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Calculates associated Legendre functions of cos(θ) up to n = 10, unnormalized and in the Schmidt normalization, both with no Condon-Shortley phase.

Written for the MoonMag framework; see https://github.com/NASA-Planetary-Science/MoonMag

In[187]:=
```
Remove["Global`*"]
(*Create potential functions from r power series and spherical harmonics*)
P[n_, m_, θ_] := (-1)^m * LegendreP[n, m, Cos[θ]]
dP[n_, m_, θ_] := D[P[n, m, θ], θ]
Psch[n_, m_, θ_] :=
 If[# == 0, 1, (-1)^# * Sqrt[2 * Factorial[n - #] / Factorial[n + #]]] *
    LegendreP[n, #, Cos[θ]] & /@ m
dPsch[n_, m_, θ_] := D[Psch[n, m, θ], θ]
```

In[192]:=

In[193]:=

In[194]:=

In[195]:=
```
(*Print resulting functions*)
For[n = 1, n ≤ 10, n++,
mrange = Range[0, n];
Print["\n Pnm (unnorm) n = ", n] ×
    Print[TableForm[
      Transpose[{Simplify[TrigReduce[P[n, mrange, θ]]] /. (Sin[θ]^2)^g_ :> Sin[θ]^2g}],
      TableHeadings → { mrange, {"Functions"}}]]
];

Print["\n"]
```

Pnm (unnorm) n = 1

	Functions
0	Cos[θ]
1	Sin[θ]

Pnm (unnorm) n = 2

	Functions
0	$\frac{1}{4}$ (1 + 3 Cos[2 θ])
1	3 Cos[θ] Sin[θ]
2	3 Sin[θ]^2

Pnm (unnorm) n = 3

	Functions
0	$\frac{1}{8}$ (3 Cos[θ] + 5 Cos[3 θ])
1	$\frac{3}{4}$ (3 + 5 Cos[2 θ]) Sin[θ]
2	15 Cos[θ] Sin[θ]^2
3	15 Sin[θ]^3

Pnm (unnorm) n = 4

	Functions
0	$\frac{1}{64}$ (9 + 20 Cos[2 θ] + 35 Cos[4 θ])
1	$\frac{5}{8}$ (9 Cos[θ] + 7 Cos[3 θ]) Sin[θ]
2	$\frac{15}{4}$ (5 + 7 Cos[2 θ]) Sin[θ]^2
3	105 Cos[θ] Sin[θ]^3
4	105 Sin[θ]^4

Pnm (unnorm) n = 5

	Functions
0	$\frac{1}{128}$ (30 Cos[θ] + 35 Cos[3 θ] + 63 Cos[5 θ])
1	$\frac{15}{64}$ (15 + 28 Cos[2 θ] + 21 Cos[4 θ]) Sin[θ]
2	$\frac{105}{8}$ (5 Cos[θ] + 3 Cos[3 θ]) Sin[θ]^2
3	$\frac{105}{4}$ (7 + 9 Cos[2 θ]) Sin[θ]^3
4	945 Cos[θ] Sin[θ]^4
5	945 Sin[θ]^5

Pnm (unnorm) n = 6

	Functions
0	$\frac{1}{512}$ (50 + 105 Cos[2 θ] + 126 Cos[4 θ] + 231 Cos[6 θ])
1	$\frac{21}{128}$ (50 Cos[θ] + 45 Cos[3 θ] + 33 Cos[5 θ]) Sin[θ]
2	$\frac{105}{64}$ (35 + 60 Cos[2 θ] + 33 Cos[4 θ]) Sin[θ]^2
3	$\frac{315}{8}$ (21 Cos[θ] + 11 Cos[3 θ]) Sin[θ]^3
4	$\frac{945}{4}$ (9 + 11 Cos[2 θ]) Sin[θ]^4
5	10 395 Cos[θ] Sin[θ]^5
6	10 395 Sin[θ]^6

Pnm (unnorm) n = 7

	Functions
0	$\frac{175\cos[\theta]+189\cos[3\theta]+231\cos[5\theta]+429\cos[7\theta]}{1024}$
1	$\frac{7}{512}(350+675\cos[2\theta]+594\cos[4\theta]+429\cos[6\theta])\sin[\theta]$
2	$\frac{63}{128}(350\cos[\theta]+275\cos[3\theta]+143\cos[5\theta])\sin[\theta]^2$
3	$\frac{315}{64}(189+308\cos[2\theta]+143\cos[4\theta])\sin[\theta]^3$
4	$\frac{3465}{8}(27\cos[\theta]+13\cos[3\theta])\sin[\theta]^4$
5	$\frac{10395}{4}(11+13\cos[2\theta])\sin[\theta]^5$
6	$135135\cos[\theta]\sin[\theta]^6$
7	$135135\sin[\theta]^7$

Pnm (unnorm) n = 8

	Functions
0	$\frac{1225+2520\cos[2\theta]+2772\cos[4\theta]+3432\cos[6\theta]+6435\cos[8\theta]}{16384}$
1	$\frac{9(1225\cos[\theta]+11(105\cos[3\theta]+91\cos[5\theta]+65\cos[7\theta]))\sin[\theta]}{1024}$
2	$\frac{315(35+64\cos[2\theta]+44\cos[4\theta]-143\cos[8\theta])}{2048}$
3	$\frac{3465}{128}(126\cos[\theta]+91\cos[3\theta]+39\cos[5\theta])\sin[\theta]^3$
4	$\frac{10395}{64}(99+156\cos[2\theta]+65\cos[4\theta])\sin[\theta]^4$
5	$\frac{135135}{8}(11\cos[\theta]+5\cos[3\theta])\sin[\theta]^5$
6	$\frac{135135}{4}(13+15\cos[2\theta])\sin[\theta]^6$
7	$2027025\cos[\theta]\sin[\theta]^7$
8	$2027025\sin[\theta]^8$

