

# 1 Introduction

## 1.1 Outline of proposed research project

The text Counterexamples in Topology by Steen and Seebach has been a fabulous resource for students and researchers in Topology since its publication in 1970. The book was the product of an undergraduate research project funded by NSF and supervised by Steen and Seebach (and including then student Gary Gruenhagen) to systematically survey important topological counterexamples. More recently James Dabbs has implemented a database on Github based on the Steen and Seebach textbook called Pi-Base (see <https://topology.pi-base.org/>) and it is currently being maintained by Dabbs and Stephen Clontz. This resource has great potential to both researchers and advanced undergraduate and graduate students at the start of their research careers. There are still big gaps in the database's subject matter, especially in relation to research in and around Frechet-Urysohn spaces. There is a significant body of work, and especially interesting counterexamples, concerning Michael's class of bisequential spaces, Arhangel'skii's alpha-i spaces and several game theoretic formulations of convergence which do not yet appear in the Pi-Base. The project has two goals. The first, and most accessible, is to give a systematic survey of the recent research which will be implemented into the Pi-Base database. The second half of the project will be devoted to open problems related to a recent class of examples defined from ladder systems (more generally on so called square-sequence) described in two

## 2 Meeting Log

### Monday April 21

1. Frechet Fan -  $S_\omega$ :  $\omega \times (\omega + 1)/\omega \times \{\infty\}$  (i.e.  $\omega \times (\omega + 1)$  with the points at infinity identified). Show that
  - $S_\omega$  is not first countable ✓
  - $S_\omega$  is Fréchet. ✓
2. Product of Fréchet spaces not always Fréchet: take  $(\omega + 1) \times S_\omega$ . Let  $A = \{(m, (m, n)) : m, n \in \omega\}$ . Show that
  - $(\omega + 1, \infty) \in \overline{A}$  ✓
  - No sequence in  $A$  converges to  $(\omega + 1, \infty)$ . ✓
3. Right way to think about sequences:  $A \subset X$  converges to  $a \in X$  if  $|A| = \aleph_0$  and for all neighbourhoods  $U_x \subset X$ ,  $|A - U_x| < \aleph_0$ .
4. Right way to think about Fréchet space: take sequential closure once same as closure.
5. Another exercise: which  $\alpha_i$  properties does  $S_\omega$  have? ✓

### Thursday May 1

1. Go over example 3.9; why is  $S_\omega$  Fréchet? ✓
2. Give example of space that is  $\alpha_2$  but not  $\alpha_1$  ✓,  $\alpha_3$  but not  $\alpha_2$  etc.
3. Under which set theoretic assumptions do such examples exist? Look at the paper by Nyikos: Subsets of  ${}^\omega\omega$  and the Fréchet Urysohn and  $\alpha_i$  properties. **Good grasp at sections 1, 2, 5; partial understanding section 3; skipped section 4, 6.**
4. For example it is consistent that both  $\alpha_1$  and countable imply first countable, and  $\alpha_2$  and countable imply  $\alpha_1$ , yet there exists a countable  $\alpha_2$  space that is not first countable.

5. We talked a little bit about topological groups (a topological space equipped with group operation that is continuous) and some of the nice structure they have: the group operation being continuous and invertible implies  $G \times G \rightarrow G$  is always a homeomorphism, in particular  $aG$  is the homeomorphic image of  $G$  by left multiplication of  $a \in G$ , which in often cases lets us define a nbhd base at the identity and "send" it to all points of  $G$  in order to define the topology.
6. Look into what it takes to contribute to the pi-base, in particular add the  $\alpha_i$  spaces. **Forked database to github account; properties are stored as markdown text files, should not be difficult to add properties as well as if/then.**  
 Things to add:
  - $\alpha_i$  and if  $\alpha_i$  then  $\alpha_{i+1}$
  - bisequential
  - v-space (w-space exists already)
  - if 1st countable then  $\alpha_1$  Frechet
  - if bisequential then Frechet
  - w-space iff Frechet and  $\alpha_2$
7. Some more exercises
  - $X$  is  $\alpha_1$  and Fréchet,  $Y$  is Fréchet, then  $X \times Y$  is Fréchet. **this is not true, take  $(\omega + 1) \times S_\omega$ .**
  - $\alpha_2$  is equivalent to  $A \cap B \neq \emptyset$  for all  $A \in \xi$  whenever  $\xi$  is countable collection of sequences at  $x$ . ✓

## Thursday May 15

1. When is the square of an  $\alpha_i$  space still  $\alpha_i$ ? What  $\alpha$  properties does  $\alpha_i \times \alpha_j$  have?
2. I asked for an example of a tower, and Paul proved that towers exist, but said that theres not really specific constructions of towers.
3. historical note: the cardinals that lay between  $\omega_1$  and  $\mathcal{P}(\omega)$  came from attempts to solve CH, for example, if  $\mathfrak{t} \leq \mathfrak{c}$  then, finding the cardinality for  $\mathfrak{t}$  would help in finding cardinality of  $\mathfrak{c}$ .
4. Proximal game
5. Uniformities
6. Homogeneity in Top group used to determine uniformity for player 1 in version of prox game for top groups
7. FUF and 2FUF
8. exercises:
  - prove lemmas used in existence proof of towers.
  - If  $\{A_\alpha : \alpha \in \gamma\}$  is a tower and  $\{\alpha_\xi : \xi < \lambda\}$  is increasing and cofinal in  $\gamma$  then  $A_{\alpha_\xi} : \xi < \lambda$  is a tower.
  - $S_\omega$  is neither FUF or 2FUF ✓
  - Proximal game is equivalent to Gruenhage game. **Proximal space implies W ✓**
  - Winning strategy in proximal game implies Frechet **I showed W and w space imply frechet**
9. note: send Paul changes to databse before pushing to github

## Wednesday May 28

### 1. More on Uniformities

- Different metrics generate different Uniformities, for example discrete metric vs  $d(m, n) = |\frac{1}{m} - \frac{1}{n}|$  on  $\omega$ , Uniformities from those metrics still generate the same (discrete) metric on  $\omega$
- star refinements generalize triangle inequality, for example  $\{B(x, \frac{1}{2^{n+1}}) : x \in X\}$  star refines  $\{B(x, \frac{1}{2^n}) : x \in X\}$  and gives triangle inequality

### 2. went over "proof" that proximal implies W, however Paul raised issue that player 2 in proximal game can use Hausdorffness to always pick points $z$ that prompt player 1 in Proximal game to choose entourages $U$ such that $U[x] \cap U[p] = \emptyset$ , where $p$ is where Gruenhage game being played. **this problem is fixed by starting Gruenhage game one round later than proximal game and player 2 pick $p$ on turn 1 of proximal game**

### 3. Exercises:

- If  $X$  semi-proximal then is FUF? **turns out this isn't known (yet)**
- Is double arrow space Proximal? ✓
- Semi Proximal is  $\alpha_2$ . ✓

### 4. Paul showed me proof that semi Proximal is Frechet

### 5. Malykhin's problem: If $G$ is a separable Frechet topological group, is it metrizable? Answer: Con(yes) and Con(No)

### 6. **Paul's main question:** Can we strengthen hypotheses of Malykhin's problem to obtain answer that is not independent of ZFC?

## Thursday June 5

### 1. Discussed implications of conditions on filter $\mathcal{F}$ when going from top space to top group

### 2. Questions:

- If  $\mathcal{F}$  is FUF, is  $\tau_{\mathcal{F}}$  semiproximal? **no??? see prop 5.11**
- If  $\tau_{\mathcal{F}}$  is semiproximal, is  $\mathcal{F}$  countable generated?
- Assuming CH, does there exist filters  $\mathcal{F}_0, \mathcal{F}_1$  both FUF neither countably generated where  $\tau_{\mathcal{F}_0}$  is semiproximal but  $\tau_{\mathcal{F}_1}$  isn't.

### 3. Exercises:

- Assume CH, construct FUF filter that is not countable generated. ✓
- What does it mean for a sequence in  $(a_n)$  to converge to a point in  $G$ , i.e.,  $(a_n) \rightarrow a$  iff what?
- Consider a play of the Proximal game which yields sequences  $(a_n)$  and  $(V_{U_n})$ . What does it mean for there to exist some  $b \in G$  such that  $b \in \bigcap_{n \in \omega} (a_n \Delta V_{U_n})$ ? What does that look like?

## Thursday June 18

### 1. We talked about stars: how can we get a handle on the choices available to player 2?

### 2. We looked at how sequences in $\tau_{\mathcal{F}}$ behave, basically we just played with things trying to get a grasp at how things look

### 3. More questions:

- In general, when is a topological group Proximal or Semi Proximal?
- If a top group is prox then is it first countable? (or contrapositive)
- In general what can we say about top groups and uniformities on them?

