filled in 8 different ways, Tenth place can be filled in 7 different ways and unit place can be filled in 6 different ways.

Thus the total number of 4 digit numbers greater than 3000 can be formed in

$7 \times 8 \times 7 \times 6 = 2352$ ways.

**Q.3 i) Suppose repetitions are not permittable, how many four digit numbers can be formed from six digits 1, 2, 3, 5, 7, 8 ? ii) How many of such numbers are less than 4000 ? iii) How many in (i) are even ? iv) How many in (ii) are odd ?**

**v) How many in (i) contain both 3 and 5.**

**vi) How many in (i) are divisible by 10.** [SPPU : Dec.-05]

**Ans. :** i) Out of 6 numbers, 4 digit numbers can be formed in ${}^6P_4$ ways.

$\therefore$ Number of ways $= \frac{6\,!}{2\,!} = 360$

ii) The four digit numbers which are less than 4000 are the numbers in which first digit is 1, 2 or 3 i.e. 1st digit can be chosen in 3 ways, 2nd digit can be any one of the remaining 5 digits. 3rd digit can be any of the remaining 4 digit and the 4th digit is any one of the remaining 3 digits.

Hence the total number of ways = $3 \times 5 \times 4 \times 3 = 180$.

iii) Those numbers ending in 2 or 8 are even numbers. Hence the last digit (4th digit) can be chosen in 2 ways (the number 2 or 8). The first digit can be chosen in any one of the remaining 5 digits, 2nd in any of the 4 digits and 3rd in any of the 3 digits. Hence the total number of ways.

$= (5 \times 4 \times 3) \times 2$

| TH | H | Ten | Unit place |
|---|---|---|---|

= 120 ways

iv) The numbers less than 4000 and are odd. The numbers ending with 1, 3, 5 or 7 are odd. The 4 digit numbers ending in 1 and less than 4000 should begin with eithor 2 or 3.

Then there are $2 \times 4 \times 3 \times 1 = 24$ such numbers. Similarly the number ending in 3 are 24. However the number ending with 5 or 7 are

TH UnitPlace

$3 \times 4 \times 3 \times 2 \quad = 72$ ways.

Hence the total number which are odd and less than 4000 are $24 + 24 + 72 = 120$





v) The digit 3 can occupy any of the 4 positions and the remaining 3 positions will be occupied by the digit 5. Hence the number of ways in which two positions are occupied by 3 and 5 will be $4 \times 3$ i.e. 12.

Now the remaining two positions will be filled by the remaining 4 numbers i.e. 1, 2, 7 and 8.

Hence out of remaining two positions one position can be occupied in 4 different ways and the remaining position will be occupied in 3 different ways.

Hence total number of 4 digit numbers in which both 3 and 5 are present $= 12 \times 4 \times 3 = 144$.

vi) Not even a single number is divisible by 10 as there is no zero at unit's place.

**Q.4 A menu card in a restaurant displays four soups, five main courses, three desserts and 5 beverages. How many different menus can a customer select if, i) He selects one item from each group without omission.**

**ii) He chooses to omit the beverages, but selects one each from the other groups.**

**iii) He choose to omit the desserts but decides to take a beverage and one item each from the remaining groups.**

**Ans. :**

i) The customer can select the soup in 4 ways, the main course in 5 ways, the dessert in 3 ways and beverages in 5 ways.

Hence by product rule, the number of ways in which he can select one item each, without omission is $4 \times 5 \times 3 \times 5 = 300$.

ii) The number of ways in which he omit beverages $= 4 \times 5 \times 3 = 60$ ways.

iii) The number of ways in which he omit desserts but he takes all other items $= 4 \times 5 \times 5 = 100$ ways.

**Q.5 10 different $M_1$ books, 3 different $M_2$ books, 5 different $M_3$ books and 7 different D.S. books are to be arranged on a shelf. How many different arrangements are possible if**

**i) The books in each subject must all be together**

**ii) Only $M_3$ books must be together.**

**Ans. :** i) $M_1$ books can be arranged among themselves in 10 ! ways, the $M_2$ books in 3 ! ways, $M_3$ books in 5 ! ways and D.S. books in 7 ! ways.

Hence the total number of arrangements = $4! \cdot 10! \cdot 3! \cdot 5! \cdot 7!$

ii) Consider the 5 $M_3$ books as a single book. Then there are 21 books which can be arranged in 2 ! ways. In each of these arrangements the $M_3$ books can be arranged among themselves in 5! ways.

Hence the number of arrangements in $5! \cdot 2!$

**Q.6 2 mathematics papers and 5 other papers are to be arranged at an examination. Find the total number of ways if,**

**i) Mathematics papers are consecutive. ii) Mathematics papers are not consecutive.**

**Ans. :** i) Both mathematics papers ($M_1$ and $M_2$) are together, consider both $M_1$ and $M_2$ as single paper.

These two papers among themselves can be arranged in 2! ways.

Now 6 papers (as $M_1$ and $M_2$ is considered as single paper) can be arranged in 6! ways.

Hence total number of arrangements = $2! \cdot 6!$

ii) If $M_1$ or $M_2$ are not consecutive than they are to be arranged between the 4 gaps or at the 2 ends.

$M_1$ or $M_2$ ▭ $M_1$ or $M_2$ ▭ $M_1$ or $M_2$ ▭ $M_1$ or $M_2$ ▭ $M_1$ or $M_2$ ▭ $M_1$ or $M_2$

Where ▭ denotes other papers

Hence there are 6 places where mathematics papers can be arranged. Therefore, 2 mathematics papers can be arranged in 6 places in ${}^6P_2$ ways. Five other papers can be arranged among themselves in 5! ways.

Therefore total number of arrangements

$$= 5! \cdot {}^6P_2 = 5! \cdot 6 \cdot 5 = (120) \cdot (30) = 3600$$

**Q.7 How many permutations can be made out of the letter of word "COMPUTER" ? How many of these i) begin with C ii) end with R iii) begin with C and end with R iv) C and R occupy the end places**

**Ans. :** There are 8 letters in the word "COMPUTER" and all are distinct

$\therefore$ The total number of permutations of these letters is 8! = 40320.





**i) Permutations begin with C :**

The first position can be filled in only one way i.e. C and the remaining 7 letters can be arranged in 7! ways.

$\therefore$ The total no. of permutations begining with C = $1 \times 7! = 5040$

**ii) Permutations end with R :**

The Last position can be filled in only one way and the remaining 7 letters can arranged in 7! ways.

$\therefore$ The total no. of permutations ending with R be = $7! \times 1 = 5040$

**iii) Permutation begin with C and end with R :**

The first position can be filled in only one way i.e. C and the end position also can be filled in only one way i.e. R and the remaining 6 letters can be arranged in 6! ways.

$\therefore$ The required no. of permutations = $1 \times 6! \times 1 = 720$

**iv) Permutation in which C and R occupy end places :**

C and R occupy end positions in 2 ! ways i.e. CR or RC and the remaining 6 letters can be arranged in 6! ways.

$\therefore$ The total no. of required permutations = $2! \times 6! = 1440$

A) Permutations with restrictions :

1) The number of permutations of n different objects taken r at a time in which p particular objects do not occur is ${}^{(n-p)}P_r$.

2) The number of permutations of n different objects taken r at a time in which p particular objects are present is ${}^{(n-p)}P_{r-p} \times {}^rP_p$.

**Q.8 Show that the number of injective functions from a set with r elements to a set with n elements is ${}^nP_r$ ; $r \le n$.**

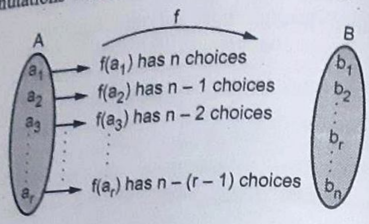
**Solution :** Let A and B be two sets with $|A| = r$ and $|B| = n$.

$A = \{a_1, a_2, \ldots a_r\}$ and $B = \{b_1, b_2, b_3 \ldots b_r, \ldots b_n\}$

Let $f : A \to B$ be an injective function

Hence by product rule, the number of injective functions from A to B is $n(n-1) \quad (n-2) \ldots (n-r+1) = {}^nP_r$

**Type II)** Permutations when all objects are not distinct




If $r_1$ objects are of one kind

$r_2$ objects are of second kind

$r_3$ objects are of third kind

⋮

$r_k$ objects are of $k^{th}$ kind, where $r_1 + r_2 + \ldots r_k = n$

Then the number of permutations when all are taken at a time (i.e. $r_1, r_2, \ldots r_k$ are taken) is $\dfrac{n!}{r_1!\, r_2!\, r_3! \ldots r_k!}$.

**Q.9 Find the number of permutations that can be made out of the letters i) MISSISSIPPI ii) ASSASSINATION**

☞[SPPU : Dec.-14]

**Ans. :** i) There are 11 letters in the word out of which S, I, P, M are distinct.

S appears 4 times, I appears 4 times

P appears 2 times, M appears 1 time

$\therefore$ The required no. of permutations $= \dfrac{11!}{4!\ 4!\ 2!\ 1!} = 34650$

ii) There are 13 letters of which A, I, N, S, T and O are different.

A appears 3 times, I appears 2 times N appears 2 times,

S appears 4 times T appears 1 time, O appears 1 time

$\therefore$ The required no. of permutations $= \dfrac{13!}{3!\ 2!\ 2!\ 4!\ 1!\ 1!} = 10810800$

**Q.10 How many ways can the letters in the word "PIONEER" be arranged so that the two E's are always together.**





**Ans. :** The word 'PIONEER' has two E's and remaining 5 letters are distinct. These distinct five letters can be arranged in 5 ways and for each such arrangement two E's can occupy any of the six remaining places. Hence the required no. of permutations are

$6 \times 5! = 6! = 720.$

**Q.11 How many seven digit numbers can be formed using digits 1, 7, 2, 7, 6, 7, 6 ?**

**Ans. :** There are 7 digits out of which 7 is repeated 3 times, 6 is repeated twice, appears 1 time and 2 appear 1 time.

$\therefore$ The total no. of permutation $= \dfrac{7!}{3! \times 2! \times 1! \times 1!} = 420$

**Type III)** Permutations with repeated objects.

The number of different permutations of n distinct objects taken r at a time when every object is allowed to repeat any number of times is given by $n^r$.

**Q.12 How many 4 digits numbers can be formed by using the digits 2, 4, 6, 8 when repetition of digits are allowed ?**

**Ans. :** We have 4 digits numbers.

No. of ways of filling unit's place = 4

No. of ways of filling ten's place = 4

No. of ways of filling hundred's place = 4

No. of ways of filling thousand's place = 4

Therefore the total number of 4 digits numbers

$= 4 \times 4 \times 4 \times 4 = 4^4 = 256$

**Q.13 How many 4 digits even numbers can be formed by using the digits 1, 3, 4, 6, 8 when repetition of digits are allowed ?**

**Ans. :** We have 3 even numbers and 2 odd numbers. Therefore,

The no. of ways of filling unit's place = 3

The no. of ways of filling ten's place = 5

The no. of ways of filling hundred's place = 5

The no. of ways of filling thousand's place = 5

Thus, the total no. of required 4 digits numbers $= 3 \times 5^3 = 3 \times 125 = 375$

**Q.14 In how many ways can 5 software projects be alloted to 6 final year students when all of five projects are not alloted to the same student.**

**Ans. :** We have 5 projects and 6 students. Each project can be alloted in 6 ways.

Thus, the number of ways of alloting 5 projects $= 6 \times 6 \times 6 \times 6 \times 6 = 6^5$

The number of ways in which all 5 projects are allotted to same student $= 6$.

Therefore, total number of ways to allocate 5 projects to 6 students

$= 6^5 - 6 = 7770$

**Q.15 A bit is either 0 or 1. A byte is a sequence of 8 bits. Find i) Number of bytes ii) Number of bytes that begin with 11 and end with 11.**

**Ans. :** i) Total number of byte is

$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8 = 256.$

ii) As the first two and last two bits are fixed i.e. 11 the remaining bits in the sequence are either 0 or 1.

$\therefore$ The required no. of total bytes $= 2^4 = 16$.

**Q.16 Prove that the number of circular permutations of n different objects is (n − 1) !**

**Sol. :** Let us consider that k be the number of required permutations. For each such circular permutation of k, there are n corresponding linear permutations. We can start from every object of n objects in the circular permutation. Thus for k circular permutations, we have kn linear permutations.

Therefore $k \cdot n = n! \Rightarrow k = \frac{n!}{n} = (n-1)!$ Hence the proof.

**Q.17 How many ways can these letters A, B, C, D, E and F be arranged in a circle ?**

**Ans. :** There are six letters. Hence the no. of ways to arrange these six letters in a circle is $(6 - 1)! = 5! = 120$.





**Q.18 In how many waye 10 programmers can s.t. on a round table to discuss the project. So that project leader and a particular programmer always sit together ?**

**Ans. :** There are 10 programmers. But project leader and particular programmer always sit together. So both become a single unit and hence there are

$(10 - 2 + 1) = 9$ remins

Thus these 9 units can be arreanged on round table in $(9 - 1)! = 8!$ ways.

The two programmer i.e. project leader and particular programmer can be arranged in 2! ways.

Therefore the required no. of ways $= 2! \times 8! = 80640$.

**Q.19 Determine the number of ways in which 5 software engineers and 6 electronics engineers can be sitted at a round table so that no two software engineers can sit together.**

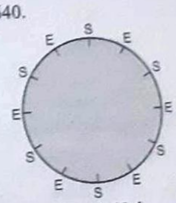



Fig. Q.19.1

**Ans. :** There are 6 electronics engineers that can be arranged round a table in $(6 - 1)!$ ways. There are 5 software engineers and they are not to sit together, so there are six places for software engineers and can be placed in 6! ways as shown in Fig. Q.19.1

Therefore the required no. of ways

$= (6 - 1)! \times 6! = 5! \times 6! = 120 \times 720 = 86400$

## 3.2 : Combination

### Important Points to Remember

A combination is a selection of some or all objects from a set of given objects where the order of the objects does not matter. In this context, we used mainly two words "selection" and "arrangement". In selection, order of objects is immaterial i.e. selection is a set. But in arrangement, the order of objects is important it is not a set. Arrangement is a n-tupple. Arrangement is associated with permutation selection with combination.

**I) Definition :** The number of combinations of n different objects taken r at a time is given by ${}^nC_r$ and defined as,

$${}^nC_r = \frac{n!}{r!\,(n-r)!} \;;\; r \le n$$

**Properties :**

1) $${}^nC_r = \frac{n!}{r!(n-r)!} = \frac{{}^nP_r}{r!}$$

2) $${}^nC_n = \frac{n!}{n!\,(n-n)!} = 1$$

3) $${}^nC_0 = \frac{n!}{0!\,(n-0)!} = \frac{n!}{n!} = 1$$

4) $${}^nC_1 = \frac{n!}{1!\,(n-1)!} = n\ ,\quad {}^nC_2 = \frac{n!}{2!\,(n-2)!} = \frac{n\,(n-1)}{2}$$

$${}^nC_2 = \frac{n\,(n-1)\,(n-2)}{3!}\ ,\quad {}^nC_4 = \frac{n\,(n-1)\,(n-2)\,(n-3)}{4!}$$

**Q.20 Find the value of n if i) $^nC_{n-2} = 10$ and**

**ii) $^{25}C_{n+2} = {}^{25}C_{2n-1}$**

**Ans. :** i) We have $^nC_{n-2} = 10 \Rightarrow \dfrac{n!}{(n-2)!\,(n-(n-2))!} = 10$

$$\therefore \quad \frac{n\,(n-1)\,(n-2)!}{(n-2)!\,(2!)} = 10$$

$$n\,(n-1) = 10 \times 2 = 20$$

$$n^2 - n - 20 = 0$$

$$\Rightarrow \quad (n-5)\,(n+4) = 0$$

$n = 5$ or $n = -4$ as $n = -4$ is not possible

$$\boxed{n = 5}$$

ii) We know that $^nC_r = {}^nC_{n-2}$

Now $^{25}C_{n-2} = {}^{25}C_{n-2}$

$\Rightarrow$ Either $n + 2 = 2n - 1$ or $(n+2) + (2n-1) = 25$

$\therefore$ either $n = 3$ or $3n = 25 - 1 = 24$

$\therefore$ $n = 3$ or $n = 8$

$\therefore$ $n = 3, 8.$





**Q.21 How many 16 bit strings are there containing exactly five 0's ?**

**Ans. :** Each string of 16 bit has 16 digits. A 16-bit string having exactly five 0's is determined if we tell which bit are 0's. So, here order is immaterial.

$\therefore$ This can be done in $^{16}C_5$ ways.

$\therefore$ The total number of 16-bit strings is,

$${}^{16}C_5 = \frac{16!}{5! \times (16-5)!} = \frac{16 \times 15 \times 14 \times 13 \times 12 \times 11!}{5 \times 4 \times 3 \times 2 \times 1 \times 11!} = \mathbf{4368}$$

**Q.22 In how many ways can 30 late admitted students be assigned to three practical batches A, B, C if A can accomodate 10 students, B - 15 students and C - 5 students only ?**

**Ans. :** The batch A can accomodate 10 students out of 30.

$\therefore$ The batch A can be assigned 10 students in $^{30}C_{10}$ ways.

then batch B can be assigned 15 students in $^{20}C_{15}$ ways.

then batch C can be assigned 5 students in $^{5}C_5$ ways.

Therefore by the product rule, the total no of ways of assigned students is

$${}^{30}C_{10} \times {}^{20}C_{15} \times {}^{5}C_5 = \frac{30!}{10!\,(20!)} \times \frac{20!}{15! \times 5!} \times 1 = \frac{30!}{10!\,15!\,5!}$$

**Q.23 How many ways can we select a software development group of 1 project leader, 15 programmers and 6 data entry operators from a group of 5 project leaders 20 programmers and 25 data entry operators.**

**Ans. :** One project leader can be selected from 5 project leaders in $^5C_1 = 5$ ways.

15 programmers can be selected from 20 programmers in $^{20}C_{15}$ ways.

6 data entry operators can be selected from 25 data entry operators in $^{25}C_6$ ways.

Therefore the total number of ways to select the software development group is

$${}^5C_1 \times {}^{20}C_{15} \times {}^{25}C_6 = 96101544000.$$

**Q.24 From 10 programmers in how many ways can 5 be selected when i) A particular programmer is included everytime ii) A particular programmer is not included at all time.**

**Ans. :** The number of ways to select 5 programmers from 10 is $^{10}C_5 = \frac{10!}{5!\,5!} = \frac{10\times 9\times 8\times 7\times 6}{5\times 4\times 3\times 2\times 1} = \mathbf{252}$

i) When a particular programmer is included every time then the remaining 5 – 1 = 4 programmers can be selected from the remaining 10 – 1 = 9 programmers. This can be done in $^9C_4$

$\therefore \quad ^9C_4 = \frac{9!}{4!\,5!} = \frac{9\times 8\times 7\times 5}{4\times 3\times 2} = 126$

ii) When a particular programmer is not included at all then the 5 programmers can be selected from the remaining 10 – 1 = 9 programmers.

