# Math137 - November 11, 2015

## Curve Sketching - Applied Max/Min Problems

**Example**

Sketch the following function: $y = (\frac{\ln(x)}{x})^2$

i) **X Intercepts:**
$\ln(x) = 0$
$x = 1$

ii) **Y Intercept:**
$y = (\frac{\ln(0)}{0})^2$
This function does not exist at 0, so there is no y intercept.

iii) **Vertical Asymptotes:**
$x = 0$ On the positive side of 0, the goes to infinity. On the left side, the function is undefined.

iv) **Horizontal End Behavior:**
As x gets big positively, $x$ overpowers $\ln(x)$, so the function tends to 0.
As x gets big negatively, The function does not exist.

v) **Critical Values:**

$$f(x) = (\frac{\ln x}{x})^2$$
$$f'(x) = 2(\frac{\ln x}{x})(\frac{(\frac{1}{x}(x) - \ln x}{x^2})$$
$$= \frac{(2)(\ln x)(1 - \ln x)}{x^3}$$

$\therefore$ Critical values are $1, e, 0$.

vi) **Increasing/Decreasing Intervals**

| | $x < 0$ | $0 < x < 1$ | $1 < x < e$ | $x > e$ |
|---|---|---|---|---|
| $2(\ln x)(1 - \ln x)$ | dne | - | + | - |
| $x^3$ | - | + | + | + |
| $f'$ | dne | - | + | - |
| $f$ | dne | Dec. | Inc. | Dec. |

$f(1) = (\frac{\ln 1}{1})^2$
$f(1) = 0$
So, min at (1, 0)

$f(e) = (\frac{\ln e}{e})^2$
$f(e) = \frac{1}{e^2}$
So, max at $(e, \frac{1}{e^2})$

vii) **Concavity**

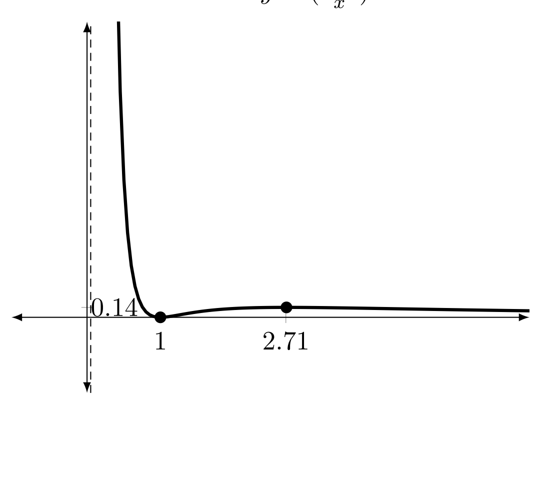
$$f'(x) = \frac{2\ln x - 2(\ln x)^2}{x^3}$$
$$f''(x) = \frac{(\frac{2}{x} - \frac{4\ln x}{x})x^3 - (2\ln x - 2(\ln x)^2)3x^2}{x^6}$$
$$= \frac{2x^2 - 4(\ln x)(x^2) - (2\ln x - 2(\ln x)^2)(3x^2)}{x^6}$$
$$= \frac{2 - 4\ln x - 6\ln x + 6(\ln x)^2}{x^4}$$
$$= \frac{2(3(\ln x)^2 - 5\ln x + 1)}{x^4}$$
$$\ln x = \frac{5 + -\sqrt{25 - 4(3)(1)}}{6}$$
$$= \frac{5 + -\sqrt{13}}{6}$$
$$x \approx e^{1.43}, e^{0.23}$$
$$\approx 4.18, 1.26$$

viii) **Intervals of Concavity and POIs**

| | $x < 0$ | $0 < x < 1.26$ | $1.26 < x < 4.18$ | $x > 4.18$ |
|---|---|---|---|---|
| Num. | dne | + | - | + |
| Denom. | + | + | + | + |
| $f''$ | dne | + | - | + |
| $f$ | dne | Conc. Up | Conc. Down | Conc.Up |

The function exists at $x = 1.26, 4.18$, and concavity changes at these values so they are points of inflection.

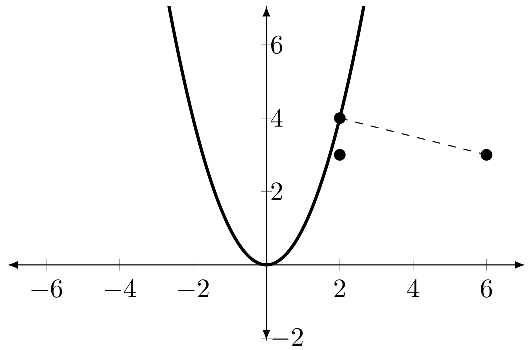
$y = (\frac{\ln x}{x})^2$



**Applied Max/Min Problems, Optimization**

**Examples:**

i) Find the point on the parabola $y = x^2$ which is closest to the point $(6, 3)$.




We want to minimize distance $d$, the length of the line segment from the nearest point on the parabola to $(6, 3)$. If we were to choose some point $P$ on our parabola, we could create a triangle connecting $P$, $(6, 3)$, and a point consisting of the x value of $P$ and the y value of $(6, 3)$. That's confusing, so look at the diagram above.

$d^2 = (x - 6)^2 + (x - 3)^2$

**Critical Points:**

$$\begin{aligned} \frac{dd^2}{dx} &= 2(x-6) + 4x(x^2-3) \\ &= 2(2x^3 - 5x - 6) \\ &= (x-2)(2x^2+4x+3) \end{aligned}$$

$2x^2 + 4x + 3 = 0$ has no real solutions, so the only critical value is $x = 2$. If we check, we see the derivative is negative before $x = 2$ and positive after $x = 2$, so $x = 2$ is the minimum.
$y = (2)^2$
$y = 4$

$\therefore$ the closest point on the function $y = x^2$ to the point $6, 3)$

ii) A rain gutter is to be constructed from a metal sheet of width 30cm by bending $\frac{1}{3}$ of the sheet up at both sides through an angle $\theta$. How should $\theta$ be chosen to maximize the amount of water contained?
I have no idea how to illustrate this, so I'll explain it. We have a 30cm flat metal sheet. We want to bend up the 10cm at both ends to maximize the amount of water contained. Bending up the 10 cm at both ends creates a trapazoid with theta representing the angle between the ground and the sheet. We want to find theta such that the area of our trapezoid is maximized.

$A(\theta) = (10h) + 10h\cos(\theta)$
As $h = 10\sin(\theta)$,
$A(\theta) = 100\sin(\theta)(1 + \cos(\theta))$
$\frac{dA}{d\theta} = 100\cos(\theta)(1 + \cos(\theta)) - 100\sin^2(\theta)$

Set derivative equal to 0 for Critical Values.

$0 = 100(\cos^2\theta - \sin^2\theta + \cos\theta)$
$0 = 2\cos^2\theta + \cos\theta - 1 \; 0 = (2\cos\theta - 1)(\cos\theta + 1)$

So, $\cos\theta = \frac{1}{2}$ or $-1$
-1 is out of our interval, so $\frac{\pi}{3}$ is our theta that maximizes area.