### 4. Check mathscinet for resources that could be helpful

## Wednesday June 25

1. Due to the implicit assumption that  $\emptyset \in a\Delta V_F \forall a \in [\omega]^{<\omega} \forall F \in \mathcal{F}$  I was wrong about  $\text{st}(p, V_F) = [\omega]^{<\omega}$ , in fact it is equal to  $z\Delta V_F$  where  $z = p - F$ .
2. Task: ✓ describe what the winning strategy for player 1 looks like in the proximal game on  $\tau_{\mathcal{F}}$  when  $\mathcal{F}$  is countable generated:
  - $\{F_n : n < \omega\}$  generated  $\mathcal{F}$
  - Game has 3 data:  $(F_n, a_n, z_n)$ , where  $z_n = a_n - F_n$
  - Player 1 player  $F_{n_{k+1}} \subset F_{n_k}$  such that  $F_{n_{k+1}} \cap a_k = \emptyset$
  - 3 cases:  $\bigcap z_n = \emptyset, < \aleph_0, = \aleph_0$
3. Prove that  $\tau_{\mathcal{F}}$  is not proximal when  $\mathcal{F}$  is uncountably generated ✓
4. with the above established, and a clearer picture of what  $\tau_{\mathcal{F}}$  looks like, attack Paul's questions.

## Monday June 30

1. "correct" way to think about strategies and counterstrategies is to not think of them as functions, but rather to think of a play of the game as an infinite tree where each branch corresponds to a possible play by either player 1 or 2: Player 1 then has a winning strategy (proximal) if every path of the tree wins and Player 2 loses (semiproximal) if there exists at least one path that wins for Player 1.
2. The goal now is to construct an uncountable filter that is semiproximal:
  - start again with the cofinite filter
  - at stage  $\alpha < \omega_1$  we have the sets added to the filter at each previous step  $\mathcal{A} = \{A_\beta : \beta < \alpha\}$  as well as a countable collection of sequences  $\mathcal{B} = \{B_\beta : \beta < \alpha\}$
  - still on stage  $\alpha$ , we look at some "partial" strategy  $\rho$  for player 2 (i.e. a function defined on the elements of the filter existing so far) and look at the tree of all possible plays of the game w.r.t.  $\rho$  and extend the filter (add a new element  $A_\alpha$ ) such that there exists a branch of the decision tree corresponding to a convergent sequence
  - Through this process we essentially account for all partial sequences
  - The resulting filter is therefore semiproximal since we know any strategy for player 2 corresponds to a partial strategy which has a branch with a convergent sequence (i.e. losing for player 2) that would have been preserved after all remaining steps of construction
  - the previous item isn't exactly clear and may require more than CH, in particular require  $\diamond$
3. Look into  $\diamond$ .

## 3 Set Theory

### 3.1 Some Interesting Cardinals

#### 3.1.1 Subsets of ${}^\omega\omega$ and Almost Disjoint Families

**Definition 3.1.** Let  $\mathcal{A}$  be an infinite family of infinite subsets of  $\omega$ . The family  $\mathcal{A}$  is said to be an almost disjoint family (ADF) of subsets of  $\omega$  if  $A \cap B$  is finite for any  $A, B \in \mathcal{A}$  with  $A \neq B$ . By Zorn's lemma we can assume  $\mathcal{A}$  lives inside of a maximal (w.r.t. inclusion) family that is also pairwise disjoint. Such a family is a maximal almost disjoint family (MADF).

**Definition 3.2.** Define an order  $\leq^*$  on  ${}^\omega\omega$  where  $f <^* g$  if  $f(n) < g(n)$  for all but finitely many  $n \in \omega$

**Definition 3.3.** Let  $(A, \prec)$  be a linearly ordered set. We say that  $B \subset A$  is unbounded in  $A$  if there does not exist  $y \in A$  such that  $x \prec y$  for all  $x \in B$ . We say that  $C \subset A$  is cofinal (or dominating) in  $A$  if for all  $x \in A$  there exists  $z \in C$  such that  $x \prec z$ .

**Definition 3.4.**

- $\mathfrak{a} = \min\{|\mathcal{A}| : \mathcal{A} \text{ is infinite MADF}\}$
- $\mathfrak{b} = \min\{|B| : B \text{ is } \leq^* \text{-unbounded subset of } {}^\omega\omega\}$
- $\mathfrak{d} = \min\{|D| : D \text{ is } \leq^* \text{-cofinal in } {}^\omega\omega\}$

Each of  $\mathfrak{a}, \mathfrak{b}, \mathfrak{d}$  are at least  $\omega_1$  but at most  $\mathcal{P}(\omega)$ .

### 3.1.2 Towers and Pseudo-Intersections

**Definition 3.5.** Define an order on  $\mathcal{P}(X)$  for some set  $X$ . We say  $A \subset^* B$  if  $B$  contains all but finitely many points of  $A$ . If  $\mathcal{F} \subset \mathcal{P}(X)$  is a collection of (infinite) sets, then  $A$  is a pseudo-intersection of  $\mathcal{F}$  if  $A \subset^* F$  for all  $F \in \mathcal{F}$ .

**Definition 3.6.** We call  $\mathcal{T} \subset [\omega]^\omega$  a tower if  $\mathcal{T}$  is well ordered by  $\subset^*$  and has no infinite intersection.

**Lemma 3.7.** If  $\mathcal{A} = \{A_n : n \in \omega\} \subset [\omega]^\omega$  such that for all  $n \in \omega$

$$A_0 \supset^* A_1 \supset^* \dots \supset^* A_n$$

then  $\mathcal{A}$  has infinite pseudo-intersection.

**Theorem 3.8.** *There exists a tower.*

*Proof.* Define  $\langle A_\alpha : \alpha < \mathfrak{c} \rangle$  as follows:

1.  $A_\alpha = \omega \setminus n$  for  $\alpha < \omega$
2. If  $\alpha$  is a successor ordinal, define  $A_\alpha$  such that  $|A_{\alpha-1} \setminus A_\alpha| = \aleph_0$
3. If  $\alpha$  is a limit ordinal then define  $A_\alpha$  such that  $A_\alpha \subset^* A_\beta$  for all  $\beta < \alpha$

Suppose  $\gamma$  is given and  $\langle A_\alpha : \alpha < \gamma \rangle$  has been defined. If  $\gamma$  is a successor then define  $A_\gamma$  as in (2). Otherwise  $\gamma$  is a limit ordinal: if (3) isn't possible then  $\langle A_\alpha : \alpha < \gamma \rangle$  is a tower; otherwise define  $A_\gamma$  as in (3) and continue.

This process eventually terminates, otherwise we obtain  $\mathfrak{c}$  many distinct subsets of  $\omega$ , a contradiction.  
**TODO: elaborate** □

**Definition 3.9.**

- $\mathfrak{t} = \min\{|\mathcal{T}| : \mathcal{T} \text{ is a tower}\}$
- $\mathfrak{p} = \min\{|\mathcal{P}| : \mathcal{P} \subset [\omega]^\omega \text{ is a family with the SFIP that has no pseudo-intersection}\}$

## 4 Topology

### 4.1 Sequential and Fréchet Spaces

**Definition 4.1.** For a topological space  $X$  and any set  $A \subset X$ , the *sequential closure* of  $A$  is

$$[A]_{\text{seq}} := \left\{ x \in X : \exists (x_n) \in A \left( \lim_{n \rightarrow \infty} x_n = x \right) \right\}.$$

In general we can repeat this operation recursively  $[[[A]_{\text{seq}}]_{\text{seq}} \dots]_{\text{seq}}$  by which is meant the *total sequential closure* of  $A$ .

**Fact 4.2.** In general it takes at most  $\omega_1$  many iterations of the sequential closure to get a closed set.

**Definition 4.3.** A space  $X$  is said to be Fréchet if  $[A]_{\text{seq}} = \overline{A}$  for all  $A \subset X$ .

**Example 4.4.** Let  $X = \omega_1 + 1$  with the order topology.  $X$  is not Fréchet, since any sequence  $(x_n) \in \omega_1$  cannot converge to  $\infty$ , as otherwise  $\omega_1 = \sup\{x_n : n \in \mathbb{N}\}$ , a contradiction.

**Definition 4.5.** A space  $X$  is sequential if any closed set  $A \subset X$  is equal to its total sequential closure.

Then a space that is Fréchet is also sequential. The following example shows that the converse is not true.

**Example 4.6.** Let  $X^* = \omega \times (\omega + 1)$  be given the order topology and let  $X = X^* \cup \{\infty\}$  where the neighbourhoods of  $\infty$  are such that there exists  $p \in \omega$  such that  $|\{(m, n) : m > p, n \in \omega + 1\} - U_\infty| < \aleph_0$ . Then  $X$  is sequential but not Fréchet. To see this, note that for all  $m \in \omega$  the sequence  $A_m = \{(m, n) : n \in \omega\}$  converges to  $(m, \omega + 1)$  and moreover  $B = \{(m, \omega + 1) : m \in \omega\}$  is a sequence that converges to  $\infty$ . Then  $A = \bigcup_{m \in \omega} A_m$  is such that  $[[A]_{\text{seq}}]_{\text{seq}} = X$ , hence  $X$  is sequential. On the other hand there is no sequence in  $A$  that converges to  $\infty$ . Suppose there were, say some  $\gamma \rightarrow \infty$ . Then for all  $m \in \omega$ ,  $U_m = X - \{(m, n) : n \in \omega + 1\}$  is a neighbourhood of  $\infty$  such that  $|\gamma - U_m| < \aleph_0$ . Hence  $\gamma$  has only finitely many terms belonging to each column. If  $\alpha_m = \max\{\gamma \cap \{(m, n) : n \in \omega\}\}$ , then  $U = X - \bigcup_{m \in \omega} \{(m, n) : n \leq \alpha_m\}$  is a neighbourhood of  $\infty$  disjoint from  $\gamma$ , a contradiction. Hence  $X$  is not Fréchet.

