SVD-Based Voltage Stability Assessment From Phasor Measurement Unit Data

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***Abstract*—Electric power systems can display a range of undesirable dynamic phenomena by which acceptable, stable operation may be lost. Quasi-steady-state operational problems such as the voltage instability phenomena are among these. Ill-conditioning of the power flow, reflected in high sensitivity of bus voltage magnitudes to load variation is an often observed precursor to such quasi-steady state operational problems. Motivated by this insight, work here will propose a voltage stability and conditioning monitor that is model-free in real-time, based solely on phasor measurement unit (PMU) data, arguing that such an approach is well suited to near-real-time application. We review model-dependent singular value analysis in voltage stability assessment, and relate these existing approaches to our proposed model-free method in real-time application. The proposed algorithm is first applied under the idealized assumption of full measurement data at all buses. This work then extends the algorithm to apply in the more practical case for which only subset of buses have available measurement data. The proposed approach is illustrated in IEEE test cases, augmented to include heavy load conditions that stress voltage stability. Algorithms for efficient computation of small numbers of singular values, as well as means to exploit low-rank updates in data, are reviewed to demonstrate opportunities for fast computation that allow these SVD methods to operate in near-real-time in large systems.**

***Index Terms*—Phasor measurement units, real-time wide area monitoring, singular value decomposition, voltage stability.**

I. Introduction

MAJOR power system outages involve a range of different phenomena, but it is often true that quasi-steady state phenomena such as voltage stability, and ultimately voltage collapse, play a role. In current utility practice, operational measures of vulnerability to voltage instability are typically based on the state estimator, that uses a network model to compute the steady state operating point of the grid, with common update rates on the order of several minutes. The dependence on accurate knowledge of network parameters and topology, and relatively infrequent update rate may be viewed as shortcomings. Among advances that can support new approaches has been a proliferation of vastly improved measurement technology in the grid. This suggests the value of developing efficient, PMU-based metrics of system performance, that may be computed in near real time, without knowledge of network parameters or topology. Our goal here will be to propose one such metric to indicate power flow conditioning, and hence grid vulnerability to quasi-steady state operational problems such as voltage instability phenomena.

A new metric proposed here is based on Singular Value Decomposition (SVD), with its underlying computation very closely related to those of the Karhunen-Loeve Transformation (KLT), Principal Component Analysis (PCA), and Proper Orthogonal Decomposition (POD) [1], [2]. SVD has been widely used in a number of engineering areas to characterize dominant system modes. In structural dynamics, SVD is used to reduce system model dimensions and to extract energy coherent structures. In structural vibration studies, [3] and [4] argue that the Proper Orthogonal Modes (POMs) computed from measured vibration data converge to the Linear Normal Modes (LNMs) of the system model for the undamped, free vibration case. Structural health monitoring in the civil engineering literature goes further to employ related methods for multidimensional damage identification [5]. In power system engineering, SVD has been employed in the context of "full-model-based" analysis to assess voltage stability by examination of the smallest singular value of the power flow Jacobian [6]–[10]. Recently, PCA-based dimension reduction techniques for PMU data have also been studied, utilizing concepts from linearized state space dynamics to justify low underlying dimensionality, and examining changes in these low dimensional subspaces as a means to detect events impacting the electromechanical dynamic response of the power system [11].

In contrast to electromechanical stability studies focusing on response of generator frequencies and phase angles, voltage stability refers to the ability of a power system to sustain voltage magnitudes at all buses after small or large disturbances [12]. Because the time frame of voltage stability may vary from seconds to many tens of minutes, short-term voltage stability is distinguished from long-term voltage stability phenomena [13]. While most research in short-term voltage stability has employed fully populated dynamic models, a model-free short-term voltage stability monitor using PMU data has been reported in [14]; it makes a use of the Lyapunov exponent as its stability metric. A model-based long-term voltage stability detector using PMU data is presented in [15], and another method based on Thevenin impedance matching condition appears in

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[16], though the former requires some data on system topology and electrical parameters. A model-free, Artificial Neural Network (ANN) based long-term voltage stability monitoring is also proposed in [17]. This method requires ANN training before online deployment. It can be seen as a weak point when the system topology changes while it is running since it requires ANN training again.

An overview of on-going research projects for long-term voltage stability monitoring methods using PMU data may be found in the reports of the North American SynchroPhasor Initiative (NASPI).[1] These may be broadly classified into two groups; one is based on machine learning techniques such as decision tree methods [18], and the other group focusing on Thevenin impedance matching [19]. Monitoring of reactive power margins at generators represents a very practical, existing approach that provides a measurement-only, (largely) model-free indicator for long-term voltage stability [6]. The work to be presented here may be viewed as an evolution of this type of model-free, long-term voltage stability assessment, seeking to make use of the broader class of data available from PMU's.

The conceptual basis of the method to be proposed here is, in simple terms, a sensitivity analysis of power flow equation. Eigenvalue or singular value analysis of Jacobian matrix has long been utilized to characterize long-term voltage stability [6]–[10]. However, such indices have traditionally been computationally expensive, and very dependent upon accurate knowledge of system parameters and topology information.

The contribution of this paper can be seen as follows: (i) a proposed method for long-term voltage stability in this paper is a model-free in real-time and requires no information of system parameters or topology information while it is running online, (ii) the proposed method can be applied with quite limited subsets of voltage magnitude and phase angle measurements, and does not require full set of measurements at a majority of buses.

