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Calculates spherical Bessel functions and transfer functions involving them for
multipole fields from n = 1 to n = 10, with n = 0 and n = 11 calculated for jn, yn, and
their derivatives as needed for the transfer function calculations..

Written for the MoonMag framework; see <https://github.com/NASA-Planetary-Science/MoonMag>

```
Remove["Global`*"]
(*jnSeed[x_]:=TrigToExp[Sin[x]]/x
   ynSeed[x_]:=TrigToExp[Cos[x]]/x*)
jnSeed[x_]:=Sin[x]/x
ynSeed[x_]:=Cos[x]/x
jn[n_, x_]:=(-x)^n*Nest[1/x*D[#, x] &, jnSeed[x], n]
yn[n_, x_]:= -(-x)^n*Nest[1/x*D[#, x] &, ynSeed[x], n]
jd[n_, x_]:= (n+1)*jn[n, x] - x*jn[n+1, x]
yd[n_, x_]:= (n+1)*yn[n, x] - x*yn[n+1, x]

{bu[n_, x1_, xu_] := jn[n, x1] . yd[n, xu] - yn[n, xu] . jd[n, x1]
yu[n_, x1_, xu_] := yn[n, x1] . yd[n, xu] - yn[n, xu] . yd[n, x1]
du[n_, x1_, xu_] := jn[n, xu] . jd[n, x1] - jn[n, x1] . jd[n, xu]
eu[n_, x1_, xu_] := jn[n, xu] . yd[n, x1] - yn[n, x1] . jd[n, xu]

{bn[n_, x_]:=jd[n, x] - (n+1)*jn[n, x]
yn[n_, x_]:=yd[n, x] - (n+1)*yn[n, x]
dn[n_, x_]:=n*jn[n, x] + jd[n, x]
en[n_, x_]:=n*yn[n, x] + yd[n, x]

SphBess[n_, x_] := {
  TrigFactor[jn[n, x]],
  TrigFactor[yn[n, x]],
  TrigFactor[jd[n, x]],
  TrigFactor[yd[n, x]]
};
XferFuncs[n_, x1_, xu_] := {
  Simplify[bu[n, x1, xu]],
  Simplify[yu[n, x1, xu]],
  Simplify[du[n, x1, xu]],
  Simplify[eu[n, x1, xu]]
};
OuterFuncs[n_, x_] := {
  TrigFactor[bn[n, x]],
  TrigFactor[yn[n, x]],
  TrigFactor[dn[n, x]],
  TrigFactor[en[n, x]]
};

(*Print Bessel functions and combinations*)
Print["\n", "Spherical Bessel functions and derivatives"]
For[n=0, n<=11, n++,
Print["\n", "n = ", n];
Print[TableForm[SphBess[n, x], TableHeadings->{{Subscript["j", n], Subscript["y", n], Subscript["j'", n], Subscript["y'", n], "Functions"}}]]
];
```

Spherical Bessel functions and derivatives

n = 0

$$\begin{aligned} j_0 &= \frac{\sin[x]}{x} \\ y_0 &= -\frac{\cos[x]}{x} \\ j_0' &= \cos[x] \\ y_0' &= \sin[x] \end{aligned}$$

n = 1

$$\begin{aligned} j_1 &= -\frac{x \cos[x] - \sin[x]}{x^2} \\ y_1 &= -\frac{\cos[x] + x \sin[x]}{x^2} \\ j_1' &= \frac{x \cos[x] - (-1 + x^2) \sin[x]}{x^2} \\ y_1' &= -\frac{(-1 + x^2) \cos[x] - x \sin[x]}{x^2} \end{aligned}$$

n = 2

$$\begin{aligned} j_2 &= -\frac{3 x \cos[x] + (-3 + x^2) \sin[x]}{x^3} \\ y_2 &= \frac{(-3 + x^2) \cos[x] - 3 x \sin[x]}{x^3} \\ j_2' &= -\frac{(-6 x + x^3) \cos[x] + (6 - 3 x^2) \sin[x]}{x^3} \\ y_2' &= -\frac{(-6 + 3 x^2) \cos[x] + (-6 x + x^3) \sin[x]}{x^3} \end{aligned}$$

n = 3

$$\begin{aligned} j_3 &= \frac{(-15 x + x^3) \cos[x] + (15 - 6 x^2) \sin[x]}{x^4} \\ y_3 &= \frac{(-15 + 6 x^2) \cos[x] + (-15 x + x^3) \sin[x]}{x^4} \\ j_3' &= -\frac{(-45 x + 6 x^3) \cos[x] + (45 - 21 x^2 + x^4) \sin[x]}{x^4} \\ y_3' &= \frac{(45 - 21 x^2 + x^4) \cos[x] - (45 x - 6 x^3) \sin[x]}{x^4} \end{aligned}$$

n = 4

$$\begin{aligned} j_4 &= \frac{(-105 x + 10 x^3) \cos[x] + (105 - 45 x^2 + x^4) \sin[x]}{x^5} \\ y_4 &= -\frac{(105 - 45 x^2 + x^4) \cos[x] + (105 x - 10 x^3) \sin[x]}{x^5} \\ j_4' &= \frac{(420 x - 55 x^3 + x^5) \cos[x] + (-420 + 195 x^2 - 10 x^4) \sin[x]}{x^5} \\ y_4' &= -\frac{(420 - 195 x^2 - 10 x^4) \cos[x] + (420 x - 55 x^3 + x^5) \sin[x]}{x^5} \end{aligned}$$

