Temperature First and Second Derivatives

Igor Yanovsky
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1 Temperature First Derivative

The hydrostatic relation is

$$h(\zeta) = \frac{g^{\oplus} {H^{\oplus}}^2}{g^{\oplus} H^{\oplus} - k \ln 10 \sum_{q=1}^{Q} f_q^T P_q(\zeta)} - H^{\oplus}$$

The **temperature** representation is of the form

$$T(\zeta) = \sum_{q=1}^{Q} \eta_q^T(\zeta) f_q^T.$$

The species incremental opacity integral is

$$\triangle\delta_{i\to i-1}^k = \int_{\zeta_i}^{\zeta_{i-1}} F^k(\zeta)\, \beta^k\, \frac{ds}{dh}\, \frac{dh}{d\zeta}\, d\zeta.$$

Substituting

$$\begin{aligned}
\frac{ds}{dh} &= \frac{R}{\sqrt{R^2 - R_t^2}} = \frac{h + H^{\oplus}}{\sqrt{(h + H^{\oplus})^2 - (h_t + H^{\oplus})^2}}, \\
\frac{dh}{d\zeta} &= \frac{(h + H^{\oplus})^2 k T \log 10}{g_0 {H^{\oplus}}^2 m},
\end{aligned}$$

into the species incremental opacity integral and differentiating with respect to f_q^T gives

$$\begin{aligned}
\frac{\partial \triangle\delta^k_{i\to i-1}}{\partial f_q^T} &= \frac{\partial}{\partial f_q^T}\int_{\zeta_i}^{\zeta_{i-1}} F^k(\zeta)\,\beta^k\big(P(\zeta),T(\zeta),\zeta\big)\,\frac{ds}{dh}\,\frac{dh}{d\zeta}\,d\zeta \\
&= \frac{\partial}{\partial f_q^T}\int_{\zeta_i}^{\zeta_{i-1}} F^k(\zeta)\,\beta^k\big(P(\zeta),T(\zeta),\zeta\big)\,\frac{h+H^\oplus}{\sqrt{(h+H^\oplus)^2-(h_t+H^\oplus)^2}}\,\frac{(h+H^\oplus)^2 kT\log 10}{g_0 H^{\oplus\,2} m}\,d\zeta \\
&= \int_{\zeta_i}^{\zeta_{i-1}}\left\{F^k\,\frac{\partial\beta^k}{\partial f_q^T}\,\frac{h+H^\oplus}{\sqrt{(h+H^\oplus)^2-(h_t+H^\oplus)^2}}\,\frac{(h+H^\oplus)^2 kT\log 10}{g_0 H^{\oplus\,2} m}\right. \\
&\quad + F^k\beta^k\frac{\partial}{\partial f_q^T}\left[\frac{(h+H^\oplus)^3}{\sqrt{(h+H^\oplus)^2-(h_t+H^\oplus)^2}}\right]\frac{kT\log 10}{g_0 H^{\oplus\,2} m} \\
&\quad \left. + F^k\beta^k\frac{(h+H^\oplus)^3}{\sqrt{(h+H^\oplus)^2-(h_t+H^\oplus)^2}}\,\frac{\partial}{\partial f_q^T}\left[\frac{kT\log 10}{g_0 H^{\oplus\,2} m}\right]\right\}d\zeta \\
&= \int_{\zeta_i}^{\zeta_{i-1}}\left\{F^k\,\underbrace{\frac{\partial\beta^k}{\partial T}\frac{\partial T}{\partial f_q^T}}_{\partial\beta^k/\partial f_q^T}\;\frac{ds}{dh}\,\frac{dh}{d\zeta}\right. \\
&\quad + F^k\beta^k\frac{2(h+H^\oplus)^2\frac{\partial h}{\partial f_q^T}-3(h_t+H^\oplus)^2\frac{\partial h}{\partial f_q^T}+(h+H^\oplus)(h_t+H^\oplus)\frac{\partial h_t}{\partial f_q^T}}{\Big((h+H^\oplus)^2-(h_t+H^\oplus)^2\Big)^{\frac{3}{2}}}(h+H^\oplus)^2\,\frac{kT\log 10}{g_0 H^{\oplus\,2} m} \\
&\quad \left. + F^k\beta^k\,\underbrace{\frac{(h+H^\oplus)^3}{\sqrt{(h+H^\oplus)^2-(h_t+H^\oplus)^2}}}_{\frac{ds}{dh}(h+H^\oplus)^2}\,\eta_q^T(\zeta)\;\underbrace{\left[\frac{k\log 10}{g_0 H^{\oplus\,2} m}\right]}_{\frac{dh}{d\zeta}/T/(h+H^\oplus)^2}\right\}d\zeta.
\end{aligned}$$

Approximation:

$$\beta^k = \beta_0^k\left(\frac{T}{T_0}\right)^{n^k}.$$

Then,

$$\frac{\partial\beta^k}{\partial T} = n^k\beta_0^k\left(\frac{T}{T_0}\right)^{n^k-1}\frac{1}{T_0} = \frac{n^k\beta^k}{T}.$$

