1 Introduction

1.1 Outline of proposed research project

The text Counterexamples in Topology by Steen and Seebach has been a fabulous resource for students and researchers in Topology since its publication in 1970. The book was the product of an undergraduate research project funded by NSF and supervised by Steen and Seebach (and including then student Gary Gruenhage) to systematically survey important topological counterexamples. More recently James Dabbs has implemented a database on Github based on the Steen and Seebach textbook called Pi-Base (see https://topology.pi-base.org/) and it is currently being maintained by Dabbs and Stephen Clontz. This resource has great potential to both researchers and advanced undergraduate and graduate students at the start of their research careers. There are still big gaps in the database's subject matter, especially in relation to research in and around Frechet-Urysohn spaces. There is a significant body of work, and especially interesting counterexamples, concerning Michael's class of bisequential spaces, Arhangel'skii's alpha-i spaces and several game theoretic formulations of convergence which do not yet appear in the Pi-Base. The project has two goals. The first, and most accessible, is to give a systematic survey of the recent research which will be implemented into the Pi-Base database. The second half of the project will be devoted to open problems related to a recent class of examples defined from ladder systems (more generally on so called square-sequence) described in two

2 Meeting Log

Monday April 21

1. Frechet Fan - S_ω: $\omega \times (\omega+1)/\omega \times \{\infty\}$ (i.e. $\omega \times (\omega+1)$ with the points at infinity identified). Show that
   - S_ω is not first countable ✓
   - S_ω is Fréchet. ✓
2. Product of Fréchet spaces not always Fréchet: take $(\omega+1) \times S_\omega$. Let $A = \{(m,(m,n)) : m,n \in \omega$. Show that
   - $(\omega+1, \infty) \in \overline{A}$ ✓
   - No sequence in A converges to $(\omega+1,\infty)$. ✓
3. Right way to think about sequences: $A \subset X$ converges to $a \in X$ if $|A| = \aleph_0$ and for all neighboourhoods $U_x \subset X$, $|A \setminus U_x| < \aleph_0$.
4. Right way to think about Fréchet space: take sequential closure once same as closure.
5. Another exercise: which α_i properties does S_ω have? ✓

Thursday May 1

1. Go over example 3.9; why is S_ω Fréchet? ✓
2. Give example of space that is α_2 but not α_1 ✓, α_3 but not α_2 etc.
3. Under which set theoretic assumptions do such examples exist? Look at the paper by Nyikos: Subsets of ${}^\omega\omega$ and the Fréchet Urysohn and α_i properties. Good grasp at sections 1, 2, 5; partial understanding section 3; skipped section 4, 6.
4. For example it is consistent that both α_1 and countable imply first countable, and α_2 and countable imply α_1, yet there exists a countable α_2 space that is not first countable.

5. We talked a little bit about topological groups (a topological space equipped with group operation that is continuous) and some of the nice structure they have: the group operation being continuous and invertible implies $G \times G \to G$ is always a homeomorphism, in particular aG is the homeomorphic image of G by left multiplication of $a \in G$, which in often cases lets us define a nbhd base at the identity and "send" it to all points of G in order to define the topology.

6. Look into what it takes to contribute to the pi-base, in particular add the α_i spaces. Forked database to github account; properties are stored as markdown text files, should not be difficult to add properties as well as if/then.

   Things to add:

   - α_i and if α_i then α_{i+1}
   - bisequential
   - v-space (w-space exists already)
   - if 1st countable then α_1 Frechet
   - if bisequential then Frechet
   - w-space iff Frechet and α_2

7. Some more exercises

   - X is α_1 and Fréchet, Y is Fréchet, then $X \times Y$ is Fréchet. this is not true, take $(\omega + 1) \times S_\omega$.
   - α_2 is equivalent to $A \cap B \neq \emptyset$ for all $A \in \xi$ whenever ξ is countable collection of sequences at x. ✓

Thursday May 15

1. When is the square of an α_i space still α_i? What α properties does $\alpha_i \times \alpha_j$ have?

2. I asked for an example of a tower, and Paul proved that towers exist, but said that theres not really specific constructions of towers.

3. historical note: the cardinals that lay between ω_1 and $\mathcal{P}(\omega)$ came from attempts to solve CH, for example, if $\mathfrak{t} \leq \mathfrak{c}$ then, finding the cardinality for $\mathfrak{t}$ would help in finding cardinality of $\mathfrak{c}$.

4. Proximal game

5. Uniformities

6. Homogeneity in Top group used to determine uniformity for player 1 in version of prox game for top groups

7. FUF and 2FUF

8. exercises:

   - prove lemmas used in existence proof of towers.
   - If $\{A_\alpha : \alpha \subset \gamma\}$ is a tower and $\{\alpha_\xi : \xi < \lambda\}$ is increasing and cofinal in γ then $A_{\alpha_\xi} : \xi < \lambda$ is a tower.
   - S_ω is neither FUF or $2FUF$ ✓
   - Proximal game is equivalent to Gruenhage game. Proximal space implies W ✓
   - Winning strategy in proximal game implies Frechet I showed W and w space imply frechet

9. note: send Paul changes to databse before pushing to github

Wednesday May 28

1. More on Uniformities
   - Different metrics generate different Uniformities, for example discrete metric vs $d(m,n) = |\frac{1}{m} - \frac{1}{n}|$ on ω, Uniformities from those metrics still generate the same (discrete) metric on ω
   - star refinements generalize triangle inequality, for example $\{\mathrm{B}(x, \frac{1}{2^{n+1}}) : x \in X\}$ star refines $\{\mathrm{B}(x, \frac{1}{2^n}) : x \in X\}$ and gives triangle inequality
2. went over "proof" that proximal implies W, however Paul raised issue that player 2 in proximal game can use Hausdorffness to always pick points z that prompt player 1 in Proximal game to choose entourages U such that $U[x] \cap U[p] = \emptyset$, where p is where Gruenhage game being played. this problem is fixed by starting Gruenhage game one round later than proximal game and player 2 pick p on turn 1 of proximal game
3. Exercises:
   - If X semi-proximal then is FUF? turns out this isn't known (yet)
   - Is double arrow space Proximal?✓
   - Semi Proximal is α_2. ✓
4. Paul showed me proof that semi Proximal is Frechet
5. Malykhin's problem: If G is a seperable Frehcet topological group, is it metrizable? Answer: Con(yes) and Con(No)
6. **Paul's main question:** Can we strengthen hypotheses of Malykhin's problem to obtain answer that is not independent of ZFC?