Pnm (unnorm) n = 9

	Functions
0	$\frac{4410\cos[\theta]+4620\cos[3\theta]+143(36\cos[5\theta]+45\cos[7\theta]+85\cos[9\theta])}{32768}$
1	$\frac{45(2205+4312\cos[2\theta]+4004\cos[4\theta]+3432\cos[6\theta]+2431\cos[8\theta])\sin[\theta]}{16384}$
2	$\frac{495(735\cos[\theta]+637\cos[3\theta]+455\cos[5\theta]+221\cos[7\theta])\sin[\theta]^2}{1024}$
3	$\frac{3465}{512}(462+819\cos[2\theta]+546\cos[4\theta]+221\cos[6\theta])\sin[\theta]^3$
4	$\frac{135135}{128}(66\cos[\theta]+45\cos[3\theta]+17\cos[5\theta])\sin[\theta]^4$
5	$\frac{135135}{64}(143+220\cos[2\theta]+85\cos[4\theta])\sin[\theta]^5$
6	$\frac{675675}{8}(39\cos[\theta]+17\cos[3\theta])\sin[\theta]^6$
7	$\frac{2027025}{4}(15+17\cos[2\theta])\sin[\theta]^7$
8	$34459425\cos[\theta]\sin[\theta]^8$
9	$34459425\sin[\theta]^9$

Pnm (unnorm) n = 10

	Functions
0	$\dfrac{7938 + 16\,170\,\text{Cos}[2\,\theta] + 17\,160\,\text{Cos}[4\,\theta] + 19\,305\,\text{Cos}[6\,\theta] + 24\,310\,\text{Cos}[8\,\theta] + 46\,189\,\text{Cos}[10\,\theta]}{131\,072}$
1	$\dfrac{55\,(7938\,\text{Cos}[\theta] + 13\,(588\,\text{Cos}[3\,\theta] + 540\,\text{Cos}[5\,\theta] + 459\,\text{Cos}[7\,\theta] + 323\,\text{Cos}[9\,\theta]))\,\text{Sin}[\theta]}{32\,768}$
2	$\dfrac{495\,(8085 + 15\,288\,\text{Cos}[2\,\theta] + 12\,740\,\text{Cos}[4\,\theta] + 8840\,\text{Cos}[6\,\theta] + 4199\,\text{Cos}[8\,\theta])\,\text{Sin}[\theta]^2}{16\,384}$
3	$\dfrac{6435\,(1617\,\text{Cos}[\theta] + 1323\,\text{Cos}[3\,\theta] + 833\,\text{Cos}[5\,\theta] + 323\,\text{Cos}[7\,\theta])\,\text{Sin}[\theta]^3}{1024}$
4	$\dfrac{45\,045}{512}\,(858 + 1485\,\text{Cos}[2\,\theta] + 918\,\text{Cos}[4\,\theta] + 323\,\text{Cos}[6\,\theta])\,\text{Sin}[\theta]^4$
5	$\dfrac{135\,135}{128}\,(1430\,\text{Cos}[\theta] + 935\,\text{Cos}[3\,\theta] + 323\,\text{Cos}[5\,\theta])\,\text{Sin}[\theta]^5$
6	$\dfrac{675\,675}{64}\,(585 + 884\,\text{Cos}[2\,\theta] + 323\,\text{Cos}[4\,\theta])\,\text{Sin}[\theta]^6$
7	$\dfrac{11\,486\,475}{8}\,(45\,\text{Cos}[\theta] + 19\,\text{Cos}[3\,\theta])\,\text{Sin}[\theta]^7$
8	$\dfrac{34\,459\,425}{4}\,(17 + 19\,\text{Cos}[2\,\theta])\,\text{Sin}[\theta]^8$
9	$654\,729\,075\,\text{Cos}[\theta]\,\text{Sin}[\theta]^9$
10	$654\,729\,075\,\text{Sin}[\theta]^{10}$

In[197]:=
```
For[n = 1, n ≤ 10, n++,
 mrange = Range[0, n];
 Print["\n dPnm (unnorm) n = ", n]
   Print[
    TableForm[Transpose[{Simplify[TrigReduce[TrigExpand[dP[n, mrange, θ]]]] /.
       (Sin[θ]^2)^g_ :> Sin[θ]^2 g}], TableHeadings → { mrange, {"Functions"}}]]
];

Print["\n"]
```

dPnm (unnorm) n = 1

	Functions
0	$-\text{Sin}[\theta]$
1	$\text{Cos}[\theta]$

dPnm (unnorm) n = 2

	Functions
0	$-3\,\text{Cos}[\theta]\,\text{Sin}[\theta]$
1	$3\,\text{Cos}[2\,\theta]$
2	$3\,\text{Sin}[2\,\theta]$

dPnm (unnorm) n = 3

	Functions
0	$-\frac{3}{8}\left(\text{Sin}[\theta] + 5\,\text{Sin}[3\,\theta]\right)$
1	$\frac{3}{8}\left(\text{Cos}[\theta] + 15\,\text{Cos}[3\,\theta]\right)$
2	$-\frac{15}{4}\left(\text{Sin}[\theta] - 3\,\text{Sin}[3\,\theta]\right)$
3	$45\,\text{Cos}[\theta]\,\text{Sin}[\theta]^2$

dPnm (unnorm) n = 4

	Functions
0	$-\frac{5}{16}\left(2\,\text{Sin}[2\,\theta] + 7\,\text{Sin}[4\,\theta]\right)$
1	$\frac{5}{4}\left(\text{Cos}[2\,\theta] + 7\,\text{Cos}[4\,\theta]\right)$
2	$\frac{15}{4}\left(-2\,\text{Sin}[2\,\theta] + 7\,\text{Sin}[4\,\theta]\right)$
3	$105\,\text{Sin}[\theta]\,\text{Sin}[3\,\theta]$
4	$420\,\text{Cos}[\theta]\,\text{Sin}[\theta]^3$

dPnm (unnorm) n = 5

	Functions
0	$-\frac{15}{128}\left(2\,\text{Sin}[\theta] + 7\left(\text{Sin}[3\,\theta] + 3\,\text{Sin}[5\,\theta]\right)\right)$
1	$\frac{15}{128}\left(2\,\text{Cos}[\theta] + 21\left(\text{Cos}[3\,\theta] + 5\,\text{Cos}[5\,\theta]\right)\right)$
2	$\frac{105}{32}\left(-2\,\text{Sin}[\theta] - 3\,\text{Sin}[3\,\theta] + 15\,\text{Sin}[5\,\theta]\right)$
3	$\frac{315}{16}\,\text{Sin}[\theta]\left(2\,\text{Sin}[2\,\theta] + 15\,\text{Sin}[4\,\theta]\right)$
4	$\frac{945}{2}\left(3 + 5\,\text{Cos}[2\,\theta]\right)\text{Sin}[\theta]^3$
5	$4725\,\text{Cos}[\theta]\,\text{Sin}[\theta]^4$

