Math136 - January 27'th, 2016

Rank, Solution Theorem, Matrices

Rank of a Matrix

Suppose A is a matrix with RREF R. The **rank** of A is the number of leading ones in R.

Theorem 2.2.4

If A is a $m \times n$ matrix, then Rank $A \leq \min(m, n)$.

Theorem 2.2.5

Let A be the coefficient matrix of a system of m linear equations in n unknowns with augmented matrix $\left[A \quad | \quad \vec{b}\right]$.

1. If the rank of A is less than the rank of the augmented matrix $\left[A \quad | \quad \vec{b}\right]$, then the system is consistent.
2. If the system $\left[A \quad | \quad \vec{b}\right]$ is consistent, then the system contains ($n-$rank A) free variables.
3. Rank $A = m$ iff the system is consistent for every $\vec{b} \in \mathbb{R}^m$

This is the **System-Rank Theorem**

Note: Suppose that A is the coefficient matrix or a system of equations with augmented matrix $\left[A \quad | \quad \vec{b}\right]$. Then if A has RREF R, then $\left[A \quad | \quad \vec{b}\right]$ will have RREF $\left[R \quad | \quad \vec{r}\right]$ for some $\vec{r}$

Theorem 2.2.6

Let $\left[A \quad | \quad \vec{b}\right]$ be a consistent system of m linear equations in n variables with RREF $\left[R \quad | \quad \vec{b}\right]$. If rant $A = k < n$, then a vector equation of the solution set of $\left[A \quad | \quad \vec{b}\right]$ has the form $\vec{x} = \vec{d} + t_1\vec{v}_1 + \cdots + t_{n-k}\vec{v}_{n-k}, t_1, \ldots, t_{n-k} \in \mathbb{R}$ where $\vec{d} \in \mathbb{R}^n$ and $\{\vec{v}_1, \ldots, \vec{v}_{n-k}\}$ is a linearly independent set of vectors in $\mathbb{R}^n$. So the solution set is an $(n-k)$-flat in $\mathbb{R}^n$

A system of linear equations is **homogeneous** if the right hand side is zero. The set of solutions to a homogeneous system is the solution space.

Note: A homogeneous system is always consistent since we get a solution by setting all vars = 0.

Matrices

A $m \times n$ matrix A is a rectangular array with m rows and n columns. We denote the entry in the ith row and jth column by a_{ij}, that is:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{m1} & & & a_{mn} \end{bmatrix}$$

Addition and Scalar Multiplication

Let $A, B \in M_{m \times n}(\mathbb{R})$ and $c \in \mathbb{R}$. We define $A + B$ and cA as:

$(A + B)_{ij} = A_{ij} + B_{ij}$

$(cA)_{ij} = c(A)_{ij}$

Theorem 3.1.1

The set of matrices in $M_{m \times n}(\mathbb{R})$ satisfy the properties analogous to those in theorem 1.1.1.