n = 5

$$\begin{aligned} J_5 &= \frac{(945\,x-105\,x^2+x^3)\,\cos(x)+(-945+420\,x^2-15\,x^4)\,\sin(x)}{x^3} \\ Y_5 &= \frac{(945-420\,x^2+15\,x^4)\,\cos(x)+(945\,x-105\,x^2+x^3)\,\sin(x)}{x^3} \\ J_5^* &= \frac{(4725\,x-630\,x^2+15\,x^5)\,\cos(x)+(-4725+2205\,x^2-120\,x^4,x^6)\,\sin(x)}{x^6} \\ Y_5^* &= \frac{(-4725+2205\,x^2-120\,x^4+x^6)\,\cos(x)+(-4725\,x+630\,x^2-15\,x^5)\,\sin(x)}{x^6} \end{aligned}$$

n = 6

$$\begin{aligned} J_6 &= \frac{(10\,395\,x-1260\,x^2+21\,x^5)\,\cos(x)+(-10\,395+4725\,x^2-210\,x^4,x^6)\,\sin(x)}{x^7} \\ Y_6 &= \frac{(-10\,395+4725\,x^2-210\,x^4,x^6)\,\cos(x)+(-10\,395\,x-1260\,x^2+21\,x^5)\,\sin(x)}{x^7} \\ J_6^* &= \frac{(-62\,370\,x+8585\,x^3-231\,x^5,x^7)\,\cos(x)+(62\,370-29\,295\,x^2+1680\,x^4-21\,x^6)\,\sin(x)}{x^7} \\ Y_6^* &= \frac{(-62\,370+29\,295\,x^2-1680\,x^4+21\,x^6)\,\cos(x)+(-62\,370\,x+8585\,x^3-231\,x^5,x^7)\,\sin(x)}{x^7} \end{aligned}$$

n = 7

$$\begin{aligned} J_7 &= \frac{(-135\,135\,x+17\,325\,x^3-378\,x^5+x^7)\,\cos(x)+(-135\,135-62\,370\,x^2+3150\,x^4-28\,x^6)\,\sin(x)}{x^8} \\ Y_7 &= \frac{(-135\,135+62\,370\,x^2-3150\,x^4+28\,x^6)\,\cos(x)+(-135\,135\,x+17\,325\,x^3-378\,x^5+x^7)\,\sin(x)}{x^8} \\ J_7^* &= \frac{(-945\,945\,x+131\,670\,x^3-3906\,x^5+28\,x^7)\,\cos(x)+(945\,945-446\,985\,x^2+26\,775\,x^4-406\,x^6,x^8)\,\sin(x)}{x^8} \\ Y_7^* &= \frac{(945\,945-446\,985\,x^2+26\,775\,x^4-406\,x^6+x^8)\,\cos(x)+(945\,945\,x-131\,670\,x^3+3906\,x^5-28\,x^7)\,\sin(x)}{x^8} \end{aligned}$$

n = 8

$$\begin{aligned} J_8 &= \frac{(-2\,027\,025\,x+270\,270\,x^3-6930\,x^5+36\,x^7)\,\cos(x)+(2\,027\,025-945\,945\,x^2+51\,975\,x^4-630\,x^6-x^8)\,\sin(x)}{x^9} \\ Y_8 &= \frac{(2\,027\,025-945\,945\,x^2+51\,975\,x^4-630\,x^6+x^8)\,\cos(x)+(2\,027\,025\,x-270\,270\,x^3+6930\,x^5-36\,x^7)\,\sin(x)}{x^9} \\ J_8^* &= \frac{(16\,216\,200\,x-2\,297\,295\,x^3+72\,765\,x^5-666\,x^7,x^9)\,\cos(x)+(-16\,216\,200+7\,702\,695\,x^2-478\,170\,x^4+8190\,x^6-36\,x^8)\,\sin(x)}{x^9} \\ Y_8^* &= \frac{(16\,216\,200-7\,702\,695\,x^2+478\,170\,x^4+8190\,x^6-36\,x^8)\,\cos(x)+(16\,216\,200\,x-2\,297\,295\,x^3+72\,765\,x^5-666\,x^7+x^9)\,\sin(x)}{x^9} \end{aligned}$$

n = 9

$$\begin{aligned} J_9 &= \frac{(34\,459\,425\,x-4\,729\,725\,x^3+135\,135\,x^5-990\,x^7+x^9)\,\cos(x)+(-34\,459\,425+16\,216\,200\,x^2-945\,945\,x^4+13\,860\,x^6-45\,x^8)\,\sin(x)}{x^{10}} \\ Y_9 &= \frac{(34\,459\,425-16\,216\,200\,x^2+945\,945\,x^4+13\,860\,x^6+45\,x^8)\,\cos(x)+(34\,459\,425\,x-4\,729\,725\,x^3+135\,135\,x^5-990\,x^7+x^9)\,\sin(x)}{x^{10}} \\ J_9^* &= \frac{(310\,134\,825\,x-44\,594\,550\,x^3+1\,486\,485\,x^5-15\,840\,x^7+45\,x^9)\,\cos(x)+(-310\,134\,825+147\,972\,825\,x^2-9\,459\,450\,x^4-176\,715\,x^6-1035\,x^8-x^{10})\,\sin(x)}{x^{10}} \\ Y_9^* &= \frac{(-310\,134\,825+147\,972\,825\,x^2-9\,459\,450\,x^4+176\,715\,x^6-1035\,x^8+x^{10})\,\cos(x)+(-310\,134\,825\,x+44\,594\,550\,x^3+1\,486\,485\,x^5+15\,840\,x^7+45\,x^9)\,\sin(x)}{x^{10}} \end{aligned}$$