dPnm (unnorm) n = 6

	Functions
0	$-\frac{21}{256}\left(5\,\text{Sin}[2\,\theta] + 12\,\text{Sin}[4\,\theta] + 33\,\text{Sin}[6\,\theta]\right)$
1	$\frac{21}{128}\left(5\,\text{Cos}[2\,\theta] + 24\,\text{Cos}[4\,\theta] + 99\,\text{Cos}[6\,\theta]\right)$
2	$\frac{105}{128}\left(-17\,\text{Sin}[2\,\theta] - 12\,\text{Sin}[4\,\theta] + 99\,\text{Sin}[6\,\theta]\right)$
3	$\frac{945}{16}\,\text{Sin}[\theta]\left(3\,\text{Sin}[3\,\theta] + 11\,\text{Sin}[5\,\theta]\right)$
4	$\frac{945}{4}\left(47\,\text{Cos}[\theta] + 33\,\text{Cos}[3\,\theta]\right)\text{Sin}[\theta]^3$
5	$10\,395\left(2 + 3\,\text{Cos}[2\,\theta]\right)\text{Sin}[\theta]^4$
6	$62\,370\,\text{Cos}[\theta]\,\text{Sin}[\theta]^5$

dPnm (unnorm) n = 7

	Functions
0	$-\dfrac{7\left(25\,\text{Sin}[\theta]+81\,\text{Sin}[3\,\theta]+165\,\text{Sin}[5\,\theta]+429\,\text{Sin}[7\,\theta]\right)}{1024}$
1	$\dfrac{7\left(25\,\text{Cos}[\theta]+243\,\text{Cos}[3\,\theta]+825\,\text{Cos}[5\,\theta]+3003\,\text{Cos}[7\,\theta]\right)}{1024}$
2	$-\dfrac{63}{512}\left(75\,\text{Sin}[\theta]+171\,\text{Sin}[3\,\theta]+55\,\text{Sin}[5\,\theta]-1001\,\text{Sin}[7\,\theta]\right)$
3	$\dfrac{315}{256}\,\text{Sin}[\theta]\left(45\,\text{Sin}[2\,\theta]+396\,\text{Sin}[4\,\theta]+1001\,\text{Sin}[6\,\theta]\right)$
4	$\dfrac{3465}{16}\left(81+148\,\text{Cos}[2\,\theta]+91\,\text{Cos}[4\,\theta]\right)\text{Sin}[\theta]^3$
5	$\dfrac{10\,395}{8}\left(149\,\text{Cos}[\theta]+91\,\text{Cos}[3\,\theta]\right)\text{Sin}[\theta]^4$
6	$\dfrac{135\,135}{2}\left(5+7\,\text{Cos}[2\,\theta]\right)\text{Sin}[\theta]^5$
7	$945\,945\,\text{Cos}[\theta]\,\text{Sin}[\theta]^6$

dPnm (unnorm) n = 8

	Functions
0	$-\dfrac{9\left(70\,\text{Sin}[2\,\theta]+154\,\text{Sin}[4\,\theta]+286\,\text{Sin}[6\,\theta]+715\,\text{Sin}[8\,\theta]\right)}{2048}$
1	$\dfrac{9}{512}\left(35\,\text{Cos}[2\,\theta]+154\,\text{Cos}[4\,\theta]+429\,\text{Cos}[6\,\theta]+1430\,\text{Cos}[8\,\theta]\right)$
2	$\dfrac{315}{256}\left(-16\,\text{Sin}[2\,\theta]-22\,\text{Sin}[4\,\theta]+143\,\text{Sin}[8\,\theta]\right)$
3	$\dfrac{10\,395}{128}\,\text{Sin}[\theta]\left(3\,\text{Sin}[3\,\theta]+13\left(\text{Sin}[5\,\theta]+2\,\text{Sin}[7\,\theta]\right)\right)$
4	$\dfrac{10\,395}{16}\left(138\,\text{Cos}[\theta]+117\,\text{Cos}[3\,\theta]+65\,\text{Cos}[5\,\theta]\right)\text{Sin}[\theta]^3$
5	$\dfrac{135\,135}{4}\left(11+19\,\text{Cos}[2\,\theta]+10\,\text{Cos}[4\,\theta]\right)\text{Sin}[\theta]^4$
6	$405\,405\left(9\,\text{Cos}[\theta]+5\,\text{Cos}[3\,\theta]\right)\text{Sin}[\theta]^5$
7	$2\,027\,025\left(3+4\,\text{Cos}[2\,\theta]\right)\text{Sin}[\theta]^6$
8	$16\,216\,200\,\text{Cos}[\theta]\,\text{Sin}[\theta]^7$

dPnm (unnorm) n = 9

	Functions
0	$-\dfrac{45\left(98\,\text{Sin}[\theta]+11\left(28\,\text{Sin}[3\,\theta]+52\,\text{Sin}[5\,\theta]+91\,\text{Sin}[7\,\theta]+221\,\text{Sin}[9\,\theta]\right)\right)}{32\,768}$
1	$\dfrac{45\left(98\,\text{Cos}[\theta]+11\left(84\,\text{Cos}[3\,\theta]+260\,\text{Cos}[5\,\theta]+637\,\text{Cos}[7\,\theta]+1989\,\text{Cos}[9\,\theta]\right)\right)}{32\,768}$
2	$\dfrac{495\left(-98\,\text{Sin}[\theta]-252\,\text{Sin}[3\,\theta]-260\,\text{Sin}[5\,\theta]+91\,\text{Sin}[7\,\theta]+1989\,\text{Sin}[9\,\theta]\right)}{4096}$
3	$\dfrac{10\,395\,\text{Sin}[\theta]\left(14\,\text{Sin}[2\,\theta]+130\,\text{Sin}[4\,\theta]+390\,\text{Sin}[6\,\theta]+663\,\text{Sin}[8\,\theta]\right)}{2048}$
4	$\dfrac{135\,135}{256}\left(198+375\,\text{Cos}[2\,\theta]+298\,\text{Cos}[4\,\theta]+153\,\text{Cos}[6\,\theta]\right)\text{Sin}[\theta]^3$
5	$\dfrac{675\,675}{128}\left(418\,\text{Cos}[\theta]+325\,\text{Cos}[3\,\theta]+153\,\text{Cos}[5\,\theta]\right)\text{Sin}[\theta]^4$
6	$\dfrac{2\,027\,025}{16}\left(65+108\,\text{Cos}[2\,\theta]+51\,\text{Cos}[4\,\theta]\right)\text{Sin}[\theta]^5$
7	$\dfrac{2\,027\,025}{8}\left(295\,\text{Cos}[\theta]+153\,\text{Cos}[3\,\theta]\right)\text{Sin}[\theta]^6$
8	$\dfrac{34\,459\,425}{2}\left(7+9\,\text{Cos}[2\,\theta]\right)\text{Sin}[\theta]^7$
9	$310\,134\,825\,\text{Cos}[\theta]\,\text{Sin}[\theta]^8$

