Math137 - November 23, 2015

Reimann Sums

NOTE: Today in class, we just went over this printed note. I have no hand written notes, so I'll just upload the printed note. I did not make this not, credit to Mike Eden aka best calc prof.

Area Under Curves, Definite Integrals and Riemann Sums

Area Under Curves (continued)

Consider the problem of trying to calculate the area under the curve $y = f(x)$ from $x = a$ to $x = b$.

Divide the interval $[a, b]$ into n equal subintervals of width $\Delta x = \dfrac{b-a}{n}$ * using points

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

with $x_1 = x_0 + \Delta x$, $x_2 = x_0 + 2\Delta x$, and so on.

Choose sample points x_1^*, x_2^*, $\cdots$, x_n^* in the intervals $[x_0, x_1]$, $[x_1, x_2]$, $\cdots$, $[x_{n-1}, x_n]$.

Consider the sum $\displaystyle\sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x.$

Fact

If f is continuous, then the area under the curve is $A = \displaystyle\lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i^*)\Delta x\right]$.

Note

We can calculate the area using any sample points in the intervals (including the left-hand and right-hand endpoints).

Example

a b $n=4$

Consider $f(x) = \sin x$ with $\frac{\pi}{2} \le x \le \pi$.

(a) Estimate the area under the curve using four subintervals and midpoints as sample points.

$x_1 = x_0 + 1\Delta x = \frac{\pi}{2} + \frac{\pi}{8} = \frac{5\pi}{8}$

Solution

Here, $a = \frac{\pi}{2}$, $b = \pi$, $n = 4$, $\Delta x = \frac{1}{4}(\pi - \frac{\pi}{2}) = \frac{\pi}{8}$. $x_1^* = \frac{1}{2}\left(\frac{\pi}{2} + \frac{5\pi}{8}\right)$ MIDPOINT

Thus, $x_0 = \frac{\pi}{2}$, $x_1 = \frac{5\pi}{8}$, $x_2 = \frac{3\pi}{4}$, $x_3 = \frac{7\pi}{8}$, $x_4 = \pi$.

The midpoints of these intervals are $x_1^* = \frac{9\pi}{16}$, $x_2^* = \frac{11\pi}{16}$, $x_3^* = \frac{13\pi}{16}$, $x_4^* = \frac{15\pi}{16}$.

Therefore, $A \approx \sin(\frac{9\pi}{16}) \cdot \frac{\pi}{8} + \sin(\frac{11\pi}{16}) \cdot \frac{\pi}{8} + \sin(\frac{13\pi}{16}) \cdot \frac{\pi}{8} + \sin(\frac{15\pi}{16}) \cdot \frac{\pi}{8} \approx 1.00645$.

(b) Write the area as a limit using right-hand endpoints.

Solution

Here, $a = \frac{\pi}{2}$, $b = \pi$, $\Delta x = \frac{1}{n}(\pi - \frac{\pi}{2}) = \frac{\pi}{2n}$. $\Delta x = \dfrac{b-a}{n}$

Thus, $x_k = x_0 + k\Delta x = \frac{\pi}{2} + \frac{\pi k}{2n}$.

