Math138 - January 11'th 2016

Trig Integration

Trig Integration - Cases

Sometimes letting x equal a trig function can simplify an integral.

We'll be looking at three main cases:

1) $\sqrt{a^2 - x^2} \implies x = a \sin\theta$
2) $\sqrt{a^2 + x^2} \implies x = a \tan\theta$
3) $\sqrt{x^2 - a^2} \implies x = a \sec\theta$

Three Warnings

1) Watch for simpler methods of integration!
2) You may need to simplify before making a trig sub.
3) Always switch back to x at the end.

Examples

1)

$$\int \frac{1}{\sqrt{x^2+4}}\, dx$$
$$= \int \frac{2\sec^2\theta}{\sqrt{4\tan^2\theta + 4}}$$
$$= \int \frac{\sec^2\theta}{\sqrt{\tan^2\theta + 1}}$$
$$= \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}$$
$$= \int \sec\theta\, d\theta$$
$$= \ln|\ \sec\theta + \tan\theta\ | + c$$
$$= \ln\left|\ \frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\ \right| + c$$

$$x = 2\tan\theta$$
$$dx = 2\sec^2\theta\, d\theta$$

NOTE: USE TRIANGLES TO GET THESE VALUES

$$\tan\theta = \frac{x}{2}$$
$$\sec\theta = \frac{\sqrt{x^2+4}}{2}$$

2)

$$\int \frac{\sqrt{9-4x^2}}{x^2}\,dx \qquad x=\frac{3}{2}\sin\theta$$
$$=2\int \frac{\sqrt{\frac{9}{4}-x^2}}{x^2} \qquad dx=\frac{3}{2}\cos\theta\,d\theta$$
$$\frac{2x}{3}=\sin\theta$$
$$=2\int \frac{\sqrt{\frac{9}{4}-\frac{9}{4}\sin\theta}}{\frac{9}{4}\sin^2\theta}\cdot\frac{3}{2}\cos\theta\,d\theta \qquad \cot\theta=\frac{\sqrt{9-4x^2}}{2x}$$
$$=2\cdot\frac{3}{2}\cdot\frac{3}{2}\cdot\frac{4}{9}\int \frac{\sqrt{1-\sin^2\theta}\cdot\cos\theta}{\sin^2\theta}$$
$$=2\int \frac{\cos^2\theta}{\sin^2\theta}$$
$$=2\int \cot^2\theta$$
$$=2\int \csc^2\theta-1$$
$$=2(-\cot\theta-\theta)+c$$
$$=2(\frac{-\sqrt{9-4x^2}}{2x}-\arcsin\frac{2x}{3})$$

3)

$$\int \frac{dx}{x^2\sqrt{x^2-4}} \qquad x=2\sec\theta$$
$$=\int \frac{2\sec\theta\tan\theta}{4\sec^2\theta\sqrt{4\sec^2\theta-4}} \qquad \frac{x}{2}=\sec\theta$$
$$=\frac{1}{4}\int \frac{\sec\theta\tan\theta}{\sec^2\theta\sqrt{\sec^2\theta-1}} \qquad \frac{2}{x}=\cos\theta$$
$$=\frac{1}{4}\int \frac{\tan\theta}{\sec\theta}$$
$$=\frac{1}{4}\int \cos\theta$$
$$=\frac{1}{4}\sin\theta$$
$$=\frac{1}{4}\frac{\sqrt{x^2-4}}{x}$$

4)

$$\begin{aligned}
&\int x\sqrt{x^2-9}\,dx && u = x^2-9\\
&= \frac{1}{2}\int \sqrt{u}\,du && du = 2x\,dx\\
&= \frac{1}{3}u^{3/2} + c && \frac{du}{2x} = dx\\
&= \frac{1}{3}(x^2-9)^{3/2} + c
\end{aligned}$$

5)

$$\begin{aligned}
&\int \frac{x^2}{(1+x^2)^2}\,dx && x = \tan\theta\\
&= \int \frac{\tan^2\theta\sec^2\theta}{\sec^4\theta} && dx = \sec^2\theta\\
&= \int \frac{\tan^2\theta}{\sec^2\theta} && \sin(2\theta) = 2\sin\theta\cos\theta\\
&= \int \left(\frac{\sin^2\theta}{\cos^2\theta}\right)\left(\cos^2\theta\right) && = \frac{2x}{1+x^2}\\
&= \int \sin^2\theta\\
&= \frac{1}{2}\int 1-\cos(2\theta)\,d\theta\\
&= \frac{1}{2}\left(\theta - \frac{\sin(2\theta)}{2}\right) + c\\
&= \frac{1}{2}\left(\arctan x - \frac{x}{1+x^2}\right) + c
\end{aligned}$$