**Proposition 4.7.** If  $X$  is first countable then  $X$  is Fréchet.

*Proof.* Let  $A \subset X$  and let  $x \in \overline{A}$ . Then  $x$  has a countable neighbourhood base  $N_x$  such that  $U \cap A \neq \emptyset$  for all  $U \in N_x$ . Enumerating the neighbourhoods of  $x$  as  $U_1, U_2, \dots$  then the sequence  $(x_n)_{n \geq 1}$  where  $x_n \in U_n \cap A$  for each  $n \in \omega$  is such that  $(x_n)_{n \geq 1}$  converges to  $x$ .  $\square$

The following example shows that the converse is not true.

**Example 4.8** (Fréchet Fan). Let  $S_\omega$  be the quotient of  $\omega \times (\omega + 1)$  obtained by identifying all the points  $\{(m, \omega + 1) : m \in \omega\}$  as  $\infty^*$ . More precisely  $S_\omega$  has the quotient topology induced by the map  $h : \omega \times (\omega + 1) \rightarrow S_\omega$  where  $h(x) = x$  for all  $x \in \omega \times \omega$  and  $h(x) = \infty^*$  for all  $x \in \omega \times \{\omega + 1\}$ . Then  $S_\omega$  is Fréchet but not 1st countable.

To see that  $S_\omega$  is Fréchet, let  $A \subset S_\omega$  such that  $\infty \in \overline{A}$ .  $A$  must meet at least one column of  $S_\omega$  in an infinite set, otherwise we could find a nbhd of  $\infty^*$  disjoint from  $A$ . Then  $A$  restricted to that column will be a convergent sequence to  $\infty$ .

Now assuming that  $S_\omega$  was countable, we would have a countable neighbourhood base at  $\infty^*$ . For each  $k \in \omega$  let  $B_k = \bigcup_{m \in \omega} \{m\} \times (f_k(m), \infty^*]$  for some  $f_k : \omega \rightarrow \omega$  determining the startpoints of each interval. Suppose  $\mathcal{B} = \{B_k : k \in \omega\}$  is a base at  $\infty^*$ , then let  $f^* : \omega \rightarrow \omega$  be defined by  $f^*(m) = f_m(m) + 1$  for all  $m \in \omega$ . Letting  $B^* = \bigcup_{m \in \omega} \{m\} \times (f^*(m), \infty^*]$  then  $B^*$  is an open neighbourhood of  $\infty^*$  but it is clear by construction that  $B_k \not\subset B^*$  for all  $k \in \omega$ . Hence  $\mathcal{B}$  cannot be a neighbourhood base and  $S_\omega$  is not first countable.

As the following example shows, the product of Fréchet spaces need not be Fréchet.

**Example 4.9.** Let  $X = (\omega + 1) \times S_\omega$ , and consider the set  $A = \{(m, (m, n)) : m, n \in \omega\}$ . If  $\infty^*$  is the identified point of  $S_\omega$ , let  $\infty = \{\omega + 1\} \times \infty^*$ . Then  $\infty \in \overline{A}$  but  $\infty \notin [A]_{\text{seq}}$ . The open neighbourhoods of  $\infty$  are of the form  $(\alpha, \omega + 1] \times (\bigcup_{m \in \omega} \{m\} \times (f(m), \infty^*])$ , which clearly always has non empty intersection with  $A$ . Hence  $\infty \in \overline{A}$ . To see that  $\infty \notin [A]_{\text{seq}}$ , suppose  $\gamma$  is a sequence in  $A$  that converges to  $\infty$ . Since the sets  $U_k = (k, \omega + 1] \times (\bigcup_{m \in \omega} \{m\} \times (f(m), \infty^*])$  are open neighbourhoods of  $\infty$  it must be the case that  $|\gamma - U_k| < \aleph_0$  for all  $k \in \omega$ . Thus  $\gamma \cap (\{k\} \times \{k\} \times (1, \omega + 1])$  is finite for every  $k$ . Let  $h : \omega \rightarrow \omega$  be defined by  $h(k) = \max\{\pi_3(\gamma \cap (\{k\} \times \{k\} \times (1, \infty^*])) + 1$  for  $k \in \omega$ . Pictorially,  $h$  is picking the point on each spine beyond which no elements of  $\gamma$  exist. Thus

$$W = (1, \omega + 1] \times \left( \bigcup_{n \in \omega} \{n\} \times (h(n), \infty^*] \right)$$

is an open neighbourhood of  $\infty$  which by construction is disjoint from  $\gamma$ . Hence  $\gamma$  cannot converge to  $\infty$  showing that  $X$  is not Fréchet.

## 4.2 $\alpha_i$ notions of convergence

**Definition 4.10.** Let  $X$  be a topological space and  $\xi$  be a countable family of sequences converging to a point  $x \in X$ . We say that  $x$  is an  $\alpha_i$  point for  $i = 1, 2, 3, 4$  if there exists a sequence  $B$  such that

- $\alpha_1$ :  $|A - B| < \aleph_0$  for every  $A \in \xi$ ;
- $\alpha_2$ :  $|A \cap B| = \aleph_0$  for every  $A \in \xi$ ;
- $\alpha_3$ :  $|A \cap B| = \aleph_0$  for infinitely many  $A \in \xi$ ;
- $\alpha_4$ :  $A \cap B \neq \emptyset$  for infinitely many  $A \in \xi$ .

Then  $X$  is an  $\alpha_i$  space if every  $x \in X$  is an  $\alpha_i$  point. Note that if a space is  $\alpha_i$  then it is  $\alpha_{i+1}$  for  $i = 1, 2, 3$ .

**Example 4.11.**  $S_\omega$  is not even  $\alpha_4$ . For each  $m \in \omega$  let  $A_m = \{m\} \times (1, \infty)$ . Then  $\xi = \{A_m : m \in \omega\}$  is a countable collection of sequences converging to  $\infty$ . Suppose  $B$  is a sequence that converges to  $\infty$  such that  $A \cap B \neq \emptyset$  for infinitely many  $A \in \xi$ . In particular let  $\alpha \leq \omega$  be such that  $A_i \cap B \neq \emptyset$  for all  $i \in \alpha$  and let  $f : \alpha \rightarrow \omega$  be defined by  $f(k) \in B \cap A_k$  for all  $k \in \alpha$ . Then

$$U = \left( \bigcup_{k \in \alpha} \{k\} \times (f(k) + 1, \infty] \right) \times \left( \bigcup_{m \in \omega - \alpha} \{m\} \times (1, \infty] \right)$$

is such that  $|B \cap U^C| = \aleph_0$ , hence  $B$  does not converge to  $\infty$ .

**Exercise 4.12.**  $X$  is  $\alpha_2$  iff whenever  $\xi$  is a countable collection of sequences converging to  $x$  there exists  $B \rightarrow x$  such  $A \cap B \neq \emptyset$  for all many  $A \in \xi$ .

**Solution.** The forward direction is obvious. Conversely, let  $\xi = \{A_1, A_2, \dots\}$  be a countable collection of sequences converging to  $x$ . For every  $n \in \omega$  let  $\{a_{nm} : m \in \omega\}$  enumerate the elements of  $A_n$  and define  $A_{nm} = A_n - \{a_{n1}, a_{n2}, \dots, a_{n,m-1}\}$  for each  $n, m \in \omega$ . Then the  $A_{nm}$  still converge to  $x$  and  $\mathcal{A} = \{A_{nm} : n, m \in \omega\}$  is a sheaf at  $x$ . By hypothesis there exists a  $B$  converging to  $x$  that meets each  $A_{nm}$ , and thus meets each  $A_n$  in an infinite set.

## 4.3 $\Psi$ -like spaces

**Definition 4.13.** Let  $D$  be a countable set and let  $\mathcal{A} \subset \mathcal{P}(D)$  be an ADF. Define a topology on  $X = D \cup \mathcal{A}$  such that the points of  $D$  are isolated and  $D$  is dense in  $X$ . Then for every  $A \in \mathcal{A}$  attach a point  $z_A$  where the nbhds of  $z_A$  consist of all sets of the form  $B(A, F) = \{z_A\} \cup (A - F)$  where  $F \subset A$  is finite.

