Radiative Transfer First and Second Derivatives

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1 Radiative Transfer First Derivative

The radiative transfer equation [1] is

$$I(\mathbf{x}) = \sum_{i=1}^{2N} \triangle B_i \mathcal{T}_i,$$

where $\triangle B_i$ is the source function in differential temperature format, and

$$\mathcal{T}_i = \exp\Big(-\sum_{j=2}^{i} \triangle\delta_{j\to j-1}\Big).$$

The derivative of the radiative transfer is

$$\begin{aligned}
\frac{\partial I(\mathbf{x})}{\partial x_k} &= \frac{\partial}{\partial x_k}\sum_{i=1}^{2N} \triangle B_i \mathcal{T}_i \\
&= \sum_{i=1}^{2N}\Big[\frac{\partial \triangle B_i}{\partial x_k}\mathcal{T}_i + \triangle B_i \frac{\partial \mathcal{T}_i}{\partial x_k}\Big] \\
&= \sum_{i=1}^{2N}\Big[\frac{\partial \triangle B_i}{\partial x_k}\exp\Big(-\sum_{j=2}^{i}\triangle\delta_{j\to j-1}\Big) + \triangle B_i \frac{\partial}{\partial x_k}\exp\Big(-\sum_{j=2}^{i}\triangle\delta_{j\to j-1}\Big)\Big] \\
&= \sum_{i=1}^{2N}\Big[\frac{\partial \triangle B_i}{\partial x_k}\exp\Big(-\sum_{j=2}^{i}\triangle\delta_{j\to j-1}\Big) - \triangle B_i \Big(\sum_{j=2}^{i}\frac{\partial \triangle\delta_{j\to j-1}}{\partial x_k}\Big)\exp\Big(-\sum_{j=2}^{i}\triangle\delta_{j\to j-1}\Big)\Big] \\
&= \sum_{i=1}^{2N}\Big[\frac{\partial \triangle B_i}{\partial x_k} - \triangle B_i \Big(\sum_{j=2}^{i}\frac{\partial \triangle\delta_{j\to j-1}}{\partial x_k}\Big)\Big]\mathcal{T}_i \\
&= \sum_{i=1}^{2N}\Big[\frac{\partial \triangle B_i}{\partial x_k} - \triangle B_i \mathcal{W}_i\Big]\mathcal{T}_i,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{W}_1 &= 0, \\
\mathcal{W}_i &= \sum_{j=2}^{i}\frac{\partial \triangle\delta_{j\to j-1}}{\partial x_k} \\
&= \sum_{j=2}^{i-1}\frac{\partial \triangle\delta_{j\to j-1}}{\partial x_k} + \frac{\partial \triangle\delta_{i\to i-1}}{\partial x_k} \\
&= \mathcal{W}_{i-1} + \frac{\partial \triangle\delta_{i\to i-1}}{\partial x_k}.
\end{aligned}$$

2 Radiative Transfer Second Derivative

The second derivative of the radiative transfer is

$$
\begin{aligned}
\frac{\partial^2 I(\mathbf{x})}{\partial x \partial y} &= \frac{\partial}{\partial y}\Big(\frac{\partial I}{\partial x}\Big) = \frac{\partial}{\partial y}\Bigg(\sum_{i=1}^{2N}\Bigg[\frac{\partial \triangle B_i}{\partial x} - \triangle B_i \sum_{j=2}^{i} \frac{\partial \triangle \delta_{j\to j-1}}{\partial x}\Bigg]\mathcal{T}_i\Bigg) \\
&= \sum_{i=1}^{2N}\Bigg(\Bigg[\frac{\partial^2 \triangle B_i}{\partial x \partial y} - \frac{\partial \triangle B_i}{\partial y}\sum_{j=2}^{i}\frac{\partial \triangle \delta_{j\to j-1}}{\partial x} - \triangle B_i \sum_{j=2}^{i}\frac{\partial^2 \triangle \delta_{j\to j-1}}{\partial x \partial y}\Bigg]\mathcal{T}_i \\
&\qquad - \Bigg[\frac{\partial \triangle B_i}{\partial x} - \triangle B_i \sum_{j=2}^{i}\frac{\partial \triangle \delta_{j\to j-1}}{\partial x}\Bigg]\sum_{j=2}^{i}\frac{\partial \triangle \delta_{j\to j-1}}{\partial y}\,\mathcal{T}_i\Bigg) \\
&= \sum_{i=1}^{2N}\Bigg[\frac{\partial^2 \triangle B_i}{\partial x \partial y} - \frac{\partial \triangle B_i}{\partial y}\sum_{j=2}^{i}\frac{\partial \triangle \delta_{j\to j-1}}{\partial x} - \frac{\partial \triangle B_i}{\partial x}\sum_{j=2}^{i}\frac{\partial \triangle \delta_{j\to j-1}}{\partial y} \\
&\qquad + \triangle B_i\Bigg(\Big(\sum_{j=2}^{i}\frac{\partial \triangle \delta_{j\to j-1}}{\partial x}\Big)\Big(\sum_{j=2}^{i}\frac{\partial \triangle \delta_{j\to j-1}}{\partial y}\Big) - \sum_{j=2}^{i}\frac{\partial^2 \triangle \delta_{j\to j-1}}{\partial x \partial y}\Bigg)\Bigg]\mathcal{T}_i,
\end{aligned}
$$