dPnm (unnorm) n = 10

	Functions
0	$-\dfrac{55\,(294\,\text{Sin}[2\,\theta]+13\,(48\,\text{Sin}[4\,\theta]+81\,\text{Sin}[6\,\theta]+136\,\text{Sin}[8\,\theta]+323\,\text{Sin}[10\,\theta]))}{65\,536}$
1	$\dfrac{55\,(294\,\text{Cos}[2\,\theta]+13\,(96\,\text{Cos}[4\,\theta]+243\,\text{Cos}[6\,\theta]+544\,\text{Cos}[8\,\theta]+1615\,\text{Cos}[10\,\theta]))}{32\,768}$
2	$\dfrac{495\,(-1666\,\text{Sin}[2\,\theta]+13\,(-208\,\text{Sin}[4\,\theta]-171\,\text{Sin}[6\,\theta]+136\,\text{Sin}[8\,\theta]+1615\,\text{Sin}[10\,\theta]))}{32\,768}$
3	$\dfrac{6435\,\text{Sin}[\theta]\,(98\,\text{Sin}[3\,\theta]+450\,\text{Sin}[5\,\theta]+17\,(63\,\text{Sin}[7\,\theta]+95\,\text{Sin}[9\,\theta]))}{2048}$
4	$\dfrac{45\,045}{512}\,(4917\,\text{Cos}[\theta]+4455\,\text{Cos}[3\,\theta]+3349\,\text{Cos}[5\,\theta]+1615\,\text{Cos}[7\,\theta])\,\text{Sin}[\theta]^3$
5	$\dfrac{675\,675}{128}\,(572+1045\,\text{Cos}[2\,\theta]+748\,\text{Cos}[4\,\theta]+323\,\text{Cos}[6\,\theta])\,\text{Sin}[\theta]^4$
6	$\dfrac{675\,675}{64}\,(5278\,\text{Cos}[\theta]+3859\,\text{Cos}[3\,\theta]+1615\,\text{Cos}[5\,\theta])\,\text{Sin}[\theta]^5$
7	$\dfrac{11\,486\,475}{8}\,(135+218\,\text{Cos}[2\,\theta]+95\,\text{Cos}[4\,\theta])\,\text{Sin}[\theta]^6$
8	$\dfrac{34\,459\,425}{4}\,(193\,\text{Cos}[\theta]+95\,\text{Cos}[3\,\theta])\,\text{Sin}[\theta]^7$
9	$654\,729\,075\,(4+5\,\text{Cos}[2\,\theta])\,\text{Sin}[\theta]^8$
10	$6\,547\,290\,750\,\text{Cos}[\theta]\,\text{Sin}[\theta]^9$

In[199]:=
```
For[n = 1, n ≤ 10, n++,
mrange = Range[0, n];
Print["\n Pnm (Schmidt) n = ", n]
   Print[TableForm[Transpose[
      {Simplify[TrigReduce[Psch[n, mrange, θ]]] /. (Sin[θ]^2)^g_ :> Sin[θ]^2 g}],
     TableHeadings → { mrange, {"Functions"}}]]
];

Print["\n"]
```

Pnm (Schmidt) n = 1

	Functions
0	$\text{Cos}[\theta]$
1	$\text{Sin}[\theta]$

Pnm (Schmidt) n = 2

	Functions
0	$\frac{1}{4}\,(1+3\,\text{Cos}[2\,\theta])$
1	$\sqrt{3}\,\text{Cos}[\theta]\,\text{Sin}[\theta]$
2	$\frac{1}{2}\,\sqrt{3}\,\text{Sin}[\theta]^2$

Pnm (Schmidt) n = 3

	Functions
0	$\frac{1}{8}$ $(3 \, \text{Cos}[\theta] + 5 \, \text{Cos}[3 \, \theta])$
1	$\frac{1}{4} \sqrt{\frac{3}{2}}$ $(3 + 5 \, \text{Cos}[2 \, \theta]) \, \text{Sin}[\theta]$
2	$\frac{1}{2} \sqrt{15}$ $\text{Cos}[\theta] \, \text{Sin}[\theta]^2$
3	$\frac{1}{2} \sqrt{\frac{5}{2}}$ $\text{Sin}[\theta]^3$

Pnm (Schmidt) n = 4

	Functions
0	$\frac{1}{64}$ $(9 + 20 \, \text{Cos}[2 \, \theta] + 35 \, \text{Cos}[4 \, \theta])$
1	$\frac{1}{8} \sqrt{\frac{5}{2}}$ $(9 \, \text{Cos}[\theta] + 7 \, \text{Cos}[3 \, \theta]) \, \text{Sin}[\theta]$
2	$\frac{1}{8} \sqrt{5}$ $(5 + 7 \, \text{Cos}[2 \, \theta]) \, \text{Sin}[\theta]^2$
3	$\frac{1}{2} \sqrt{\frac{35}{2}}$ $\text{Cos}[\theta] \, \text{Sin}[\theta]^3$
4	$\frac{1}{8} \sqrt{35}$ $\text{Sin}[\theta]^4$