**Proposition 4.14.** Let  $X$  be as above. Then  $X$  is Hausdorff, first countable and locally compact. If  $X$  is constructed as above but  $\mathcal{A}$  is MADF then  $X$  is also pseudo-compact (i.e. all continuous real functions on  $X$  are bounded).

*Proof.* To check Hausdorffness it suffices to show  $Z = X - D$  is Hausdorff. Let  $z_1, z_2 \in Z$ . Then  $z_1, z_2$  correspond to sets  $A_1, A_2$  such that  $A_1 \cap A_2 = I$  is finite. Then  $B(A_1, I)$  and  $B(A_2, I)$  are disjoint open sets containing  $z_1, z_2$  respectively.

To see that  $X$  is locally compact, note that  $\{x\}$  is a compact neighbourhood base for all  $x \in D$ . Local compactness of  $Z$  follows from the fact that by design each  $z_A \in Z$  is the limit of  $A$  viewed as a sequence. In other words, whenever  $B(A_z, F)$  is a basic nbhd of  $z_A$ , any open cover will have a smaller nbhd  $B(A_z, F')$  that omits only finitely many points.  $\square$

**Proposition 4.15.** The one point compactification of a  $\Psi$ -like space of cardinality  $< \mathfrak{a}$  is Fréchet.

## 4.4 Constructing Examples of $\alpha_i$ Spaces

### 4.4.1 Squares of $\alpha_i$ Spaces

**Example 4.16.** If  $X, Y$  are  $\alpha_1$ , then so is  $X \times Y$ . Let  $\xi = \langle \xi_n : n \in \omega \rangle$  be a sheaf at some  $x \times y \in X \times Y$ . Then  $\pi_i(\xi) := \langle \pi_i(\xi_n) : n \in \omega \rangle$  is a sheaf at  $\pi_i(x \times y)$  in  $X, Y$  respectively for  $i = 1, 2$ . Hence there exists  $\alpha_1$  sequences  $B_x, B_y$  for  $\pi_i(\xi)$ . Letting  $M_n = \max\{k \in \omega : x_{n,k} \notin \pi_1(\xi_n) \vee y_{n,k} \notin \pi_2(\xi_n)\}$  for each  $n \in \omega$ , the sequence  $B = \bigcup_{n \in \omega} B_x \cap (\xi_n - x_{n1}, \dots, x_{n,M_n}) \times B_y \cap (\xi_n - y_{n1}, \dots, y_{n,M_n})$  is as desired.

#### 4.4.2 Misc $\alpha_i$ Spaces

**Example 4.17.** Let  $D = \omega \times \omega$ , let  $\mathcal{F} = \{f_\alpha : \alpha < \mathfrak{b}\} \subset {}^\omega\omega$  be  $<^*$ -well-ordered and  $<^*$ -unbounded, and let  $\mathcal{C} = \{C_n = \{n\} \times \omega : n \in \omega\}$ . Construct a  $\Psi$ -like from  $X = \mathcal{F} \cup \mathcal{C}$ , where we attach compactification points  $\{n\} \times (\omega + 1)$  for each  $C_n$  and compactification points  $p_\alpha$  for each  $f_\alpha$ . Then the resulting one point compactification,  $X + \infty$ , is  $\alpha_2$  and Fréchet but not  $\alpha_1$ .  $X + \infty$  becomes  $\alpha_1$  if  $\mathcal{F}$  is cofinal and well ordered.

### 4.5 Frechet-Urysohn for Finite Sets

**Definition 4.18.** A  $\pi$ -network at a point  $x \in X$  is a collection  $\mathcal{F} \subset \mathcal{P}(X)$  such that for all open nbhds  $U_x$  there is  $F \in \mathcal{F}$  such that  $F \subset U_x$ .

**Definition 4.19.** We say that an infinite family of sets  $\langle F_n \subset \mathcal{P}(X) : n \in \omega \rangle$  converges to a point  $x \in X$  if  $|\{F_n : F_n \not\subset U_x\}| < \aleph_0$  for every open nbhd  $U_x$ .

**Definition 4.20.**  $X$  is Fréchet-Urysohn Finite (FUF) if for all  $x \in X$  whenever  $\mathcal{F} \subset [X]^{<\omega}$  is a  $\pi$ -network at  $x$  there exists  $\langle F_n : n \in \omega \rangle \subset \mathcal{F}$  such that the  $F_n$  converge to  $x$ . We say that  $X$  is  $n$ FUF if the condition holds for all  $\pi$ -networks  $\mathcal{F} \subset [X]^n$  and we say that  $X$  is boundedly FUF if  $X$  is  $n$ FUF for all  $n \in \omega$ .

**Example 4.21.**  $S_\omega$  is not 2FUF (and hence not FUF). Let  $F_m^n = \{(1, m), (m + 1, n)\}$  and let  $\mathcal{F} = \{F_m^n : m, n \in \omega\}$ . Then  $\mathcal{F}$  is a  $\pi$ -network at  $\infty$  with no convergent subcollection.

To see that  $\mathcal{F}$  is a  $\pi$ -network, let  $U$  be an open nbhd of  $\infty$ . Then  $U = \bigcup_{k \in \omega} \{k\} \times (f(k), \infty]$  for some  $f \in {}^\omega\omega$ . Letting  $m = f(1) + 1$  and  $n = f(m) + 1$ , then  $F_m^n \subset U$ .

Suppose  $\mathcal{G} \subset \mathcal{F}$  converges to  $\infty$ . Then since  $V_n = S_\omega - (\{1\} \times (1, n])$  is an open nbhd of  $\infty$  for each  $n \in \omega$ , we see that the sets  $\{G \in \mathcal{G} : G \not\subset V_n\}$  are each finite. In particular, this shows that there are only finitely many  $G \in \mathcal{G}$  that meet each column, hence for each  $n \in \omega$  there exists  $a_n \in \omega$  such that for all  $b \geq a_n$ ,  $F_n^b \notin \mathcal{G}$ . Letting  $g : \omega \rightarrow \omega$  be defined by  $g(n) = a_n$  we obtain the open nbhd  $W = \bigcup_{n \in \omega} \{n\} \times (g(n), \infty]$  which is such that  $G \not\subset W$  for all  $G \in \mathcal{G}$ , i.e.  $\{G \in \mathcal{G} : G \not\subset W\}$  is infinite so that  $\mathcal{G}$  does not converge to  $\infty$ .

**Proposition 4.22.** If  $X$  is first countable then  $X$  is FUF.

*Proof.* Suppose  $X$  is first countable, let  $x \in X$ , and let  $\mathcal{F}$  be a  $\pi$ -network at  $x$ . Let  $\mathcal{U} = \{U_0, U_1, \dots\}$  be a nbhd base at  $x$  and WLOG assume that  $U_0 \supset U_1 \supset \dots$  (since otherwise we could just take  $U_0, U_0 \cap U_1, \dots$ ). Then for each  $n \in \omega$  there is a  $F_n \in \mathcal{F}$  such that  $F_n \subset U_n$  and the  $F_n$  converge to  $x$ .  $\square$

**Definition 4.23.**  $\mathcal{A} \subset \mathcal{P}(X)$  is a network if for all  $x \in X$  and for all nbhds  $U \ni x$  there is  $A \in \mathcal{A}$  such that  $x \in A \subset U$ . If  $\mathcal{F}$  is a filter, then we might also refer to a network  $\mathcal{A}$  for  $\mathcal{F}$  where every  $F \in \mathcal{F}$  contains some  $A \in \mathcal{A}$ .

**Remark 4.24.** A network is a generalization of a base since we no longer require the elements to be open sets.

**Definition 4.25** (FUF filters). A filter  $\mathcal{F}$  on  $\omega$  is FUF if whenever  $A \subset [\omega]^{<\omega}$  is a network for  $\mathcal{F}$  then there exists  $B \subset A$  such that  $B$  converges wrt to  $\mathcal{F}$ .

**Lemma 4.26.** Let  $\mathcal{F}$  be a filter on  $[\omega]^\omega$ . If  $\mathcal{F}$  is countably generated then  $\mathcal{F}$  has pseudo-intersection.

**Proposition 4.27.** Assume CH. Then there exists a FUF filter  $\mathcal{F}$  that is not countably generated.

*Proof.* Starting with  $\mathcal{F}_0$  the cofinite filter on  $\omega$ , we will recursively construct a filter by adding a new set to our filter at each stage such that the FUF condition is preserved. We assume that  $\mathcal{P}([\omega]^{<\omega}) = \{A_n : n < \omega_1\}$  and will construct our filter in  $\omega_1$  many steps.

On step 0, take  $A_0 \subset [\omega]^{<\omega}$ . If it's a network for  $\mathcal{F}_0$  then we find  $B_0 \subset A_0$  such that  $B_0 \rightarrow \mathcal{F}_0$ , which exists since  $\omega$  is first countable. Otherwise, if  $A_0$  isn't a network let  $B_0$  be the empty sequence and proceed to the next step.

