1 Introduction

1.1 Outline of proposed research project

The text Counterexamples in Topology by Steen and Seebach has been a fabulous resource for students and researchers in Topology since its publication in 1970. The book was the product of an undergraduate research project funded by NSF and supervised by Steen and Seebach (and including then student Gary Gruenhage) to systematically survey important topological counterexamples. More recently James Dabbs has implemented a database on Github based on the Steen and Seebach textbook called Pi-Base (see https://topology.pi-base.org/) and it is currently being maintained by Dabbs and Stephen Clontz. This resource has great potential to both researchers and advanced undergraduate and graduate students at the start of their research careers. There are still big gaps in the database's subject matter, especially in relation to research in and around Frechet-Urysohn spaces. There is a significant body of work, and especially interesting counterexamples, concerning Michael's class of bisequential spaces, Arhangel'skii's alpha-i spaces and several game theoretic formulations of convergence which do not yet appear in the Pi-Base. The project has two goals. The first, and most accessible, is to give a systematic survey of the recent research which will be implemented into the Pi-Base database. The second half of the project will be devoted to open problems related to a recent class of examples defined from ladder systems (more generally on so called square-sequence) described in two

2 Meeting Log

Monday April 21

1. Frechet Fan - S_ω: $\omega \times (\omega+1)/\omega \times \{\infty\}$ (i.e. $\omega \times (\omega+1)$ with the points at infinity identified). Show that
   - S_ω is not first countable ✓
   - S_ω is Fréchet. ✓
2. Product of Fréchet spaces not always Fréchet: take $(\omega+1) \times S_\omega$. Let $A = \{(m,(m,n)) \,:\, m,n \in \omega$. Show that
   - $(\omega+1,\infty) \in \overline{A}$ ✓
   - No sequence in A converges to $(\omega+1,\infty)$. ✓
3. Right way to think about sequences: $A \subseteq X$ converges to $a \in X$ if $|A| = \aleph_0$ and for all neighboourhoods $U_x \subseteq X$, $|A \setminus U_x| < \aleph_0$.
4. Right way to think about Fréchet space: take sequential closure once same as closure.
5. Another exercise: which α_i properties does S_ω have? ✓

Thursday May 1

1. Go over example 3.9; why is S_ω Fréchet?
2. Give example of space that is α_2 but not α_1, α_3 but not α_2 etc.
3. Under which set theoretic assumptions do such examples exist? Look at the paper by Nyikos: Subsets of ${}^\omega\omega$ and the Fréchet Urysohn and α_i properties.
4. For example it is consistent that both α_1 and countable imply first countable, and α_2 and countable imply α_1, yet there exists a countable α_2 space that is not first countable.

5. We talked a little bit about topological groups (a topological space equipped with group operation that is continuous) and some of the nice structure they have: the group operation being continuous and invertible implies $G \times G \to G$ is always a homeomorphism, in particular aG is the homeomorphic image of G by left multiplication of $a \in G$, which in often cases lets us define a nbhd base at the identity and "send" it to all points of G in order to define the topology.

6. Look into what it takes to contribute to the pi-base, in particular add the α_i spaces.

7. Some more exercises
   - X is α_1 and Fréchet, Y is Fréchet, then $X \times Y$ is Fréchet.
   - α_2 is equivalent to $A \cap B \neq \emptyset$ for all $A \in \xi$ whenever ξ is countable collection of sequences at x. ✓

3 Set Theory

3.1 Some Interesting Cardinals

Definition 3.1. Let $\mathcal{A}$ be an infinite family of infinite subsets of ω. The family $\mathcal{A}$ is said to be an almost disjoint family (ADF) of subsets of ω if $A \cap B$ is finite for any $A, B \in \mathcal{A}$ with $A \neq B$. By Zorn's lemma we can assume $\mathcal{A}$ lives inside of a maximal (w.r.t. inclusion) family that is also pairwise disjoint. Such a family is a maximal almost disjoint family (MADF).

Definition 3.2. Define an order $\leq^*$ on ${}^\omega\omega$ where $f <^* g$ iff there exists n such that $f(k) < g(k)$ for all $k \geq n$

Definition 3.3. Let $(A, \prec)$ be a linearly ordered set. Then we say that $B \subseteq A$ is undominating in A if there does not exist $y \in A$ such that $x \prec y$ for all $x \in B$. We say that $C \subseteq A$ is cofinal (or dominating) in A if for all $x \in A$ there exists $z \in C$ such that $x \prec x$.

Definition 3.4.