Thursday June 5

1. Discussed impliciations of conditions on filter $\mathcal{F}$ when going from top space to top group
2. Questions:
   - If $\mathcal{F}$ is FUF, is $\tau_{\mathcal{F}}$ semiproximal?
   - If $\tau_{\mathcal{F}}$ is semiproximal, is $\mathcal{F}$ countable generated?
   - Assuming CH, does there exist filters $\mathcal{F}_0, \mathcal{F}_1$ both FUF neither countably generated where $\tau_{\mathcal{F}_0}$ is semiproximal but $\tau_{\mathcal{F}_1}$ isn't.
3. Exercises:
   - Assume CH, construct FUF filter that is not countable generated. ✓
   - What does it mean for a sequence in (a_n) to converge to a point in G, i.e., $(a_n) \to a$ iff what?
   - Consider a play of the Proximal game which yields sequences (a_n) and (V_{U_n}). What does it mean for there to exist some $b \in G$ such that $b \in \bigcap_{n\in\omega}(a_n \Delta V_{U_n})$? What does that look like?

3 Set Theory

3.1 Some Interesting Cardinals

3.1.1 Subsets of ${}^{\omega}\omega$ and Almost Disjoint Familes

Definition 3.1. Let $\mathcal{A}$ be an infinite family of infinite subsets of ω. The family $\mathcal{A}$ is said to be an almost disjoint family (ADF) of subsets of ω if $A \cap B$ is finite for any $A, B \in \mathcal{A}$ with $A \neq B$. By Zorn's lemma we can assume $\mathcal{A}$ lives inside of a maximal (w.r.t. inclusion) family that is also pairwise disjoint. Such a family is a maximal almost disjoint family (MADF).

Definition 3.2. Define an order $\leq^*$ on ${}^\omega\omega$ where $f <^* g$ if $f(n) < g(n)$ for all but finitely many $n \in \omega$

Definition 3.3. Let $(A, \prec)$ be a linearly ordered set. We say that $B \subset A$ is unbounded in A if there does not exist $y \in A$ such that $x \prec y$ for all $x \in B$. We say that $C \subset A$ is cofinal (or dominating) in A if for all $x \in A$ there exists $z \in C$ such that $x \prec x$.

Definition 3.4.

- $\mathfrak{a} = \min\{|\mathcal{A}| \,:\, \mathcal{A} \text{ is infinte MADF}\}$
- $\mathfrak{b} = \min\{|B| \,:\, B \text{ is } \leq^* \text{-unbounded subset of } {}^\omega\omega\}$
- $\mathfrak{d} = \min\{|D| \,:\, D \text{ is } \leq^* \text{-cofinal in } {}^\omega\omega\}$

Each of $\mathfrak{a}, \mathfrak{b}, \mathfrak{d}$ are at least ω_1 but at most $\mathcal{P}(\omega)$.

3.1.2 Towers and Pseudo-Intersections

Definition 3.5. Define an order on $\mathcal{P}(X)$ for some set X. We say $A \subset^* B$ if B contains all but finitely many points of A. If $\mathcal{F} \subset \mathcal{P}(X)$ is a collection of (infinite) sets, then A is a pseudo-intersection of $\mathcal{F}$ if $A \subset^* F$ for all $F \in \mathcal{F}$.

Definition 3.6. We call $\mathcal{T} \subset [\omega]^\omega$ a tower if $\mathcal{T}$ is well ordered by $\subset^*$ and has no infinite intersection.

Lemma 3.7. If $\mathcal{A} = \{A_n \,:\, n \in \omega\} \subset [\omega]^\omega$ such that for all $n \in \omega$

$$A_0 \supset^* A_1 \supset^* \ldots \supset^* A_n$$

then $\mathcal{A}$ has infintie pseudo-intersection.

Theorem 3.8. *There exists a tower.*

Proof. Define $\langle A_\alpha \,:\, \alpha < \mathfrak{c}\rangle$ as follows:

1. $A_\alpha = \omega \setminus n$ for $\alpha < \omega$
2. If α is a successor ordinal, define A_α such that $|A_{\alpha-1} \setminus A_\alpha| = \aleph_0$
3. If α is a limite ordinal then define A_α such that $A_\alpha \subset^* A_\beta$ for all $\beta < \alpha$

Suppose γ is given and $\langle A_\alpha \,:\, \alpha < \gamma\rangle$ has been defined. If γ is a successor then define A_γ as in (2). Otherwise γ is a limite ordinal: if (3) isn't possible then $\langle A_\alpha \,:\, \alpha < \gamma\rangle$ is a tower; otherwise define A_γ as in (3) and continue.

This process eventually terminates, otherwise we obtain $\mathfrak{c}$ many distint subsets of ω, a contradiction. TODO: elaborate $\square$

Definition 3.9.

- $\mathfrak{t} = \min\{|\mathcal{T}| \,:\, \mathcal{T} \text{ is a tower}\}$
- $\mathfrak{p} = \min\{|\mathcal{P}| \,:\, \mathcal{P} \subset [\omega]^\omega \text{ is a family with the SFIP that has no pseudo-intersection}\}$

4 Topology

4.1 Sequential and Fréchet Spaces

Definition 4.1. For a topological space X and any set $A \subset X$, the *sequential closure* of A is

$$[A]_{\text{seq}} := \left\{x \in X \,:\, \exists (x_n) \in A \left(\lim_{n\to\infty} x_n = x\right)\right\}.$$

In general we can repeat this operation recursively $[[[A]_{\text{seq}}]_{\text{seq}} \ldots]_{\text{seq}}$ by which is meant the *total sequential closure* of A.

