Math137 - November 9'th, 2015

Intro to Curve Sketching

Curve Sketching - General Procedure

All necessary information for sketching $y = f(x)$ can be obtained from the following:

1) $f(x)$
   - Determine domain of the function.
   - x, y Intercepts
   - Asymptotes
2) $f'(x)$
   - Determine intervals of increase and decrease by applying the Increasing/Decreasing Function Test (listed below).
   - Determine local minimums and maximums by applying the First Derivative Test (listed below).
3) $f''(x)$
   - Determine Concavity of f by applying the Concavity Test (listed below)
   - Determine POI's (Points of Inflection).

Increasing/Decreasing Function Test

a) If $f'(x) > 0$ on some interval I, then $f(x)$ is increasing on the interval I.

b) If $f'(x) < 0$ on some interval I, then $f(x)$ is decreasing in the interval I.

First Derivative Test

Suppose c is a critical point of a continuous function f.

a) If f' changes from positive to negative at c, then f has a local maximum at c.

b) If f' changes from negative to positive at c, then f has a local minimum at c.

c) If f' does not change sign at c, then there is no local maximum or minimum at c.

Concavity Test

a) If $f''(x) > 0$ for all $x \in$ an interval I, then $f(x)$ is concave up on I.

b) If $f''(x) < 0$ for all $x \in$ an interval I, then $f(x)$ is concave down on I.

Inflection Points

Inflection points occur where $f''(x)$ changes sign and f is continuous. If $f''(c) = 0$, c is a **possible** inflection point. We still must check that the sign is different on the left and right side of c to confirm.

Example Sketch

Sketch the following function: $y = f(x) = \frac{x^3}{(1+x)}$

i) Lets begin by finding out zero's (x intercepts):

$$x^3 = 0$$
$$x = 0$$

ii) Now the y intercept:

$$y = \frac{0^3}{(1+0)}$$
$$= 0$$

iii) Vertical Asymptotes:

$$(1+x) = 0$$
$$x = -1$$

For asymptote behavior, we'll plug in a number very close to the asymptote on either side and see if we have a positive or negative number. We're only concerned with positive-ness and negative-ness (The actual value is irrelevant to us) so I'll represent these numbers by + signs or - signs

First, lets try just to the left of our asymptote, $x = -1.01$

$$f(x) = \frac{(-1.01)^3}{(1+(-1.01))}$$
$$= \frac{-}{-}$$
$$= +$$

So as x approaches -1 from the left, y tends to ∞. Now we'll try the other side.

$$f(x) = \frac{(-0.99)^3}{(1+(-0.99))}$$
$$= \frac{-}{+}$$
$$= -$$

As x approaches -1 from the right side, y tends to $-\infty$.

iv) Critical Points:

$$f(x) = \frac{x^3}{1+x}$$
$$f'(x) = \frac{(3x^2)(1+x) + (x^3)(1)}{(1+x)^2}$$
$$= \frac{x^2(3+2x)}{(1+x)^2}$$

$\therefore$ Critical values at $x = 0, -1, \frac{-3}{2}$

Now we'll check positiveness/negativeness of our derivative to find increasing/decreasing intervals of our original function.

	$x < \frac{-3}{2}$	$\frac{-3}{2} < x < -1$	$-1 < x < 0$	$x > 0$
$x^2(3+2x)$	-	+	+	+
$(1+x)^2$	+	+	+	+
f'	-	+	+	+
f	dec.	inc.	inc.	inc.

From this, we can see the intervals on which our function is increasing and decreasing. We can also find local maxima and minima anywhere where our derivative changes sign **and** the function exists.
We see we have one derivative sign change at $x = \frac{-3}{2}$. Our original function f is continuous so long as $x \neq -1$, so the value is included in our function. Therefore, we can conclude $x = \frac{-3}{2}$ is a local minimum. We'll want the y value as well, so let's calculate that.

$$\begin{aligned} f(\frac{-3}{2}) &= \frac{(\frac{-3}{2})^3}{1 + \frac{-3}{2}} \\ &= (\frac{-27}{8})(\frac{2}{-1}) \\ &= \frac{27}{4} \end{aligned}$$

$\therefore$ We have a minimum at $(-1.5, 6.75)$.

v) Now we'll do concavity and points of inflection.

$$\begin{aligned} f'(x) &= \frac{(x^2)(3+2x)}{(1+x)^2} \\ f''(x) &= \frac{[(2x)(3+2x) + (x^2)(2)] - [(x^2)(3+2x)(2)(1+x)]}{(1+x)^4} \\ &= \frac{(6x + 4x^2 + 2x^2)(1 + 2x + x^2) - (3x^2 + 2x^3)(2+2x)}{(1+x)^4} \\ &= \frac{6x + 12x^2 + 6x^3 + 4x^2 + 8x^3 + 4x^4 + 2x^2 + 4x^3 + 2x^4 - 6x^2 - 6x^3 - 4x^3 - 4x^4}{(1+x)^4} \\ &= \frac{2x^4 + 8x^3 + 12x^2 + 6x}{(1+x)^4} \\ &= \frac{2x(x^3 + 4x^2 + 6x + 3)}{(1+x)^4} \\ &= \frac{(2x)(x+1)(x^2+3x+3)}{(1+x)^4} \\ &= \frac{(2x)(x^2+3x+3)}{(1+x)^3} \end{aligned}$$

Lovely. The critical values of our second derivative are: $x = 0, -1$. Now we do the same thing as we did with the first derivative: Find where this function is increasing and decreasing using the critical values as interval endpoints.

	$x < -1$	$-1 < x < 0$	$x > 0$
$(2x)(x^2+3x+3)$	-	-	+
$(1+x)^3$	-	+	+
$f''(x)$	+	-	+
$f(x)$	conc. up	conc. down	conc. up

We have possible points of inflection at $x = -1, 0$.
The function does not exist at $x = -1$, so it cannot be a point of inflection.
The function exists and is continuous at $x = 0$, so $x = 0$ is a POI.

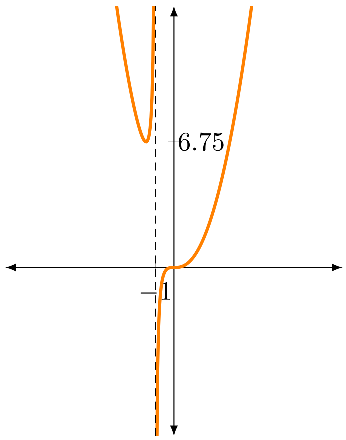
Now, lets recap all we've learned about our function.

x intercepts: $x = 0$
y intercept: $y = 0$
V asymptote: $x = -1$
Behavior:
From the left, y approaches ∞
From the right, y approaches $-\infty$

Increasing:
$x \in (-\infty, \frac{-3}{2}) \cup (-, \infty)$
Decreasing:
$x \in (\frac{-3}{2}, -1) \cup (-1, 0)$
Local min at $(-1.5, 6.75)$

Concave Up:
$x \in (-\infty, -1) \cup (0, \infty)$
Concave Down:
$x \in (-1, 0)$
POI at $x = 0$

That's a lot of information. Wow. Now we can create a very accurate graph of this function.



This is what our graph should look like. Don't mind the asymptote x marker, this was my first graph in latex, I'll figure out how to fine tune the graphs eventually!