

Temperature Derivatives

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Contents

1	Temperature First Derivative	1
2	Temperature Second Derivative	4
3	The Planck Radiation Function Derivative	5
3.1	Analytical Expression for the Source Function Derivative for Temperature	5
3.2	Approximation of the Source Function Derivative for Temperature	6

1 Temperature First Derivative

Using notations from [1], the hydrostatic relation is

$$h(\zeta) = \frac{g^{\oplus} H^{\oplus 2}}{g^{\oplus} H^{\oplus} - k \ln 10 \sum_{q=1}^Q f_q^T P_q(\zeta)} - H^{\oplus}$$

The temperature representation is of the form

$$T(\zeta) = \sum_{q=1}^Q f_q^T \eta_q^T(\zeta).$$

The species incremental opacity integral is

$$\Delta \delta_{i \rightarrow i-1}^k = \int_{\zeta_i}^{\zeta_{i-1}} F^k(\zeta) \beta^k \frac{ds}{dh} \frac{dh}{d\zeta} d\zeta.$$

Substituting

$$\begin{aligned} \frac{ds}{dh} &= \frac{R}{\sqrt{R^2 - R_t^2}} = \frac{h + H^{\oplus}}{\sqrt{(h + H^{\oplus})^2 - (h_t + H^{\oplus})^2}}, \\ \frac{dh}{d\zeta} &= \frac{(h + H^{\oplus})^2 k T \log 10}{g_0 H^{\oplus 2} m}, \end{aligned}$$

into the species incremental opacity integral and differentiating with respect to f_q^T gives

$$\begin{aligned}
\frac{\partial \Delta \delta_{i \rightarrow i-1}^k}{\partial f_q^T} &= \frac{\partial}{\partial f_q^T} \int_{\zeta_i}^{\zeta_{i-1}} F^k(\zeta) \beta^k(P(\zeta), T(\zeta), \zeta) \frac{ds}{dh} \frac{dh}{d\zeta} d\zeta \\
&= \frac{\partial}{\partial f_q^T} \int_{\zeta_i}^{\zeta_{i-1}} F^k(\zeta) \beta^k(P(\zeta), T(\zeta), \zeta) \frac{h + H^\oplus}{\sqrt{(h + H^\oplus)^2 - (h_t + H^\oplus)^2}} \frac{(h + H^\oplus)^2 kT \log 10}{g_0 H^{\oplus 2} m} d\zeta \\
&= \int_{\zeta_i}^{\zeta_{i-1}} \left\{ F^k \frac{\partial \beta^k}{\partial f_q^T} \frac{h + H^\oplus}{\sqrt{(h + H^\oplus)^2 - (h_t + H^\oplus)^2}} \frac{(h + H^\oplus)^2 kT \log 10}{g_0 H^{\oplus 2} m} \right. \\
&\quad + F^k \beta^k \frac{\partial}{\partial f_q^T} \left[\frac{(h + H^\oplus)^3}{\sqrt{(h + H^\oplus)^2 - (h_t + H^\oplus)^2}} \right] \frac{kT \log 10}{g_0 H^{\oplus 2} m} \\
&\quad \left. + F^k \beta^k \frac{(h + H^\oplus)^3}{\sqrt{(h + H^\oplus)^2 - (h_t + H^\oplus)^2}} \frac{\partial}{\partial f_q^T} \left[\frac{kT \log 10}{g_0 H^{\oplus 2} m} \right] \right\} d\zeta \\
&= \int_{\zeta_i}^{\zeta_{i-1}} \left\{ F^k \underbrace{\frac{\partial \beta^k}{\partial T} \frac{\partial T}{\partial f_q^T}}_{\partial \beta^k / \partial f_q^T} \frac{ds}{dh} \frac{dh}{d\zeta} \right. \\
&\quad + F^k \beta^k \frac{2(h + H^\oplus)^2 \frac{\partial h}{\partial f_q^T} - 3(h_t + H^\oplus)^2 \frac{\partial h}{\partial f_q^T} + (h + H^\oplus)(h_t + H^\oplus) \frac{\partial h_t}{\partial f_q^T}}{\left((h + H^\oplus)^2 - (h_t + H^\oplus)^2 \right)^{\frac{3}{2}}} (h + H^\oplus)^2 \frac{kT \log 10}{g_0 H^{\oplus 2} m} \\
&\quad \left. + F^k \beta^k \underbrace{\frac{(h + H^\oplus)^3}{\sqrt{(h + H^\oplus)^2 - (h_t + H^\oplus)^2}}}_{\frac{ds}{dh} (h + H^\oplus)^2} \underbrace{\eta_q^T(\zeta) \left[\frac{k \log 10}{g_0 H^{\oplus 2} m} \right]}_{\frac{dh}{d\zeta} / T / (h + H^\oplus)^2} \right\} d\zeta.
\end{aligned}$$