This can be done in $^9C_5$ ways.

$\therefore \quad ^9C_5 = \frac{9!}{4!\,5!} = 126$

**Q.25 A committee of 5 people is to be formed from a group of 4 men and 7 women. How many possible committees can be formed if at least 3 women are on the committee ?** **☞[SPPU : May-14]**

**Ans. :** If at least three women are on committee, it means committee with 3 women or 4 women or 5 women.

i) 3 women can be selected in $^7C_3$ ways.

2 men can be selected in $^4C_2$ ways.

The no. of ways this can be done is $^7C_3 \times {}^4C_2 = 210$ ways.

ii) 4 women and 1 man can be selected in $^7C_4$ and $^4C_1$ ways respectively.

$\therefore$ The number of ways to form a committee is $^7c_4 \times {}^4c_1 = 140$ ways.

iii) 5 women can be selected in $^7C_5$ ways.

$\therefore$ The no. of ways to form a committee is $^7C_5 = 21$ ways

Hence the total no. of ways a committee can be formed with at least 3 women is 210 + 140 + 21 = 371.





**Q.26 How many automobile license plates can be made if each plate contains two different letters followed by three different digits. Solve the problem if the first digit can not be zero.** **☞[SPPU : May-06]**

**Ans. :** The first position is a letter and can be selected from 26 letters in $^{26}C_1 = 26$ ways.

The second position is a letter and can be selected from 26 – 1 = 25 letters in $^{25}C_1 = 25$ ways.

For digits :

1) The first digit can be selected from 10 digits in $^{10}C_1 = 10$ ways.

2) The second digit can be selected from 9 digits in $^9C_1 = 9$ ways

3) The third digit can be selected from 8 digits in $^8C_1 = 8$ ways.

Therefore the total number of license plates

$= 26 \times 25 \times 10 \times 9 \times 8 = 468000$

Now, in license plate, the first digit can not be 0 then the first position can selected from 9 digits in $^9C_1$ ways.

The second digit can be zero, but one digit is already selected for the first position. Hence the second digit can be selected in $^9C_1$ ways. The third digit can be selected in $^8C_1$ ways.

Hence the total no. of required license plates are

$26 \times 25 \times 9 \times 9 \times 8 = 421200.$

**Q.27 In the discrete structure paper, there are 10 questions. In how many ways can an examiner select five questions in all the first question is compulsory.**

**Ans. :** The first question is compulsory, so the examiner has to select 4 questions from the remaining 9 questions.

$\therefore$ The number of ways to select five questions is

$1\times {}^9C_4 = \frac{9!}{4!\,5!} = \frac{9\times 8\times 7\times 6}{4\times 3\times 2} = 126$ ways.

**Q.28 Determine the number of triangles that are formed by selecting points from a set of 12 points out of which 5 are colinear.**

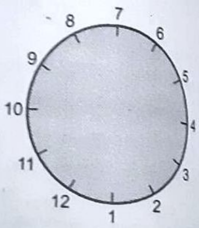
**Ans. :** By using 12 points, the number of triangles formed is $^{12}C_3$. As five points are colinear i.e. lie on same line, they do not form any triangle. Thus $^{5}C_3$ triangles are lost.

$\therefore$ The total number of triangles produced is

$$^{12}C_3 - {}^{5}C_3 = \frac{12!}{3! \times 9!} - \frac{5!}{3!\ 2!} = \frac{12 \times 11 \times 10}{3 \times 2} - \frac{5 \times 4}{2} = 220 - 10 = 210$$

**Q.29 How many lines can be drawn through 12 points on a circle and line passes through exactly two points ?**

**Ans. :** As all points on the circle are not colinear, thus no line will lost.




The total no. of lines drawn through 12 points is

$$^{12}C_2 = \frac{12!}{2!\ 10!} = \frac{12 \times 11}{2 \times 1} = 66$$

**Q.30 Determine the number of diagonals that can be drawn by joining the nodes of octgon.**

**Ans. :** The number of lines that can be drawn by 2 points out of 8 points of octagon is $^{8}C_2 = 28$. Out of these 28 lines, 8 are the sides of the octagon.

$\therefore$ The number of diagonals = 28 − 8 = 20.

**Q.31 In a box, there are 40 floppy disks of which 4 are defective. Determine**

**i) In how many ways we can select five floppy disks ?**

**ii) In how many ways we can select five non defective floppy disks ?**

**iii) In how many ways we can select five floppy disks containing exactly three defective disks ?**

**iv) In how many ways we can select five floppy disks containing at least 1 defective disk ?**

**Ans. :** i) There are 40 floppy disks out of which we have to select 5 floppy disks in $^{40}C_5$ ways.

$$^{40}C_5 = \frac{40!}{5! \times (40-5)!} = \frac{40 \times 39 \times 38 \times 37 \times 36}{5 \times 4 \times 3 \times 2 \times 1} = 658008$$





ii) There are 40 − 4 = 36 nondefective floppy disks out of which we have to select 5. This can be done in $^{36}C_5$ ways.

$$\therefore \quad ^{36}C_5 = \frac{36!}{5! \times (31)!} = \frac{36 \times 35 \times 34 \times 33 \times 32}{5 \times 4 \times 3 \times 2 \times 1} = 376992$$

iii) To select exactly three defective floppy disks out of 4 disks, we have $^{4}C_3$ ways and the remaining two floppy disks can be selected from 36 disks in $^{36}C_2$ ways.

Therefore, the required no. of ways = $^{4}C_3 \times {}^{36}C_2$

$$= \frac{4!}{3! \times 1!} \times \frac{36!}{2!\ 34!} = 4 \times \frac{36 \times 35}{2} = 2520$$

iv) There are 4 defective floppy disks out of which at least one must be selected. We know that the total number of ways to select 5 disks from 40 disks is $^{40}C_5$.

Also the number of ways to select 5 floppy disks with no defective is $^{36}C_5$ way.

Therefore the required no. of ways

$$= {}^{40}C_5 - {}^{36}C_5 = 658008 - 376992 = 281016$$

**II) Combination with Repetitions :** The number of ways to fill r slots from n categories with repetition allowed is C (n + r − 1, r) = C (n + r − 1, n − 1)

Therefore we have

$$^{(n+r-1)}C_r = {}^{(n+r-1)}C_{n-1} = C(n+r-1, r)$$

and C (n + r − 1, r) = The number of non-negative integers solution of $x_1 + x_2 + \ldots + x_n = r$

= The number of ways of placing r indistinguishable balls in n numbered boxes.

= The number of binary numbers with n−1 one's and r zero's.

= C (n + r − 1 ; n − 1)

= The number of r-combinations of $\{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_n\}$

**Q.32 How many 4 combinations of {1, 2, 3, 4, 5, 6} are there with unlimited repetition ?**

**Ans. :** We have r = 4, n = 6

∴ The number of 4 combinations of {1, 2, 3, 4, 5, 6} are

$$C(6+4-1, 4) = C(9, 4) = \frac{9!}{4!\ 5!} = \frac{9\times 8\times 7\times 6\times 5!}{4\times 3\times 2\times 1\times 5!} = 126.$$

**Q.33 Find the number of 3-combinations of $\{\infty\cdot a_1, \infty\cdot a_2, \infty\cdot a_3, \infty\cdot a_4\}$.**

**Ans. :** We have n = 4, r = 3

∴ The number of 3-combinations of the given set is

$$C(4+3-1, 3) = C(6, 3) = \frac{6!}{3!\ 3!} = \frac{6\times 5\times 4\times 3!}{3\times 2\times 3!} = 20$$

**Q.34 The number of non negative integer solutions to $x_1 + x_2 + x_3 + x_4 = 20$**

**Ans. :** we have r = 20, n = 4

∴ The number of non negative integer solutions

$$= C(4+20-1, 20) = C(23, 20) = \frac{23!}{20!\times 3!} = \frac{23\times 22\times 21}{3\times 2} = 1771$$

**Q.35 The number of ways of placing 8 similar balls in 5 numbered boxes.**

**Ans. :** The number of ways of placing r = 8 similar balls in n = 5 boxes is

$$C(5+8-1, 8) = C(12, 8) = 495.$$

**Q.36 Find the number of binary numbers with six 1's and 4 zero's.**

**Ans. :** The number of binary numbers with 6 one's and 4

$$\text{zero's} = C(6+4, 4) = C(10, 4) = \frac{10\times 9\times 8\times 7\times 6!}{4\times 3\times 2\times 1\times 6!} = 210$$

**Q.37 How many integral solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 = 16$ where each $x_i \geq 2$ ?**

**Ans. :** Let $x_i = y_i + 2$ where $y_i \geq 0$.

we have $x_1 + x_2 + x_3 + x_4 + x_5 = 16$

iff $y_1 + 2 + y_2 + 2 + y_3 + 2 + y_4 + 2 + y_5 + 2 = 16$

Iff $y_1 + y_2 + y_3 + y_4 + y_5 = 16 - 10 = 6$





Thus the number of integral solutions of given equation is the same as the number of integral solutions of $y_1 + y_2 + y_3 + y_4 + y_5 = 6$

There are C (5 − 1 + 6, 6) = C (10, 6) such solutions

$$\therefore \quad C(10, 6) = \frac{10!}{6!\times 4!} = \frac{10\times 9\times 8\times 7\times 6!}{4\times 3\times 2\times 1\times 6!} = 210$$

**Q.38 How many integral solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 = 30$ where $x_1 \geq 2, x_2 \geq 3, x_3 \geq 4, x_4 \geq 2, x_5 \geq 0$.**

**Ans. :** Let $x_1 = y_1 + 2,\ x_2 = y_2 + 3,\ x_3 = y_3 + 4,\ x_4 = y_4 + 2,\ x_5 = y_5 + 0$

$$\therefore \quad x_1 + x_2 + x_3 + x_4 + x_5 = 30$$

$$\Rightarrow \quad y_1 + 2 + y_2 + 3 + y_3 + 4 + y_4 + 2 + y_5 = 30$$

$$y_1 + y_2 + y_3 + y_4 + y_5 = 19$$

∴ The required number of integral solutions are

$$C(5-1+19, 19) = C(23, 19) = \frac{23\times 22\times 21\times 20}{4\times 3\times 2\times 1} = 8855$$

**Theorem 1 :** The number of integer solutions to $a_1 + a_2 + a_3 + \ldots + a_n = r$ when $a_1 \geq b_1, a_2 \geq b_2, a_3 \geq b_3, \ldots a_n \geq b_n$ is $C(n + r - 1 - b_1 - b_2 - b_3 - \ldots b_n, r - b_1 - b_2 - b_3 - \ldots - b_n)$

**Theorem 2 :** The number of ways to select r things from n categories with b total restrictions on the r things is C (n + r − 1 − b, r − b)

**Theorem 3 :** The number of ways to select r things from n categories with atleast 1 thing from each category is C (r − 1, r − n) (∵ b = n)

## 3.3 : Generation of Permutation and Combination

### Important Points to Remember

**I) Generation of permutations :** Suppose we want to generate n! permutations of n distinct objects. For n = 1, 2, 3, it is simple but when n is large it is difficult to keep track of what we have written and make sure that we shall write down all permutations with no repetition or omissions.

An interesting problem is to find a systematic procedure for generating all n! permutations of a set with n distinct elements.

Suppose from the initial permutation 1, 2, 3, ... n by using the next permutation procedure repeatedly we shall obtain all the permutations of 1, 2, 3, ..., n. The last permutation is n, (n – 1), (n – 2), ... 4, 3, 2, 1.

Procedure for next permutation :

**Step 1 :** Given a permutation $a_1, a_2, a_3, \ldots a_n$ of 1, 2, 3, ..., n

**Step 2 :** Scan from right to left (L ← R). Find the first m such that $a_m < a_{m+1}$

**Step 3 :** $\alpha = \min \{a_k | k = m+1, m+2, m+3, \ldots n, a_k > a_m\}$

**Step 4 :** The next permutation is

$a_1 a_2 a_3 \ldots a_{m-1}, \alpha, x, x, x \ldots$

where x, x, x ... are the remaining numbers arranged in the increasing order.

**Algorithms :**

I) Algorithm for generating the next Permutation in Lexicographic order :

Next permutation $[(a_1, a_2, \ldots, a_n)]$ : Permutation of {1, 2, 3, ...n} not equal to (n, (n – 1) (n – 2) ... 3 2 1)]

i : = n – 1

while $a_i > a_k$

k : = k – 1 {$a_k$ is the smallest integer greater than $a_i$ to the right of $a_i$} interchange $a_i$ and $a_k$

r : = n

s : = j + 1

while r > s

begin

interchange $a_r$ and $a_s$

r : = r – 1

s = s + 1

end

{This puts the tail end of permutation after the $i^{th}$ position in increasing order}

**Q.39 Let n = 6 and given permutation is 125364 find the next permutation.**





**Solution : Step 1 :** Given permutation is 125364

**Step 2 :** Scan from right to left and find the first m such that. $a_m < a_{m+1}$

Now start with $5^{th}$ position ; As 6 > 4, m ≠ 5

As 3 < 6 and position of 3 is $4^{th}$ ∴ m = 4.

**Step 3 :** $\alpha = \min \{a_k | a_k > a_m\} = \min \{6, 4\} = 4$

**Step 4 :** Replace element $a_m$ by α i.e. 3 by 4, keep previous elements as its i.e. 125 and write all remaining elements in increasing order i.e. 36.

∴ The next permutation is 125436.

**Q.40 Find the next two permutations of 125436.**

**Ans. :** i) Given that 1254 3 6

m = 5 and α = min { 6 } = 6

∴

∴ The next permutation is 125463.

ii) Find the next permutation of 1254 63

m = 4, α = min {6} = 6

∴ The next permutation is 125634.

**Q.41 Generate all permutations for n = 3 by next permutation method.**

**Ans. :** Consider the following table to generate all permutations for n = 3.

| Sr. No. | Given permutation | Next permutation |
|---|---|---|
| 1) | 1 2 3 ↑ m = 2, α = 3 | 1 3 2 |
| 2) | 1 3 2 ↑ m = 1, α = 2 | 2 1 3 |
| 3) | 2 1 3 ↑ m = 2, α = 3 | 2 3 1 |
| 4) | 2 3 1 ↑ m = 1, α = 3 | 3 1 2 |
| 5) | 3 1 2 ↑ m = 2, α = 2 | 3 2 1 Last permutation. |

$\therefore$ The set of all permutations for $n = 3$ is

{(1 2 3) (1 3 2) (2 1 3) (2 3 1) (3 1 2) (3 2 1)}

**Procedure to generate subsets of {1, 2, 3, ... n}**

Let $\{a_1, a_2, a_3, \dots a_k\}$ be a subset of size k of $\{1, 2, 3, \dots n\}$ with $a_1 < a_2 < a_3 < \dots < a_k$.

The maximum possible value of $a_k$ is n

The maximum possible value of $a_{k-1} = n - 1$

In general the maximum possible value of $a_i$ is $n - k + i$. Consider the subset $\{1, 2, 3, \dots k-1, k\}$ If $k \neq n$, its maximum value then increase k by 1, so that the next subset $\{1, 2, 3, \dots k-1, k+1\}$ is generated. We continue this procedure till we reached to $\{1, 2, 3 \dots (k-1), n\}$. Now repeat the procedure for $k - 1$, if $k - 1 \neq n - 1$ then increase it by 1 and continue this process with $k - 1$ till we reached to $\{1, 2, 3, \dots (k-2)\ (n-1), n\}$ Then move to $(k - 2)$ and repeat the same process. In this manner, moving from right to left we finally reach to an element $a_j$ such that $a_j$ can be increased to $a_{j+1}$ but no $a_j$ with $i > j$ can be increased which means that at some stage $a_i$ is equal to its maximum value $n-k+i$. This procedure terminates when $a_1$ reaches to its maximum value.

Algorithm for generating the next r-combination in lexicographic order.

Next r-combination $(\{a_1, a_2, a_3, \dots a_r\})$ : proper subset of $\{1, 2, 3, \dots n\}$ not equal to

$\{n-r+1, n-r+2, \dots n\}$ : with $a_1 < a_2 < a_3 < \dots a_r)$

$i ; = r$

while $a_i = n - r + i$

$i : = i - 1$

$a_i = a_2 + 1$

for $j = i + 1$ to r

$a_j = a_i + j - i$

**Q.42 Generate all subsets of size 4 of {1, 2, 3, 4, 5, 6}.**

**Ans. :** Let us begin with {1, 2, 3, 4}. We know that for any subset $\{a_1, a_2, a_3, a_4\}$ with $a_1 < a_2 < a_3 < a_4$ the maximum possible value of $a_4$ is 6, $a_3$ is 5, $a_2$ is 4 and $a_1$ is 3.





i) For $a_4 = 4$ :

Hence increasing 4 by 1 we obtain a subset {1, 2, 3, 5}. Since $a_4$ has not still reached to 6.

$\therefore$ Again increase 5 by 1, we get {1, 2, 3, 6}.

ii) For $a_3 = 3$ :

The maximum value of $a_3$ is 5.

$\therefore$ Increase $a_3$ successively by 1 still, we reach to 5.

$\therefore \{1, 2, 4, \underset{\uparrow}{5}\}$ , $\{1, 2, \underset{\uparrow}{4}\ 6\}$ , $\{1, 2, 5, 6\}$

iii) For $a_2 = 2$ : The maximum value of $a_2$ is 4

$\therefore$ We get $(1, \underset{\uparrow}{3}, 4, 5\}$, $\{1, 3, 4, \underset{\uparrow}{6}\}$, $\{1, 3, \underset{\uparrow}{5}, 6\}$, $\{1, 3, \underset{\uparrow}{4}, 6\}$

iv) For $a_1 = 1$ : Maximum value of $a_1$ is 3.