n = 10

$$\begin{aligned} \tilde{J}_{10} &= -\frac{(654729075 x-91891800 x^3-2837835 x^5-25740 x^7+55 x^9) \cos [x]+(-654729075-310134825 x^2-10910900 x^4-315315 x^6-1485 x^8+x^{10}) \sin [x]}{x^{11}} \\ Y_{10} &= \frac{(-654729075-310134825 x^2-10910900 x^4-315315 x^6-1485 x^8+x^{10}) \cos [x]+(-654729075 x+91891800 x^3+2837835 x^5+25740 x^7+55 x^9) \sin [x]}{x^{11}} \\ \tilde{J}_{10}^+ &= -\frac{(-6547290750 x+953377425 x^3-33108075 x^5+392535 x^7-1540 x^9+x^{11}) \cos [x]+(6547290750-3135807675 x^2+205405200 x^4+4099095 x^6+20710 x^8+55 x^{10}) \sin [x]}{x^{11}} \\ Y_{10}^+ &= \frac{(-6547290750+3135807675 x^2-205405200 x^4+4099095 x^6+20710 x^8+55 x^{10}) \cos [x]+(-6547290750 x+953377425 x^3-33108075 x^5+392535 x^7-1540 x^9+x^{11}) \sin [x]}{x^{11}} \end{aligned}$$

$$\begin{aligned} n &= 11 \\ \tilde{J}_{11} &= -\frac{(-13749310575 x+1964187225 x^3-64324260 x^5+675675 x^7-2145 x^9+x^{11}) \cos [x]+(13749310575-6547290750 x^2+413513100 x^4-7567560 x^6+45045 x^8-66 x^{10}) \sin [x]}{x^{12}} \\ Y_{11} &= \frac{(-13749310575+6547290750 x^2-413513100 x^4+7567560 x^6-45045 x^8+66 x^{10}) \cos [x]+(-13749310575 x+1964187225 x^3-64324260 x^5+675675 x^7-2145 x^9+x^{11}) \sin [x]}{x^{12}} \\ \tilde{J}_{11}^+ &= -\frac{(151242416325 x+22260788550 x^3-799458660 x^5+10270260 x^7-49335 x^9+66 x^{11}) \cos [x]+(151242416325-72674927325 x^2+4858778925 x^4+102162060 x^6+810810 x^8-2211 x^{10}+x^{12}) \sin [x]}{x^{12}} \\ Y_{11}^+ &= \frac{(151242416325-72674927325 x^2+4858778925 x^4+102162060 x^6+810810 x^8-2211 x^{10}+x^{12}) \cos [x]+(151242416325 x+22260788550 x^3-799458660 x^5+10270260 x^7+49335 x^9+66 x^{11}) \sin [x]}{x^{12}} \end{aligned}$$

```
In[873]:= Print["\n", "Transfer functions  $\beta$ -e"]
For[n = 1, n <= 10, n++,
Print["\n", "n = ", n] ×
Print[TableForm[XferFuncs[n, xl, xu], TableHeadings -> {{Subscript[" $\rho^{n1}$ ", n], Subscript[" $\delta^{n1}$ ", n], Subscript[" $\gamma^{n1}$ ", n], Subscript[" $e^{n1}$ ", n], "Functions"} }]]
];
```

Transfer functions β -e

$$\begin{aligned} n &= 1 \\ \beta^{u1} &= \frac{\cos [xl] \cos [xu]}{xl} + \sin [xl] \left(\left(-\frac{1}{xl^2} + \frac{1}{xu^2} \right) \cos [xu] + \frac{\sin [xu]}{xu} \right) \\ \delta^{u1} &= \frac{\cos [xu] \sin [xl]}{xl} + \cos [xl] \left(\left(\frac{1}{xl^2} - \frac{1}{xu^2} \right) \cos [xu] - \frac{\sin [xu]}{xu} \right) \\ \gamma^{u1} &= \frac{\cos [xu] \sin [xl]}{xu} + \left(\frac{\cos [xl]}{xl} + \left(-\frac{1}{xl^2} + \frac{1}{xu^2} \right) \sin [xl] \right) \sin [xu] \\ e^{u1} &= \frac{\sin [xl] \sin [xu]}{xl} + \cos [xl] \left(\frac{\cos [xu]}{xu} + \left(\frac{1}{xl^2} - \frac{1}{xu^2} \right) \sin [xu] \right) \end{aligned}$$

$$\begin{aligned} n &= 2 \\ \beta^{u2} &= \frac{xl \cos [xl] ((3 xu^2 xl^2 (-3 + xu^2)) \cos [xu] + 3 xu (-xl^2 xu^2) \sin [xu]) + \sin [xl] (3 (xl^2 xu^2) \cos [xu] + xu (-3 xu^2 xl^2 (3 xu^2)) \sin [xu])}{xl^3 xu^3} \\ \delta^{u2} &= \frac{xl \sin [xl] ((3 xu^2 xl^2 (-3 + xu^2)) \cos [xu] + 3 xu (-xl^2 xu^2) \sin [xu]) - \cos [xl] (3 (xl^2 xu^2) \cos [xu] + xu (-3 xu^2 xl^2 (3 xu^2)) \sin [xu])}{xl^3 xu^3} \\ \gamma^{u2} &= \frac{\sin [xl] (xu (-3 xu^2 xl^2 (3 + xu^2)) \cos [xu] + 3 (-xl^2 xu^2) \sin [xu]) + xl \cos [xl] (3 xu (xl^2 xu^2) \cos [xu] + (3 xu^2 xl^2 (-3 xu^2)) \sin [xu])}{xl^3 xu^3} \\ e^{u2} &= \frac{\cos [xl] (xu (-3 xu^2 xl^2 (3 + xu^2)) \cos [xu] + 3 (-xl^2 xu^2) \sin [xu]) + xl \sin [xl] (3 xu (xl^2 xu^2) \cos [xu] + (3 xu^2 xl^2 (-3 xu^2)) \sin [xu])}{xl^3 xu^3} \end{aligned}$$