Preliminary work examining the proposed method was introduced in [20] under the idealized assumption of voltage magnitude and phase angle measurements assumed available at every bus. A key goal in the work here will be consideration of the more practical scenario in which voltage phase angle and magnitude measurements are available from a limit subset of buses. Section IV of this paper will examine the characteristics displayed by time-series matrices of PMU data, demonstrating that significant numbers of components of the dominant singular vectors have magnitude close to zero. Here this feature will be exploited to inform the selection of small subsets of measurements (corresponding to rows in the PMU data matrix) that capture the information necessary to approximate the dominant singular value(s), and hence the voltage stability measure. Examples to follow will typically consider the case of measurements available for 10–15% of all buses. Further, work here will study the proposed method's performance with heavy load conditions, to test the voltage stability indicator under very stressed conditions. Also, studies here compare results of the method when employing down-sampled data, versus outcomes when a 30 Hz reporting rate is employed.

[1][Online]. Available: https://www.naspi.org

It is clear that computation cost for SVD-based methods increases substantially as the number of measurements expands. If unaddressed, this could prove a limiting factor for use of such methods in real time. To address this issue, the work here will review well-established algorithms to reduce SVD computation time [21]. In particular, Lanczos algorithms and power methods will help achieve near real time assessment in large data sets.

II. Background: Power Flow, Load Behavior, and Singular Value Decomposition (SVD)

Power flow equations define the equilibrium operating condition for a synchronous power grid. Viewed abstractly, the physical power system may be viewed as an analog power flow solution engine, taking power injection variations as inputs, and "computing" bus voltage phasors as outputs. In its standard formulation, the power flow is a square mapping, with an equal number of input arguments (the active and reactive power injections) and output results (magnitudes and angles of bus voltage phasors). A key assumption in our formulation is a separation of load variation into two time scales. The first, dominant component represents the slowly varying bulk consumption, evolving over 10's of minutes to hours, displaying periodic behavior on a daily cycle.

However, both first principles and more detailed electrical measurements suggest that load demand must also display faster time scale, random behavior. Load at a bulk distribution bus aggregates the behavior of potentially millions of individual power consuming devices, displaying on-off behavior governed by human users or local control systems. As a result, one may expect the fast variation in load (time scale of seconds) to display small magnitude random jump behavior, filtered by the electrical characteristics of the distribution system [22]. A key premise for this work emerges: the vector of driving inputs contains a large signal component, that slowly moves the operating point, and a small-signal, randomly varying component, that persistently excites the system about its operating point. To model the fast time scale stochastic variation in load, early studies tended to employ white noise; more recent studies have adopted Ornstein-Uhlenbeck process models [23], [24], and the latter will be used here.

As in many branches of circuit analysis, this split of large signal and small signal components suggests the usefulness of linearized approximations. The nominal operating point is set by the large signal component, with the impact of the small signal component analyzed via linearization about this (slowly varying) operating point. Note, however, that even with this time scale separation, our approach tracks the quasi-static motion of the power flow solution in its small signal analysis, rather than treating very fastest time scale of dynamic phenomena. In this context, the linearization of interest will be represented by the familiar Jacobian of the power flow, that (to local, linear approximation) predicts the quasi-static motion of the power flow solution in response to small, random variations in load and power injections. Our approach then seeks to measure nearness of the power flow Jacobian to singularity, reflected its smallest singular value, or in the largest singular value of its inverse.

With the motivation of measuring nearness of the power flow Jacobian to singularity, it is useful to review the SVD [25]–[27].

SVD factors an arbitrary matrix into products of three matrices, and offers geometric insight into the behavior of the matrix as a linear operator. Equation (1) shows the general structure of SVD for an arbitrary matrix A, and for a square, nonsingular A, (2) is a closely related expression for its inverse.

$$A = USV^T \tag{1}$$

$$A^{-1} = VS^{-1}U^T = \sum_{i=1}^{n} \frac{u_i^T v_i}{s_{ii}} \tag{2}$$

For A an $m \times n$ matrix of real-valued entries, U is a unitary $m \times m$ matrix whose columns are the left singular vectors, and V is an $n \times n$ unitary matrix whose rows are the right singular vectors. S is an $m \times n$ matrix, in which an upper-left square block has diagonal entries termed singular values, and all other entries are zeros. In geometric terms, the singular values identify the "gain" of the linear operator, acting on orthogonal axes in the domain ("input"), determined by the right singular vectors, and reflected in the range ("output") along the orthogonal axes determined by the left singular vectors. In the case of a square matrix, the smallest singular value provides a measure of the distance between the matrix and the nearest singular matrix. For a square, nonsingular matrix, dyad expansion in (2) offers another perspective on the role of the smallest singular value of the original matrix, as setting a "maximum gain" (i.e., the induced euclidean norm) for the inverse [27].

III. Voltage Stability Test

The "forward" power flow equations, linearized about an operating point can be written as shown in (3). Viewing the physical power system as a power flow solver, it is useful to invert this forward form, treating loads and power injections as inputs, and the output response being phasor angles and voltage magnitudes (as measured by PMUs). Then, (4) provides the desired input-output relationship. Letting the Jacobian play the role of A in (2), as the smallest s_{ii} approaches zero, small variations in power have the potential to yield large response in bus voltage magnitude and angle variations. Indeed, the smallest singular value of Jacobian matrix has been specifically proposed as an index of vulnerability voltage collapse in a variety of past works [6]–[10].