$\therefore$ We get $\{\underset{\uparrow}{2}, 3, 4, 5\}$, $\{2, 3, 4, \underset{\uparrow}{6}\}$, $\{2, 3, \underset{\uparrow}{5}, 6\}$, $\{2, 4, 5, 6\}$, $\{3, 4, 5, 6\}$

Thus we obtain the following 15 subsets

{ {1, 2, 3, 4} {1, 2, 3, 5} {1, 2, 3, 6}

{1, 2, 4, 5} {1, 2, 4, 6} {1, 2, 5, 6}

{1, 3, 4, 5} {1, 3, 4, 6}, {1, 3, 5, 6} {1, 4, 5, 6}

{2, 3, 4, 5}, {2, 3, 4, 6}, {2, 3, 5, 6} {2, 4, 5, 6} {3, 4, 5, 6} }

## 3.4 : Binomial Theorem

**Q.43 What is the coefficient of $x^{13} y^7$ in the expansion of $(x+y)^{20}$ ?**

**Ans. :** we know that the coefficient of $x^{n-r} y^r$ in the expansion of $(x+y)^n$ is

$$^n c_r = \binom{n}{r}$$

Here $n = 20$, $r = 7$, $n - r = 13$

$\therefore$ The required coefficient is

$$\binom{20}{7} = \frac{20\,!}{7\,! \times 13\,!} = \frac{20 \times 19 \times 18 \times 17 \times 16 \times 15 \times 14 \times 13!}{7 \times 6 \times 5 \times 4 \times 2 \times 13!} = 77520$$

**Q.44 What is the coefficient of $x^{12}\, y^{13}$ in the expansion of $(x-y)^{25}$ ?**

**Ans. : By the binomial theorem, the required coefficients**

$$-\binom{25}{12} = \frac{-(25\,!)}{12!\ 13!} = -\ 5{,}200300$$

**Q.45 What is the coefficient of $x^{101}\, y^{99}$ in the expansion of $(2x-3y)^{200}$ ?**

**Ans. :** We have

$$(2x-3y)^{200} = \sum_{r=0}^{200} \binom{200}{r} (2x)^r\, (-3y)^{n-r}$$

Put $r = 101$ at R.H.S. and $x = 1 = y$

$\therefore$ The required coefficient of $x^{101}\, y^{99}$ is

$$= \binom{200}{101} 2^{101}\, (-3)^{99} = -\binom{200}{101} 2^{101}\, (3)^{99}$$

**Q.46 What is the coefficient of $x^{10}$ in the expansion of $(x+\frac{1}{x})^{100}$ ?**

**Ans. :** We have

$$(x+\frac{1}{x})^{100} = \sum_{r=0}^{100} \binom{100}{r} x^r \left(\frac{1}{x}\right)^{100-r} = \sum_{r=10}^{100} \binom{100}{r}\, x^{2r-100}$$

put $2r - 100 = 10 \Rightarrow 2r = 110 \Rightarrow r = 55$

$\therefore$ The coefficient of $x^{10}$ is $\binom{100}{55}$

**Q.47 What is the coefficient of $x^k$ in the expansion of $(x^2 - \frac{1}{x})^{100}$ ? Hence find the coefficient of $x^{51}$**

**Ans. :** We have

$$(x^2 - \frac{1}{x})^{100} = \sum_{r=0}^{100} \binom{100}{r} (x^2)^r\, (-\frac{1}{x})^{100-r}$$





$$= \sum_{r=0}^{100} \binom{100}{r} x^{2r} \left(-\frac{1}{x}\right)^{100-r} = \sum_{r=0}^{100} \binom{100}{r} (-1)^{100-r}\, (x)^{3r-100}$$

Put $3r - 100 = k \Rightarrow r = \frac{1}{3}(k + 100)$

$$\therefore\ (x^2 - \frac{1}{x})^{100} = \sum_{k=-100}^{200} \binom{100}{\frac{1}{3}(k+100)} (-1)^{100-\frac{1}{3}(k+100)}\, (x)^k$$

Thus the coefficient of $x^k$ in $(x^2 - \frac{1}{x})^{100}$ is

$$= \binom{100}{\frac{1}{3}(k+100)} (-1)^{100-\frac{1}{3}(k+100)}$$

**Q.48 What is the coefficient of $x^{20}$ and $x^{21}$ in $(x^2 - \frac{1}{x})^{100}$ ?**

**Ans. :** i) Coefficient of $x^{20}$ :

put $k = 20$ in example (13)

$\therefore$ The required coefficient is $= \binom{100}{40} (-1)^{100-40} = \binom{100}{40}$

ii) Coefficient of $x^{21}$ :

put $k = 21$ in example (13), we get

$$k = 3r - 100 \Rightarrow 3r = k - 100$$

$$3r = 21 - 100 = -\ 79$$

$$r = -\frac{79}{3}$$ which is not possible because $r \in N$ (natural number)

$\therefore$ The required coefficient is zero.

**Q.49 Let n be a nonnegative integer, then S.T. $\sum_{r=0}^{n} \binom{n}{r} = 2^n$ by i) Binomial theorem ii) Combinatorial method**

**Ans. :** i) By Binomial theorem

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n} x^0 y^n$$

Put $x = y = 1$, we get

$$(1+1)^n = 2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$$

$$2^n = \sum_{r=0}^{n} \binom{n}{r}$$

ii) Let A be a set with n elements. Then the power set of A has $2^n$ elements. Now find all subsets by different method.

There are subsets of A with zero elements 1, 2, 3...n elements ($\phi$) is a subset of A with no element.

So there are $\binom{n}{0}$ subsets with no element (zero element) $\binom{n}{1}$ subsets with one element,

$\binom{n}{2}$ subsets with 2 elements... and $\binom{n}{n}$ subset

With n elements $\therefore$ By addition principle, the total number of subsets are given as follows :

$$2^n = \binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n} = \sum_{r=0}^{n} \binom{n}{r}$$

**Q.50 Prove that if n and r integers with $1 \le r \le n$ then**

**$r\binom{n}{r} = n\binom{n-1}{r-1}$ combinatorially.**

**Ans. :** A good mathematician can prove this identity by connecting it to some real life example of set. Suppose we have a set A with n elements, count the number of ways to select a subset with r elements from A and then element of this set by two different methods.

i) The number of ways to select subsets with r elements from set A with n element is $\binom{n}{r}$. The number of ways to select an element of this set is r.

Thus the total number of elements $= r\binom{n}{r}$

ii) The number of ways to select one element from set A is n.

The number of ways to select r - 1 elements from n - 1 elements is $\binom{n-1}{r-1}$





$\therefore$ The total no. of ways $= n\binom{n-1}{r-1}$

Hence $r\binom{n}{r} = n\binom{n-1}{r-1}$

**Q.51 Let n be a non negative integer then $\sum_{r=0}^{n} 2^r \binom{n}{r} = 3^n$**

**Ans. :** By Binomial theorem, putting x = 1, y = 2, we get

$$(1+2)^n = \sum_{r=0}^{n} \binom{n}{r} 1^{n-r}\, 2^r = \sum_{r=0}^{n} 2^r \binom{n}{r}$$

Hence $\sum_{r=0}^{n} 2^r \binom{n}{r} = 3^n$

**Q.52 Let n and r be positive integers with $r \le n$ then prove that**

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

**This is called as Pascal's Identity.**

**Ans. :** Let A be a set with n+1 elements. The number of ways to select subsets with r elements from A is $\binom{n+1}{r}$

Let $x \in A$ and $B = A - \{x\}$

There are two possibilities to select subset with r elements from set B with n elements.

i) The subset of A with r elements contain x with r - 1 elements of B

So there are $\binom{n}{r-1}$ subsets of A that contain x.

ii) The subset of A with r elements without x i.e. select r elements from the set

$B = A - \{x\}$ with n elements.

This can be done in $\binom{n}{r}$ ways.

Thus $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$

**Q.53 Let m, n and r be non negative integers with $0 \le r \le m$ and $0 \le r \le n$ Then $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$**

**This is known as Vandermonde's Identity.**

**Ans. :** Let A and B two disjoint sets with $|A| = m$ and $|B| = n$

$\therefore \quad |A \cup B| = |A| + |B| = m + n$

The total number of ways to select r elements from $A \cup B$ is $\binom{m+n}{r}$

Alternately, to select r elements from $A \cup B$ is to select k elements from set B and $r-k$ elements from set B is $\binom{n}{k}$. The number of ways to select r-k elements from A is $\binom{m}{r-k}$

Therefore by product rule, this can be done in $\binom{m}{r-k}\binom{n}{k}$ ways.

Hence the total number of ways to select r elements from $A \cup B$ is also equals to

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

**Q.54 Let n be a non negative integer then P.T. $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$**

**Ans. :** Put m = n = r in Vandermorde's identity, we get

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2 \qquad \left(\because \binom{n}{n-k} = \binom{n}{k}\right)$$

***END...***



**Unit IV**

# 4 Graph Theory

## 4.1 : Graph Terminology

**Q.1 Define graphs with examples.**

**Ans. : Graphs :** A graph is an ordered pair (V(G), E(G)) where

i) V(G) is non empty finite set of elements known as vertices or nodes. V(G) is called the vertex set.

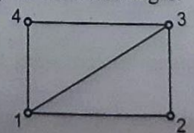
ii) E(G) is a family of unordered pairs (not necessarily distinct) of elements of v, known as edges or arc or branches of G. E(G) is known as edge set.

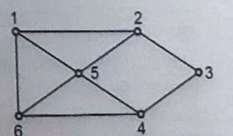
Graphs are so named because they can be represented diagrammatically in the plane.

It is denoted by V(G, E).

a) Each vertex of G is represented by a point or small circle in the plane. In practical examples vertex set may be the set of states or cities or objects etc.

b) Every edge is represented by a continuous curve or straight line segment. Edges may be the route among states or cities or relation among objects etc. Diagrams of road maps, electrical circuits, chemical compounds, job scheduling family trees, all have two objects common namely vertices and edges.

1) 


2) 


Let us consider the following examples of graphs with V(G) and E(G).

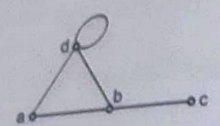
$V(G_1) = \{1, 2, 3, 4\}$

$E(G_1) = \{(1, 2), (1, 3)\ (1, 4)\ (2, 3), (3, 4)\}$

$V(G_2) = \{1, 2, 3, 4, 5, 6\}$

$E(G_2) = \{(1, 2), (1, 5), (1, 6), (2, 5), (2, 3)\ (3, 4), (4, 5), (4, 6), (5, 6)\}$

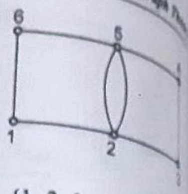
3)




$V(G_3) = \{a, b, c, d\}$

$E(G_3) = \{(a, b), (a, d)\ (b, c)\ (b, d), (d, d)\}$

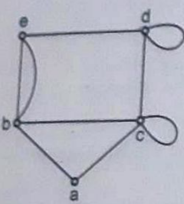
4)




$V(G_2) = \{1, 2, 3, 4, 5, 6\}$

$E(G_4) = \{(1, 2), (1, 6), (2, 3), (2,5), (2, 5), (3, 4), (4, 5), (5, 6)\}$

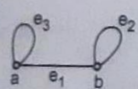
5)




$V(G_5) = \{a, b, c, d\}$

$E(G_5) = \{(a, b), (a, c), (b, e)\ (b, e), (b, c)\ (c, c)\ (c, d)\ (d, d), (d, e)\}$

6)




$V(G_6) = \{a, b\}$

$E(G_6) = \{a, b)\ (a, a)\ (b, b) = \{e_1, e_2, e_3\}$

i) If x and y are two vertices of a graph G and unordered pair $\{x, y\} = (x, y) = e$ is an edge then we say that edge e joins x and y or e is incident to both vertices x and y.

In this case, vertices x and y are said to be incident one e.g. In example (1), $e = (2, 3)$ $\therefore$ e is incident at 2 and 3 and vertices 2, 3 are one incident on $e = (2, 3)$.

ii) Two vertices x and y are said to be adjacent to each other if the pair (x, y) is an edge of G.

If $e = (x, y)$ is an edge of G then x and y are said to be end vertices of e and we can say that e is incident at x and y.

iii) Two edges $e_1$ and $e_2$ are said to be adjacent if they have a common vertex i.e. If $e_1$ and $e_2$ are adjacent then $e_1 = \{x, y\}$ and $e_2 = \{y, z\}$.

iv) An edge joining a vertex to itself is called a loop. E.g. In example (5) there are 2 loops (c, c) and (d, d).





v) A pair of vertices of a graph is joined by two or more edges, such edges are called as multiple or parallel edges.

In example (4) (2, 5) (2, 5) are multiple edges.

## 4.2 : Special Types of Graphs

**Q.2 Explain different types of graphs.**

1) **Multigraph :** A graph in which a pair of vertices is joined by two or more edges is called a multigraph or multiple graph.

i.e. A graph having multiple edges is called a multigraph. In examples (1), (2), (3), (6), graphs are not multigraphs and graphs in examples (4), (5) are multigraphs.

2) **Pseudograph :** A graph having loops but no multiple edges is called a Pseudograph.

Graphs in examples (3), (5) and (6) are pseudographs. A graph having only loops is called a Haary graph. For example :

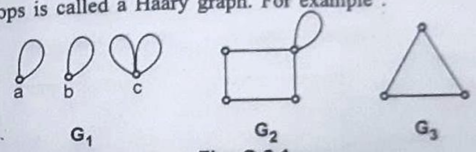
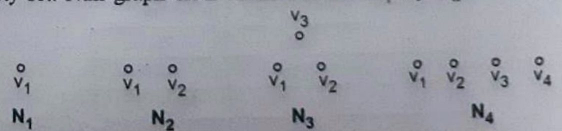



**Fig. Q.2.1**

Graph G is a pseudograph is well as Haary graph. Graph $G_2$ is Pseudograph but not Haary graph. Graph $G_3$ is neither Pseudo not Haary graphs.

3) **Simple graph :** A graph without loops and multiple edges is called a simple graph. Graphs in examples (1) and (2) are simple graphs. Graphs in examples (3), (4), (5), (6) are not simple graphs.

4) **Null graph :** A graph G (V, E) is said to be null graph if E is a empty set. Null graph on n vertices is denoted by $N_n$.




5) **Finite graph :** A graph G (V, E) in which V (x) and E (x) are finite sets is called a finite graph. Otherwise infinite graph.

6) **Directed graph :** A graph G (V, E) is said to be directed graph if the elements of E are an ordered pairs of vertices. E.g.

E = {(a, b), (b, c), (a, c)}

Here (a, c) ≠ (c, a), (c, a) ∉ E (G).

A graph which is not directed is called non-directed graph or graph.

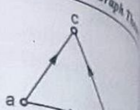



**Fig. Q.2.2 Directed graph**

**7) Weighted graph :** A graph G (V, E) in which some weight is assigned to every edge of G, is called weighted graph.

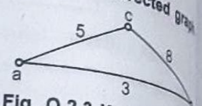



**Fig. Q.2.3 Weighted graph**

**8) Degree of a vertex :**

**a)** In a directed graph G the number of edges ending at vertex v is called the indegree of v. It is denoted by deg $G^+$ (v) or $d^+$ (v)

**b) Outdegree :** In a directed graph G, the number of edges beginning at vertex v is called the outdegree of v. It is denoted by deg $G^-$ (v) or $d^-$ (v).

**c)** The number of edges incident at a vertex v of a graph G with self loops counted twice is called the degree of the vertex v. It is denoted by d(v). A vertex of degree one is called pendent vertex. A vertex of degree zero is called isolated vertex. An edge incident at pendent vertex is called pendent edge.

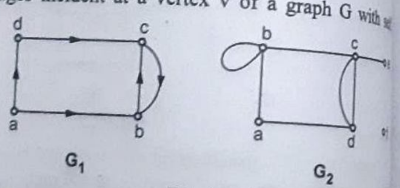



**Fig. Q.2.4**

e.g.

In graph $G_1$,

| Vertices | Indegree | Outdegree |
|---|---|---|
| a | 0 | 2 |
| b | 2 | 1 |
| c | 2 | 1 |
| d | 1 | 1 |

In graph $G_2$,

d(a) = 2 , d(b) = 2 + 2 = 4 , d(c) = 4 , d(e) = 1





d(d) = 3 , d(f) = 0

∴ f is an isolated vertex. e is a pendent vertex. An edge {c, e} is a pendent edge.

9) **Order and size of graph :** The number of vertices in a finite graph G is called the order of G. The number of edges in a finite graph G is called size of the graph. A graph of order n and size m is called (n, m) graph.

If G is a (p, q) graph then G has p vertices and q edges.

10) **Degree sequence of a graph :**

Let G be a graph with vertex set V = $\{v_1, v_2, v_3, \ldots v_n\}$ and $d_i$ = deg $(v_i)$ then the sequence $(d_1, d_2, d_3, \ldots d_n)$ in any order is called the degree sequence of G.

**Note :** 1) Vertices of G are ordered so that degree sequence is monotonically increasing.

2) Two graphs with same degree sequence are called to be degree equivalent. e.g.

d $(v_1)$ = 4, d $(v_2)$ = 3,

d $(v_3)$ = 2, d $(v_5)$ = 5,

d $(v_6)$ = 3), d $(v_4)$ = 1

∴Its degree seq. is (4, 3, 2, 1, 5, 3)

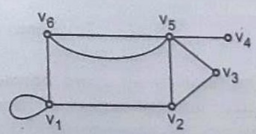



**Fig. Q.2.5**

By relabelling vertices we may write degree sequence as (1, 2, 3, 3, 4, 5).

## 4.3 : Handshaking Lemma

**Q.3 Handshaking lemma : Let G (V, E) be any graph then $\sum_{v \in V} d(v) = 2q$ where q denotes the number of edges of G.**

**Ans. : Proof :** Let us argue by induction on q. Suppose q = 0 i.e. G has no edge i.e. E is an empty set. So d(v) = 0, ∀ v ∈ V.

$\therefore \sum d(v) = 2q = 0.$

Let G be a graph with q > 0 edges. Choose any edge e = {u, v} of G. Consider the graph $G_1$ obtained from G as follows :

i) The vertex set of $G_1$ is same as the vertex set of G i.e.

V $(G_1)$ = V (G) = V = p

ii) The edges of $G_1$ are all edges of G except e.

In other words, $G_1$ is obtained from G by deleting the edge e.

$\therefore$ x is a (p, q − 1) graph.

$\therefore$ By induction principle, result is true for q−1 edges

i.e. $\sum_{x \in V} d(x) = 2(q-1)$

The degree of a vertex x other than u or v in $G_1$ is same as that in G. And the degree of u in $G_1$ is one less than the degree of u in G.

i.e. $d_{G_1}(u) = d_G(u) - 1$

Similarly $d_{G_1}(v) = d_G(v) - 1$

Hence equation (1) becomes

$$\sum_{\substack{x \in V \\ x \neq u, v}} d(x) + d_{G_1}(u) + d_{G_1}(v) = 2(q-1)$$

$$\sum_{\substack{x \in V \\ x \neq u, v}} d(x) + d_{G_1}(u) - 1 + d_G(v) - 1 = 2(q-2)$$

$$\therefore \quad \sum_{x \in V} d(x) = 2q \quad \text{Hence the proof.}$$

The result is so named because it implies that if several people shake hands, the total number of hands shaken must be even as two hands are involved in one handshake.

**Note :** If $\sum_{v \in V} d(v)$ = Odd number then there does not exist any graph with this degree sequence.

## 4.4 : Representing Graphs

**Q.4 Explain matrix representation of a graph with suitable examples.**

**☞ [SPPU : Dec.-09, 10, May-1**

**Ans. : 1. Adjacency Matrix :** Let G be a graph with n vertices and no parallel edges. The adjacency matrix of x is denoted by,

$A(G) = [a_{ij}]_{n \times n}$ and defined as

$a_{ij} = 1$ if $v_i$ and $v_j$ are adjacent

$= 0$ if $v_i$ and $v_j$ are not adjacent.

