Math137 - November 227, 2015

Change of Variable - More Examples

Ex. Let $F(x) = \int_{2x}^{x^3} e^{-t} dt$. Find $F'(2)$.

This looks very similar to examples we did in the last note with FTC1. We have an issue though. To use FTC1 to find $F'(x)$, we need to have a constant lower bound and a function upperbound in terms of x. Here, we have two bounds both functions of x. Using integral property 4, we can introduce some arbitrary constant and rewrite our integral in terms of it as follows:

$$
\begin{aligned}
F(x) &= \int_{2x}^{0} e^{-t} dt + \int_{0}^{x^3} e^{-t} dt \\
&= -\int 0^{2x} e^{-t} dt + \int_{0}^{x^3} e^{-t} dt \\
F'(x) &= -(e^{2x}) \cdot \frac{d}{dx} 2x + e^{-x^3} \cdot \frac{d}{dx} x^3 \\
&= -2e^{-2x} + ex^2 e^{-x^3} \\
F'(2) &= -2e^{-2(2)} + e(2)^2 e^{-8} \\
&= -2e^{-4} + 12e^{-8}
\end{aligned}
$$

Using a Change of Variable to Evaluate Anti-Derivatives

Ex. Evaluate:

a) $\int_0^1 (\sqrt{x^2+x+3})(2x+1)dx$

Let $u = x^2 + x + 3$
$\frac{du}{dx} = 2x + 1$
$du = (2x+1)dx$
BOUNDS:
When $x = 0, u = x^2 + 0 + 3 = 3$
When $x = 1, u = 1^2 + 1 + 3 = 5$

$$
\begin{aligned}
&\int_0^1 (\sqrt{x^2+x+3})(2x+1)dx \\
&= \int_3^5 \sqrt{u}\ \ du \\
&= \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_3^5 \\
&= \left[\frac{2}{3}\sqrt{u^3}\right]_3^5 \\
&= \frac{2}{3}(\sqrt{125} - \sqrt{27})
\end{aligned}
$$

SIDE WORK/THOUGHT PROCESS
Notice we don't know the function that gave this derivative. We cannot expand because of the square root, and we don't have a product rule for integration
We will try a new method: Change of Variable. We will let some expression $= u$ such that the expression become simple enough for us to anti-differentiate.
Figuring out the correct value is tough, and requires practice.
Let $u = x^2 + x + 3$
$\frac{du}{dx} = 2x + 1$
$du = (2x+1)dx$
This is what we want. This substitution will give us a du which we need, and will eliminate all other terms. Lovely.
We will also need to change our bounds of integration, since we are working with u and not x anymore. Just sub the x value into our equation for u to get the new bounds.

b) $\int \frac{x}{\sqrt{9x^2+4}} dx$

Gross. Again, we don't know a function that differentiates to this, so lets try a variable substitution.

Let $u = 9x^2 + 4$
$\frac{du}{dx} = 18x$
$du = 18x \cdot dx$
Notice how substituting in u will replace the denominator of our fraction, but leave x and dx. We should manipulate the derivative of u in a way that it will eliminate those.
$du = 18x \cdot dx$
$x \cdot dx = \frac{1}{18} \cdot du$
We are ready to attempt the substitution.

$$\int \frac{x}{\sqrt{9x^2+4}} dx$$
$$= \int \frac{1}{\sqrt{u}} \cdot \frac{1}{18} \cdot du$$
$$= \frac{1}{18} \int u^{\frac{-1}{2}} \cdot du$$
$$= \frac{1}{18} \cdot \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C$$
$$= \frac{1}{9}\sqrt{u} + C$$
$$= \frac{1}{9}\sqrt{9x^2+4} + C$$

c) $\int 5xe^{-x^2} dx$

Let $u = x^2$
$du = 2x \cdot dx$
$\frac{5}{2} du = 5x \cdot dx$

$$\int e^{-u} \cdot (\frac{5}{2} du)$$
$$= \frac{5}{2} \int e^{-u}$$
$$= \frac{5}{2} \frac{e^{-u}}{-1} + C$$
$$= \frac{-5}{2} e^{-x^2} + C$$

d) $\int \frac{e^x}{1+e^{2x}} dx$

Let $u = e^x$
$\frac{du}{dx} = e^x$
$du = e^x \cdot dx$

$$\int \frac{e^x}{1+e^{2x}} dx$$
$$= \int \frac{1}{1+u^2} du$$
$$= \arctan(u) + C$$
$$= \arctan(e^x) + C$$