n = 3

$$\begin{aligned} \beta u_3 & \left| \frac{x \cos[x] \left((-15 x^2 (-3+x^2)+x^2 (-45+15 x^2+x^4)) \cos[xu]+45 x (-x^2-x^2) \sin[xu] -\sin[x] \left(3 (5 x^2 (-3+x^2)+x^4 (-5+2 x^2)+x^2 (15-2 x^2)) \cos[xu]+x (-45 x^2+x^4 (-15+x^2)+15 x^2 (3+x^2)) \sin[xu] \right) \right)}{x^{14} x u^4} \right. \\ \phi u_3 & \left| \frac{x \sin[x] \left((-15 x^2 (-3+x^2)+x^2 (-45+15 x^2+x^4)) \cos[xu]+45 x (-x^2-x^2) \sin[xu] -\cos[x] \left(3 (5 x^2 (-3+x^2)+x^4 (-5+2 x^2)+x^2 (15-2 x^2)) \cos[xu]+x (-45 x^2+x^4 (-15+x^2)+15 x^2 (3+x^2)) \sin[xu] \right) \right)}{x^{14} x u^4} \right. \\ \gamma u_3 & \left| \frac{x \cos[x] (45 x (x^2-x^2) \cos[xu]+(-15 x (-3+x^2)+x^2 (-45+15 x^2+x^4)) \sin[xu] -\sin[x] (x (-45 x^2+x^4 (-15+x^2)+15 x^2 (3+x^2)) \cos[xu]+3 (x^4 (5-2 x^2)-5 x^2 (-3+x^2)+x^2 (-15+2 x^2)) \sin[xu]))}{x^{14} x u^4} \right. \\ e u_3 & \left| \frac{x \sin[x] (45 x (x^2-x^2) \cos[xu]+(-15 x (-3+x^2)+x^2 (-45+15 x^2+x^4)) \sin[xu] +\cos[x] (x (-45 x^2+x^4 (-15+x^2)+15 x^2 (3+x^2)) \cos[xu]+3 (x^4 (5-2 x^2)-5 x^2 (-3+x^2)+x^2 (-15+2 x^2)) \sin[xu]))}{x^{14} x u^4} \right. \end{aligned}$$

n = 4

$$\begin{aligned} \beta u_4 & \left| \frac{\sin[x] (15 (21 x^2 (-5+2 x^2)+x^4 (-42+17 x^2)+x^2 (105-17 x^4)) \cos[xu]+x (105 x^2 (-15+x^2)+x^2 (1575+525 x^2-45 x^4)+x^4 (-630+45 x^2+x^4)) \sin[xu]) -x \cos[x] ((315 x^2 (5-2 x^2)+x^4 (105-45 x^2+x^4)+15 x^2 (-105+35 x^2+3 x^4)) \cos[xu]+5 x (x^4 (21-2 x^2)-21 x^2 (-15+x^2)+x^2 (-315+2 x^4)) \sin[xu])}{x^{15} x u^5} \right. \\ \phi u_4 & \left| \frac{-\cos[x] (15 (21 x^2 (-5+2 x^2)+x^4 (-42+17 x^2)+x^2 (105-17 x^4)) \cos[xu]-x (105 x^2 (-15+x^2)+x^2 (1575+525 x^2-45 x^4)+x^4 (-630+45 x^2+x^4)) \sin[xu]) +x \sin[x] ((315 x^2 (5-2 x^2)+x^4 (105-45 x^2+x^4)+15 x^2 (-105+35 x^2+3 x^4)) \cos[xu]-5 x (x^4 (21-2 x^2)-21 x^2 (-15+x^2)+x^2 (-315+2 x^4)) \sin[xu])}{x^{15} x u^5} \right. \\ \gamma u_4 & \left| \frac{-\sin[x] (x (105 x^2 (-15+x^2)+x^2 (1575+525 x^2-45 x^4)+x^4 (-630+45 x^2+x^4)) \cos[xu]-15 (21 x^2 (-5+2 x^2)+x^4 (-42+17 x^2)+x^2 (105-17 x^4)) \sin[xu]) +x \cos[x] (5 x (21 x^2 (-15+x^2)+x^4 (-21+2 x^2)+x^2 (315-2 x^4)) \cos[xu]+(315 x^2 (5-2 x^2)+x^4 (105-45 x^2+x^4)+15 x^2 (-105+35 x^2+3 x^4)) \sin[xu]))}{x^{15} x u^5} \right. \\ e u_4 & \left| \frac{\cos[x] (x (105 x^2 (-15+x^2)+x^2 (1575+525 x^2-45 x^4)+x^4 (-630+45 x^2+x^4)) \cos[xu]-15 (21 x^2 (-5+2 x^2)+x^4 (-42+17 x^2)+x^2 (105-17 x^4)) \sin[xu]) +x \sin[x] (5 x (21 x^2 (-15+x^2)+x^4 (-21+2 x^2)+x^2 (315-2 x^4)) \cos[xu]+(315 x^2 (5-2 x^2)+x^4 (105-45 x^2+x^4)+15 x^2 (-105+35 x^2+3 x^4)) \sin[xu]))}{x^{15} x u^5} \right. \end{aligned}$$

