Digital Signal Processing SS 2022: 1. Assignment

Analog Signals and Systems

Group number 34

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1. Task

Computations with complex numbers.

Transformations/pre work:

$$\begin{aligned}
&c_1 = -3 + j5 : \\
&r = \sqrt{(-3)^2 + 5^2} = \sqrt{34} \\
&tanMinus1 = tan^{-1}\left(\frac{5}{-3}\right) = -59.0362\circ \\
&\theta = tanMinus1 + 180\circ = 120.9638\circ \\
&\theta_{euler} = \theta/180\circ = 0.6720211111 \\
&\text{It follows:} c_1 = \sqrt{34}e^{\theta_{euler}\pi j} = \sqrt{34}e^{2.111216586j}
\end{aligned}$$

$$\begin{aligned}
&c_2 = \sqrt{2}e^{\frac{j3\pi}{4}} : \\
&\text{For} \frac{-j3\pi}{4} \text{we get:} \frac{-j3 \cdot 180\circ}{4} = -135\circ \\
&cos(-135\circ) = sin(-135\circ) = -\frac{\sqrt{2}}{2} \\
&\text{It follows:} c_2 = \sqrt{2}\left(-\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2 \cdot 2}}{2} - j\frac{\sqrt{2 \cdot 2}}{2} = -1 - j
\end{aligned}$$

$$\begin{aligned}
&c_3 = \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1 + j) : \\
&\theta_{c_3} = tan^{-1}\left(\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right) = tan^{-1}(1) = 45\circ
\end{aligned}$$

Calculations:

$$\begin{aligned}
&c_4 = c_1 + c_2 = (-3 + j5) + (-1 - j) = -4 + j(5 - 1) = -4 + 4j \\
&c_5 = c_1 \cdot c_2 = (-3 + j5) \cdot (-1 - j) = 3 + 3j - 5j - 5j^2 = 3 - 2j + 5 = 8 - 2j \\
&c_6 = |c_3|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1 \\
&c_7 = arg(c_3) = \theta_{c_3} = 45\circ \\
&c_8 = \frac{c_1}{c_2} = \frac{\sqrt{34}e^{0.6720211111\pi j}}{\sqrt{2}e^{\frac{j3\pi}{4}}} = \sqrt{17}e^{(0.6720211111+0.75)\pi j} = \sqrt{17}e^{1.4202021111\pi j} \\
&\sqrt{17}cos(1.4202021111 \cdot 180\circ) = -1.022725813 \approx -1 \\
&\sqrt{17}sin(1.4202021111 \cdot 180\circ) = -3.994249856 \approx -4 \\
&c_8 = -1 - j4 \\
&c_9 = c_1 \cdot c_1* = (-3 + 5j) \cdot (-3 - 5j) = 9 + 15j - 15j - 25j^2 = 9 + 25 = 34
\end{aligned}$$

2. Task

Fourier Transform

Objective: Prove the Fourier transform pair of the cosine function, as presented in the lecture (p. 39 of DSP 2). The result for the complex sine function (p. 40 of DSP 2) and Euler's formula are used.

From

$$\begin{aligned}
x(t) &= \int_{-\infty}^{+\infty} A \cdot \delta(f - f_0) \cdot e^{j2\pi ft} \cdot df = A \cdot e^{j2\pi f_0 t} \\
\longleftrightarrow X(f) &= A \cdot \delta(f - f_0) = \int_{-\infty}^{+\infty} A \cdot e^{j2\pi f_0 t} \cdot e^{-j2\pi ft} \cdot dt,
\end{aligned}$$