**Note :** i) A (G) is asymmetric - binary matrix.

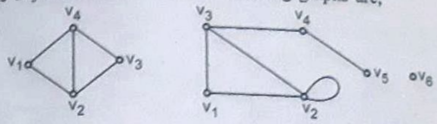




ii) The principal diagonal entries are all zeros if G has no loops.

iii) The $i^{th}$ row sum = $i^{th}$ column sum = $d(v_i)$

e.g. 1) The adjacent matrices of the following graphs are,




$G_1$ $G_2$

Fig. Q.4.1

$$A(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}_{4 \times 4} \end{array}, \quad A(G_2) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6 \times 6} \end{array}$$

The adjacency matrix for a multigraph G is a n × n Matrix $A(G) = [a_{ij}]_{n \times n}$ where

$a_{ij}$ = Number of edges joining $v_i$ and $v_j$

The adjacency matrix of the following graph is

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 & 1 \end{bmatrix} \end{array}$$

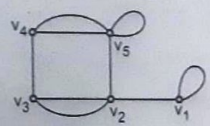



Fig. Q.4.2

**2. Incidence Matrix :** Let G be a graph with n vertices and m edges without self loops. The incidence matrix is denoted by X (G) or I (G) and defined as

$X(G) = [x_{ij}]_{n \times m}$ where

$x_{ij} = 1$ if $j^{th}$ edge is incident on $i^{th}$ vertex $v_i$.

$= 0$ otherwise.

X (G) is a n × m matrix whose n rows correspond to the n vertices m columns correspond to m edges. The graph and it's incidence are given below

$$X(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5 \times 7} \end{array}$$

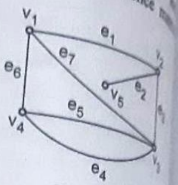



Fig. Q.4.3

**Properties of. Incidence Matrix**

1) It contains only 0 and 1.
2) Each column in the incidence matrix has exactly two 1's appearing in that column.
3) The sum of elements in a row is equal to the degree of the corresponding vertex.
4) Two identical columns correspond to the parallel edges in graph.
5) A row with all 0's represents an isolated vertex.
6) A row with single 1 represents a pendent vertex.

The incidence matrix of a graph with loop is given as follows :

$$X(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{3 \times 3} \end{array}$$

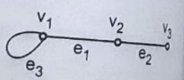



Fig. Q.4.4

**3. Adjacency Matrix of a Diagraph (Directed graph)**

Let G be a directed graph with n vertices and without parallel edges. The adjacency matrix is denoted by,

$$A(D) = [a_{ij}]_{n \times n}$$

where, $a_{ij} = 1$ if there is an edge directed from $v_i$ to $v_j$

$= 0$ otherwise

In network flow, adjacency matrix is also known as connection matrix or transition matrix. The adjancency matrix and diagraph are given below :





$$A(D) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6 \times 6} \end{array}$$

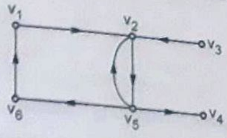



Fig. Q.4.5

**4. Incidence Matrix of Diagraph :** The incidence matrix of a diagraph with n vertices m edges and no self loops is a matrix.

$$X(G) = [x_{ij}]_{n \times m} \text{ where}$$

$x_{ij} = 1$ if $j^{th}$ edge $e_j$ is incident out of $i^{th}$ vertex $v_i$.

$= -1$ if $j^{th}$ edge $e_j$ is incident into of vertex $v_i$.

$= 0$ otherwise.

A graph and its incidence matrix are given below :

$$X(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}_{6 \times 6} \end{array}$$

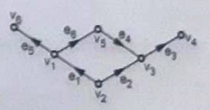



Fig. Q.4.6

**Note :** i) Sum of elements in each column of incidence matrix is zero.

**Examples :**

**Q.5 Show that in a graph the number of vertices of odd degree is even.**

**Ans. :** Let G be a (p – q) graph.

By handshaking lemma $\sum_{v_i \in V} d(v_i) = 2q$

Now separate out vertices of even degree and odd degree

$$\therefore \quad \sum_{v_i \in V} d(v_i) = \sum_{\substack{x \in V \\ \text{even degree}}} d(x) + \sum_{\substack{y \in V \\ \text{odd degree}}} d(y) = 2q$$

$$\therefore \sum_{\substack{v_i \in V \\ \text{odd degree}}} d(v_i) = 2q - \sum_{\substack{x \in V \\ \text{even degree}}} d(x) = \text{Even number}$$

∴ The sum of vertices of odd degree is even.

Hence the number of vertices of odd degree is even.

**Q.6 Show that the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.**

**Ans. :** Let G be a graph with n vertices m edges

∴ By handshaking lemma

$$\sum_{v \in V} d(v) = 2m \rightarrow$$

Let $x \in V$ ∴ x must be adjacent to remaining $(x - 1)$ vertices

$\therefore d(x) = n - 1.\ \forall x \in V$

∴ Equation (1) $\Rightarrow (n-1) + (n-1) + \ldots$ n times $= 2m$

$$n(n-1) = 2m$$

$$m = \frac{n(n-1)}{2}$$

Hence the maximum number of edges in any simple graph with n vertices is $\frac{n(n-1)}{2}$

**Q.7 Determine the number of edges in a graph with 6 nodes, 2 of degree 4 and 4 of degree 2. Draw to such graphs.** ☞[SPPU : Dec.-09]

**Ans. :** Let G be the required graph with 6 nodes and m edges.

∴ By handshaking lemma

$$\sum_{v \in G} d(v) = 2m$$

$$d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) + l(v_6) = 2m$$

$$4 + 4 + 2 + 2 + 2 + 2 = 2m$$

$$2m = 16$$

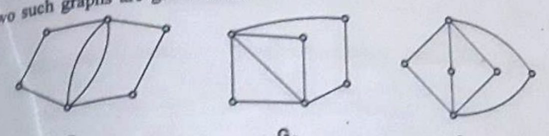
$$m = 8$$

Hence 8 edges are required.





Two such graphs are given below :



$G_1$ $G_2$ $G_3$

**Fig. Q.7.1**

**Q.8 Is it possible to construct a graph with 12 nodes such that 2 of the nodes have degree 3 and the remaining have degree 4.** ☞[SPPU : Dec.-10]

**Ans. :** Let G be the required graph with 12 vertices.

By handshaking lemma.

$$\sum_{v \in V(G)} d(v) = 2m$$

$$(3 + 3) + (4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4) = 2m$$

$$6 + 40 = 2m$$

$$\Rightarrow \quad m = 23$$

∴ It is possible to construct such graph.

**Q.9 Is graph exist for the degree sequence 4, 4, 3, 3, 2, 2, 1.**

**Ans. :** Now apply handshaking lemma

$$\sum d(v) = 2m = \text{Even}$$

$$4 + 4 + 3 + 3 + 2 + 2 + 1 = \text{Even}$$

$$19 = \text{Even which is impossible}$$

∴ Such graph does not exist.

**Q.10 How many simple labelled graphs with n vertices are there ?** ☞[SPPU : May-10]

**Ans. :** We know that a simple graph with n vertices has maximum possible number of edges $\frac{n(n-1)}{2} = m$ (say).

To construct a simple graph with e edges and n vertices, can be done in $\binom{m}{e}$ ways.

i.e. ${}^{m}C_e$ ways where $m = \frac{n(n-1)}{2}$ and $0 \le e \le m$

Hence the total number of ways to construct such graphs is given by,

$$\binom{m}{0}+\binom{m}{1}+\binom{m}{2}+\ldots+\binom{m}{m} = 2^m = 2^{\left(\frac{n(n-1)}{2}\right)} \text{ ways.}$$

**Q.11 Show that a simple graph of order 4 and size 7 does not exist.**

**Ans. :** Let G be a simple graph with 4 vertices.

Then G has at most $\frac{n(n-1)}{2} = \frac{4 \times 3}{2} = 6$ edges.

But given that G has 7 edges which is contradiction.

$\therefore$ there can not be a simple graph with 4 vertices and 7 edges.

**Q.12 Explain i) Regular graph ii) Complete graph iii) Bipartite graph iv) Complete bipartite graph.**

**Ans. : I) Regular Graph :** A graph G is said to be r-regular graph if every vertex of G has degree r.

i) Regular graph of degree zero is called null graph.

ii) A regular graph of degree 3 is called cubic graph.

**e.g.**

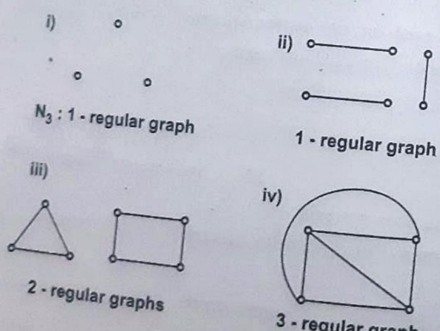



Fig. Q.12.1

**II) Complete Graph :** A simple graph G in which every pair of distinct vertices are adjacent is called a complete graph. If G is a complete graph on n vertices then it is denoted by $K_n$.





In a complete graph, there is an edge between every pair of distinct vertices.

In graph $K_n$, every vertex is adjacent to remaining $n - 1$ vertices so degree of each vertex is $n - 1$.

Thus $K_n$ is a $(n - 1)$ - regular graph.

$K_n$ has exactly $\frac{n(n-1)}{2}$ edges.

Consider the following examples :

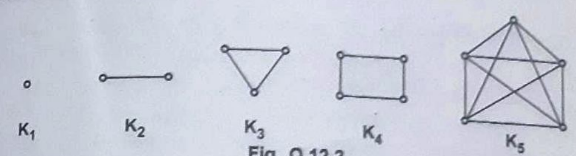



Fig. Q.12.2

**III) Bipartite Graph :** A graph G (v, E) is said to be bipartite graph if its vertex set can be partitioned into two disjoint subsets say $v_1$ and $v_2$ such that $v_1 \cup v_2 = v$ and $v_1 \cap v_2 = \phi$ and every edge of G joins a vertex of $v_1$ to a vertex of $v_2$.

In Bipartite graph, vertices of $v_1$ should not be adjacent. It is free from loops.

Following graphs are bipartite graphs,

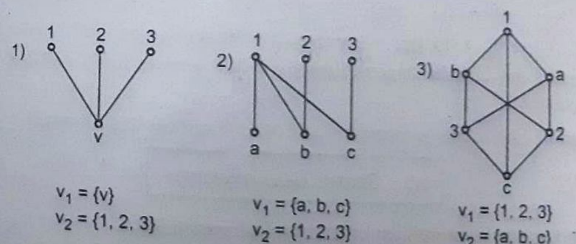



Fig. Q.12.3

**IV) Complete Bipartite Graph :** A bipartite graph G (v, E), $v_1 \cup v_2 = v$ and $v_1 \cap v_2 = \phi$ is said to be complete Bipartite graph if each vertex of $V_1$ is joined to every vertex of $v_2$ by a unique edge.

If $|v_1| = m$, $|v_2| = n$, then the complete bipartite graph $G(v_1 \cup v_2, E)$ ... denoted by $K_{m,n}$

**Examples :**

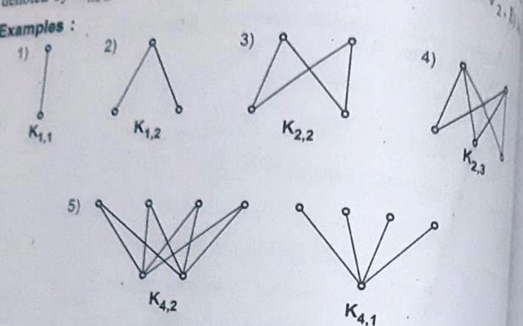



**Fig. Q.12.4**

The graph $K_{1,n}$ is called as star graph.

**Q.13 Is there exist any complete bipartite graph with 7 vertices and 14 edges ?**

**Ans. :** First find all possible bipartitions of 7. They are 6 + 1, 5 + 2, 4 + 3.

We know that, if $G(v_1 \cup v_2, E)$ is a bipartite graph then the number of edges in G is equal to $|v_1| \cdot |v_2|$

i.e. $\quad |E| = |v_1| \cdot |v_2|$

Here $\quad |E| = 14$ But $6 \cdot 1 = 6$, $5 \cdot 2 = 10$, $4 \cdot 3 = 12$

Therefore the complete bipartite graphs with 7 vertices has 6 or 10 or 12 edges only.

Therefore any complete bipartite graph with 7 vertices and 14 edges.

## 4.5 : Graph Isomorphism

**Q.14 Explain isomorphism of graphs with examples.**

**[SPPU : Dec.-12]**

**Ans. :** In real life we come across so many similar objects or figures with respect to size, shape or orientation. Similarly there are a few concepts in graph theory which deal with the similarity of graphs w.r.t. number of vertices or number of edges, number of regions and so on.





Among all such similarities the most important one is an isomorphism of graphs.

**Definition :** Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs. $G_1$ and $G_2$ are said to be isomorphic graphs if

i) There exists a bijective function $\phi : V_1 \rightarrow V_2$

ii) There exists a bijective function $\psi : E_1 \rightarrow E_2$ such that $e = (x, y)$ is an edge in $G_1$ iff $(\phi(x), \phi(y))$ is an edge in $G_2$.

The pair of functions $\phi$ and $\psi$ is called an isomorphism of $G_1$ and $G_2$. It is denoted by $G_1 \cong G_2$.

Suppose two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are isomorphic graphs. Then it is clear that,

i) $|V_1| = |V_2|$ i.e. $G_1$ and $G_2$ must have same number of vertices.

ii) $|E_1| = |E_2|$ i.e. $G_1$ and $G_2$ must have same number of edges.

iii) $G_1$ and $G_2$ must have an equal number of vertices with the same degree.

iv) $G_1$ and $G_2$ must have an equal number of loops.

v) $G_1$ and $G_2$ must have same number of pendent.

vi) $G_1$ and $G_2$ must have same number of pendent edges.

vii) If u and v are adjacent in $G_1$ then the corresponding vertices in $G_2$ are also adjacent.

In general it is easier to prove two graphs are not isomorphic by proving that any one of the above property fails.

Consider the following example

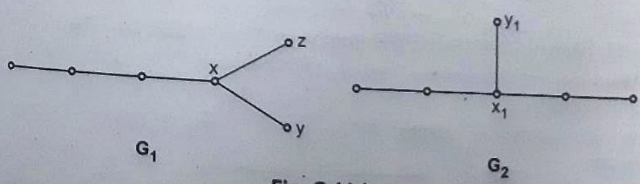



**Fig. Q.14.1**

These graphs have i) The same number of vertices.

ii) The same number of edges.

iii) An equal number of vertices of degree k.

These conditions are necessary for two graphs to be isomorphic but not sufficient.

In graph $G_1$, d (x) = 3, d (y) = 1, d (z) = 1 and x and y are adjacent to vertex x.

In graph $G_2$, d $(x_1)$ = 3, d $(y_1)$ = 1

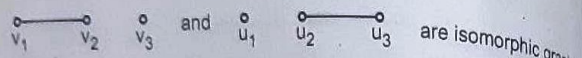
there is only one pendent vertex adjacent to $x_1$. Hence adjacency is not preserved. Therefore $G_1$ is not isomorphic to $G_2$ i.e. $G_1 \not\cong G_2$.

**Note :** Isomorphism of graphs is an equivalence relation.

**Examples :**

**Q.15 Draw all isomorphic graphs on vertices 2 and 3.**

**Ans. : i) For 3 vertices.**

a) 

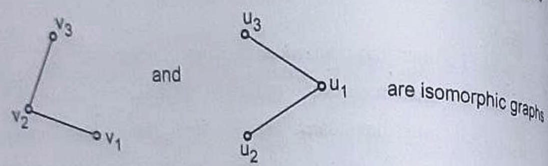

b) 


Fig. Q.15.1

**ii) For two vertices.**

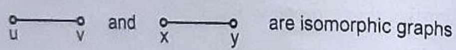



Fig. Q.15.1 (a)

**Q.16 Draw all non isomorphic graphs on 2 and 3 and 4 vertices.**

**Ans. :** All non-isomorphic graphs on 2 vertices are

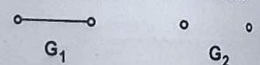



Fig. Q.16.1

All non isomorphic graphs on 3 vertices.

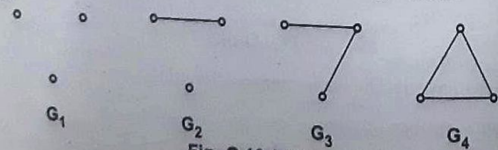



Fig. Q.16.1 (a)





All non isomorphic graphs on 4 vertices.

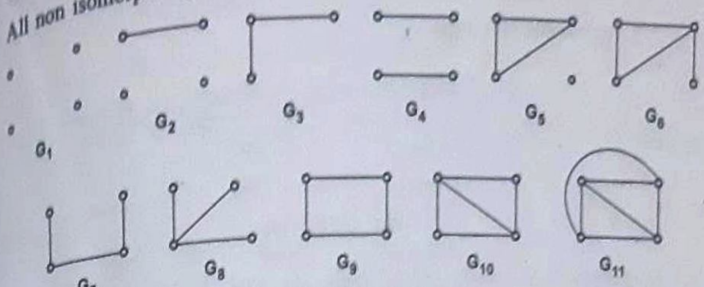



Fig. Q.16.1 (b)

**Q.17 Draw all nonisomorphic graphs on 5 vertices and 5 edges.**

**Ans. :** The following are nonisomorphic graphs with 5 vertices and 5 edges.

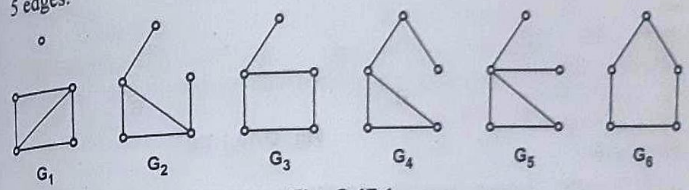



Fig. Q.17.1

**Q.18 Find whether the following pairs of graphs are isomorphic or not.**

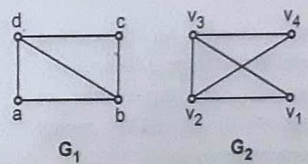



Fig. Q.18.1

**Ans. : i)** Both the graphs have 4 vertices and 5 edges.

Both have 2 vertices of degree 3 and 2 vertices of degree 2.

$\therefore \phi : \{a, b, c, d\} \rightarrow \{v_1, v_2, v_3, v_4\}$ such that

$a \rightarrow v_1$

$c \rightarrow v_4$

$b \rightarrow v_2$

$d \rightarrow v_3$

$\phi$ preserves adjacency and non-adjacency of vertices.

$\phi$ is bijective.