- $\mathfrak{a} = \min\{|\mathcal{A}| : \mathcal{A} \text{ is infinte MADF}\}$
- $\mathfrak{b} = \min\{|F| : F \text{ is } \leq^* \text{-undominated subset of } {}^\omega\omega\}$
- $\mathfrak{d} = \min\{|F| : F \text{ is } \leq^* \text{-cofinal in } {}^\omega\omega\}$

Each of $\mathfrak{a}, \mathfrak{b}, \mathfrak{d}$ are at least ω_1 but at most $\mathcal{P}(\omega)$.

4 Topology

4.1 Basics

- $x \in \overline{A}$ iff $U \cap A \neq \emptyset$ for all open sets U containing x.

Definition 4.1 (Seperation Axioms)**.** Let X be a topological space. X is said to be

- T_0: whenever $x \neq y$ there exists an open $U \subseteq X$ with $x \in U$ but $y \notin U$ (or $y \in U$ and $x \notin U$).
- T_1: whenever $x \neq y$ there exists open sets U_x, U_y with $x \notin U_y$ and $y \notin U_x$
- T_2: whenever $x \neq y$ there exists open sets U_x, U_y with $x \notin U_y$ and $y \notin U_x$ and $U_x \cap U_y = \emptyset$
- Regular: $x \in X$ and closed $A \subseteq X$ with $x \notin A$, there exists open disjoint $U, V \subseteq X$ with $x \in U$ and $A \subseteq V$
- T_3: regular + T_1
- Completely regular: $x \in X$ and closed $A \subseteq X$ with $x \notin A$, there exists a continuous $f : X \to \mathbb{R}$ with $f(x) = a$ and $f(A) = b$ $(a \neq b)$

- Tychonoff ($T_{3\frac{1}{2}}$): Completely regular + T_1
- Normal: Closed $A, B \subseteq X$ with $A \cap B = \emptyset$, there exists open disjoint $U, V \subseteq X$ with $A \subseteq U$ and $b \subseteq V$
- Completely Normal: for all $A \subseteq X$, A is normal.

4.2 Sequential and Fréchet Spaces

Definition 4.2. For a topological space X and any set $A \subset X$, the *sequential closure* of A is

$$[A]_{\text{seq}} := \left\{ x \in X \ : \ \exists (x_n) \in A \left(\lim_{n\to\infty} x_n = x \right) \right\}.$$

In general we can repeat this operation recursively $[[[A]_{\text{seq}}]_{\text{seq}} \ldots]_{\text{seq}}$ by which is meant the *total sequential closure* of A.

Fact 4.3. In general it takes at most ω_1 many iterations of the sequential closure to get a closed set.

Definition 4.4. A space X is said to be Fréchet if $[A]_{\text{seq}} = \overline{A}$ for all $A \subseteq X$.

Example 4.5. Let $X = \omega_1 + 1$ with the order topology. X is not Fréchet, since any sequence $(x_n) \in \omega_1$ cannot converge to ∞, as otherwise $\omega_1 = \sup\{x_n \ : \ n \in \mathbb{N}\}$, a contradiction.

Definition 4.6. A space X is sequential if any closed set $A \subseteq X$ is equal to its total sequential closure.

Then a space that is Fréchet is also sequential. The following example shows that the converse is not true.

Example 4.7. Let $X^* = \omega \times (\omega + 1)$ be given the order topology and let $X = X^* \cup \{\infty\}$ where the neighboourhoods of ∞ are such that there exists $p \in \omega$ such that $|\{(m, n) : m > p, n \in \omega{+}1\} \backslash U_\infty| < \aleph_0$. Then X is sequential but but not Fréchet. To see this, note that for all $m \in \omega$ the sequence $A_m = \{(m, n) \ : \ n \in \omega\}$ converges to $(m, \omega + 1)$ and moreover $B = \{(m, \omega + 1) \ : \ m \in \omega\}$ is a sequence that converges to ∞. Then $A = \bigcup_{m\in\omega} A_m$ is such that $[[A]_{\text{seq}}]_{\text{seq}} = X$, hence X is sequential. On the other hand there is no sequence in A that converges to ∞. Suppose there were, say some $\gamma \to \infty$. Then for all $m \in \omega$, $U_m = X \backslash \{(m, n) : n \in \omega{+}1\}$ is a neighboourhood of ∞ such that $|\gamma \setminus U_m| < \aleph_0$. Hence γ has only finitely many terms belonging to each column. If $\alpha_m = \max\{\gamma \cap \{(m, n) \ : \ n \in \omega\}\}$, then $U = X \setminus \bigcup_{m\in\omega} \{(m, n) \ : \ n \leq \alpha_m\}$ is a neighbourhood of ∞ disjoint from γ, a contradiction. Hence X is not Fréchet.

Proposition 4.8. If X is first countable then X is Fréchet.

