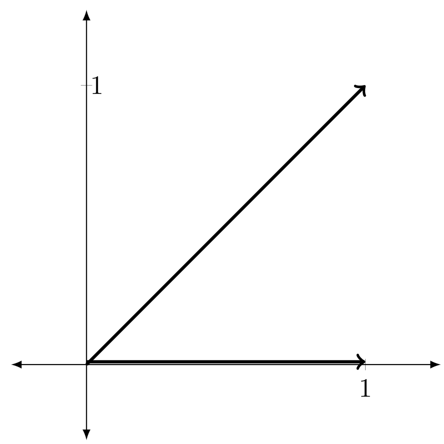
Math136 - January 15'th, 2016

Angles and Cross Product

Angle

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. The **angle** between $\vec{x}$ and $\vec{y}$ is any angle θ such that $\vec{x} \cdot \vec{y} = || \vec{x} || \, || \vec{y} || \cos\theta$

E.g. In $\mathbb{R}^2$, if $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, θ is the angle between the two vectors below:



Clearly, the angle subtended between the two vectors should be $\frac{\pi}{4}$. Filling in the angle formula would also return $\frac{\pi}{4}$.

$\theta = \frac{7\pi}{4}$ is also a correct answer.

Constructing Orthogonal Vectors

Given $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, how can we construct an orthogonal vector?

$\vec{0}$ works, but that's stupid.

We could also take $\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$ since $\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (-x_2)(x_1) + (x_1)(x_2) = 0$

What about a vector that's orthogonal to TWO vectors SIMULTANEOUSLY in $\mathbb{R}^3$?

Cross Product

Let $\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in \mathbb{R}^3$.

The cross product of $\vec{v}$ and $\vec{w}$ is:

$$\vec{v} \times \vec{w} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$$

Important property: $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$.

Theorem 1.3.5

Suppose $\vec{v}, \vec{w}, \vec{y} \in \mathbb{R}^3, c \in \mathbb{R}$

1) If $\vec{n} = \vec{v} \times \vec{w}$, then for any $\vec{y} \in \text{span}\{\vec{v}, \vec{w}\}$, we have $\vec{y} \cdot \vec{n} = 0$.
2) $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$
3) $\vec{v} \times \vec{v} = \vec{0}$
4) $\vec{v} \times \vec{w} = \vec{0}$ iff either $\vec{v} = \vec{0}$ or $\vec{w} = \vec{0}$ or $\vec{w}$ is a scalar multiple of $\vec{v}$.
5) $\vec{v} \times (\vec{w} + \vec{y}) = \vec{v} \times \vec{w} + \vec{v} \times \vec{y}$
6) $(c\vec{v}) \times \vec{w} = c(\vec{v} \times \vec{w})$
7) $|| \vec{v} \times \vec{w} || = || \vec{v} || \, || \vec{w} || \, | \sin\theta |$ where θ is the angle between $\vec{v}$ and $\vec{w}$.