Math138 - January 13'th, 2016

Partial Fractions Continued

Examples:

1)

$$\int \frac{x+3}{x^4+9x^2}\,dx$$

$$\begin{aligned}
\frac{x+3}{x^2(x^2+9)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+9} \\
\implies x+3 &= Ax^3 + 9Ax + Bx^2 + 9B + Cx^3 + Dx^2 \\
&= (A+C)x^3 + (B+D)x^2 + 9AX + 9B
\end{aligned}$$

By Substitution...

$$\begin{aligned}
0 &= A + C \\
0 &= B + D \\
1 = 9A \implies A &= \frac{1}{9} \\
3 = 9B \implies B &= \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
\int \frac{x+3}{x^4+9x^2}\,dx &= \frac{1}{9x} + \frac{1}{3x^2} + \frac{\frac{-1}{9}x - \frac{1}{3}}{x^2+9}\,dx \\
&= \frac{1}{9}\ln|\,x\,| - \frac{1}{3}x - \frac{1}{9}\int \frac{x}{x^2-9} - \frac{1}{3}\int \frac{1}{x^2-9} \quad \text{(Use U-sub)} \\
&= \frac{1}{9}\ln|\,x\,| - \frac{1}{3x} - \frac{1}{9}\left(\frac{1}{3}\ln|\,x^2+9\,|\right) - \frac{1}{3}\left(\frac{1}{3}\arctan(\frac{x}{3})\right) + c
\end{aligned}$$

2)

$$\int \frac{x^3-2x}{x^2+3x+2}\,dx$$

Notice deg(num) $>$ deg(denom). We must long divide. After long dividing we get:

$$\begin{aligned}
\frac{x^3-2x}{x^2+3x+2} &= x - 3 + \frac{5x+6}{x^2+3x+2} \\
\frac{5x-6}{(x+2)(x+1)} &= \frac{A}{x+2} + \frac{B}{x+1}
\end{aligned}$$

We follow the same steps as before to get:

$$\begin{aligned}
A &= 4 \\
B &= 1
\end{aligned}$$

$$\begin{aligned}
\int \frac{x^3-2x}{x^2+3x+2} &= \int (x-3) + \frac{4}{x+2} + \frac{1}{x+1}\,dx \\
&= \frac{x^2}{2} - 3x + 4\ln|\,x+2\,| + \ln|\,x+1\,| + c
\end{aligned}$$

Strategy for Integration

#1) Try an algebraic manipulation (ex. Expanding, long division, factoring, identities)

$$a) \int \frac{\tan^2\theta}{\sec^2\theta}\, d\theta$$
$$= \int \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos^2\theta\, d\theta$$
$$= \int \sin^2\theta$$
$$= \int \frac{1}{2}\left(1 - \cos(2\theta)\right)$$

At this point, it's a simple integral.

$$b) \int \frac{x+1}{x^2+4x+3}\, dx$$
$$= \int \frac{x+1}{(x+1)(x+3)}$$
$$= \int \frac{1}{x+3}$$
$$= \ln \mid x+3 \mid + c$$

#2) Look for a substitution!

- Let a troublesome term $= u$
- Let something inside an ugly power $= u$
- Let a function inside a function $= u$

$$a) \int \frac{\ln x}{x} \quad u = \ln x$$
$$= \int u\, du$$
$$= \frac{u^2}{2}$$
$$= \frac{(\ln x)^2}{2}$$

$$b) \int e^{\sqrt{x}} \quad u = \sqrt{x}$$
$$= 2\int e^u\, du$$

Now use IBP

$$c) \int \frac{x^2}{x^3+7}\, dx \quad u = x^3 + 7$$
$$= \int \frac{1}{3u^{9/17}}\, du$$
$$= \frac{1}{3}\int u^{-9^{1}7}\, du$$

Now use power rule.

#3) Look at major attributes of the integrand

- Powers of sin/cos or sec/tan or csc/cot?
  - Use appropriate u-sub
- Rational Functions?
  - Long division
  - Partial Fractions
- Radicals?
  - Completing the square and trig sub.
- Products of unrelated functions?
  - IBP (ILATE)

Can we Integrate Any Elementary Function?

Nope.

For example, you we cannot find the anti-derivative of:

$\frac{e^x}{x}, e^(x^2), \frac{\sin x}{x}, \frac{\cos x}{x}, \sin x^2, \frac{1}{\ln x}, \cos(x^2)$