$$\begin{bmatrix} \Delta \underline{P} \\ \Delta \underline{Q} \end{bmatrix} = \begin{bmatrix} \frac{\partial \underline{P}(\underline{\delta}, |\underline{V}|)}{\partial \underline{\delta}} & \frac{\partial \underline{P}(\underline{\delta}, |\underline{V}|)}{\partial |\underline{V}|} \\ \frac{\partial \underline{Q}(\underline{\delta}, |\underline{V}|)}{\partial \underline{\delta}} & \frac{\partial \underline{Q}(\underline{\delta}, |\underline{V}|)}{\partial |\underline{V}|} \end{bmatrix} \begin{bmatrix} \Delta \underline{\delta} \\ \Delta |\underline{V}| \end{bmatrix} \tag{3}$$

$$\begin{bmatrix} \Delta \underline{\delta} \\ \Delta |\underline{V}| \end{bmatrix} = J^{-1} \begin{bmatrix} \Delta \underline{P} \\ \Delta \underline{Q} \end{bmatrix} = \sum_{i=1}^{n} \frac{u_i^T v_i}{s_{ii}} \begin{bmatrix} \Delta \underline{P} \\ \Delta \underline{Q} \end{bmatrix} \tag{4}$$

To characterize the role of each block of the Jacobian in its inverse, denote:

$$A = \frac{\partial \underline{P}}{\partial \underline{\delta}},\; B = \frac{\partial \underline{P}}{\partial |\underline{V}|},\; C = \frac{\partial \underline{Q}}{\partial \underline{\delta}},\; D = \frac{\partial \underline{Q}}{\partial |\underline{V}|} \tag{5}$$

Traditional methods of power flow analysis often seek to exploit approximate decoupling between angle-active power, versus voltage magnitude-reactive power. Such decoupling approximation might suggest that with a goal of considering voltage stability, one might focus exclusively on the D block above. However, we argue against such a decoupling approximation here, in part because systems under highly stressed operating conditions, for which risk of voltage instability may be most severe, are precisely those for which decoupling approximations may be least accurate. To make this argument more rigorous, consider operating conditions for which D is nonsingular (even if it may be approaching singularity), and consider the Schur complement of J with respect to D:

$$\frac{J}{D} = A - BD^{-1}C \tag{6}$$

The Schur-Banachiewicz inverse [28] then re-expresses the inverse in block form as:

$$J^{-1} = \begin{bmatrix} \left(\frac{J}{D}\right)^{-1} & \left(\frac{J}{D}\right)^{-1} BA^{-1} \\ -D^{-1}C\left(\frac{J}{D}\right)^{-1} & D^{-1} + D^{-1}C\left(\frac{J}{D}\right)^{-1} BD^{-1} \end{bmatrix} \tag{7}$$

Examination of the lower right block of (7) confirms that variation in voltage magnitude, $\Delta|V|$, is not driven by $D^{-1} = [\partial Q/\partial|V|]^{-1}$ alone, even in the case that there is no variation of active power, ΔP. Simply put, for the objectives of characterizing voltage sensitivity to load variation, one needs more complete information beyond that captured in D^{-1} alone, and hence the approach here advocates a method that seeks to approximate through measurements the largest singular value of the complete power flow Jacobian inverse, not a sub-block. More generally, reordering of the $\Delta\delta$, $\Delta|V|$ and ΔP, ΔQ terms allow the block matrix $[\partial Q/\partial|V|]$ to be associated with any of the block "locations" within the Jacobian matrix such as A, B, C, or D. However, even with reordering, it remains impossible to isolate a single term of the form $D^{-1} = [\partial Q/\partial|V|]^{-1}$. Hence, the authors believe that consideration of the full Jacobian matrix behavior is more appropriate in the application here.

Fig. 1 is intended to serve as a simple, low-dimensional visualization of this input-output relationship under the effect of small, random load variations. The upper figure displays sampled variations in active and reactive loads at a particular bus, applied around a nominal operating point in a 14-bus test power system model. The random variation in P-Q plane (upper figure) is generated by selecting ΔP and ΔQ values that are described in polar coordinates. That is, the magnitude of the complex quantity $(\Delta P + j\Delta Q)$ is selected from a uniform distribution on the open interval (0, 0.02); its angle is from a uniform distribution on the interval $[0, 2\pi)$. Repeatedly solving the power flow in this model, these input variations (upper figure) are then mapped to points in the output domain (bottom figure), that of bus voltage magnitude and angle. The geometry of output points obtained from a full nonlinear solution closely approximates what one expects from the linear case: the sphere of points in the input space maps (nearly) to an ellipse in the output space. This matches the standard geometric interpretation of the singular values: largest singular value sets the length of the major axis of the "output ellipse," smallest singular value sets the length of the minor axis.

IV. Jacobian Conditioning/Voltage Stability Assessment Using PMU Data

Because the proposed algorithm seeks a model-free analysis in real-time application, the information assumed available will