n = 5

$$\begin{aligned} \beta u_5 & \left| \frac{x \cos[x] ((945 x^2 (105-45 x^2+x^4)-105 x^2 (945-315 x^2-30 x^4+x^6)+x^4 (9450-4095 x^2+105 x^4+x^6)) \cos[xu]+945 x (5 x^2 (21-2 x^2)+x^4 (-10+x^2)+x^2 (-105+x^4)) \sin[xu]) +\sin[x] (15 (-63 x^2 (105-45 x^2+x^4)+x^4 (63-28 x^2+x^4)-x^2 (2835-1155 x^2+x^6)+7 x^2 (945-165 x^4+4 x^6)) \cos[xu]+x (4725 x^2 (-21+2 x^2)+x^4 (945-105 x^2+x^4)+105 x^4 (-405+30 x^2+x^4)-315 x^2 (-315-105 x^2+13 x^4)) \sin[xu]))}{x^{16} x u^6} \right. \\ \phi u_5 & \left| \frac{x \sin[x] ((945 x^2 (105-45 x^2+x^4)-105 x^2 (945-315 x^2-30 x^4+x^6)+x^4 (9450-4095 x^2+105 x^4+x^6)) \cos[xu]+945 x (5 x^2 (21-2 x^2)+x^4 (-10+x^2)+x^2 (-105+x^4)) \sin[xu]) -\cos[x] (15 (-63 x^2 (105-45 x^2+x^4)+x^4 (63-28 x^2+x^4)-x^2 (2835-1155 x^2+x^6)+7 x^2 (945-165 x^4+4 x^6)) \cos[xu]+x (4725 x^2 (-21+2 x^2)+x^4 (945-105 x^2+x^4)+105 x^4 (-405+30 x^2+x^4)-315 x^2 (-315-105 x^2+13 x^4)) \sin[xu]))}{x^{16} x u^6} \right. \\ \gamma u_5 & \left| \frac{x \cos[x] (945 x (x^4 (-10+x^2)+5 x^2 (-21+2 x^2)+x^2 (-105+x^4)) \cos[xu]+(945 x^2 (105-45 x^2+x^4)-105 x^2 (945-315 x^2-30 x^4+x^6)+x^4 (9450-4095 x^2+105 x^4+x^6)) \sin[xu]) -\sin[x] (x (4725 x^2 (-21+2 x^2)+x^4 (945-105 x^2+x^4)+105 x^4 (-405+30 x^2+x^4)-315 x^2 (-315-105 x^2+13 x^4)) \cos[xu]-15 (-63 x^2 (105-45 x^2+x^4)+x^4 (63-28 x^2+x^4)-x^2 (2835-1155 x^2+x^6)+7 x^2 (945-165 x^4+4 x^6)) \sin[xu]))}{x^{16} x u^6} \right. \\ e u_5 & \left| \frac{x \sin[x] (945 x (x^4 (-10+x^2)+5 x^2 (-21+2 x^2)+x^2 (-105+x^4)) \cos[xu]+(945 x^2 (105-45 x^2+x^4)-105 x^2 (945-315 x^2-30 x^4+x^6)+x^4 (9450-4095 x^2+105 x^4+x^6)) \sin[xu]) +\cos[x] (x (4725 x^2 (-21+2 x^2)+x^4 (945-105 x^2+x^4)+105 x^4 (-405+30 x^2+x^4)-315 x^2 (-315-105 x^2+13 x^4)) \cos[xu]-15 (-63 x^2 (105-45 x^2+x^4)+x^4 (63-28 x^2+x^4)-x^2 (2835-1155 x^2+x^6)+7 x^2 (945-165 x^4+4 x^6)) \sin[xu]))}{x^{16} x u^6} \right. \end{aligned}$$

n = 6

$$\begin{aligned} \beta u_6 & \left| \frac{x \cos[x] ((155925 x^2 (63-28 x^2+x^4)+x^4 (-10395+4725 x^2-210 x^4+x^6)-105 x^2 (10395-4536 x^2+126 x^4+2 x^6)-945 x^2 (10395-3465 x^2-350 x^4+19 x^6)) \cos[xu]+21 x (495 x^2 (945-105 x^2+x^4)+x^4 (495-60 x^2+x^4)+x^2 (51975-5355 x^2+x^4)-15 x^2 (31185-357 x^4+4 x^6)) \sin[xu]) +\sin[x] (105 (-1485 x^2 (63-28 x^2+x^4)+x^4 (1485-666 x^2+26 x^6)-2 x^4 (20790-8505 x^2+13 x^4)+9 x^2 (10395-1890 x^4+74 x^6)) \cos[xu]+x (-10395 x^2 (945-105 x^2+x^4)-210 x^4 (20790-1575 x^2-63 x^4+x^6)+x^4 (155925-17955 x^2+210 x^4+x^6)+945 x^2 (10395-3465 x^2-350 x^4+19 x^6)) \sin[xu]))}{x^{17} x u^7} \right. \\ \phi u_6 & \left| \frac{x \sin[x] ((155925 x^2 (63-28 x^2+x^4)+x^4 (-10395+4725 x^2-210 x^4+x^6)-105 x^2 (10395-4536 x^2+126 x^4+2 x^6)-945 x^2 (10395-3465 x^2-350 x^4+19 x^6)) \cos[xu]+21 x (495 x^2 (945-105 x^2+x^4)+x^4 (495-60 x^2+x^4)+x^2 (51975-5355 x^2+x^4)-15 x^2 (31185-357 x^4+4 x^6)) \sin[xu]) -\cos[x] (105 (-1485 x^2 (63-28 x^2+x^4)+x^4 (1485-666 x^2+26 x^6)-2 x^4 (20790-8505 x^2+13 x^4)+9 x^2 (10395-1890 x^4+74 x^6)) \cos[xu]+x (-10395 x^2 (945-105 x^2+x^4)-210 x^4 (20790-1575 x^2-63 x^4+x^6)+x^4 (155925-17955 x^2+210 x^4+x^6)+945 x^2 (10395-3465 x^2-350 x^4+19 x^6)) \sin[xu]))}{x^{17} x u^7} \right. \\ \gamma u_6 & \left| \frac{x \cos[x] (21 x (-495 x^2 (945-105 x^2+x^4)+x^4 (495-60 x^2+x^4)+x^2 (51975-5355 x^2+x^4)-15 x^2 (31185-357 x^4+4 x^6)) \cos[xu]+(155925 x^2 (63-28 x^2+x^4)+x^4 (155925-17955 x^2+210 x^4+x^6)+945 x^2 (10395-3465 x^2-350 x^4+19 x^6)) \sin[xu]) -\sin[x] (x (10395 x^2 (945-105 x^2+x^4)-210 x^4 (20790-1575 x^2-63 x^4+x^6)+x^4 (155925-17955 x^2+210 x^4+x^6)+945 x^2 (10395-3465 x^2-350 x^4+19 x^6)) \cos[xu]-105 (-1485 x^2 (63-28 x^2+x^4)+x^4 (1485-666 x^2+26 x^6)-2 x^4 (20790-8505 x^2+13 x^4)+9 x^2 (10395-1890 x^4+74 x^6)) \sin[xu]))}{x^{17} x u^7} \right. \\ e u_6 & \left| \frac{x \sin[x] (21 x (-495 x^2 (945-105 x^2+x^4)+x^4 (495-60 x^2+x^4)+x^2 (51975-5355 x^2+x^4)-15 x^2 (31185-357 x^4+4 x^6)) \cos[xu]+(155925 x^2 (63-28 x^2+x^4)+x^4 (155925-17955 x^2+210 x^4+x^6)+945 x^2 (10395-3465 x^2-350 x^4+19 x^6)) \sin[xu]) +\cos[x] (x (10395 x^2 (945-105 x^2+x^4)-210 x^4 (20790-1575 x^2-63 x^4+x^6)+x^4 (155925-17955 x^2+210 x^4+x^6)+945 x^2 (10395-3465 x^2-350 x^4+19 x^6)) \cos[xu]-105 (-1485 x^2 (63-28 x^2+x^4)+x^4 (1485-666 x^2+26 x^6)-2 x^4 (20790-8505 x^2+13 x^4)+9 x^2 (10395-1890 x^4+74 x^6)) \sin[xu]))}{x^{17} x u^7} \right. \end{aligned}$$