Approximation:

$$\beta^k = \beta_0^k \left(\frac{T}{T_0} \right)^{n^k}.$$

Then,

$$\frac{\partial \beta^k}{\partial T} = n^k \beta_0^k \left(\frac{T}{T_0} \right)^{n^k - 1} \frac{1}{T_0} = \frac{n^k \beta^k}{T}.$$

Here, n^k is calculated from

$$\begin{aligned}
\frac{\beta_1^k}{\beta_2^k} &= \left(\frac{T_1}{T_2} \right)^{n^k}, \quad \text{or} \\
n^k &= \frac{\log \frac{\beta^k(T_0 + \epsilon, P, \nu)}{\beta^k(T_0 - \epsilon, P, \nu)}}{\log \left(\frac{T_0 + \epsilon}{T_0 - \epsilon} \right)},
\end{aligned}$$

with, for example, $T_0 = 230$ and $\epsilon = 5$.

Thus, the **temperature derivative of the incremental opacity** equation is

$$\begin{aligned} \frac{\partial \Delta \delta_{i \rightarrow i-1}^k}{\partial f_q^T} &= \int_{\zeta_i}^{\zeta_{i-1}} \left\{ F^k \beta^k \frac{n^k}{T} \eta_q^T(\zeta) \frac{ds}{dh} \frac{dh}{d\zeta} \right. \\ &\quad + F^k \beta^k \frac{2(h + H^\oplus)^2 \frac{\partial h}{\partial f_q^T} - 3(h_t + H^\oplus)^2 \frac{\partial h}{\partial f_q^T} + (h + H^\oplus)(h_t + H^\oplus) \frac{\partial h_t}{\partial f_q^T}}{\left((h + H^\oplus)^2 - (h_t + H^\oplus)^2 \right)^{\frac{3}{2}}} \frac{dh}{d\zeta} \\ &\quad \left. + F^k \beta^k \frac{\eta_q^T(\zeta)}{T} \frac{ds}{dh} \frac{dh}{d\zeta} \right\} d\zeta. \end{aligned}$$

$$\begin{aligned} \frac{\partial h}{\partial f_q^T} &= \frac{g^\oplus H^{\oplus 2} k \log 10 P_q(\zeta)}{\left[g^\oplus H^\oplus - k \log 10 \sum_{q=1}^Q f_q^T P_q(\zeta) \right]^2}, \\ \frac{\partial h_t}{\partial f_q^T} &= \frac{g^\oplus H^{\oplus 2} k \log 10 P_q(\zeta_t)}{\left[g^\oplus H^\oplus - k \log 10 \sum_{q=1}^Q f_q^T P_q(\zeta_t) \right]^2}. \end{aligned}$$

Since **incremental opacity** is given by

$$\Delta \delta_{i \rightarrow i-1} = \Delta \delta_{i \rightarrow i-1}^{O_3} + \Delta \delta_{i \rightarrow i-1}^{N_2},$$

we have

$$\frac{\partial \Delta \delta_{i \rightarrow i-1}}{\partial f_q^T} = \frac{\partial \Delta \delta_{i \rightarrow i-1}^{O_3}}{\partial f_q^T} + \frac{\partial \Delta \delta_{i \rightarrow i-1}^{N_2}}{\partial f_q^T}.$$