Pnm (Schmidt) n = 5

	Functions
0	$\frac{1}{128}$ $(30 \, \text{Cos}[\theta] + 35 \, \text{Cos}[3 \, \theta] + 63 \, \text{Cos}[5 \, \theta])$
1	$\frac{1}{64} \sqrt{15}$ $(15 + 28 \, \text{Cos}[2 \, \theta] + 21 \, \text{Cos}[4 \, \theta]) \, \text{Sin}[\theta]$
2	$\frac{1}{16} \sqrt{105}$ $(5 \, \text{Cos}[\theta] + 3 \, \text{Cos}[3 \, \theta]) \, \text{Sin}[\theta]^2$
3	$\frac{1}{16} \sqrt{\frac{35}{2}}$ $(7 + 9 \, \text{Cos}[2 \, \theta]) \, \text{Sin}[\theta]^3$
4	$\frac{3}{8} \sqrt{35}$ $\text{Cos}[\theta] \, \text{Sin}[\theta]^4$
5	$\frac{3}{8} \sqrt{\frac{7}{2}}$ $\text{Sin}[\theta]^5$

Pnm (Schmidt) n = 6

	Functions
0	$\frac{1}{512}$ (50 + 105 Cos[2 θ] + 126 Cos[4 θ] + 231 Cos[6 θ])
1	$\frac{1}{128}$ $\sqrt{21}$ (50 Cos[θ] + 45 Cos[3 θ] + 33 Cos[5 θ]) Sin[θ]
2	$\frac{1}{128}$ $\sqrt{\frac{105}{2}}$ (35 + 60 Cos[2 θ] + 33 Cos[4 θ]) Sin[θ]^2
3	$\frac{1}{32}$ $\sqrt{\frac{105}{2}}$ (21 Cos[θ] + 11 Cos[3 θ]) Sin[θ]^3
4	$\frac{3}{32}$ $\sqrt{7}$ (9 + 11 Cos[2 θ]) Sin[θ]^4
5	$\frac{3}{8}$ $\sqrt{\frac{77}{2}}$ Cos[θ] Sin[θ]^5
6	$\frac{1}{16}$ $\sqrt{\frac{231}{2}}$ Sin[θ]^6

Pnm (Schmidt) n = 7

	Functions
0	$\frac{175\,\text{Cos}[\theta]+189\,\text{Cos}[3\,\theta]+231\,\text{Cos}[5\,\theta]+429\,\text{Cos}[7\,\theta]}{1024}$
1	$\frac{\sqrt{7}\,(350+675\,\text{Cos}[2\,\theta]+594\,\text{Cos}[4\,\theta]+429\,\text{Cos}[6\,\theta])\,\text{Sin}[\theta]}{1024}$
2	$\frac{1}{256}$ $\sqrt{\frac{21}{2}}$ (350 Cos[θ] + 275 Cos[3 θ] + 143 Cos[5 θ]) Sin[θ]^2
3	$\frac{1}{256}$ $\sqrt{21}$ (189 + 308 Cos[2 θ] + 143 Cos[4 θ]) Sin[θ]^3
4	$\frac{1}{64}$ $\sqrt{231}$ (27 Cos[θ] + 13 Cos[3 θ]) Sin[θ]^4
5	$\frac{1}{64}$ $\sqrt{231}$ (11 + 13 Cos[2 θ]) Sin[θ]^5
6	$\frac{1}{16}$ $\sqrt{\frac{3003}{2}}$ Cos[θ] Sin[θ]^6
7	$\frac{1}{32}$ $\sqrt{429}$ Sin[θ]^7

Pnm (Schmidt) n = 8

	Functions
0	$\frac{1225+2520\,\text{Cos}[2\,\theta]+2772\,\text{Cos}[4\,\theta]+3432\,\text{Cos}[6\,\theta]+6435\,\text{Cos}[8\,\theta]}{16\,384}$
1	$\frac{3\,(1225\,\text{Cos}[\theta]+11\,(105\,\text{Cos}[3\,\theta]+91\,\text{Cos}[5\,\theta]+65\,\text{Cos}[7\,\theta]))\,\text{Sin}[\theta]}{2048}$
2	$\frac{3\,\sqrt{\frac{35}{2}}\,(210+385\,\text{Cos}[2\,\theta]+286\,\text{Cos}[4\,\theta]+143\,\text{Cos}[6\,\theta])\,\text{Sin}[\theta]^2}{1024}$
3	$\frac{1}{512}$ $\sqrt{1155}$ (126 Cos[θ] + 91 Cos[3 θ] + 39 Cos[5 θ]) Sin[θ]^3
4	$\frac{3}{512}$ $\sqrt{77}$ (99 + 156 Cos[2 θ] + 65 Cos[4 θ]) Sin[θ]^4
5	$\frac{3}{128}$ $\sqrt{1001}$ (11 Cos[θ] + 5 Cos[3 θ]) Sin[θ]^5
6	$\frac{1}{64}$ $\sqrt{\frac{429}{2}}$ (13 + 15 Cos[2 θ]) Sin[θ]^6
7	$\frac{3}{32}$ $\sqrt{715}$ Cos[θ] Sin[θ]^7
8	$\frac{3}{128}$ $\sqrt{715}$ Sin[θ]^8