it follows that

$$\begin{aligned}
x(t) &= A \cdot \cos(2\pi f_0 t) \\
\longleftrightarrow X(f) &= \int_{-\infty}^{+\infty} A \cdot \cos(2\pi f_0 t) \cdot e^{-j2\pi ft} \cdot dt \\
&= A \cdot \int_{-\infty}^{+\infty} \frac{1}{2} \cdot \left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right) \cdot e^{-j2\pi ft} \cdot dt \\
&= \frac{A}{2} \cdot \int_{-\infty}^{+\infty} \left(e^{j2\pi f_0 t} \cdot e^{-j2\pi ft} + e^{-j2\pi f_0 t} \cdot e^{-j2\pi ft}\right) \cdot dt \\
&= \frac{A}{2} \cdot \int_{-\infty}^{+\infty} \left(e^{-j2\pi(f-f_0)t} + e^{-j2\pi(f+f_0)t}\right) \cdot dt \\
&= \frac{A}{2} \cdot \left(\int_{-\infty}^{+\infty} e^{-j2\pi(f-f_0)t} \cdot dt + \int_{-\infty}^{+\infty} e^{-j2\pi(f+f_0)t} \cdot dt\right) \\
&= \frac{A}{2} \cdot \left(\delta(f - f_0) + \delta(f + f_0)\right) \\
&= \frac{A}{2} \cdot \delta(f - f_0) + \frac{A}{2} \cdot \delta(f + f_0),
\end{aligned}$$

which was to be shown.

3. Task

a) Computing ϕ_i for different frequencies f_i for the time shifted signal $cos(2\pi f_i(t - 0.1s))$. Furthermore, verifying the time shift theorem (page 44 lecture slides 2) for this case.

Computing ϕ_i:

$$\begin{aligned}
cos(2\pi f_i(t - 0.1s)) &= cos(2\pi f_i t + \phi) \\
2\pi f_i(t - 0.1s) &= 2\pi f_i t + \phi \\
f_i t - f_i 0.1s &= f_i t + \frac{\phi}{2\pi} \\
-f_i 0.1s &= \frac{\phi}{2\pi} \\
-f_i \pi 0.2s &= \phi = -f_i 2\pi\tau
\end{aligned}$$

$$\begin{aligned}
\phi_1 &= -\frac{1}{s}\pi 0.2s = -0.2\pi \\
\phi_2 &= -\frac{2}{s}\pi 0.2s = -0.4\pi \\
\phi_3 &= -\frac{3}{s}\pi 0.2s = -0.6\pi
\end{aligned}$$

Verifying time shift theorem:

We already know $cos(2\pi f_i(t - 0.1s)) = cos(2\pi f_i t + \phi)$. Therfore, we can also have a look at the phase shifted version. So for the shifted version we get by page 39 lecture slide set 2:

$$\begin{aligned}
cos(2\pi f_i t + \phi) : X_{shifted}(f) &= \frac{1}{2}e^{j\phi}\delta(f - f_i) + \frac{1}{2}e^{-j\phi}\delta(f + f_i) \\
&= \frac{1}{2}e^{j(-f_i 2\pi\tau)}\delta(f - f_i) + \frac{1}{2}e^{-j(-f_i 2\pi\tau)}\delta(f + f_i) \\
&= \frac{1}{2}e^{-jf_i 2\pi\tau}\delta(f - f_i) + \frac{1}{2}e^{jf_i 2\pi\tau}\delta(f + f_i)
\end{aligned}$$

As you can see, we have $e^{-jf_i 2\pi\tau}$ x-axis=f_i and $e^{jf_i 2\pi\tau}$ for x-axis=$-f_i$.
When only considering time shift $t - \tau$ then we consider part $\frac{1}{2}e^{-jf_i 2\pi\tau}\delta(f - f_i)$ of X(f)

for the unshifted version:
$\phi = -\tau 2\pi f_i = -0 \cdot 2\pi f_i = 0$

$$\begin{aligned}
x(t) = cos(2\pi f_i t + \phi) : X_{unshifted}(f) &= cos(2\pi f_i t + 0) \\
&= \frac{1}{2}e^0\delta(f - f_i) + \frac{1}{2}e^0\delta(f + f_i) \\
&= \frac{1}{2}\delta(f - f_i) + \frac{1}{2}\delta(f + f_i)
\end{aligned}$$

Here, $e^{(+/-)jf_i 2\pi\tau}$ from above are missing. However, the rest stays the same!

