1 Introduction

1.1 Outline of proposed research project

The text Counterexamples in Topology by Steen and Seebach has been a fabulous resource for students and researchers in Topology since its publication in 1970. The book was the product of an undergraduate research project funded by NSF and supervised by Steen and Seebach (and including then student Gary Gruenhage) to systematically survey important topological counterexamples. More recently James Dabbs has implemented a database on Github based on the Steen and Seebach textbook called Pi-Base (see https://topology.pi-base.org/) and it is currently being maintained by Dabbs and Stephen Clontz. This resource has great potential to both researchers and advanced undergraduate and graduate students at the start of their research careers. There are still big gaps in the database's subject matter, especially in relation to research in and around Frechet-Urysohn spaces. There is a significant body of work, and especially interesting counterexamples, concerning Michael's class of bisequential spaces, Arhangel'skii's alpha-i spaces and several game theoretic formulations of convergence which do not yet appear in the Pi-Base. The project has two goals. The first, and most accessible, is to give a systematic survey of the recent research which will be implemented into the Pi-Base database. The second half of the project will be devoted to open problems related to a recent class of examples defined from ladder systems (more generally on so called square-sequence) described in two

2 Meeting Log

Monday April 21

1. Frechet Fan - S_ω: $\omega \times (\omega+1)/\omega \times \{\infty\}$ (i.e. $\omega \times (\omega+1)$ with the points at infinity identified). Show that
   - S_ω is not first countable ✓
   - S_ω is Fréchet. ✓
2. Product of Fréchet spaces not always Fréchet: take $(\omega+1) \times S_\omega$. Let $A = \{(m,(m,n)) : m,n \in \omega$. Show that
   - $(\omega+1, \infty) \in \overline{A}$ ✓
   - No sequence in A converges to $(\omega+1,\infty)$. ✓
3. Right way to think about sequences: $A \subseteq X$ converges to $a \in X$ if $|A| = \aleph_0$ and for all neighboourhoods $U_x \subseteq X$, $|A \setminus U_x| < \aleph_0$.
4. Right way to think about Fréchet space: take sequential closure once same as closure.
5. Another exercise: which α_i properties does $\times S_\omega$ have? ✓

3 Topology

3.1 Basics

- $x \in \overline{A}$ iff $U \cap A \neq \emptyset$ for all open sets U containing x.

Definition 3.1 (Seperation Axioms)**.** Let X be a topological space. X is said to be

- T_0: whenever $x \neq y$ there exists an open $U \subseteq X$ with $x \in U$ but $y \notin U$ (or $y \in U$ and $x \notin U$).
- T_1: whenever $x \neq y$ there exists open sets U_x, U_y with $x \notin U_y$ and $y \notin U_x$
- T_2: whenever $x \neq y$ there exists open sets U_x, U_y with $x \notin U_y$ and $y \notin U_x$ and $U_x \cap U_y = \emptyset$

- Regular: $x \in X$ and closed $A \subseteq X$ with $x \notin A$, there exists open disjoint $U, V \subseteq X$ with $x \in U$ and $A \subseteq V$
- T_3: regular + T_1
- Completely regular: $x \in X$ and closed $A \subseteq X$ with $x \notin A$, there exists a continuous $f : X \to \mathbb{R}$ with $f(x) = a$ and $f(A) = b$ $(a \neq b)$
- Tychonoff ($T_{3\frac{1}{2}}$): Completely regular + T_1
- Normal: Closed $A, B \subseteq X$ with $A \cap B = \emptyset$, there exists open disjoint $U, V \subseteq X$ with $A \subseteq U$ and $b \subseteq V$
- Completely Normal: for all $A \subseteq X$, A is normal.

3.2 Sequential and Fréchet Spaces

Definition 3.2. For a topological space X and any set $A \subset X$, the *sequential closure* of A is

$$[A]_{\text{seq}} := \left\{ x \in X \ : \ \exists (x_n) \in A \left(\lim_{n\to\infty} x_n = x \right) \right\}.$$

In general we can repeat this operation recursively $[[[A]_{\text{seq}}]_{\text{seq}} \ldots]_{\text{seq}}$ by which is meant the *total sequential closure* of A.

Fact 3.3. In general it takes at most ω_1 many iterations of the sequential closure to get a closed set.

