Math135 - November 23, 2015

Polynomials

Polynomials in the field $\mathbb{F}$

A polynomial in x over the field $\mathbb{F}$ is al algebraic expression of the form:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $n \geq 0$ is an integer.

- x is the indeterminate or value
- The numbers $a_0, a_1, \ldots, a_n$ are called coefficients

The coefficients a_i belong to field $\mathbb{F}$.

We use $\mathbb{F}[x]$ to denote the set of all polynomials over a set $\mathbb{F}$, where typically $\mathbb{F}$ is either $\mathbb{C}, \mathbb{R}, \mathbb{Q}$, or $\mathbb{Z}_p$

i) $(2i + \pi)z^3 - \sqrt{5}z + \frac{55}{4}i$ is a polynomial over the field $\mathbb{C}$.
ii) $\frac{5}{2}x^5 + \sqrt{2}x^3 + x$ is a polynomial over $\mathbb{R}$ or $\mathbb{C}$
iii) $x^2 + x + \frac{1}{x}$ Is not a polynomial as the term $\frac{1}{x} = x^{-1}$ and-1 is not ≥ 0
iv) $x = \sqrt{x}$ is not a polynomial as the term $\sqrt{x} = x^{\frac{1}{2}}$ and $\frac{1}{2}$ is not an integer.

Degree of a polynomial

Let $n \geq 0$ be an integer. If $a_n \neq 0$ in the polynomial:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

Then the polynomial is said to have degree n. In other words, the degree of a polynomial is the largest element of x that has a non-zero coefficient.

The zero polynomial has al[[]]l of its coefficients = 0 and its degree is not defined. Polynomials of degree 1 are called linear polynomials. Degree 2 polynomials are called quadratics. Degree 3 polynomials are called cubics.

Equality of Polynomials

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and $g(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$ both be polynomials in $\mathbb{F}[x]$
The polynomials $f(x)$ and $g(x)$ are equal iff $a_i = b_i$ for all i.

Operations on Polynomials

i) **Adding and Subtracting**
We add and subtract term by term. Collect like terms, add/subtract the co-efficients.

Ex. Calculate $(2x^4 + 4x + 1) + (2x^3 + 3x + 4) - (3x^3 + 4)$ in $\mathbb{Z}_5$

$= 2x^4 - x^3 + 7x + 1$
$= 2x^4 + 4x^3 + 2x + 1$

ii) **Multiplication**
When we multiply, we expand and collect like terms.

Ex. Calculate $(2ix^2 + (3 + i))(5x^2 - i)$ in $\mathbb{C}[x]$

$= 10ix^4 - 2i^2x^2 + (15 + 5i)x^2 - (3i + 1)$
$= 10ix^4 - (17 - 5i)x^2 + (1 - 3i)$

Ex. $(x^5 + x^2 + 1)(x + 1) + x^3 + x + 1$ in $\mathbb{Z}_2$

$= x^6 + x^5 + x^3 + x^2 + x + 1 + x^3 + x + 1$
$= x^6 + x^5 + 2x^3 + x^2 + 2x + 1$
$= x^6 + x^5 + x^2 + 1$

Ex. Prove that $(ax + b)(x^2 + x + 1)$ is the zero polynomial in $\mathbb{R}[x]$ iff $a = b = 0$

$(ax + b)(x^2 + x + 1)$
$= ax^3 + ax^2 + ax + bx^2 + bx + b$
$= ax^3 + (a + b)x^2 + (a + b)x + b$

This is the zero polynomial iff all coefficients are zero. We require $a = a + b = b = 0$. This is only possible if $a = b = 0$.

Ex. Show that there does not exist a polynomial in $\mathbb{R}[x]$ $ax + b$ such that $(x + 1)(ax + b) = x^2 + 1$

Expanding, we get:
$= ax^2 + bx + ax + b = ax^2 + (a + b)x + b$
To get x_1^2, we require:
$a = 1$
$b = 1$
$(a + b) = 0$
These 4 conditions are impossible to meet simultaneously.

Division Algorithm for Polynomials (DAP)

If $f(x)$ and $g(x)$ are polynomials in $\mathbb{F}[x]$ and $g(x)$ is not the zero polynomial, then there exists unique polynomials $q(x)$ and $r(x)$ in $\mathbb{F}[x]$ such that

$$f(x) = q(x)g(x) + r(x) \text{ where deg } r(x) < \text{ deg } g(x) \text{ or } r(x) = 0$$

The polynomial $q(x)$ is called the quotient polynomial. The polynomial $r(x)$ is called the remainder polynomial. If $r(x) = 0$, we say that $g(x)$ divides $f(x)$ or $g(x)$ is a factor of $f(x)$ and we write $g(x)|f(x)$.

To find quotients and remainders, we use long division.