The **temperature derivative** of radiance is given as

$$\frac{\partial I(\mathbf{x})}{\partial f_q^T} = \sum_{i=1}^{2N} \left[\frac{\partial \Delta B_i}{\partial f_q^T} - \Delta B_i \mathcal{W}_i \right] \mathcal{T}_i,$$

where

$$\begin{aligned} \mathcal{W}_{1,q} &= 0, \\ \mathcal{W}_{i,q} &= \sum_{j=2}^i \frac{\partial \Delta \delta_{j \rightarrow j-1}}{\partial f_q^T} \\ &= \sum_{j=2}^{i-1} \frac{\partial \Delta \delta_{j \rightarrow j-1}}{\partial f_q^T} + \frac{\partial \Delta \delta_{i \rightarrow i-1}}{\partial f_q^T} \\ &= \mathcal{W}_{i-1,q} + \frac{\partial \Delta \delta_{i \rightarrow i-1}}{\partial f_q^T}. \end{aligned}$$

2 Temperature Second Derivative

$$\begin{aligned}
\frac{\partial^2 \Delta \delta_{i \rightarrow i-1}^k}{\partial f_q^T \partial f_q^T} &= \frac{\partial}{\partial f_q^T} \int_{\zeta_i}^{\zeta_{i-1}} \left\{ F^k \beta^k \frac{n^k}{T} \eta_q^T(\zeta) \frac{ds}{dh} \frac{dh}{d\zeta} \right. \\
&\quad \left. + F^k \beta^k \frac{2(h + H^\oplus)^2 \frac{\partial h}{\partial f_q^T} - 3(h_t + H^\oplus)^2 \frac{\partial h}{\partial f_q^T} + (h + H^\oplus)(h_t + H^\oplus) \frac{\partial h_t}{\partial f_q^T}}{\left((h + H^\oplus)^2 - (h_t + H^\oplus)^2 \right)^{\frac{3}{2}}} \frac{dh}{d\zeta} \right. \\
&\quad \left. + F^k \beta^k \frac{\eta_q^T(\zeta)}{T} \frac{ds}{dh} \frac{dh}{d\zeta} \right\} d\zeta \\
&= \int_{\zeta_i}^{\zeta_{i-1}} \left\{ \frac{\partial}{\partial f_q^T} \left[F^k \beta^k \frac{n^k}{T} \eta_q^T(\zeta) \frac{ds}{dh} \frac{dh}{d\zeta} \right] d\zeta \right. \\
&\quad \left. + \frac{\partial}{\partial f_q^T} \left[F^k \beta^k \frac{2(h + H^\oplus)^2 \frac{\partial h}{\partial f_q^T} - 3(h_t + H^\oplus)^2 \frac{\partial h}{\partial f_q^T} + (h + H^\oplus)(h_t + H^\oplus) \frac{\partial h_t}{\partial f_q^T}}{\left((h + H^\oplus)^2 - (h_t + H^\oplus)^2 \right)^{\frac{3}{2}}} \frac{dh}{d\zeta} \right] \right. \\
&\quad \left. + \frac{\partial}{\partial f_q^T} \left[F^k \beta^k \frac{\eta_q^T(\zeta)}{T} \frac{ds}{dh} \frac{dh}{d\zeta} \right] \right\} d\zeta
\end{aligned}$$