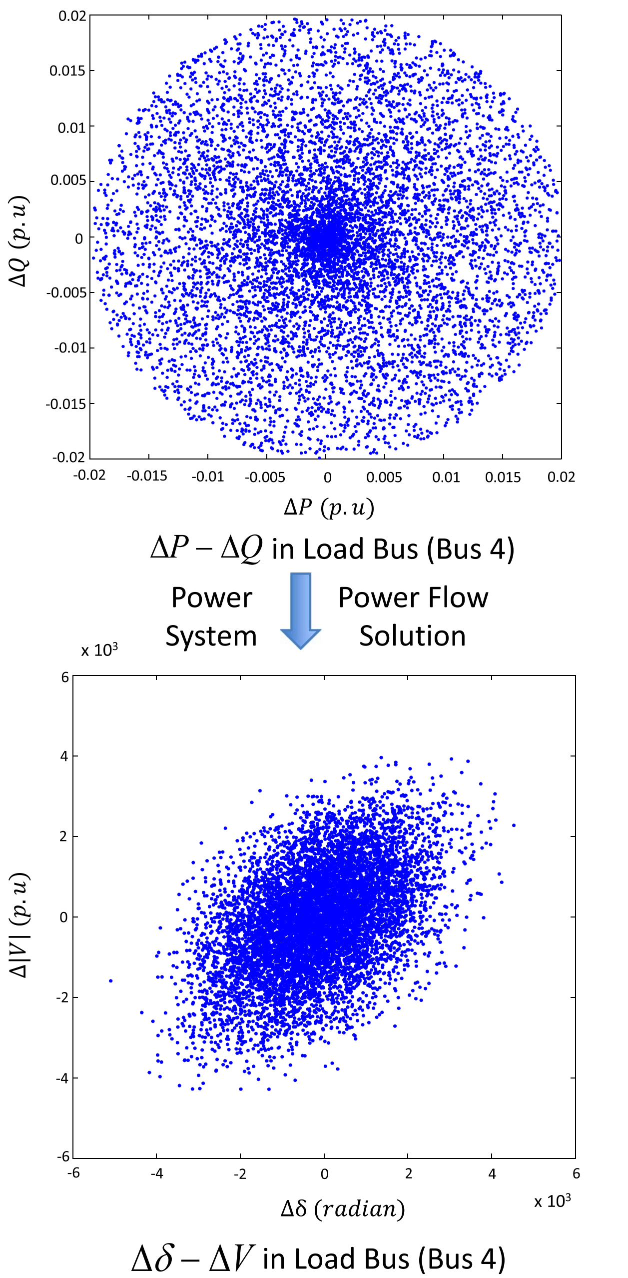

Fig. 1. Graphical illustration of power flow input-output variation about an operating point for IEEE 14-bus test system.

be limited to sampled measurements; here we restrict attention to voltage magnitudes and phase angles from the buses in the power system. Under this assumption on available measurements, the goal of the proposed algorithm is straightforward: Estimate the major axis of the "output ellipse" through use of only measurement information, and track how the quantity evolves in response to load variations. The algorithm employed to estimate the major axis is quite simple, and is closely analogous to the use of SVD tools in other streaming data applications [29]: after subtracting a base state from the measured PMU data, and possible filtering/bad-data-correction, construct a sliding, windowed array of the streaming data, up to the most recent measurement.

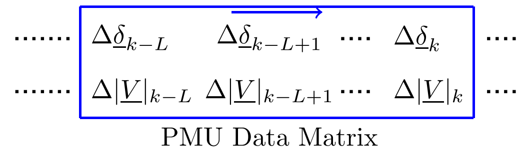

Fig. 2. L-column of the data matrix. $\Delta\underline{\delta}$ and $\Delta|\underline{V}|$ represent bus voltage angles and bus voltage magnitudes deviations of buses from the base states.

For estimating power-flow conditioning, the base state is chosen as a low stress operating point; typically a lowest normal loading condition. From this array, one computes the largest singular value, or several largest singular values, and associated left singular vector(s). Fig. 2 shows the graphical description of sliding windowed PMU data matrix. $\Delta\underline{\delta}$ and $\Delta|\underline{V}|$ stand for a vector of the voltage phase angle and a vector of the voltage magnitude after subtracting the base state, respectively, and the subscript k correspond to the kth time point. Also, the variable L denotes the length of time window in PMU data array. At each new data sample update, the data matrix discards the column corresponding to the oldest data sample, and appends the column corresponding to the new data sample. The largest singular value of the PMU data array with time points from 1 to kth is proposed as a means to estimate the largest singular value of the inverse power flow Jacobian corresponding to the operating point at time point kth, and therefore, also a measure of vulnerability to voltage instability at that time point.

The window length selected for the PMU data matrix has significant impact on the result. Algebraically, a large window size would seem preferable, to provide a larger number of output vectors from which to estimate the major axis. However, if the window is so large that the system can experience significant change in operating point over the time interval, the underlying assumption of capturing linearized, "small signal" behavior comes into question. This consideration also suggests that the reporting time between samples should not be allowed to be too long. For convenience, many of illustrative computations here employ a reporting rate of 0.1 Hz on the samples, with comparison to performance when a reporting rate of 30 Hz is used. This 0.1 Hz choice proves fast enough to allow adequate numbers of columns without excessively long time windows. However, very slow reporting rates (time scale of a minute or more between samples) would be inadvisable, because adequate numbers of columns would then yield very long time windows, over which the system operating point is likely to change significantly. Arguing for a smaller window size would be the viewpoint that the SVD result delivers a more recent status of the system, with lower computational cost. The discussion here assumes a simple rectangular window, with no time-dependent weighting of data; consideration of more sophisticated windows awaits future work.

An analogous question may be posed regarding the number of rows that significantly contribute to the data matrix; that is, to characterize the number of components of a singular vector that are significantly different from zero. This is often termed the "main dimension." In particular, consider a main dimension defined as the least number of components of normalized left singular vector such that the sum of squares (s.o.s) of these

TABLE I
MAIN DIMENSION OF SINGULAR VECTOR ASSOCIATED WITH THE LARGEST SINGULAR VALUE IN IEEE 14-, 118-, AND 300-BUS SYSTEMS

Bus Type	Data Type	Mean of Main Dim. (s.o.s>0.9)	Total Dim.
14 Bus	$Full$ PMU	12.60	26
	V_{mag} PMU	6.77	13
	V_{angle} PMU	7.10	13
118 Bus	$Full$ PMU	62.63	234
	V_{mag} PMU	28.23	117
	V_{angle} PMU	61.93	117
300 Bus	$Full$ PMU	236.00	598
	V_{mag} PMU	50.30	299
	V_{angle} PMU	236.00	299

components is greater than a predetermined threshold value, μ. For the numerical examples examined here, the main dimension proves to be relatively small relative to the overall dimension of the measurement vector; i.e., many components are essentially zero.

