

Radiative Transfer First and Second Derivatives

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1 Radiative Transfer First Derivative

The radiative transfer equation [1] is

$$I(\mathbf{x}) = \sum_{i=1}^{2N} \Delta B_i \mathcal{T}_i,$$

where ΔB_i is the source function in differential temperature format, and

$$\mathcal{T}_i = \exp \left(- \sum_{j=2}^i \Delta \delta_{j \rightarrow j-1} \right).$$

The derivative of the radiative transfer is

$$\begin{aligned} \frac{\partial I(\mathbf{x})}{\partial x_k} &= \frac{\partial}{\partial x_k} \sum_{i=1}^{2N} \Delta B_i \mathcal{T}_i \\ &= \sum_{i=1}^{2N} \left[\frac{\partial \Delta B_i}{\partial x_k} \mathcal{T}_i + \Delta B_i \frac{\partial \mathcal{T}_i}{\partial x_k} \right] \\ &= \sum_{i=1}^{2N} \left[\frac{\partial \Delta B_i}{\partial x_k} \exp \left(- \sum_{j=2}^i \Delta \delta_{j \rightarrow j-1} \right) + \Delta B_i \frac{\partial}{\partial x_k} \exp \left(- \sum_{j=2}^i \Delta \delta_{j \rightarrow j-1} \right) \right] \\ &= \sum_{i=1}^{2N} \left[\frac{\partial \Delta B_i}{\partial x_k} \exp \left(- \sum_{j=2}^i \Delta \delta_{j \rightarrow j-1} \right) - \Delta B_i \left(\sum_{j=2}^i \frac{\partial \Delta \delta_{j \rightarrow j-1}}{\partial x_k} \right) \exp \left(- \sum_{j=2}^i \Delta \delta_{j \rightarrow j-1} \right) \right] \\ &= \sum_{i=1}^{2N} \left[\frac{\partial \Delta B_i}{\partial x_k} - \Delta B_i \left(\sum_{j=2}^i \frac{\partial \Delta \delta_{j \rightarrow j-1}}{\partial x_k} \right) \right] \mathcal{T}_i \\ &= \sum_{i=1}^{2N} \left[\frac{\partial \Delta B_i}{\partial x_k} - \Delta B_i \mathcal{W}_i \right] \mathcal{T}_i, \end{aligned}$$

where

$$\begin{aligned} \mathcal{W}_1 &= 0, \\ \mathcal{W}_i &= \sum_{j=2}^i \frac{\partial \Delta \delta_{j \rightarrow j-1}}{\partial x_k} \\ &= \sum_{j=2}^{i-1} \frac{\partial \Delta \delta_{j \rightarrow j-1}}{\partial x_k} + \frac{\partial \Delta \delta_{i \rightarrow i-1}}{\partial x_k} \\ &= \mathcal{W}_{i-1} + \frac{\partial \Delta \delta_{i \rightarrow i-1}}{\partial x_k}. \end{aligned}$$

2 Radiative Transfer Second Derivative

The second derivative of the radiative transfer is

$$\begin{aligned}
\frac{\partial^2 I(\mathbf{x})}{\partial x \partial y} &= \frac{\partial}{\partial y} \left(\frac{\partial I}{\partial x} \right) = \frac{\partial}{\partial y} \left(\sum_{i=1}^{2N} \left[\frac{\partial \Delta B_i}{\partial x} - \Delta B_i \sum_{j=2}^i \frac{\partial \Delta \delta_{j \rightarrow j-1}}{\partial x} \right] \mathcal{T}_i \right) \\
&= \sum_{i=1}^{2N} \left(\left[\frac{\partial^2 \Delta B_i}{\partial x \partial y} - \frac{\partial \Delta B_i}{\partial y} \sum_{j=2}^i \frac{\partial \Delta \delta_{j \rightarrow j-1}}{\partial x} - \Delta B_i \sum_{j=2}^i \frac{\partial^2 \Delta \delta_{j \rightarrow j-1}}{\partial x \partial y} \right] \mathcal{T}_i \right. \\
&\quad \left. - \left[\frac{\partial \Delta B_i}{\partial x} - \Delta B_i \sum_{j=2}^i \frac{\partial \Delta \delta_{j \rightarrow j-1}}{\partial x} \right] \sum_{j=2}^i \frac{\partial \Delta \delta_{j \rightarrow j-1}}{\partial y} \mathcal{T}_i \right) \\
&= \sum_{i=1}^{2N} \left[\frac{\partial^2 \Delta B_i}{\partial x \partial y} - \frac{\partial \Delta B_i}{\partial y} \sum_{j=2}^i \frac{\partial \Delta \delta_{j \rightarrow j-1}}{\partial x} - \frac{\partial \Delta B_i}{\partial x} \sum_{j=2}^i \frac{\partial \Delta \delta_{j \rightarrow j-1}}{\partial y} \right. \\
&\quad \left. + \Delta B_i \left(\left(\sum_{j=2}^i \frac{\partial \Delta \delta_{j \rightarrow j-1}}{\partial x} \right) \left(\sum_{j=2}^i \frac{\partial \Delta \delta_{j \rightarrow j-1}}{\partial y} \right) - \sum_{j=2}^i \frac{\partial^2 \Delta \delta_{j \rightarrow j-1}}{\partial x \partial y} \right) \right] \mathcal{T}_i,
\end{aligned}$$

