MATH 137 The Fundamental Theorem of Calculus Fall 2015

FTC - Part 1

If f is continuous on the interval $[a, b]$, the integral function $g(x)$ defined by

$$g(x) = \int_a^x f(t)\ \mathrm{dt}, \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b) and

$$g'(x) = f(x)$$

FTC - Part 2

If f is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)\ \mathrm{dx} = F(b) - F(a),$$

where F is the antiderivative of f (ie. $F' = f$)

Proof:

Let $g(x) = \int_a^x f(t)\ \mathrm{dt}$. We know from FTC 1 that $g'(x) = f(x)$; that is, g is an antiderivative of f. If F is any other antiderivative of f on $[a, b]$, then we know that

$$F(x) = g(x) + c$$

for $a < x < b$ by the corollary of the Constant Function Theorem.
Since both F and g are continuous on $[a, b]$, we see that $F(x) = g(x) + c$ also holds when $x = a$ and $x = b$ by taking one-sided limits (as $x \to a^+$ and $x \to b^-$).
Evaluating $F(b) - F(a)$, we have

$$\begin{aligned} F(b) - F(a) &= [g(b) + c] - [g(a) + c] \\ &= g(b) - g(a) \\ &= \int_a^b f(t)\ \mathrm{dt} - \int_a^a f(t)\ \mathrm{dt} \\ &= \int_a^b f(t)\ \mathrm{dt} - 0 \\ &= \int_a^b f(t)\ \mathrm{dt} \qquad \square \end{aligned}$$

Example 1

The function g is defined by $g(x) = \displaystyle\int_0^x (t - t^2)\ \mathrm{dt}$, for all $x > 0$.
Calculate $g'(1)$ and find the location of any inflection points of g.

Example 2

Determine $\dfrac{dy}{dx}$ given that $y = \displaystyle\int_{1+3x^2}^{4} \frac{1}{2 + e^t}\ \mathrm{dt}$.

Example 3

Evaluate each of the following definite integrals.

(a) $\displaystyle\int_1^3 \left(u + \frac{2}{u}\right)\ \text{du}$

(b) $\displaystyle\int_0^{ln 8} \left(2e^{-2t}\right)\ \text{dt}$

(c) $\displaystyle\int_0^1 (2u+1)^2\ \text{du}$

(d) $\displaystyle\int_{-\frac{1}{2}}^0 (\cos \pi x)\ \text{dx}$