

Math137 - November 12, 2015

Introduction To Complex Numbers

Number Sets We Know So Far

The first number set we learned was the Naturals (\mathbb{N}). After that, we moved to a superset of the Naturals, the Integers (\mathbb{Z}). Then, to a superset of the Integers, the Rationals (\mathbb{Q}). And finally, a superset of the Rationals, the Reals (\mathbb{R}). Today, we will the basics of a superset of the reals: The Complex Number System (\mathbb{C}). Hooray.

Complex Numbers

A **Complex Number** in **standard form** is an expression of the form $x + yi$, where $x, y \in \mathbb{R}$.

The set of all complex numbers is written as $\mathbb{C} = \{x + yi \mid x, y \in \mathbb{R}\}$

The real part of a complex number z , x , is denoted as $Re(z)$.

The imaginary part of a complex number z , yi , is denoted as $Im(z)$

Examples of Complex Numbers

i) $z = -1 + 2i$:

$$Re(z) = -1, Im(z) = 2$$

ii) $z = \pi + \sqrt{7}i$:

$$Re(z) = \pi, Im(z) = \sqrt{7}$$

iii) $z = 5 + 0i$:

$$Re(z) = 5, Im(z) = 0$$

Note: $Im(z) = 0$. When this is the case, we say $z \in \mathbb{R}$

iv) $z = 0 - 3i$:

$$Re(z) = 0, Im(z) = -3$$

Note: $Re(z) = 0$. When this is the case, we say z is purely imaginary.

Complex numbers are equal iff the real parts are equal **and** the imaginary parts are equal.

Addition

Addition in the complex number is defined as:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$\text{Example: } (3 + 2i) + (-1 + 4i) = (2 + 6i)$$

Multiplication

Multiplication is defined as :

$$(a + bi)(c + di) = (ac - bd) + (ad + cb)i$$

This looks really confusing. Turns out, we don't need this formula at all if we keep in mind $i^2 = -1$. Using this fact, we can just FOIL complex number products in the same way we do with reals.

Example:

$$\begin{aligned}(4 - 3i)(2 + i) &= 8 + ri - 6i - 3i^2 \\ &= 8 - 2i + 3 \\ &= 11 - 2i\end{aligned}$$

Solving Equations

Example: Solve $x^2 + 2x + 5 = 0$.

$$\begin{aligned}x &= \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} \\&= \frac{-2 \pm \sqrt{-16}}{2} \\&= \frac{-2 \pm 4\sqrt{-1}}{2} \\&= 1 \pm 2\sqrt{-1}\end{aligned}$$

Previously, we would stop here (Or when we noticed -16 under the square root) and state that there is no real solution. However when working in the complex number system, we can fully solve for x.

$$\begin{aligned}x &= -1 \pm 2\sqrt{-1} \\&= -1 \pm 2i\end{aligned}$$

Subbing these two solutions back into our original equation would produce an answer of 0.

Additive Identity

The additive identity in the complex number system is $0 + 0i$.

$$z + 0 + 0i = z$$

Additive Inverse

$$x + yi + (-x - yi) = 0 + 0i$$

So, the additive inverse $-z$ of z (Defined as $(x + yi)$) is $(-x - yi)$.

Subtraction

Subtraction is defined as:

$$z - w = z + (-w)$$

Subtraction in the complex number system works exactly the same as subtraction in the real number system.

Multiplicative Identity

The multiplicative identity is $1 + 0i$

$$(z)(1 + 0i) = z$$

Multiplicative Inverse

$$(x + yi)^{-1} = \frac{x + yi}{x^2 + y^2} = \frac{(x - yi)}{(x + yi)(x - yi)}$$

Division

The easy way to do division in the complex number system is to multiply by the conjugate of the denominator over the conjugate of the denominator.

Examples:

i) Express the following in standard form:

$$\begin{aligned}& \frac{(1 - 2i) - (3 + 4i)}{(5 - 6i)} \\&= \frac{(-2 - 6i)}{5 - 6i} \frac{5 + 6i}{5 + 6i} \\&= \frac{-10 - 12i - 30i + 36i^2}{25 - 36i^2} \\&= \frac{-46 - 42i}{61} \\&= \frac{46}{61} - \frac{42i}{61}\end{aligned}$$

ii) Simplify:

$$\begin{aligned}& i^{2015} \\&= (i^2)^{1007} i \\&= (-1)^{1007} i \\&= -i\end{aligned}$$