1 Introduction

1.1 Outline of proposed research project

The text Counterexamples in Topology by Steen and Seebach has been a fabulous resource for students and researchers in Topology since its publication in 1970. The book was the product of an undergraduate research project funded by NSF and supervised by Steen and Seebach (and including then student Gary Gruenhage) to systematically survey important topological counterexamples. More recently James Dabbs has implemented a database on Github based on the Steen and Seebach textbook called Pi-Base (see https://topology.pi-base.org/) and it is currently being maintained by Dabbs and Stephen Clontz. This resource has great potential to both researchers and advanced undergraduate and graduate students at the start of their research careers. There are still big gaps in the database's subject matter, especially in relation to research in and around Frechet-Urysohn spaces. There is a significant body of work, and especially interesting counterexamples, concerning Michael's class of bisequential spaces, Arhangel'skii's alpha-i spaces and several game theoretic formulations of convergence which do not yet appear in the Pi-Base. The project has two goals. The first, and most accessible, is to give a systematic survey of the recent research which will be implemented into the Pi-Base database. The second half of the project will be devoted to open problems related to a recent class of examples defined from ladder systems (more generally on so called square-sequence) described in two

2 Meeting Log

Monday April 21

1. Frechet Fan - S_ω: $\omega \times (\omega+1)/\omega \times \{\infty\}$ (i.e. $\omega \times (\omega+1)$ with the points at infinity identified). Show that
   - S_ω is not first countable ✓
   - S_ω is Fréchet. ✓
2. Product of Fréchet spaces not always Fréchet: take $(\omega+1) \times S_\omega$. Let $A = \{(m,(m,n)) \ : \ m,n \in \omega$. Show that
   - $(\omega+1, \infty) \in \overline{A}$
   - No sequence in A converges to $(\omega+1,\infty)$.
3. Right way to think about sequences: $A \subseteq X$ converges to $a \in X$ if $|A| = \aleph_0$ and for all neighboourhoods $U_x \subseteq X$, $|A \setminus U_x| < \aleph_0$.
4. Right way to think about Fréchet space: take sequential closure once same as closure.
5. Another exercise: which α_i properties does S_ω have?

3 Topology

3.1 Basics

3.2 Sequential and Fréchet Spaces

Definition 3.1. [?AH90] For a topological space X and any set $A \subset X$, the *sequential closure* of A is

$$[A]_{\text{seq}} := \left\{ x \in X \ : \ \exists (x_n) \in A \left(\lim_{n\to\infty} x_n = x \right) \right\}.$$

The *total sequential closure* of A is the operation of recursively taking the sequential closure of A until

$$[[[A]_{\text{seq}}]_{\text{seq}} \dots]_{\text{seq}} = \overline{A}.$$

Fact 3.2. In general it takes at most ω_1 many iterations of the sequential closure to get a closed set.

Definition 3.3.

1. [?EK89] A space X is *sequential* if

$$A \subset X \text{ is closed } \iff \forall (x_n) \in A \left(\lim_{n\to\infty} x_n \in A \right).$$

2. A space X is said to sequential if sequentially closed sets are closed.

Definition 3.4. A space X is said to be Fréchet if $[A]_{\text{seq}} = \overline{A}$ for all $A \subseteq X$.

Example 3.5. Let $X = \omega_1 + 1$ with the order topology. X is not Fréchet, since any sequence $(x_n) \in \omega_1$ cannot converge to ∞, as otherwise $\omega_1 = \sup\{x_n : n \in \mathbb{N}\}$, a contradiction.

Clearly then, if a space is Fréchet then it is also Sequential. The following example shows that the converse is not true.

Example 3.6. Let $X^* = \omega \times (\omega + 1)$ be given the order topology and let $X = X^* \cup \{\infty\}$ where the neighboourhoods of ∞ are such that there exists $p \in \omega$ such that $|\{(m, n) : m > p, n \in \omega+1\} \setminus U_\infty| < \aleph_0$. Then X is sequential but but not Fréchet. To see this, note that for all $m \in \omega$ the sequence $A_m = \{(m, n) : n \in \omega\}$ converges to $(m, \omega + 1)$ and moreover $B = \{(m, \omega + 1) : m \in \omega\}$ is a sequence that converges to ∞. Then $A = \bigcup_{m\in\omega} A_m$ is such that $[[A]_{\text{seq}}]_{\text{seq}} = X$, hence X is sequential. On the other hand there is no sequence in A that converges to ∞. Suppose there were, say some $\gamma \to \infty$. Then for all $m \in \omega$, $U_m = X \setminus \{(m, n) : n \in \omega+1\}$ is a neighboourhood of ∞ such that $|\gamma \setminus U_m| < \aleph_0$. Hence γ has only finitely many terms belonging to each column. If $\alpha_m = \max\{\gamma \cap \{(m, n) : n \in \omega\}\}$, then $U = X \setminus \bigcup_{m\in\omega}\{(m, n) : n \leq \alpha_m\}$ is a neighbourhood of ∞ disjoint from γ, a contradiction. Hence X is not Fréchet.

