Sequential Spaces and Infinite Topological Games

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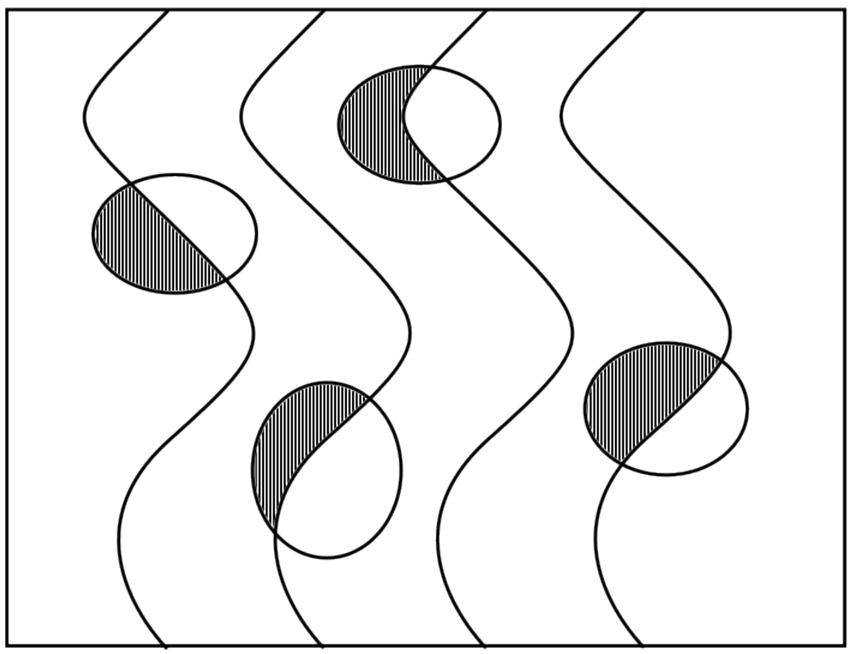

York University
August 2025

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1 Introduction

The following notes document my progress learning the topology and set theory related to the study of sequential spaces in the summer of 2025. This was made possible by a NSERC USRA award and the generous help of my advisor Paul Szeptycki who met with me frequently to answer my questions and share with me his knowledge. I learned an incredible amount and hope to continue learning more in the future. The notes are mostly incomplete since I was often jumping from one topic to another and would abruptly cut an idea short. I wouldn't be surprised if there are many mistakes as well. Also I didn't bother to write *everything* since I wasn't trying to write a self contained article. So if there's something missing its probably because its easy to find in a textbook. My goal however was to capture most of the important definitions, and figure out for myself the proofs of certain examples of properties or spaces, or in the case where I was confronted with a difficult (or cryptically written) result, rewrite the proof in my own words. Paul also shared with me some questions that were interesting to him, and I did my best to work on those as well. Maybe this will lead to some new results, which is pretty cool, but if not, ultimately I am still very happy with all that I have learned.

2 Meeting Log

Monday April 21

1. Frechet Fan - S_ω: $\omega \times (\omega+1)/\omega \times \{\infty\}$ (i.e. $\omega \times (\omega+1)$ with the points at infinity identified). Show that
   - S_ω is not first countable ✓
   - S_ω is Fréchet. ✓
2. Product of Fréchet spaces not always Fréchet: take $(\omega+1) \times S_\omega$. Let $A = \{(m,(m,n)) : m,n \in \omega$. Show that
   - $(\omega+1,\infty) \in \overline{A}$ ✓
   - No sequence in A converges to $(\omega+1,\infty)$. ✓
3. Right way to think about sequences: $A \subset X$ converges to $a \in X$ if $|A| = \aleph_0$ and for all neighboourhoods $U_x \subset X$, $|A - U_x| < \aleph_0$.
4. Right way to think about Fréchet space: take sequential closure once same as closure.
5. Another exercise: which α_i properties does S_ω have? ✓

Thursday May 1

1. Go over example 3.9; why is S_ω Fréchet? ✓
2. Give example of space that is α_2 but not α_1 ✓, α_3 but not α_2 etc.
3. Under which set theoretic assumptions do such examples exist? Look at the paper by Nyikos: Subsets of ${}^\omega\omega$ and the Fréchet Urysohn and α_i properties. Good grasp at sections 1, 2, 5; partial understanding section 3; skipped section 4, 6.
4. For example it is consistent that both α_1 and countable imply first countable, and α_2 and countable imply α_1, yet there exists a countable α_2 space that is not first countable.
5. We talked a little bit about topological groups (a topological space equipped with group operation that is continuous) and some of the nice structure they have: the group operation being continuous and invertible implies $G \times G \to G$ is always a homeomorphism, in particular aG is the homeomorphic image of G by left multiplication of $a \in G$, which in often cases lets us define a nbhd base at the identity and "send" it to all points of G in order to define the topology.

6. Look into what it takes to contribute to the pi-base, in particular add the α_i spaces. Forked database to github account; properties are stored as markdown text files, should not be difficult to add properties as well as if/then.

   Things to add:

   - α_i and if α_i then α_{i+1}
   - bisequential
   - if 1st countable then α_1 Frechet
   - if bisequential then Frechet
   - w-space iff Frechet and α_2

7. Some more exercises

   - X is α_1 and Fréchet, Y is Fréchet, then $X \times Y$ is Fréchet. this is not true, take $(\omega + 1) \times S_\omega$.
   - α_2 is equivalent to $A \cap B \neq \emptyset$ for all $A \in \xi$ whenever ξ is countable collection of sequences at x. ✓

Thursday May 15

1. When is the square of an α_i space still α_i? What α properties does $\alpha_i \times \alpha_j$ have?

2. I asked for an example of a tower, and Paul proved that towers exist, but said that theres not really specific constructions of towers.

3. historical note: the cardinals that lay between ω_1 and $\mathcal{P}(\omega)$ came from attempts to solve CH, for example, if $\mathfrak{t} \leq \mathfrak{c}$ then, finding the cardinality for $\mathfrak{t}$ would help in finding cardinality of $\mathfrak{c}$.

4. Proximal game

5. Uniformities

6. Homogeneity in Top group used to determine uniformity for player 1 in version of prox game for top groups

7. FUF and 2FUF

8. exercises:

   - prove lemmas used in existence proof of towers.
   - If $\{A_\alpha : \alpha \subset \gamma\}$ is a tower and $\{\alpha_\xi : \xi < \lambda\}$ is increasing and cofinal in γ then $A_{\alpha_\xi} : \xi < \lambda$ is a tower.
   - S_ω is neither FUF or $2FUF$ ✓
   - Proximal game is equivalent to Gruenhage game. Proximal space implies W ✓
   - Winning strategy in proximal game implies Frechet I showed W and w space imply frechet

9. note: send Paul changes to databse before pushing to github

Wednesday May 28

1. More on Uniformities

   - Different metrics generate different Uniformities, for example discrete metric vs $d(m, n) = |\frac{1}{m} - \frac{1}{n}|$ on ω, Uniformities from those metrics still generate the same (discrete) metric on ω
   - star refinements generalize triangle inequality, for example $\{\mathrm{B}(x, \frac{1}{2^{n+1}}) : x \in X\}$ star refines $\{\mathrm{B}(x, \frac{1}{2^n}) : x \in X\}$ and gives triangle inequality

2. went over "proof" that proximal implies W, however Paul raised issue that player 2 in proximal game can use Hausdorffness to always pick points z that prompt player 1 in Proximal game to choose entourages U such that $U[x] \cap U[p] = \emptyset$, where p is where Gruenhage game being played. this problem is fixed by starting Gruenhage game one round later than proximal game and player 2 pick p on turn 1 of proximal game

3. Exercises:
   - Is double arrow space Proximal?✓
   - Semi Proximal is α_2. ✓

4. Paul showed me proof that semi Proximal is Frechet

5. Malykhin's problem: If G is a seperable Frehcet topological group, is it metrizable? Answer: Con(yes) and Con(No)

6. **Paul's main question:** Can we strengthen hypotheses of Malykhin's problem to obtain answer that is not independent of ZFC?

Thursday June 5

1. Discussed impliciations of conditions on filter $\mathcal{F}$ when going from top space to top group

2. Questions:
   - If $\mathcal{F}$ is FUF, is $\tau_{\mathcal{F}}$ semiproximal? ~~no??? see prop 5.11~~
   - If $\tau_{\mathcal{F}}$ is semiproximal, is $\mathcal{F}$ countable generated?
   - Assuming CH, does there exist filters $\mathcal{F}_0, \mathcal{F}_1$ both FUF neither countably generated where $\tau_{\mathcal{F}_0}$ is semiproximal but $\tau_{\mathcal{F}_1}$ isn't.

3. Exercises:
   - Assume CH, construct FUF filter that is not countable generated. ✓
   - What does it mean for a sequence in (a_n) to converge to a point in G, i.e., $(a_n) \to a$ iff what?
   - Consider a play of the Proximal game which yields sequences (a_n) and (V_{U_n}). What does it mean for there to exist some $b \in G$ such that $b \in \bigcap_{n\in\omega}(a_n \Delta V_{U_n})$? What does that look like?

Thursday June 18

1. We talked about stars: how can we get a handle on the choices available to player 2?

2. We looked at how sequences in $\tau_{\mathcal{F}}$ behave, basically we just played with things trying to get a grasp at how things look

3. More questions:
   - In general, when is a topological group Proximal or Semi Proximal?
   - If a top group is prox then is it first countable? (or contrapositive)
   - In general what can we say about top groups and uniformities on them?

4. Check mathscinet for resources that could be helpful

Wednesday June 25

1. Due to the imlicit assumption that $\emptyset \in a\Delta V_F \forall a \in [\omega]^{<\omega} \forall F \in \mathcal{F}$ I was wrong about $\mathrm{st}(p, V_F) = [\omega]^{<\omega}$, in fact it is equal to $z\Delta V_F$ where $z = p - F$.

2. Task: ✓describe what the winning strategy for player 1 looks like in the proximal game on $\tau_{\mathcal{F}}$ when $\mathcal{F}$ is countable generated:

   - $\{F_n : n < \omega\}$ generated $\mathcal{F}$
   - Game has 3 data: (F_n, a_n, z_n), where $z_n = a_n - F_n$
   - Player 1 player $F_{n_{k+1}} \subset F_{n_k}$ such that $F_{n_{k+1}} \cap a_k = \emptyset$
   - 3 cases: $\bigcap z_n = \emptyset, < \aleph_0, = \aleph_0$

3. Prove that $\tau_{\mathcal{F}}$ is not proximal when $\mathcal{F}$ is uncountably generated ✓

4. with the above established, and a clearer picture of what $\tau_{\mathcal{F}}$ looks like, attack Paul's questions.

Monday June 30

1. "correct" way to think about strategies and counterstrategies is to not think of them as functions, but rather to think of a play of the game as an infinite tree where each branch corresponds to a possible play by either player 1 or 2: Player 1 then has a winning strategy (proximal) if every path of the tree wins and Player 2 loses (semiproximal) if there exists at leat one path that wins for Player 1.