Fact 4.2. In general it takes at most ω_1 many iterations of the sequential closure to get a closed set.

Definition 4.3. A space X is said to be Fréchet if $[A]_{\text{seq}} = \overline{A}$ for all $A \subset X$.

Example 4.4. Let $X = \omega_1 + 1$ with the order topology. X is not Fréchet, since any sequence $(x_n) \in \omega_1$ cannot converge to ∞, as otherwise $\omega_1 = \sup\{x_n : n \in \mathbb{N}\}$, a contradiction.

Definition 4.5. A space X is sequential if any closed set $A \subset X$ is equal to its total sequential closure.

Then a space that is Fréchet is also sequential. The following example shows that the converse is not true.

Example 4.6. Let $X^* = \omega \times (\omega + 1)$ be given the order topology and let $X = X^* \cup \{\infty\}$ where the neighboourhoods of ∞ are such that there exists $p \in \omega$ such that $|\{(m,n) : m > p, n \in \omega{+}1\} \backslash U_\infty| < \aleph_0$. Then X is sequential but but not Fréchet. To see this, note that for all $m \in \omega$ the sequence $A_m = \{(m,n) : n \in \omega\}$ converges to $(m, \omega+1)$ and moreover $B = \{(m, \omega+1) : m \in \omega\}$ is a sequence that converges to ∞. Then $A = \bigcup_{m\in\omega} A_m$ is such that $[[A]_{\text{seq}}]_{\text{seq}} = X$, hence X is sequential. On the other hand there is no sequence in A that converges to ∞. Suppose there were, say some $\gamma \to \infty$. Then for all $m \in \omega$, $U_m = X \backslash \{(m,n) : n \in \omega{+}1\}$ is a neighboourhood of ∞ such that $|\gamma \setminus U_m| < \aleph_0$. Hence γ has only finitely many terms belonging to each column. If $\alpha_m = \max\{\gamma \cap \{(m,n) : n \in \omega\}\}$, then $U = X \setminus \bigcup_{m\in\omega} \{(m,n) : n \leq \alpha_m\}$ is a neighbourhood of ∞ disjoint from γ, a contradiction. Hence X is not Fréchet.

Proposition 4.7. If X is first countable then X is Fréchet.

Proof. Let $A \subset X$ and let $x \in \overline{A}$. Then x has a countable neighbourhood base N_x such that $U \cap A \neq \emptyset$ for all $U \in N_x$. Enumerating the neighboourhoods of x as $U_1, U_2, \ldots$ then the sequence $(x_n)_{n\geq 1}$ where $x_n \in U_n \cap A$ for each $n \in \omega$ is such that $(x_n)_{n\geq 1}$ converges to x. □

The following example shows that the converse is not true.

Example 4.8 (Fréchet Fan)**.** Let S_ω be the quotient of $\omega \times (\omega + 1)$ obtained by identifying all the points $\{(m, \omega+1) : m \in \omega\}$ as ∞^*. More precisely S_ω has the quotient topology induced by the map $h : \omega \times (\omega+1) \to S_\omega$ where $h(x) = x$ for all $x \in \omega \times \omega$ and $h(x) = \infty^*$ for all $x \in \omega \times \{\omega+1\}$. Then S_ω is Fréchet but not 1st countable.

To see that S_ω is Fréchet, let $A \subset S_\omega$ such that $\infty \in \overline{A}$. A must meet at least one column of S_ω in an infinite set, otherwise we could find a nbhd of ∞^* disjoint from A. Then A restricted to that column will be a convergent sequence to ∞.

Now asssuming that S_ω was countable, we would have a countable neighbourhood base at ∞^*. For each $k \in \omega$ let $B_k = \bigcup_{m\in\omega} \{m\} \times (f_k(m), \infty^*]$ for some $f_k : \omega \to \omega$ determining the startpoints of each interval. Suppose $\mathcal{B} = \{B_k : k \in \omega\}$ is a base at ∞^*, then let $f^* : \omega \to \omega$ be defined by $f^*(m) = f_m(m) + 1$ for all $m \in \omega$. Letting $B^* = \bigcup_{m\in\omega} \{m\} \times (f^*(m), \infty^*]$ then B^* is an open neighboourhood of ∞^* but it is clear by construction that $B_k \not\subset B^*$ for all $k \in \omega$. Hence $\mathcal{B}$ cannot be a neighboourhood base and S_ω is not first countable.

As the following example shows, the product of Fréchet spaces need not be Fréchet.

Example 4.9. Let $X = (\omega+1) \times S_\omega$, and consider the set $A = \{(m,(m,n)) : m, n \in \omega\}$. If ∞^* is the identified point of S_ω, let $\infty = \{\omega+1\} \times \infty^*$. Then $\infty \in \overline{A}$ but $\infty \notin [A]_{\text{seq}}$. The open neighboourhoods of ∞ are are of the form $(\alpha, \omega+1] \times \left(\bigcup_{m\in\omega} \{m\} \times (f(m), \infty^*]\right)$, which clearly always has non emtpy intersection with A. Hence $\infty \in \overline{A}$. To see that $\infty \notin [A]_{\text{seq}}$, suppose γ is a sequence in A that converges to ∞. Since the sets $U_k = (k, \omega+1] \times \left(\bigcup_{m\in\omega} \{m\} \times (f(m), \infty^*]\right)$ are open neighbourhoods of ∞ it must be the case that $|\gamma \setminus U_k| < \aleph_0$ for all $k \in \omega$. Thus $\gamma \cap (\{k\} \times \{k\} \times (1, \omega+1])$ is finite for every k. Let $h : \omega \to \omega$ be defined by $h(k) = \max\{\pi_3(\gamma \cap (\{k\} \times \{k\} \times (1, \infty^*])\} + 1$ for $k \in \omega$. Pictorially, h is picking the point on each spine beyond which no elements of γ exist. Thus

$$W = (1, \omega+1] \times \left(\bigcup_{n\in\omega} \{n\} \times (h(n), \infty^*] \right)$$

is an open neighbourhood of ∞ which by construction is disjoint from γ. Hence γ cannot converge to ∞ showing that X is not Fréchet.