In Table I, mean of main dimension is displayed for a threshold of $\mu = 0.9$, with mean taken over 8640 samples of synthetically generated PMU data in the IEEE 14, 118, and 300-bus test systems. Numerical experience with the IEEE 14, 118, and 300-bus system indicates that large numbers of components of the singular vector associated with voltage phase angle tend to play a role in the main dimension; in keeping with the qualitative observation that voltage-reactive behavior is more localized in power systems, for larger systems, far fewer components of the singular vector associated with voltage magnitude tend to contribute to the main dimension. Also, these numerical examples suggest that the main dimension maintains a nearly fixed value for measurement matrices for families of operating points associated with load varying over a fairly wide range.

The main dimension computation is clearly related to the problem of identifying core subspaces, as was considered for PMU data in [11]. There, the authors examined the m principal components of covariance matrix of PMU data, with m is selected to preserve a cumulative covariance $\sum_{i=1}^{m} var_i \geq \tau$. There τ is design variable, with a role analogous to that of μ in main dimension computations here. Both provide indications that the significant information content in PMU data is relatively sparse, provided the threshold measuring significant information context is set relatively "loosely" (e.g., μ above was selected only as 0.9, suggesting the resulting value of main dimension represents the number of measurement components necessary to capture 90% of the vector's energy). This implies that the "direction" of the largest singular value of the PMU data matrix can be approximated with a subset of the measurement data. Details about the selection of the subset of measurements will be discussed in Section VI.

V. NUMERICAL STUDY RESULTS

To examine the behavior of the proposed voltage stability measure, and to be able to compare against full, exact computation of the power flow Jacobian inverse and its largest singular value, we generate synthetic load data for a 24-hour cycle in the IEEE 118-bus and 300-bus test systems. To limit the volume of data to be processed, initial computations will employ down-sampling to a 0.1 Hz data reporting rate; comparisons to assess added information available from higher reporting rates available in PMU standards (e.g., 30 Hz) will follow. In the 118-bus system, a sinusoidal 24-hour load cycle displaying a peak at noon and minimum at midnight is used. For the IEEE 300-bus system case, the pattern is shifted to yield a peak at 6 pm and a minimum at 6 am. These cycles represent the slow load variation, that moves the operating point. In keeping with the approach described in Section II, a faster time scale, random variation generated by an Ornstein-Uhlenbeck Process model is added to every load component.

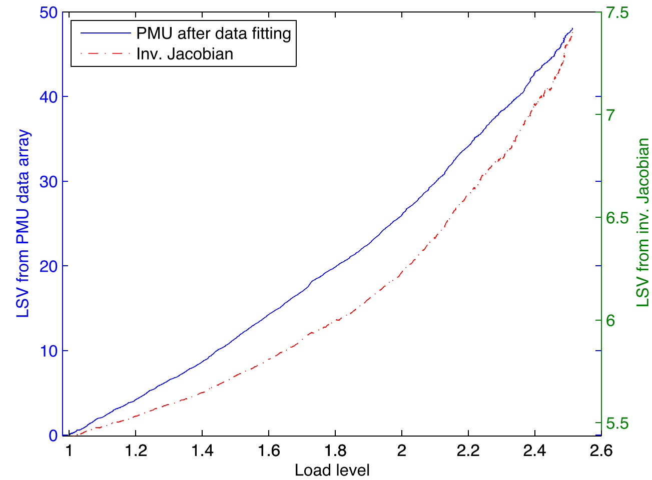

Fig. 3. IEEE 118-bus example, stressed by increasing load until system does not have any equilibrium point.

For the initial studies in [20], full availability of power flow measurement data at every bus was assumed. Under the assumption of PMU measurement at every bus, IEEE 118-bus system with heavy load is studied to see its performance in such a condition and to provide voltage stability indicator threshold. Also, IEEE 300-bus system with full availability of measurement data is tested at 0.1 Hz and check the applicability of the proposed method at 30 Hz reporting rate. Again, we must emphasize that in the numerical studies considered here, the bus voltage magnitudes and angles are "pseudo-PMU measurements," computed from time sequenced power flow. In the ultimate target application of the algorithm, these quantities would be obtained from physical PMU measurements.

A. IEEE 118-bus System

Because of the strong correlation of largest singular value of the Jacobian inverse, and that of the measurement array, both off-line system study and simpler engineering judgment can offer insight into selection of a voltage instability threshold. While the approach is model-free in real-time, the off-line calculation of a threshold will require a modest number of studies to be performed using system model populated with power flow data, over a range of loading levels. This is necessary to identify the two parameters that characterize the affine relation between largest singular value of the power flow Jacobian inverse (computed in the off-line study cases) and the largest singular value of the PMU data matrix.

For IEEE 118-bus system case shown in Fig. 3, the system losses its steady-state solution (to within the computational accuracy of a full Newton Power Flow computation) at a loading level one step "beyond" that at which the largest singular value of the power flow Jacobian inverse is equal to 7.402. The corresponding largest singular value of PMU data array is 48.11, and the loading level is 2.514 times that of the nominal operating point. This type of off-line study suggests one method of selecting an appropriate threshold for our proposed measure, but also offers insight into simpler, engineering-based judgment that may be used, without extensive off-line study. Of course, for any given system, the method here will require some limited amount of off-line, model-based study to perform the affine fit between the largest singular value of the power flow Jacobian inverse, and the largest singular value of the PMU data matrix. However, once that affine fit is established, on-line computation of the the largest singular value of the PMU data matrix then provides an estimate for the largest singular value for the power flow Jacobian inverse, without any on-line calculation of that Jacobian.