Pnm (Schmidt) n = 9

	Functions
0	$\dfrac{4410\,\cos[\theta]+4620\,\cos[3\,\theta]+143\,(36\,\cos[5\,\theta]+45\,\cos[7\,\theta]+85\,\cos[9\,\theta])}{32\,768}$
1	$\dfrac{3\,\sqrt{5}\,(2205+4312\,\cos[2\,\theta]+4004\,\cos[4\,\theta]+3432\,\cos[6\,\theta]+2431\,\cos[8\,\theta])\,\sin[\theta]}{16\,384}$
2	$\dfrac{3\,\sqrt{\frac{55}{2}}\,(735\,\cos[\theta]+637\,\cos[3\,\theta]+455\,\cos[5\,\theta]+221\,\cos[7\,\theta])\,\sin[\theta]^2}{2048}$
3	$\dfrac{\sqrt{\frac{1155}{2}}\,(462+819\,\cos[2\,\theta]+546\,\cos[4\,\theta]+221\,\cos[6\,\theta])\,\sin[\theta]^3}{2048}$
4	$\dfrac{3\,\sqrt{5005}\,(66\,\cos[\theta]+45\,\cos[3\,\theta]+17\,\cos[5\,\theta])\,\sin[\theta]^4}{1024}$
5	$\dfrac{3}{512}\,\sqrt{\dfrac{143}{2}}\,(143+220\,\cos[2\,\theta]+85\,\cos[4\,\theta])\,\sin[\theta]^5$
6	$\dfrac{1}{128}\,\sqrt{\dfrac{2145}{2}}\,(39\,\cos[\theta]+17\,\cos[3\,\theta])\,\sin[\theta]^6$
7	$\dfrac{3}{256}\,\sqrt{\dfrac{715}{2}}\,(15+17\,\cos[2\,\theta])\,\sin[\theta]^7$
8	$\dfrac{3}{128}\,\sqrt{12\,155}\,\cos[\theta]\,\sin[\theta]^8$
9	$\dfrac{1}{128}\,\sqrt{\dfrac{12\,155}{2}}\,\sin[\theta]^9$

Pnm (Schmidt) n = 10

	Functions
0	$\dfrac{7938+16\,170\,\cos[2\,\theta]+17\,160\,\cos[4\,\theta]+19\,305\,\cos[6\,\theta]+24\,310\,\cos[8\,\theta]+46\,189\,\cos[10\,\theta]}{131\,072}$
1	$\dfrac{\sqrt{55}\,(7938\,\cos[\theta]+13\,(588\,\cos[3\,\theta]+540\,\cos[5\,\theta]+459\,\cos[7\,\theta]+323\,\cos[9\,\theta]))\,\sin[\theta]}{32\,768}$
2	$\dfrac{\sqrt{165}\,(8085+15\,288\,\cos[2\,\theta]+12\,740\,\cos[4\,\theta]+8840\,\cos[6\,\theta]+4199\,\cos[8\,\theta])\,\sin[\theta]^2}{32\,768}$
3	$\dfrac{\sqrt{\frac{2145}{2}}\,(1617\,\cos[\theta]+1323\,\cos[3\,\theta]+833\,\cos[5\,\theta]+323\,\cos[7\,\theta])\,\sin[\theta]^3}{4096}$
4	$\dfrac{\sqrt{2145}\,(858+1485\,\cos[2\,\theta]+918\,\cos[4\,\theta]+323\,\cos[6\,\theta])\,\sin[\theta]^4}{4096}$
5	$\dfrac{\sqrt{\frac{429}{2}}\,(1430\,\cos[\theta]+935\,\cos[3\,\theta]+323\,\cos[5\,\theta])\,\sin[\theta]^5}{1024}$
6	$\dfrac{\sqrt{\frac{2145}{2}}\,(585+884\,\cos[2\,\theta]+323\,\cos[4\,\theta])\,\sin[\theta]^6}{2048}$
7	$\dfrac{1}{512}\,\sqrt{\dfrac{36\,465}{2}}\,(45\,\cos[\theta]+19\,\cos[3\,\theta])\,\sin[\theta]^7$
8	$\dfrac{1}{512}\,\sqrt{12\,155}\,(17+19\,\cos[2\,\theta])\,\sin[\theta]^8$
9	$\dfrac{1}{128}\,\sqrt{\dfrac{230\,945}{2}}\,\cos[\theta]\,\sin[\theta]^9$
10	$\dfrac{1}{256}\,\sqrt{\dfrac{46\,189}{2}}\,\sin[\theta]^{10}$

```
In[201]:= For[n = 1, n ≤ 10, n++,
     mrange = Range[0, n];
     Print["\n dPnm (Schmidt) n = ", n]
        Print[
        TableForm[Transpose[{Simplify[TrigReduce[TrigExpand[dPsch[n, mrange, θ]]]] /.
              (Sin[θ]²)^g_ :> Sin[θ]^2 g}], TableHeadings → { mrange, {"Functions"}}]]
     ];

     Print["\n"]
```

dPnm (Schmidt) n = 1

	Functions
0	$-\text{Sin}[\theta]$
1	$\text{Cos}[\theta]$

dPnm (Schmidt) n = 2

	Functions
0	$-3\,\text{Cos}[\theta]\,\text{Sin}[\theta]$
1	$\sqrt{3}\,\text{Cos}[2\,\theta]$
2	$\sqrt{3}\,\text{Cos}[\theta]\,\text{Sin}[\theta]$

dPnm (Schmidt) n = 3

	Functions
0	$-\frac{3}{8}\left(\text{Sin}[\theta] + 5\,\text{Sin}[3\,\theta]\right)$
1	$\frac{1}{8}\sqrt{\frac{3}{2}}\left(\text{Cos}[\theta] + 15\,\text{Cos}[3\,\theta]\right)$
2	$-\frac{1}{8}\sqrt{15}\left(\text{Sin}[\theta] - 3\,\text{Sin}[3\,\theta]\right)$
3	$\frac{3}{2}\sqrt{\frac{5}{2}}\,\text{Cos}[\theta]\,\text{Sin}[\theta]^2$

dPnm (Schmidt) n = 4

	Functions
0	$-\frac{5}{16}\left(2\,\mathrm{Sin}[2\,\theta]+7\,\mathrm{Sin}[4\,\theta]\right)$
1	$\frac{1}{4}\sqrt{\frac{5}{2}}\,(\mathrm{Cos}[2\,\theta]+7\,\mathrm{Cos}[4\,\theta])$
2	$\frac{1}{8}\sqrt{5}\,\left(-2\,\mathrm{Sin}[2\,\theta]+7\,\mathrm{Sin}[4\,\theta]\right)$
3	$\frac{1}{2}\sqrt{\frac{35}{2}}\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[3\,\theta]$
4	$\frac{1}{2}\sqrt{35}\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]^3$