4.2 α_i notions of convergence

Definition 4.10. Let X be a topological space and ξ be a countable family of sequences converging to a point $x \in X$. We say that x is an α_i point for $i = 1, 2, 3, 4$ if there exists a sequence B such that

- α_1: $|A \setminus B| < \aleph_0$ for every $A \in \xi$;
- α_2: $|A \cap B| = \aleph_0$ for every $A \in \xi$;
- α_3: $|A \cap B| = \aleph_0$ for infinitely many $A \in \xi$;
- α_4: $A \cap B \neq \emptyset$ for infinitely many $A \in \xi$.

Then X is an α_i space if every $x \in X$ is an α_i point. Note that if a space is α_i then it is α_{i+1} for $i = 1, 2, 3$.

Example 4.11. S_ω is not even α_4. For each $m \in \omega$ let $A_m = \{m\} \times (1, \infty)$. Then $\xi = \{A_m : m \in \omega\}$ is a countable collection of sequences converging to ∞. Suppose B is a sequence that converges to ∞ such that $A \cap B \neq \emptyset$ for infinitely many $A \in \xi$. In particular let $\alpha \leq \omega$ be such that $A_i \cap B \neq \emptyset$ for all $i \in \alpha$ and let $f : \alpha \to \omega$ be defined by $f(k) \in B \cap A_k$ for all $k \in \alpha$. Then

$$U = \left(\bigcup_{k \in \alpha} \{k\} \times (f(k) + 1, \infty] \right) \times \left(\bigcup_{m \in \omega \setminus \alpha} \{m\} \times (1, \infty] \right)$$

is such that $|B \cap U^C| = \aleph_0$, hence B does not converge to ∞.

Exercise 4.12. X is α_2 iff whenever ξ is a countable collection of sequences converging to x there exists $B \to x$ such $A \cap B \neq \emptyset$ for all many $A \in \xi$.

Solution. The forward direction is obvious. Conversely, let $\xi = \{A_1, A_2, \dots\}$ be a countable collection of sequences converging to x. For every $n \in \omega$ let $\{a_{nm} : m \in \omega\}$ enumerate the elements of A_n and define $A_{nm} = A_n \setminus \{a_{n1}, a_{n2}, \dots, a_{n,m-1}\}$ for each $n, m \in \omega$. Then the A_{nm} still convege to x and $\mathcal{A} = \{A_{nm} : n, m \in \omega\}$ is a sheaf at x. By hypothesis there exists a B converging to x that meets each A_{nm}, and thus meets each A_n in an infinite set.

4.3 Ψ-like spaces

Definition 4.13. Let D be a countable set and let $\mathcal{A} \subset \mathcal{P}(D)$ be an ADF. Define a topology on $X = D \cup \mathcal{A}$ such that the points of D are isolated and D is dense in X. Then for every $A \in \mathcal{A}$ attach a point z_A where the nbhds of z_A consist of all sets of the form $B(A, F) = \{z_A\} \cup (A \setminus F)$ where $F \subset A$ is finite.

Proposition 4.14. Let X be as above. Then X is Hausdorff, first countable and locally compact. If X is constructed as above but $\mathcal{A}$ is MADF then X is also pseudo-compact (i.e. all continuous real functions on X are bounded).

Proof. To check Hausdorffness it suffices to show $Z = X \setminus D$ is Hausdorff. Let $z_1, z_2 \in Z$. Then z_1, z_2 correspond to sets A_1, A_2 such that $A_1 \cap A_2 = I$ is finite. Then $B(A_1, I)$ and $B(A_2, I)$ are disjoint open sets containing z_1, z_2 respectively.

To see that X is locally compact, note that $\{x\}$ is a compact neighbourhood base for all $x \in D$. Local compactness of Z follows from the fact that by design each $z_A \in Z$ is the limit of A viewed as a sequence. In other words, whenever $B(A_z, F)$ is a basic nbhd of z_A, any open cover will have a smaller nbhd $B(A_z, F')$ that omits only finitely many points. $\square$

Proposition 4.15. The one point compactification of a Ψ-like space of cardinality $< \mathfrak{a}$ is Fréchet.

4.4 Constructing Examples of α_i Spaces

4.4.1 Squares of α_i Spaces

Example 4.16. If X, Y are α_1, then so is $X \times Y$. Let $\xi = \langle \xi_n : n \in \omega \rangle$ be a sheaf at some $x \times y \in X \times Y$. Then $\pi_i(\xi) := \langle \pi_i(\xi_n) : n \in \omega \rangle$ is a sheaf at $\pi_i(x \times y)$ in X, Y respectively for $i = 1, 2$. Hence there exists α_1 sequences B_x, B_y for $\pi_i(\xi)$. Letting $M_n = \max\{k \in \omega : x_{n,k} \notin \pi_1(\xi_n) \vee y_{n,k} \notin \pi_2(\xi_n)\}$ for each $n \in \omega$, the sequence $B = \bigcup_{n \in \omega} B_x \cap (\xi_n \setminus x_{n1}, \dots, x_{n,M_n}) \times B_y \cap (\xi_n \setminus y_{n1}, \dots, y_{n,M_n})$ is as desired.

4.4.2 Misc α_i Spaces

Example 4.17. Let $D = \omega \times \omega$, let $\mathcal{F} = \{f_\alpha : \alpha < \mathfrak{b}\} \subset {}^\omega\omega$ be $<^*$-well-ordered and $<^*$-unbounded, and let $\mathcal{C} = \{C_n = \{n\} \times \omega : n \in \omega\}$. Construct a Ψ-like from $X = \mathcal{F} \cup \mathcal{C}$, where we attach compactification points $\{n\} \times (\omega + 1)$ for each C_n and compactification points p_α for each f_α. Then the resulting one point compactification, $X + \infty$, is α_2 and Fréchet but not α_1. $X + \infty$ becomes α_1 if $\mathcal{F}$ is cofinal and well ordered.

