Math137 - November 13'th, 2015

Newton's Method - Integral Calculus

Newton's Method

Newton's Method is an efficient algorithm to find roots of $f(x)$. That is, the x values where $f(x) = 0$

To use Newton's Method, begin by making an root approximation, x_0. A closer approximation will mean Newton's Method will converge faster, though an accurate approximation is not necessary. We run our approximation through a simple algorithm n times to achieve a close approximation. Usually, an accurate estimation can be reached within 3 iterations of Newtons Method.

When we have a x_i, we calculate x_{i+1} using the following formula.

$$x_{i+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{So long as } f'(x_n) \neq 0$$

Example: Use Newton's Method to find the root in $(0, 2)$ to $f(x) = e^x - 2\cos x = 0$

Start by computing $f'(x)$.

$f'(x) = e^x + 2\sin x$

Now we need to choose an estimation. Lets choose $x_0 = 1.5$

We obtain the following sequence of approximations using the Newton's Method algorithm listed above:

$x_0 = 1.5, x_1 = 0.830, x_2 = 0.580, x_3 = 0.541$

The actual root was $c \approx 0.540$ Newton's Method is not exact but a very good approximation.

Integral Calculus

The basis of integral calculus comes from the area problem. Suppose you're given a continuous function $f(x)$ that is positive on some interval $[a, b]$. Find the area between $f(x)$ and the x axis between a and b.

We could approximate the area beneath the curve by dividing the interval into n equal subintervals, drawing vertical lines through our function. We can then tally up the area of these rectangles and have a good estimation of our area.

As $n \to \infty$ that the approximation of the area $\to A$, our actual area.

Define: Riemann Sum: Let $f(x)$ be defined on $[a, b]$ and let Δ be a partition of $[a, b]$, given by:

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

Where $\Delta x_i = x_i - x_{i-1}, i = 1, 2, \ldots, n$ (Δx_i represents the width of the i'th partition/sub-interval)

Let $c_i \in [x_{i-1}, x_i]$, then

$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

Is called a Riemann sum of $f(x)$ for the partition Δ. It represents an approximation of the area A under the curve.

Remark: If each subinterval is of equal length, then $\Delta x_i = \Delta x = \frac{b-a}{n}$

And $x_j = a + (\frac{b-1}{n})j, j = 0, \ldots, n$

Also, as $n \to \infty, \Delta x \to 0$

Example: Estimate or approximate the area between $y = f(x) = \sqrt{1 - x^2}$ and the x axis between $x = 0$ and $x = 1$

Choose the subintervals to be of equal length.
$\Delta x = \frac{1-0}{n} = \frac{1}{n}$

$$A \approx \sum_{i=1}^{b} f(c_i)\Delta x = \sum_{i=1}^{n} \sqrt{1 - c_i{}^2} \cdot \frac{1}{n}, x_{i-1} < c_i < x$$

If we choose $c_i = \frac{i}{n}$, we'll get a lower approximation.

$$A_L = \sum_{i=1}^{n} \frac{1}{n} \cdot \sqrt{1 - (\frac{i}{n})^2}$$

If we choose $c_i = x_{i-1} = \frac{i-1}{n}$, we obtain an upper estimate.

$$A_U = \sum_{i=1}^{n} \frac{1}{n} \cdot \sqrt{1 - (\frac{i-1}{n})^2}$$

Thus, $A_L < A < A_U$ for all n

n	A_L	A_U
4	0.6239	0.8739
10	0.7261	0.8261
100	0.7801	0.7901
1000	0.7848	0.7858