Proof. Let $A \subseteq X$ and let $x \in \overline{A}$. Then x has a countable neighbourhood base N_x such that $U \cap A \neq \emptyset$ for all $U \in N_x$. Enumerating the neighboourhoods of x as $U_1, U_2, \ldots$ then the sequence $(x_n)_{n\geq 1}$ where $x_n \in U_n \cap A$ for each $n \in \omega$ is such that $(x_n)_{n\geq 1}$ converges to x. □

The following example shows that the converse is not true.

Example 4.9 (Fréchet Fan)**.** Let S_ω be the quotient of $\omega \times (\omega + 1)$ obtained by identifying all the points $\{(m, \omega + 1) \ : \ m \in \omega\}$ as ∞^*. More precisely S_ω has the quotient topology induced by the map $h : \omega \times (\omega + 1) \to S_\omega$ where $h(x) = x$ for all $x \in \omega \times \omega$ and $h(x) = \infty^*$ for all $x \in \omega \times \{\omega + 1\}$. Then S_ω is Fréchet but not 1st countable. To see that S_ω is Fréchet, note that by definition of the quotient topology, the open neighbourhoods of ∞^* are those sets $U \subset S_\omega$ such that $\infty^* \in U$ and $h^{-1}(U)$ is open in $\omega \times (\omega + 1)$. As $\{(m, \omega + 1) \ : \ m \in \omega\} \subset h^{-1}(U)$ we see that U is an open neighbourhood of ∞^* iff $h^{-1}(U)$ is an open neighbourhood of $(m, \omega + 1]$ for all $m \in \omega$. Hence the open neighboourhoods of ∞^* are of the form $\bigcup_{m\in\omega} \{m\} \times (f(m), \infty^*]$ where $f : \omega \to \omega$ is just some mapping that indicates the startpoint of each interval. It follows that for all $m \in \omega$ the sequence $A_m = \{(m, n) \ : \ n \in \omega\}$ converges to ∞^* so that $A = \bigcup_{m\in\omega} A_m$ is such that $[A]_{\text{seq}} = S_\omega$. On the other hand it's obvious that $\overline{A} = S_\omega$, so that S_ω is indeed Fréchet. Now asssuming that S_ω was countable, we would have a countable neighbourhood base at ∞^*. For each $k \in \omega$ let $B_k = \bigcup_{m\in\omega} \{m\} \times (f_k(m), \infty^*]$ for some $f_k : \omega \to \omega$ determining the startpoints of each interval. Suppose $\mathcal{B} = \{B_k \ : \ k \in \omega\}$ is a base at ∞^*, then let $f^* : \omega \to \omega$ be defined by $f^*(m) = f_m(m) + 1$ for all $m \in \omega$. Letting $B^* = \bigcup_{m\in\omega} \{m\} \times (f^*(m), \infty^*]$ then B^* is an open neighboourhood of ∞^* but it is clear by construction that $B_k \not\subset B^*$ for all $k \in \omega$. Hence $\mathcal{B}$ cannot be a neighboourhood base and S_ω is not first countable.

As the following example shows, the product of Fréchet spaces need not be Fréchet.

Example 4.10. Let $X = (\omega+1) \times S_\omega$, and consider the set $A = \{(m,(m,n)) : m,n \in \omega\}$. If ∞^* is the identified point of S_ω, let $\infty = \{\omega+1\} \times \infty^*$. Then $\infty \in \overline{A}$ but $\infty \notin [A]_{\text{seq}}$. The open neighboourhoods of ∞ are are of the form $(\alpha, \omega+1] \times \left(\bigcup_{m\in\omega} \{m\} \times (f(m), \infty^*]\right)$, which clearly always has non emtpy intersection with A. Hence $\infty \in \overline{A}$. To see that $\infty \notin [A]_{\text{seq}}$, suppose γ is a sequence in A that converges to ∞. Since the sets $U_k = (k, \omega+1] \times \left(\bigcup_{m\in\omega} \{m\} \times (f(m), \infty^*]\right)$ are open neighbourhoods of ∞ it must be the case that $|\gamma \setminus U_k| < \aleph_0$ for all $k \in \omega$. Thus $\gamma \cap (\{k\} \times \{k\} \times (1, \omega+1])$ is finite for every k. Let $h : \omega \to \omega$ be defined by $h(k) = \max\{\pi_3(\gamma \cap (\{k\} \times \{k\} \times (1, \infty^*])\} + 1$ for $k \in \omega$. Pictorially, h is picking the point on each spine beyond which no elements of γ exist. Thus

$$W = (1, \omega+1] \times \left(\bigcup_{n\in\omega} \{n\} \times (h(n), \infty^*]\right)$$

is an open neighbourhood of ∞ which by construction is disjoint from γ. Hence γ cannot converge to ∞ showing that X is not Fréchet.