$\therefore$ $G_1$ and $G_2$ are isomorphic graphs.

**Q.22 Explain how to obtain new graphs from old graphs with examples.** [SPPU : May-07, Dec.-09, ...]

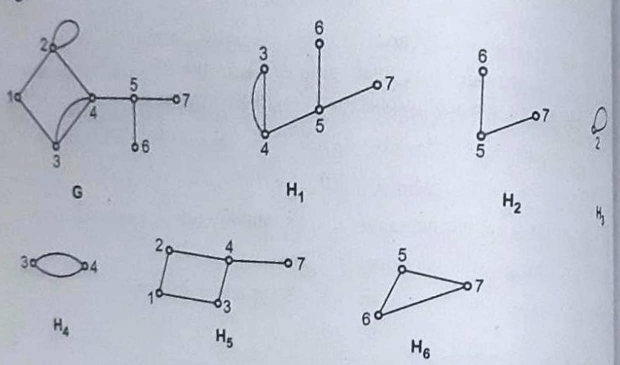
**Ans. :** A good mathematical theory must contain sufficient number of models and examples. Moreover it must have methods to generate new objects from old ones.

In this section we derive new graphs from old graphs.

**1) Subgraphs :** Let G (V, E) be any graph. A graph H $(V_1, E_1)$ is said to be subgraph of G if $V_1 \subseteq V$ and $E_1 \subseteq E$.

We also say that G is a supergraph of H.

e.g.



Graphs $H_1$, $H_2$, $H_3$ and $H_4$ are subgraphs of G. But graphs $H_5$ and $H_6$ are not subgraph as $(4, 7) \in E(H_5)$ but $(4, 7) \notin E(G)$ and $(6, 7) \in E(H_6)$ but $(6, 7) \notin E(G)$.

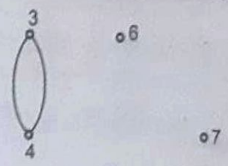
**Properties :**

1) Each graph is a subgraph of itself.
2) A subgraph of a subgraph of a graph G is a subgraph of G.
3) A graph G - {v} is a subgraph of G which is obtained from G by removing the vertex $v \in G$ and also the edges which are incident at v.
4) If $e \in (G)$ then G - e is a subgraph of G obtained from G by deleting the edge e.

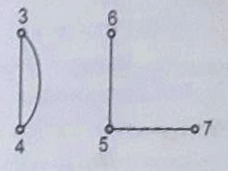




In above example $H_1$ - {5} is



and $H_1$ - (4, 5) is given by



**2) Edge Disjoint Subgraphs :** Two subgraphs $H_1$ and $H_2$ of the graph G are said to be edge disjoint subgraphs of a graph G if there is no edge common between $H_1$ and $H_2$ i.e. $E(H_1) \cap E(H_2) = \phi$ It may have common vertex.

**3) Vertex Disjoint Subgraphs :** Two subgraphs $H_1$ and $H_2$ of the graph G are said to be vertex disjoint subgraphs if there is no vertex common between $H_1$ and $H_2$ i.e. $V(H_1) \cap V(H_2) = \phi$.

**Note :** 1) All vertex disjoint subgraphs are edge disjoint subgraphs.

**4) Spanning Subgraph :** Let G (V, E) be any graph. A subgraph H of a graph G is said to be spanning subgraph if V (G) = V (H).

**Example :** Let G be the following graph :

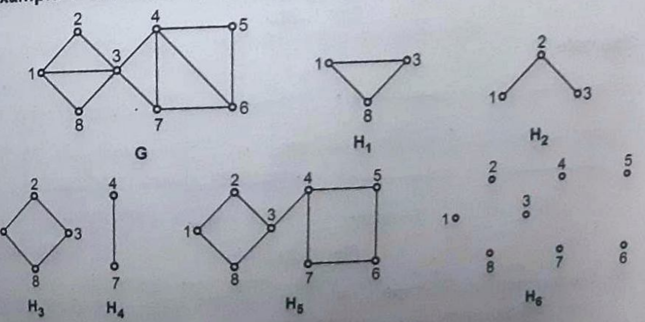


**Fig. Q.22.1**

$\psi \to E_1 \to E_2$ Is bijective.

$\therefore G_1$ is isomorphic to Graph $G_2$.

ii)

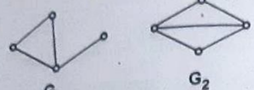



Fig. Q.18.1 (a)

As $G_1$ has 4 edges and $G_2$ has 5 edges, $G_1$ and $G_2$ are not isomorphic graphs.

iii) $G_1$ And $G_2$ are not isomorphic graphs because in $G_1$ vertices $v_1$ and $v_3$ of 4 degree are non adjacent while in $G_2$, the vertices x and y of degree 4 are adjacent.

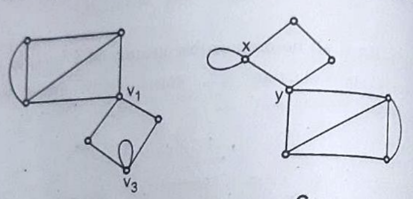



Fig. Q.18.1 (b)

**Q.19 Identify whether the given graphs are isomorphic or not**

☞[SPPU : Dec.-12]

**Ans. :** In graph $G_1$, there are 2 vertices of degree 3. But in $G_2$, there is only one vertex of degree 3. So $G_1$ and $G_2$ are not isomorphic graphs.

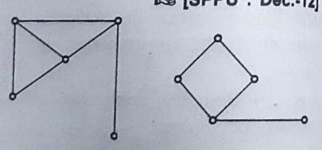



Fig. Q.19.1

ii)

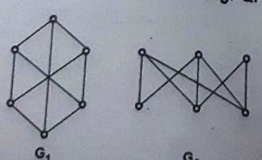



Fig. Q.19.1 (a)





Graph $G_1$ has 9 edges and $G_2$ has 8 edges.

Therefore $G_1$ and $G_2$ are not isomorphic graphs.

**Q.20 Show that the following graphs are isomorphic.**

☞ [SPPU : May-14]

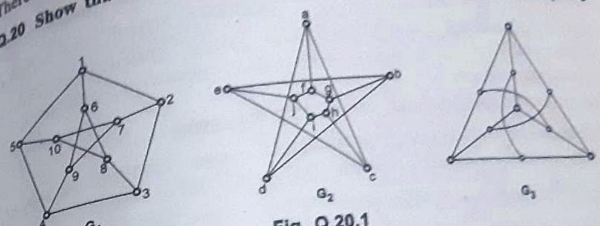



Fig. Q.20.1

**Ans. :** All graphs $G_1$, $G_2$ and $G_3$ have 10 vertices and 15 edges. All these graphs are 3-regular graphs. Also they preserve adjacency. Hence all graphs are isomorphic. Isomorphism is given by

$1 \to f \quad 2 \to g \quad 3 \to h \quad 4 \to i \quad 5 \to j \quad 6 \to a$

$7 \to b \quad 8 \to c \quad 9 \to d \quad 10 \to e$

In the similar way, we can show that $G_1$ and $G_3$ are isomorphic graphs.

**Q.21 Are the graphs isomorphic ? Why ?**

**Ans. :** Given graphs $G_1$ and $G_2$ have 8 vertices and 10 edges.

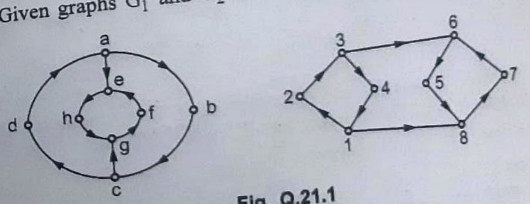



Fig. Q.21.1

Both the graphs have 4 vertices of degree 2 and 4 vertices of degree 3. Also the adjacency is preserved.

$\phi : v(G_1) \to v(G_2)$ is defined as

$a \to 1,\ b \to 2,\ c \to 3,\ d \to 4,\ e \to 8,\ f \to 5,\ g \to 6,\ h \to 7,$

$\phi$ is bijective.

$\therefore$ $G_1$ and $G_2$ are isomorphic graphs.

**Q.22 Explain how to obtain new graphs from old graphs with examples.** [SPPU : May-07, Dec.-09, 12]

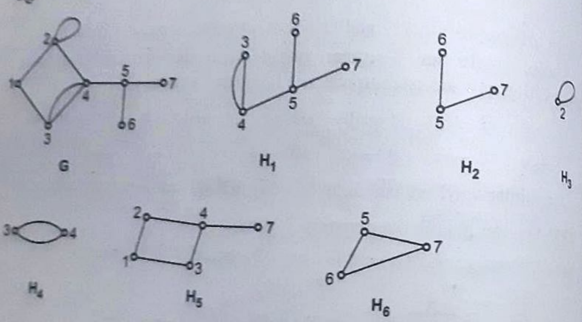
**Ans. :** A good mathematical theory must contain sufficient number of models and examples. Moreover it must have methods to generate new objects from old ones.

In this section we derive new graphs from old graphs.

1) **Subgraphs :** Let G (V, E) be any graph. A graph H $(V_1, E_1)$ is said to be subgraph of G if $V_1 \subseteq V$ and $E_1 \subseteq E$.

We also say that G is a supergraph of H.

e.g.




Graphs $H_1$, $H_2$, $H_3$ and $H_4$ are subgraphs of G. But graphs $H_5$ and $H_6$ are not subgraph as $(4, 7) \in E(H_5)$ but $(4, 7) \notin E(G)$ and $(6, 7) \in E(H_6)$ but $(6, 7) \notin E(G)$.

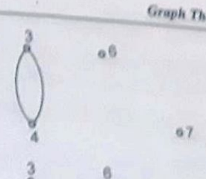
**Properties :**

1) Each graph is a subgraph of itself.
2) A subgraph of a subgraph of a graph G is a subgraph of G.
3) A graph G - {v} is a subgraph of G which is obtained from G by removing the vertex $v \in G$ and also the edges which are incident at v.
4) If $e \in (G)$ then G - e is a subgraph of G obtained from G by deleting the edge e.

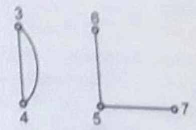




In above example $H_1$ - {5} is



and $H_1$ - (4, 5) is given by



2) **Edge Disjoint Subgraphs :** Two subgraphs $H_1$ and $H_2$ of the graph G are said to be edge disjoint subgraphs of a graph G if there is no edge common between $H_1$ and $H_2$ i.e. $E(H_1) \cap E(H_2) = \phi$ It may have common vertex.

3) **Vertex Disjoint Subgraphs :** Two subgraphs $H_1$ and $H_2$ of the graph G are said to be vertex disjoint subgraphs if there is no vertex common between $H_1$ and $H_2$ i.e. $V(H_1) \cap V(H_2) = \phi$.

**Note :** 1) All vertex disjoint subgraphs are edge disjoint subgraphs.

4) **Spanning Subgraph :** Let G (V, E) be any graph. A subgraph H of a graph G is said to be spanning subgraph if V (G) = V (H).

**Example :** Let G be the following graph :

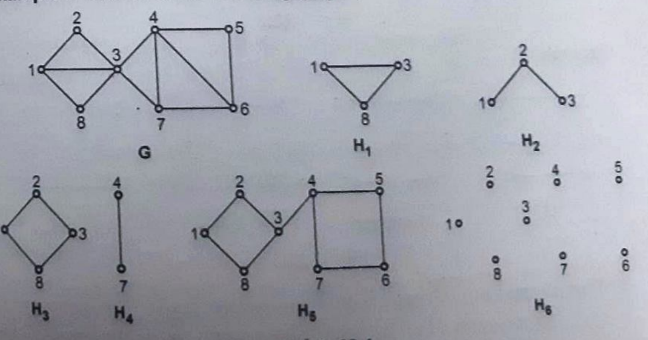



**Fig. Q.22.1**

Graphs $H_1, H_2, \ldots H_6$ are subgraphs of G.

$H_1$ and $H_2$ are edge disjoint subgraphs but not vertex disjoint subgraphs.

$H_3$ and $H_4$ are vertex disjoint subgraphs as well as edge disjoint subgraphs.

Subgraphs $H_5$ and $H_6$ are spanning subgraphs of G as $V(H_5) = V(H_6) = V(G)$.

5) **Factors of a Graph :** Let G be any graph. A k-factor of a graph G is defined to be a spanning subgraph of the graph with the degree of each of its vertex being K. i.e. K-factor is a K-regular graph.

When G has a 1-factor, say $G_1$, if the number of vertices are even and edges of G are point disjoint.

In particular, $K_{2n+1}$ can not have a 1-factor but $K_{2n}$ can have 1-factor of graph.

**Example 1)**

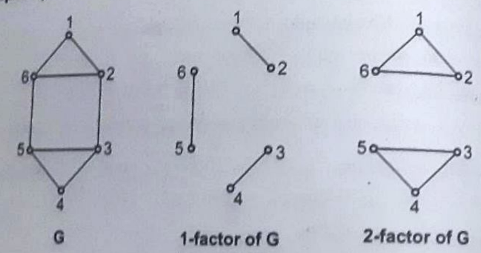



Fig. Q.22.2

**Example 2)**

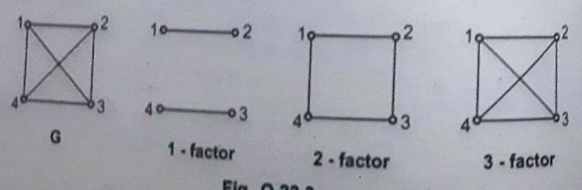



Fig. Q.22.3

6) **Complement of a Graph :** Let G be a simple graph. The complement of G is denoted by $\overline{G}$ is the graph whose vertex set is the same as the

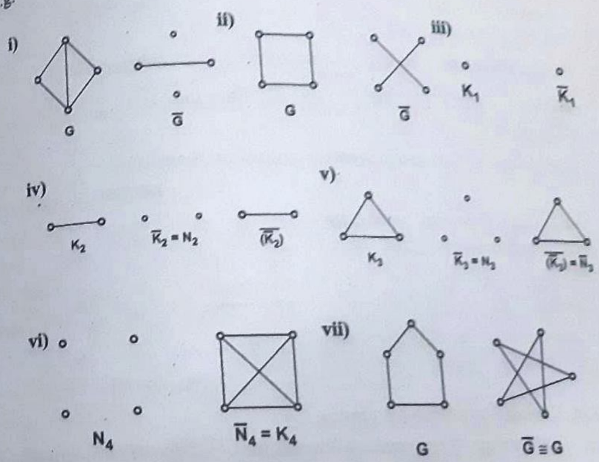




vertex set of G and in which two vertices are adjacent if and only if they are not adjacent in G.

A graph is said to be self complementary graph if it is isomorphic to its complement.

e.g.




G is isomorphic to $\overline{G}$. $\therefore$ G is self complementary graph.

**Note :**

1) For any graph G, $\overline{(\overline{G})} = G$

2) The complement of the null graph on n vertices is the complete graph $K_n$ on n vertices and vice versa.

3) $K_1$ is self complementary graph.

**Examples :**

**Q.23 For the following graphs, determine whether H (V', $E'$) is a subgraph of G where**

**i) V′ = {A, B, C}, E′ = {(A, B), (A, F)}**

ii) $V' = \{B, C, D\}$, $E' = \{(B, C), (B, D)\}$

iii) $V' = \{A, B, C, D\}$,

$E' = \{(A, C)\}$

☞[SPPU : May-07, Dec.09]

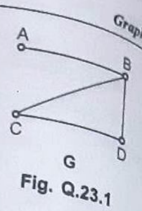


Fig. Q.23.1

**Ans. :** i) H is not a subgraph of G because $F \in V(H)$ but $F \notin V(G)$, so $V(H) \not\subset V(G)$

ii) Here $V' \subset V(G)$, $E' \subset E(G)$, so $H(V', E')$ is a subgraph of G.

iii) Here $V' \subset v(G)$, but $E' \not\subset E(G)$ Therefore $H(V', E')$ is not a subgraph of G.

**Q.24 Draw all self complementary graphs on 5 vertices.**

☞[SPPU : Dec.-12]

**Ans. :** The following graphs are self complementary graphs on 5 vertices.

Here $\overline{G_1} = G_2$ and $\overline{G_2} = G_1$

$\therefore$ $G_1$ as well as $G_2$ are self complementary graphs.

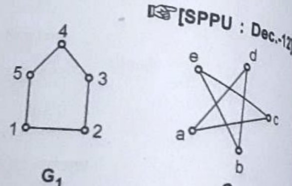


Fig. Q.24.1

**Q.25 Explain operations on graphs**

**Ans. :** We define some standard operations of graphs like intersection, union, ringsum etc.

**A) Intersection of Two Graphs :** The intersection of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ is a graph G (V, E) whose vertex set is $V = V_1 \cap V_2$ and edge set is $E = E_1 \cap E_2$. The intersection of $G_1$ and $G_2$ is denoted by $G_1 \cap G_2$.

e.g.

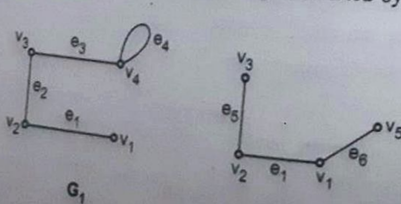


Fig. Q.25.1





$V_1 = \{v_1, v_2, v_3, v_4\}$ $V_2 = \{v_1, v_2, v_3, v_5\}$

$E_1 = \{e_1, e_2, e_3, e_4\}$ $E_2 = \{e_1, e_5, e_6\}$

Therefore $G = G_1 \cap G_2$ (v, E) where

$V = V_1 \cap V_2 = \{v_1, v_2, v_3\}$,

$E = E_1 \cap E_2 = \{e_1\}$

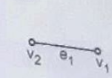


Fig. Q.25.2

**B) Union of Two Graphs :** Let $G_1(V_1, E_1)$, $G_2(V_2, E_2)$ be two graphs. The union of $G_1$ and $G_2$ is denoted by $G_1 \cup G_2 = G$ (v, E) and it is a graph whose vertex set is $V = V_1 \cup V_2$ and Edge set is

$E = E_1 \cup E_2$

Consider the graphs $G_1$ and $G_2$ as shown in above example :

The union of $G_1$ and $G_2$ is given by G (v, E) where $V = V_1 \cup V_2 = \{v_1, v_2, v_3, v_4, v_5\}$

$V = V_1 \cup V_2 = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

**Note :** Both graphs $G_1$ and $G_2$ are subgraphs of $G_1 \cup G_2$.

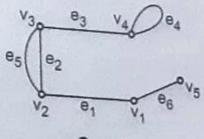


Fig. Q.25.3

**C) The Ring Sum of Two Graphs :** The ring sum of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ is denoted by $G = G_1 \oplus G_2$ (V, E) whose vertex set is $V = V_1 \cup V_2$ and the edge set consists of those edges which are either in $E_1$ or in $E_2$ but not in both i.e. $E = (E_1 \cup E_2) - (E_1 \cap E_2)$.

