Math136 - January 8'th, 2016

Linear Dependency and Basis

Proof Continued From Last Class

$(\implies)$ We can express $\vec{v}_i$ as

$$\begin{aligned}\vec{v}_i &= 0\vec{v}_1 + \cdots + 1\vec{v}_i + 0\vec{v}_{i+1} + \cdots + \vec{v}_k \in \text{ span}(\{\vec{v}_1, \ldots, \vec{v}_k\}) \\ &= \text{span}(\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_k\})\end{aligned}$$

And so $\vec{v}$ is a linear combination of $\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_K$ by the definition of span.

E.g. Try to reduce span($\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$)

$$\begin{aligned}&= \text{span}(\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}) && \text{since } \begin{bmatrix}2\\0\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix} + 0\begin{bmatrix}1\\3\end{bmatrix} + 0\begin{bmatrix}0\\1\end{bmatrix} \\ &= \text{span}(\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}) && \text{since } \begin{bmatrix}1\\3\end{bmatrix} = 1\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix}\end{aligned}$$

Of course, there is no way to create either vector from a scalar multiple of the other in this case, so we're done.

Linear Dependency

A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is **linearly dependent** if there are scalars $c_1, \ldots, c_k \in \mathbb{R}$ which are not all 0 such that $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = 0$

E.g. (Referring to previous e.g.) we have:

$$2\begin{bmatrix}1\\0\end{bmatrix} + (-1)\begin{bmatrix}2\\0\end{bmatrix} + 0\begin{bmatrix}1\\3\end{bmatrix} + 0\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$\therefore \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$ is linearly dependent.

Note that this linear dependency allowed us to show that $\begin{bmatrix}2\\0\end{bmatrix}$ could be removed.

Linear Independency

If $\{\vec{v}_1, \ldots, \vec{x}_k\}$ is not linearly dependent, then it is **linearly independent**.

E.g. The set of $\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$ is linearly independent. To check:

Suppose that $c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$, so $c_1 = 0, c_2 = 0$

Hence, only the trivial linear combination (The solution where everything is 0) of these vectors gives $\vec{0}$

Note: Any set containing $\vec{0}$ is linearly dependent.

Spanning Set and Basis

If $S \subseteq \mathbb{R}^n$ satisfies $S = \text{span}(\{\vec{v}_1, \dots, \vec{v}_k\}$ then $\{\vec{v}_1, \dots, \vec{v}_k\}$ is called a **spanning set** for S. Also, S is spanned by $\{\vec{v}_1, \dots, \vec{v}_k\}$. If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is also linearly independent, then it is a **basis** for S.

Note: if the spanning set is not linearly independent, we can use Theorem 2 to reduce to a smaller spanning set, and eventually a basis.

Standard Basis

The **standard basis** for $\mathbb{R}^n$ consists of $\vec{c}_1, \dots, \vec{e}^n$ where $\vec{v}_i \in \mathbb{R}^n$ has it's i'th coordinate equal to 1 and all other coordinates equal to 0.

E.g. $\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$ is the standard basis for $\mathbb{R}^3$

E.g. Find a basis for $S = \text{span}(\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix} \})$

Let's check for linear dependency.

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\implies \begin{bmatrix} c_1 + 2c_2 + 0c_3 \\ c_1 + c_2 - c_3 \\ c_1 - c_2 - 3c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$c_1 + 2c_2 = 0$ (Equation 1)
$c_1 + c_2 - c_3 = 0$ (Equation 2)
$c_1 - c_2 - 3_3 = 0$ (Equation 3)

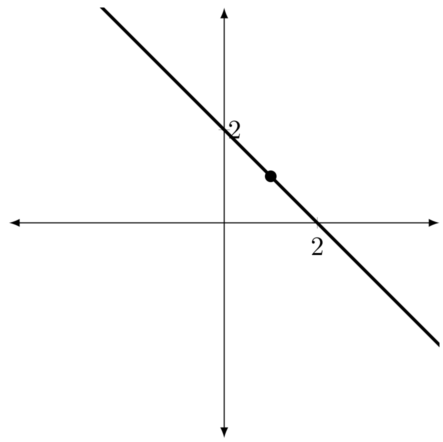
Let's solve this with some high-school math.

From eq. 1, $c_1 = -2c_2$
Substitute into eq. 2
$-2c_2 + c_2 - c_3 = 0 \implies c_3 - c_2$
Substitute both into eq. 3
$(-2c_2) - c_2 - 3(-c_2) = 0$
So, in fact the solutions are given by $\begin{bmatrix} c_1 \\ c_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ -t \end{bmatrix}$ for any $t \in \mathbb{R}$

Suppose we looked at $t = 1$. We would see that:
$$-2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
So we rearrange to get:
$$2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$$
$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \tfrac{1}{2} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + \tfrac{1}{2} \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix}$$
So by theorem 2, we can remove $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and we find $S = \text{span}\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix} \}$ which is a basis for S.

E.g. Describe the set $\begin{bmatrix}1\\1\end{bmatrix} + \tau \begin{bmatrix}1\\-1\end{bmatrix} \mid \tau \in \mathbb{R}$ geometrically.

This is a line with slope (-1) drawn through $\begin{bmatrix}1\\1\end{bmatrix}$ as shown below.



<u>Line in $\mathbb{R}^n$</u>

A **line** in $\mathbb{R}^n$ is given by the vector equation:

$\vec{x} = c_1\vec{v} + \vec{b}$

For $c_1 \in \mathbb{R}, \vec{v} \in \mathbb{R}^n, \vec{b} \in \mathbb{R}^n$ and fixed.