Math138 - January 04, 2016

Integration by Parts

The Formula

Let u and v be functions of x.

The product rule says $\frac{d}{dx}(u)(v) = \frac{du}{dx}v + u\frac{dv}{dx}$.

If we integrate, we get:

$$\begin{aligned}
\int \frac{d}{dx}(u\ v)dx &= \int \frac{du}{dx} \cdot (v)(dx) + \int u \cdot \frac{du}{dx}dx \\
(u)(v) &= \int v \cdot (du) + \int u \cdot (dv) \\
&= \int u \cdot dv = (u)(v) - \int v \cdot du \quad \text{(REMEMBER THIS FORMULA IT'S VERY IMPORTANT!)}
\end{aligned}$$

This looks a little bit confusing, and I was confused at first so I'll try to explain. What we've just proven is this: If we have a function that is very difficult or impossible to integrate with our current rules, we can split it into the product of two functions u and dv and use Integration by Parts to integrate.

To do this, first we need a u and dv function! Even though dv looks like a derivative, don't worry about this. Choose u and dv such that u times dv equals the function you're trying to integrate. Below are tips for choosing effective functions. Then you'll want to *differentiate* u and call the derivative du. Next, *integrate* dv and call the integral v. Now by Integration by Parts, the original integration we were trying to solve is equal to $(u)(v) - \int v \cdot du$.

Strategies for Choosing Functions

When integrating the product of two functions using Integration by Parts, we need to choose u and v effectively. We will be differentiating u and integrating v.

- Pick dv to be the most complicated part of the integral that you know how to integrate.
- Pick u so it gets simpler when you take the derivative.

ILATE Rule

The above tips may seem ambiguous and confusing. Thankfully, the 'ILATE' rule make choosing u and dv as simple as memorization.

- Pick u to be the first function that appears in this list:
  - **I**nverse trig functions
  - **L**ogarithms
  - **A**lgebraic (Polynomials)
  - **T**rig
  - **E**xponentials

Examples

a)

$$\int x^2 \ln x \, dx \qquad u = \ln x$$

$$= \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} dx \qquad du = \frac{1}{x} dx$$

$$= \frac{x^3}{3} \ln x - \int \frac{x^2}{3} dx \qquad dv = x^2 dx$$

$$= \frac{x^3}{3} \ln x - \frac{x^3}{9} + c \qquad v = \frac{x^3}{3}$$

b)

$$\int xe^x \, dx \qquad u = x$$

$$= xe^x - \int e^x \, dx \qquad du = \frac{1}{x} dx$$

$$= xe^x - e^x + c$$

$$dv = x^2 dx$$

$$v = \frac{x^3}{3}$$

c)

$$\int \ln x \, dx \qquad u = \ln x$$

$$= x \ln x - \int x \frac{1}{x} \, dx \qquad du = \frac{1}{x} dx$$

$$dv = dx$$

$$= x \ln x - x + c \qquad v = x$$

d)

$$\int x^2 \cos x \, dx \qquad u = x^2, \; du = 2x \, dx \quad \text{FIRST USAGE}$$

$$dv = \cos x, \; v = \sin x$$

$$= x^2 \sin x - \int 2x \sin x \, dx \qquad u = 2x, \; du = 2 \, dx \quad \text{SECOND USAGE}$$

$$dv = 2 \, dx, \; v = -\cos x$$

$$= x^2 \sin x - ((2x)(-\cos x) - 2 \int \cos x \, dx$$

$$= x^2 \sin x + 2x \cos x + 2 \sin x + c$$

e)

$$\int e^x \cos x \, dx \qquad u = \cos x, \; du = -\sin x \, dx \quad \text{FIRST USAGE}$$

$$dv = e^x, \; v = e^x$$

$$= \cos x e^x - \int e^x (-\sin x) \, dx \qquad u = \sin x, \; du = \cos x \quad \text{SECOND USAGE}$$

$$dv = e^x, \; v = e^x$$

$$= e^x \cos x + \int e^x \sin x \, dx$$

$$= e^x \cos x + e^x \sin x - \int e^x \cos x$$

Notice the integral is the same as our original integral.

Let the original integral be I

$$I = e^x \cos x + e^x \sin x - I$$

$$2I = e^x \cos x + e^x \sin x$$

$$I = \frac{1}{2}(e^x \cos x + e^x \sin x)$$

f)

$$\int_1^3 x^3 \ln x \, dx$$
$$= \left[\frac{x^4}{4}\ln x\right]_1^3 - \int_1^3 \frac{x^4}{4}\frac{1}{x}\, dx$$
$$= \left[\frac{x^4}{4}\ln x\right]_1^3 - \int_1^3 \frac{x^3}{4}\, dx$$
$$= \left[\frac{x^4}{4}\ln x - \frac{x^4}{16}\right]_1^3$$
$$= \frac{81}{4}\ln 3 - \frac{81}{16} - (0 - \frac{1}{16})$$
$$= 18\frac{\ln 3}{4} - \frac{80}{16}$$

$$u = \ln x$$
$$du = \frac{1}{x}dx$$

$$dv = x^3 dx$$
$$v = \frac{x^4}{4}$$

Trig Identities

In this section, we have two goals: integrating powers of Sin/Cos, and Sec/Tan. When these are multiplied together, they don't reduce into simpler trig functions like most other combinations.

How to Integrate Powers of Sin/Cos

e.g. $\int \sin^m(x)\cos^n(x)\, dx$

Case 1: m and n are both even.

- Use trig formulas to reduce to lower powers.
- Then use these trig identites:
  - $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$
  - $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$

Case 2: m or n is odd.

Then substitute $u =$ the base of the even power.

Case 3: Both are odd.

Let $u =$ the base of the higher power.

Case 4: m or n isn't an integer.

Let u be the base of the ugliest power.