Math135 - November 25, 2015

More Polynomials: RT and FT

Example:

Let $f(x)$ and $g(x)$ be polynomials in a field $\mathbb{F}$.

If $f(x) \mid g(x)$ and $g(x) \mid f(x)$, show that $f(x) = cg(x)$ for some c in the field $\mathbb{F}$.

Proof: Assume $f(x) \mid g(x)$ and $g(x) \mid f(x)$
$g(x) =_1 (x)f(x)$ for some $g_q(x) \in \mathbb{F}[x]$
$f(x) = q_2 g(x)$ for some $q_2(x) \in \mathbb{F}[x]$

By substitution, $g(x) = q_1(x)q_2(x)g(x)$
We know the degree of $q_1(x)q_2(x) = 0$ (Since $(q_1(x)q_2(x) = 1)$
So, the degree of $q_1(x) = 0$, and the degree of $q_2(x) = 0$
This means $q_1(x)$ and $q_2(x)$ are constants.
$q_2(x) = c$, for some $c \in \mathbb{F}$.

$\therefore f(x) = cg(x)$

Polynomial Equation

A polynomial equation is an equation of the form:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

Which will often be written as $f(x) = 0/$

An element $c \in \mathbb{F}$ is called a root or zero of $f(x)$ iff $f(c) = 0$.

Remainder Theorem

The remainder when the polynomial $f(x)$ is divided by $(x - c)$ is $f(c)$.

Ex. i) From yesterday:

$f(z) = iz^3 + (3+i)z^2 + (3+5i)z + (-2+2i)$
$f(z) = (iz^2 + 4z + (-1-i))(z + (1+i)) + 2i$

RT tells us $f(-1-i) = 2i$

ii) $f(x) = x^2 + 1 = (x+1)(x-1) + 2$

RT tells us $f(-1) = 2$, $f(1) = 2$.

Ex. In $\mathbb{Z}_7$, what is the remainder when $f(x) = 4x^3 + 2x + 5$ is divided by $x + 6$

The dumb way:

$$\begin{aligned} f(x) &= f(-6) \\ &= 4(-6)^3 + 2(-6) + 5 \\ &= -864 + 2 + 5 \\ &= 4 \end{aligned}$$

The smart way:

$$\begin{aligned} f(-6) &= f(1) \\ &= 4 + 2 + 5 \\ &= 4 \end{aligned}$$

Factor Theorem (FT)

The linear polynomial $(x - c)$ is a factor of the polynomial $f(x)$ iff $f(c) = 0$.