4.5 Frechet-Urysohn for Finite Sets

Definition 4.18. A π-network at a point $x \in X$ is a collection $\mathcal{F} \subset \mathcal{P}(X)$ such that for all open nbhds U_x there is $F \in \mathcal{F}$ such that $F \subset U_x$.

Definition 4.19. We say that an infinite family of sets $\langle F_n \subset \mathcal{P}(X) : n \in \omega \rangle$ converges to a point $x \in X$ if $|\{F_n : F_n \not\subset U_x\}| < \aleph_0$ for every open nbhd U_x.

Definition 4.20. X is Fréchet-Urysohn Finite (FUF) if for all $x \in X$ whenever $\mathcal{F} \subset [X]^{<\omega}$ is a π-network at x there exists $\langle F_n : n \in \omega \rangle \subset \mathcal{F}$ such that the F_n converge to x. We say that X is nFUF if the condition holds for all π-networks $\mathcal{F} \subset [X]^n$ and we say that X is boundedly FUF if X is nFUF for all $n \in \omega$.

Example 4.21. S_ω is not 2FUF (and hence not FUF). Let $F_m^n = \{(1, m), (m + 1, n)\}$ and let $\mathcal{F} = \{F_m^n : m, n \in \omega\}$. Then $\mathcal{F}$ is a π-network at ∞ with no convergent subcollection.

To see that $\mathcal{F}$ is a π-network, let U be an open nbhd of ∞. Then $U = \bigcup_{k \in \omega} \{k\} \times (f(k), \infty]$ for some $f \in {}^\omega\omega$. Letting $m = f(1) + 1$ and $n = f(m) + 1$, then $F_m^n \subset U$.

Suppose $\mathcal{G} \subset \mathcal{F}$ converges to ∞. Then since $V_n = S_\omega \setminus (\{1\} \times (1, n])$ is an open nbhd of ∞ for each $n \in \omega$, we see that the sets $\{G \in \mathcal{G} : G \not\subset V_n\}$ are each finite. In particular, this shows that there are only finitely many $G \in \mathcal{G}$ that meet each column, hence for each $n \in \omega$ there exists $a_n \in \omega$ such that for all $b \geq a_n$, $F_n^b \notin \mathcal{G}$. Letting $g : \omega \to \omega$ be defined by $g(n) = a_n$ we obtain the open nbhd $W = \bigcup_{n \in \omega} \{n\} \times (g(k), \infty]$ which is such that $G \notin W$ for all $G \in \mathcal{G}$, i.e. $\{G \in \mathcal{G} : G \not\subset W\}$ is infinite so that $\mathcal{G}$ does not converge to ∞.

Proposition 4.22. If X is first countable then X is FUF.

Proof. Suppose X is first countable, let $x \in X$, and let $\mathcal{F}$ be a π-network at x. Let $\mathcal{U} = \{U_0, U_1, \ldots\}$ be a nbhd base at x and WLOG assume that $U_0 \supset U_1 \supset \ldots$ (since otherwise we could just take $U_0, U_0 \cap U_1, \ldots$). Then for each $n \in \omega$ there is a $\mathcal{F} \ni F_n \subset U_n$ and the F_n converge to x. ☐

Definition 4.23. $\mathcal{A} \subset \mathcal{P}(X)$ is a network if for all $x \in X$ and for all nhds $U \ni x$ there is $A \in \mathcal{A}$ such that $x \in A \subset U$. If $\mathcal{F}$ is a filter, then we might also refer to a network $\mathcal{A}$ for $\mathcal{F}$ where every $F \in \mathcal{F}$ contains some $A \in \mathcal{A}$.

Remark 4.24. A network is a generalization of a base since we no longer require the elements to be open sets.

Definition 4.25 (FUF filters)**.** A filter $\mathcal{F}$ on ω is FUF if whenever $A \subset [\omega]^{<\omega}$ is a network for $\mathcal{F}$ then there exists $B \subset A$ such that B converges wrt to $\mathcal{F}$.

Lemma 4.26. Let $\mathcal{F}$ be a filter on $[\omega]^\omega$. If $\mathcal{F}$ is countably generated then $\mathcal{F}$ has pseudo-intersection.

Proposition 4.27. Assume CH. Then there exists a FUF filter $\mathcal{F}$ that is not countably generated.

Proof. Starting with $\mathcal{F}_0$ the cofinite filter on ω, we will recursively construct a filter by adding a new set to our filter at each stage such that the FUF condition is preserved. We assume that $\mathcal{P}([\omega]^{<\omega}) = \{A_n : n < \omega_1\}$ and will construct our filter in ω_1 many steps.

On step 0, take $A_0 \subset [\omega]^{<\omega}$. If it's a network for $\mathcal{F}_0$ then we find $B_0 \subset A_0$ such that $B_0 \to \mathcal{F}_0$, which exists since ω is first countable. Otherwise, if A_0 isn't a network, we proceed to the next step.

On step α, let $\mathcal{F}^*_\alpha = \mathcal{F}_0 \cup \left(\bigcup_{i<\alpha}\{X_i\}\right)$ and let $\mathcal{F}_\alpha = F^*_\alpha \cup \{X_\alpha\}$ where X_α will be defined in such a way as to preserve any previous convergent sequences, which we denote $\{B_\beta : \beta < \alpha\}$. Note that $\mathcal{F}^*_\alpha$ is countably generated, as $\mathcal{F}_0$ is countably generated and we have progressed only countably many steps. So we have $F_0 \supset F_1 \supset \dots$ which generate $\mathcal{F}^*_\alpha$. Let $X_\alpha = \bigcup_{i<\alpha} F_i \cap (\cup B_i)$. Since each F_i contains a tail of every B_β, we have that $X_\alpha \supset^* B_\beta$ for all β. On the other hand, none of the F_n generate X_α since $F_1 \cap B_1 \not\supset F_k$ for all $k \geq 1$ and in general $F_n \cap B_n \not\supset F_k$ for all $n \in \omega, k \geq n$. Hence $X_\alpha \notin \mathcal{F}^*_\alpha$ and $\mathcal{F}_\alpha$ is as desired.

