Math135 - November 20, 2015

Roots of Exponential Numbers

Complex n'th Roots

If a is a complex number and n is a natural number, then the complex numbers that solve $z^n = a$ are called the complex n'th roots.

Ex. Find all complex 6'th roots of -64.

We need to solve $z^6 = -64$.
Let $z = r(cos\theta + -\sin\theta)$
$r^6(\cos(6\theta) + i\sin(6\theta)) = 64(\cos\pi + i\sin\pi)$

Two complex numbers in polar form are equal iff their moduli are equal and the difference between their arguments is a multiple of 2π. (Including 0 and negatives)

Thus,

$$r^6 = 64 \qquad\qquad 6\theta - \pi = 2k\pi, k \in \mathbb{Z}$$
$$4 = 2 \qquad\qquad 6\theta = (2k+1)\pi$$
$$\theta = \frac{\pi}{6} + \frac{\pi}{3}k$$

Note: If we start with $k = 0$ and count up, θ will begin to repeat when $k = 6$ since $6(\frac{\pi}{3}) = 2\pi$, and sin / cos of $2\pi = 0$.

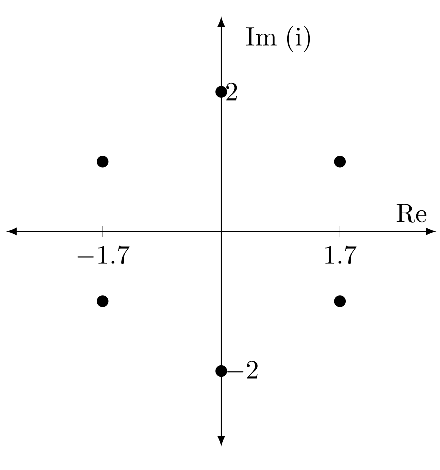
We get 6 roots with arguments:

$$\frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{9\pi}{6}, \frac{11\pi}{6}$$

$\therefore$ the roots will be:

$$\sqrt{3}+1,\ 2i\ , -\sqrt{3}+i, -\sqrt{3}-i, -2i, \sqrt{3}-i$$

Notice what happens when we plot these roots on the complex plane.



Complex n'th Roots Theorem (CNRT)

Let $n \in \mathbb{N}$. If $r(\cos\theta + i\sin\theta)$ is the polar form of a complex number a, then the solutions to $z^n = a$ are:

$$\sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right)\text{for } k = 0, 1, 2, \ldots, n-1$$

Note

1. Any non-zero complex number will have n distinct $n'th$ roots.
2. The roots graphed on a complex plane will lie on a circle of radius r centered on the pole (origin) and they will be evenly spaced with angles of $\frac{2\pi}{n}$ between them.

n'th Roots of Unity

Let $n \in \mathbb{N}$ An n'th root of unity is a complex number that solves:

$$z^n = 1$$

Ex. Find all 8'th roots of unity.

First, note that the complex number 1 has $r = 1$ and $\theta = 0$ (Think about where 1 lies on the complex plane)

So, we will use CNRT where $n = 8, a = 1, r = 1, \theta = 0$

$$\begin{aligned}
z &= \sqrt[8]{1}\left(\cos\frac{0 + 2k\pi}{8} + i\sin\frac{0 + 2k\pi}{8}\right) \\
&= \cos\frac{k\pi}{4} + i\sin\frac{k\pi}{4}, k = 0, 1, \ldots, 7 \\
&= 1, \frac{sqrt2}{2} + \frac{\sqrt{2}}{2}i, i, -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, -1, -\frac{\sqrt{2}2}{-}\frac{\sqrt{2}}{2}i, -i, \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i
\end{aligned}$$

Ex. $z^5 = -16\overline{z}$

$z = r(\cos\theta + i\sin\theta)$

$r^5(\cos(5\theta) + i\sin(5\theta)) = 16(\cos\pi + i\sin\pi)(r(\cos(-\theta) + i\sin(-\theta)))$
$r^5(\cos(5\theta) + i\sin(5\theta)) = 16r(\cos(\pi - \theta) + i\sin(\pi - \theta))$

$$\begin{aligned}
r^5 &= 16r \\
r^5 - 16r &= 0 \\
r(r^4 - 16) &= 0 \\
r = 0&,\ r = 2
\end{aligned}$$

$$\begin{aligned}
5\theta &= \pi - \theta + 2k\pi \\
6\theta &= \pi + 2k\pi \\
\theta &= \frac{\pi + 2k\pi}{6} \\
\theta &= \frac{\pi}{6} + \frac{k\pi}{3}
\end{aligned}$$

If $r = 0$, $z = 0 + 0i$

$z = \sqrt{3} + i, 2i, -\sqrt{3} + i, -\sqrt{3} - i, -2i, \sqrt{3} - i, 0$