For the numerical example of off-line study illustrated in Fig. 3, suppose one selected a threshold at approximately 80% of that reached at the maximum loading point. This yields a threshold of approximately 7 for largest singular value of the power flow Jacobian inverse, corresponding to a threshold of 40 for largest singular value of the PMU data matrix. Consider the engineering interpretation of this threshold, in light of the concepts illustrated in Fig. 1. Recall the interpretation of largest singular value as the maximum 2-norm gain of the powerflow solution operator. Suppose one identifies a maximum credible deviation of load $(\Delta P, \Delta Q)$ (as measured in 2-norm), and a maximum acceptable deviation in the power flow solution quantities $(\Delta\delta, \Delta V)$ (again, as measured in 2-norm). The ratio between these determines a worst-case allowable sensitivity of the power flow solution to changes in load; the largest singular value of the power flow Jacobian inverse is precisely this worst-case sensitivity. For the illustrative example in the IEEE-118 bus system here, our selection a threshold of 7 has the engineering interpretation as a bound on the maximum ratio allowed between the load variation (in per unit) and power solution variation (in radian angle and per unit voltage magnitude). Again, the reader is reminded that the actual quantity calculated on-line will be the "model-free" PMU data matrix's largest singular value, which in this example would have a corresponding threshold of 40.

B. IEEE 300-bus System

IEEE 300-bus system with full availability of measurement data is tested and the correlation of largest singular value of the Jacobian inverse, and that of the measurement array is studied with a time sequence of operating points generated over hypothetical 24-hour pattern of load and generation variation, displayed in Fig. 4. The relationship between the measurement-based largest singular value and that of the inverse power flow Jacobian shows a somewhat non-linear characteristic. We hypothesize that some of the difference is inherent to comparing the linearized information of the Jacobian inverse, to the full nonlinear behavior of the power flow. In particular, the 300-bus test case had the property that a number of generators reach reactive power limits at high load levels. As is typical power flow studies, this is represented by a change in the constitutive relations for the generator's reactive power/voltage magnitude behavior, from P-V bus to P-Q bus, which implies a discontinuity in some terms of the power flow Jacobian. For the studies, here, we adopted a smoothed approximation to the transition, to avoid the discontinuity in the Jacobian. None-the-less, in the vicinity of operating points at which generators encounter reactive limits, it is reasonable to assume that the linearized approximation to the power flow becomes much less accurate, and may account for some of the discrepancy between the pseudo-measurement-based singular value result, versus that based on the Jacobian inverse. Optimistically, one might argue that the measurement-based measure could prove a better measure of the conditioning of an operating point, but this hypothesis will require careful future study.

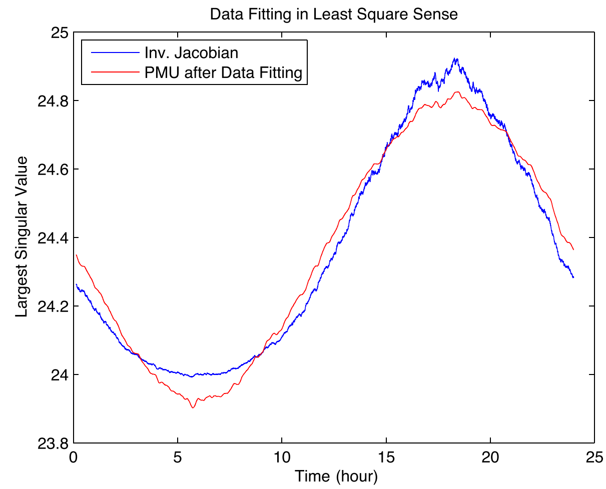

Fig. 4. Largest singular value comparison between inverse Jacobian and full PMU matrix in IEEE 300-bus System. A 0.1-Hz data reporting rate is assumed.

Because handling of 24-hour data at the 30 Hz reporting rate is cumbersome (yielding more than 2.5 million sample points), a down-sampled data reporting rate, equal to 0.1 Hz, is used for most test cases. However, 30 Hz data reporting rate case is shown in Fig. 5, using one representative hour. This figure illustrates a fairly close match between singular value as computed via the down-sampled measurement data, versus that for a 30 Hz reporting rate. Numerical study with similar loading condition as Fig. 4 is shown in Fig. 5 with 30 Hz reporting case. These results suggest that while computations employing data with a 30 Hz reporting does track the largest singular value of power flow Jacobian inverse somewhat more closely than down-sampled case, the differences are not too significant. Moreover, it should be noted that a two degree of freedom affine fit over a 24-hour interval (as displayed in Fig. 4), will inevitably be somewhat less accurate than such a fit applied over a 1-hour window, as displayed in Fig. 5.