We continue step α by taking $A_\alpha \subset [\omega]^\omega$. If it's a network for $\mathcal{F}_\alpha$, find another sequence $B_\alpha \subset A_\alpha$ that converges with respect to $\mathcal{F}_\alpha$, otherwise proceed to the next step.

Let $\mathcal{F}$ be the filter obtained after recursively iterating through all $A_n \subset [\omega]^{<\omega}$, i.e. $\mathcal{F} = \mathcal{F}_0 \cup \left(\bigcup_{\alpha<\omega_1}\{X_\alpha\}\right)$. By construction $\mathcal{F}$ is FUF and to see that its uncountably generated let $\mathcal{G} \subset \mathcal{F}$ be a generating set. Then for some $\alpha < \omega_1$ there's $\mathcal{G}_\alpha \subset \mathcal{G}$ which generates $\mathcal{F}^*_\alpha$. But $\mathcal{F}^*_\alpha$ was also generated by $F_0 \supset F_1 \supset \dots$ and X_α was such that $X_\alpha \not\supset F_n$ for all $n \in \omega$. Since the F_n would necessarily also generate $\mathcal{G}_\alpha$ this implies that $X_\alpha \not\supset G_n$ for all $n \in \omega$. Hence $\mathcal{G}_\alpha$ generates $\mathcal{F}^*_\alpha$ but not $\mathcal{F}_\alpha$. Furthermore, $\mathcal{G}_\alpha$ cannot generate $\mathcal{F}_\eta$ either, for $\alpha < \eta < \omega_1$, which follows from the definition of the X_η. Thus as $\mathcal{F}$ is obtained from F_α by adding uncountably many new sets, each of which not generated by $\mathcal{G}_\alpha$, it must be the case that $\mathcal{G}$ is uncountable. $\square$

4.6 Uniformities

In general uniformities provide an alternate way to generalize metric spcaes. Topological spaces generalize metric spaces while preserving the notion of continuous functions whereas uniform spaces preserve uniformly continuous functions. Given a set X there are two ways to contruct a uniformity $\mathcal{D}$ on X.

4.6.1 Diagonal Uniformities

A diagonal uniformity on a set X is a principal filter $\mathcal{D}$ on the square $X \times X$ generated by the diagonal satisfying

1. For all $D \in \mathcal{D}$ there exists $E \circ E \subset D$ for some $E \in \mathcal{D}$
2. $D^{-1} \in \mathcal{D}$ for all $D \in \mathcal{D}$

where $E \circ E = \{(x, y) \in E \times E : \exists z \in E\,((x, z) \in E \wedge (z, y) \in E)\}$ and $D^{-1} = \{(y, x) : (x, y) \in D\}$. Note that $E \circ E$ will often be abreviated E^2 and we may at times go so far as to use $E^4 = (E^2)^2 = E \circ E \circ E \circ E$. Also, whenever $D \in \mathcal{D}$ is such that $D = D^{-1}$ we will say that D is symmetric. A diagonal uniformity can be generated by filterbase consisting of symmetric sets D each containing the diagonal. The pair $(X, \mathcal{D})$ will be called a uniform space.

If $(X, \mathcal{D})$ is a uniform space then for any $x \in X$ and $D \in \mathcal{D}$ we say that $D[x] = \{y \in X : (x, y) \in D\}$ is the section of D at x. More generally, we can also talk about $D[A] = \{y \in X : (x, y) \in D, x \in A\}$. Although the former is used quite frequantily because the collection $\tau_{\mathcal{D}} = \{D[x] : x \in X, D \in \mathcal{D}\}$ is a base for a topology on X, and the most natural way to pass from uniform to topological spaces.

4.6.2 Uniform Covers

4.7 Topological Groups

Definition 4.28. A topological group G is a group whose underlying set has a topology such that

- the group operation is continuous;
- the map that sends $x \mapsto x^{-1}$ is continuous.

For sets $A, B \subset G$ and any point $g \in G$ we denote $AB = \{a \cdot b \, : \, a \in A, b \in B\}$ and $gA = \{g \cdot a \, : \, a \in A\}$.

We will often construct topological groups from topological space. Take $X = \{\infty\} \cup \omega$ with the points of ω isolated and let $\mathcal{F}$ a filter on ω. Then $G = [\omega]^{<\omega}$ with the operation symmetric difference $\Delta : G \times G \to G$ defined by $a \Delta b = a \setminus b \cup b \setminus a$ and $\emptyset$ the identity will be our group. For each $U \in \mathcal{F}$ set $V_U \subset [\omega]^{<\omega}$ where $V_U = \{a \in [\omega]^{<\omega} : a \subset U\}$. Then $\mathcal{V} = \{V_U : U \in \mathcal{F}\}$ is a nbhd base at $\emptyset$ and $\tau_{\mathcal{F}} = \{a \Delta V_U : a \in [\omega]^{<\omega}, U \in \mathcal{F}\}$ is a base for a topology on G.

Proposition 4.29. If $\mathcal{F}$ is FUF then $\tau_{\mathcal{F}}$ is Frechet.

Proof. Let $\mathcal{F}$ be FUF and let $A \subset \tau_{\mathcal{F}}$ be such that $x \in \overline{A}$. WLOG assume that $x = \emptyset$. Then $\mathcal{A} = \{a \in A \cap [F]^{\leq \omega} : F \in \mathcal{F}$ is a network for $\mathcal{F}$ and hence there is a $B \subset \mathcal{A}$ such that $B \to \mathcal{F}$, i.e., $\underline{B \to x}$.

Conversely, let $\mathcal{A}$ be a network for $\mathcal{F}$ and let $[\mathcal{A}]^{<\omega} = \{[a]^{<\omega} : a \in \mathcal{A}\}$. Then $\emptyset \in \overline{[\mathcal{A}]^{<\omega}}$ since there is $a \subset F$ implies $[a]^{<\omega} \subset [F]^{<\omega}$ hence $[\mathcal{A}]^{<\omega} \cap [F]^{<\omega} \neq \emptyset$ for all $F \in \mathcal{F}$. Thus there is a $B \subset [\mathcal{A}]^{<\omega}$ that converges to $\emptyset$ and hence $B \to \mathcal{F}$. $\square$

Proposition 4.30. $\mathcal{F}$ is countably generated $\iff$ G is secound countable $\iff$ G is metrizable.