On step  $\alpha$ , let  $\mathcal{F}_\alpha^* = \mathcal{F}_0 \cup (\bigcup_{i < \alpha} \{X_i\})$  and let  $\mathcal{F}_\alpha = \mathcal{F}_\alpha^* \cup \{X_\alpha\}$  where  $X_\alpha$  will be defined in such a way as to preserve any previous convergent sequences, which we denote  $\{B_\beta : \beta < \alpha\}$ . Note that  $\mathcal{F}_\alpha^*$  is countably generated, as  $\mathcal{F}_0$  is countably generated and we have progressed only countably many steps. So we have  $F_0 \supset F_1 \supset \dots$  which generate  $\mathcal{F}_\alpha^*$ . Let  $X_\alpha = \bigcup_{i < \alpha} F_i \cap (\cup B_i)$ . Since each  $F_i$  contains a tail of every  $B_\beta$ , we



have that  $X_\alpha \supset^* B_\beta$  for all  $\beta$ . On the other hand, none of the  $F_n$  generate  $X_\alpha$  since  $F_1 \cap B_1 \not\supset F_k$  for all  $k \geq 1$  and in general  $F_n \cap B_n \not\supset F_k$  for all  $n \in \omega, k \geq n$ . Hence  $X_\alpha \notin \mathcal{F}_\alpha^*$  and  $\mathcal{F}_\alpha$  is as desired.

We continue step  $\alpha$  by taking  $A_\alpha \subset [\omega]^\omega$ . If it's a network for  $\mathcal{F}_\alpha$ , find another sequence  $B_\alpha \subset A_\alpha$  that converges with respect to  $\mathcal{F}_\alpha$ , otherwise if  $A_\alpha$  isn't a network let  $B_\alpha$  be the empty sequence proceed to the next step.

Let  $\mathcal{F}$  be the filter obtained after recursively iterating through all  $A_n \subset [\omega]^{<\omega}$ , i.e.  $\mathcal{F} = \mathcal{F}_0 \cup (\bigcup_{\alpha < \omega_1} \{X_\alpha\})$ . By construction  $\mathcal{F}$  is FUF and to see that its uncountably generated let  $\mathcal{G} \subset \mathcal{F}$  be a generating set. Then for some  $\alpha < \omega_1$  there's  $\mathcal{G}_\alpha \subset \mathcal{G}$  which generates  $\mathcal{F}_\alpha^*$ . But  $\mathcal{F}_\alpha^*$  was also generated by  $F_0 \supset F_1 \supset \dots$  and  $X_\alpha$  was such that  $X_\alpha \not\supset F_n$  for all  $n \in \omega$ . Since the  $F_n$  would necessarily also generate  $\mathcal{G}_\alpha$  this implies that  $X_\alpha \not\supset G_n$  for all  $n \in \omega$ . Hence  $\mathcal{G}_\alpha$  generates  $\mathcal{F}_\alpha^*$  but not  $\mathcal{F}_\alpha$ . Furthermore,  $\mathcal{G}_\alpha$  cannot generate  $\mathcal{F}_\eta$  either, for  $\alpha < \eta < \omega_1$ , which follows from the definition of the  $X_\eta$ . Thus as  $\mathcal{F}$  is obtained from  $F_\alpha$  by adding uncountably many new sets, each of which not generated by  $\mathcal{G}_\alpha$ , it must be the case that  $\mathcal{G}$  is uncountable.  $\square$

## 4.6 Uniformities

In general uniformities provide an alternate way to generalize metric spaces. Topological spaces generalize metric spaces while preserving the notion of continuous functions whereas uniform spaces preserve uniformly continuous functions. Given a set  $X$  there are two ways to construct a uniformity  $\mathcal{D}$  on  $X$ .

### 4.6.1 Diagonal Uniformities

A diagonal uniformity on a set  $X$  is a principal filter  $\mathcal{D}$  on the square  $X \times X$  generated by the diagonal satisfying

1. For all  $D \in \mathcal{D}$  there exists  $E \circ E \subset D$  for some  $E \in \mathcal{D}$
2.  $D^{-1} \in \mathcal{D}$  for all  $D \in \mathcal{D}$

where  $E \circ E = \{(x, y) \in E \times E : \exists z \in E ((x, z) \in E \wedge (z, y) \in E)\}$  and  $D^{-1} = \{(y, x) : (x, y) \in D\}$ . Note that  $E \circ E$  will often be abbreviated  $E^2$  and we may at times go so far as to use  $E^4 = (E^2)^2 = E \circ E \circ E \circ E$ . Also, whenever  $D \in \mathcal{D}$  is such that  $D = D^{-1}$  we will say that  $D$  is symmetric. A diagonal uniformity can be generated by filterbase consisting of symmetric sets  $D$  each containing the diagonal. The pair  $(X, \mathcal{D})$  will be called a uniform space.

If  $(X, \mathcal{D})$  is a uniform space then for any  $x \in X$  and  $D \in \mathcal{D}$  we say that  $D[x] = \{y \in X : (x, y) \in D\}$  is the section of  $D$  at  $x$ . More generally, we can also talk about  $D[A] = \{y \in X : (x, y) \in D, x \in A\}$ . Although the former is used quite frequently because the collection  $\tau_{\mathcal{D}} = \{D[x] : x \in X, D \in \mathcal{D}\}$  is a base for a topology on  $X$ , and the most natural way to pass from uniform to topological spaces.

### 4.6.2 Uniform Covers

## 4.7 Topological Groups

**Definition 4.28.** A topological group  $G$  is a group whose underlying set has a topology such that

- the group operation is continuous;
- the map that sends  $x \mapsto x^{-1}$  is continuous.

For sets  $A, B \subset G$  and any point  $g \in G$  we denote  $AB = \{a \cdot b : a \in A, b \in B\}$  and  $gA = \{g \cdot a : a \in A\}$ .

We will often construct topological groups from topological space. Take  $X = \{\infty\} \cup \omega$  with the points of  $\omega$  isolated and let  $\mathcal{F}$  a filter on  $\omega$ . Then  $G = [\omega]^{<\omega}$  with the operation symmetric difference  $\Delta : G \times G \rightarrow G$  defined by  $a \Delta b = a - b \cup b - a$  and  $\emptyset$  the identity will be our group. For each  $U \in \mathcal{F}$  set  $V_U \subset [\omega]^{<\omega}$  where  $V_U = \{a \in [\omega]^{<\omega} : a \subset U\}$ . Then  $\mathcal{V} = \{V_U : U \in \mathcal{F}\}$  is a nbhd base at  $\emptyset$  and  $\tau_{\mathcal{F}} = \{a \Delta V_U : a \in [\omega]^{<\omega}, U \in \mathcal{F}\}$  is a base for a topology on  $G$ .

**Proposition 4.29.** If  $\mathcal{F}$  is FUF then  $\tau_{\mathcal{F}}$  is Frechet.

*Proof.* Let  $\mathcal{F}$  be FUF and let  $A \subset \tau_{\mathcal{F}}$  be such that  $x \in \overline{A}$ . WLOG assume that  $x = \emptyset$ . Then  $\mathcal{A} = \{a \in A \cap [F]^{\leq \omega} : F \in \mathcal{F} \text{ is a network for } \mathcal{F} \text{ and hence there is a } B \subset \mathcal{A} \text{ such that } B \rightarrow \mathcal{F}, \text{ i.e., } B \rightarrow x.\}$

Conversely, let  $\mathcal{A}$  be a network for  $\mathcal{F}$  and let  $[\mathcal{A}]^{< \omega} = \{[a]^{< \omega} : a \in \mathcal{A}\}$ . Then  $\emptyset \in [\mathcal{A}]^{< \omega}$  since there is  $a \subset F$  implies  $[a]^{< \omega} \subset [F]^{< \omega}$  hence  $[\mathcal{A}]^{< \omega} \cap [F]^{< \omega} \neq \emptyset$  for all  $F \in \mathcal{F}$ . Thus there is a  $B \subset [\mathcal{A}]^{< \omega}$  that converges to  $\emptyset$  and hence  $B \rightarrow \mathcal{F}$ .  $\square$

**Proposition 4.30.**  $\mathcal{F}$  is countably generated  $\iff \tau_{\mathcal{F}}$  is second countable  $\iff \tau_{\mathcal{F}}$  is metrizable.

**Example 4.31.** If  $\mathcal{F}$  is the cofinite filter on  $\omega$  then in the group  $\tau_{\mathcal{F}}$  the sequence defined by  $x_n^m = \{n, n+1, \dots, n+m\}$ , for any  $m < \omega$ , converges to  $\emptyset$ . Letting  $m = 1$  and using the homogeneity of  $\tau_{\mathcal{F}}$  we obtain a simple recipe to construct sequences converging to any point  $p \in [\omega]^{< \omega}$ ; just take  $y_n = x_n^1 \Delta p$  for all  $n \in \omega$ .