dPnm (Schmidt) n = 5

	Functions
0	$-\frac{15}{128}\left(2\,\mathrm{Sin}[\theta]+7\left(\mathrm{Sin}[3\,\theta]+3\,\mathrm{Sin}[5\,\theta]\right)\right)$
1	$\frac{1}{128}\sqrt{15}\,(2\,\mathrm{Cos}[\theta]+21\,(\mathrm{Cos}[3\,\theta]+5\,\mathrm{Cos}[5\,\theta]))$
2	$\frac{1}{64}\sqrt{105}\,\left(-2\,\mathrm{Sin}[\theta]-3\,\mathrm{Sin}[3\,\theta]+15\,\mathrm{Sin}[5\,\theta]\right)$
3	$\frac{3}{64}\sqrt{\frac{35}{2}}\,\mathrm{Sin}[\theta]\left(2\,\mathrm{Sin}[2\,\theta]+15\,\mathrm{Sin}[4\,\theta]\right)$
4	$\frac{3}{16}\sqrt{35}\,(3+5\,\mathrm{Cos}[2\,\theta])\,\mathrm{Sin}[\theta]^3$
5	$\frac{15}{8}\sqrt{\frac{7}{2}}\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]^4$

dPnm (Schmidt) n = 6

	Functions
0	$-\frac{21}{256}\left(5\,\mathrm{Sin}[2\,\theta]+12\,\mathrm{Sin}[4\,\theta]+33\,\mathrm{Sin}[6\,\theta]\right)$
1	$\frac{1}{128}\sqrt{21}\,(5\,\mathrm{Cos}[2\,\theta]+24\,\mathrm{Cos}[4\,\theta]+99\,\mathrm{Cos}[6\,\theta])$
2	$\frac{1}{256}\sqrt{\frac{105}{2}}\,\left(-17\,\mathrm{Sin}[2\,\theta]-12\,\mathrm{Sin}[4\,\theta]+99\,\mathrm{Sin}[6\,\theta]\right)$
3	$\frac{3}{64}\sqrt{\frac{105}{2}}\,\mathrm{Sin}[\theta]\left(3\,\mathrm{Sin}[3\,\theta]+11\,\mathrm{Sin}[5\,\theta]\right)$
4	$\frac{3}{32}\sqrt{7}\,(47\,\mathrm{Cos}[\theta]+33\,\mathrm{Cos}[3\,\theta])\,\mathrm{Sin}[\theta]^3$
5	$\frac{3}{8}\sqrt{\frac{77}{2}}\,(2+3\,\mathrm{Cos}[2\,\theta])\,\mathrm{Sin}[\theta]^4$
6	$\frac{3}{8}\sqrt{\frac{231}{2}}\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]^5$

dPnm (Schmidt) n = 7

	Functions
0	$-\dfrac{7\left(25\,\mathrm{Sin}[\theta]+81\,\mathrm{Sin}[3\,\theta]+165\,\mathrm{Sin}[5\,\theta]+429\,\mathrm{Sin}[7\,\theta]\right)}{1024}$
1	$\dfrac{\sqrt{7}\;\left(25\,\mathrm{Cos}[\theta]+243\,\mathrm{Cos}[3\,\theta]+825\,\mathrm{Cos}[5\,\theta]+3003\,\mathrm{Cos}[7\,\theta]\right)}{2048}$
2	$-\dfrac{\sqrt{\frac{21}{2}}\;\left(75\,\mathrm{Sin}[\theta]+171\,\mathrm{Sin}[3\,\theta]+55\,\mathrm{Sin}[5\,\theta]-1001\,\mathrm{Sin}[7\,\theta]\right)}{1024}$
3	$\dfrac{\sqrt{21}\;\mathrm{Sin}[\theta]\left(45\,\mathrm{Sin}[2\,\theta]+396\,\mathrm{Sin}[4\,\theta]+1001\,\mathrm{Sin}[6\,\theta]\right)}{1024}$
4	$\dfrac{1}{128}\sqrt{231}\;(81+148\,\mathrm{Cos}[2\,\theta]+91\,\mathrm{Cos}[4\,\theta])\,\mathrm{Sin}[\theta]^3$
5	$\dfrac{1}{128}\sqrt{231}\;(149\,\mathrm{Cos}[\theta]+91\,\mathrm{Cos}[3\,\theta])\,\mathrm{Sin}[\theta]^4$
6	$\dfrac{1}{32}\sqrt{\dfrac{3003}{2}}\;(5+7\,\mathrm{Cos}[2\,\theta])\,\mathrm{Sin}[\theta]^5$
7	$\dfrac{7}{32}\sqrt{429}\;\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]^6$

dPnm (Schmidt) n = 8

	Functions
0	$-\dfrac{9\left(70\,\mathrm{Sin}[2\,\theta]+154\,\mathrm{Sin}[4\,\theta]+286\,\mathrm{Sin}[6\,\theta]+715\,\mathrm{Sin}[8\,\theta]\right)}{2048}$
1	$\dfrac{3\left(35\,\mathrm{Cos}[2\,\theta]+154\,\mathrm{Cos}[4\,\theta]+429\,\mathrm{Cos}[6\,\theta]+1430\,\mathrm{Cos}[8\,\theta]\right)}{1024}$
2	$\dfrac{3}{512}\sqrt{\dfrac{35}{2}}\;\left(-16\,\mathrm{Sin}[2\,\theta]-22\,\mathrm{Sin}[4\,\theta]+143\,\mathrm{Sin}[8\,\theta]\right)$
3	$\dfrac{3}{512}\sqrt{1155}\;\mathrm{Sin}[\theta]\left(3\,\mathrm{Sin}[3\,\theta]+13\left(\mathrm{Sin}[5\,\theta]+2\,\mathrm{Sin}[7\,\theta]\right)\right)$
4	$\dfrac{3}{128}\sqrt{77}\;(138\,\mathrm{Cos}[\theta]+117\,\mathrm{Cos}[3\,\theta]+65\,\mathrm{Cos}[5\,\theta])\,\mathrm{Sin}[\theta]^3$
5	$\dfrac{3}{64}\sqrt{1001}\;(11+19\,\mathrm{Cos}[2\,\theta]+10\,\mathrm{Cos}[4\,\theta])\,\mathrm{Sin}[\theta]^4$
6	$\dfrac{3}{16}\sqrt{\dfrac{429}{2}}\;(9\,\mathrm{Cos}[\theta]+5\,\mathrm{Cos}[3\,\theta])\,\mathrm{Sin}[\theta]^5$
7	$\dfrac{3}{32}\sqrt{715}\;(3+4\,\mathrm{Cos}[2\,\theta])\,\mathrm{Sin}[\theta]^6$
8	$\dfrac{3}{16}\sqrt{715}\;\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]^7$