5 Topological Games

5.1 Gruenhage Game

Let $x \in X$ be a point chosen before the start of game. On turn zero, player 1 chooses an open nbhd U_0 of x and player 2 chooses a point $x_0 \in U_0$; on the n'th turn player one chooses an open nbhd U_n of x and player 2 chooses a point $x_n \in U_n$. Player 1 wins if the sequence of points $\{x_n \, : \, n \in \omega\}$ converges to x.

Formally, we can describe a strategy for player 1 as a mapping $\sigma : [X]^{<\omega} \to \mathcal{O}_x$ from finite sequences in X to the collection of open sets containing x where $\sigma(\emptyset) = X$. In particular we say that $\langle x_n \, : \, n \in \omega \rangle$ is a σ-sequence provided $x_{n+1} \in \sigma(\{x_1, \ldots, x_n\})$ for all n. We say that σ is a winning strategy if every σ-sequence converges to x.

A strategy for play 2 is a mapping $\tau : \mathcal{S}(x) \times \mathcal{O}_x \to X$ such that $\tau(F, U) \in U$ for all $F \in \mathcal{S}(x)$ and $U \in \mathcal{O}_x$.

Definition 5.1. We say that X is a W-space if there is a winning strategy for player 1 at every point $x \in X$. We say that X is a w-space if for every strategy for player 2 there exists a counterstrategy for player 1 that wins.

Definition 5.2. Let $x \in X$ and let $\tau : \mathcal{S}(X) \times \mathcal{O}_x(X) \to X$ be a strategy for player 2 at x. Say that $\mathcal{F}_\tau = \langle F_n \, : \, n \in \omega \rangle$ is a τ-chain if for all $n \in \omega$ we have $F_n \subset F_{n+1}$ and $F_{n+1} = F_n \cup \{\tau(F_n, U)\}$ for some $U \in \mathcal{O}_x$.

Proposition 5.3. X is a w-space iff it is α_2 and Fréchet.

Proof. The forward direction is similar to above result, with the difference that we define a strategy τ for player 2 such that $\tau(F, U_x) \in U_x \cap A$ on the one hand, and $\tau(F, U_y) \in U_y \cap \xi_{|F|}$ on the other hand; then since X is a w-space, we obtain a counter strategy that wins for player 1, i.e., the desired convergent sequence.

Now suppose X is α_2 and Fréchet and let $\tau : \mathcal{S}(X) \times \mathcal{O}_x(X) \to X$ be a strategy for player 2 at x. For each $F \in \mathcal{S}(X)$ let $\tau_F = \{\tau(F, U) \, : \, U \in \mathcal{O}_x\}$ and note that $x \in \overline{\tau_F}$ for all F. By Fréchetness, each τ_F contains an A_F that converges to x and $\xi = \langle A_F \, : \, F \in \mathcal{S}(X)$ is a sheaf at x. Let B be a sequence that converges to x which meets each A_F and let $\sigma : [X]^{<\omega} \to \mathcal{O}_x$ be a counterstrategy for player 1 such that $\tau(F, \sigma(F)) \in B \cap A_F$ for all $F \in [X]^{<\omega}$. Then σ wins for player 1. This assumes X is countable.

Suppose X is α_2 and Fréchet and let $x \in X$ be a point. If $\tau : \mathcal{S}(X) \times \mathcal{O}_x(X) \to X$ is a strategy for player 2 at x, then for each $F \in \mathcal{S}(X)$ let $\tau_F = \{\tau(F, U) \, : \, U \in \mathcal{O}_x\}$ and note that $x \in \overline{\tau_F}$, which by Fréchetness of X, implies the existence of a $A_F \subset \tau_F$ that converges to x for each $F \in \mathcal{S}(X)$. Let $\mathfrak{F}$ be a collection of τ-chains such that for each $y \in X$ there exists exactlly one $\mathcal{F}_{\tau_y} \in \mathfrak{F}$ such that $\{y\} \in \mathcal{F}_{\tau_y}$ (AC). Observe that to each $\mathcal{F}_{\tau_y} \in \mathfrak{F}$ there corresponds a countable sheaf $\xi_y = \langle A_F \, : \, F \in \mathcal{F}_{\tau_y} \rangle$ and as X is α_2, we therefore obtain for each $\mathcal{F}_{\tau_y}$ a sequence B_y that meets each $A \in \xi_y$ and converges to x.

Let $\sigma : \mathcal{S}(X) \to \mathcal{O}_x$ be a counter strategy for player 1 defined recursively such that $\tau(\{y\}, \sigma(\{y\})) \in B_y \cap A_{\{y\}}$ and $\tau(F, \sigma(F)) \in B_y \cap A_F$ if $F \in \mathcal{F}_{\tau_y}$. To see that σ wins for player 1, let $\langle x_n : n \in \omega\rangle$ be a σ-sequence. Letting $F_n = \{x_1, \ldots, x_n\}$ for all $n \in \omega$ we observe that $F_1 = \{x_1\}$, $x_{n+1} \in \sigma(F_n)$, and $x_{n+1} = \tau(F_n, \sigma(F_n))$. Thus the collection $\langle F_n : n \in \omega\rangle$ forms the τ-chain $\mathcal{F}_{\tau_{x_1}}$ which implies $\langle x_n\rangle = B_{x_1}$, hence $\langle x_n : n \in \omega\rangle$ converges to x. □

5.2 Proximal Game

This game is due to Bell and is described as follows. Let $(X, \mathcal{D})$ be a uniform space. On turn zero, player 1 chooses an entourage $D_0 \in \mathcal{D}$ and player 2 chooses a point $x_0 \in X$. On the next turn, player 1 chooses $D_1 \in \mathcal{D}$ with $D_1 \subset D_0$ and player 2 chooses $x_1 \in D_0[x_0]$. The game continues and in general on the n'th turn player 1 chooses $D_{n+1} \in \mathcal{D}$ with $D_{n+1} \subset D_n$ and player 2 chooses $x_{n+1} \in D_n[x_n]$. Player 1 wins if either:

1. there exists $x \in X$ such that the $\langle x_n : n \in \omega\rangle$ converge to x;
2. $\bigcap_{n\in\omega} D_n[x_n] = \emptyset$.

