Math135 - November 30, 2015

Proof of CJRT - More Polynomials

Proof of CJRT

Recall: CJRT states that if n is a root of a complex polynomial, then $\overline{n}$ is also a root, so long as $f(x)$ has real coefficients.

Proof: Assume $f(x)$ is a polynomial with real coefficients, and that $c \in \mathbb{C}$ is a root of $f(x)$.

So,

$$f(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0 = 0$$
$$\overline{a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0} = \overline{0}$$
$$\overline{a_n c}^n + \overline{a_{n-1} c}^{n-1} + \cdots + \overline{a_1 c} + \overline{a_0} = 0$$
$$a_n \overline{c}^n + a_{n-1}\overline{c}^{n-1} + \cdots + a_1 \overline{c} + a_0 = 0$$

(Because a_i is a real number, and the conjugate of a real is itself.)

$\therefore f(\overline{c}) = 0$, so $\overline{c}$ is a root of f.

Ex. Write $f(x) = x^4 - 5x^3 + 16x^2 - 9x - 13$ as a product of irreducible factors in $\mathbb{C}[x]$, given that $2 - 3i$ is a root.

We know $2 - 3i$ is a root, so by CJRT, $2 + 3i$ is a root. $\therefore (x - 2 - 3i)(x - 2 + 3i)$ is a factor of $f(x)$
$(x - 2 - 3i)(x - 2 + 3i)$
$= x^2 - 2x + 3xi - 2x + 4 - 6i - 3ix + 6i + 9$
$= x^2 - 4x + 13$
Now, by long division, we see that $f(x) = (x^2 - 4x + 13)(x^2 - x - 1) = (x-2-3i)(x-2+3i)(x - \frac{1}{2} - \frac{\sqrt{5}}{2})(x - \frac{1}{2} + \frac{sqrt5}{2})$

$\therefore f(x)$ written as irreducible factors in $\mathbb{C}[x]$ is $f(x) = (x - 2 - 3i)(x - 2 + 3i)(x - \frac{1}{2} - \frac{\sqrt{5}}{2})(x - \frac{1}{2} + \frac{sqrt5}{2})$

Ex. Prove that any real polynomial of odd degree has a real root.

Proof by contradiction.
Assume that $f(x)$ is a polynomial of k degree and k is odd and $f(x)$ has no real roots.

Thus, all of its roots are complex numbers with non-zero imaginary parts.

By CJRT, if c is a root, then $\bar{c}$ is also root. We know $c \neq \bar{c}$ since all the roots have non-zero imaginary parts.

Thus, since all roots are imaginary, there must be an even number of roots.

but, by CPN, there are k complex roots for a complex polynomial of degree k, and k is odd.

Contradiction!