## 5 Topological Games

### 5.1 Gruenhage Game

Let  $x \in X$  be a point chosen before the start of game. On turn zero, player 1 chooses an open nbhd  $U_0$  of  $x$  and player 2 chooses a point  $x_0 \in U_0$ ; on the  $n$ 'th turn player one chooses an open nbhd  $U_n$  of  $x$  and player 2 chooses a point  $x_n \in U_n$ . Player 1 wins if the sequence of points  $\{x_n : n \in \omega\}$  converges to  $x$ .

Formally, we can describe a strategy for player 1 as a mapping  $\sigma : [X]^{< \omega} \rightarrow \mathcal{O}_x$  from finite sequences in  $X$  to the collection of open sets containing  $x$  where  $\sigma(\emptyset) = X$ . In particular we say that  $\langle x_n : n \in \omega \rangle$  is a  $\sigma$ -sequence provided  $x_{n+1} \in \sigma(\{x_1, \dots, x_n\})$  for all  $n$ . We say that  $\sigma$  is a winning strategy if every  $\sigma$ -sequence converges to  $x$ .

A strategy for play 2 is a mapping  $\tau : \mathcal{S}(x) \times \mathcal{O}_x \rightarrow X$  such that  $\tau(F, U) \in U$  for all  $F \in \mathcal{S}(x)$  and  $U \in \mathcal{O}_x$ .

**Definition 5.1.** We say that  $X$  is a W-space if there is a winning strategy for player 1 at every point  $x \in X$ . We say that  $X$  is a w-space if for every strategy for player 2 there exists a counterstrategy for player 1 that wins.

**Definition 5.2.** Let  $x \in X$  and let  $\tau : \mathcal{S}(X) \times \mathcal{O}_x(X) \rightarrow X$  be a strategy for player 2 at  $x$ . Say that  $\mathcal{F}_{\tau} = \langle F_n : n \in \omega \rangle$  is a  $\tau$ -chain if for all  $n \in \omega$  we have  $F_n \subset F_{n+1}$  and  $F_{n+1} = F_n \cup \{\tau(F_n, U)\}$  for some  $U \in \mathcal{O}_x$ .

**Proposition 5.3.**  $X$  is a w-space iff it is  $\alpha_2$  and Fréchet.

*Proof.* The forward direction is similar to above result, with the difference that we define a strategy  $\tau$  for player 2 such that  $\tau(F, U_x) \in U_x \cap A$  on the one hand, and  $\tau(F, U_y) \in U_y \cap \xi_{|F|}$  on the other hand; then since  $X$  is a w-space, we obtain a counter strategy that wins for player 1, i.e., the desired convergent sequence.

Now suppose  $X$  is  $\alpha_2$  and Fréchet and let  $\tau : \mathcal{S}(X) \times \mathcal{O}_x(X) \rightarrow X$  be a strategy for player 2 at  $x$ . For each  $F \in \mathcal{S}(X)$  let  $\tau_F = \{\tau(F, U) : U \in \mathcal{O}_x\}$  and note that  $x \in \overline{\tau_F}$  for all  $F$ . By Fréchetness, each  $\tau_F$  contains an  $A_F$  that converges to  $x$  and  $\xi = \langle A_F : F \in \mathcal{S}(X) \rangle$  is a sheaf at  $x$ . Let  $B$  be a sequence that converges to  $x$  which meets each  $A_F$  and let  $\sigma : [X]^{< \omega} \rightarrow \mathcal{O}_x$  be a counterstrategy for player 1 such that  $\tau(F, \sigma(F)) \in B \cap A_F$  for all  $F \in [X]^{< \omega}$ . Then  $\sigma$  wins for player 1. **This assumes  $X$  is countable.**

Suppose  $X$  is  $\alpha_2$  and Fréchet and let  $x \in X$  be a point. If  $\tau : \mathcal{S}(X) \times \mathcal{O}_x(X) \rightarrow X$  is a strategy for player 2 at  $x$ , then for each  $F \in \mathcal{S}(X)$  let  $\tau_F = \{\tau(F, U) : U \in \mathcal{O}_x\}$  and note that  $x \in \overline{\tau_F}$ , which by Fréchetness of  $X$ , implies the existence of a  $A_F \subset \tau_F$  that converges to  $x$  for each  $F \in \mathcal{S}(X)$ . Let  $\mathfrak{F}$  be a collection of  $\tau$ -chains such that for each  $y \in X$  there exists exactly one  $\mathcal{F}_{\tau_y} \in \mathfrak{F}$  such that  $\{y\} \in \mathcal{F}_{\tau_y}$  (AC). Observe that to each  $\mathcal{F}_{\tau_y} \in \mathfrak{F}$  there corresponds a countable sheaf  $\xi_y = \langle A_F : F \in \mathcal{F}_{\tau_y} \rangle$  and as  $X$  is  $\alpha_2$ , we therefore obtain for each  $\mathcal{F}_{\tau_y}$  a sequence  $B_y$  that meets each  $A \in \xi_y$  and converges to  $x$ .

Let  $\sigma : \mathcal{S}(X) \rightarrow \mathcal{O}_x$  be a counter strategy for player 1 defined recursively such that  $\tau(\{y\}, \sigma(\{y\})) \in B_y \cap A_{\{y\}}$  and  $\tau(F, \sigma(F)) \in B_y \cap A_F$  if  $F \in \mathcal{F}_{\tau_y}$ . To see that  $\sigma$  wins for player 1, let  $\langle x_n : n \in \omega \rangle$  be a  $\sigma$ -sequence. Letting  $F_n = \{x_1, \dots, x_n\}$  for all  $n \in \omega$  we observe that  $F_1 = \{x_1\}$ ,  $x_{n+1} \in \sigma(F_n)$ , and  $x_{n+1} = \tau(F_n, \sigma(F_n))$ . Thus the collection  $\langle F_n : n \in \omega \rangle$  forms the  $\tau$ -chain  $\mathcal{F}_{\tau_{x_1}}$  which implies  $\langle x_n \rangle = B_{x_1}$ , hence  $\langle x_n : n \in \omega \rangle$  converges to  $x$ .  $\square$

## 5.2 Proximal Game

This game is due to Bell and is described as follows. Let  $(X, \mathcal{D})$  be a uniform space. On turn zero, player 1 chooses an entourage  $D_0 \in \mathcal{D}$  and player 2 chooses a point  $x_0 \in X$ . On the next turn, player 1 chooses  $D_1 \in \mathcal{D}$  with  $D_1 \subset D_0$  and player 2 chooses  $x_1 \in D_0[x_0]$ . The game continues and in general on the  $n$ 'th turn player 1 chooses  $D_{n+1} \in \mathcal{D}$  with  $D_{n+1} \subset D_n$  and player 2 chooses  $x_{n+1} \in D_n[x_n]$ . Player 1 wins if either:

1. there exists  $x \in X$  such that the  $\langle x_n : n \in \omega \rangle$  converge to  $x$ ;
2.  $\bigcap_{n \in \omega} D_n[x_n] = \emptyset$ .

**Definition 5.4.** A strategy for player 1 in the Proximal game is a mapping  $\rho : \mathcal{S}(X) \rightarrow \mathcal{D}$  such that  $x_{n+1} \in \rho(F)[x_n]$  for all  $n \in \omega$  and  $\rho(F) \supset \rho(G)$  if  $G \subset F$ . Note that  $F \in \mathcal{S}(X)$  represents player 2's first  $n$  choices if  $|F| = n$  and  $\rho(F)$  represents player 1's choice on turn  $n + 1$ . We will denote  $\rho(\emptyset)$  as  $X \times X$ .

**Definition 5.5.** We say that  $X$  is Proximal if player 1 has a strategy  $\rho$  such that every  $\rho$ -sequence converges. We say that  $X$  is Semi-Proximal if for every strategy for player 2, player 1 has a counter strategy that wins.

**Remark 5.6.** Paul explained the game to me in a slightly different way.  $X$  is assumed to be compact. On turn one player 1 begins by playing a finite open cover (FOC)  $\mathcal{U}_1$  and player 2 continues by picking a point  $x_1 \in X$ . On the second turn player 1 chooses a FOC  $\mathcal{U}_2$  such that  $\mathcal{U}_2 \prec \mathcal{U}_1$  and player 2 picks a point  $x_2 \in \text{st}(x_1, \mathcal{U}_2)$ . Then player 1 wins if either  $\bigcap_{n \in \omega} \text{st}(x_n, \mathcal{U}_n) = \emptyset$  or the  $\langle x_n \rangle$  converge to some point.

**Example 5.7.** The double arrow space is  $X = [0, 1] \times \{0, 1\}$  with the order topology; it is not Semi-Proximal. As  $X$  is compact, there exists a unique uniformity that generates the topology. Player 1 plays FOCs of the form  $\mathcal{U} = \{(a, b] \times \{0\} \cup [a, b) \times \{1\} : a, b \in [0, 1]\}$  and player 2 picks points from alternating rows every turn. Player 2's strategy will always be viable, since the star of any point wrt to any FOC will always intersect both rows.