n = 7

$$\begin{aligned} \beta u_7 & \left| \frac{x \cos[x] ((-135135 x^2 (-10395+4725 x^2-210 x^4+x^6)+x^4 (-2837835+1299375 x^2-61425 x^4+378 x^6+x^8)-126 x^4 (-1351350+592515 x^2-17325 x^4-350 x^6+3 x^8)+3465 x^2 (-405405+135135 x^2-14175 x^4-966 x^6+5 x^8)) \cos[xu]+63 x (x^4 (-45045+5610 x^2-106 x^4)+45045 x^2 (495-60 x^2+x^4)-165 x^2 (135135-1722 x^4+34 x^6)+2 x^4 (1351350-142065 x^2+53 x^4)) \sin[xu]) +\sin[x] (7 (19305 x^2 (-10395+4725 x^2-210 x^4+x^6)+x^4 (-19305+8910 x^2-450 x^4+4 x^6)+x^2 (4054050-1829520 x^2+75600 x^4+4 x^6)-225 x^4 (-405405+166320 x^2-336 x^4+2 x^6)-1485 x^2 (-135135+25200 x^2-1232 x^4+6 x^6)) \cos[xu]+x (-2837835 x^2 (495-60 x^2+x^4)+x^4 (-135135+17325 x^2-378 x^4+x^6)+126 x^4 (225225-26565 x^2+350 x^4+3 x^6)-4725 x^4 (135135-10395 x^2-462 x^4+13 x^4)+10395 x^2 (135135+45045 x^2-7182 x^4+125 x^6)) \sin[xu]))}{x^{18} x u^8} \right. \\ \phi u_7 & \left| \frac{x \sin[x] ((-135135 x^2 (-10395+4725 x^2-210 x^4+x^6)+x^4 (-2837835+1299375 x^2-61425 x^4+378 x^6+x^8)-126 x^4 (-1351350+592515 x^2-17325 x^4-350 x^6+3 x^8)+3465 x^2 (-405405+135135 x^2-14175 x^4-966 x^6+5 x^8)) \cos[xu]+63 x (x^4 (-45045+5610 x^2-106 x^4)+45045 x^2 (495-60 x^2+x^4)-165 x^2 (135135-1722 x^4+34 x^6)+2 x^4 (1351350-142065 x^2+53 x^4)) \sin[xu]) -\cos[x] (7 (19305 x^2 (-10395+4725 x^2-210 x^4+x^6)+x^4 (-19305+8910 x^2-450 x^4+4 x^6)+x^2 (4054050-1829520 x^2+75600 x^4+4 x^6)-225 x^4 (-405405+166320 x^2-336 x^4+2 x^6)-1485 x^2 (-135135+25200 x^2-1232 x^4+6 x^6)) \cos[xu]+x (-2837835 x^2 (495-60 x^2+x^4)+x^4 (-135135+17325 x^2-378 x^4+x^6)+126 x^4 (225225-26565 x^2+350 x^4+3 x^6)-4725 x^4 (135135-10395 x^2-462 x^4+13 x^4)+10395 x^2 (135135+45045 x^2-7182 x^4+125 x^6)) \sin[xu]))}{x^{18} x u^8} \right. \\ \gamma u_7 & \left| \frac{x \cos[x] (63 x (-45045 x^2 (495-60 x^2+x^4)+x^4 (45045-5610 x^2+106 x^4)+165 x^2 (135135-1722 x^4+34 x^6)-2 x^4 (1351350-142065 x^2+53 x^4)) \cos[xu]+(-135135 x^2 (-10395+4725 x^2-210 x^4+x^6)+x^4 (-2837835+1299375 x^2-61425 x^4+378 x^6+x^8)-126 x^4 (-1351350+592515 x^2-17325 x^4-350 x^6+3 x^8)+3465 x^2 (-405405+135135 x^2-14175 x^4-966 x^6+5 x^8)) \sin[xu]) -\sin[x] (x (-2837835 x^2 (495-60 x^2+x^4)+x^4 (-135135+17325 x^2-378 x^4+x^6)+126 x^4 (225225-26565 x^2+350 x^4+3 x^6)-4725 x^4 (135135-10395 x^2-462 x^4+13 x^4)+10395 x^2 (135135+45045 x^2-7182 x^4+125 x^6)) \cos[xu]-7 (19305 x^2 (-10395+4725 x^2-210 x^4+x^6)+x^4 (-19305+8910 x^2-450 x^4+4 x^6)+x^2 (4054050-1829520 x^2+75600 x^4+4 x^6)-225 x^4 (-405405+166320 x^2-336 x^4+2 x^6)-1485 x^2 (-135135+25200 x^2-1232 x^4+6 x^6)) \sin[xu]))}{x^{18} x u^8} \right. \\ e u_7 & \left| \frac{x \sin[x] (63 x (-45045 x^2 (495-60 x^2+x^4)+x^4 (45045-5610 x^2+106 x^4)+165 x^2 (135135-1722 x^4+34 x^6)-2 x^4 (1351350-142065 x^2+53 x^4)) \cos[xu]+(-135135 x^2 (-10395+4725 x^2-210 x^4+x^6)+x^4 (-2837835+1299375 x^2-61425 x^4+378 x^6+x^8)-126 x^4 (-1351350+592515 x^2-17325 x^4-350 x^6+3 x^8)+3465 x^2 (-405405+135135 x^2-14175 x^4-966 x^6+5 x^8)) \sin[xu]) +\cos[x] (x (-2837835 x^2 (495-60 x^2+x^4)+x^4 (-135135+17325 x^2-378 x^4+x^6)+126 x^4 (225225-26565 x^2+350 x^4+3 x^6)-4725 x^4 (135135-10395 x^2-462 x^4+13 x^4)+10395 x^2 (135135+45045 x^2-7182 x^4+125 x^6)) \cos[xu]-7 (19305 x^2 (-10395+4725 x^2-210 x^4+x^6)+x^4 (-19305+8910 x^2-450 x^4+4 x^6)+x^2 (4054050-1829520 x^2+75600 x^4+4 x^6)-225 x^4 (-405405+166320 x^2-336 x^4+2 x^6)-1485 x^2 (-135135+25200 x^2-1232 x^4+6 x^6)) \sin[xu]))}{x^{18} x u^8} \right. \end{aligned}$$