Here, n^k is calculated from

$$\begin{aligned}
\frac{\beta_1^k}{\beta_2^k} &= \left(\frac{T_1}{T_2}\right)^{n^k}, \qquad \text{or} \\
n^k &= \frac{\log\dfrac{\beta^k(T_0+\epsilon,P,\nu)}{\beta^k(T_0-\epsilon,P,\nu)}}{\log\left(\dfrac{T_0+\epsilon}{T_0-\epsilon}\right)},
\end{aligned}$$

with, for example, $T_0 = 230$ and $\epsilon = 5$.
Thus, the **temperature derivative of the incremental opacity** equation is

$$\begin{aligned}\frac{\partial\triangle\delta^k_{i\to i-1}}{\partial f^T_q} &= \int_{\zeta_i}^{\zeta_{i-1}}\Bigg\{F^k\beta^k\,\frac{n^k}{T}\,\eta^T_q(\zeta)\,\frac{ds}{dh}\,\frac{dh}{d\zeta} \\ &\quad +F^k\beta^k\frac{2(h+H^\oplus)^2\dfrac{\partial h}{\partial f^T_q}-3(h_t+H^\oplus)^2\dfrac{\partial h}{\partial f^T_q}+(h+H^\oplus)(h_t+H^\oplus)\dfrac{\partial h_t}{\partial f^T_q}}{\Big((h+H^\oplus)^2-(h_t+H^\oplus)^2\Big)^{\frac{3}{2}}}\,\frac{dh}{d\zeta} \\ &\quad +F^k\beta^k\,\frac{\eta^T_q(\zeta)}{T}\frac{ds}{dh}\,\frac{dh}{d\zeta}\Bigg\}\,d\zeta.\end{aligned}$$

$$\begin{aligned}\frac{\partial h}{\partial f^T_q} &= \frac{g^\oplus {H^\oplus}^2 k\ \log 10\ P_q(\zeta)}{\Big[g^\oplus H^\oplus - k\log 10\sum_{q=1}^Q f^T_q P_q(\zeta)\Big]^2}, \\ \frac{\partial h_t}{\partial f^T_q} &= \frac{g^\oplus {H^\oplus}^2 k\ \log 10\ P_q(\zeta_t)}{\Big[g^\oplus H^\oplus - k\log 10\sum_{q=1}^Q f^T_q P_q(\zeta_t)\Big]^2}.\end{aligned}$$

Since **incremental opacity** is given by

$$\triangle\delta_{i\to i-1} = \triangle\delta^{O_3}_{i\to i-1} + \triangle\delta^{N_2}_{i\to i-1},$$

we have

$$\frac{\partial\triangle\delta_{i\to i-1}}{\partial f^T_q} = \frac{\partial\triangle\delta^{O_3}_{i\to i-1}}{\partial f^T_q} + \frac{\partial\triangle\delta^{N_2}_{i\to i-1}}{\partial f^T_q}.$$

The **temperature derivative** of radiance is given as

$$\frac{\partial I(\mathbf{x})}{\partial f^T_q} = \sum_{i=1}^{2N}\left[\frac{\partial\triangle B_i}{\partial f^T_q} - \triangle B_i\mathcal{W}_i\right]\mathcal{T}_i,$$

where

$$\begin{aligned}\mathcal{W}_{1,q} &= 0, \\ \mathcal{W}_{i,q} &= \sum_{j=2}^{i}\frac{\partial\triangle\delta_{j\to j-1}}{\partial f^T_q} \\ &= \sum_{j=2}^{i-1}\frac{\partial\triangle\delta_{j\to j-1}}{\partial f^T_q} + \frac{\partial\triangle\delta_{i\to i-1}}{\partial f^T_q} \\ &= \mathcal{W}_{i-1,q} + \frac{\partial\triangle\delta_{i\to i-1}}{\partial f^T_q}.\end{aligned}$$

