Area Under Curves, Definite Integrals and Riemann Sums

Area Under Curves (continued)

Consider the problem of trying to calculate the area under the curve $y = f(x)$ from $x = a$ to $x = b$.

Divide the interval $[a, b]$ into n equal subintervals of width $\Delta x = \dfrac{b-a}{n}$ * using points

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

with $x_1 = x_0 + \Delta x$, $x_2 = x_0 + 2\Delta x$, and so on.

Choose sample points $x_1^*, x_2^*, \cdots, x_n^*$ in the intervals $[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n]$.

Consider the sum $\displaystyle\sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$.

Fact

If f is continuous, then the area under the curve is $A = \displaystyle\lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i^*)\Delta x\right]$.

Note

We can calculate the area using any sample points in the intervals (including the left-hand and right-hand endpoints).

Example

Consider $f(x) = \sin x$ with $\frac{\pi}{2} \le x \le \pi$. a b n=4

(a) Estimate the area under the curve using four subintervals and midpoints as sample points.

Solution

$x_1 = x_0 + 1\Delta x = \frac{\pi}{2} + \frac{\pi}{8} = \frac{5\pi}{8}$

Here, $a = \frac{\pi}{2}$, $b = \pi$, $n = 4$, $\Delta x = \frac{1}{4}(\pi - \frac{\pi}{2}) = \frac{\pi}{8}$.

$x_1^* = \frac{1}{2}\left(\frac{\pi}{2} + \frac{5\pi}{8}\right)$ MIDPOINT

Thus, $x_0 = \frac{\pi}{2}$, $x_1 = \frac{5\pi}{8}$, $x_2 = \frac{3\pi}{4}$, $x_3 = \frac{7\pi}{8}$, $x_4 = \pi$.

The midpoints of these intervals are $x_1^* = \frac{9\pi}{16}$, $x_2^* = \frac{11\pi}{16}$, $x_3^* = \frac{13\pi}{16}$, $x_4^* = \frac{15\pi}{16}$.

Therefore, $A \approx \sin(\frac{9\pi}{16}) \cdot \frac{\pi}{8} + \sin(\frac{11\pi}{16}) \cdot \frac{\pi}{8} + \sin(\frac{13\pi}{16}) \cdot \frac{\pi}{8} + \sin(\frac{15\pi}{16}) \cdot \frac{\pi}{8} \approx 1.00645$.

(b) Write the area as a limit using right-hand endpoints.

Solution

Here, $a = \frac{\pi}{2}$, $b = \pi$, $\Delta x = \frac{1}{n}(\pi - \frac{\pi}{2}) = \frac{\pi}{2n}$. $\Delta x = \frac{b-a}{n}$

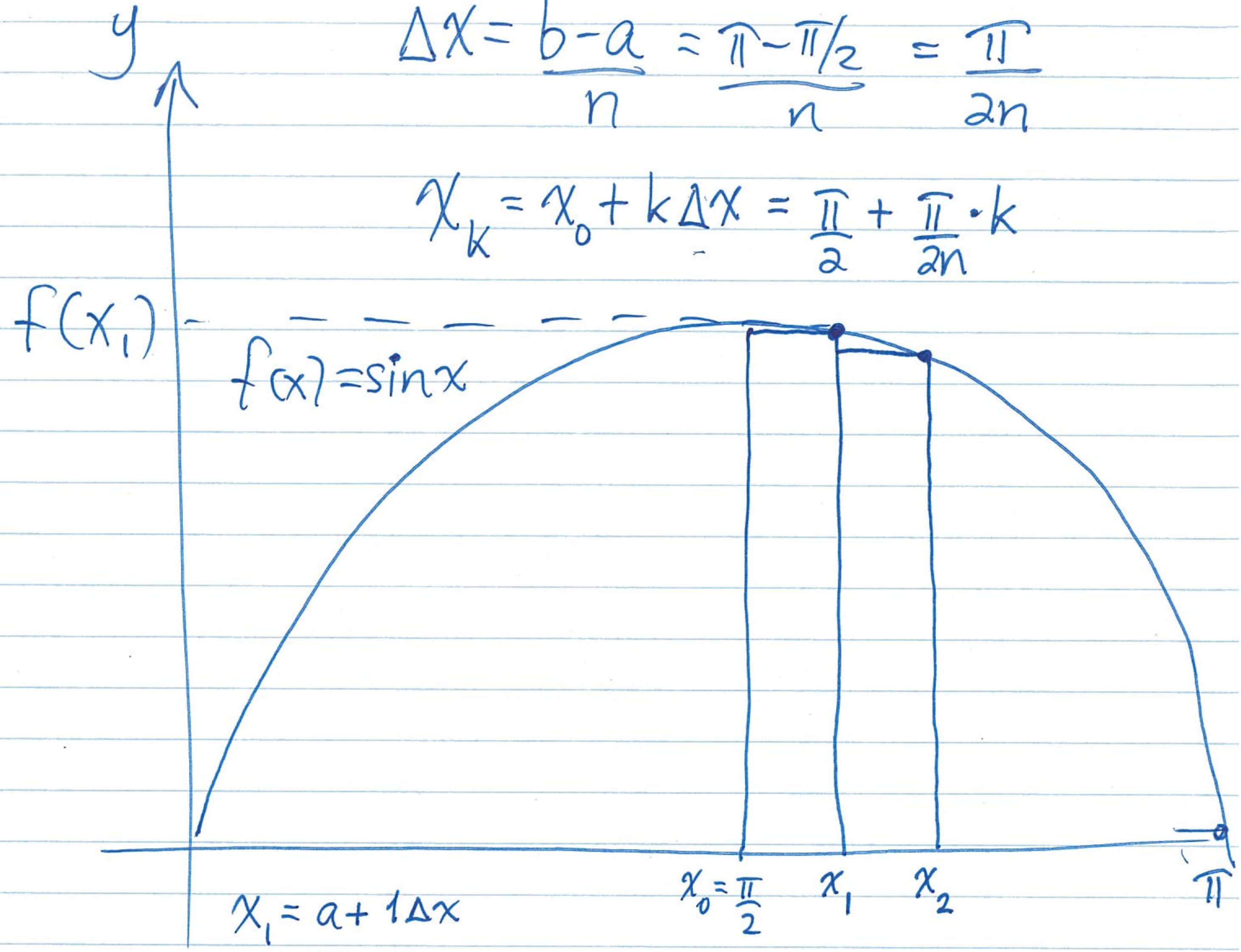
Thus, $x_k = x_0 + k\Delta x = \frac{\pi}{2} + \frac{\pi k}{2n}$.

Therefore, $A = \displaystyle\lim_{n\to\infty}\left[\sum_{k=1}^{n} f(x_k)\Delta x\right] = \lim_{n\to\infty}\left[\sum_{k=1}^{n} \sin(\tfrac{\pi}{2} + \tfrac{\pi k}{2n}) \cdot \tfrac{\pi}{2n}\right]$.

Q. What if you were told to use left-hand endpoints? $\displaystyle\sum_{k=0}^{n-1}$

$$\Delta x = \frac{b-a}{n} = \frac{\pi - \pi/2}{n} = \frac{\pi}{2n}$$

$$x_k = x_0 + k\Delta x = \frac{\pi}{2} + \frac{\pi}{2n}\cdot k$$



$$x_1 = a + 1\Delta x = \frac{\pi}{2} + \frac{\pi}{2n}$$

$$x_2 = a + 2\Delta x$$

$$x_k = a + k\Delta x$$