2. The goal now is to construct an uncountable filter that is semiproximal:

   - start again with the cofinite filter
   - at stage $\alpha < \omega_1$ we have the sets added to the filter at each previous step $\mathcal{A} = \{A_\beta : \beta < \alpha\}$ as well as a countable collection of sequences $\mathcal{B} = \{B_\beta : \beta < \alpha\}$
   - still on stage α, we look at some "partial" strategy ρ for player 2 (i.e. a function defined on the elements of the filter existing so far)and look at the tree of all possible plays of the game w.r.t. ρ and extend the filter (add a new element A_α) such that there exists a bramnch of the decision tree corresponding to a a convergent sequence
   - Thhrough this process we essentially acountb for all partial sequences
   - The resulting filter is therefore semiproxiaml since we know any strategy for player 2 corresponds to a partial strategy which has a branch with a convergent seqeunces (i.e. losing for player 2) that would have been preserved after all remaining steps of construction
   - the previous item isn't exactly clear and may require more than CH, in partocular require $\diamondsuit$

3. Learn about $\diamondsuit$.

Wednesday July 2

1. Ultimately $\diamondsuit$ isn't necesaary for the construction of the uncountably generated semiproximal filter

2. start with countably generated filter, enumerate the strategies defined only on the sequences on the generateing

3. CH is enough here since any strategy on full filter will extend one of the inital strategies

4. Discussion led to Paul conjecturing that it is possible to use MA to build uncountably generated filter that isn't semi proximal

5. Learn about MA ✓

Monday July 7

1. We discussed the construction of the uncountably generated SP filter: everything seems fine
2. We talked about MA and the Baire category theorem
3. Exercise: Use MA_{ω_1} to show that uncountably generated filters have pseudo intersection. (Paul gave me much of the details I just need to verify them)
4. Once I show Paul I understand the basic ideas and constructions using MA we will be able to talk about the ideas he has regarding the uncoutnably generated not SP filter
5. Also we talked a bit about what a my poster presentation might look like for the NSERC conference: probably just give some definitions and pictures describing convergence, give some examples of sequences and neat spaces, like the Frechet fan, maybe I can describe the gruenhage game as well.

Friday July 11

1. We chatted a bit about the nice community of set theorists in Toronto. The weekly set theory seminar at the Fields institute was started by Franklin Tall which played and continues to play an important role in the community of Set Theoriests in Toronto.
2. Increasignly Paul thinks that it must be the case that FUF implies SP
3. there is a result somewhere, maybe by Nyikos, that says that if $\kappa < \mathfrak{c}$ and $\mathcal{F}$ is a κ-generated filter, then $\mathcal{F}$ is FUF.
4. Question: If player 2 has a winning strategy, then is it the case that the winning strategy can be translated so that the points picked by player 2 never "stick out" of the sets chosen by playuer 1
5. learn about γ-sets ✓
6. MA implies every $X \subset \mathbb{R}$ such that $|X| < \mathfrak{c}$ is a γ-set
7. 2^ω with the product topology is the Cantor set and can essentially be view as a subset of $\mathbb{R}$, thus $X \subset \mathbb{R}$ would be a γ-set
8. exercise: there exists $f : 2^\omega \to [0, 1]$ that is at most 2 to 1.
9. fun fact: the cantor set is constructed by deleting intervals. Howqever the endpoints of those intervals remain, for example $\frac{1}{9}$ and $\frac{2}{3}$ are in the cantor set. By identifying the enpoints of each removed interval together (for example identify $\frac{1}{3}$ with $\frac{2}{3}$) we obtain a quotient space that homeomorphic to the unit interval. Whats weird about this is that the cantor set has no interior, but by glueing those points together we recover the lost interior. Strange.
10. Two different topologies on 2^ω
    (1) an open set of $f \in 2^\omega$ looks like $U_f = \{g \in 2^\omega : f =^* g\} \cup \{g \in 2^{<\omega} : g =^* f\}$. The points of 2^ω are isolated. Note that the points of 2^ω are just given the product topology sincve they must agree at only finitely many coordinates.
    (2) the open sets of $f \in 2^\omega$ looks like $U_f(n) = \{f \upharpoonright [n, \omega)\}$ ranging over the n. Essentially taking the "final segments" of f. Note that this space is Hausdorff ($f \neq g \implies \exists m(f(m) \neq g(m))$, just take $U_f(m)$ and $U_g(m)$ as disjoint open sets). Its also locally compact, as each of the elements form a convergent space.
11. Take the 2nd topology above. This is the space we now care about. We form a kind of Ψ-space from it by taking the 1 point compactification of it. The nbhds of ∞ are those sets who are compact: removing finitely many $f \in 2^\omega$ plus the intial α levels of the tree would make an open nbhd of ∞.

12. Let $X \subset 2^\omega$ and let I_X be the collection of all sets containing finitely many $f \in X$ plus the intial α levels of the tree, i.e. sets of the form $\{A \cup 2^\alpha : A \subset X \text{ is finite}, \alpha < \omega\}$. This is an ideal. Let $\mathcal{F}_X = I_X^*$ be its dual ideal. In his paper on the Cantor tree, Nyikos showed that this filter is FUF iff X is a γ-set. does this answers point 2?

13. Let $X \subset 2^\omega$ be a γ-set and let $\tau_{\mathcal{F}_X}$ be the topological group created from $\mathcal{F}_X$ as defined above. Is $\tau_{\mathcal{F}_X}$ Semi Proximal?

Friday July 18

1. γ-sets to seems to be really important. Something to look at now is $C_p(X)$, the space of continuous real valued functions from X with the topology of pointwise converge (or $\mathbb{R}^X$ with the product topology, same thing)

2. Paul showed me one direction of proof that $C_p(X)$ is Frechet iff X is a γ-set.

3. The proof above involves an example of so called "selection" principles, which has to do with taking a collection of covers and selecting one set per cover in such a way as to produce a new cover.

4. There exists (yet another) game that has to so with ω-covers: In the n'th inning playr 1 picks $\mathcal{U}_n$, an ω-cover of X, and player 2 picks $A_n \in \mathcal{U}_n$. Player 2 wins if $\{A_n\}_{n<\omega}$ is *not* an ω-cover of X.

5. This led Paul to the conjecture that the Proximal game on a $\tau_\mathcal{F}$ might be equivalent to the ω-game.

6. In fact, due to the similarity between many of the ideas we have seen Paul thinks it might be the case for any toplogical group that frechet is equivalent to semi proximal

7. Otherwise, we talked talked about the future of the project, as the summmer is coming slowly to an end. He said that we could work throughtout the upcoming school year.

3 Set Theory

3.1 Ideals, Filters, Almost Disjoint Families

An ideal $\mathcal{I}$ on a set X is a non-empty collection of subsets such that it is closed under finite unions and the taking of finite unions and that $X \notin \mathcal{I}$. Similarly, a filter is a collection of sets closed under finite intersections and the taking of supersets and doesn't contain the empty set.

If $\mathcal{I}$ is an ideal, then we call $\mathcal{I}^*$ the dual filter, where $\mathcal{I}^* = \{X \setminus A : A \in \mathcal{I}\}$ and we call $\mathcal{I}^+$ the collection of $\mathcal{I}$-positive sets, where $\mathcal{I}^+ = \mathcal{P}(X) \setminus \mathcal{I}$.

Definition 3.1. Let $\mathcal{A}$ be an infinite family of infinite subsets of ω. The family $\mathcal{A}$ is called an almost disjoint (AD) family of subsets of ω if $A \cap B$ is finite for any $A, B \in \mathcal{A}$ with $A \neq B$.

By the Zorn-Kuratowski lemma we can assume $\mathcal{A}$ lives inside of a maximal AD family $\mathcal{A}'$, where by maximal it is meant: if $B \notin \mathcal{A}'$ then $A \cap B$ is infinite for all $A \in \mathcal{A}'$. Such a family is a maximal almost disjoint (MAD) family.

Let $\mathcal{A}$ be an AD family. We call $\mathcal{I}(\mathcal{A}) = \{X \subset \omega : X \subset^* \bigcup \mathcal{B} \text{ for some } \mathcal{B} \in [\mathcal{A}]^{<\omega}\}$ the ideal generated by $\mathcal{A}$. If it is the case that for all $X \in \mathcal{I}^+(\mathcal{A}) = \mathcal{P}(\omega) \setminus \mathcal{I}(\mathcal{A})$ there is an infinite subset $B \subset X$ such that B is almost disjoint from every $A \in \mathcal{A}$, then $\mathcal{A}$ is said to be *nowehere MAD*.

3.2 Some Interesting Cardinals

3.2.1 Subsets of ${}^\omega\omega$ and Almost Disjoint Familes

Definition 3.2. Define an order $\leq^*$ on ${}^\omega\omega$ where $f <^* g$ if $f(n) < g(n)$ for all but finitely many $n \in \omega$

Definition 3.3. Let $(A, \prec)$ be a linearly ordered set. We say that $B \subset A$ is unbounded in A if there does not exist $y \in A$ such that $x \prec y$ for all $x \in B$. We say that $C \subset A$ is cofinal (or dominating) in A if for all $x \in A$ there exists $z \in C$ such that $x \prec x$.

Definition 3.4.

- $\mathfrak{a} = \min\{|\mathcal{A}| \,:\, \mathcal{A} \text{ is infinte MAD family}\}$
- $\mathfrak{b} = \min\{|B| \,:\, B \text{ is } \leq^* \text{-unbounded subset of } {}^\omega\omega\}$
- $\mathfrak{d} = \min\{|D| \,:\, D \text{ is } \leq^* \text{-cofinal in } {}^\omega\omega\}$

Each of $\mathfrak{a}, \mathfrak{b}, \mathfrak{d}$ are at least ω_1 but at most $\mathcal{P}(\omega)$.

3.2.2 Towers and Pseudo-Intersections

Definition 3.5. Define an order on $\mathcal{P}(X)$ for some set X. We say $A \subset^* B$ if B contains all but finitely many points of A. If $\mathcal{F} \subset \mathcal{P}(X)$ is a collection of (infinite) sets, then A is a pseudo-intersection of $\mathcal{F}$ if $A \subset^* F$ for all $F \in \mathcal{F}$.

Definition 3.6. We call $\mathcal{T} \subset [\omega]^\omega$ a tower if $\mathcal{T}$ is well ordered by $\subset^*$ and has no infinite intersection.

Lemma 3.7. If $\mathcal{A} = \{A_n \,:\, n \in \omega\} \subset [\omega]^\omega$ such that for all $n \in \omega$

$$A_0 \supset^* A_1 \supset^* \ldots \supset^* A_n$$

then $\mathcal{A}$ has infintie pseudo-intersection.

Theorem 3.8. *There exists a tower.*

Proof. Define $\langle A_\alpha \,:\, \alpha < \mathfrak{c} \rangle$ as follows:

1. $A_\alpha = \omega \setminus n$ for $\alpha < \omega$
2. If α is a successor ordinal, define A_α such that $|A_{\alpha-1} \setminus A_\alpha| = \aleph_0$

3. If α is a limite ordinal then define A_α such that $A_\alpha \subset^* A_\beta$ for all $\beta < \alpha$

Suppose γ is given and $\langle A_\alpha : \alpha < \gamma \rangle$ has been defined. If γ is a successor then define A_γ as in (2). Otherwise γ is a limite ordinal: if (3) isn't possible then $\langle A_\alpha : \alpha < \gamma \rangle$ is a tower; otherwise define A_γ as in (3) and continue.

This process eventually terminates, otherwise we obtain $\mathfrak{c}$ many distint subsets of ω, a contradiction. TODO: elaborate $\square$

Definition 3.9.

- $\mathfrak{t} = \min\{|\mathcal{T}| : \mathcal{T} \text{ is a tower}\}$
- $\mathfrak{p} = \min\{|\mathcal{P}| : \mathcal{P} \subset [\omega]^\omega \text{ is a family with the SFIP that has no pseudo-intersection}\}$

3.3 Martin's Axiom

Most of the following definitions are just copied from Kunen.