**Proposition 5.8.** If  $X$  is semi-proximal then  $X$  is  $\alpha_2$  Fréchet.

*Proof.* Suppose  $X$  is not Fréchet. Then there exists some  $A \subset X$  with  $p \in \overline{A}$  such that no sequence in  $A$  converges to  $p$ . Then player 2 can adopt a strategy as follows. On turn one, player 1 picks  $U_1$  and player 2 picks  $x_1 = p$ . On turn 2, player 1 picks  $U_2$  and player 2 picks  $x_2 \in U_2 \cap A[p]$ , which by symmetry allows player 2 to pick  $p \in U_2 \cap A[x_2]$  on the third turn. Thus in general, player 2 picks points in  $A$  such that  $p$  always remains a valid play on the next turn, and continues to pick  $p$  ever other turn. By symmetry,  $p \in \bigcap U_i[x_i]$ , and by assumption, the sequence chosen by player 2 cannot converge, hence player 1 loses so that  $X$  is not semi-proximal.

Similarly, if  $X$  is not  $\alpha_2$ , then we can find a point  $p$  and a sheaf  $\xi$  at  $p$  such that any sequence that meets each  $\xi_i$  doesn't converge to  $p$ . As before, player 2 picks  $p$  on turn one, but on turn 2, picks  $x_2 \in U_2 \cap \xi_1$  and in general picks  $x_{2n} \in U_{2n} \cap \xi_n$ .  $\square$

**Proposition 5.9.** If player 1 (player 2) wins (loses) in the proximal game,

*Proof.* Let  $(X, \mathcal{D})$  be a uniform spaces and let  $\rho$  be a winning strategy for player 1 in the proximal game. Let  $\tau$  be a topology on  $X$  generated by the basis  $\mathcal{B} = \{D[x] : D \in \mathcal{D}, x \in X\}$ , let  $x \in X$ , and let  $\sigma : \mathcal{S}(X) \rightarrow \mathcal{O}_x$  be a strategy for player 1 in the Gruenhage game played at  $x$  defined by  $\sigma(F) = \rho(F)[x]$  for all  $F \in \mathcal{S}(X)$ .

If  $\langle x_n : n \in \omega \rangle$  is a  $\sigma$ -sequence, i.e.,  $x_{n+1} \in \sigma(\langle x_1, \dots, x_n \rangle)$  for all  $n \in \omega$ , then  $x_{n+1} \in \rho(\langle x_1, \dots, x_n \rangle)[x]$  and  $x_n \in \rho(\langle x_1, \dots, x_{n-1} \rangle)[x]$ . As the elements of  $\mathcal{D}$  are symmetric, we have both  $x \in \rho(\langle x_1, \dots, x_n \rangle)[x_{n+1}]$  and  $x \in \rho(\langle x_1, \dots, x_{n-1} \rangle)[x_n]$ , which implies  $x_{n+1} \in (\rho(\langle x_1, \dots, x_n \rangle) + \rho(\langle x_1, \dots, x_{n-1} \rangle))[x_n]$ . Thus  $x_{n+1} \in \rho(\langle x_1, \dots, x_{n-1} \rangle)[x_n]$  as  $\rho(\langle x_1, \dots, x_{n-1} \rangle) \supset \rho(\langle x_1, \dots, x_n \rangle)$  since  $\rho$  is a well defined strategy. Hence  $\langle x_n : n \in \omega \rangle$  is a  $\rho$ -sequence and  $x \in \bigcap_{n \in \omega} \rho(\langle x_1, \dots, x_{n-1} \rangle)[x_n]$ , which implies that  $\langle x_n \in \omega : n \in \omega \rangle$  must converge to  $x$ , which follows by our assumption that  $X$  is Hausdorff.  $\square$

## 5.3 Proximal Game on Topological Groups

The game is played the same but we note that the uniformities chosen by player 1 are formed as follows. On any given turn player 1 picks  $U \in \mathcal{F}$  which corresponds to  $D_U = \bigcup_{a \in [\omega]^{<\omega}} (a \Delta V_U)^2$ . In terms of uniform covers we might also talk about  $\mathcal{U}_U = \{(a \Delta V_U) : a \in [\omega]^{<\omega}\}$ . The game then proceeds as usual.

**Example 5.10.**  $(\mathbb{R}, +)$  is Proximal. A winning strategy for player 1 consists of picking uniform covers of radius  $\frac{1}{2^k}$  on turn  $k$ .

## 6 Musings

The point of this section is to keep track of thoughts relating to problems whose solutions are maybe not be clear or known (yet).

### 6.1 Behaviour ...

#### ... of Sequences

- $x_n \rightarrow x, \in \tau_{\mathcal{F}} \iff a\Delta x_n \rightarrow a\Delta x$
- WLOG we can assume  $x = \emptyset$
- $x_n \rightarrow x \iff \{x_n\}$  is a network for  $\mathcal{F}$ .

**Proposition 6.1.** Let  $\mathcal{F}$  be cofinite filter on  $\omega$  and let  $\langle x_n : n < \omega \rangle \subset [\omega]^{<\omega}$  such that  $x_n \rightarrow x$  for some  $x \in [\omega]^{<\omega}$ . There exists  $k \in \omega$  such that  $x = \bigcap_{n>k} x_n$ . Let  $\mathcal{F}$  be cofinite filter on  $\omega$  and let  $\langle x_n : n < \omega \rangle \subset [\omega]^{<\omega}$  such that  $x_n \rightarrow x$  for some  $x \in [\omega]^{<\omega}$ . There exists  $k \in \omega$  such that  $x = \bigcap_{n>k} x_n$ .

*Proof.* We show first that there exists  $k \in \omega$  such that  $x \subset \bigcap_{n>k} x_n$  and then show that the inclusion cannot be proper. Let  $p \in x$  and assume there is no  $k$  such that  $x \subset \bigcap_{n>k} x_n$ . Then we can find a subsequence  $y_n$  such that  $p \notin y_n$  for all  $n$ . Let  $F = \omega - p$ . Then  $p \in a$  for all  $a \in x\Delta V_F$ , so that  $y_n \notin x\Delta V_F$  for all  $n \in \omega$ , contradiction.

We have a  $k$  such that  $x \subset \bigcap_{n>k} x_n$ . Suppose there's a  $b \in \bigcap_{n>k} x_n$  but  $b \notin x$ . Let  $F = \omega - \bigcap_{n>k} x_n$ . Then  $b \notin a$  for all  $a \in x\Delta V_F$  so that  $x_n \not\subset x\Delta V_F$  for all  $n > k$ .  $\square$

**Remark 6.2.** The above proposition goes only one way. Take

$$x_n = \begin{cases} \{1\} & n \text{ is odd} \\ \{1, 2\} & n \text{ is even.} \end{cases}$$

Then  $\bigcap_{n>k} x_n = \{1\}$  for all  $k < \omega$  but  $x_n \not\rightarrow \{1\}$  (or  $\{1, 2\}$ ). In fact  $x_n$  doesn't converge to anything. Firstly, take  $F = \omega - \{1, 2\}$ , then  $2 \notin a$  for all  $a \in \{1\}\Delta V_F$  but  $2 \in a$  for all  $a \in \{1, 2\}\Delta V_F$  so that neither  $\langle x_n \rangle \subset^* \{1\}\Delta V_F$  nor  $\langle x_n \rangle \subset^* \{1, 2\}\Delta V_F$ . Moreover, it's clear that  $\langle x_n \rangle \not\subset^* p\Delta V_F$  for any other  $p \in [\omega]^{<\omega}$ .

#### ... of Stars

Let  $p \in [\omega]^{<\omega}$ , let  $F \in \mathcal{F}$ . What does  $S = \text{st}(p, \mathcal{U}_F)$  look like?

If  $p \subset F$ , then

- $p \in a\Delta V_F \iff p \not\subset a$
- $[\omega - p]^{<\omega} \subset S$  (take  $x \in [\omega - p]^{<\omega}$  and let  $a = x \cap F^C, b = x \cap F$ , then  $x = a\Delta b$  and  $b \in V_F$  so  $x \in S$ ).

If  $p \not\subset F$ , then

- $p \in a\Delta V_F \iff p \subset a$
- $S = [\omega]^{<\omega}$  ( $p\Delta V_F = [F \cup p]^{<\omega} = V_{F \cup p} \in S$ , so for any  $x \in [\omega]^{<\omega}$  letting  $a = x \cap (V_{F \cup p})^C$  then  $x = a\Delta F_{F \cup p} \in S$ )

Actually for any  $F \in \mathcal{F}$  and any  $p \in [\omega]^{<\omega}$ ,  $\text{st}(p, V_F) = [\omega]^{<\omega}$ : let  $\alpha = p \cap F^C$ , then  $p \in \alpha\Delta V_F = [F \cup \alpha]^{<\omega} = V_{F \cup \alpha} \in \text{st}(p, V_F)$ . For  $x \in [\omega]^{<\omega}$ , take  $a = x \cap (F \cup \alpha)^C$  so that  $x \in a\Delta V_{F \cup \alpha}$ .

