Math137 - October 29'th, 2015

L'Hopitals Rule - Examples:

i) Evaluate $\lim_{x\to 0} \frac{\ln \cos x}{\sin x}$

Let $f(x) = \ln \cos x$
Let $g(x) = \sin x$
Then, $f(0) = g(0) = 0$

$f'(x) = \frac{-\sin x}{\cos x}$ (Chain rule)
$g'(x) = \cos x$

Since $\lim_{x\to 0} \frac{f'(x)}{g'(x)} = -\lim_{x\to 0} \frac{\sin x}{\cos^2 x} = 0$ (By Quotient Rule)
$\implies \lim_{x\to 0} \frac{\ln \cos x}{\sin x} = 0$ (By L'Hopitals Rule)

ii) Evaluate $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$

Rewrite $\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \frac{x - \sin x}{x \sin x}$ (This is an indeterminate limit (0/0) so we apply L'Hoptials Rule)

Let $f(x) = x - \sin x$
Let $g(x) = x \sin x$
$f'(x) = 1 - \sin x$
$g'(x) = \sin x + x \cos x$

But, $\lim_{x\to 0} f'(x) = 0$, $\lim_{x\to 0} g'(x) = 0$
Our limit is still indeterminate, so we apply L'Hopitals rule again.

$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{f''(x)}{g''(x)}$

$f''(x) = sinx$
$g''(x) = 2\cos x - x \sin x$

$\lim_{x\to 0} \frac{f''(x)}{g''(x)} = \lim_{x\to 0} \frac{\sin x}{2\cos x \sin x} = \frac{0}{2} = 0$ (Limit Quotient Rule)

$\therefore \lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = 0$ (By L'Hoptials Rule)

iii) Evaluate $\lim_{x\to 0^+} x \ln x$

Rewrite $x \ln x = \frac{lnx}{\frac{1}{x}}$ ($\frac{-\infty}{\infty}$ Indeterminate limit, so we use L'Hopitals rule)

Let $f(x) = \ln x$
Let $g(x) = \frac{1}{x}$
So, $f'(x) = \frac{1}{x}$
and $g'(x) = \frac{-1}{x^2}$

$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} \frac{x^2}{-x} = \lim_{x\to 0^+} -x = 0$

$\therefore \lim_{x\to 0^+} x \ln x = 0$ (By L'Hopitals Rule)

Note: Similarily, we can show that for any $a > 0$, $\lim_{x\to 0^+} x^a \ln x = 0$

iv) Evaluate $\lim_{x\to\infty} x^{\frac{1}{x}}$

$$\begin{aligned} Simplify : x^{\frac{1}{x}} &= e^{\ln x^{\frac{1}{x}}} \\ &= e^{\frac{1}{x}\ln x} \\ &= e^{\frac{\ln x}{x}} \end{aligned}$$

$$\begin{aligned} \lim_{x\to\infty} x^{\frac{1}{x}} &= \lim_{x\to\infty} e^{\frac{\ln x}{x}} \\ &= e^{\lim_{x\to\infty} \frac{\ln x}{x}} \end{aligned}$$

So, we must find $\lim_{x\to\infty} \frac{\ln x}{x}$
This is an indeterminate limit, so we use L'Hopitals Rule. Let $f(x) = \ln x$
Let $g(x) = x$
$f'(x) = \frac{1}{x}$
$g'(x) = 1$

$$\begin{aligned} \lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{f'(x)}{g'(x)} \\ &= \lim_{x\to\infty} \frac{\frac{1}{x}}{x} \\ &= 0 \text{ (By L'Hopitals Rule)} \end{aligned}$$

$$\begin{aligned} \lim_{x\to\infty} x^{\frac{1}{x}} &= e^{\lim_{x\to\infty} \frac{\ln x}{x}} \\ &= e^0 \\ &= 1 \end{aligned}$$

Extreme Value Theorem (EVT)

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some points $c, d \in [a, b]$