Definition 3.10. A Forcing Poset $\mathbb{P}$ is a poset with maximum element $\mathbb{1}$. In other words, a kind of upside down rooted tree.

Definition 3.11. A filter on a forcing poset $\mathbb{P}$ is a set $G \subset \mathbb{P}$ such that

(1) $\mathbb{1} \in G$

(2) $p, q \in G \implies \exists r \in G$ such that $r \leq p, q$

(3) $p \in G$ and $p \leq q \implies q \in G$.

Definition 3.12. A set $D \subset \mathbb{P}$ is dense if for all $p \in \mathbb{P}$ there is $q \in D$ such that $q \leq p$.

Definition 3.13. $p, q \in \mathbb{P}$ are said to be comparable if there exists $r \in \mathbb{P}$ such that $r \leq p, q$. If no such r exists then p, q are called incomparable.

$A \subset \mathbb{P}$ has the countable chain condition (ccc) if the elements of A are pairwise incomparable.

The term antichain is quite picturesque and comes from the poset $(\mathcal{P}(X), \subset)$ where incomparable just means disjoint. So anti chain is literally the opposite of chain.

Definition 3.14 (Martin's Axiom)**.**

- MA_κ is the statement "For any forcing poset with the ccc, if $\mathcal{D}$ is collection of κ many dense sets, then there exists a filter G that meets every $D \in \mathcal{D}$."
- MA is the statement "MA_κ holds for all $\kappa < \mathfrak{c}$.

The filter in the above definition has a special name: it is called "Generic with respect to $\mathcal{D}$"

The usefulness of MA is that is allows to carry out constructions using "finite approximation". In contrast, the $\diamondsuit$ principle allows "approximate" recursion, in the sense that we can give properties to ω_1 many objects in only ω steps. MA is similar, in that we use it to build objects with ω_1 many properties, but this without recurison. Instead, we use forcing posets and familes of dense subsets (which guarantee some kind of conditions) and use the existence of generic filter to obtain our desired object.

Example 3.15. MA_{ω_1} implies that every uncountably generated filter has pseudo-intersection.

Proof. Let $\mathcal{F}$ be uncountably generated filter. Let $P = \{(s, A) : s \in [\omega]^{<\omega}, A \in [\omega]^\omega\}$ and say that $(s, A) \leq (t, B)$ if

(1) $s \supset t$

(2) $A \supset B$

(3) $s - t \subset \bigcap_{\beta \in B} F_\beta$.

Let $E_n = \{(s, A) : |s| > n\}$ and let $D_\alpha = \{(s, A) : \alpha \in A\}$. Each E_n is dense since for any $p = (s, A) \in \mathbb{P}$, if $|s| < n$ then we can pick enough points from $\omega - s$ and add them to s to obtain s' so that $|s'| > n$. THen $E_n \ni p' = (s', A) < p$. Similary, the $D'_\alpha s$ are dense since $D_\alpha \ni p' = (s, A') < p$ where $A' = A \cup \{\alpha\}$.

By MA there is a filter G in $\mathbb{P}$ such that G meets each E_n and D_α. Letting $X = \{s : \exists A((s, A))$ $\square$

3.4 Trees

Trees are similar to posets, the difference being that for any element, the set of initial segments must form a chain. In other words

Definition 3.16. A tree T is poset $(T, <_T)$ such that for any $t \in T$ the set $t^{\uparrow} = \{t' \in T : t' \leq t\}$ is a chain. We refer often to the levels of a tree where the α'th level is $\mathcal{L}_\alpha = \{t \in T : \text{type}(t) = \alpha\}$.

Example 3.17. Let $T = 2^{<\omega}$ be the set of all finite binary strings. We can identify the cantor set C with 2^ω if in the ternary expansion of any element $p \in C$ we rplace the 2s with 1s. The *Cantor Tree* is $C_T = T \cup C$. We can view an point of C as sitting "above" the chain of finite partial sequences which converges to that point in the regular sense.

There are several different ways to topologize a tree, but the most common is to use the *interval* (or *tree*) topology which has as a base all sets of the form $(s, t] = \{s < t' \leq t$ or $\{t\}$ in the case where t is a minimal element of T.

Example 3.18. Let C_T be the Cantor tree with the interval topology and consider $C_T + \infty$, its one point compactification. As usual, nbhds of ∞ look like compliments of compact sets, so in this case a nbhd will look like the compliment of a finite collection of branches or levels.

Another topology is the Cone Topology, where instead of taking the tails of branches, we instead take the "cones" of succesors. This topology has as a base sets of the form

4 Topology

4.1 Sequential and Fréchet Spaces

Definition 4.1. For a topological space X and any set $A \subset X$, the *sequential closure* of A is

$$[A]_{\mathrm{seq}} := \left\{ x \in X \ : \ \exists (x_n) \in A \left(\lim_{n \to \infty} x_n = x \right) \right\}.$$

In general we can repeat this operation recursively $[[[A]_{\mathrm{seq}}]_{\mathrm{seq}} \dots]_{\mathrm{seq}}$ by which is meant the *total sequential closure* of A.

Fact 4.2. In general it takes at most ω_1 many iterations of the sequential closure to get a closed set.

Definition 4.3. A space X is Fréchet if $[A]_{\mathrm{seq}} = \overline{A}$ for all $A \subset X$.

Example 4.4. Let $X = \omega_1 + 1$ with the order topology. X is not Fréchet, since any sequence $(x_n) \in \omega_1$ cannot converge to ∞, as otherwise $\omega_1 = \sup\{x_n \ : \ n \in \mathbb{N}\}$, a contradiction.

Example 4.5. The one point compactification of the Cantor Tree $C_T + \infty$ is Frechet.

Definition 4.6. A space X is sequential if any closed set $A \subset X$ is equal to its total sequential closure.

Then a space that is Fréchet is also sequential. The following example shows that the converse is not true.

Example 4.7. Let $X^* = \omega \times (\omega + 1)$ be given the order topology and let $X = X^* \cup \{\infty\}$ where the neighboourhoods of ∞ are such that there exists $p \in \omega$ such that $|\{(m,n) : m > p, n \in \omega{+}1\} - U_\infty| < \aleph_0$. Then X is sequential but but not Fréchet. To see this, note that for all $m \in \omega$ the sequence $A_m = \{(m,n) \ : \ n \in \omega\}$ converges to $(m, \omega + 1)$ and moreover $B = \{(m, \omega + 1) \ : \ m \in \omega\}$ is a sequence that converges to ∞. Then $A = \bigcup_{m \in \omega} A_m$ is such that $[[A]_{\mathrm{seq}}]_{\mathrm{seq}} = X$, hence X is sequential. On the other hand there is no sequence in A that converges to ∞. Suppose there were, say some $\gamma \to \infty$. Then for all $m \in \omega$, $U_m = X - \{(m,n) : n \in \omega{+}1\}$ is a neighboourhood of ∞ such that $|\gamma - U_m| < \aleph_0$. Hence γ has only finitely many terms belonging to each column. If $\alpha_m = \max\{\gamma \cap \{(m,n) \ : \ n \in \omega\}\}$, then $U = X - \bigcup_{m \in \omega} \{(m,n) \ : \ n \leq \alpha_m\}$ is a neighbourhood of ∞ disjoint from γ, a contradiction. Hence X is not Fréchet.

Proposition 4.8. If X is first countable then X is Fréchet.

Proof. Let $A \subset X$ and let $x \in \overline{A}$. Then x has a countable neighbourhood base N_x such that $U \cap A \neq \emptyset$ for all $U \in N_x$. Enumerating the neighboourhoods of x as $U_1, U_2, \dots$ then the sequence $(x_n)_{n \geq 1}$ where $x_n \in U_n \cap A$ for each $n \in \omega$ is such that $(x_n)_{n \geq 1}$ converges to x. $\square$

The following example shows that the converse is not true.

Example 4.9 (Fréchet Fan)**.** Let S_ω be the quotient of $\omega \times (\omega + 1)$ obtained by identifying all the points $\{(m, \omega + 1) \ : \ m \in \omega\}$ as ∞^*. More precisely S_ω has the quotient topology induced by the map $h : \omega \times (\omega + 1) \to S_\omega$ where $h(x) = x$ for all $x \in \omega \times \omega$ and $h(x) = \infty^*$ for all $x \in \omega \times \{\omega + 1\}$. Then S_ω is Fréchet but not 1st countable.

To see that S_ω is Fréchet, let $A \subset S_\omega$ such that $\infty \in \overline{A}$. A must meet at least one column of S_ω in an infinite set, otherwise we could find a nbhd of ∞^* disjoint from A. Then A restricted to that column will be a convergent sequence to ∞.

Now asssuming that S_ω was countable, we would have a countable neighbourhood base at ∞^*. For each $k \in \omega$ let $B_k = \bigcup_{m \in \omega} \{m\} \times (f_k(m), \infty^*]$ for some $f_k : \omega \to \omega$ determining the startpoints of each interval. Suppose $\mathcal{B} = \{B_k \ : \ k \in \omega\}$ is a base at ∞^*, then let $f^* : \omega \to \omega$ be defined by $f^*(m) = f_m(m) + 1$ for all $m \in \omega$. Letting $B^* = \bigcup_{m \in \omega} \{m\} \times (f^*(m), \infty^*]$ then B^* is an open neighboourhood of ∞^* but it is clear by construction that $B_k \not\subset B^*$ for all $k \in \omega$. Hence $\mathcal{B}$ cannot be a neighboourhood base and S_ω is not first countable.

As the following example shows, the product of Fréchet spaces need not be Fréchet.

Example 4.10. Let $X = (\omega+1) \times S_\omega$, and consider the set $A = \{(m,(m,n)) \,:\, m,n \in \omega\}$. If ∞^* is the identified point of S_ω, let $\infty = \{\omega+1\} \times \infty^*$. Then $\infty \in \overline{A}$ but $\infty \notin [A]_{\text{seq}}$. The open neighboourhoods of ∞ are are of the form $(\alpha,\omega+1] \times \left(\bigcup_{m\in\omega}\{m\} \times (f(m),\infty^*]\right)$, which clearly always has non emtpy intersection with A. Hence $\infty \in \overline{A}$. To see that $\infty \notin [A]_{\text{seq}}$, suppose γ is a sequence in A that converges to ∞. Since the sets $U_k = (k,\omega+1] \times \left(\bigcup_{m\in\omega}\{m\} \times (f(m),\infty^*]\right)$ are open neighbourhoods of ∞ it must be the case that $|\gamma - U_k| < \aleph_0$ for all $k \in \omega$. Thus $\gamma \cap (\{k\} \times \{k\} \times (1,\omega+1])$ is finite for every k. Let $h : \omega \to \omega$ be defined by $h(k) = \max\{\pi_3(\gamma \cap (\{k\} \times \{k\} \times (1,\infty^*])\} + 1$ for $k \in \omega$. Pictorially, h is picking the point on each spine beyond which no elements of γ exist. Thus

$$W = (1,\omega+1] \times \left(\bigcup_{n\in\omega}\{n\} \times (h(n),\infty^*]\right)$$

is an open neighbourhood of ∞ which by construction is disjoint from γ. Hence γ cannot converge to ∞ showing that X is not Fréchet.