With $(t - \tau) = (t - 0.1s)$ we describe a shift to the right, therefore $\delta(f - f_i)$ is of interest. Therfore have only at this parts a look of X(f):

$$\begin{gathered}
X_{shifted}(f) : xs = \frac{1}{2}e^{-jf_i 2\pi\tau}\delta(f - f_i) \\
X_{unshifted}(f) : xu = \frac{1}{2}\delta(f - f_i) \\
\text{so we get: } xs = xu \cdot e^{-jf_i 2\pi\tau} \\
\text{It follows: If } x(t) \circ\!-\!\bullet\, X_{unshifted}(f) \text{ then } x(t - \tau) \circ\!-\!\bullet\, X_{unshifted}(f)e^{-jf_i 2\pi\tau}
\end{gathered}$$

Therefore, when comparing the original with the time shifted signal, it follows the Time Shift theorem:

$$x(t - \tau) \circ\!-\!\bullet\, X(f)e^{-jf_i 2\pi\tau}$$

b) Plotting task. Relating to the frequency $f_2 = 2Hz$ the original signal $cos(2\pi f_2 t)$ and time delayed signal $cos(2\pi f_2(t - 0.1s))$ shall be plotted. Not to forget the phase shifted signal $cos(2\pi f_2 t + \phi)$.

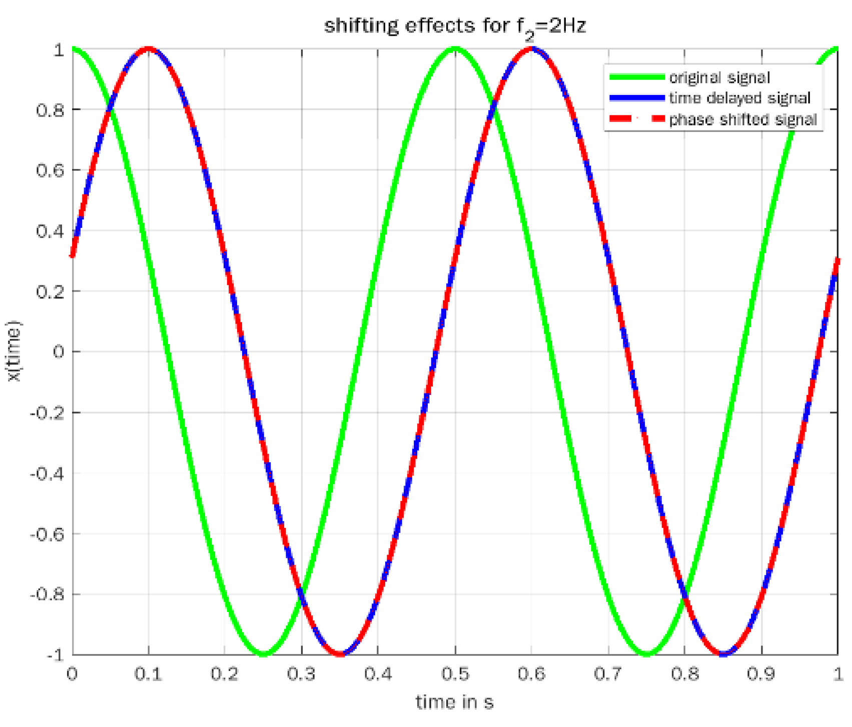

Figure 1: plot of $cos(2\pi f_2 t)$ and its 0.1 s time delayed version.

c) Plotting all three ϕ_i and showing that they lay on a line passing the origin of coordinate system.