VI. Numerical Study Results for Singular Vector-Based PMU Placements

The singular vector associated with the largest singular value indicates which direction of change would experience the

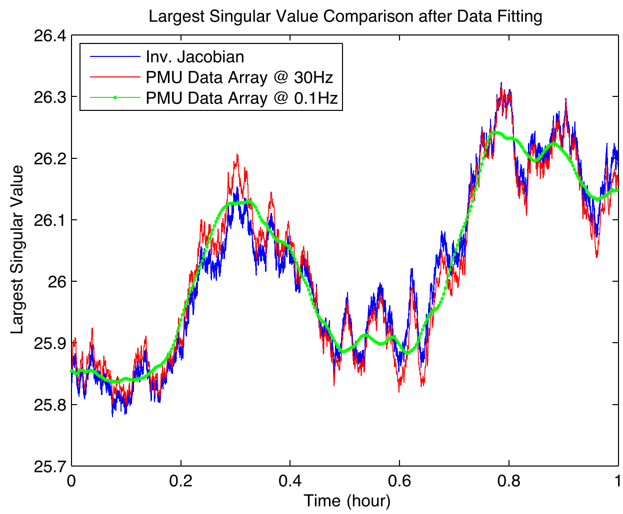

Fig. 5. Largest singular value comparison among power flow Jacobian inverse and PMU matrix with 30-Hz reporting rate and down-sampled 0.1-Hz reporting rate in IEEE 300-bus system.

highest "gain" be most sensitive through a matrix. This suggests that selection of measurement placement for PMUs may be informed by identifying those components of measurements that contribute most to the singular vector direction. Such a selection process would simply place measurements in those bus locations associated with large magnitude components of sub-$U_{J^{-1}}$ vector, either in the voltage magnitude part or the voltage angle part.

Using the information from the U vector of inverse Jacobian matrix, the most sensitive buses to the largest singular value can be identified. The reduced PMU data matrix is constructed by selecting the 10% most sensitive buses on voltage magnitude or phase angle. The 10% comes from the fact that 10% of buses in the system currently have been installed PMU. Least Square Error (LSE) is calculated for each case in order to evaluate the performance.

It should be stressed that this approach to placing PMU measurements has a different, and inherently less demanding, goal relative to placement algorithms based on state estimator observability; optimal placement of measurements for the purpose of this paper's algorithm will not correspond to optimal placement for state estimator observability. Simply put, the state estimator observability problem attempts to reconstruct the full state of all $\underline{\delta}'s$ and $|\underline{V}|'s$, from measurements [30]. This is very different from the problem addressed in this paper, that in its most basic form seeks to estimate only a single positive real quantity: the largest singular value of power flow Jacobian inverse. Therefore, it should not be surprising that the method here can perform acceptably with far fewer measurements than are typically required for state estimator observability.

A. IEEE 118-bus System

In IEEE 118-bus system, the measurement placement based on singular vector information shows a bit of improvement over a like number of randomly placed measurements.

While random placement has LSE equal to 0.6987, LSE of the singular vector based placement is 0.5571 as shown in Fig. 6.

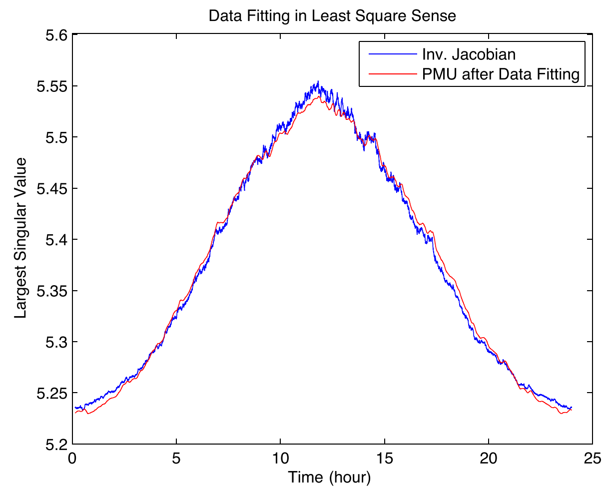

Fig. 6. Least square error performance for case of the IEEE 118-bus system. LSE $= 0.5571$.

TABLE II
BUS NUMBERS AND LSES OF THE OPTIMAL PMU PLACEMENT AND RANDOM PMU PLACEMENT IN IEEE 300-BUS SYSTEM

	30 Optimal PMU placement	30 Random PMU placement
Bus Number	31, 32, 43, 44, 45, 265, 267, 268, 270, 271, 274, 275, 277, 278, 279, 280, 281, 282, 283, 284, 285, 286, 287, 288, 289, 290, 292, 296, 297	4, 9, 19, 20, 23, 67, 89, 99, 101,106, 107, 108, 110, 130, 138, 139, 141, 161, 168, 182, 189, 208, 212 214, 221, 225, 226, 230, 233, 296
LSE (1^{st} case)	4.8732	4.9051
LSE (2^{nd} case)	16.6307	17.6484

Based on the observed numerical study case, it implies that the proposed method can have slightly better performance if PMUs are placed according to the information from singular vector. Also, although the performance with data set of randomly selected buses is degraded a bit, it is still able to be utilized to assess the voltage stability from measurement data. Though interesting, the indicator computed from the random placements has smoother time behavior than that produced from the singular-vector-based placements. It can be thought of as the buses to contribute the voltage stability most are selected by using singular vector associated with the largest singular value.

B. IEEE 300-bus System

In IEEE 300-bus, two different cases are investigated. The first case corresponds to the load variation scenario without reactive power compensators; the second includes the effect of reactive power compensators switched in under high load conditions. Through numerical experiment, there is little difference between the singular vector-based placement and the random placement in the case without reactive compensation. In Table II, LSE of singular based placement is 4.8732, compared to that of random placement is 4.9051.

The second case, which includes the reactive power compensation in high load condition, shows similar characteristics that the singular vector-based placement can have some improvement over the random placement. As shown in Table II, the LSEs are 16.6307 with singular vector-based placement and 17.6484 with random placement. Based on the numerical studies in IEEE 118-bus and 300-bus system, the algorithmic placement is a bit more responsive to the abrupt changes of the system, but random placement still has a comparable result for voltage stability assessment.