4.2 α_i notions of convergence

Definition 4.11. Let X be a topological space and ξ be a countable family of sequences converging to a point $x \in X$. We say that x is an α_i point for $i = 1,2,3,4$ if there exists a sequence B such that

- α_1: $|A - B| < \aleph_0$ for every $A \in \xi$;
- α_2: $|A \cap B| = \aleph_0$ for every $A \in \xi$;
- α_3: $|A \cap B| = \aleph_0$ for infinitely many $A \in \xi$;
- α_4: $A \cap B \neq \emptyset$ for infinitely many $A \in \xi$.

Then X is an α_i space if every $x \in X$ is an α_i point. Note that if a space is α_i then it is α_{i+1} for $i = 1,2,3$.

Example 4.12. S_ω is not even α_4. For each $m \in \omega$ let $A_m = \{m\} \times (1,\infty)$. Then $\xi = \{A_m \,:\, m \in \omega\}$ is a countable collection of sequences converging to ∞. Suppose B is a sequence that converges to ∞ such that $A \cap B \neq \emptyset$ for infinitely many $A \in \xi$. In particular let $\alpha < \omega$ be such that $A_i \cap B \neq \emptyset$ for all $i \in \alpha$ and let $f : \alpha \to \omega$ be defined by $f(k) \in B \cap A_k$ for all $k \in \alpha$. Then

$$U = \left(\bigcup_{k\in\alpha}\{k\} \times (f(k)+1,\infty]\right) \times \left(\bigcup_{m\in\omega-\alpha}\{m\} \times (1,\infty]\right)$$

is such that $|B \cap U^C| = \aleph_0$, hence B does not converge to ∞.

Exercise 4.13. X is α_2 iff whenever ξ is a countable collection of sequences converging to x there exists $B \to x$ such $A \cap B \neq \emptyset$ for all many $A \in \xi$.

Solution. The forward direction is obvious. Conversely, let $\xi = \{A_1, A_2, \dots\}$ be a countable collection of sequences converging to x. For every $n \in \omega$ let $\{a_{nm} \,:\, m \in \omega\}$ enumerate the elements of A_n and define $A_{nm} = A_n - \{a_{n1}, a_{n2}, \dots, a_{n,m-1}\}$ for each $n,m \in \omega$. Then the A_{nm} still convege to x and $\mathcal{A} = \{A_{nm} \,:\, n,m \in \omega\}$ is a sheaf at x. By hypothesis there exists a B converging to x that meets each A_{nm}, and thus meets each A_n in an infinite set.

4.3 Examples of Spaces

4.3.1 Ψ spaces

Definition 4.14. Let D be a countable set and let $\mathcal{A}$ be an AD family on D. Define a topology on $X = D \cup \mathcal{A}$ such that the points of D are isolated and the nbhds of any point $A \in \mathcal{A}$ looks like $B(A,F) = \{A\} \cup (A - F)$ where $F \subset A$ is finite. Such a space is called a Ψ space (or also a Mrowka space) and is denoted $\Psi(D,\mathcal{A})$, or just $\Psi(\mathcal{A})$ if D is clear from context.

Proposition 4.15. Let X be as above. Then X is Hausdorff, first countable and locally compact. If X is constructed as above but $\mathcal{A}$ is MADF then X is also pseudo-compact (i.e. all continuous real functions on X are bounded).

Proof. To check Hausdorffness it suffices to show $Z = X - D$ is Hausdorff. Let $z_1, z_2 \in Z$. Then z_1, z_2 correspond to sets A_1, A_2 such that $A_1 \cap A_2 = I$ is finite. Then $B(A_1, I)$ and $B(A_2, I)$ are disjoint open sets containing z_1, z_2 respectively.

To see that X is locally compact, note that $\{x\}$ is a compact neighbourhood base for all $x \in D$. Local compactness of Z follows from the fact that by design each $z_A \in Z$ is the limit of A viewed as a sequence. In other words, whenever $B(A_z, F)$ is a basic nbhd of z_A, any open cover will have a smaller nbhd $B(A_z, F')$ that omits only finitely many points. □

The one point compactification of a Ψ space $\Psi(\mathcal{A})$ is called Franklin Compactum and is denoted $F(\mathcal{A})$.

Proposition 4.16. Let $\mathcal{A}$ be an AD family and let $F(\mathcal{A})$ be its Franklin Compactum. The following are equivalent:

a) $F(\mathcal{A})$ is Fréchet.

b) $\mathcal{A}$ is nowhere MAD.

c) $|F(\mathcal{A})| < \mathfrak{a}$.

In 4.10 was given an example of a space that is the product of two Frechet spaces yet not Frechet. However one of the factors, S_ω, is not compact since the cover consisiting of the sets $U_n = S_\omega - \{(m, 0) : m \in \omega, m \neq n\}$ has no finite subcover. Simon's "Barely Frechet" Compacta is serves as interesting example for two reasons: (1) its is the product of two compact Frechet spaces that is not Frechet, and (2) both factors are α_4 but not α_3. First a Lemma:

Lemma 4.17. Let $\mathcal{A}$ be a MAD family. There exists a partition of $\mathcal{A}$ such that $\mathcal{A} = \mathcal{A}_0 \sqcup \mathcal{A}_1$ such that both $\mathcal{A}_1$ and $\mathcal{A}_2$.

Example 4.18. Let $\mathcal{A}$ be a MAD family and partition it into two pieces $\mathcal{A}_0$ and $\mathcal{A}_1$ as in the lemma. Then $F(\mathcal{A}_i)$ is Frechet by 4.16 and α_4 but not α_3 for $i = 0, 1$. Simon's "Barely Frechet" Compacta is the product $X = F(\mathcal{A}_0) \times F(\mathcal{A}_1)$ and it is not Frechet.

4.3.2 The α_i Properties are distinct

We have already seen examples of spaces that are either α_1 or not even α_4. Also, in the construction of Simons "Barely Frechet" Compactum we encountered a space that is α_4 but not α_3. The following is a space that is α_3 but not α_2.

Proposition 4.19. Let $C_T + \infty$ be the one point compactification of the Cantor Tree. It is α_3 but not α_2.

Proof. Let D be a countable dense subset of the Cantor set (it exists since seperability is hereditary in metric spaces). If $d \in D$, let d_n be the partial sequence in T of length n so that $\langle d_n : n < \omega \rangle$ converges to d. For each $d \in D$ recursively define a sequence $\langle x_n : n < \omega \rangle$ where $x_0 = d_0$ and $x^d_{n+1} = d_n{}^\frown(-d_{n+1})$ where $-d_i = 1$ iff $d_i = 0$. So each x^d_n is the succesor of d_n that went down the other branch. Now the Cantor set set is first category and so the Baire Category Theorem applies, we use this fact to show that if σ is a sequence converging to ∞ then there is at least one d for which $\sigma \cap x^d_n < \aleph_0$. For each n let

$$A_n = \{t \in C : V(t, n) \cap \sigma = \emptyset\} \quad \text{where } V(t, n) = t^\uparrow \cap \left(\bigcup_{\alpha > n} \mathcal{L}(\alpha) \right).$$

Note that $p \in C \setminus \bigcup A_n$ would imply that $\sigma \to p$ which contradicts that C_T is Hausdorff, hence $C = \bigcup A_n$. The BCT implies that not all the A_n can be nowhere dense, so there is an inteval $J \subset C$ and an n such that A_n is dense in A_n. Now for any $d \in D \cap J$, $\sigma \cap x^d_n$ must be finite since they can meet only above the n'th level of C_T.

□

4.3.3 Squares of α_i Spaces

Example 4.20. If X, Y are α_1, then so is $X \times Y$. Let $\xi = \langle \xi_n : n \in \omega \rangle$ be a sheaf at some $x \times y \in X \times Y$. Then $\pi_i(\xi) := \langle \pi_i(\xi_n) : n \in \omega \rangle$ is a sheaf at $\pi_i(x \times y)$ in X, Y respectively for $i = 1, 2$. Hence there exists α_1 sequences B_x, B_y for $\pi_i(\xi)$. Letting $M_n = \max\{k \in \omega : x_{n,k} \notin \pi_1(\xi_n) \vee y_{n,k} \notin \pi_2(\xi_n)\}$ for each $n \in \omega$, the sequence $B = \bigcup_{n\in\omega} B_x \cap (\xi_n - x_{n1}, \ldots, x_{n,M_n}) \times B_y \cap (\xi_n - y_{n1}, \ldots, y_{n,M_n})$ is as desired.

4.3.4 Misc α_i Spaces

Example 4.21. Let $D = \omega \times \omega$, let $\mathcal{F} = \{f_\alpha : \alpha < \mathfrak{b}\} \subset {}^\omega\omega$ be $<^*$-well-ordered and $<^*$-unbounded, and let $\mathcal{C} = \{C_n = \{n\} \times \omega : n \in \omega\}$. Construct a Ψ-like from $X = \mathcal{F} \cup \mathcal{C}$, where we attach compactification points $\{n\} \times (\omega + 1)$ for each C_n and compactification points p_α for each f_α. Then the resulting one point compactification, $X + \infty$, is α_2 and Fréchet but not α_1. $X + \infty$ becomes α_1 if $\mathcal{F}$ is cofinal and well ordered.

4.4 Frechet-Urysohn for Finite Sets

Definition 4.22. A π-network at a point $x \in X$ is a collection $\mathcal{F} \subset \mathcal{P}(X)$ such that for all open nbhds U_x there is $F \in \mathcal{F}$ such that $F \subset U_x$.

Definition 4.23. We say that an infinite family of sets $\langle F_n \subset \mathcal{P}(X) : n \in \omega \rangle$ converges to a point $x \in X$ if $|\{F_n : F_n \not\subset U_x\}| < \aleph_0$ for every open nbhd U_x.

Definition 4.24. X is Fréchet-Urysohn Finite (FUF) if for all $x \in X$ whenever $\mathcal{F} \subset [X]^{<\omega}$ is a π-network at x there exists $\langle F_n : n \in \omega \rangle \subset \mathcal{F}$ such that the F_n converge to x. We say that X is nFUF if the condition holds for all π-networks $\mathcal{F} \subset [X]^n$ and we say that X is boundedly FUF if X is nFUF for all $n \in \omega$.

Example 4.25. S_ω is not 2FUF (and hence not FUF). Let $F^n_m = \{(1, m), (m + 1, n)\}$ and let $\mathcal{F} = \{F^n_m : m, n \in \omega\}$. Then $\mathcal{F}$ is a π-network at ∞ with no convergent subcollection.

To see that $\mathcal{F}$ is a π-network, let U be an open nbhd of ∞. Then $U = \bigcup_{k\in\omega}\{k\} \times (f(k), \infty]$ for some $f \in {}^\omega\omega$. Letting $m = f(1) + 1$ and $n = f(m) + 1$, then $F^n_m \subset U$.

Suppose $\mathcal{G} \subset \mathcal{F}$ converges to ∞. Then since $V_n = S_\omega - (\{1\} \times (1, n])$ is an open nbhd of ∞ for each $n \in \omega$, we see that the sets $\{G \in \mathcal{G} : G \not\subset V_n\}$ are each finite. In particular, this shows that there are only finitely many $G \in \mathcal{G}$ that meet each column, hence for each $n \in \omega$ there exists $a_n \in \omega$ such that for all $b \geq a_n$, $F^b_n \notin \mathcal{G}$. Letting $g : \omega \to \omega$ be defined by $g(n) = a_n$ we obtain the open nbhd $W = \bigcup_{n\in\omega}\{n\} \times (g(k), \infty]$ which is such that $G \notin W$ for all $G \in \mathcal{G}$, i.e. $\{G \in \mathcal{G} : G \not\subset W\}$ is infinite so that $\mathcal{G}$ does not converge to ∞.

