Math136 - January 11'th, 2016

F-Flats, Hyperplanes and Subspaces

Line

A **line** in $\mathbb{R}^3$ through $\vec{b} \in \mathbb{R}$ with direction vector $v \in \mathbb{R}$ is the set $\{c_1\vec{v}+\vec{b}\}$ which we often write as a vector equation:

$\vec{x} = c_1\vec{v} + \vec{v},\ c_1 \in \mathbb{R} \quad (\vec{v} \neq \vec{0})$

Plane

A **plane** in $\mathbb{R}^n$ is given by the vector equation:

$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \vec{b},\ \ c_1, c_2 \in \mathbb{R}$
(Where $\vec{v}_1, \vec{v}_2, \vec{b}$ are fixed vectors and c_1, c_2 vary over $\vec{b}$) where $\{\vec{v}_1, \vec{v}_2\}$ are linearly independent (L.I)

K-Flat

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in mathbbR^n$ be L.I. vectors and $\vec{b} \in \mathbb{R}^n$. We call the set with vector equation

$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k + \vec{b},\ \ c_1, c_2, \ldots, c_k \in \mathbb{R}$

A **k-flat** through $\vec{b}$

Hyperplane

A $(n-1)$-flat in $\mathbb{R}^n$ is called a **hyperplane**.

E.g. The vector equation:

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad c_1, c_2, c_3 \in \mathbb{R}$$

Defines a 3-flat in $\mathbb{R}^4$ which is a hyperplane.

Note: Before we call this a 3-flat, we must check all vectors are L.I.

Subspaces

A subspace of $\mathbb{R}^n$ is a subset $S \subset \mathbb{R}^n$ which satisfies the following 10 properties:

S_1 $\vec{x} + \vec{y} \in S\ \forall \vec{x}, \vec{y} \in S$
S_2 $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})\ \forall \vec{x}, \vec{y}, \vec{w} \in S$
S_3 $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
S_4 There is $\vec{0} \in S$ with $\vec{x} + \vec{0} = \vec{x}\ \forall \vec{x} \in S$
S_5 For any $\vec{x} \in S$, there is $(-\vec{x}) \in S$ with $\vec{x} + (-\vec{x}) = \vec{0}$
S_6 $c\vec{x} \in S \forall c \in \mathbb{R}, \vec{x} \in S$
S_7 $c(d\vec{x}) = (cd)\vec{x}\ \forall c, d \in \mathbb{R}, \vec{x} \in S$
S_8 $(c + d)\vec{x} = c\vec{x} + d\vec{x}\ \forall c, d \in \mathbb{R}$
S_9 $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}\ \forall c \in \mathbb{R}, \vec{x}, \vec{y} \in S$
S_10 $1\vec{x} = \vec{x}\ \forall \vec{x} \in S$

Theorem 1.2.1 - Subspace Test

Let $S \subset \mathbb{R}^n$ be a non-empty subset of $\mathbb{R}^n$ that is closed under addition and scaler multiplication (I.e. $\vec{x}+\vec{y} \in S\ \forall \vec{x},\vec{y} \in S$ and $c\vec{x} \in S\ \forall c \in \mathbb{R}, \vec{x} \in S$

Then, S is a subspace of $\mathbb{R}^n$

E.g. Is $S = \text{span}\{\begin{bmatrix}1\\0\end{bmatrix}\}$ a subspace of $\mathbb{R}^2$?

First, note that $\begin{bmatrix}1\\1\end{bmatrix} = 0\begin{bmatrix}1\\0\end{bmatrix} \in S$, so S is non empty.

Now we must check that S satisfies closure under addition.
If $\vec{x},\vec{y} \in S$, then by definition $\vec{x} = c_1\begin{bmatrix}1\\0\end{bmatrix}, \vec{y} = c_2\begin{bmatrix}1\\0\end{bmatrix}$ for some $c_1, c_2 \in \mathbb{R}$
Then $\vec{x}+\vec{y} = c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}1\\0\end{bmatrix} = (c_1+c_2)\begin{bmatrix}1\\0\end{bmatrix} \in S$

$\therefore S$ is closed under addition ✓

Now we must check closure under scalar multiplication.
Suppose $\vec{x} \in S,\ d \in \mathbb{R}$
We need to check that $d\vec{x} \in S$

$d\vec{x} = d(c\begin{bmatrix}1\\0\end{bmatrix}) = dc\begin{bmatrix}1\\0\end{bmatrix} \in S$
$\therefore S$ is closed under scalar multiplication.
$\therefore S$ is a subspace of $\mathbb{R}^n$

E.g. Is $T = \{\begin{bmatrix}x_1\\x_2\end{bmatrix} \in \mathbb{R}^2 \mid x_1 + x_2^2 = 0\}$ a subset of $\mathbb{R}^3$?

Trick question! $T \in \mathbb{R}^2$, so it can't be a subspace of $\mathbb{R}^3$

Is T a subspace of $\mathbb{R}^2$?

It is non empty because $\begin{bmatrix}0\\0\end{bmatrix} \in T$

Closed under multiplication?
Lets try $\begin{bmatrix}-1\\1\end{bmatrix} \in T$ since $(-1)+1^2 = 0$
But $2\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-2\\2\end{bmatrix} \notin T$ since $(-2)+2^2 \neq 0$
Since T is not closed under multiplication it is not a subspace of $\mathbb{R}^2$

E.g. is $u = \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$ a subspace of $\mathbb{R}^3$?

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in U \checkmark$

Closed under addition?

If $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in U$, then

$(x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) = (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) = 0 + 0 = 0$

So $\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in U$

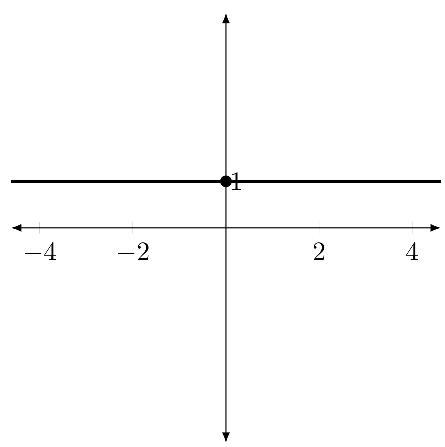
Exercise: Check closure under scalar multiplication.

Theorem 1.2.2

If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{R}^n$, then $S = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a subspace of $\mathbb{R}^n$

E.g. Is the line with vector equation $\vec{x} = c \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c \in \mathbb{R}$ a subspace of $\mathbb{R}^2$?

If we were to graph this vector equaton, we would see this:



Notice this never passes through the $\vec{0}$ vector. Clearly this cannot be a subspace of $\mathbb{R}^2$