Math135 - November 13, 2015

Working With Complex Numbers

Properties of Arithmetic in $\mathbb{C}$

Let $u, v, w \in \mathbb{C}$

1) Associativity of addition: $(u+v)+z = u(vz)$
2) Commutativity of addition: $u+v = v+u$
3) Additive identity: $0 = 0+0i$ has the property that $z+0=z$
4) Additive inverse: If $z = x+yi$ then there exists an *additive inverse* of z, written $-z$ with the property that $z+(-z)=0$ The additive inverse of $z=x+yi$ is $-z=-x-yi$
5) Associativity of multiplication: $(u \cdot v) \cdot z = u \cdot (v \cdot z)$
6) Commutativity of multiplication: $u \cdot v = v \cdot u$
7) Multiplicative identity: $1 = 1+0i$ has the property that $z \cdot 1 = z$
8) Multiplication inverses: If $z = x+yi \neq 0$ then there exists a *multiplicative inverse* of z, written z^{-1}, with the property that $z \cdot z^{-1} = 1$. The multiplicative inverse of $z = x+yi$ is $z^{-1} = \frac{x-yi}{x^2+y^2}$
9) Distributivity: $z \cdot (u+v) = z \cdot u + z \cdot v$

Prove Number 2: $u+v=v+u$

Let $u = x = yi, x, y \in \mathbb{R}$
Let $v = a+bi, a, b \in \mathbb{R}$

Left Side:
$= x+yi+a+bi$
$= (x+a)+(y+b)i$
$= (a+x)+(b+y)i$
$= a+bi+x+yi$
$= v+u$
$=$ RS

Exponents

By definition: $z^0 = 1+0i$ and $z^n = z \cdot z \cdot \cdots \cdot z$ n times.
Given these definitions, we can define the following: (If $m, n \in \mathbb{N}$)

$z^m \cdot z^n = z^{m+n}$
$z^{mn} = z^{mn}$

Example: Find a real solution to $6z^3 + (1+3\sqrt{2}i)z^2 - (11-2\sqrt{2}i)z - 6 = 0$

$$6z^3 + z^2 + 3\sqrt{2}z^2 i - 11z + 2\sqrt{2}zi - 6 = 0$$
$$6x^3 + z^2 - 11z - 6 + (3\sqrt{2}z^2 + 2\sqrt{2}z)i = 0$$

$$3\sqrt{2}z^2 + 2\sqrt{2}z = 0$$
$$\sqrt{2}z(3z+2) = 0$$
$$z = 0, \frac{-3}{2}$$

Now we'll sub these values into the other function.

$$\begin{aligned} 6(0)^3 + 0^2 - 11(0) - 6 &= 0 \\ -6 &= 0 \\ 6(\frac{-2}{3})^3 + (\frac{-2}{3})^2 + 11\frac{-2}{3} - 6 &= 0 \\ 0 &= 0 \end{aligned}$$

$\therefore$ Our solution is $z = 6$

Complex Conjugate

The complex conjugate of $z = x + yi$ is the complex number $x - yi$, denoted by $\bar{z}$.

Example: $z = 1 + 2i, \bar{z} = 1 - 2i$

We can express z^{-1} in terms of $\bar{z}$

If $z = x + yi$

$$z^{-1} = \frac{x - yi}{x^2 + y^2}$$
$$z^{-1} = \frac{x - yi}{(x - yi)(x + yi)}$$
$$z^{-1} = \frac{\overline{z}}{z\overline{z}}$$

Properties of Conjugates

1) $\overline{z + w} = \bar{z} + \bar{w}$
2) $\overline{zw} = \bar{z} \cdot \bar{w}$
3) $\bar{\bar{z}} = z$
4) $z + \overline{z} = 2Re(z)$
5) $z - \overline{z} = 2iIe(z)$
6) $\overline{(\frac{1}{z})} = \frac{1}{z}$
7) $\overline{(\frac{z}{w})} = \frac{\bar{z}}{\bar{w}}$

Real Numbers and Conjugates

Prove that $z \in \mathbb{R}$ iff $z = \overline{z}$.

$\implies$ Assume z is a real value:
$z = x + 0i$
$\overline{z} = x - 0i$
$z = \overline{z}$

$\impliedby$ Assume $z = \overline{z}$:
$x + yi = x - yi$
$y = -y$
$y = 0$
So, $z = x + 0i$, $z \in \mathbb{R}$.

Note: z is purely imaginary iff $z = -\overline{z}$