Math135 - November 16, 2015

Complex Modulus - Polar Coordinates

Complex Modulus

The modulus of the complex number $z = x + yi$ is the non-negative real number:

$$\mid z \mid = \mid x + yi \mid = \sqrt{x^2 + y^2}$$

We cannot order complex numbers, but using the modulus gives us a way to compare them.

Ex. If $z = 3 - 5i$, then
$|z| = \sqrt{3^2 + (-5)^2}$
$|z| = \sqrt{34}$

Properties of Modulus

If z and w are complex numbers, then

1. $|z| = 0$ if and only if$z = 0$
2. $|\overline{z}| = |z|$
3. $\overline{z}z = |z|^2$
4. $|zw| = |z||w|$
5. $|z + w| \leq |z| + |w|$

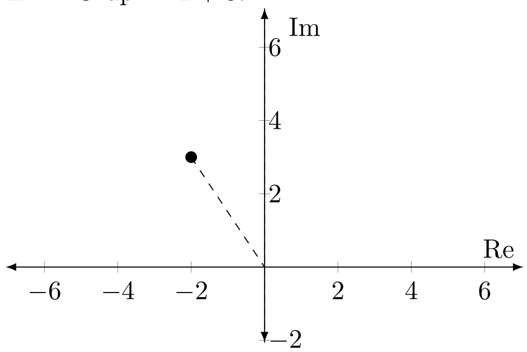
Ex. Let $z \in \mathbb{C}$ such that $z \neq \pm i$
Prove $\dfrac{z}{1+x^2} \in \mathbb{R} \iff z \in \mathbb{R}$ or $|z| = 1$
Assume $\dfrac{z}{1+z^2} \in \mathbb{R}$.

$$\begin{aligned}
\frac{z}{1+z^2} &= \overline{\frac{z}{1+z^2}} \\
&= \frac{\overline{z}}{1+(\overline{z})^2} \\
z(1+\overline{z}^2) &= \overline{z}(1+z^2) \\
z + z\overline{z}^2 &= \overline{z} + \overline{z}z^2 \\
0 &= z - \overline{z} + z\overline{z}^2 - \overline{z}z^2 \\
&= z - \overline{z} - z\overline{z}(\overline{z} + z) \\
&= (z - \overline{z})(1 - z\overline{z})
\end{aligned}$$

So, $z = z\overline{z} \in \mathbb{R}$ or $|z| = 1$ Since all steps taken in this proof are reversible, we don't have to prove the iff the other way.

The Complex Plane

Ex. Graph $-2+3i$.

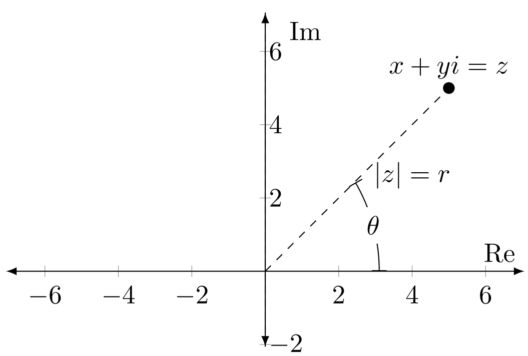


This is called an Argand Diagram.

Thinking graphically, the **conjugate** is a reflection in the real axis.
The **modulo** is the distance from the origin (0,0).
Addition is like vector addition. You draw the line from the origin to one point, draw a line from the origin to the other point, then append one line on the end of the other line to

Polar Coordinates

Polar coordinates offer a way to think of complex numbers in terms of a magnitude and an angle. To construct a polar coordinate, imagine the magnitude r as a point on the positive real axis. Then, rotate this point θ radians around a circle centered at the origin with a radius r. When working with polar coordinates, the origin is called the **pole**, and the positive real axis is called the **polar axis**.



$r=|x|=\sqrt{x^2+y^2}$
$\theta=$the counter-clockwise angle of rotation from the polar axis measured in radians.
(r,θ) represents a number in the complex number system. Using this notation, every number can be represented an infinite number of ways, because:

$(r,\theta)=(r,\theta+2k\pi), k\in\mathbb{Z}$