4.3 α_i notions of convergence

Definition 4.11. Let X be a topological space and ξ be a countable family of sequences converging to a point $x \in X$. We say that x is an α_i point for $i = 1,2,3,4$ if there exists a sequence B such that

- α_1: $|A \setminus B| < \aleph_0$ for every $A \in \xi$;
- α_2: $|A \cap B| = \aleph_0$ for every $A \in \xi$;
- α_3: $|A \cap B| = \aleph_0$ for infinitely many $A \in \xi$;
- α_4: $A \cap B \neq \emptyset$ for infinitely many $A \in \xi$.

Then X is an α_i space if every $x \in X$ is an α_i point. Note that if a space is α_i then it is α_{i+1} for $i = 1,2,3$.

Example 4.12. S_ω is not even α_4. For each $m \in \omega$ let $A_m = \{m\} \times (1, \infty)$. Then $\xi = \{A_m : m \in \omega\}$ is a countable collection of sequences converging to ∞. Suppose B is a sequence that converges to ∞ such that $A \cap B \neq \emptyset$ for infinitely many $A \in \xi$. In particular let $\alpha \leq \omega$ be such that $A_i \cap B \neq \emptyset$ for all $i \in \alpha$ and let $f : \alpha \to \omega$ be defined by $f(k) \in B \cap A_k$ for all $k \in \alpha$. Then

$$U = \left(\bigcup_{k\in\alpha} \{k\} \times (f(k)+1, \infty]\right) \times \left(\bigcup_{m\in\omega\setminus\alpha} \{m\} \times (1, \infty]\right)$$

is such that $|B \cap U^C| = \aleph_0$, hence B does not converge to ∞.

Exercise 4.13. X is α_2 iff whenever ξ is a countable collection of sequences converging to x there exists $B \to x$ such $A \cap B \neq \emptyset$ for all many $A \in \xi$.

Solution. The forward direction is obvious. Conversely, let $\xi = \{A_1, A_2, \dots\}$ be a countable collection of sequences converging to x. For every $n \in \omega$ let $\{a_{nm} : m \in \omega\}$ enumerate the elements of A_n and define $A_{nm} = A_n \setminus \{a_{n1}, a_{n2}, \dots, a_{n,m-1}\}$ for each $n, m \in \omega$. Then the A_{nm} still convege to x and $\mathcal{A} = \{A_{nm} : n, m \in \omega\}$ is a sheaf at x. By hypothesis there exists a B converging to x that meets each A_{nm}. Thus B meets each A_n in an infinite set.

4.4 Ψ-like spaces

Definition 4.14. Let D be a countable set and let $\mathcal{A} \subseteq \mathcal{P}(D)$ be an ADF. Define a topology on $X = D \cup \mathcal{A}$ such that the points of D are isolated and D is dense in X. Then for every $A \in \mathcal{A}$ attach a point z_A where the nbhds of z_A consist of all sets of the form $B(A, F) = \{z_A\} \cup (A \setminus F)$ where $F \subseteq A$ is finite.

Proposition 4.15. Let X be as above. Then X is Hausdorff, first countable and locally compact. If X is constructed as above but $\mathcal{A}$ is MADF then X is also pseudo-compact (i.e. all continuous real functions on X are bounded).

Proof. To check Hausdorffness it suffices to show $Z = X \setminus D$ is Hausdorff. Let $z_1, z_2 \in Z$. Then z_1, z_2 correspond to sets A_1, A_2 such that $A_1 \cap A_2 = I$ is finite. Then $B(A_1, I)$ and $B(A_2, I)$ are disjoint open sets containing z_1, z_2 respectively.

To see that X is locally compact, note that $\{x\}$ is a compact neighbourhood base for all $x \in D$. Local compactness of Z follows from the fact that by design each $z_A \in Z$ is the limit of A viewed as a sequence. In other words, whenever $B(A_z, F)$ is a basic nbhd of z_A, any open cover will have a smaller nbhd $B(A_z, F')$ that omits only finitely many points. □

Proposition 4.16. The one point compactification of a Ψ-like space of cardinality $< \mathfrak{a}$ is Fréchet.

4.5 Constructing Examples of α_i Spaces

Example 4.17. Let $D = \omega \times \omega$, let $\mathcal{F} = \{f_\alpha : \alpha < \mathfrak{b}\} \subseteq {}^\omega\omega$ be $<^*$-well-ordered and $<^*$-unbounded, and let $\mathcal{C} = \{C_n = \{n\} \times \omega : n \in \omega\}$. Construct a Ψ-like from $X = \mathcal{F} \cup \mathcal{C}$, where we attach compactification points $\{n\} \times (\omega + 1)$ for each C_n and compactification points p_α for each f_α. Then the resulting one point compactification, $X + \infty$, is α_2 and Fréchet but not α_1. $X + \infty$ becomes α_1 if $\mathcal{F}$ is cofinal and well ordered.

4.6 Bisequential Spaces

Definition 4.18.