Math135 - November 19, 2015

Proof of DMT - Complex Exponentials

Imaginary Inverse Property:

$$z^{-n} = (z^{-1})^n = (z^n)^{-1} = \frac{1}{z^n}$$

Proof of DMT

Reminder: DMT States that $(\cos\theta + i\sin\theta)^n = (\cos(n\theta) + i\sin(n\theta))$

Case 1: $\mathbf{n = 0}$

LS: $(\cos\theta + i\sin\theta)^0$
$= 1$

RS: $\cos 0 + i\sin 0$
$= 1 + 0i$
$= 1$
$= LS$

Case 2: $\mathbf{n > 0}$

Proof by induction:

Base Case: $n = 1$

LS: $(\cos\theta + i\sin\theta)^1$
$= \cos\theta + i\sin\theta$

RS: $\cos((1)\theta) + i\sin((1)\theta)$
$= \cos\theta + i\sin\theta$
$= LS$

Inductive Hypothesis:

Assume $(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta)$ for some $k > 0$

Inductive Conclusion:

Need to show: $(\cos\theta + i\sin\theta)^{k+1} = \cos((k+1)\theta) + i\sin((k+1)\theta)$

LS:

$$\begin{aligned}
&(\cos\theta + i\sin\theta)^{k+1} \\
&= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) \\
&= (\cos(k\theta) + i\sin(k\theta))(\cos\theta + i\sin\theta) \quad \text{(By Inductive Hypothesis)} \\
&= \cos(k\theta + k) + i\sin(k\theta + k) \quad \text{(By PMCT)} \\
&= \cos((k+1)\theta) + i\sin((k+1)\theta) \\
&= RS
\end{aligned}$$

$\therefore$ by POMI, $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ for all $n > 0$.

Case 3: $\mathbf{n < 0}$

Let $n = -m, m > 0$

$$\begin{aligned}
(\cos\theta + i\sin\theta)^n &= (\cos\theta + i\sin\theta)^{-m} \\
&= \frac{1}{(\cos\theta + i\sin\theta)^m} \\
&= \frac{1}{(\cos(m\theta) + i\sin(m\theta))} \quad \text{(By Case 2)} \\
&= \frac{\cos(m\theta) - i\sin(m\theta)}{\cos^2(m\theta) + \sin^2(m\theta)} \\
&= \cos(-m\theta) + i\sin(-m\theta) \quad \text{(By symmetry of sin/cos)} \\
&= \cos(n\theta) + i\sin(n\theta)
\end{aligned}$$

QED.

A Proof with Imaginary Numbers

Ex. Prove that if w is complex, $|w| = 1$ and θ is an argument of w, then $\frac{-i}{2}(w^n - w^{-n}) = \sin(n\theta)\ \forall\, n \in \mathbb{Z}$

$$\begin{aligned}
\text{LS: } &= \frac{-i}{2}\left(w^n - w^{-n}\right) \\
&= \frac{-i}{2}\left((\cos\theta + i\sin\theta)^n - (\cos\theta + i\sin\theta)^{-n}\right) \\
&= \frac{-i}{2}\left(\cos(n\theta) + i\sin(n\theta) - (\cos(-n\theta) + i\sin(-n\theta))\right) \\
&= \frac{i}{2}\left(\cos(n\theta) + i\sin(n\theta) - \cos(n\theta) + \sin(n\theta)\right) \\
&= \frac{-i}{2}\left(2i\sin(2\theta)\right) \\
&= \sin(n\theta) \\
&= \text{RS} \quad \square
\end{aligned}$$

Complex Exponential

We will define complex exponential functions as:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

We can write the polar form of any complex number as $z = 4e^{i\theta}$, where $r = |z|$, and θ is an argument of z.

Ex. Write $(2e^{i(\frac{11\pi}{6})})^6$ in standard form.

$$\begin{aligned}
(2e^{i(\frac{11\pi}{6})})^6 &= (2(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}))^6 \\
&= 2^6(\cos 11\pi + i\sin 11\pi) \\
&= 64(\cos\pi + i\sin\pi) \\
&= 64(-1 + 0i) \\
&= -64
\end{aligned}$$

Awesome Properties of Complex Exponential Form

Notice how $(2e^{i\frac{11\pi}{6}})^6 = 2^6 e^{i11\pi}$.

Exponential form is useful because most of the exponent laws from the reals still apply.

Ex. $(e^{i\theta})^n = e^{in\theta}$
$(e^{i\theta})(e^{i\phi}) = e^{i(\theta+\phi)}$

Derivatives also work similarly in complex exponential form.

$\frac{d}{d\theta}(e^{i\theta}) = i(e^{i\theta})$

The Craziest Equation In Math

Notice what happens when we set $r = 1$, $\theta = \pi$.

$$e^{i\pi} = \cos\pi + i\sin\pi$$
$$e^{i\pi} = -1$$
$$e^{i\pi} + 1 = 0$$