Math137 - December 03, 2015

Final Exam Review I

Limits

Evaluate each of the following:

a)

$$\lim_{x\to 5^-} \frac{|5+4x-x^2|}{1-|4-x|}$$
$$= \lim_{x\to 5^-} \frac{|(-1)(x^2-4x-5)}{1-|4-x|}$$
$$= \lim_{x\to 5^-} \frac{|(x-5)(x+1)|}{1-|4-x|}$$
$$= \lim_{x\to 5^-} \frac{|x-5|\,|x+1|}{1-|4-x|}$$
$$= \lim_{x\to 5^-} \frac{-(x-5)(x+1)}{-(x-5)}$$
$$= 6$$

b) $\lim_{x\to 0} x^2(1+\sin\frac{1}{x})$

$-1 \le \sin\frac{1}{x} \le 1 \quad \forall x \ne 0$

$0 \le 1+\sin\frac{1}{x} \le 2$

$0 \le x^2(1+\sin\frac{1}{x}) \le 2x^2$

$\lim_{x\to 0} 0 \le \lim_{x\to 0} x^2(1+\sin\frac{1}{x}) \le \lim_{x\to 0} 2x^2$ (By the squeeze theorem)

$0 \le \lim_{x\to 0} x^2(1+\sin\frac{1}{x}) \le 0$

$\therefore \lim_{x\to 0} x^2(1+\sin\frac{1}{x}) = 0$

c) $\lim_{x\to\infty} x^{e^{-x}}$

$$= \lim_{x\to\infty} e^{\ln x^{e^{ex}}}$$
$$= \lim_{x\to\infty} e^{e^{-x}\ln x}$$
$$= e^{\lim_{x\to\infty} e^{-x}\ln x}$$

We must evaluate this new limit.

$$= \lim_{x\to\infty} e^{-x}\ln x$$
$$= \lim_{x\to\infty} \frac{\ln x}{e^x}$$
$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{e^x}$$
$$= \lim_{x\to\infty} \frac{1}{xe^x}$$
$$= 0$$
$$\therefore \lim_{x\to\infty} x^{e^{-x}} = e^{\lim_{x\to\infty} e^{-x}\ln x} = e^0 = 1$$

Continuity

Let $f(x) = \begin{cases} 1 - \ln x & x < 1 \\ c \cdot \arctan x & x \geq 1 \end{cases}$

Determine the value of c so that f is a continuous function $\forall x > 0$

f is a continuous function $\forall x > 0, x \neq 1$
For f to be continuous at 1, we need:
$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x) = f(1)$
$\therefore \lim\limits_{x \to 1^-} 1 - \ln x = \lim\limits_{x \to 1^+} c \cdot \arctan x = c \cdot \arctan 1$
$1 = c \cdot \arctan 1 = c \cdot \arctan 1$
$1 = c \cdot \pi/4 = c \cdot \pi/4$
$c = \frac{4}{\pi}$

Differentiability

Using the definition of the derivative, determine any points where $y(x)$ is not differentiable.

1) $x < 0, x(x-3) > 0 \implies |x(x-3)| = x(x-3)$
2) $0 \leq x < 3, x(x-3) < 0 \implies |x(x-3)| = -x(x-3)$
3) $x \geq 3, x(x-3) > 0 \implies |x(x-3)| = x(x-3)$

$\therefore y(x) = \begin{cases} x(x-3) & x < 0 \cup x \geq 3 \\ -x(x-3) & 0 \leq x < 3 \end{cases}$

Now we know $y(x)$ is differentiable for all $x \neq 0, 3$.

Now we need to prove differentiability at 0 and 3. I'm very tired though, so I'm not going to type it up. TLDR, use the limit definition of a derivative ay a point to check if the derivative exists at 0 and 3.