Definition 3.4. A space X is said to be Fréchet if $[A]_{\text{seq}} = \overline{A}$ for all $A \subseteq X$.

Example 3.5. Let $X = \omega_1 + 1$ with the order topology. X is not Fréchet, since any sequence $(x_n) \in \omega_1$ cannot converge to ∞, as otherwise $\omega_1 = \sup\{x_n \ : \ n \in \mathbb{N}\}$, a contradiction.

Definition 3.6. A space X is sequential if any closed set $A \subseteq X$ is equal to its total sequential closure.

Then a space that is Fréchet is also sequential. The following example shows that the converse is not true.

Example 3.7. Let $X^* = \omega \times (\omega + 1)$ be given the order topology and let $X = X^* \cup \{\infty\}$ where the neighboourhoods of ∞ are such that there exists $p \in \omega$ such that $|\{(m, n) : m > p, n \in \omega{+}1\} \setminus U_\infty| < \aleph_0$. Then X is sequential but but not Fréchet. To see this, note that for all $m \in \omega$ the sequence $A_m = \{(m, n) \ : \ n \in \omega\}$ converges to $(m, \omega + 1)$ and moreover $B = \{(m, \omega + 1) \ : \ m \in \omega\}$ is a sequence that converges to ∞. Then $A = \bigcup_{m\in\omega} A_m$ is such that $[[A]_{\text{seq}}]_{\text{seq}} = X$, hence X is sequential. On the other hand there is no sequence in A that converges to ∞. Suppose there were, say some $\gamma \to \infty$. Then for all $m \in \omega$, $U_m = X \setminus \{(m, n) : n \in \omega{+}1\}$ is a neighboourhood of ∞ such that $|\gamma \setminus U_m| < \aleph_0$. Hence γ has only finitely many terms belonging to each column. If $\alpha_m = \max\{\gamma \cap \{(m, n) \ : \ n \in \omega\}\}$, then $U = X \setminus \bigcup_{m\in\omega}\{(m, n) \ : \ n \leq \alpha_m\}$ is a neighbourhood of ∞ disjoint from γ, a contradiction. Hence X is not Fréchet.

Proposition 3.8. If X is first countable then X is Fréchet.

Proof. Let $A \subseteq X$ and let $x \in \overline{A}$. Then x has a countable neighbourhood base N_x such that $U \cap A \neq \emptyset$ for all $U \in N_x$. Enumerating the neighboourhoods of x as $U_1, U_2, \ldots$ then the sequence $(x_n)_{n\geq 1}$ where $x_n \in U_n \cap A$ for each $n \in \omega$ is such that $(x_n)_{n\geq 1}$ converges to x. □

The following example shows that the converse is not true.

Example 3.9 (Fréchet Fan)**.** Let S_ω be the quotient of $\omega \times (\omega + 1)$ obtained by identifying all the points $\{(m, \omega + 1) \ : \ m \in \omega\}$ as ∞^*. More precisely S_ω has the quotient topology induced by the map $h : \omega \times (\omega + 1) \to S_\omega$ where $h(x) = x$ for all $x \in \omega \times \omega$ and $h(x) = \infty^*$ for all $x \in \omega \times \{\omega + 1\}$. Then S_ω is Fréchet but not 1st countable. To see that S_ω is Fréchet, note that by definition of the quotient topology, the open neighbourhoods of ∞^* are those sets $U \subset S_\omega$ such that $\infty^* \in U$ and $h^{-1}(U)$ is open in $\omega \times (\omega + 1)$. As $\{(m, \omega + 1) \ : \ m \in \omega\} \subset h^{-1}(U)$ we see that U is an open neighbourhood of ∞^* iff $h^{-1}(U)$ is an open neighbourhood of $(m, \omega + 1]$ for all $m \in \omega$. Hence the open neighboourhoods of ∞^* are of the form $\bigcup_{m\in\omega}\{m\} \times (f(m), \infty^*]$ where $f : \omega \to \omega$ is just some mapping that indicates the startpoint of each interval. It follows that for all $m \in \omega$ the sequence $A_m = \{(m,n) : n \in \omega\}$ converges to ∞^* so that $A = \bigcup_{m\in\omega} A_m$ is such that $[A]_{\text{seq}} = S_\omega$. On the other hand it's obvious that $\overline{A} = S_\omega$, so that S_ω is indeed Fréchet. Now asssuming that S_ω was countable, we would have a countable neighbourhood base at ∞^*. For each $k \in \omega$ let $B_k = \bigcup_{m\in\omega} \{m\} \times (f_k(m), \infty^*]$ for some $f_k : \omega \to \omega$ determining the startpoints of each interval. Suppose $\mathcal{B} = \{B_k : k \in \omega\}$ is a base at ∞^*, then let $f^* : \omega \to \omega$ be defined by $f^*(m) = f_m(m) + 1$ for all $m \in \omega$. Letting $B^* = \bigcup_{m\in\omega} \{m\} \times (f^*(m), \infty^*]$ then B^* is an open neighboourhood of ∞^* but it is clear by construction that $B_k \not\subset B^*$ for all $k \in \omega$. Hence $\mathcal{B}$ cannot be a neighboourhood base and S_ω is not first countable.