VII. Algorithms for SVD Computational Cost Reduction

Since voltage stability index from the proposed algorithm should be calculated in near time, low computational cost for SVD computation is essential. The best algorithms for full SVD computation of m-by-n matrix is $O(nm^2 + n^3)$ [27]. The orthogonalization/factorization based algorithm is implemented in LAPACK [27], [31]. However, iterative method for computing SVD such as Lanczos algorithm and Arnoldi's algorithm can substantially reduce its computation time if a small number of the largest singular values (or eigenvalues) and singular vectors (or eigenvectors) in an array is of our main concern. Lanczos algorithm is originated from power-method to find eigenvalues and eigenvectors of a square matrix or singular values and singular vectors of a rectangular matrix. The power method for computing eigenvalues/vectors of n-by-n B can be summarized as follows: 1) Start with random vector $\underline{x}_0 \in \Re^n$. 2) Compute $\underline{x}_{i+1} = B\underline{x}_i$. 3) Iterate step 2) until $\underline{x}_{i+1} - \underline{x}_i < tolerance$. Then, $\frac{\underline{x}_{i+1}}{\|\underline{x}_i\|}$ is the normalized eigenvector corresponding to the largest eigenvalue, which is 2-norm of $B\frac{\underline{x}_{i+1}}{\|\underline{x}_i\|}$. For next largest eigenvalue and eigenvector, go back to step 1) but start with any $\underline{y}_0 \in \Re^n$ that is orthogonal to previous eigenvector. For singular value/vector of a rectangular matrix, A, use the fact that AA^T and A^TA are square matrices. Lanczos algorithm and Arnoldi's algorithm save the vector $\underline{x}_i$ in step 2) for Gram-Schmidt process to re-orthogonalize them into a basis spanning Kylov subspace. They are implemented in ARPACK [21].

Lanczos algorithm is a little bit slower than the orthogonalization/factorization algorithm for full SVD on the relatively small size of matrix. However, as the matrix size increases, the power method based algorithm outperforms the the orthogonalization/factorization based algorithm as shown in Table III. It shows the computation time required to compute a series of the singular values from 8640 PMU data. "svds" (Matlab function) is based on the lanczos algorithm in ARPACK, and "svd" (Matlab function) uses the orthogonalization/factorization algorithm in LAPACK. The third column in Table III shows the case of computing only the largest singular value/vector with power-method.

It is possible to have a convergence problem if the magnitude of eigenvalues are similar. However, based on simple numerical study with IEEE 118-bus system, the difference between the first few singular values increases as the system gets heavily loaded. For example, when system loads varies from the nominal loading to 1.8 times of the nominal loading, the largest singular value of PMU data array with 24 data set (12 PMUs assumed) varies from 0.0175 to 6.1040, the second largest singular value varies from 0.0022 to 0.0237, and the third largest singular value varies from 0.0010 to 0.0064. As a reference, the largest singular value of power flow Jacobian inverse varies from 5.1958 to 5.4787, the second largest singular value varies from 3.1076 to 3.2502, and the third largest singular value varies from 1.1350 to 1.1949.

TABLE III
Comparison of SVD Computation Time (s) for Processing 8640 PMU Data Samples in the Different Algorithms

System	CPU/RAM	6 SVDs svds	Full SVD svd	Power Method Programmed Here
IEEE 14	Core 2/4GB	49.13	5.82	1.25
	i5/6GB	34.937	2.74	0.7
IEEE 118	Core 2/4GB	164.87	74.94	31.72
	i5/6GB	97.72	38.869	9.092
IEEE 300	Core 2/4GB	298.597	415.33	225.092
	i5/6GB	146.6	208.434	83.476

For computing a series of a small number of largest singular values from arrays sequentially updated by a rank-one modification, even lower computation cost can be achieved. The authors in [32] compare SVD updating with rank-one modification to a Lanczos algorithm. The scheme in [32] is about 10 times faster than Lanczos algorithm in 1000-by-1000 matrix, with even better performance as matrix dimension increases. This scheme could be applied to our problem, suggesting further SVD computational cost reduction could be achieved for very high dimensional PMU data array.

VIII. Conclusion/Future Direction

It is generally accepted that a number of trends today, from markets to intermittent renewable integration, are causing the operating conditions in electric power networks to become more volatile with respect to time. In this context, there exists a strong need for estimating system stability on fast time scale. The algorithm presented in this paper can be a solution for approximating a voltage stability indicator in near real time, using only measurement information that is becoming widely available through PMUs. This work has presented with the case of heavy load condition to provide the stability indicator threshold and the case of 30 Hz reporting rate to show its applicability in one of the standard PMU reporting rates. The simulation results suggest that the proposed "model-free in real-time" measure closely tracks information that would be obtained from a computation of the largest singular value of the power flow Jacobian inverse (i.e., using full information of the model). However, even with increasing PMU penetration, they will be available on a modest percentage of the total number of system buses. We have demonstrated that singular vector information can assist with this aspect of the measurement design problem. Also, SVD computation cost can be a limiting factor to apply the method to very large system in real time. To address this issue, well-established algorithms are reviewed to exploit special characteristics to reduce SVD computation time. In particular, Lanczos algorithms and power methods were studied to help achieve voltage stability assessment, in near real time for large data sets.

The proposed method also shows promise as a means to detect topology changes in the electric power system, with such changes in network structure reflected by jumps in the value of the largest singular value. This will be a valuable subject for future work, which should also focus on characterization of the affine fit between the largest singular value of the power flow Jacobian, and that of the PMU data matrix, under circumstances in which topology changes occur during the study window.

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