Math135 - December 02, 2015

RQF and RFRP

Real Quadratic Factors (RQF)

Let $f(x)$ be a polynomial with real coefficients. If $c \in \mathbb{C}$, $Im(c) \neq 0$ is a root of $f(x)$, then there exists a real quadratic factor of $f(x)$ with c as a root.

Proof:

Let $f(x)$ be a polynomial with real coefficients, let $c \in \mathbb{C}$ be a root of f where $Im(c) \neq 0$.

By CJRT, $\bar{c}$ is also a root.

So, $(x-c)(x-\bar{c})$ is a factor of $f(x)$. We know these are different factors since the imaginary parts $\neq 0$.

$((x-c)(x-\bar{c})$
$= x^2 - cx - \bar{c}x + c\bar{c}$
$= x^2 - (c - \bar{c})x + c\bar{c}$
$= x^2 - 2Re(c)x + |c|^2$

$2Re(c) \in \mathbb{R}$, $|c|^2 \in \mathbb{R}$, so we have a real quadratic factor with c as a root, QED.

Ex. $f(x) = x^4 - 5x^3 + 16x^2 - 9x - 13$
$= (x - 2 + 3i)(x - 2 - 3i)(x - \frac{1}{2} + \frac{\sqrt{5}}{2})(x - \frac{1}{2} - \frac{\sqrt{5}}{2})$
$= (x^2 - 4x + 13)(x - \frac{1}{2} + \frac{\sqrt{5}}{2})(x - \frac{1}{2} - \frac{\sqrt{5}}{2})$

Ex. $f(x) = x^4 + 2x^2 + 1$
$= (x^2)^2 + 2x^2 + 1$
$= (x^2 + 1)^2$
$= (x - i)^2(x + i)^2$

Real Quadratic Factors (RQF)

Let $f(x)$ be a polynomial with real coefficients. Then, $f(x)$ can be written as a product of real linear and real quadratic factors.

Proof:

Let $f(x)$ be a polynomial of degree n with real coefficients.
Consider factoring $f(x)$ over $\mathbb{C}[x]$. We will get n complex roots, and be able to express $f(x)$ as a product of n linear factors.
Some of these factors may be real, and some of them may be non-real.
Since f has real coefficients, any imaginary roots come in pairs. (CJRT) Multiplying these pairs of imaginary roots together will produce a quadratic with real coefficients. So, any real polynomial can be broken down into linear and quadratic terms.

This is the last of the material that will be covered on the final exam. :D