Math138 - January 08, 2016

Sin/Cos and Tan/Sec Integration Rules

Rules

In the last lecture, we went over the 4 cases for sin/cos integration. We refer to these cases when solving the following examples.

P.s. If the letters not being on the same line as the math bothers you, sorry! I know there are ways to fix it but I've been typesetting for hours and I don't care anymore :D

Examples

a)

$$\begin{aligned} &\int \sin^6 x \cos^3 x \, dx \\ &= \int u^6 \cos^2 x \, du \\ &= \int u^6 (1-u^2) \, du \\ &= \int u^6 - u^8 \, du \\ &= \frac{u^7}{7} - \frac{u^9}{9} + c \end{aligned}$$

$$\begin{aligned} u &= \sin x \\ du &= \cos x \, dx \end{aligned}$$

Side note: Can they take marks off if you write $-c$?

b)

$$\begin{aligned} &\int \cos^1 5x \sin^5 x \, dx \\ &= -\int u^1 5 \sin^4 x \, du \\ &= -\int u^1 5 (1-u^2)^2 \, du \\ &= -\int u^1 5 (1 - 2u^2 + u^4) \, du \\ &= -\int u^1 6 - 2u^1 6 + u^1 9 \\ &= -(\frac{u^1 6}{16} - \frac{u^1 8}{9} + \frac{u^2 0}{20}) + c \\ &= \frac{\cos^1 6x}{16} - \frac{\cos^1 8x}{9} + \frac{\cos^2 0x}{20} + c \end{aligned}$$

$$\begin{aligned} u &= \cos x \\ du &= -\sin x \, dx \\ \sin^4 x &= (1-\cos^2 x)^2 \\ (\sin^2 x)^2 &= (1-u^2)^2 \end{aligned}$$

c)

$$
\begin{aligned}
&\int \sin^2 x \cos^2 x \; dx \\
&= \int \left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right) \\
&= \frac{1}{4}\int 1-\cos^2(2x) \; dx \\
&= \frac{1}{4}\int \sin^2(2x) \; dx \\
&= \frac{1}{4}\int \left(\frac{1-\cos(4x)}{2}\right) \; dx \\
&= \frac{1}{8}\int 1-\cos 4x \; dx \\
&= \frac{1}{8}\left(x-\frac{\sin(4x)}{4}\right)+c
\end{aligned}
\qquad
\begin{aligned}
u &= \cos x \\
du &= -\sin x \; dx \\
\sin^4 x &= (1-\cos^2 x)^2 \\
(\sin^2 x)^2 &= (1-u^2)^2
\end{aligned}
$$

Powers of Sec/Tan (Same as Csc/Cot)

Say we are trying to integrate $\int \tan^m x \cdot \sec^n x$. The rules work very similarly to sin/cos, but the cases are different.

Case 1: If n is even and m is anything, let $u = \tan x$ and use the identity $\sec^2 x = 1 + \tan^2 x$

Case 2: If m is odd and n is anything, let $u = \sec x$ and use the identity $\tan^2 x = \sec^2 x - 1$
Note: If n is even and m is odd, use the higher power for u.

Case 3: if m is even and n is odd, use $\tan^2 x = \sec^2 x - 1$ to write the entire integral in terms of sec, then use the formula for $\int \sec^n x \; dx$ from Assignment 1.

Examples

a)

$$
\begin{aligned}
&\int \tan^2 x \sec^4 x \; dx \\
&= \int u^2 \sec^2 x \; dx \\
&= \int u^2(1+\tan^2 x) \; du \\
&= \int u^2(1+u^2) \; du \\
&= \int u^2+u^4 \; du \\
&= \frac{u^3}{3}+\frac{u^5}{5} \\
&= \frac{\tan^3 x}{3}+\frac{\tan^5 x}{5}
\end{aligned}
\qquad
\begin{aligned}
u &= \tan x \\
du &= \sec^2 x \; dx
\end{aligned}
$$

b)

$$\int \tan^3 x \sec^3 x \; dx \qquad\qquad u = \sec x$$

$$du = \sec x \tan x; dx$$

$$= \int u^2 \tan^2 x \; du$$

$$= u^2(\sec^2 x - 1) \; du$$

$$= u^2(u^2 - 1) \; du$$

$$= \int u^4 - u^2 \; du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + c$$

$$= \frac{\sec^5 x}{5} - \frac{\sec{}^3 x}{3}$$

c)

$$\int \tan^3 x \sec^{14} x \; dx \qquad\qquad u = \sec x$$

$$du = \sec x \tan x \; dx$$

$$= \int u^{13} \tan^2 x \; du$$

$$= \int u^{13}(\sec^2 x - 1) \; du$$

$$= \int u^{13}(u^2 - 1) \; du$$

$$= \int u^{26} - u^{13} \; du$$

$$= \frac{u^{27}}{27} - \frac{u^{14}}{14} + c$$

$$= \frac{\sec^{27} x}{27} - \frac{\sec^{14} x}{14} + c$$

d)

$$\int \tan^4 x \sec^3 x \; dx$$

$$= \int (\sec^2 x - 1)^2 \sec^3 x \; dx$$

$$= \int (\sec^4 + 2\sec^2 x + 1) \sec^3 x \; dx$$

$$= \int sec^6 x + 2\sec^5 x + \sec^3 x \; dx$$

Now use the reduction formula for secant to reduce (From Assignment 1)