The ring sum of above graphs $G_1$ and $G_2$ is given by $G(V, E) = G_1 \oplus G_2$

$V = \{v_1, v_2, v_3, v_4, v_5\} = V_1 \cup V_2$

$E = (E_1 \cup E_2) - (E_1 \cap E_2)$

$= \{e_2, e_3, e_4, e_5, e_6\}$

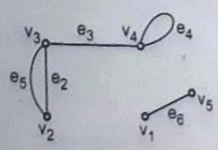


Fig. Q.25.4

**D) Sum of Two Graphs :** The sum of two vertex disjoint graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ is denoted by $G_1 + G_2 = G$ (V, E) is defined as the graph whose vertex set is $V(G_1 \cup G_2)$ and consisting of edges which are in $G_1$ or $G_2$ together with the edges obtained by joining each vertex of $G_1$ to each vertex of $G_2$. Thus $G_1 + G_2$ is nothing but the graph $G_1 \cup G_2$ in which each vertex of $G_1$ is joined to each vertex of $G_2$ by an edge.

e.g. If

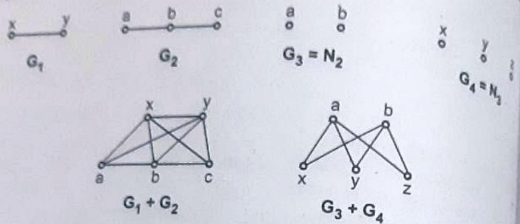



Fig. Q.25.5

**Note :** The sum $N_m + N_n$ of null graphs is nothing but the complete bipartite graph $K_{m, n}$.

**E) Product of Two Graphs :** Let $G_1 (V_1, E_1)$ and $G_2 (V_2, E_2)$ be two vertex disjoint graphs then the product of $G_1$ and $G_2$ is denoted by $G_1 \times G_2 = G (V, E)$ is a graph whose vertex set is $V = V_1 \times V_2$ and two edges $(x_1, x_2)$ and $(y_1, y_2)$ are adjacent if $x_1 = y_1$ and $x_2$ is adjacent to $y_2$ in $G_2$ or $x_2 = y_2$ and $x_1$ is adjacent to $y_1$ in $G_1$.

e.g. If

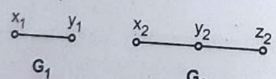



Fig. Q.25.6

Then $G_1 \times G_2$ is given below :

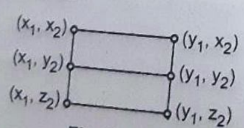



Fig. Q.25.7

**F) Decomposition :** A graph G is said to have been decomposed into two subgraphs H and K if $H \cup K = G$ and $H \cap K =$ Null graph i.e. each edge of G occurs either in H or in K but not in both. But vertices may occur in both. In this context isolated vertices are not considered.





e.g. The decomposition of G into H and K is given below :

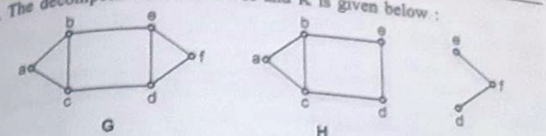



Fig. Q.25.8

**G) Fusion of Vertices :** A pair of vertices a and b in a graph G are said to be fused if a and b are replaced by a single new vertex say c such that every edge that was incident on either a or b or both is incident on the new vertex c. The fusion of two vertices do not change the number of edges but reduced number of vertices by 1.

e.g.

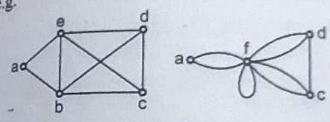



Fig. Q.25.9

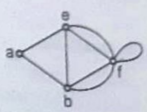



Fig. Q.25.10

**Q.26 Paths and circuits with examples.**

**1) Path :** An alternating sequence of vertices and edges $v_0 - e_1 - e_2 - e_3 - \ldots v_{n-1} - e_n - v_n$ beginning are ending with vertices in which each edge is incident with the two vertices immediately preceding if and following it is called a path.

The vertices $v_0$ and $v_n$ are called terminal vertices and $v_1, v_2, \ldots v_{n-1}$ are called its interior vertices.

e.g. Let G be the following graph.

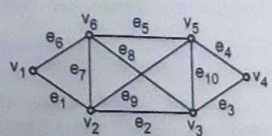



Fig. Q.26.1

Following are some examples of path

i) $v_1 - e_1 - v_2 - v_3$

ii) $v_2 - e_2 - v_3 - e_3 - v_4 - e_4 - v_5 - e_{10} - v_3 - e_2 - v_2$

iii) $v_6 - e_5 - v_5 - e_{10} - v_3 - e_8 - v_6$

iv) $v_1 - e_6 - v_6$

There are so many paths between every distinct pair of vertices of graph G. Depending upon the nature of terminal vertices, there are types at path.

A path in which terminal vertices are equal is called a **closed path**. closed path is known as circuit. A path in which terminal vertices are distinct, is called an **open path**.

In above examples, paths in (i) and (iv) are open paths and (ii) and (iii) are closed paths.

**1. Simple Path :** A path in a graph G is said to be a simple path if the edges do not repeat in the path. Vertices may be repeated.

e.g. i) $v_1 - e_1 - v_2 - e_2 - v_3 - e_{10} - v_5$ is a simple path.

ii) $v_1 - e_1 - v_2 - e_2 - v_3 - e_{10} - v_5 - e_9 - v_2 - e_7 - v_6$ is a simple path in which $v_3$ is repeated.

iii) $v_3 - e_3 - v_4 - e_4 - v_5 - e_{10} - v_3 - e_8 - v_6$ is a simple path in which $v_3$ is repeated.

iv) $v_3 - e_3 - v_4 - e_4 - v_5 - e_{10} - v_3 - e_3 - v_4$ is not a simple path as an edge $e_3$ is repeated.

**2. Elementary Path :** A path in a graph G is said to be elementary path if vertices do not repeat in the path. Every elementary path is a simple path.

e.g. i) $v_1 - e_1 - v_2 - e_2 - v_3$ is an elementary path.

ii) $v_1 - e_1 - v_2 - e_2 - v_3 - e_8 - e_6 - e_7 - v_2$ is not an elementary path. But it is simple path.

**3. Simple and Elementary Circuits :** A closed path is known as circuit.

A simple path which is closed is called a simple circuit of graph.

In other words, A circuit in a graph G is said to be simple circuit if all edges of a circuit are distinct.

A circuit in a graph G is said to be elementary circuit if all vertices of a circuit are distinct except the terminal vertices i.e. the first and last vertices. The number of edges in any circuit (or path) is called the length of the circuit (or path).

In above graph G.

e.g. i) $v_1 - e_1 - v_2 - e_2 - v_3 - e_{10} - v_5 - e_9 - v_2 - e_1 - v_1$ is a circuit with $e_1$ repeated twice and $v_2$ is also repeated twice.





ii) $v_1 - e_1 - v_2 - e_2 - v_3 - e_{10} - v_5 - e_9 - v_2 - e_7 - v_6 - e_6 - v_1$ is a simple circuit but not elementary circuit as $v_2$ is repeated.

iii) $v_1 - e_1 - v_2 - e_7 - v_6 - e_6 - v_1$ is an elementary circuit.

**Q.27 Define connected and disconnected graphs**

**Ans. :** A graph G is said to be connected graph if there exists a path between every pair of vertices. A graph which is not connected is called the disconnected graph.

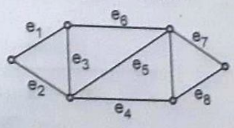
A disconnected graph consists of two or more parts called components or blocks if each of which is connected and there is no path between two vertices if they belong to different components.

## 4.6 : Connectivity

**Q.28 Explain edge and vertex connectivity.**

**Ans. : Edge Connectivity :** A set of edges of a connected graph G whose removal disconnects G is called a disconnecting set of G. A cutset is defined as a minimal disconnecting set i.e. A minimal set of edges whose removal from G gives a disconnected graph is called a cutset.

If a cutset of a graph contains only one edge, then that edge is called as an isthmus or bridge. The number of edges in the smallest cutset of G is called the edge connectivity of G. It is denoted by $\lambda$ (G).



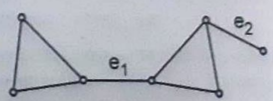

G

Fig. Q.28.1

e.g. Consider the following graph G.

Cutsets of G are as follows :

i) $\{e_4, e_5, e_6\}$, ii) $\{e_1, e_3, e_6\}$, iii) $\{e_1, e_2\}$,




$G_1$

Fig. Q.28.2

A set $\{e_1, e_2, e_3\}$ is not a cutset because its subset $\{e_1, e_2\}$ is a cutset. The edge connectivity of graph G is 2. i.e. $\lambda (G) = 2$.

Consider the following graph $G_1$.

Graph $G_1$ has edge connectivity 1 as $G_1 - \{e_1\}$ is a disconnected graph. $\therefore$ $e_1$ is an isthmus or Bridge.

$G - e_2$ is also disconnected graph. $\therefore$ $e_2$ is also isthmus.

**2) Vertex Connectivity :** The vertex connectivity k(G) of a simple connected graph G is defined as the smallest number of vertices whose removal disconnects the graph.

In graph G, the sets $\{v_2, v_5, v_4\}$, $\{v_2, v_3, v_4, v_5\}$, $\{v_2, v_3\}$ disconnect graph G. The smallest set is $\{v_2, v_3\}$

$\therefore$ k (G) = 2.

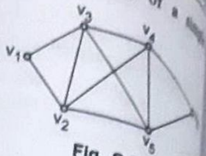



**Fig. Q.28.3**

1) A graph G is said to be k-connected if its vertex connectivity is k.

2) A graph G is said to be seperable graph if its vertex connectivity is one.

3) A vertex v of a connected graph G is said to be cut vertex if G - {v} is a disconnected graph.

4) $k(G) \leq \lambda(G) \leq \delta$

i.e. vertex connectivity ≤ edge connectivity ≤ minimum degree of a vertex in G and

$$\lambda(G) \leq \left[\frac{2e}{n}\right]$$

where e = Number of edges in G. n = Number of vertices in G.

## 4.7 : Single Source Shortest Path Algorithm and Dijkstra's Algorithm

**Q.29 Explain shortest path algorithm and Dijkstra's algorithm.**

**☞[SPPU : May-05, 07, 14, 15, Dec.-06, 07, 12, 13, 14, 15]**

**Ans. :** Suppose there is associated to each edge e of a graph a real number w(e). w(e) is called the weight of e. A weighted graph is a graph in which each edge has a weight. The weight of graph G is the sum of weight of all edges of G. Weighted graph has many applications in communication networks. Given a railway network connecting several cities, determine a shortest route between two cities.

We consider the weighted graph where the vertices are the towns, rail roads are the edges and the weight represent the distance between directly linked cities. Therefore weights are non negative integers. The problem is to find a path of minimum weight connectivity two given cities. Of course, this is possible theoretically. One has to list all paths, find their weights and select minimum one. But for large networks (large number of





vertices and edges) this may not be efficient. So we required different method to take such problems. The algorithm was found by Dijkshtra (1959) and is known as Dijkshtra's algorithm.

A) Dijkshtra's algorithm to find the shortest path from the vertex a to vertex z of a graph G. Let G (v, E) be a simple graph and a, z ∈ V.

Suppose L(x) is the label of the vertex which represents the length of the shortest path from the vertex a. $W_{ij}$ = Weight of an edge $e_{ij} = (v_i, v_j)$

Consider the following steps :

**Step 1 :** Let P be the set of those vertices which have permanent labels and T = set of all vertices of G.

Set L (a) = 0, L (x) = ∞ ; ∀ x ∈ T and x ≠ a

P = ϕ and T = v.

**Step 2 :** Select the vertex v in T which has the smallest label. This label is called the permanent label of v. Also set P as P ∪ {v} and T as T - {v}.

If v = z then L (z) is the length of the shortest path from the vertex a to z and stop the procedure.

**Step 3 :** If v ≠ z, then revise the labels of the vertices of T. i.e. The vertices which do not have permanent labels.

The new label of x in T is given by

$$L(x) = \min \{\text{old } L(x), L(v) + w(v, x)\}$$

where w (v, x) is the weight of the edge joining v and x. If there is no edge joining v and x then take w (v, x) = ∞.

**Step 4 :** Repeat the steps 2 and 3 until z gets the permanent label.

**Examples :**

**Q.30 Use Dijkstra's algorithm to find the shortest path between a and z.**

**☞[SPPU : May-05, 14, 8 Marks, Dec.-06, 6 Marks]**

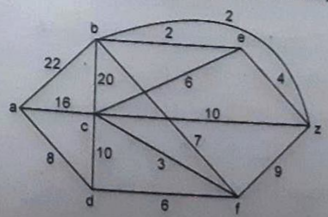



**Fig. Q.30.1**

**Ans. :**

**Step 1 :** $P = \varnothing, \quad T = \{a, b, c, d, e, f, z\}$

$L\{a\} = 0$

$L\{x\} = \infty, \quad \forall\, x \in T, \quad x \neq a$

**Step 2 :** v = a, the permanent label of a is 0

$P = \{a\}, \quad T = \{b, c, d, e, f, z\}$

$L\{b\} = \min\{\text{old } L(b), L(a) + w(a, b)\}$

$= \min\{\infty, 0 + 22\} = 22$

$L\{c\} = \min\{\infty, 0 + 16\} = 16$

$L\{d\} = \min\{\infty, 0 + 8\} = 8$

$L\{e\} = \min\{\infty, 0 + \infty\} = \infty$

$L\{f\} = \min\{\infty, 0 + \infty\} = \infty$

$L\{z\} = \min\{\infty, 0 + \infty\} = \infty$

$\therefore$ $L\{d\} = 8$ is the minimum label.

**Step 3 :** v = d, the permanent label of d is 8.

$P = \{a, d\}, \quad T = \{b, c, e, f, z\}$

$L\{b\} = \min\{\text{old } L(b), L(d) + w(d, b)\}$

$= \min\{22, 8 + \infty\} = 22$

$L\{c\} = \min\{16, 8 + 10\} = 16$

$L\{e\} = \min\{\infty, 8 + \infty\} = \infty$

$L\{f\} = \min\{\infty, 8 + 6\} = 14$

$L\{z\} = \min\{\infty, 8 + \infty\} = \infty$

$\therefore$ $L\{f\} = 14$ is the minimum label.

**Step 4 :** v = f, the permanent label of f is 14.

$P = \{a, d, f\}, \quad T = \{b, c, e, z\}$

$L\{b\} = \min\{\text{old } L(b), L(f) + w(b, f)\}$

$= \min\{22, 14 + 7\} = 21$

$L\{c\} = \min\{16, 14 + 3\} = 16$

$L\{e\} = \min\{\infty, 14 + \infty\} = \infty$





$L\{z\} = \min\{\infty, 14 + 9\} = 23$

$L\{c\} = 16$ is the minimum label.

$\therefore$

**Step 5 :** v = c, the permanent lable of c is 16.

$P = \{a, d, f, c\}, \quad T = \{b, e, z\}$

$L\{b\} = \min\{\text{old } L(b), L(f) + w(f, b)\}$

$= \min\{21, 16 + 20\} = 21$

$L\{e\} = \min\{\infty, 16 + 6\} = 22$

$L\{z\} = \min\{23, 16 + 10\} = 23$

$L\{b\} = 2$ is the minimum label.

$\therefore$

**Step 6 :** v = b, the permanent label of b is 21.

$P = \{a, d, f, c, b\}, \quad T = \{e, z\}$

$L\{e\} = \min\{\text{old } L(e), L(b) + w(e, b)\}$

$= \min\{22, 21 + 2\} = 22$

$L\{z\} = \min\{23, 21 + 2\} = 23$

$L\{e\} = 22$ is the minimum label.

$\therefore$

**Step 7 :** v = e, the permanent label of e is 22.

$P = \{a, d, f, c, b, e\}, \quad T = \{z\}$

$L(z) = \min\{\text{old } L(z), L(e) + w(e, z)\}$

$= \min\{23, 22 + 4\} = 23$ which is the minimum label

**Step 8 :** v = z, the permanent label of z is 23.

Hence the length of the shortest path from a to z is 23.

The shortest path is adfz or adfbz.

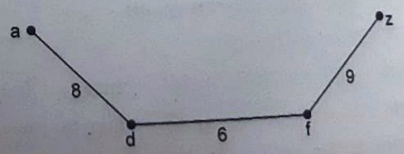



**Fig. Q.30.1 (a)**

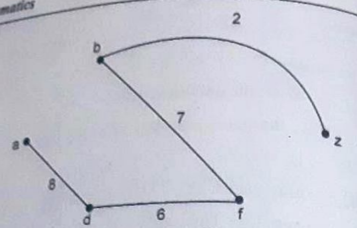



Fig. Q.30.1 (b)

**Q.31 Find the shortest path from a-z in the given graph using Dijkstra's algorithm.**

**☞[SPPU : May-07, Dec.-07, 10]**

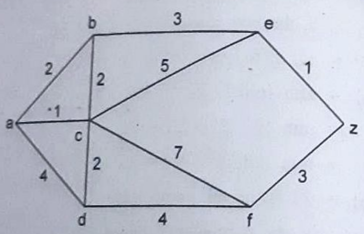



Fig. Q.31.1

**Ans. : Step 1 :**

Set $P = \varnothing$, $T = \{a, b, c, d, e, f, z\}$

$L\{a\} = 0$

$L\{x\} = \infty, \ \forall x \in T, \ x \neq a$

**Step 2 :** $v = a$, the permanent label of a is 0.

$P = \{a\}$, $T = \{b, c, d, e, f, z\}$

$L\{b\} = \min\{\text{old } L(b), L(a) + w(a, b)\}$

$= \min\{\infty, 0 + 2\} = 2$

$L\{c\} = \min\{\infty, 0 + 1\} = 1$ $\quad L\{d\} = \min\{\infty, 0 + 4\} = 4$

$L\{e\} = \min\{\infty, 0 + \infty\} = \infty$ $\quad L\{f\} = \min\{\infty, 0 + \infty\} = \infty$

$L\{z\} = \min\{\infty, 0 + \infty\} = \infty \therefore L\{c\} = 1$ is the minimum label.





**Step 3 :** $v = c$, the permanent label of c is 1.

$P = \{a, c\}$, $T = \{b, d, e, f, z\}$

$L\{b\} = \min\{2, 1 + 2\} = 2$ $\quad L\{d\} = \min\{4, 1 + 2\} = 3$

$L\{e\} = \min\{\infty, 1 + 5\} = 6$ $\quad L\{f\} = \min\{\infty, 1 + 7\} = 8$

$L\{z\} = \min\{\infty, 1 + \infty\} = \infty \therefore L\{b\} = 2$ is the minimum label.

