Math136 - January 18'th, 2016

Scalar Equations of Vector Planes

Recall

A Plane is given by a vector equation:

$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \vec{b}, c_1, c_2 \in \mathbb{R}$

Where $\vec{v}_1, \vec{v}_2, \vec{b}$ are fixed vectors with $\{\vec{v}_1, \vec{v}_2\}$ being linearly independent.

Theorem 1.3.6

The above plane can be described as the set of vectors satisfying:

$(\vec{x} - \vec{b}) \cdot \vec{n} = 0$, Where $\vec{n} = \vec{v}_1 \times \vec{v}_2$

Scalar Equation and Normal Vector

In the above theorem, the equation $(\vec{x} - \vec{b}) \cdot \vec{n} = 0$ can be rearranged into:

$$\vec{x} \cdot \vec{n} = \vec{b} \cdot \vec{n}$$

And that is the {Scalar Equation for the plane. The vector $\vec{n}$ is a **Normal Vector** to the plane.

Ex. Find a scalar equation for the plane with vector equation:

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, c_2, c_2 \in \mathbb{R}$$

First, we need the normal vector by taking the cross product (See formula on last note).

$$\begin{aligned} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} &= \begin{bmatrix} 1 \cdot 2 - 1 \cdot 1 \\ -(1 \cdot 2 - 1 \cdot 0) \\ 1 \cdot 1 - 1 \cdot 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \end{aligned}$$

$$\vec{x} \cdot \vec{n} = \vec{b} \cdot \vec{n}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$x_1 - 2x_2 + x_3 = 0$ $\square$

If we were to generate some point on this plane and plug it into this equation, we would find the equation satisfied. :)

Generalizing to Hyperplanes

More generally, if $\{\vec{v}_1, \dots, \vec{v}_{m-1}\}$ is a L.I set of vectors in $\mathbb{R}^m$, and $\vec{b} \in \mathbb{R}^m$, then we consider the hyperplane with vector equation:

$\vec{x} = c_1\vec{v}_1 + \cdots + c_{m-1}\vec{v}_{m-1} \quad , c_1, \dots, c_{m-1} \in \mathbb{R}$

Then, if $\vec{n} \in \mathbb{R}^m$ is a non-zero vector that is orthogonal to each of $\vec{v}_1, \dots, \vec{v}_{m-1}$, then that same hyperplane has a scalar equation from expanding $\vec{x} \cdot \vec{n} = \vec{b} \cdot \vec{n}$ to:

$n_1x_1 + n_2x_2 + \cdots + n_mx_m = n_1b_1 + n_2b_2 + \cdots + n_mb_m$

Where $\vec{n} = \begin{bmatrix} n_1 \\ \vdots \\ n_m \end{bmatrix}$

(Later, we will show that such an $\vec{n}$ always exists and how to find it.)

Projections

The **Projection** of $\vec{u}$ onto $\vec{v}$ is:

$\text{Proj }_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2}\right)\vec{v}$

The **Perpendicular** of $\vec{u}$ onto $\vec{v}$ is:

$\text{Perp }_{\vec{v}}(\vec{u}) = \vec{u} - \text{Proj}_{\vec{v}}(\vec{u})$