Proposition 4.26. If X is first countable then X is FUF.

Proof. Suppose X is first countable, let $x \in X$, and let $\mathcal{F}$ be a π-network at x. Let $\mathcal{U} = \{U_0, U_1, \ldots\}$ be a nbhd base at x and WLOG assume that $U_0 \supset U_1 \supset \ldots$ (since otherwise we could just take $U_0, U_0 \cap U_1, \ldots$). Then for each $n \in \omega$ there is a $\mathcal{F} \ni F_n \subset U_n$ and the F_n converge to x. □

Definition 4.27. $\mathcal{A} \subset \mathcal{P}(X)$ is a network if for all $x \in X$ and for all nhds $U \ni x$ there is $A \in \mathcal{A}$ such that $x \in A \subset U$. If $\mathcal{F}$ is a filter, then we might also refer to a network $\mathcal{A}$ for $\mathcal{F}$ where every $F \in \mathcal{F}$ contains some $A \in \mathcal{A}$.

Remark 4.28. A network is a generalization of a base since we no longer require the elements to be open sets.

Definition 4.29 (FUF filters)**.** A filter $\mathcal{F}$ on ω is FUF if whenever $A \subset [\omega]^{<\omega}$ is a network for $\mathcal{F}$ then there exists $B \subset A$ such that B converges wrt to $\mathcal{F}$.

Lemma 4.30. Let $\mathcal{F}$ be a filter on $[\omega]^\omega$. If $\mathcal{F}$ is countably generated then $\mathcal{F}$ has pseudo-intersection.

Proposition 4.31. Assume CH. Then there exists a FUF filter $\mathcal{F}$ that is not countably generated.

Proof. Starting with $\mathcal{F}_0$ the cofinite filter on ω, we will recursively construct a filter by adding a new set to our filter at each stage such that the FUF condition is preserved. We assume that $\mathcal{P}([\omega]^{<\omega}) = \{A_n : n < \omega_1\}$ and will construct our filter in ω_1 many steps.

On step 0, take $A_0 \subset [\omega]^{<\omega}$. If it's a network for $\mathcal{F}_0$ then we find $B_0 \subset A_0$ such that $B_0 \to \mathcal{F}_0$, which exists since ω is first countable. Otherwise, if A_0 isn't a network let B_0 be the empty sequence and proceed to the next step.

On step α, let $\mathcal{F}^*_\alpha = \mathcal{F}_0 \cup \left(\bigcup_{i<\alpha}\{X_i\}\right)$ and let $\mathcal{F}_\alpha = F^*_\alpha \cup \{X_\alpha\}$ where X_α will be defined in such a way as to preserve any previous convergent sequences, which we denote $\{B_\beta : \beta < \alpha\}$. Note that $\mathcal{F}^*_\alpha$ is countably generated, as $\mathcal{F}_0$ is countably generated and we have progressed only countably many steps. So we have $F_0 \supset F_1 \supset \dots$ which generate $\mathcal{F}^*_\alpha$. Let $X_\alpha = \bigcup_{i<\alpha} F_i \cap (\cup B_i)$. Since each F_i contains a tail of every B_β, we have that $X_\alpha \supset^* B_\beta$ for all β. On the other hand, none of the F_n generate X_α since $F_1 \cap B_1 \not\supset F_k$ for all $k \geq 1$ and in general $F_n \cap B_n \not\supset F_k$ for all $n \in \omega, k \geq n$. Hence $X_\alpha \notin \mathcal{F}^*_\alpha$ and $\mathcal{F}_\alpha$ is as desired.

We continue step α by taking $A_\alpha \subset [\omega]^\omega$. If it's a network for $\mathcal{F}_\alpha$, find another sequence $B_\alpha \subset A_\alpha$ that converges with respect to $\mathcal{F}_\alpha$, otherwise if A_α isn't a network let B_α be the empty sequence proceed to the next step.

Let $\mathcal{F}$ be the filter obtained after recursively iterating through all $A_n \subset [\omega]^{<\omega}$, i.e. $\mathcal{F} = \mathcal{F}_0 \cup \left(\bigcup_{\alpha<\omega_1}\{X_\alpha\}\right)$. By construction $\mathcal{F}$ is FUF and to see that its uncountably generated let $\mathcal{G} \subset \mathcal{F}$ be a generating set. Then for some $\alpha < \omega_1$ there's $\mathcal{G}_\alpha \subset \mathcal{G}$ which generates $\mathcal{F}^*_\alpha$. But $\mathcal{F}^*_\alpha$ was also generated by $F_0 \supset F_1 \supset \dots$ and X_α was such that $X_\alpha \not\supset F_n$ for all $n \in \omega$. Since the F_n would necessarily also generate $\mathcal{G}_\alpha$ this implies that $X_\alpha \not\supset G_n$ for all $n \in \omega$. Hence $\mathcal{G}_\alpha$ generates $\mathcal{F}^*_\alpha$ but not $\mathcal{F}_\alpha$. Furthermore, $\mathcal{G}_\alpha$ cannot generate $\mathcal{F}_\eta$ either, for $\alpha < \eta < \omega_1$, which follows from the definition of the X_η. Thus as $\mathcal{F}$ is obtained from F_α by adding uncountably many new sets, each of which not generated by $\mathcal{G}_\alpha$, it must be the case that $\mathcal{G}$ is uncountable. $\square$

4.5 Uniformities

In general uniformities provide an alternate way to generalize metric spcaes. Topological spaces generalize metric spaces while preserving the notion of continuous functions whereas uniform spaces preserve uniformly continuous functions. Given a set X there are two ways to contruct a uniformity $\mathcal{D}$ on X.

4.5.1 Diagonal Uniformities

A diagonal uniformity on a set X is a principal filter $\mathcal{D}$ on the square $X \times X$ generated by the diagonal satisfying

1. For all $D \in \mathcal{D}$ there exists $E \circ E \subset D$ for some $E \in \mathcal{D}$
2. $D^{-1} \in \mathcal{D}$ for all $D \in \mathcal{D}$

where $E \circ E = \{(x, y) \in E \times E : \exists z \in E\,((x, z) \in E \wedge (z, y) \in E)\}$ and $D^{-1} = \{(y, x) : (x, y) \in D\}$. Note that $E \circ E$ will often be abreviated E^2 and we may at times go so far as to use $E^4 = (E^2)^2 = E \circ E \circ E \circ E$. Also, whenever $D \in \mathcal{D}$ is such that $D = D^{-1}$ we will say that D is symmetric. A diagonal uniformity can be generated by filterbase consisting of symmetric sets D each containing the diagonal. The pair $(X, \mathcal{D})$ will be called a uniform space.

If $(X, \mathcal{D})$ is a uniform space then for any $x \in X$ and $D \in \mathcal{D}$ we say that $D[x] = \{y \in X : (x, y) \in D\}$ is the section of D at x. More generally, we can also talk about $D[A] = \{y \in X : (x, y) \in D, x \in A\}$. Although the former is used quite frequantily because the collection $\tau_{\mathcal{D}} = \{D[x] : x \in X, D \in \mathcal{D}\}$ is a base for a topology on X, and the most natural way to pass from uniform to topological spaces.

4.6 Covers and Selection Principles

Definition 4.32. Let X be a space and let $\mathcal{U}$ be open cover of X.

a) If every finite subset of X is contained in some element of $\mathcal{U}$, then $\mathcal{U}$ is called an ω-cover.

b) If every every point of X is contained in all but finitely many elements of $\mathcal{U}$, then $\mathcal{U}$ is called an γ-cover

c) If every ω-cover on X has γ-subcover then X is called a γ-set.

The definitions of ω/γ-covers and γ-sets were introduced by Gerlitz and Nagy in their paper that investigated properties of $C_p(X)$ (the space of continuous real valued functions from X with the topology of pointwise convergence). We can also talk about n-covers (where each set of cardinality n is contained in an element of the cover): Then a cover is an ω-cover iff it is an n-cover for all n. Thus the term ω-cover is analagous to the term FUF, since the FUF property holds on X iff X is n-FUF for all n. The term γ-set/cover gets its namesake from the fact that it was initially listed as property γ from a list of properties α through ϵ.

One of the intersting results from that paper is that $C_p(X)$ is Frechet iff X is a γ-set. It required the following definition and lemma.

Definition 4.33 (Strictly Frechet)**.** Let $p \in X$ and suppose $A_n \subset X$ with $p \in \overline{A_n}$ for all $n \in \omega$. If there exists points $a_n \in A_n$ for all n such that $a_n \to p$ then X is striclty Frecht at p. If this property holds at all $p \in X$ then X is Strictly Frechet.

Note that if $p \in \overline{A}$, then by taking the sequence $A_n = A - \{a_1, \ldots, a_{n-1}\}$ in the definition of Striclty Frechet we obtain a sequence $\langle a_n : n < \omega \rangle \subset A$ that covnerges to p. Thus Strictly Frechet implies Frechet.

Lemma 4.34. If X is a γ-set, then whenever $\langle \mathcal{U}_n : n < \omega \rangle$ is a sequence of ω-covers of X there exists a sequence $\langle A_n : n < \omega, A_n \in \mathcal{U}_n \rangle$ that is a γ-cover of X

Proposition 4.35. If X is normal, then $C_p(X)$ is Frechet iff X is a γ-set.

Proof. If $C_p(X)$ is Frechet, let $\mathcal{U}$ be an ω-cover of X and let

$$\Phi = \{f \in C_p(X) : \text{ there exists } U \in \mathcal{U} \text{ such that if } |f(x)| < 1 \text{ then } x \in U\}.$$

Let $\mathbf{0}$ be the constant zero function, and let V be a nbhd of $\mathbf{0}$, which is of the form $V = \{f : |f(x)| < \varepsilon \, \forall x \in F\}$ for $\varepsilon > 0$ and F a finite (hence closed) subset of X. There exists a $U \in \mathcal{U}$ that contains F, and by normality there is $f \in C_p(X)$ where

$$f(x) = \begin{cases} 0 & x \in F \\ 1 & x \in X - U. \end{cases}$$

So $f \in \Phi \cap V$ which implies that $f \in \overline{\Phi}$. By Frechetness of $C_p(X)$ there exists $\langle f_n \in \Phi : n < \omega \rangle$ that converges to $\mathbf{0}$. Pick $U_n \in \mathcal{U}$ such that $|f_n(x)| < 1$ implies $x \in U_n$. The pointwise convergence of f_n implies that $\langle U_n : n < \omega \rangle$ is a γ-set.

On the other hand, suppose that X is a γ-set. We show $C_p(X)$ is Strictly Frechet. Pick $\Phi_n \in C_p(X)$ such that $\mathbf{0} \in \overline{\Phi}$ for $n \in \omega$. For each n let $\mathcal{U}_n = \{\{x \in X : |f(x)| < \frac{1}{2^n}\} : f \in \Phi_n\}$ and note that $\mathcal{U}_n$ is an ω-cover. Indeed, for $F \subset X$ finite, the nbhd $V(F, \frac{1}{2^n})$ of $\mathbf{0}$ meets Φ_n (since $\mathbf{0} \in \overline{\Phi_n}$) so that $F \in \mathcal{U}_n$. By the above lemma, we can choose $A_n \in \mathcal{U}_n$ such that $\langle A_n \in \mathcal{U}_n : n < \omega, \rangle$ that is a γ-cover of X. Each A_n is of the form $\{x \in X : |f(x)| < \frac{1}{2^n}\}$ for $f \in \Phi_n$ hence there is $f_n \in \Phi_n$ such that $f_n(x) < \frac{1}{2^n}$ for all $x \in A_n$. Then the f_n converge to $\mathbf{0}$. $\square$

Remark 4.36. $C_p(X)$ is a topological group and thus homogeneous. Thus we can assume without loss of generality that any convergent sequence is a seqeunce converging to $\mathbf{0}$. This is why in the above proof we look only at $\mathbf{0}$.