Definition 5.4. A strategy for player 1 in the Proximal game is a mapping $\rho : \mathcal{S}(X) \to \mathcal{D}$ such that $x_{n+1} \in \rho(F)[x_n]$ for all $n \in \omega$ and $\rho(F) \supset \rho(G)$ if $G \subset F$. Note that $F \in \mathcal{S}(X)$ represents player 2's first n choices if $|F| = n$ and $\rho(F)$ represents player 1's choice on turn $n + 1$. We will denote $\rho(\emptyset)$ as $X \times X$.

Definition 5.5. We say that X is Proximal if player 1 has a strategy ρ such that every ρ-sequence converges. We say that X is Semi-Proximal if for every strategy for player 2, player 1 has a counter strategy that wins.

Remark 5.6. Paul explained the game to me in a slightly different way. X is assumed to be compact. On turn one player 1 begins by playing a finite open cover (FOC) $\mathcal{U}_1$ and player 2 continues by picking a point $x_1 \in X$. On the second turn player 1 chooses a FOC $\mathcal{U}_2$ such that $\mathcal{U}_2 \prec \mathcal{U}_1$ and player 2 picks a point $x_2 \in \mathrm{st}(x_1, \mathcal{U}_2)$. Then player 1 wins if either $\bigcap_{n\in\omega} \mathrm{st}(x_n, \mathcal{U}_n) = \emptyset$ or the $\langle x_n\rangle$ converge to some point.

Example 5.7. The double arrow space is $X = [0, 1] \times \{0, 1\}$ with the order topology; it is not Semi-Proximal. As X is compact, there exists a unique uniformity that generates the topology. Player 1 plays FOCs of the form $\mathcal{U} = \{(a, b] \times \{0\} \cup [a, b) \times \{1\} : a, b \in [0, 1]\}$ and player 2 picks points from alternating rows every turn. Player 2's strategy will always be viable, since the star of any point wrt to any FOC will always intersect both rows.

Proposition 5.8. If X is semi-proximal then X is α_2 Fréchet.

Proof. Suppose X is not Fréchet. Then there exists some $A \subset X$ with $p \in \overline{A}$ such that no sequence in A converges to p. Then player 2 can addopt a strategy as follows. On turn one, player 1 picks U_1 and player 2 picks $x_1 = p$. On turn 2, player 1 picks U_2 and player 2 picks $x_2 \in U_2 \cap A[p]$, which by symmetry allows player 2 to pick $p \in U_2 \cap A[x_2]$ on the tird turn. Thus is general, player 2 picks points in A such that p always remains a valid play on the next turn, and continues to pick p ever other turn. By symmetry, $p \in \bigcap U_i[x_i]$, and by assumption, the sequence chosen by player 2 cannot converge, hence player 1 loses so that X is not semi-proximal.

Similarly, if X is not α_2, then we can find can find a point p and a sheaf ξ at p such that any sequence that meets each ξ_i doesn't converge to p. As before, player 2 picks p on turn one, but on turn 2, picks $x_2 \in U_2 cap \xi_1$and in general picks $x_{2n} \in U_{2n} \cap \xi_n$. □

Proposition 5.9. If player 1 (player 2) wins (loses) in the proximal game,

Proof. Let $(X, \mathcal{D})$ be a uniform spaces and let ρ be a winning strategy for player 1 in the proximal game. Let τ be a topology on X generated by the basis $\mathcal{B} = \{D[x] : D \in \mathcal{D}, x \in X\}$, let $x \in X$, and let $\sigma : \mathcal{S}(X) \to \mathcal{O}_x$ be a strategy for player 1 in the Gruenhage game played at x defined by $\sigma(F) = \rho(F)[x]$ for all $F \in \mathcal{S}(X)$.

If $\langle x_n : n \in \omega\rangle$ is a σ-sequence, i.e., $x_{n+1} \in \sigma(\langle x_1, \ldots, x_n\rangle)$ for all $n \in \omega$, then $x_{n+1} \in \rho(\langle x_1, \ldots, x_n\rangle)[x]$ and $x_n \in \rho(\langle x_1, \ldots, x_{n-1}\rangle)[x]$. As the elements of $\mathcal{D}$ are symmetric, we have both $x \in \rho(\langle x_1, \ldots, x_n\rangle)[x_{n+1}]$ and $x \in \rho(\langle x_1, \ldots, x_{n-1}\rangle)[x_n]$, which implies $x_{n+1} \in (\rho(\langle x_1, \ldots, x_n\rangle) + \rho(\langle x_1, \ldots, x_{n-1}\rangle))[x_n]$. Thus $x_{n+1} \in \rho(\langle x_1, \ldots, x_{n-1}\rangle)[x_n]$ as $\rho(\langle x_1, \ldots, x_{n-1}\rangle) \supset \rho(\langle x_1, \ldots, x_n\rangle)$ since ρ is a well defined strategy. Hence $\langle x_n : n \in \omega\rangle$ is a ρ-sequence and $x \in \bigcap_{n\in\omega} \rho(\langle x_1, \ldots, x_{n-1}\rangle)[x_n]$, which implies that $\langle x_n \in \omega : n \in \omega\rangle$ must converge to x, which follows by our assumption that X is Hausdorff. □

Remark 5.10 (Proximal Game on Topological Groups)**.** The game is played the same but we note that the uniformities chosen by player 1 are formed as follows. On any given turn player 1 picks $U \in \mathcal{F}$ which corresponds to $D_U = \bigcup_{a \in [\omega]^{<\omega}} (a\Delta V_U)^2$. In terms of FOCs, we might can also talk about $\mathcal{U}_U = \{(a\Delta V_U) : a \in [\omega]^{<\omega}\}$. The game then proceeds as usual.