or

$$\frac{\partial^2 I(\mathbf{x})}{\partial x \partial y} = \sum_{i=1}^{2N} \left[\frac{\partial^2 \Delta B_i}{\partial x \partial y} - \frac{\partial \Delta B_i}{\partial y} \mathcal{W}_i^x - \frac{\partial \Delta B_i}{\partial x} \mathcal{W}_i^y + \Delta B_i \left(\mathcal{W}_i^x \mathcal{W}_i^y - \partial \mathcal{W}_i^{x,y} \right) \right] \mathcal{T}_i,$$

where

$$\begin{aligned}
\mathcal{W}_1^x &= 0, \quad \mathcal{W}_1^y = 0, \quad \partial \mathcal{W}_1^{x,y} = 0, \\
\mathcal{W}_i^x &= \sum_{j=2}^i \frac{\partial \Delta \delta_{j \rightarrow j-1}}{\partial x} = \sum_{j=2}^{i-1} \frac{\partial \Delta \delta_{j \rightarrow j-1}}{\partial x} + \frac{\partial \Delta \delta_{i \rightarrow i-1}}{\partial x} \\
&= \mathcal{W}_{i-1}^x + \frac{\partial \Delta \delta_{i \rightarrow i-1}}{\partial x}, \\
\mathcal{W}_i^y &= \sum_{j=2}^i \frac{\partial \Delta \delta_{j \rightarrow j-1}}{\partial y} = \sum_{j=2}^{i-1} \frac{\partial \Delta \delta_{j \rightarrow j-1}}{\partial y} + \frac{\partial \Delta \delta_{i \rightarrow i-1}}{\partial y} \\
&= \mathcal{W}_{i-1}^y + \frac{\partial \Delta \delta_{i \rightarrow i-1}}{\partial y}, \\
\partial \mathcal{W}_i^{x,y} &= \sum_{j=2}^i \frac{\partial^2 \Delta \delta_{j \rightarrow j-1}}{\partial x \partial y} \\
&= \partial \mathcal{W}_{i-1}^{x,y} + \frac{\partial^2 \Delta \delta_{i \rightarrow i-1}}{\partial x \partial y}.
\end{aligned}$$

For $x = y$, the second derivative of the radiative transfer is

$$\frac{\partial^2 I(\mathbf{x})}{\partial x^2} = \sum_{i=1}^{2N} \left[\frac{\partial^2 \Delta B_i}{\partial x^2} - 2 \frac{\partial \Delta B_i}{\partial x} \sum_{j=2}^i \frac{\partial \Delta \delta_{j \rightarrow j-1}}{\partial x} + \Delta B_i \left(\left(\sum_{j=2}^i \frac{\partial \Delta \delta_{j \rightarrow j-1}}{\partial x} \right)^2 - \sum_{j=2}^i \frac{\partial^2 \Delta \delta_{j \rightarrow j-1}}{\partial x^2} \right) \right] \mathcal{T}_i,$$

or

$$\frac{\partial^2 I(\mathbf{x})}{\partial x^2} = \sum_{i=1}^{2N} \left[\frac{\partial^2 \Delta B_i}{\partial x^2} - 2 \frac{\partial \Delta B_i}{\partial x} \mathcal{W}_i + \Delta B_i \left(\mathcal{W}_i^2 - \frac{\partial \mathcal{W}_i}{\partial x} \right) \right] \mathcal{T}_i,$$

where

$$\begin{aligned}
\mathcal{W}_1^x &= 0, \quad \partial \mathcal{W}_1^{x,x} = 0, \\
\mathcal{W}_i^x &= \sum_{j=2}^i \frac{\partial \Delta \delta_{j \rightarrow j-1}}{\partial x} = \sum_{j=2}^{i-1} \frac{\partial \Delta \delta_{j \rightarrow j-1}}{\partial x} + \frac{\partial \Delta \delta_{i \rightarrow i-1}}{\partial x} \\
&= \mathcal{W}_{i-1}^x + \frac{\partial \Delta \delta_{i \rightarrow i-1}}{\partial x}, \\
\partial \mathcal{W}_i^{x,x} &= \sum_{j=2}^i \frac{\partial^2 \Delta \delta_{j \rightarrow j-1}}{\partial x^2} \\
&= \partial \mathcal{W}_{i-1}^{x,x} + \frac{\partial^2 \Delta \delta_{i \rightarrow i-1}}{\partial x^2}.
\end{aligned}$$

References

- [1] I. Yanovsky. Radiative transfer and the forward model for the microwave limb sounder (mls). *MLS iy-001*, 2010.