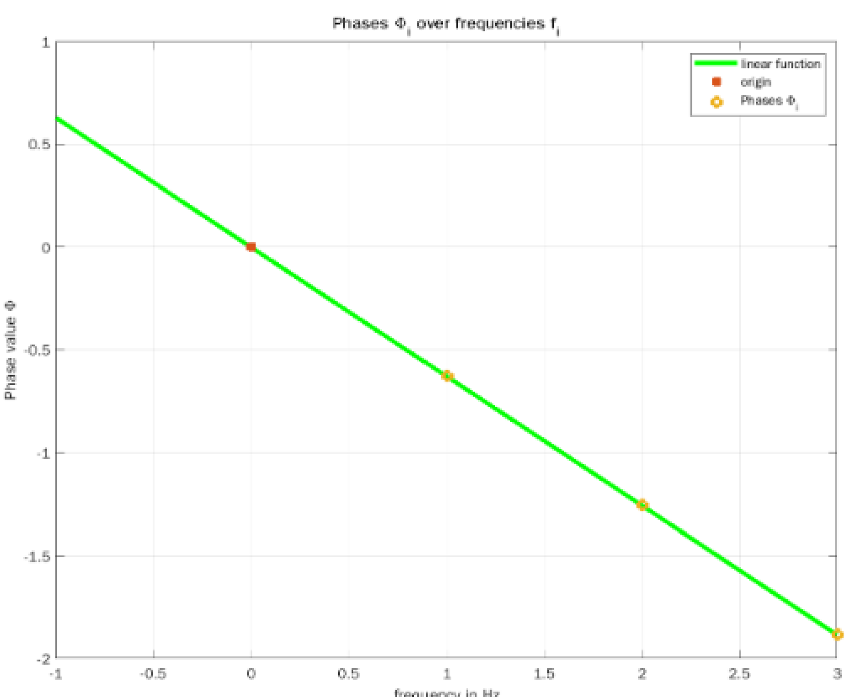

Figure 2: Plotting $\phi_i = -\tau 2\pi f_i$ with $f_i = \{1, 2, 3\}$ Hz.

As you can see in the plot all ϕ_i lay on a line which goes through the origin of the coordinate system. However, to be sure in saying that this is the case, we also calculated it (**see assignment1_3.m**). For this purpose we designed a linear function which shall be our line:
linear_func = @(freq) slope * freq + 0;
'$+0$' means that our function goes through the origin. 'freq' describes the inputted x-axis-value frequency. The slope was calculated by $\frac{\phi_1 - \phi_2}{1-2}$.
For all ϕ_i values with i={1,2,3} we tested whether linear_func(i) has the same value. That was the case. Therefore, all three ϕ_i lay on this line/linear function.

4. Task

Linearity and Time Invariance

Objective: Prove, or disprove, linearity and time invariance of a given system $\mathcal{S}$. The definitions for linearity (p. 52 of DSP 2) and time invariance (p. 53 of DSP 2) are used. The obtained outputs $y'(t)$ are compared with the assumptions $y''(t)$.

$$y(t) = \mathcal{S}\{x(t)\} = \left(x(t)\right)^2$$

For linearity:

$$\begin{aligned}
y_1(t) &= \mathcal{S}\{x_1(t)\} = \left(x_1(t)\right)^2 \\
y_2(t) &= \mathcal{S}\{x_2(t)\} = \left(x_2(t)\right)^2 \\
x'(t) &= \alpha \cdot x_1(t) + \beta \cdot x_2(t) \\
y'(t) &= \mathcal{S}\{x'(t)\} = \left(x'(t)\right)^2 = \left(\alpha \cdot x_1(t) + \beta \cdot x_2(t)\right)^2 \\
&= \alpha^2 \cdot \left(x_1(t)\right)^2 + 2 \cdot \alpha \cdot \beta \cdot x_1(t) \cdot x_2(t) + \beta^2 \cdot \left(x_2(t)\right)^2 \\
y''(t) &= \alpha \cdot y_1(t) + \beta \cdot y_2(t) = \alpha \cdot \left(x_1(t)\right)^2 + \beta \cdot \left(x_2(t)\right)^2
\end{aligned}$$

Since $y'(t) \neq y''(t)$, system $\mathcal{S}$ is non-linear.

For time invariance:

$$\begin{aligned}
x'(t) &= x(t - \tau) \\
y'(t) &= \mathcal{S}\{x'(t)\} = \left(x'(t)\right)^2 = \left(x(t - \tau)\right)^2 \\
y''(t) &= y(t - \tau) = \left(x(t - \tau)\right)^2
\end{aligned}$$

Since $y'(t) = y''(t)$, system $\mathcal{S}$ is time-invariant.