As the following example shows, the product of Fréchet spaces need not be Fréchet.

Example 3.10. Let $X = (\omega + 1) \times S_\omega$, and consider the set $A = \{(m, (m, n)) : m, n \in \omega\}$. If ∞^* is the identified point of S_ω, let $\infty = \{\omega + 1\} \times \infty^*$. Then $\infty \in \overline{A}$ but $\infty \notin [A]_{\text{seq}}$. The open neighboourhoods of ∞ are are of the form $(\alpha, \omega + 1] \times \left(\bigcup_{m\in\omega} \{m\} \times (f(m), \infty^*]\right)$, which clearly always has non emtpy intersection with A. Hence $\infty \in \overline{A}$. To see that $\infty \notin [A]_{\text{seq}}$, suppose γ is a sequence in A that converges to ∞. Since the sets $U_k = (k, \omega + 1] \times \left(\bigcup_{m\in\omega} \{m\} \times (f(m), \infty^*]\right)$ are open neighbourhoods of ∞ it must be the case that $|\gamma \setminus U_k| < \aleph_0$ for all $k \in \omega$. Thus $\gamma \cap (\{k\} \times \{k\} \times (1, \omega + 1])$ is finite for every k. Let $h : \omega \to \omega$ be defined by $h(k) = \max\{\pi_3(\gamma \cap (\{k\} \times \{k\} \times (1, \infty^*]))\} + 1$ for $k \in \omega$. Pictorially, h is picking the point on each spine beyond which no elements of γ exist. Thus

$$W = (1, \omega + 1] \times \left(\bigcup_{n\in\omega} \{n\} \times (h(n), \infty^*] \right)$$

is an open neighbourhood of ∞ which by construction is disjoint from γ. Hence γ cannot converge to ∞ showing that X is not Fréchet.

3.3 α_i notions of convergence

Definition 3.11. Let X be a topological space and ξ be a countable family of sequences converging to a point $x \in X$. We say that x is an α_i point for $i = 1, 2, 3, 4$ if there exists a sequence B such that

- α_1: $|A \setminus B| < \aleph_0$ for every $A \in \xi$;
- α_2: $|A \cap B| = \aleph_0$ for every $A \in \xi$;
- α_3: $|A \cap B| = \aleph_0$ for infinitely many $A \in \xi$;
- α_4: $A \cap B \neq \emptyset$ for infinitely many $A \in \xi$.

Then X is an α_i space if every $x \in X$ is an α_i point. Note that if a space is α_i then it is α_{i+1} for $i = 1, 2, 3$.

Example 3.12. S_ω is not even α_4. For each $m \in \omega$ let $A_m = \{m\} \times (1, \infty)$. Then $\xi = \{A_m : m \in \omega\}$ is a countable collection of sequences converging to ∞. Suppose B is a sequence that converges to ∞ such that $A \cap B \neq \emptyset$ for infinitely many $A \in \xi$. In particular let $\alpha \leq \omega$ be such that $A_i \cap B \neq \emptyset$ for all $i \in \alpha$ and let $f : \alpha \to \omega$ be defined by $f(k) \in B \cap A_k$ for all $k \in \alpha$. Then

$$U = \left(\bigcup_{k\in\alpha} \{k\} \times (f(k) + 1, \infty] \right) \times \left(\bigcup_{m\in\omega\setminus\alpha} \{m\} \times (1, \infty] \right)$$

is such that $|B \cap U^C| = \aleph_0$, hence B does not converge to ∞.

3.4 Bisequential Spaces

Definition 3.13.