**Step 4 :** $v = b$, the permanent label of b is 2.

$P = \{a, c, b\}$, $T = \{d, e, f, z\}$

$L\{d\} = \min\{3, 2 + \infty\} = 3$ $\quad L\{e\} = \min\{6, 2 + 3\} = 5$

$L\{f\} = \min\{8, 2 + \infty\} = 8$ $\quad L\{z\} = \min\{\infty, 2 + \infty\} = \infty$

$\therefore L\{d\} = 3$ is the minimum label.

**Step 5 :** $v = d$, the permanent label of d is 3.

$P = \{a, c, b, d\}$, $T = \{e, f, z\}$

$L\{e\} = \min\{5, 3 + \infty\} = 5$ $\quad L\{f\} = \min\{8, 3 + 4\} = 7$

$L\{z\} = \min\{\infty, 3 + \infty\} = \infty \therefore$ $\quad L\{e\} = 5$ is the minimum label.

**Step 6 :** $v = e$, the permanent label of e is 5.

$P = \{a, c, b, d, e\}$, $\quad T = \{f, z\}$

$L\{f\} = \min\{7, 5 + \infty\} = 7$ $\quad L\{z\} = \min\{\infty, 5 + 1\} = 6$

$\therefore L\{z\} = 6$ is the minimum label.

**Step 7 :** $v = z$, the permanent label of z is 6.

Hence the length of shortest path from a to z is 6.

The shortest path is a b e z.

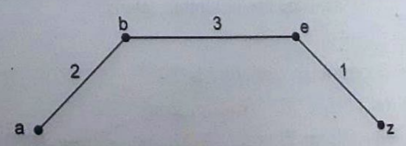



Fig. Q.31.1 (a)

**Q.32 Find the shortest path between a-z for the given graph : using Dijkstra's algorithm.**

**☞[SPPU : Dec.-12, 13, 14, May-15, Marks 8]**

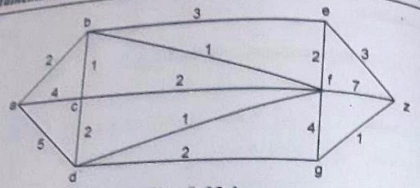



Fig. Q.32.1

**Ans. :**

**Step 1 :** Set $P = \emptyset$, $T = \{a, b, c, d, e, f, g, z\}$

$L\{a\} = 0$

$L\{x\} = \infty, \quad \forall\, x \in T, \quad x \neq a.$

**Step 2 :** v = a, the permanent label of a is 0.

$P = \{a\}, \quad T = \{b, c, d, e, f, g, z\}$

$L\{b\} = \min\{\text{old } L(b),\ L(a) + w(a, b)\} = \min\{\infty, 0 + 2\} = 2$

$L\{c\} = \min\{\infty, 0 + 4\} = 4 \qquad L\{d\} = \min\{\infty, 0 + 5\} = 5$

$L\{e\} = L\{f\} = L\{g\} = L\{z\} = \infty$

$\therefore$ $L\{b\} = 2$ is the minimum label. The permanent label of b is 2.

**Step 3 :** v = b

$P = \{a, b\}, \quad T = \{c, d, e, f, g, z\}$

$L\{c\} = \min\{L(c), L(b) + w\{b, c\}\} = \min\{4, 2 + 1\} = 3$

$L\{d\} = \min\{5, 2 + \infty\} = 5 \qquad L\{e\} = \min\{\infty, 2 + 3\} = 5$

$L\{f\} = \min\{\infty, 2 + 1\} = 3 \qquad L\{g\} = L\{z\} = \infty$

$\therefore$ $L\{c\} = L\{f\} = 3$ are the minimum labels.

**Step 4 :** v = c or f

Let v = f, permanent label of f is 3.

$P = \{a, b, f\}, \quad T = \{c, d, e, g, z\}$

$L\{c\} = \min\{3, 3 + 2\} = 3 \qquad L\{d\} = \min\{5, 3 + 1\} = 4$

$L\{e\} = \min\{5, 3 + 2\} = 5 \qquad L\{g\} = \min\{\infty, 3 + 4\} = 7$

$L\{z\} = \min\{\infty, 3 + 7\} = 10$

$\therefore$ $L\{c\} = 3$ is the minimum label.





**Step 5 :** v = c, permanent label of c = 3.

$P = \{a, b, f, c\}, \quad T = \{d, e, g, z\}$

$L\{d\} = \min\{4, 3 + 2\} = 4 \qquad L\{e\} = \min\{5, 3 + \infty\} = 5$

$L\{g\} = \min\{7, 3 + \infty\} = 7 \qquad L\{z\} = \min\{10, 3 + \infty\} = 10$

$\therefore$ $L\{d\} = 4$ is the minimum label.

**Step 6 :** v = d, permanent label of d = 4.

$P = \{a, b, f, c, d\}, \quad T = \{e, g, z\}$

$L\{e\} = \min\{5, 4 + \infty\} = 5 \qquad L\{g\} = \min\{7, 4 + 2\} = 6$

$L\{z\} = \min\{10, 4 + \infty\} = 10$ $\therefore$ $L\{e\} = 5$ is the minimum label.

**Step 7 :** v = e, permanent label of e is 5.

$P = \{a, b, f, c, d, e\}, \quad T = \{g, z\}$

$L\{g\} = \min\{6, 5 + \infty\} = 6 \qquad L\{z\} = \min\{10, 5 + 3\} = 8$

$\therefore$ $L\{g\} = 6$ is the minimum label.

**Step 8 :** v = g, permanent label of g is 6.

$P = \{a, b, f, c, d, e, g\}, \quad T = \{z\}$

$L\{z\} = \min\{8, 6 + 1\} = 7$ which is the minimum label.

**Step 9 :** v = z, permanent label of z is 7.

Hence the length of shortest path from a to z is 7.

The shortest path is a b f d g z.

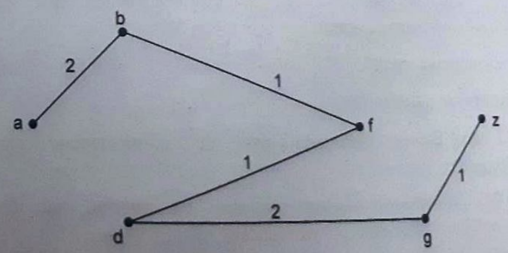



Fig. Q.32.1 (a)

## 4.8 : Eulerian and Hamilton Paths

**Q.33 Define eulerian path and circuit.**

**Ans. :** A path is called an Eulerian path if every edge of graph G appears exactly once in the path.

A circuit of a graph which contains every edge of graph exactly once is called the Eulerian circuit.

A graph which has an Eulerian circuit is called as Eulerian graph.

The problem of find Eulerian path is the same as the problem of drawing a network without lifting the pencil off the paper and without retracing an edge.

Consider the following graphs :

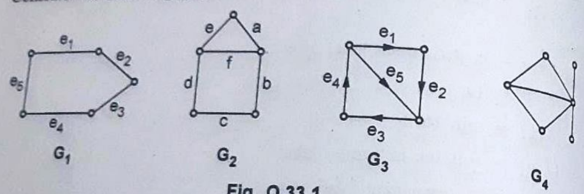



**Fig. Q.33.1**

In graph $G_1$, Eulerian circuit is $e - e_2 - e_3 - e_4 - e_5 - e_1$

$\therefore G_1$ is an Eulerian graph.

In graph $G_2$, Eulerian circuit does not exist.

$\therefore G_2$ is not Eulerian graph.

In graph $G_3$, Eulerian path is $e_1 \to e_2 \to e_3 \to e_4 \to e_4 \to e_5$ but $G_3$ does not have any Eulerian circuit.

$\therefore G_3$ is not Eulerian graph.

$G_4$ is also not an Eulerian graph.

The existence of Eulerian paths and circuits in a graph depends upon the degree of vertices.

**Theorem 1 :** An undirected graph possesses an Eulerian path iff it is connected and has either zero or two vertices of odd degree.

**Theorem 2 :** An undirected graph possesses an Eulerian circuit iff it is connected and its vertices are all of even degree.





**Theorem 3 :** A directed graph possesses an Eulerian circuit iff it is connected and incoming degree of every vertex is equal to its outgoing degree.

**Examples :**

**Q.34 Find under what conditions $K_{m,n}$ the complete bipartite graph will have an Eulerian circuit** [SPPU : Dec.-09]

**Ans. :** In $K_{m,n}$ consider the following cases.

**Case 1 :** m = n and both m and n are even :

In this case, degree of each vertex is even, Hence by theorem 1, $K_{m,n}$ will have an Eulerian circuit. For example $K_{1,2}$ and $K_{4,4}$

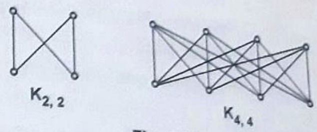



**Fig. Q.34.1**

**Case 2 :** If m = n and m, n are odd :

In this case degree of each vertex is odd. Hence Eulerian circuit will not exist.

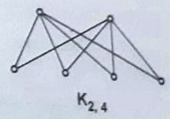



**Fig. Q.34.2**

**Case 3 :** If m ≠ n but m and n are even :

In this case, degree of each vertex is even. So there exists an Eulerian circuit.

**Case 4 :** If m ≠ n and either m is odd or n is odd or both are odd : Then graph will have vertices of odd degree. Hence Eulerian circuit does not exist. e.g. $K_{2,3}$.

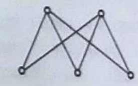


**Fig. Q.34.3**

**Q.35 Define Hamiltonian graphs.** [SPPU : Dec.-04, 09, 10, 12, 15]

**Ans. :** In this section, we introduce a class of graphs which possess a strinking similarity to Eulerian graphs.

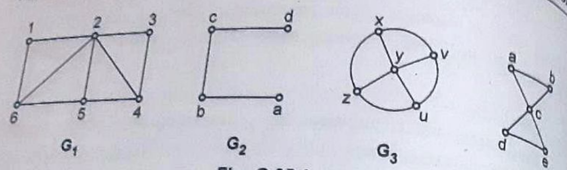
We will now define Hamiltonian path and circuits of the connected graph.

A path in a connected graph G is called a Hamiltonian path if it contains every vertex of G exactly once.

A circuit in a connected graph G is called a Hamiltonian circuit if it contains every vertex of G exactly once.

A graph which has a Hamiltonian circuit is called a Hamiltonian graph.

Consider the following graphs :




$G_1$ $G_2$ $G_3$ $G_4$

Fig. Q.35.1

In graph $G_1$, Hamiltonian circuit is 1-2-3-4-5-6-1

$\therefore G_1$ is a Hamiltonian graph.

In graph $G_2$, Hamiltonian path is a-b-c-d but Hamiltonian circuit does not exist.

$\therefore G_2$ is not Hamiltonian graph.

In graph $G_3$, Hamiltonian circuit is x-y-z-u-v-x

$\therefore G_3$ is Hamiltonian graph but it is not Eulerian.

In graph $G_4$, Hamiltonian path is a-b-c-d-e but Hamiltonian circuit does not exist.

$\therefore G_4$ is not Hamiltonian but Eulerian graph.

**Important theorems :**

**Theorem 1 :** Let G be a simple connected graph on n vertices. If the sum of the degree of each pair of vertices in G is $(n - 1)$ or large then there exists a Hamiltonian path in G.

**Theorem 2 :** If G (v, E) is a simple connected graph on n vertices and $d(v) = \frac{n}{2}$ ; $\forall v \in V$ then G will contain a Hamiltonian circuit.

This condition is sufficient condition but not necessary.

**Theorem 3 :** Let G (v, E) be a connected simple graph. If G has a Hamiltonian circuit then for every proper non empty subsets of v, the components in the graph G-S is less than or equal to the number of vertices in S.

**Theorem 4 :** A Hamiltonian graph contains no cut vertices and hence is 2-connected.





**Q.36 Show that the complete bipartite graph $K_{m,n}$ is Hamiltonian for m = n and for m ≠ n, $K_{m,n}$ is not Hamiltonian graph.**

**Ans. :** In a complete bipartite graph $K_{m,n}$ for m = n i.e. $K_{n,n}$, degree of each vertex is n.

Therefore $d(v) \geq \frac{n}{2}$ for all $v \in v\ (K_{n,n})$

By theorem 2, G contains a Hamiltonian circuit.

Hence $K_{n,n}$ is a Hamiltonian graph.

If m ≠ n, Let $V_1$ and $V_2$ be the partitions of the vertex set of $k_{m,n}$ where $|v_1| = m$ and $|v_2| = n$. Without loss of generality assume that m < n.

The graph $K_{m,n} - V_1$ is a null graph on n vertices.

Hence it is a disconnected graph with n components.

Therefore the number of components in $K_{m,n} - V_1 = n$ which is greater than the number of vertices in $V_1$

Hence by theorem 3, $K_{m,n}$ does not contain a Hamiltonian circuit when m ≠ n.

**Q.37 Show that the complete graph $K_n$ ($n \geq 3$) is a Hamiltonian graph. What is the length of that circuit ? How many circuits exist in $K_n$ ?**

**[SPPU : Dec.-09]**

**Ans. :** The complete graph $K_n$ has n vertices, n ≥ 3 and degree of each vertex is n - 1. As n ≥ 3.

$$d(v) = n - 1 \geq \frac{n}{2} ;\ \forall v \in V(K_n)$$

Therefore by theorem 2, $K_n$ has a Hamiltonian circuit. Hence $K_n$ is a Hamiltonian graph.

Hamiltonian circuit contains all vertices of graph and length of circuit is the number of vertices present it. Hence in $K_n$, the length of the Hamiltonian circuit is n and there are $\frac{(n-1)!}{2}$ Hamiltonian circuits in $K_n$.

**Q.38 Find the Hamiltonian path and circuit in $K_{4,3}$ ?**

**[SPPU : Dec.-04]**

**Ans. :** The complete bipartite graph $K_{4,3}$ is given by, In $K_{4,3}$, $4 \neq 3$ Hence it does not contain Hamiltonian circuit. Here degree of each vertex is either 3 or 4.

$\therefore$ For x, y any two vertices in $K_{4,3}$ $d(x) + d(y) = 7.1 = 6$

Hence by theorem 1, the graph $K_{4,3}$ has a Hamiltonian path. It is given by x a y b z c w.

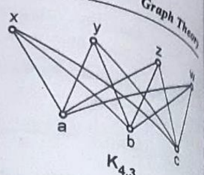



Fig. Q.38.1

**Q.39 Give an example of the following graphs**
**a) Eulerian but not Hamiltonian. b) Hamiltonian but not Eulerian. c) Eulerian as well as Hamiltonian. d) Neither Eulerian nor Hamiltonian.**

**Ans. : a) Eulerian but not Hamiltonian graph.**

Eulerian circuit : a b c d e c a

No Hamiltonian circuit.

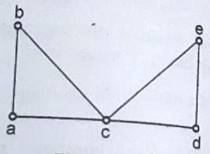



Fig. Q.39.1

**b) Hamiltonian but not Eulerian**

Hamiltonian circuit : abcdea

No Eulerian circuit because d (b) = 3.

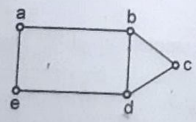



Fig. Q.39.1 (a)

**c) Eulerian and Hamiltonian graph.**

Hamiltonian circuit :
a-b-c-a, 1-2-3-4-1

Eulerian circuit :
a-b-c-a, 1-2-3-4-1

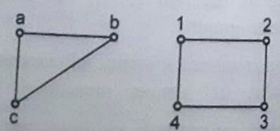



Fig.Q.39.1 (b)





d) Neither Eulerian nor Hamiltonian

No Hamiltonian circuit and no Eulerian circuit.

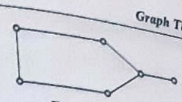


Fig. Q.39.1 (c)

**Q.40 Determine, if the following graphs are having the Hamiltonian circuit or path. Justify your answer.** [SPPU : Dec.-12]

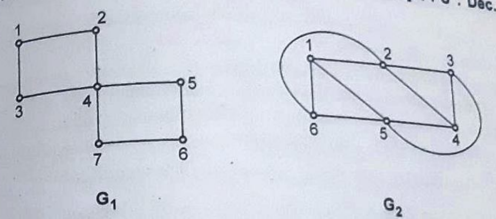



Fig. Q.40.1

**Ans. :** In graph $G_1$, there are n = 7 vertices.

d (4) = 4 and all remaining vertices is 2.

So to draw Hamiltonian circuit, we have to visit vertex 4 twice. Which is not possible in Hamiltonian path. $G_1$ has no Hamiltonian circuit. Hamiltonian path is 1-2-3-4-5-6-7.

In graph $G_2$, there are 6 vertices and degree of each vertex is 3 or 4.

If we consider two vertices of lowest degree then also their sum is 6 which is equal to the number of vertices. So there exists a Hamiltonian path in $G_2$. $\therefore$ Path is 1-2-3-4-5-6.

In graph $G_2$, $d(x) = \frac{6}{2} = 3$ ; $x \in V(G_2)$

$\therefore$ By theorem 2, $\exists$ a Hamiltonian circuit

$\therefore$ Hamiltonian circuit is 1-2-3-4-5-6-1.

**Q.41 Which of the following have a Euler circuit or path or Hamiltonian cycle ? Write the path or circuit.** [SPPU : Dec.-10]

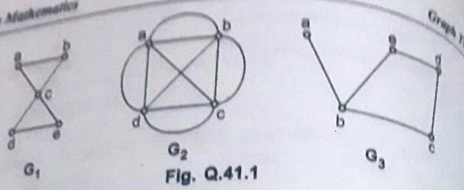



Fig. Q.41.1

Ans. : In graph $G_1$, degree of each vertex is an even so $\exists$ an Eulerian circuit which is a-b-c-d-e-c-a.

In graph G, there are 5 vertices and degree sum of every pair of vertices is 4 or greater than 4. Hence there exists a Hamiltonian path in $G_1$ which is given by a-b-c-d-e. But there is no any Hamiltonian circuit as vertex c is a vertex. In graph $G_2$, degree of each vertex is 5 which is odd integer so there is no Eulerian path in $G_2$, degree of each vertex is $5 > \frac{4}{2}$. Hence there exixts a Hamiltonian circuit which is given by a-b-c-d-a.

In $G_3$, Eulerian path is a-b-c-d-e-b No Eulerian circuit as d (a) = 1. Hamiltonian path is a-b-c-d-e.

No Hamiltonian cyle because b is a cut vertex.

**Q.42 Define the Travelling Salesman Problem (TSP).**

**Ans. :** A salesman is required to travel a number of cities during a trip. Given the distance among cities, in what order should he travel so that he travels as minimum distance as possible ? This is known as Travelling Salesman Problem (TSP).

In terms of graph theory, the TSP is to a Hamiltonian circuit with the smallest weight. In the case of $K_n$ the problem can be solved theoretically by listing all the possible Hamiltonian circuits and select one which has least weight. But this method is highly impractical for the large graphs. In fact no efficient algorithm is there to solve TSP. It is therefore desirable to obtain a reasonably good but not an optimal solution.