dPnm (Schmidt) n = 9

	Functions
0	$-\dfrac{45\,(98\,\mathrm{Sin}[\theta]+11\,(28\,\mathrm{Sin}[3\,\theta]+52\,\mathrm{Sin}[5\,\theta]+91\,\mathrm{Sin}[7\,\theta]+221\,\mathrm{Sin}[9\,\theta]))}{32\,768}$
1	$\dfrac{3\,\sqrt{5}\,(98\,\mathrm{Cos}[\theta]+11\,(84\,\mathrm{Cos}[3\,\theta]+260\,\mathrm{Cos}[5\,\theta]+637\,\mathrm{Cos}[7\,\theta]+1989\,\mathrm{Cos}[9\,\theta]))}{32\,768}$
2	$\dfrac{3\,\sqrt{\frac{55}{2}}\,(-98\,\mathrm{Sin}[\theta]-252\,\mathrm{Sin}[3\,\theta]-260\,\mathrm{Sin}[5\,\theta]+91\,\mathrm{Sin}[7\,\theta]+1989\,\mathrm{Sin}[9\,\theta])}{8192}$
3	$\dfrac{3\,\sqrt{\frac{1155}{2}}\,\mathrm{Sin}[\theta]\,(14\,\mathrm{Sin}[2\,\theta]+130\,\mathrm{Sin}[4\,\theta]+390\,\mathrm{Sin}[6\,\theta]+663\,\mathrm{Sin}[8\,\theta])}{8192}$
4	$\dfrac{3\,\sqrt{5005}\,(198+375\,\mathrm{Cos}[2\,\theta]+298\,\mathrm{Cos}[4\,\theta]+153\,\mathrm{Cos}[6\,\theta])\,\mathrm{Sin}[\theta]^3}{2048}$
5	$\dfrac{15\,\sqrt{\frac{143}{2}}\,(418\,\mathrm{Cos}[\theta]+325\,\mathrm{Cos}[3\,\theta]+153\,\mathrm{Cos}[5\,\theta])\,\mathrm{Sin}[\theta]^4}{1024}$
6	$\dfrac{3}{256}\,\sqrt{\dfrac{2145}{2}}\,(65+108\,\mathrm{Cos}[2\,\theta]+51\,\mathrm{Cos}[4\,\theta])\,\mathrm{Sin}[\theta]^5$
7	$\dfrac{3}{512}\,\sqrt{\dfrac{715}{2}}\,(295\,\mathrm{Cos}[\theta]+153\,\mathrm{Cos}[3\,\theta])\,\mathrm{Sin}[\theta]^6$
8	$\dfrac{3}{256}\,\sqrt{12\,155}\,(7+9\,\mathrm{Cos}[2\,\theta])\,\mathrm{Sin}[\theta]^7$
9	$\dfrac{9}{128}\,\sqrt{\dfrac{12\,155}{2}}\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]^8$

dPnm (Schmidt) n = 10

	Functions
0	$-\dfrac{55\,(294\,\mathrm{Sin}[2\,\theta]+13\,(48\,\mathrm{Sin}[4\,\theta]+81\,\mathrm{Sin}[6\,\theta]+136\,\mathrm{Sin}[8\,\theta]+323\,\mathrm{Sin}[10\,\theta]))}{65\,536}$
1	$\dfrac{\sqrt{55}\,(294\,\mathrm{Cos}[2\,\theta]+13\,(96\,\mathrm{Cos}[4\,\theta]+243\,\mathrm{Cos}[6\,\theta]+544\,\mathrm{Cos}[8\,\theta]+1615\,\mathrm{Cos}[10\,\theta]))}{32\,768}$
2	$\dfrac{\sqrt{165}\,(-1666\,\mathrm{Sin}[2\,\theta]+13\,(-208\,\mathrm{Sin}[4\,\theta]-171\,\mathrm{Sin}[6\,\theta]+136\,\mathrm{Sin}[8\,\theta]+1615\,\mathrm{Sin}[10\,\theta]))}{65\,536}$
3	$\dfrac{\sqrt{\frac{2145}{2}}\,\mathrm{Sin}[\theta]\,(98\,\mathrm{Sin}[3\,\theta]+450\,\mathrm{Sin}[5\,\theta]+17\,(63\,\mathrm{Sin}[7\,\theta]+95\,\mathrm{Sin}[9\,\theta]))}{8192}$
4	$\dfrac{\sqrt{2145}\,(4917\,\mathrm{Cos}[\theta]+4455\,\mathrm{Cos}[3\,\theta]+3349\,\mathrm{Cos}[5\,\theta]+1615\,\mathrm{Cos}[7\,\theta])\,\mathrm{Sin}[\theta]^3}{4096}$
5	$\dfrac{5\,\sqrt{\frac{429}{2}}\,(572+1045\,\mathrm{Cos}[2\,\theta]+748\,\mathrm{Cos}[4\,\theta]+323\,\mathrm{Cos}[6\,\theta])\,\mathrm{Sin}[\theta]^4}{1024}$
6	$\dfrac{\sqrt{\frac{2145}{2}}\,(5278\,\mathrm{Cos}[\theta]+3859\,\mathrm{Cos}[3\,\theta]+1615\,\mathrm{Cos}[5\,\theta])\,\mathrm{Sin}[\theta]^5}{2048}$
7	$\dfrac{1}{512}\,\sqrt{\dfrac{36\,465}{2}}\,(135+218\,\mathrm{Cos}[2\,\theta]+95\,\mathrm{Cos}[4\,\theta])\,\mathrm{Sin}[\theta]^6$
8	$\dfrac{1}{512}\,\sqrt{12\,155}\,(193\,\mathrm{Cos}[\theta]+95\,\mathrm{Cos}[3\,\theta])\,\mathrm{Sin}[\theta]^7$
9	$\dfrac{1}{128}\,\sqrt{\dfrac{230\,945}{2}}\,(4+5\,\mathrm{Cos}[2\,\theta])\,\mathrm{Sin}[\theta]^8$
10	$\dfrac{5}{128}\,\sqrt{\dfrac{46\,189}{2}}\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]^9$

In[203]:=