# Math137 - November 12, 2015

## Introduction To Complex Numbers

**Number Sets We Know So Far**

The first number set we learned was the Naturals ($\mathbb{N}$). After that, we moved to a superset of the Naturals, the Integers ($\mathbb{Z}$). Then, to a superset of the Integers, the Rationals ($\mathbb{Q}$). And finally, a superset of the Rationals, the Reals($\mathbb{R}$). Today, we will the basics of a superset of the reals: The Complex Number System ($\mathbb{C}$). Hooray.

**Complex Numbers**

A **Complex Number** in **standard form** is an expression of the form $x + yi$, where $x, y \in \mathbb{R}$.
The set of all complex numbers is written as $\mathbb{C} = \{x + yi \mid x, y \in \mathbb{R}\}$
The real part of a complex number $z$, $x$, is denoted as $Re(z)$.
The imaginary part of a complex number $z$, $yi$, is denoted as $Ie(z)$

**Examples of Complex Numbers**

i) $z = -1 + 2i$:
$Re(z) = -1$, $Im(z) = 2$

ii) $z = \pi + \sqrt{7}i$:
$Re(z) = 5$, $Im(z) = 0$

iii) $z = 5 + 0i$:
$Re(z) = 5$, $Im(z) = 0$
Note: $Im(z) = 0$. When this is the case, we say $z \in \mathbb{R}$

iv) $z = 0 - 3i$:
$Re(z) = 0$, $Im(z) = 3$
Note: $Re(z) = 0$. When this is the case, we say $z$ is purely imaginary.

Complex numbers are equal iff the real parts are equal **and** the imaginary parts are equal.

**Addition**

Addition in the complex number is defined as:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Example: $(3 + 2i) + (-1 + 4i) = (2 + 6i)$

**Multiplication**

Multiplication is defined as :

$$(a + bi)(c + di) = (ac - bd) + (ad + cb)i$$

This looks really confusing. Turns out, we don't need this formula at all if we keep in mind $i^2 = -1$. Using this fact, we can just FOIL complex number products in the same way we do with reals.

Example:

$$\begin{aligned}(4 - 3i)(2 + i) &= 8 + ri - 6i - 3i^2 \\ &= 8 - 2i + 3 \\ &= 11 - 2i\end{aligned}$$

**Solving Equations**

Example: Solve $x^2 + 2x + 5 = 0$.

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{-16}}{2} \\ &= \frac{-2 \pm 4\sqrt{-1}}{2} \\ &= 1 \pm 2\sqrt{-1} \end{aligned}$$

Previously, we would stop here (Or when we noticed -16 under the square root) and state that there is no real solution. However when working in the complex number system, we can fully solve for x.

$$\begin{aligned} x &= -1 \pm 2\sqrt{-1} \\ &= -1 \pm 2i \end{aligned}$$

Subbing these two solutions back into our original equation would produce an answer of 0.

**Additive Identity**

The additive identity in the complex number system is $0 + 0i$.

$z + 0 + 0i = z$

**Additive Inverse**

$x + yi + (-x - yi) = 0 + 0i$

So, the additive inverse $-z$ of $z$ (Defined as $(x + yi)$) is $(-x - yi)$.

**Subtraction**

Subtraction is defined as:

$z - w = z + (-w)$

Subtraction in the complex number system works exactly the same as subtraction in the real number system.

**Multiplicative Identity**

The multiplicative identity is $1 + 0i$

$(z)(1 + 0i) = z$

**Multiplicative Inverse**

$$(x + yi)^{-1} = \frac{x + yi}{x^2 + y^2} = \frac{(x - yi)}{(x + yi)(x - yi)}$$

**Division**

The easy way to do division in the complex number system is to multiply by the congigate of the denominator over the congigate of the denominator.

**Examples:**

i) Express the following in standard form:

$$\frac{(1-2i)-(3+4i)}{(5-6i)}$$
$$= \frac{(-2-6i)}{5-6i}\frac{5+6i}{5+6i}$$
$$= \frac{-10-12i-30i+36i^2}{25-36i^2}$$
$$= \frac{-46-42i}{61}$$
$$= \frac{46}{61}-\frac{42i}{61}$$

ii) Simplify:

$$i^{2015}$$
$$= (i^2)^{1007}i$$
$$= (-1)^{1007}i$$
$$= -i$$