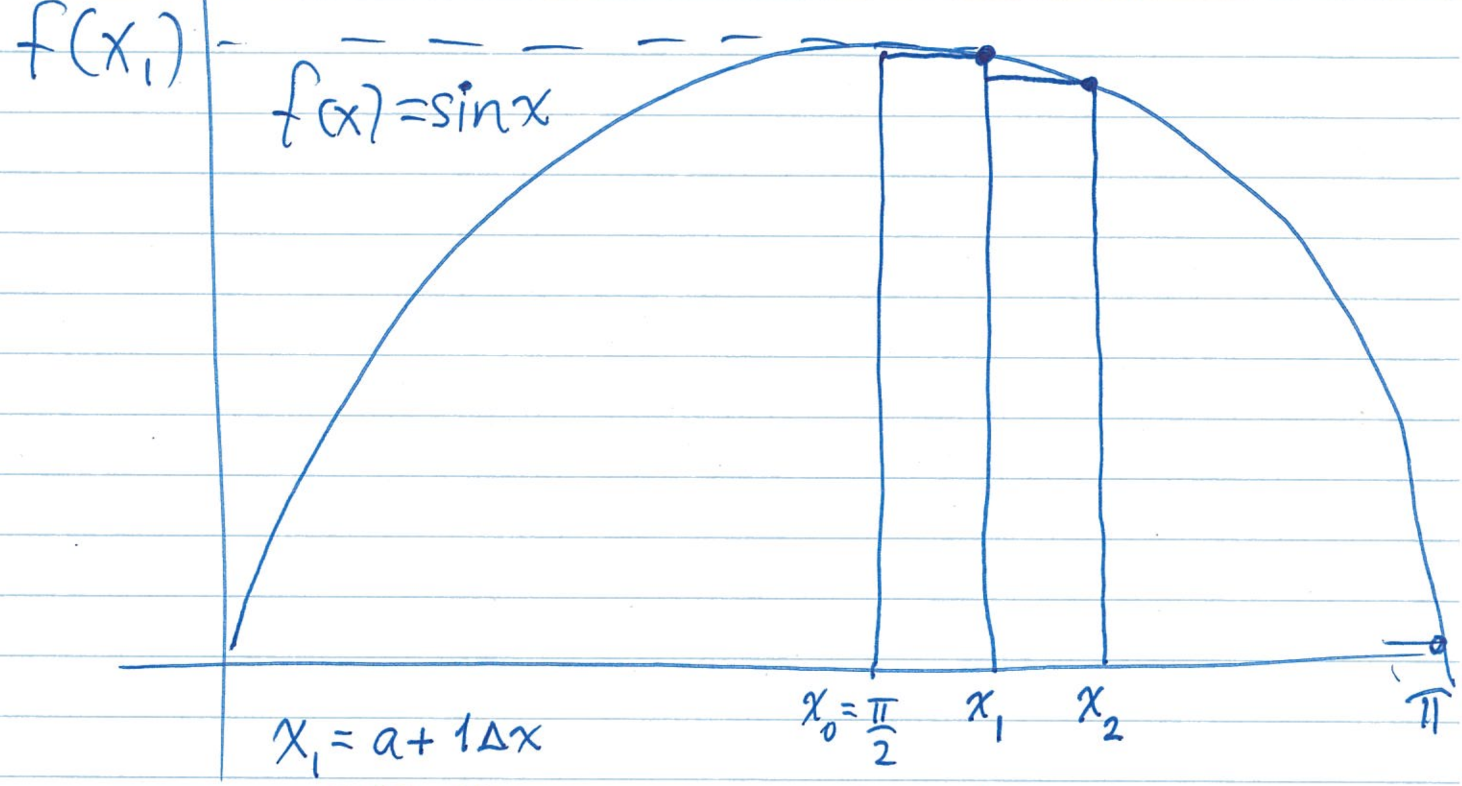
Therefore, $A = \displaystyle\lim_{n\to\infty}\left[\sum_{k=1}^{n} f(x_k)\Delta x\right] = \lim_{n\to\infty}\left[\sum_{k=1}^{n} \sin(\tfrac{\pi}{2} + \tfrac{\pi k}{2n}) \cdot \tfrac{\pi}{2n}\right]$.

Q. What if you were told to use left-hand endpoints? $\displaystyle\sum_{k=0}^{n-1}$

$$\Delta x = \frac{b-a}{n} = \frac{\pi - \pi/2}{n} = \frac{\pi}{2n}$$

$$x_k = x_0 + k\Delta x = \frac{\pi}{2} + \frac{\pi}{2n} \cdot k$$

y



$x_1 = a + 1\Delta x$

$= \frac{\pi}{2} + \frac{\pi}{2n}$

$\vdash \Delta x \dashv$

$= \frac{\pi}{2n}$

$x_2 = a + 2\Delta x$

$x_k = a + k\Delta x$