Lemma 4.34 is an example of a *Selection Principle*. In fact it is a specific case of the selection principle $S_1(\mathcal{A}, \mathcal{B})$ which says that if $\mathcal{A}$ and $\mathcal{B}$ are collections of covers, given a sequence of covers$\langle A_n : n < \omega \rangle$ in $\mathcal{A}$ there exists an exists a set $\{B_n \in A_n : n < \omega\}$ that is an element of $\mathcal{B}$. Thus, if Ω (Γ) denotes the set of all ω-covers (γ-covers), then Lemma 4.34 can be restated: "If X is a γ-set then X has the property $S_1(\Omega, \Gamma)$". The topic of selection principles has been covered extensively by M. Scheepers in a series of papers (at least 8) titled "The Combinatorics of Open Covers". Other families of covers that are studied are $\mathcal{O}$ and Λ, the classes of all open covers and "Large" covers. (Large means each point is in infinitely many elements of the cover).

4.7 Topological Groups

Definition 4.37. A topological group G is a group whose underlying set has a topology such that

- the group operation is continuous;
- the map that sends $x \mapsto x^{-1}$ is continuous.

For sets $A, B \subset G$ and any point $g \in G$ we denote $AB = \{a \cdot b \,:\, a \in A, b \in B\}$ and $gA = \{g \cdot a \,:\, a \in A\}$.

We will often construct topological groups from topological space. Take $X = \{\infty\} \cup \omega$ with the points of ω isolated and let $\mathcal{F}$ a filter on ω. Then $G = [\omega]^{<\omega}$ with the operation symmetric difference $\Delta : G \times G \to G$ defined by $a\Delta b = a - b \cup b - a$ and $\emptyset$ the identity will be our group. For each $U \in \mathcal{F}$ set $V_U \subset [\omega]^{<\omega}$ where $V_U = \{a \in [\omega]^{<\omega} : a \subset U\}$. Then $\mathcal{V} = \{V_U : U \in \mathcal{F}\}$ is a nbhd base at $\emptyset$ and $\tau_{\mathcal{F}} = \{a\Delta V_U : a \in [\omega]^{<\omega}, U \in \mathcal{F}\}$ is a base for a topology on G.

Proposition 4.38. If $\mathcal{F}$ is FUF then $\tau_{\mathcal{F}}$ is Frechet.

Proof. Let $\mathcal{F}$ be FUF and let $A \subset \tau_{\mathcal{F}}$ be such that $x \in \overline{A}$. WLOG assume that $x = \emptyset$. Then $\mathcal{A} = \{a \in A \cap [F]^{\leq\omega} : F \in \mathcal{F}$ is a network for $\mathcal{F}$ and hence there is a $B \subset \mathcal{A}$ such that $B \to \mathcal{F}$, i.e., $\underline{B \to x}$.

Conversely, let $\mathcal{A}$ be a network for $\mathcal{F}$ and let $[\mathcal{A}]^{<\omega} = \{[a]^{<\omega} : a \in \mathcal{A}\}$. Then $\emptyset \in \overline{[\mathcal{A}]^{<\omega}}$ since there is $a \subset F$ implies $[a]^{<\omega} \subset [F]^{<\omega}$ hence $[\mathcal{A}]^{<\omega} \cap [F]^{<\omega} \neq \emptyset$ for all $F \in \mathcal{F}$. Thus there is a $B \subset [\mathcal{A}]^{<\omega}$ that converges to $\emptyset$ and hence $B \to \mathcal{F}$. $\square$

Proposition 4.39. $\mathcal{F}$ is countably generated $\iff$ $\tau_{\mathcal{F}}$ is secound countable $\iff$ $\tau_{\mathcal{F}}$ is metrizable.

Example 4.40. If $\mathcal{F}$ is the cofinite filter on ω then in the group $\tau_{\mathcal{F}}$ the sequence defined by $x_n^m = \{n, n+1, \ldots, n+m\}$, for any $m < \omega$, converges to $\emptyset$. Letting $m = 1$ and using the homogeneity of $\tau_{\mathcal{F}}$ we obtain a simple recipe to construct sequences converging to any point $p \in [\omega]^{<\omega}$; just take $y_n = x_n^1 \Delta p$ for all $n \in \omega$.

Example 4.41. Recall C_T the Cantor Tree and $C_T + \infty$ its one point compactification. Take $A \subset C$ and let $T \cup \{\infty\}$ be a subspace of $T \cup A + \infty$. Denote by τ_A the topological group (made in the same way as before) with the points being finite subsets of T and the base consisting of all finite subsets of the nbhds of ∞ in $T \cup \{\infty\}$.

Proposition 4.42. Let $A \subset C$ and consider τ_A as above. Then τ_A is Frechet iff A is a γ-set.

5 Infinite Games

5.1 Gruenhage Game

Let $x \in X$ be a point chosen before the start of game. On turn zero, player 1 chooses an open nbhd U_0 of x and player 2 chooses a point $x_0 \in U_0$; on the n'th turn player one chooses an open nbhd U_n of x and player 2 chooses a point $x_n \in U_n$. Player 1 wins if the sequence of points $\{x_n \,:\, n \in \omega\}$ converges to x.

Formally, we can describe a strategy for player 1 as a mapping $\sigma : [X]^{<\omega} \to \mathcal{O}_x$ from finite sequences in X to the collection of open sets containing x where $\sigma(\emptyset) = X$. In particular we say that $\langle x_n \,:\, n \in \omega\rangle$ is a σ-sequence provided $x_{n+1} \in \sigma(\{x_1, \ldots, x_n\})$ for all n. We say that σ is a winning strategy if every σ-sequence converges to x.

A strategy for play 2 is a mapping $\tau : \mathcal{S}(x) \times \mathcal{O}_x \to X$ such that $\tau(F, U) \in U$ for all $F \in \mathcal{S}(x)$ and $U \in \mathcal{O}_x$.

Definition 5.1. We say that X is a W-space if there is a winning strategy for player 1 at every point $x \in X$. We say that X is a w-space if for every strategy for player 2 there exists a counterstrategy for player 1 that wins.

Definition 5.2. Let $x \in X$ and let $\tau : \mathcal{S}(X) \times \mathcal{O}_x(X) \to X$ be a strategy for player 2 at x. Say that $\mathcal{F}_\tau = \langle F_n \,:\, n \in \omega\rangle$ is a τ-chain if for all $n \in \omega$ we have $F_n \subset F_{n+1}$ and $F_{n+1} = F_n \cup \{\tau(F_n, U)\}$ for some $U \in \mathcal{O}_x$.

Proposition 5.3. X is a w-space iff it is α_2 and Fréchet.

Proof. The forward direction is similar to above result, with the difference that we define a strategy τ for player 2 such that $\tau(F, U_x) \in U_x \cap A$ on the one hand, and $\tau(F, U_y) \in U_y \cap \xi_{|F|}$ on the other hand; then since X is a w-space, we obtain a counter strategy that wins for player 1, i.e., the desired convergent sequence.

Now suppose X is α_2 and Fréchet and let $\tau : \mathcal{S}(X) \times \mathcal{O}_x(X) \to X$ be a strategy for player 2 at x. For each $F \in \mathcal{S}(X)$ let $\tau_F = \{\tau(F, U) : U \in \mathcal{O}_x\}$ and note that $x \in \overline{\tau_F}$ for all F. By Fréchetness, each τ_F contains an A_F that converges to x and $\xi = \langle A_F : F \in \mathcal{S}(X)$ is a sheaf at x. Let B be a sequence that converges to x which meets each A_F and let $\sigma : [X]^{<\omega} \to \mathcal{O}_x$ be a counterstrategy for player 1 such that $\tau(F, \sigma(F)) \in B \cap A_F$ for all $F \in [X]^{<\omega}$. Then σ wins for player 1. This assumes X is countable.

Suppose X is α_2 and Fréchet and let $x \in X$ be a point. If $\tau : \mathcal{S}(X) \times \mathcal{O}_x(X) \to X$ is a strategy for player 2 at x, then for each $F \in \mathcal{S}(X)$ let $\tau_F = \{\tau(F, U) : U \in \mathcal{O}_x\}$ and note that $x \in \overline{\tau_F}$, which by Fréchetness of X, implies the existence of a $A_F \subset \tau_F$ that converges to x for each $F \in \mathcal{S}(X)$. Let $\mathfrak{F}$ be a collection of τ-chains such that for each $y \in X$ there exists exactlly one $\mathcal{F}_{\tau_y} \in \mathfrak{F}$ such that $\{y\} \in \mathcal{F}_{\tau_y}$ (AC). Observe that to each $\mathcal{F}_{\tau_y} \in \mathfrak{F}$ there corresponds a countable sheaf $\xi_y = \langle A_F : F \in \mathcal{F}_{\tau_y}\rangle$ and as X is α_2, we therefore obtain for each $\mathcal{F}_{\tau_y}$ a sequence B_y that meets each $A \in \xi_y$ and converges to x.

Let $\sigma : \mathcal{S}(X) \to \mathcal{O}_x$ be a counter strategy for player 1 defined recursively such that $\tau(\{y\}, \sigma(\{y\})) \in B_y \cap A_{\{y\}}$ and $\tau(F, \sigma(F)) \in B_y \cap A_F$ if $F \in \mathcal{F}_{\tau_y}$. To see that σ wins for player 1, let $\langle x_n : n \in \omega\rangle$ be a σ-sequence. Letting $F_n = \{x_1, \ldots, x_n\}$ for all $n \in \omega$ we observe that $F_1 = \{x_1\}$, $x_{n+1} \in \sigma(F_n)$, and $x_{n+1} = \tau(F_n, \sigma(F_n))$. Thus the collection $\langle F_n : n \in \omega\rangle$ forms the τ-chain $\mathcal{F}_{\tau_{x_1}}$ which implies $\langle x_n\rangle = B_{x_1}$, hence $\langle x_n : n \in \omega\rangle$ converges to x. $\square$

5.2 Proximal Game

This game is due to Bell and is described as follows. Let $(X, \mathcal{D})$ be a uniform space. On turn zero, player 1 chooses an entourage $D_0 \in \mathcal{D}$ and player 2 chooses a point $x_0 \in X$. On the next turn, player 1 chooses $D_1 \in \mathcal{D}$ with $D_1 \subset D_0$ and player 2 chooses $x_1 \in D_0[x_0]$. The game continues and in general on the n'th turn player 1 chooses $D_{n+1} \in \mathcal{D}$ with $D_{n+1} \subset D_n$ and player 2 chooses $x_{n+1} \in D_n[x_n]$. Player 1 wins if either:

1. there exists $x \in X$ such that the $\langle x_n : n \in \omega\rangle$ converge to x;
2. $\bigcap_{n\in\omega} D_n[x_n] = \emptyset$.

Definition 5.4. A strategy for player 1 in the Proximal game is a mapping $\rho : \mathcal{S}(X) \to \mathcal{D}$ such that $x_{n+1} \in \rho(F)[x_n]$ for all $n \in \omega$ and $\rho(F) \supset \rho(G)$ if $G \subset F$. Note that $F \in \mathcal{S}(X)$ represents player 2's first n choices if $|F| = n$ and $\rho(F)$ represents player 1's choice on turn $n + 1$. We will denote $\rho(\emptyset)$ as $X \times X$.

