Math137 - November 16, 2015

Connection between Differential and Integral Calculus

Consider an arbitrary $f(x)$ which is continuous on the closed interval $[a, b]$.

The area function $A(x)$ is defined to be the area under the curve from the point a to an arbitrary point x in the interval $[a, b]$.

Consider:

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

Then, for $h > 0$, $A(x+h) - A(x)$ represents the difference of two areas. This can be approximated by the area of a rectangle with base h and height $f(c)$ where c is some point in the interval $[x, x+h]$.

Thus: $\dfrac{A(x+h) - A(x)}{h} \approx \dfrac{hf(c)}{h} = f(c)$

In the limit, as $h \to 0$, we obtain:

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = f(c)$$

Computing Areas From First Principles

Ex. Find the area A lying under the line $y = x + 1$ above the x axis and between $x = 0$ and $x = 2$.
By inspection, the area is equal to $\frac{1}{2}(2)(1+3) = 4$
We verify this result as follows:
Divide $[0, 2]$ into n subintervals, each of length $\frac{2}{n}$.
Take height of each rectangle to be that of the right endpoint.
Area of the i'th rectangle $A_i = f(x_i)\Delta x_i$ where $f(x_i) = x_i + 1 = 1 + i\frac{2}{n}$, $\Delta x_i = \Delta x = \frac{2}{n}$.
Then,

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} A_i = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2}{n}(1 + i\frac{2}{n})$$

$$A = \lim_{n \to \infty} \left[\sum_{i=1}^{n} \frac{2}{n} + i\frac{4}{n^2} \right] = \lim_{n \to \infty} \left[\frac{2}{n} \sum_{i=1}^{n} 1 + \frac{4}{n^2} \sum_{i=1}^{n} i \right]$$

$$A = \lim_{n \to \infty} \left[2 + \frac{4}{n^2}(\frac{n^2+n}{2}) \right] = 2 + \lim_{n \to \infty} \left[2(1 + \frac{1}{n}) \right]$$

$$A = 2 + 2 = 4$$

Wow.

Summation Formulas

a) $\displaystyle\sum_{i=1}^{n} 1 = 1 + 1 + \cdots + 1\ n$

b) $\displaystyle\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

c) $\displaystyle\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}(n)(n+1)(2n+1)$

d) $\displaystyle\sum i = 1ni^3 = 1^3 + 2^3 + \cdots + n^3 = (\frac{n}{2}(n+1))^2$

e) $\sum_{i=1}^{n} r^{i-1} = 1 + r + r^2 + \cdots + r^{n-1} = \dfrac{r^n - 1}{r - 1}$