**Remark 6.3.** The above shows that the only way for player 1 to win in the proximal game on  $\tau_{\mathcal{F}}$  is for the sequence of points picked by player 2 to converge, i.e., the intersection of the stars will just always be the entire space.

**Remark 6.4.** Actually  $\tau_{\mathcal{F}}$  is not Proximal (at least when  $\mathcal{F}$  is the cofinite filter). Since the star of any point w.r.t. any  $F \in \mathcal{F}$  is always equal to the entire space, there is nothing to prevent player 2 from picking a divergent sequence. So there is a first countable topological group that is not proximal.

### THE ABOVE IS ALL WRONG

1. Due to the implicit assumption that  $\emptyset \in a\Delta V_F \forall a \in [\omega]^{<\omega} \forall F \in \mathcal{F}$  I was wrong about  $\text{st}(p, \mathcal{U}_F) = [\omega]^{<\omega}$
2. Actually for  $p \in [\omega]^{<\omega}$  and  $F \in \mathcal{F}$ ,  $\text{st}(p, \mathcal{U}_F) = z\Delta V_F$  where  $z = p - F$
3. Note that for  $F \in \mathcal{F}$ ,  $V_F$  is a subgroup of  $\tau_{\mathcal{F}}$  and thus  $\mathcal{U}_F = \{a\Delta V_F\}$  is a partition of  $\tau_F$  (read set of cosets of  $\tau_F$ ).
4. thus when player 2 picks points  $a_{n+1} \in \text{st}(a_n, \mathcal{U}_{F_n})$  it is really the case that player 2 is picking  $a_{n+1}$  such that  $a_{n+1} \equiv z_n \pmod{F_n}$  where  $z_n = a_n - F_n$

### ... of Intersections

Suppose  $\langle a_n : n < \omega \rangle \subset [\omega]^{<\omega}$  and  $\langle F_n : n < \omega \rangle \subset \mathcal{F}$  and  $b \in \bigcap_{n < \omega} a_n \Delta V_{F_n}$ .

- this does not imply that  $a_n \rightarrow b$ . For example  $b \not\subset a_n$  and  $b \subset F_n$  for all  $n \in \omega$  is possible but we still have  $b \in a_n \Delta V_{F_n}$  for all  $n$ .
- On the other hand, if  $b \in \bigcap_{n < \omega} a_n \Delta V_{F_n}$  and we want  $a_n \rightarrow b$ , this would only be possible if  $b \not\subset F_n$  for all  $n > k$  for some  $k$ .

## 6.2 (partial) Results

**Observation 6.5.** If  $a \in [\omega]^{<\omega}$  and  $F \in \mathcal{F}$  and  $z = a - F$ , then  $\text{st}(a, \mathcal{U}_F) = z\Delta V_F$ . If  $p \in \text{st}(a, \mathcal{U}_F)$  then  $p \equiv z \pmod{F}$ .

**Observation 6.6.** The play of the Proximal game on  $\tau_{\mathcal{F}}$  is the triple  $(F_n, a_n, z_n)$ .

We know that second countable and metrizable implies proximal, thus if  $\mathcal{F}$  is a filter on  $\omega$  generated by  $\mathcal{A} = \langle F_n : n < \omega \rangle$ , it must be the case that  $\tau_F$  is Proximal.

Define a strategy for player 1 in the proximal game on  $\tau_F$  as follows:

<b>Player I</b>	$F_0 \in \mathcal{A}$		$F_1 \in \mathcal{A}, F_1 \cap a_0 = \emptyset$		...
<b>Player II</b>		$a_0$		$a_1 \equiv z_0 \pmod{F_0}$	...

In general, **Player I** plays covers from  $\mathcal{A}$  and on turn  $k + 1$  **Player I** plays  $F_{k+1} \subset F_k$  such that  $F_{k+1} \cap p_n = \emptyset$ .

**Lemma 6.7.** Let  $\mathcal{F}$  be countable generated filter on  $\omega$ , let  $\langle F_n : n < \omega \rangle$  be a sequence of generating sets, and let  $\langle a_n : n < \omega \rangle \subset [\omega]^{<\omega}$ . If the following holds for all  $n < \omega$ :

- $F_{n+1} \cap a_n = \emptyset$ ;
- $a_{n+1} \equiv z_n \pmod{F_n}$  where  $z_n = a_n - F_n$

then

- a) if  $\bigcup z_n$  is finite, then  $\langle a_n \rangle$  is a convergent sequence;
- b) if  $\bigcup z_n$  is infinite, then  $\bigcap \text{st}(a_n, \mathcal{U}_{F_n}) = \emptyset$ .

*Proof.*

- a) If  $\bigcap z_n = \emptyset$ , then
- b) Since  $z_n \cap F_n = \emptyset$  for all  $n \in \omega$ ,  $b \in z_n \Delta F_n$  means that  $z_n \subset b$ . Thus if there exists  $b \in \bigcap \text{st}(a_n, \mathcal{U}_{F_n}) = \bigcap z_n \Delta F_n$ , then  $z_n \subset b$  for all  $n$ . Since the  $z_n$  are increasing and their union is infinite we have that  $b$  must be infinite, a contradiction since  $b \in [\omega]^{<\omega}$ .

□

**Proposition 6.8.** When  $\mathcal{F}$  is a countably generated filter on  $\omega$  then  $\tau_{\mathcal{F}}$  is Proximal.

*Proof.* By the lemma, the strategy for **Player I** described above wins. **TODO: elaborate: why is there only the cases  $z_n$  finite/infinite?** □

**Proposition 6.9.** If  $\mathcal{F}$  is an uncountably generated filter on  $\omega$  then  $\tau_{\mathcal{F}}$  is not Proximal.

*Proof.* Let  $\sigma : [[\omega]^{<\omega}]^{<\omega} \rightarrow \mathcal{F}$  be any strategy for **Player I**. On turn 0 **Player II** can pick any  $p_0 \in [\omega]^{<\omega}$ , and therefore has countably many choices; we can enumerate them as  $\mathcal{B}_0 = \{p_0^i : i < \omega\}$ . In response to any choice by **Player II**, say  $p_0^k$ , **Player I** responds by playing  $\sigma(p_0^k)$ . Hence the possible responses by **Player I** are  $\mathcal{A}_0 = \{\sigma(p_0^i) : i < \omega\}$ .

On the  $n+1$ 'th turn, **Player II** picks from  $\text{st}(p_n, \mathcal{U}_{F_n}) \subset [\omega]^{<\omega}$  and again has countably many choices  $\mathcal{B}_n = \{p_n^i : i < \omega\}$  and the possible responses from **Player I** are  $\mathcal{A}_n = \{\sigma(p_n^i) : i < \omega\}$ .

Since  $\mathcal{A} = \bigcup \mathcal{A}_n$  is countable it doesn't generate  $\mathcal{F}$  and there is  $F^* \in \mathcal{F}$  such that  $F^* \not\supset A$  for all  $A \in \mathcal{A}$ . Consequently the set  $X = \bigcup A_n \setminus F^*$  is infinite.

**Player II** can now use the following counter strategy to win. On turn 0 **Player II** picks  $a_0 \in [X]^{<\omega}$  and **Player I** plays  $F_0 = \sigma(a_0)$ .

If  $F_0 \neq [\omega]^{<\omega}$ , then **Player II** picks  $a_1$  such that  $a_1 \cap X \neq \emptyset$  and  $a_1 - F_0 \neq \emptyset$ . Then for the rest of the game **Player II** always picks  $a_{n+1} \subset F_n$  with  $a_{n+1} \cap X \neq \emptyset$ .

Otherwise, if  $F_0 = [\omega]^{<\omega}$ , **Player II** plays  $a_1$  such that  $a_1 \cap X \neq \emptyset$  and  $a_1 \subset F_0$ . On turn  $n+1$ , if  $F_n = [\omega]^{<\omega}$  **Player II** plays  $a_{n+1}$  such that  $a_{n+1} \cap X \neq \emptyset$  and  $a_{n+1} \subset F_n$ . Otherwise if  $F_n \neq [\omega]^{<\omega}$  then **Player II** plays  $a_{n+1}$  such that  $a_{n+1} \cap X \neq \emptyset$  and  $a_{n+1} - F_n \neq \emptyset$  and for the rest of the game picks  $a_k \subset F_k$ .

Since each  $a_n$  meets  $X$  it's clear that  $a_n \not\subset F^*$  for all  $n$  so that  $a_n$  cannot converge. On the other hand, either  $F_n = [\omega]^{<\omega}$  for all  $n < \omega$  and  $\bigcap \text{st}(a_n, \mathcal{U}_{F_n}) \neq \emptyset$ , or there is a minimum  $k$  such that  $F_k \neq [\omega]^{<\omega}$ , in which case  $z = a_k - F_k \neq \emptyset$  and  $a_n \subset F_n$  for all  $n > k$  and  $z \in \bigcap \text{st}(a_n, \mathcal{U}_{F_n})$ . **TODO: clean up and make more readable** □