Proposition 3.7. If X is first countable then X is Fréchet.

Proof. Let $A \subseteq X$ and let $x \in \overline{A}$. Then x has a countable neighbourhood base N_x such that $U \cap A \neq \emptyset$ for all $U \in N_x$. Enumerating the neighboourhoods of x as $U_1, U_2, \dots$ then the sequence $(x_n)_{n\geq 1}$ where $x_n \in U_n \cap A$ for each $n \in \omega$ is such that $(x_n)_{n\geq 1}$ converges to x. □

The following example shows that the converse is not true.

Example 3.8 (Fréchet Fan)**.** Let S_ω be the quotient of $\omega \times (\omega + 1)$ obtained by identifying all the points $\{(m, \omega + 1) : m \in \omega\}$. More precisely S_ω has the quotient topology induced by the map $h : \omega \times (\omega + 1) \to S_\omega$ where $h(x) = x$ for all $x \in \omega\times\omega$ and $h(x) = \infty$ for all $x \in \omega\times\{\omega+1\}$. Then S_ω is Fréchet but not 1st countable. To see that S_ω is Fréchet, note that by definition of the quotient topology, the open neighbourhoods of ∞ are those sets $U \subset S_\omega$ such that $\infty \in U$ and $h^{-1}(U)$ is open in $\omega \times (\omega + 1)$. As $\{(m, \omega + 1) : m \in \omega\} \subset h^{-1}(U)$ we see that U is an open neighbourhood of ∞ iff $h^{-1}(U)$ is an open neighbourhood of $(m, \omega + 1)$ for all $m \in \omega$. Note that $h^{-1}(m \times (f(m), \infty]) = m \times (f(m), \omega + 1)$ is an open neighboourhood of $(m, \omega + 1)$ for each $m \in \omega$ where $f : \omega \to \omega$ is just some mapping that indicates the startpoint of each interval. Hence the open neighboourhoods of ∞ are of the form $\bigcup_{m\in\omega} m \times (f(m), \infty]$. It follows that for all $m \in \omega$ the sequence $A_m = \{(m, n) : n \in \omega\}$ converges to ∞ so that $A = \bigcup_{m\in\omega} A_m$ is such that $[A]_{\text{seq}} = S_\omega$. On the other hand it's obvious that $\overline{A} = S_\omega$, so that S_ω is indeed Fréchet. Now asssuming that S_ω was countable, we would have a countable neighbourhood base at ∞. For each $k \in \omega$ let $B_k = \bigcup_{m\in\omega} m \times (f_k(m), \infty]$ for some $f_k : \omega \to \omega$ determining the startpoints of each interval. Suppose $\mathcal{B} = \{B_k : k \in \omega\}$ is a base at ∞, then let $f^* : \omega \to \omega$ be defined by $f^*(m) = f_m(m) + 1$ for all $m \in \omega$. Letting $B^* = \bigcup_{m\in\omega} m \times (f^*(m), \infty]$ it is clear by construction that $B_k \not\subset B^*$ for all $k \in \omega$. Thus $\mathcal{B}$ cannot be a neighboourhood base, i.e., S_ω is not first countable.

Definition 3.9. [?NY92] Let X be a topological space and ξ be a countable family of sequences converging to a point $x_0 \in X$. We say that x_0 is an α_i point for $i = 1, 2, 3, 4$ if there exists a sequence β such that

- α_1: $|\gamma \setminus \beta| < \aleph_0$ for every $\gamma \in \xi$;
- α_2: $|\gamma \cap \beta| = \aleph_0$ for every $\gamma \in \xi$;

- α_3: $|\gamma \cap \beta| = \aleph_0$ for infinitely many $\gamma \in \xi$;
- α_4: $\gamma \cap \beta \neq \emptyset$ for infinitely many $\gamma \in \xi$.

Then X is an α_i space if every $x \in X$ is an α_i point.

3.3 Bisequential Spaces

Definition 3.10. Let X be a topological space. If for all $x \in X$ and $\mathcal{F} \subset \mathcal{P}(X)$ an ultrafilter which clusters at x there exists a sequence of sets $A_1 \supseteq A_2 \supseteq A_3 \ldots$ where each $A_i \in \mathcal{F}$

3.4 Topological Games

3.4.1 Two Player Convergence Game [?GH76]

Let X be a topological space and designate a point $x_0 \in X$. The two player game is defined as follows:

- On turn one player I chooses an open set U_1 containing x_0 and player II then chooses a point $x_1 \in U_1$;
- On the n'th turn player I chooses an open set U_n containing x_0 and player II then chooses a point $x_n \in U_n$.

Player I wins the game if the sequence x_n converges to x_0.

4 Examples

References

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