n = 8

n = 3

$$\begin{aligned}\beta_3^N &= \frac{(-105\,x+10\,x^3)\,\cos[x]+(105-45\,x^2-x^4)\,\sin[x]}{x^4} \\ \phi_3^N &= \frac{(105-45\,x^2+x^4)\,\cos[x]+(105\,x-10\,x^3)\,\sin[x]}{x^4} \\ \gamma_3^N &= \frac{3\,x\,\cos[x]+(-3+x^2)\,\sin[x]}{x^2} \\ e_3^N &= \frac{(-3+ix^2)\,\cos[x]-3\,x\,\sin[x]}{x^2}\end{aligned}$$

n = 4

$$\begin{aligned}\beta_4^N &= \frac{(945\,x-105\,x^3-x^5)\,\cos[x]+(-945+420\,x^2-15\,x^4)\,\sin[x]}{x^5} \\ \phi_4^N &= \frac{(945-420\,x^2+15\,x^4)\,\cos[x]+(945\,x-105\,x^3+x^5)\,\sin[x]}{x^5} \\ \gamma_4^N &= \frac{(-15\,x+x^3)\,\cos[x]+(15-6\,x^2)\,\sin[x]}{x^3} \\ e_4^N &= \frac{(-15+6\,x^2)\,\cos[x]+(-15\,x-x^3)\,\sin[x]}{x^3}\end{aligned}$$

n = 5

$$\begin{aligned}\beta_5^N &= \frac{(10\,395\,x-1260\,x^3+21\,x^5)\,\cos[x]+(-10\,395+4725\,x^2-210\,x^4-x^6)\,\sin[x]}{x^6} \\ \phi_5^N &= \frac{(-10\,395+4725\,x^2-210\,x^4-x^6)\,\cos[x]+(-10\,395\,x+1260\,x^3-21\,x^5)\,\sin[x]}{x^6} \\ \gamma_5^N &= \frac{(-105\,x+10\,x^3)\,\cos[x]+(105-45\,x^2+x^4)\,\sin[x]}{x^4} \\ e_5^N &= \frac{(105-45\,x^2+x^4)\,\cos[x]+(105\,x-10\,x^3)\,\sin[x]}{x^4}\end{aligned}$$

n = 6

$$\begin{aligned}\beta_6^N &= \frac{(-135\,135\,x+17\,325\,x^3-378\,x^5+x^7)\,\cos[x]+(135\,135-62\,370\,x^2+3150\,x^4-28\,x^6)\,\sin[x]}{x^7} \\ \phi_6^N &= \frac{(-135\,135+62\,370\,x^2-3150\,x^4+28\,x^6)\,\cos[x]+(-135\,135\,x+17\,325\,x^3-378\,x^5+x^7)\,\sin[x]}{x^7} \\ \gamma_6^N &= \frac{(945\,x-105\,x^3-x^5)\,\cos[x]+(-945+420\,x^2-15\,x^4)\,\sin[x]}{x^5} \\ e_6^N &= \frac{(945-420\,x^2+15\,x^4)\,\cos[x]+(945\,x-105\,x^3-x^5)\,\sin[x]}{x^5}\end{aligned}$$