or

$$
\frac{\partial^2 I(\mathbf{x})}{\partial x \partial y} = \sum_{i=1}^{2N}\Bigg[\frac{\partial^2 \triangle B_i}{\partial x \partial y} - \frac{\partial \triangle B_i}{\partial y}\mathcal{W}_i^x - \frac{\partial \triangle B_i}{\partial x}\mathcal{W}_i^y + \triangle B_i\Big(\mathcal{W}_i^x\,\mathcal{W}_i^y - \partial\mathcal{W}_i^{x,y}\Big)\Bigg]\mathcal{T}_i,
$$

where

$$
\begin{aligned}
\mathcal{W}_1^x &= 0, \quad \mathcal{W}_1^y = 0, \quad \partial\mathcal{W}_1^{x,y} = 0, \\
\mathcal{W}_i^x &= \sum_{j=2}^{i}\frac{\partial \triangle \delta_{j\to j-1}}{\partial x} = \sum_{j=2}^{i-1}\frac{\partial \triangle \delta_{j\to j-1}}{\partial x} + \frac{\partial \triangle \delta_{i\to i-1}}{\partial x} \\
&= \mathcal{W}_{i-1}^x + \frac{\partial \triangle \delta_{i\to i-1}}{\partial x}, \\
\mathcal{W}_i^y &= \sum_{j=2}^{i}\frac{\partial \triangle \delta_{j\to j-1}}{\partial y} = \sum_{j=2}^{i-1}\frac{\partial \triangle \delta_{j\to j-1}}{\partial y} + \frac{\partial \triangle \delta_{i\to i-1}}{\partial y} \\
&= \mathcal{W}_{i-1}^y + \frac{\partial \triangle \delta_{i\to i-1}}{\partial y}, \\
\partial\mathcal{W}_i^{x,y} &= \sum_{j=2}^{i}\frac{\partial^2 \triangle \delta_{j\to j-1}}{\partial x \partial y} \\
&= \partial\mathcal{W}_{i-1}^{x,y} + \frac{\partial^2 \triangle \delta_{i\to i-1}}{\partial x \partial y}.
\end{aligned}
$$

For $x = y$, the second derivative of the radiative transfer is

$$
\frac{\partial^2 I(\mathbf{x})}{\partial x^2} = \sum_{i=1}^{2N}\Bigg[\frac{\partial^2 \triangle B_i}{\partial x^2} - 2\frac{\partial \triangle B_i}{\partial x}\sum_{j=2}^{i}\frac{\partial \triangle \delta_{j\to j-1}}{\partial x} + \triangle B_i\Bigg(\Big(\sum_{j=2}^{i}\frac{\partial \triangle \delta_{j\to j-1}}{\partial x}\Big)^2 - \sum_{j=2}^{i}\frac{\partial^2 \triangle \delta_{j\to j-1}}{\partial x^2}\Bigg)\Bigg]\mathcal{T}_i,
$$

or

$$
\frac{\partial^2 I(\mathbf{x})}{\partial x^2} = \sum_{i=1}^{2N}\Bigg[\frac{\partial^2 \triangle B_i}{\partial x^2} - 2\frac{\partial \triangle B_i}{\partial x}\mathcal{W}_i + \triangle B_i\Big(\mathcal{W}_i^2 - \frac{\partial \mathcal{W}_i}{\partial x}\Big)\Bigg]\mathcal{T}_i,
$$

where

$$\begin{aligned}
\mathcal{W}_1^x &= 0, \qquad \partial\mathcal{W}_1^{x,x} = 0, \\
\mathcal{W}_i^x &= \sum_{j=2}^{i} \frac{\partial \triangle\delta_{j\to j-1}}{\partial x} = \sum_{j=2}^{i-1} \frac{\partial \triangle\delta_{j\to j-1}}{\partial x} + \frac{\partial \triangle\delta_{i\to i-1}}{\partial x} \\
&= \mathcal{W}_{i-1}^x + \frac{\partial \triangle\delta_{i\to i-1}}{\partial x}, \\
\partial\mathcal{W}_i^{x,x} &= \sum_{j=2}^{i} \frac{\partial^2 \triangle\delta_{j\to j-1}}{\partial x^2} \\
&= \partial\mathcal{W}_{i-1}^{x,x} + \frac{\partial^2 \triangle\delta_{i\to i-1}}{\partial x^2}.
\end{aligned}$$

References

[1] I. Yanovsky. Radiative transfer and the forward model for the microwave limb sounder (mls). *MLS iy-001*, 2010.