Math136 - January 13'th, 2016

Dot Product and Orthogonality

Fact:

A K-Flat in $\mathbb{R}^n$ that passes through the origin is a subspace.

E.g. Find a basis for the subspace $S = \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R} \mid x_1 + x_+ x_3 = 0\}$

Note: if $x_1 + x_2 + x_3 = 0$, then $x_3 = -x - 1 - x_2$

So,

$$S = \{ \begin{bmatrix} x_1 \\ x_2 \\ -1_1 - x_2 \end{bmatrix} \in \mathbb{R}^3 \mid x_1, x_2 \in \mathbb{R}\}$$

$$= \{ \begin{bmatrix} x_1 \\ 0 \\ -x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ -x_2 \end{bmatrix} \mid x_1, x_2 \in \mathbb{R}\}$$

$$= \{x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\}$$

$$= \text{span}\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\}$$

So if we find that this span is L.I. then it will be a basis. We will use the fact that a set of two vectors is L.I. iff neither vector is a scalar multiple of eachother. You should prove this, but we can see of course this is the case, so we have a basis.

Dot Product

Recall in $\mathbb{R}^2, \mathbb{R}^3$ we have dot products:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 y_1 + x_2 y_2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = x_1 y_1 + x_2 y_2 = +x_3 y_3$$

We can now generalize this to $\mathbb{R}^n$

The **Dot Product** of $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ and $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$ is:

$$\vec{x} + \vec{y} = x_1 y_1 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

Note: The dot product is also called the standard inner product or the scalar product.

E.g. $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ -3 \\ -4 \end{bmatrix} = 1 \cdot 2 + 1 \cdot (-1) + 1 \cdot (-3) + 1 \cdot (-4) = -6$

Notice the dot product always gives a **scalar**.

Theorem 1.3.2

If $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$, then:

1) $\vec{x} \cdot \vec{x} \geq 0$ and $\vec{x} \cdot \vec{x} = \vec{0}$ iff $\vec{x} = \vec{0}$
2) $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$
3) $\vec{x} \cdot (s\vec{x} + t\vec{z}) = s(\vec{x} \cdot \vec{y}) + t(\vec{x} \cdot \vec{z})$

Norm

The **length** or **norm** of $\vec{x} \in \mathbb{R}^n$ is $|| \, \vec{x} \, || = \sqrt{\vec{x} \cdot \vec{x}}$

Orthogonal

$\vec{x}, \vec{y} \in \mathbb{R}^n$ are **orthogonal** if $\vec{x} \cdot \vec{y} = 0$

Note: $\vec{0}$ is orthogonal to any $\vec{x} \in \mathbb{R}^n$

Orthogonal Set

A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in $\mathbb{R}^n$ is an orthogonal set iff $\vec{v}_i \cdot \vec{v}_j = 0$ for all $i \neq j, i, j \in \{1, \ldots, k\}$

For example, the standard basis for $\mathbb{R}^n$ is an orthogonal set.

Unit Vector

A vector $\vec{x} \in \mathbb{R}^n$ with $|| \, \vec{x} \, || = 1$ is called a unit vector.

Theorem 1.3.3

If $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$, then:

1) $|| \, \vec{x} \, || \geq 0$, and $|| \, \vec{x} \, || = 0$ iff$\vec{x} = \vec{0}$
2) $|| \, c\vec{x} \, || = | \, c \, | \; || \, \vec{x} \, ||$
3) $(\vec{x} \cdot \vec{y})^2 \leq || \, \vec{x} \, ||^2 \; || \, \vec{y} \, ||^2$
4) $|| \, \vec{x} + \vec{y} \, || \leq || \, \vec{x} \, || + || \, \vec{y} \, ||$