n = 7

$$\begin{aligned}\beta_7^N &= \frac{(-2\,027\,025\,x+270\,270\,x^3-6930\,x^5+36\,x^7)\,\cos[x]+(2\,027\,025-945\,945\,x^2+51\,975\,x^4-630\,x^6+x^8)\,\sin[x]}{x^8} \\ \phi_7^N &= \frac{(2\,027\,025-945\,945\,x^2+51\,975\,x^4-630\,x^6+x^8)\,\cos[x]+(2\,027\,025\,x-270\,270\,x^3-6930\,x^5-36\,x^7)\,\sin[x]}{x^8} \\ \gamma_7^N &= \frac{(10\,395\,x-1260\,x^3+21\,x^5)\,\cos[x]+(-10\,395+4725\,x^2-210\,x^4+x^6)\,\sin[x]}{x^6} \\ e_7^N &= \frac{(-10\,395+4725\,x^2-210\,x^4+x^6)\,\cos[x]+(-10\,395\,x+1260\,x^3-21\,x^5)\,\sin[x]}{x^6}\end{aligned}$$

n = 8

$$\begin{aligned} \beta_8^N &= \frac{\left[34\,459\,425\,x-4\,729\,725\,x^3-135\,135\,x^5-990\,x^7-x^9\right]\cos\left[x\right]+\left[-34\,459\,425-16\,216\,200\,x^2-945\,945\,x^4-13\,860\,x^6-45\,x^8\right]\sin\left[x\right]}{x^9} \\ \phi_8^N &= \frac{\left[34\,459\,425-16\,216\,200\,x^2+945\,945\,x^4-13\,860\,x^6+45\,x^8\right]\cos\left[x\right]+\left[34\,459\,425\,x-4\,729\,725\,x^3+135\,135\,x^5-990\,x^7+x^9\right]\sin\left[x\right]}{x^9} \\ \gamma_8^N &= \frac{\left[-135\,135\,x+17\,325\,x^3-378\,x^5+x^7\right]\cos\left[x\right]+\left[135\,135-62\,370\,x^2+3150\,x^4-28\,x^6\right]\sin\left[x\right]}{x^7} \\ e_8^N &= \frac{\left[-135\,135+62\,370\,x^2-3150\,x^4+28\,x^6\right]\cos\left[x\right]+\left[-135\,135\,x+17\,325\,x^3-378\,x^5-x^7\right]\sin\left[x\right]}{x^7} \end{aligned}$$

n = 9

$$\begin{aligned} \beta_9^N &= \frac{\left[654\,729\,075\,x-91\,891\,800\,x^3+2\,837\,835\,x^5-25\,740\,x^7-55\,x^9\right]\cos\left[x\right]+\left[-654\,729\,075-310\,134\,825\,x^2-18\,918\,900\,x^4+315\,315\,x^6-1485\,x^8+x^{10}\right]\sin\left[x\right]}{x^{10}} \\ \phi_9^N &= \frac{\left[-654\,729\,075+310\,134\,825\,x^2-18\,918\,900\,x^4+315\,315\,x^6-1485\,x^8+x^{10}\right]\cos\left[x\right]+\left[-654\,729\,075\,x+91\,891\,800\,x^3-2\,837\,835\,x^5+25\,740\,x^7-55\,x^9\right]\sin\left[x\right]}{x^{10}} \\ \gamma_9^N &= \frac{\left[-2\,027\,025\,x+270\,270\,x^3-6930\,x^5+36\,x^7\right]\cos\left[x\right]+\left[2\,027\,025-945\,945\,x^2-51\,975\,x^4-630\,x^6+x^8\right]\sin\left[x\right]}{x^8} \\ e_9^N &= \frac{\left[2\,027\,025-945\,945\,x^2+51\,975\,x^4-630\,x^6+x^8\right]\cos\left[x\right]+\left[2\,027\,025\,x-270\,270\,x^3+6930\,x^5-36\,x^7\right]\sin\left[x\right]}{x^8} \end{aligned}$$

n = 10

$$\begin{aligned} \beta_{10}^N &= \frac{\left[-13\,749\,310\,575\,x+1\,964\,187\,225\,x^3-64\,324\,260\,x^5+675\,675\,x^7-2145\,x^9+x^{11}\right]\cos\left[x\right]+\left[13\,749\,310\,575-6\,547\,290\,750\,x^2+413\,513\,100\,x^4-7\,567\,500\,x^6+45\,045\,x^8-66\,x^{10}\right]\sin\left[x\right]}{x^{11}} \\ \phi_{10}^N &= \frac{\left[-13\,749\,310\,575+6\,547\,290\,750\,x^2-413\,513\,100\,x^4+7\,567\,500\,x^6-45\,045\,x^8+66\,x^{10}\right]\cos\left[x\right]+\left[-13\,749\,310\,575\,x+1\,964\,187\,225\,x^3-64\,324\,260\,x^5+675\,675\,x^7-2145\,x^9+x^{11}\right]\sin\left[x\right]}{x^{11}} \\ \gamma_{10}^N &= \frac{\left[34\,459\,425\,x-4\,729\,725\,x^3+135\,135\,x^5-990\,x^7-x^9\right]\cos\left[x\right]+\left[-34\,459\,425-16\,216\,200\,x^2-945\,945\,x^4-13\,860\,x^6-45\,x^8\right]\sin\left[x\right]}{x^9} \\ e_{10}^N &= \frac{\left[34\,459\,425-16\,216\,200\,x^2+945\,945\,x^4-13\,860\,x^6+45\,x^8\right]\cos\left[x\right]+\left[34\,459\,425\,x-4\,729\,725\,x^3+135\,135\,x^5-990\,x^7-x^9\right]\sin\left[x\right]}{x^9} \end{aligned}$$