Math138 - January 25'th, 2016

Improper Integrals

Two Types

1. Integrals of discontinuous functions
2. Integrals over unbounded or infinite intervals.

Type 1) Unbounded or infinite interval

Integrals that look like:

$\int_a^\infty f(x)\,dx$, $\int_{-\infty}^a f(x)\,dx$, etc. where $f(x)$ is continuous in the region.

Type 2) $f(x)$ has an (infinite) discontinuity in the interval.

How to Solve Improper Integrals

Type 1) Idea: Replace $\pm\infty$ with a letter and take a limit.

- $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^b f(x)\,dx$
- $\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty} \int_b^a f(x)\,dx$
- $\int_{-\infty}^\infty f(x)\,dx = \lim_{t_1\to-\infty} \int_{b_1}^0 f(x) + \lim_{t_2\to\infty} \int_0^{b_2} f(x)$

Type 2) Say $f(x)$ has an (infinite) discontinuity inside the interval. We can replace the problematic value with a letter and take a limit instead.

Consider $\int_a^b f(x)\,dx$

- If $f(x)$ is not continuous at $x = a$, then:
  $\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)$
- If $f(x)$ is not continuous at $x = b$, then:
  $\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$
- If $f(x)$ is not continuous at $x = c$ where c is between $x = a$ and $x = b$, then
  $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ Then use limits in the first two cases.

Convergence and Divergence

The integral **converges** if the limit exists and is finite.

The integral **diverges** if even one limit does not exist or is $\pm\infty$