3 The Planck Radiation Function Derivative

3.1 Analytical Expression for the Source Function Derivative for Temperature

The Planck radiation function B expressed in temperature units is given by

$$B(\zeta) = \frac{h\nu}{k[\exp(\frac{h\nu}{kT(\zeta)}) - 1]}, \quad (1)$$

where h is Planck's constant, k is Boltzmann's constant, ν is the radiation frequency, and T is temperature. The temperature derivative of the Plack radiation function is

$$\begin{aligned} \frac{dB(\zeta)}{dT} &= -\frac{h\nu}{k[\exp(\frac{h\nu}{kT}) - 1]^2} \cdot \frac{-h\nu}{kT^2} \exp\left(\frac{h\nu}{kT}\right) \\ &= \frac{B^2 \exp(\frac{h\nu}{kT})}{T^2}, \end{aligned} \quad (2)$$

which can also be written as

$$\frac{dB(\zeta)}{dT} = \frac{B}{T^2} \left(\frac{h\nu}{k} + B \right). \quad (3)$$

In discrete form

$$B_i = \frac{h\nu}{k[\exp(\frac{h\nu}{kT_i}) - 1]},$$

The derivative of B with respect to ζ can be approximated by

$$\frac{dB}{d\zeta} \approx \frac{\Delta B}{\Delta \zeta}.$$

The differential source function can be approximated as $dB \approx \Delta B$:

$$\begin{aligned} \Delta B_1 &= \frac{B_1 + B_2}{2}, \\ \Delta B_i &= \frac{B_{i+1} - B_{i-1}}{2}, \\ \Delta B_{2N} &= I_0 - \frac{B_{2N-1} + B_{2N}}{2}, \end{aligned}$$

where $I_0 = 2.7$ is the background cosmic radiance incident on the atmosphere. Note that

$$\sum_{i=1}^{2N} \Delta B_i = I_0.$$

The source function Jacobian for temperature is

$$\frac{\partial \Delta B_i}{\partial f_q^T} = \frac{\frac{\partial B_{i+1}}{\partial f_q^T} - \frac{\partial B_{i-1}}{\partial f_q^T}}{2}, \quad (4)$$

where

$$\begin{aligned} \frac{\partial B_i}{\partial f_q^T} &= -\frac{h\nu}{k[\exp(\frac{h\nu}{kT_i}) - 1]^2} \cdot \frac{-h\nu}{kT_i^2} \exp\left(\frac{h\nu}{kT_i}\right) \eta_q^T(\zeta_i) \\ &= \frac{B_i^2 \exp(\frac{h\nu}{kT_i})}{T_i^2} \eta_q^T(\zeta_i) \\ &= \frac{B_i}{T_i^2} \left(\frac{h\nu}{k} + B_i \right) \eta_q^T(\zeta_i). \end{aligned} \quad (5)$$

3.2 Approximation of the Source Function Derivative for Temperature

The Planck radiation function B can be approximated by T :

$$B(\zeta) = \frac{h\nu}{k[\exp(\frac{h\nu}{kT(\zeta)}) - 1]} \approx T(\zeta).$$

Hence,

$$\begin{aligned}\Delta B_1 &= \frac{T_1 + T_2}{2}, \\ \Delta B_i &= \frac{T_{i+1} - T_{i-1}}{2}, \\ \Delta B_{2N} &= I_0 - \frac{T_{2N-1} + T_{2N}}{2},\end{aligned}$$

where $I_0 = 2.7$. Since $\frac{\partial T(\zeta)}{\partial f_q^T} = \eta_q^T(\zeta)$, the source function Jacobian for temperature can be approximated by

$$\begin{aligned}\frac{\partial \Delta B_1}{\partial f_q^T} &= \frac{\eta_q^T(\zeta_1) + \eta_q^T(\zeta_2)}{2}, \\ \frac{\partial \Delta B_i}{\partial f_q^T} &= \frac{\eta_q^T(\zeta_{i+1}) - \eta_q^T(\zeta_{i-1})}{2}, \\ \frac{\partial \Delta B_{2N}}{\partial f_q^T} &= -\frac{\eta_q^T(\zeta_{2N-1}) + \eta_q^T(\zeta_{2N})}{2}.\end{aligned}$$

References

- [1] I. Yanovsky. Radiative transfer and the forward model for the Microwave Limb Sounder (MLS). *MLS iy-001*, 2010.