Definition 5.5. We say that X is Proximal if player 1 has a strategy ρ such that every ρ-sequence converges. We say that X is Semi-Proximal if for every strategy for player 2, player 1 has a counter strategy that wins.

Remark 5.6. Paul explained the game to me in a slightly different way. X is assumed to be compact. On turn one player 1 begins by playing a finite open cover (FOC) $\mathcal{U}_1$ and player 2 continues by picking a point $x_1 \in X$. On the second turn player 1 chooses a FOC $\mathcal{U}_2$ such that $\mathcal{U}_2 \prec \mathcal{U}_1$ and player 2 picks a point $x_2 \in \mathrm{st}(x_1, \mathcal{U}_2)$. Then player 1 wins if either $\bigcap_{n\in\omega} \mathrm{st}(x_n, \mathcal{U}_n) = \emptyset$ or the $\langle x_n\rangle$ converge to some point.

Example 5.7. The double arrow space is $X = [0, 1] \times \{0, 1\}$ with the order topology; it is not Semi-Proximal. As X is compact, there exists a unique uniformity that generates the topology. Player 1 plays FOCs of the form $\mathcal{U} = \{(a, b] \times \{0\} \cup [a, b) \times \{1\} : a, b \in [0, 1]\}$ and player 2 picks points from alternating rows every turn. Player 2's strategy will always be viable, since the star of any point wrt to any FOC will always intersect both rows.

Proposition 5.8. If X is semi-proximal then X is α_2 Fréchet.

Proof. Suppose X is not Fréchet. Then there exists some $A \subset X$ with $p \in \overline{A}$ such that no sequence in A converges to p. Then player 2 can addopt a strategy as follows. On turn one, player 1 picks U_1 and player 2 picks $x_1 = p$. On turn 2, player 1 picks U_2 and player 2 picks $x_2 \in U_2 \cap A[p]$, which by symmetry allows player 2 to pick $p \in U_2 \cap A[x_2]$ on the tird turn. Thus is general, player 2 picks points in A such that p always remains a valid play on the next turn, and continues to pick p ever other turn. By symmetry, $p \in \bigcap U_i[x_i]$, and by assumption, the sequence chosen by player 2 cannot converge, hence player 1 loses so that X is not semi-proximal.

Similarly, if X is not α_2, then we can find can find a point p and a sheaf ξ at p such that any sequence that meets each ξ_i doesn't converge to p. As before, player 2 picks p on turn one, but on turn 2, picks $x_2 \in U_2 cap \xi_1$and in general picks $x_{2n} \in U_{2n} \cap \xi_n$. □

Proposition 5.9. If player 1 (player 2) wins (loses) in the proximal game,

Proof. Let $(X, \mathcal{D})$ be a uniform spaces and let ρ be a winning strategy for player 1 in the proximal game. Let τ be a topology on X generated by the basis $\mathcal{B} = \{D[x] : D \in \mathcal{D}, x \in X\}$, let $x \in X$, and let $\sigma : \mathcal{S}(X) \to \mathcal{O}_x$ be a strategy for player 1 in the Gruenhage game played at x defined by $\sigma(F) = \rho(F)[x]$ for all $F \in \mathcal{S}(X)$.

If $\langle x_n : n \in \omega \rangle$ is a σ-sequence, i.e., $x_{n+1} \in \sigma(\langle x_1, \ldots, x_n \rangle)$ for all $n \in \omega$, then $x_{n+1} \in \rho(\langle x_1, \ldots, x_n \rangle)[x]$ and $x_n \in \rho(\langle x_1, \ldots, x_{n-1} \rangle)[x]$. As the elements of $\mathcal{D}$ are symmetric, we have both $x \in \rho(\langle x_1, \ldots, x_n \rangle)[x_{n+1}]$ and $x \in \rho(\langle x_1, \ldots, x_{n-1} \rangle)[x_n]$, which implies $x_{n+1} \in (\rho(\langle x_1, \ldots, x_n \rangle) + \rho(\langle x_1, \ldots, x_{n-1} \rangle))[x_n]$. Thus $x_{n+1} \in \rho(\langle x_1, \ldots, x_{n-1} \rangle)[x_n]$ as $\rho(\langle x_1, \ldots, x_{n-1} \rangle) \supset \rho(\langle x_1, \ldots, x_n \rangle)$ since ρ is a well defined strategy. Hence $\langle x_n : n \in \omega \rangle$ is a ρ-sequence and $x \in \bigcap_{n \in \omega} \rho(\langle x_1, \ldots, x_{n-1} \rangle)[x_n]$, which implies that $\langle x_n \in \omega : n \in \omega \rangle$ must converge to x, which follows by our assumption that X is Hausdorff. □

5.3 Proximal Game on Topological Groups

The game is played the same but we note that the uniformities chosen by player 1 are formed as follows. On any given turn player 1 picks $U \in \mathcal{F}$ which corresponds to $D_U = \bigcup_{a \in [\omega]^{<\omega}} (a \Delta V_U)^2$. In terms of uniform covers we might can also talk about $\mathcal{U}_U = \{(a \Delta V_U) : a \in [\omega]^{<\omega}\}$. The game then proceeds as usual.

Example 5.10. $(\mathbb{R}, +)$ is Proximal. A winning strategy for player 1 consists of picking uniform covers of radius $\frac{1}{2^k}$ on turn k.

5.4 Selection Principle Game

Another game is $G_1(\mathcal{A}, \mathcal{B})$, which is based on the selection principle $S_1(\mathcal{A}, \mathcal{B})$. It is played as follows: in the $n'th$ inning **Player I** plays an element $A_n \in \mathcal{A}$ and **Player II** plays an element of $s_n \in A_n$. **Player 2** wins if the set $\{s_n : n \in \omega\}$ is an element of $\mathcal{B}$. In particular, we care about the game $G_1(\Omega, \Omega)$. By Lemma **??**

6 Musings

The point of this section is to keep track of thoughts relating to problems whose solutions are maybe not be clear or known (yet).

6.1 Behaviour of . . .

. . . of Sequences

- $x_n \to x, \in \tau_{\mathcal{F}} \iff a\Delta x_n \to a\Delta x$
- WLOG we can assume $x = \emptyset$
- $x_n \to x \iff \{x_n\}$ is a network for $\mathcal{F}$.

Proposition 6.1. Let $\mathcal{F}$ be cofinite filter on ω and let $\langle x_n : n < \omega\} \subset [\omega]^{<\omega}$ such that $x_n \to x$ for some $x \in [\omega]^{<\omega}$. There exists $k \in \omega$ such that $x = \bigcap_{n>k} x_n$. Let $\mathcal{F}$ be cofinite filter on ω and let $\langle x_n : n < \omega\} \subset [\omega]^{<\omega}$ such that $x_n \to x$ for some $x \in [\omega]^{<\omega}$. There exists $k \in \omega$ such that $x = \bigcap_{n>k} x_n$.

Proof. We show first that there exists $k \in \omega$ such that $x \subset \bigcap_{n>k} x_n$ and then show that the inclusion cannot be proper. Let $p \in x$ and assume there is no k such that $x \in \bigcap_{n>k} x_n$. Then we can find a subsequence y_n such that $p \notin y_n$ for all n. Let $F = \omega - p$. Then $p \in a$ for all $a \in x\Delta V_F$, so that $y_n \notin x\Delta V_F$ for all $n \in \omega$, contradiction.

We have a k such that $x \subset \bigcap_{n>k} x_n$. Suppose there's a $b \in \bigcap_{n>k} x_n$ but $b \notin x$. Let $F = \omega - \bigcap_{n>k} x_n$. Then $b \notin a$ for all $a \in x\Delta V_F$ so that $x_n \notin x\Delta V_F$ for all $n > k$. □

Remark 6.2. The above proposition goes only one way. Take

$$x_n = \begin{cases} \{1\} & n \text{ is odd} \\ \{1,2\} & n \text{ is even.} \end{cases}$$

Then $\bigcap_{n>k} x_n = \{1\}$ for all $k < \omega$ but $x_n \not\to \{1\}$ (or $\{1,2\}$) . In fact x_n doesn't converge to anything. Firstly, take $F = \omega - \{1,2\}$, then $2 \notin a$ for all $a \in \{1\}\Delta V_F$ but $2 \in a$ for all $a \in \{1,2\}\Delta V_F$ so that neither $\langle x_n\rangle \subset^* \{1\}\Delta V_F$ nor $\langle x_n\rangle \subset^* \{1,2\}\Delta V_F$. Moreover, its clear that $\langle x_n\rangle \not\subset^* p\Delta V_F$ for any other $p \in [\omega]^{<\omega}$.

. . . of Stars

Let $p \in [\omega]^{<\omega}$, let $F \in \mathcal{F}$. What does $S = \mathrm{st}(p, \mathcal{U}_F)$ look like?

If $p \subset F$, then

- $p \in a\Delta V_F \iff p \not\subset a$
- $[\omega - p]^{<\omega} \subset S$ (take $x \in [\omega - p]^{<\omega}$ and let $a = x \cap F^C, b = x \cap F$, then $x = a\Delta b$ and $b \in V_F$ so $x \in S$).

If $p \not\subset F$, then

- $p \in a\Delta V_F \iff p \subset a$
- $S = [\omega]^{<\omega}$ ($p\Delta V_F = [F \cup p]^{<\omega} = V_{F\cup p} \in S$, so for any $x \in [\omega]^{<\omega}$ letting $a = x \cap (V_{F\cup a})^C$ then $x = a\Delta F_{F\cup a} \in S$)

Actually for any $F \in \mathcal{F}$ and any $p \in [\omega]^{<\omega}$, $\mathrm{st}(p, V_F) = [\omega]^\omega$: let $\alpha = p \cap F^C$, then $p \in \alpha\Delta V_F = [F \cup \alpha]^{<\omega} = V_{F\cup\alpha} \in \mathrm{st}(p, V_F)$. For $x \in [\omega]^{<\omega}$, take $a = x \cap (F \cup \alpha)^C$ so that $x \in a\Delta V_{F\cup\alpha}$.

Remark 6.3. The above shows that the only way for player 1 to wins in the proximal game on $\tau_{\mathcal{F}}$ is for the sequence of points picked by player 2 to converges, i.e., the intersection of the stars will just always be the entire space.

Remark 6.4. Actually $\tau_{\mathcal{F}}$ is not Proximal (at least when $\mathcal{F}$ is the cofinite filter). Since the star of any point w.r.t. any $F \in \mathcal{F}$ is always equal the entire space, there is nothing to prevent player 2 from picking a divergent sequence. So there is a first countable topological group that is not proximal.

THE ABOVE IS ALL WRONG

1. Due to the imlicit assumption that $\emptyset \in a\Delta V_F \forall a \in [\omega]^{<\omega} \forall F \in \mathcal{F}$ I was wrong about $\mathrm{st}(p, \mathcal{U}_F) = [\omega]^{<\omega}$
2. Actually for $p \in [\omega]^{<\omega}$ and $F \in \mathcal{F}$, $\mathrm{st}(p, \mathcal{U}_F) = z\Delta V_F$ where $z = p - F$
3. Note that for $F \in \mathcal{F}$, V_F is a subgroup of $\tau_{\mathcal{F}}$ and thus $\mathcal{U}_F = \{a\Delta V_F\}$ is a partition of τ_F (read set of cosets of τ_F).
4. thus when player 2 picks points $a_{n+1} \in \mathrm{st}(a_n, \mathcal{U}_{F_n})$ it is really the case that player 2 is picking a_{n+1} such that $a_{n+1} \equiv z_n \pmod{F_n}$ where $z_n = a_n - F_n$

. . . of Intersections

Suppose $\langle a_n : n < \omega \rangle \subset [\omega]^{<\omega}$ and $\langle F_n : n < \omega \rangle \subset \mathcal{F}$ and $b \in \bigcap_{n<\omega} a_n \Delta V_{F_n}$.