2 Temperature Second Derivative

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\begin{aligned}
\frac{\partial^2 \triangle\delta^k_{i\to i-1}}{\partial f^T_q \partial f^T_{\tilde{q}}} &= \frac{\partial}{\partial f^T_{\tilde{q}}} \int_{\zeta_i}^{\zeta_{i-1}} \Bigg\{ F^k\beta^k \, \frac{n^k}{T} \, \eta^T_q(\zeta) \, \frac{ds}{dh} \, \frac{dh}{d\zeta} \\
&\quad + F^k\beta^k \frac{2(h+H^{\oplus})^2 \dfrac{\partial h}{\partial f^T_q} - 3(h_t+H^{\oplus})^2 \dfrac{\partial h}{\partial f^T_q} + (h+H^{\oplus})(h_t+H^{\oplus}) \dfrac{\partial h_t}{\partial f^T_q}}{\Big((h+H^{\oplus})^2-(h_t+H^{\oplus})^2\Big)^{\frac{3}{2}}} \, \frac{dh}{d\zeta} \\
&\quad + F^k\beta^k \, \frac{\eta^T_q(\zeta)}{T} \frac{ds}{dh} \, \frac{dh}{d\zeta} \Bigg\} d\zeta \\
&= \int_{\zeta_i}^{\zeta_{i-1}} \Bigg\{ \frac{\partial}{\partial f^T_{\tilde{q}}} \left[F^k\beta^k \, \frac{n^k}{T} \, \eta^T_q(\zeta) \, \frac{ds}{dh} \, \frac{dh}{d\zeta} \right] d\zeta \\
&\quad + \frac{\partial}{\partial f^T_{\tilde{q}}} \left[F^k\beta^k \frac{2(h+H^{\oplus})^2 \dfrac{\partial h}{\partial f^T_q} - 3(h_t+H^{\oplus})^2 \dfrac{\partial h}{\partial f^T_q} + (h+H^{\oplus})(h_t+H^{\oplus}) \dfrac{\partial h_t}{\partial f^T_q}}{\Big((h+H^{\oplus})^2-(h_t+H^{\oplus})^2\Big)^{\frac{3}{2}}} \, \frac{dh}{d\zeta} \right] \\
&\quad + \frac{\partial}{\partial f^T_{\tilde{q}}} \left[F^k\beta^k \, \frac{\eta^T_q(\zeta)}{T} \frac{ds}{dh} \, \frac{dh}{d\zeta} \right] \Bigg\} d\zeta
\end{aligned}
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3 The Planck Radiation Function Derivative

3.1 Analytical Expression for the Source Function Derivative for Temperature

The Planck radiation function B expressed in temperature units is given by

$$B(\zeta) \;=\; \frac{\mathrm{h}\nu}{k\big[\exp\big(\frac{\mathrm{h}\nu}{kT(\zeta)}\big)-1\big]},$$

or, in discrete form

$$B_i \;=\; \frac{\mathrm{h}\nu}{k\big[\exp\big(\frac{\mathrm{h}\nu}{kT_i}\big)-1\big]},$$

where h is Planck's constant, k is Boltzmann's constant, ν is the radiation frequency, and T is temperature.

The derivative of B with respect to ζ can be approximated by

$$\frac{dB}{d\zeta} \;\approx\; \frac{\triangle B}{\triangle\zeta}.$$

The differential source function can be approximated as $dB \approx \triangle B$:

$$\begin{aligned}
\triangle B_1 &= \frac{B_1+B_2}{2},\\
\triangle B_i &= \frac{B_{i+1}-B_{i-1}}{2},\\
\triangle B_{2N} &= I_0 - \frac{B_{2N-1}+B_{2N}}{2},
\end{aligned}$$

where $I_0 = 2.7$ is the background cosmic radiance incident on the atmosphere. Note that $\sum_{i=1}^{2N}\triangle B_i = I_0$.

The source function Jacobian for temperature is

$$\frac{\partial\triangle B_i}{\partial f_q^T} \;=\; \frac{\dfrac{\partial B_{i+1}}{\partial f_q^T} - \dfrac{\partial B_{i-1}}{\partial f_q^T}}{2}, \tag{1}$$

where

$$\begin{aligned}
\frac{\partial B_i}{\partial f_q^T} &= -\frac{\mathrm{h}\nu}{k\big[\exp\big(\frac{\mathrm{h}\nu}{kT_i}\big)-1\big]^2}\cdot\frac{-\mathrm{h}\nu}{kT_i^2}\exp\left(\frac{\mathrm{h}\nu}{kT_i}\right)\eta_q^T(\zeta_i)\\
&= \frac{B_i^2\exp\big(\frac{\mathrm{h}\nu}{kT_i}\big)}{T_i^2}\,\eta_q^T(\zeta_i).
\end{aligned}$$

3.2 Approximation of the Source Function Derivative for Temperature

The Planck radiation function B can be approximated by T:

$$B(\zeta) \;=\; \frac{\mathrm{h}\nu}{k\big[\exp\big(\frac{\mathrm{h}\nu}{kT(\zeta)}\big)-1\big]} \;\approx\; T(\zeta).$$

Hence,

$$\begin{aligned}
\triangle B_1 &= \frac{T_1+T_2}{2},\\
\triangle B_i &= \frac{T_{i+1}-T_{i-1}}{2},\\
\triangle B_{2N} &= I_0 - \frac{T_{2N-1}+T_{2N}}{2},
\end{aligned}$$

where $I_0 = 2.7$. Since $\dfrac{\partial T(\zeta)}{\partial f_q^T} = \eta_q^T(\zeta)$, the source function Jacobian for temperature can be approximated by

$$
\begin{aligned}
\frac{\partial \triangle B_1}{\partial f_q^T} &= \frac{\eta_q^T(\zeta_1) + \eta_q^T(\zeta_2)}{2}, \\
\frac{\partial \triangle B_i}{\partial f_q^T} &= \frac{\eta_q^T(\zeta_{i+1}) - \eta_q^T(\zeta_{i-1})}{2}, \\
\frac{\partial \triangle B_{2N}}{\partial f_q^T} &= -\frac{\eta_q^T(\zeta_{2N-1}) + \eta_q^T(\zeta_{2N})}{2}.
\end{aligned}
$$