One possible approach is to first find a Hamiltonian cycle and search for other Hamiltonian cycles of lesser weight. The simple method is as follows :

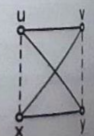



Fig. Q.42.1

Let C be the Hamiltonian circuit of a graph G.





Let further uv and xy be two no-adjacent edges of C such that the vertices u, v, x and y occur in that order in C.

If ux and vy are edges such that v (u x) + w (y v) < w (uv) + w (yx) then replace the edges uv and xy in (by ux and vy). The new cycle C' would still be Hamiltonian cycle and w (C') < w (C). This process can be continued until one gets a reasonably good Hamiltonian cycle.

**Q.43 Explain nearest neighbour method.**

**Ans. :** In this method, we start with any arbitrary vertex and find the vertex which is nearest to it. Continuing this way and coming back to the starting vertex by travelling through all the vertices exactly once, we will get Hamiltonian cycle or circuit.

Consider the following steps to find Hamiltonian cycle by this method.

**Step 1 :** Start with any arbitrary vertex say $v_1$, choose the vertex closest to $v_1$ to form an initial path of one edge. Construct this path by selecting different vertices as described in step 2.

**Step 2 :** Let $v_n$ be the latest vertex that was added to the path. Select the vertex $v_{n+1}$ closest to $v_n$ from all vertices that are not in the path and add this vertex to the path. Select those vertices which will not form a circuit in this stage.

**Step 3 :** Repeat step (2) till all the vertices of G are included in the path.

**Step 4 :** Lastly form a circuit by adding the edge connecting to $v_1$ and the last added vertex.

The circuit obtained using the nearest neighbour method will be the required Hamiltonian circuit.

**Note :** If we start with an arbitrary vertex in TSP then we may or may not minimum Hamiltonian circuit, But if we start with a vertex whose incident edge has the minimum weight in graph then we will get minimum Hamiltonian circuit as compared with arbitary starting vertex. For more details see Q.44.

**Examples :**

**Q.44 Use nearest neighbour method to find the Hamiltonian circuit starting from a in the following graph. Find its weight.**

[SPPU : Dec.-15]

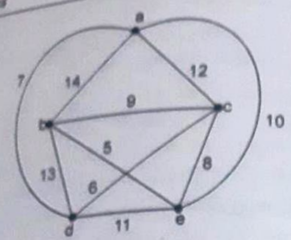



Fig. Q.44.1

**Ans. : Step 1 :** Let a be the starting vertex. Vertex a is adjacent to b, c, d, e. But minimum path is {a, d} which is the initial path.

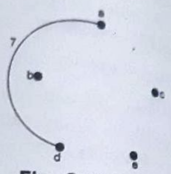



Fig. Q.44.1 (a)

**Step 2 :** There are three vertices adjacent to d. but closest one is C

∴ The path is {a, d, c}.

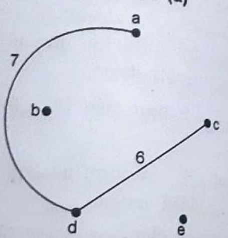



Fig. Q.44.1 (b)

**Step 3 :** There are 4 vertices adjacent to c. but closest is e

∴ The path is {a, d, c, e}.

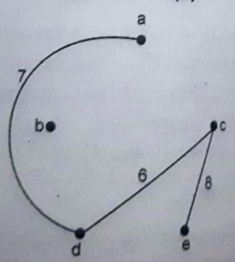



Fig. Q.44.1 (c)





**Step 4 :** There are 4 vertices adjacent to e. but closest is b

∴ The path is {a, d, c, e, b}.

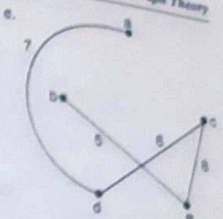



Fig. Q.44.1 (d)

**Step 5 :** Here are all vertices are corered so to complete Hamiltonian an circuit there should be a path from b to a.

∴ Hamiltonian circuit is {a, d, c, e, b, a}

Weight of the Hamiltonian circuit = 40.

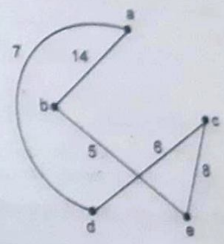



Fig. Q.44.1 (e)

## 4.9 : Planar Graphs

**Q.45 Explain planar graphs.** ☞[SPPU : May-06, Dec.-08, 09, 10, 13]

**Ans. : Definition :** A graph is said to be planar graph if it can be drawn on a plane such that no edges intersect or cross in a point other than their end vertices.

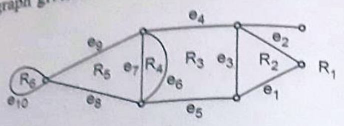
A graph G is said to be non-planar if it is not possible to draw graph be without crossing.

**1. Regions :** A plane representation of a graph divides the plane into parts or regions. They are also known as faces or windows or meshes.

A region or face is characterised by the set edges forming it's boundary.

A region is said to be finite if it's area is finite. A region is said to be infinite or unbounded if it's area is infinite. Every planar graph has an infinite region.

Consider the graph given below :




G

**Fig. Q.45.1**

The graph G has 6 regions, 7 vertices and 11 edges. Region $R_1$ is an infinite region known as exterior region. We have

$$R_2 = \{e_1, e_2, e_3\} = \text{Region bounded by } e_1, e_2, e_3$$

$$R_3 = \{e_3, e_4, e_5, e_6\}$$

$$R_4 = \{e_6, e_7\}, \quad R_6 = \{e_{10}\}.$$

It is observed that $n = 7$, $e = 11$, $r = 6$

$\therefore \quad n + r - 2 = 7 + 6 - 2 = 11 = e$

Now let us define Euler's formula.

**Q.46 State and prove Euler's formula.**

**Statement :** For any connected planar graph G, with v number of vertices, e number of edges and r number of regions

$$\boxed{v - e + r = 2}$$

or $v + r - 2 = e$

**Proof :** Let G be a connected planar graph with v vertices, e edges and r regions. We shall prove the theorem by induction on e.

**Step 1 :** For $e = 0$, we get $v = r = 1$. Thus

$$v - e + r = 1 - 0 + 1 = 2$$

Hence result is true for $e = 0$

**Step 2 :** Let $e \geq 1$. Assume that the result is true for all connected planar graphs with less than e edges. Let G be a graph with v vertices, e edges and r regions.

**Step 3 :** Case 1 : If G has a pendent vertex say x then $G - \{x\}$ is a connected graph with $v - 1$ vertices, $e - 1$ edges and r regions.





So by induction hypothesis

$$(v - 1) - (e - 1) + r = 2$$

$$v - e + r = 2$$

Case 2 : If G has no pendent vertex the G is a connected graph with circuit. Let $e_1$ be the edge of a circuit in G. Then $G - \{e_1\}$ is a connected graph with v vertices, $e - 1$ edges and $r - 1$ regions {If we remove edge from a circuit, then it reduces region by 1}.

By induction hypothesis

$$v - (e - 1) + (r - 1) = 2$$

$$\Rightarrow \quad v - e + r = 2$$

Thus by the principle of mathematical induction the result is true for all e.

**Q.47 If G (V, E) is a simple connected planar graph with v vertices and e edges then $e \leq 3v - 6$.**

**Ans. : Proof :** Give that, G is a simple planar graph, so each region of G is bounded by three or more edges.

If G has r number of regions then the total number of edges in G is $e \geq 3r$.

Also each edge of G is included in exactly two regions of G. therefore $2e \geq 3r$

$$\Rightarrow \quad \frac{2e}{3} \geq r$$

Substitute these values in Euler's theorem, we get

$$v - e + r = 2$$

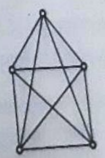
$$v - e + \frac{2e}{3} \geq 2$$

$$3v - e \geq 6$$

$$e \leq 3v - 6 \text{ Hence the proof.}$$

**Corollary 2 :** Prove that, $K_5$ (the complete graph on 5 vertices) is not planar.

**Proof :** The complete graph on 5 vertices $K_5$ is given below :



$K_5$

**Fig. Q.47.1**

$K_5$ has 5 vertices and 10 edges. i.e. $v = 5$ and $e = 10$.

Now $3v - 6 = 15 - 6 = 9$

By corollary 1, $e \leq 3v - 6$

$10 \leq 9$ which is impossible.

Therefore $K_5$ is not planar graph.

$K_5$ is the smallest planar graph with respect to number of vertices.

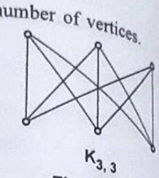
Consider the graph $K_{3,3}$

Here $v = 6, e = 9,$

$3v - 6 = 18 - 6 = 12 > 9 = e$

i.e. $e \leq 3v - 6$

But $K_{3,3}$ is not a planar graph.



$K_{3,3}$

**Fig. Q.47.2**

$\therefore$ The graph $K_{3,3}$ is the smallest non planar graph with respect to number of edges.

The graph $K_5$ is called the Kuratowski's first graph and $K_{3,3}$ is called the Kuratowski's second graph.

In 1930, Kuratowski gave a necessary and sufficient condition for a graph to be planar.

**Kuratowski's Theorem :** A graph G is a planar if G does not contain any subgraph that is isomorphic to within vertices of degree two to either $K_5$ or $K_{3,3}$

Two graphs are said to be isomorphic to within vertices of degree two if they are isomorphic or they can be reduced to isomorphic graphs by repeated insertion of vertices of degree 2 or by merging the edges which have exactly one common vertex of degree 2.

For example the following graph are isomorphic to within vertices of degree 2. (Homeomorphic)





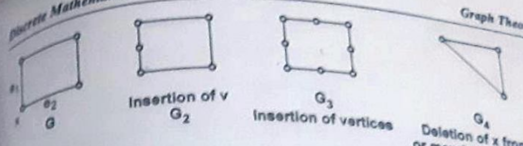



**Fig. Q.47.3**

**Q.48 Draw a planar representation of graphs given below if possible.**

**[SPPU : May-06, Dec.-09]**

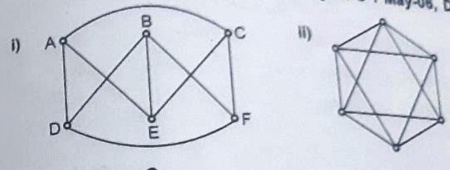



**Fig. Q.48.1**

**Ans. :** The planar representation of $G_1$ and $G_2$ is as follows :

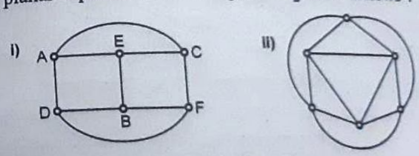



**Fig. Q.48.1 (a)**

**Q.49 Identify whether the graphs are planar or not Justify ?**

**[SPPU : Dec.-08]**

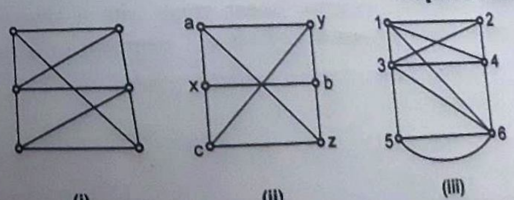



**Fig. Q.49.1**

**Ans. :** i)

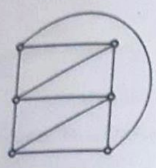


Fig. Q.49.1 (a)

Given graph is planar graph.

ii) Given graph is isomorphic to $K_{3,3}$

$\therefore$ Given graph is not planar.

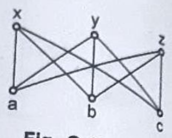



Fig. Q.49.1 (b)

iii)

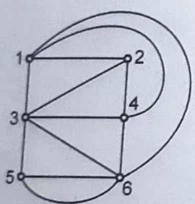



Fig. Q.49.1 (c)

$\therefore$ Given graph is planar.

**Q.50 Show that in a connected planar graph with 6 vertices, 12 edges each of region is bounded by 3 edges.** ☞[SPPU : Dec.-10, 13]

**Ans. :** According to Eulers theorem for planar graphs.

$v - e + r = 2$

Here $v = 6, e = 12$

$6 - 12 + r = 2 \Rightarrow r = 8$

... that, each edge contributed twice in a regions we have 12





So $12 \times 2 = 24$ edges are distributed among 8 regions.

$\Rightarrow \frac{24}{8} = 3$ edges for each region.

So each region is bounded by 3 edges.

**Q.51 Prove that $K_{3,3}$ is not planar graph.**

**Ans. :** $K_{3,3}$ has 6 vertices and 9 edges. Suppose $K_{3,3}$ is planar, then the boundary of each region has at least 4 edges because it is bipartite and contains no triangles. Each edge lies on boundary of two regions.

Therefore, $2e \geq \sum_{i=1}^{r}$ (the number of edges in the $i^{th}$ region)

$$2e \geq 4r$$

$$2e \geq 4(2 + e - v)$$

$$e \leq 2v - 4 \quad \text{But} \quad e = 9 \text{ and } v = 6$$

$\Rightarrow$ $9 \leq 12 - 4 = 8$ which is impossible.

$\therefore$ Hence $K_{3,3}$ is not planar graph.

## 4.10 : Graphs Colouring

**Q.52 Explain coloring of graphs.**

**Ans. :** The coloring of all vertices of a connected graph such that adjacent vertices have different colors is called a proper coloring or vertex coloring or simply a coloring of graphs.

A graph G is said to be properly colored graph if each vertex of G is colored according to a proper coloring.

e.g. 1) Consider the following graphs with proper coloring

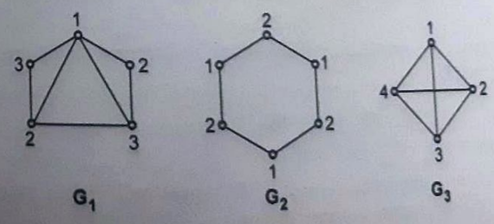



Fig. Q.52.1

**1. Chromatic Number of Graph :** The chromatic number of a graph G is denoted by X(G) and defined as the minimum number of colors required to color the vertices of G so that the adjacent vertices have different colors.

A graph G is said to be K-colorable if all vertices of G can be properly colored using at most K different colors. Obviously, a K-colorable graph is K+1 colorable.

If G is k-colorable then $X(G) \leq K$.

e.g. In above example (1) $X(G) = 3$, $X(G_2) = 2$, $G(G_3) = 4$. If G is any graph with $x(G) = K$ then the addition or deletion of loops or multiple edges do not change the chromatic number of that graph. Thus hereafter for a coloring of problem we consider only simple connected graphs.

**2. Chromatic Polynomial :** We have studied the properly coloring of graph in many different ways using a sufficiently large number of colors. The chromatic polynomial of a graph is denoted by $P_n(\lambda)$ and defined as the number of ways of properly coloring of graph using $\lambda$ or fewer colors.

e.g. 1) The chromatic polynomial of the complete graph $K_1$ is $P_1(\lambda) = \lambda$.

2) The chromatic polynomial of the complete graph $K_n$ on n vertices is

$$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$$

**3. Coloring of Planar Graph :** A map or atlas is a plane representation of a connected planar graph. Two regions of a planar graph G are said to be adjacent if they have an edge common.

The coloring of a planar graph or map means an assignment of a color to each region of a planar graph G such that adjacent regions have different colors.

A planar graph is - n colorable if minimum n different colors are required to color graph G.

**Theorem 1 :** (Four Color Theorem)

Every planar graph is 4 - colorable.

Initially it was a conjecture, but in 1979 Appel and Haken proved this. That's why this conjecture became theorem.





**4. Open Problem of Coloring :** A lot of research is done in the coloring of planar graphs, particularly coloring of vertices, or edges or regions of a planar graph.

The following open problem is stated by Dr. H. R. Bhapkar and proved partially first time.

**Open Problem :**

How many minimum colors will be required to color planar graph such that

i) Adjacent vertices have different colors

ii) Incident edges have different colors.

iii) Adjacent regions have different colors.

iv) A region, boundary edges and boundary vertices of that region have different colors.

This type of coloring is known as perfect coloring of G and denoted by PC (G).

We list some observations of perfect coloring of planar graph as follows :

i) If G is a null graph then PC (G) = 2

ii) If G is a chain graph when n vertices then,

$$PC(G) = \Delta(G) + 2$$

where $\Delta(G)$ = Highest degree of a vertex in G.

**Q.53 Explain web graph.**

**Ans. :** A directed graph whose nodes correspond to static pages on the web and whose arcs correspond to links among these pages, is called web graph.

There are severel reasons for the developement of this graph, few of them are as follows :

i) Designing crawl strategies on the web.

ii) Understanding of the sociology of content creation on the web.

iii) Analyzing the behavior of web algorithms that make use of link information.

iv) Predicting the evolution of web structures such as bipartite cores, webrings and developing better algorithms for organizing ... discovering them.

v) Predicting the emergence of very important new phenomena in the web graph.

We detail a number of experiments on a web crawl of approximately 200 million pages and 1.5 billion links. So the scale of this experiment is very large.

**Q.54 Explain graph database.**

**Ans. :** In computing, a graph database is database that uses graph structures for semantic queries with nodes, edges and properties to represent and store data. A key concept of the system is the graph (or edges or relationships) which directly relates data items in the store.

Alternately, Graph databases are based on graph theory graph databases employ nodes, edges and properties.

i) **Nodes :** Nodes representes entities such as people businesses, accounts or any other item you might want to keep track of. They are roughly the equivalent of the record, relations, the documents in a document database.

ii) Edges are known as relationships are the lines that connect nodes to other nodes, they represent the relationship among them. Edges are the key concepts in graph database, representing an abstraction that is not directly implemented in other systems.

iii) Properties are pertinent information that relate to nodes.

**Examples :**

1) The set at all the users who's phone number contains the area code "141".

2) The set of all trees in the world with similar properties.

*END...*



**Unit V**

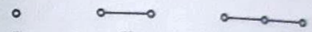
# 5 Trees

## 5.1 : Introduction

**Q.1 Define tree with examples.**

**Ans. :** A tree is a connected graph without any circuit i.e. tree is a connected acyclic graph. The collection or set of an acyclic graphs. (not necessarily connected) is called a forest.

**Examples :**

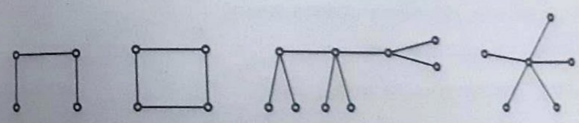


(i) (ii) (iii)

**Example 1**

All these three graphs are trees. There are unique tree on one vertex, 2 vertices and 3 vertices.

**Example 2**



(i) (ii) (iii) (iv)

Graphs (i), (iii) and (iv) are trees but (ii) is not a tree as it has a cycle.

A) A vertex of degree 1 in a tree is called a **leaf** or a **terminal node**. A vertex of a degree greater than one is called a **branch node** or **internal node.**

e.g. In a tree

c, d and e are leaves or terminal nodes and a, b, c are branch nodes.