- this does not imply that $a_n \to b$. For example $b \not\subset a_n$ and $b \subset F_n$ for all $n \in \omega$ is possible but we still have $b \in a_n \Delta V_{F_n}$ for all n.
- On the other hand, if $b \in \bigcap_{n<\omega} a_n \Delta V_{F_n}$ and we want $a_n \to b$, this would only be possible if $b \not\subset F_n$ for all $n > k$ for some k.

6.2 (partial) Results

Observation 6.5. If $a \in [\omega]^{<\omega}$ and $F \in \mathcal{F}$ and $z = a - F$, then $\mathrm{st}(a, \mathcal{U}_F) = z\Delta V_F$. If $p \in \mathrm{st}(a, \mathcal{U}_F)$ then $p \equiv z \pmod{F}$.

Observation 6.6. The play of the Proximal game on $\tau_{\mathcal{F}}$ is the triple (F_n, a_n, z_n).

We know that secound countable and metrizable implies proximal, thus if $\mathcal{F}$ is a filter on ω generated by $\mathcal{A} = \langle F_n : n < \omega \rangle$, it must be the case that τ_F is Proximal.

Define a strategy for player 1 in the proximal game on τ_F as follows:

Player I	$F_0 \in \mathcal{A}$		$F_1 \in \mathcal{A}, F_1 \cap a_0 = \emptyset$			...
Player II		a_0		$a_1 \equiv z_0 \pmod{F_0}$		...

In general, **Player I** plays covers from $\mathcal{A}$ and on turn $k+1$ **Player I** plays $F_{k+1} \subset F_k$ such that $F_{k+1} \cap p_n = \emptyset$.

Lemma 6.7. Let $\mathcal{F}$ be countable generated filter on ω, let $\langle F_n : n < \omega \rangle$ be a sequence of generating sets, and let $\langle a_n : n < \omega \rangle \subset [\omega]^{<\omega}$. If the following holds for all $n < \omega$:

- $F_{n+1} \cap a_n = \emptyset$;
- $a_{n+1} \equiv z_n \pmod{F_n}$ where $z_n = a_n - F_n$

then

a) if $\bigcup z_n$ is finite, then $\langle a_n \rangle$ is a convergent sequence;

b) if $\bigcup z_n$ is infinite, then $\bigcap \mathrm{st}(a_n, \mathcal{U}_{F_n}) = \emptyset$.

Proof.

a) If $\bigcap z_n = \emptyset$, then $a_n \subset F_n$ for all $n < \omega$ and converge to $\emptyset$, as the F_n are a (decreasing) neighboourhood base at $\emptyset$. By similar reasoning, if $\bigcup z_n = b \neq \emptyset$ then the a_n converges to b by definition of the nbhd base at b.

b) Since $z_n \cap F_n = \emptyset$ for all $n \in \omega$, $b \in z_n \Delta F_n$ means that $z_n \subset b$. Thus if there exists $b \in \bigcap \mathrm{st}(a_n, \mathcal{U}_{F_n}) = \bigcap z_n \Delta F_n$, then $z_n \subset b$ for all n. Since the z_n are increasing and their union is infinite we have that b must be infinite, a contradiction since $b \in [\omega]^{<\omega}$.

□

Proposition 6.8. When $\mathcal{F}$ is a countably generated filter on ω then $\tau_{\mathcal{F}}$ is Proximal.

Proof. By the lemma, the strategy for **Player 1** described above wins. TODO: elaborate: why is there only the cases z_n finite/infinte? □

Proposition 6.9. If $\mathcal{F}$ is an uncountably generated filter on ω then $\tau_{\mathcal{F}}$ is not Proximal.

Proof. Let $\sigma : [[\omega]^{<\omega}]^{<\omega} \to \mathcal{F}$ be any strategy for **Player I**. On turn 0 **Player II** can pick any $p_0 \in [\omega]^{<\omega}$,and therefore has countably many choices; we can enumerate them as $\mathcal{B}_0 = \{p_0^i : i < \omega\}$. In response to any choice by **Player II**, say p_0^k, **Player I** responds by playing $\sigma(p_0^k)$. Hence the possible responses by **Player I** are $\mathcal{A}_0 = \{\sigma(p_0^i) : i < \omega\}$.

On the $n+1$'th turn, **Player II** picks from $\mathrm{st}(p_n, \mathcal{U}_{F_n}) \subset [\omega]^{<\omega}$ and again has countably many choices $\mathcal{B}_n = \{p_n^i : n < \omega\}$ and the possible responses from **Player I** are $\mathcal{A}_n = \{\sigma(p_n^i) : i < \omega\}$.

Since $\mathcal{A} = \bigcup \mathcal{A}_n$ is countable it doesn't generate $\mathcal{F}$ and there is $F^* \in \mathcal{F}$ such that $F^* \not\supset A$ for all $A \in \mathcal{A}$. Consequently the set $X = \bigcup A_n \setminus F^*$ is infinite.

Player II can now use the following counter strategy to win. On turn 0 **Player II** picks $a_0 \in [X]^{<\omega}$ and **Player I** plays $F_0 = \sigma(a_0)$.

If $F_0 \neq [\omega]^{<\omega}$, then **Player II** picks a_1 such that $a_1 \cap X \neq \emptyset$ and $a_1 - F_0 \neq \emptyset$. Then for the rest of the game **Player II** always picks $a_{n+1} \subset F_n$ with $a_{n+1} \cap X \neq \emptyset$.

Otherwise, if $F_0 = [\omega]^{<\omega}$, **Player II** plays a_1 such that $a_1 \cap X \neq \emptyset$ and $a_1 \subset F_0$. On turn $n+1$, if $F_n = [\omega]^{<\omega}$ **Player II** plays a_{n+1} such that $a_{n+1} \cap X \neq \emptyset$ and $a_{n+1} \subset F_n$. Otherwise if $F_n \neq [\omega]^{<\omega}$ then **Player II** plays a_{n+1} such that $a_{n+1} \cap X \neq \emptyset$ and $a_{n+1} - F_n \neq \emptyset$ and for the rest of the game picks $a_k \subset F_k$.

Since each a_n meets X its clear that $a_n \not\subset F^*$ for all n so that a_n cannot converge. On the other hand, either $F_n = [\omega]^{<\omega}$ for all $n < \omega$ and $\bigcap \mathrm{st}(a_n, \mathcal{U}_{F_n}) \neq \emptyset$, or there is a minimum k such that $F_k \neq [\omega]^{<\omega}$, in which case $z = a_k - F_k \neq \emptyset$ and $a_n \subset F_n$ for all $n > k$ and $z \in \bigcap \mathrm{st}(a_n, \mathcal{U}_{F_n})$. TODO:clean up and make more readable □

Proposition 6.10. CH implies the existence of an uncountably generated Semi Proximal filter.

Proof. Start with $\mathcal{F}_0$ any countably generated filter on ω, where $\mathcal{F}_0$ is generated by $\mathcal{E}_0 = \langle E_n^0 : n < \omega \rangle$. Let Σ be the set of all of strategies for **Player II** on $\tau_{\mathcal{F}_0}$ restricted to $\mathcal{E}_0$, where Σ is enumerated as $\{\sigma_\alpha : \alpha < \omega_1\}$. We build $\mathcal{F}$ by recursion on Σ.

Fix $\sigma_0 \in \Sigma$ and note that it loses for **Player II** since $\mathcal{F}_0$ is Proximal and **Player I** has the winning strategy of playing inside of $\mathcal{E}_0$. More precisely, there is a sequence $\langle E_{n_k}^0 : k < \omega \rangle \subset \mathcal{E}_0$ such that $\langle \sigma_0 \left(E_{n_k}^0 \right) : k < \omega \rangle$ converges with respect to $\mathcal{F}_0$. Let B_0 denote the convergent sequence and let $A_0 = E_0^0 \cap B_0$, which is not an element of $\mathcal{F}_0$.

At stage α assume that:

(1) $\mathcal{A}_\alpha = \{A_\beta : \beta < \alpha\}$ is such that $A_i \not\subset A_j$ whenever $i < j$;

(2) $\mathcal{F}_\beta = \mathcal{F}_0 \cup (\mathcal{A}_\alpha \setminus \{A_\gamma\}_{\gamma \geq \beta})$ for all $\beta < \alpha$ and each $\mathcal{F}_\beta$ is generated by $\mathcal{E}_\beta = \langle E_n^\beta : n < \omega \rangle$;

(3) $\mathcal{B}_\alpha = \{B_\beta : \beta < \alpha\}$ where $B_\beta = \langle \sigma_\beta(E_{n_k}^\beta) : k < \omega \rangle$ converges to $\mathcal{F}_\beta$ and $B_\beta \subset^* A_\gamma$ for any $\beta \leq \gamma < \alpha$.

Let $\mathcal{F}_\alpha = \mathcal{F}_0 \cup \mathcal{A}_\alpha$ and let $\Sigma_\alpha = \{\text{strategies on } \tau_{\mathcal{F}_\alpha}\}$. $\mathcal{F}_\alpha$ is generated by $\mathcal{E}_\alpha = \langle E_n^\alpha : n < \omega \rangle$ and **Player I** can play from $\mathcal{E}_\alpha$ to defeat any $\sigma' \in \Sigma_\alpha$ since $\mathcal{F}_\alpha$ is Proximal. However, in order to guarantee that **Player I** has enough winning counter strategies available in the future, we will insist that **Player I** plays from $\mathcal{E}_\beta$ to defeat **Player II** any time σ' extends a $\sigma_\beta \in \Sigma$ from a previous stage β. In other words, if $\beta < \alpha$ and $\sigma' \restriction \mathcal{F}_0 = \sigma_\beta$, then $B_\beta \to \mathcal{F}_\beta$ exists and hence σ' loses for **Player II** on $\mathcal{F}_\beta$. By (3), we know that **Player I** can stick to their previous strategy of playing inside $\mathcal{E}_\beta$ to defeat σ' on $\mathcal{F}_\alpha$.

Let $\sigma_\alpha \in \Sigma$. We continue the construction by fixing $\sigma \in \Sigma_\alpha$ such that σ extends σ_α. Let $\langle E^\alpha_{n_k} : k < \omega \rangle \subset \mathcal{E}_\alpha$ be such that $B_\alpha = \langle \sigma(E^\alpha_{n_k}) : k < \omega \rangle$ converges to $\mathcal{F}_\alpha$ and let $A_\alpha = \bigcup_{n \in \omega} B_n \cap E^\alpha_n$ which is not in $\mathcal{F}_\alpha$.

Let $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ and let $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{A}$ be the resulting filter. Now any strategy σ for **Player II** on $\tau_{\mathcal{F}}$ must agree with some σ_α on $\mathcal{F}_0$, so there is a B_α that converges with respect to $\mathcal{F}_\alpha$, and hence also $\mathcal{F}$. So by playing inside of $\mathcal{E}_\alpha$ **Player I** defeats σ on $\mathcal{F}$.

$\square$