Digital Signal Processing SS 2022: 3.Assignment

The Discrete-Time Fourier Transform

Group number 34

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1. Task

Working with LTI system with the output: $y[n] = (1-g) \cdot x[n] + g \cdot y[n-1]$

a) Find impulse response h[n]

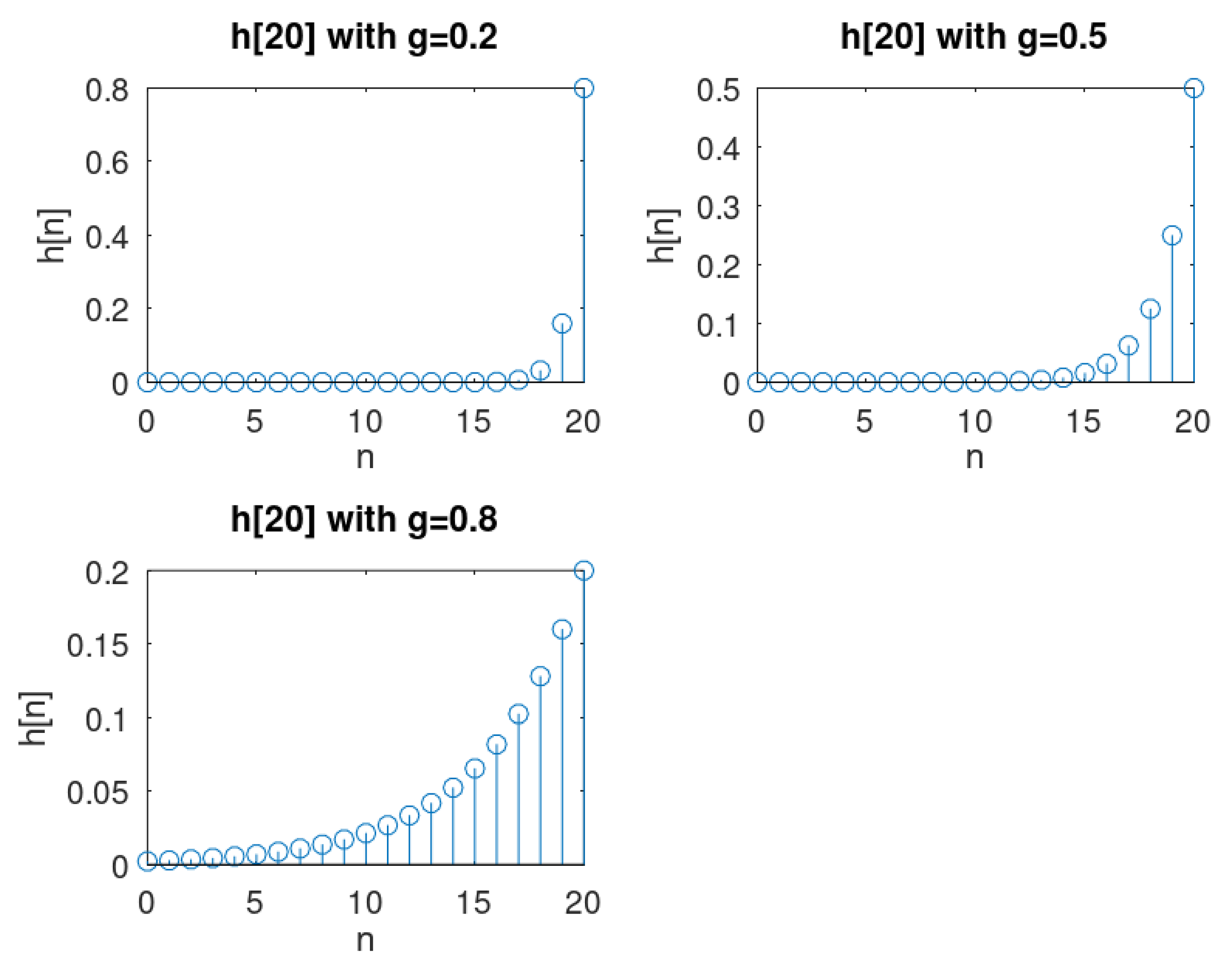
First, we split up y[n] to get an understanding of the formula. Furthermore, we know from lecture 4 slide 37 that we can use the factors in front of all x[...] to collect a set we is defining h[n] for y[n].

$$\begin{aligned} y[n] &= (1-g) \cdot x[n] + g \cdot y[n-1] \\ &= (1-g) \cdot x[n] + g \cdot ((1-g) \cdot x[n-1] + g \cdot y[n-2]) \\ &= (1-g) \cdot x[n] + g \cdot ((1-g) \cdot x[n-1] + g \cdot ((1-g) \cdot x[n-2] + g \cdot y[n-3])) \\ &= \ldots \end{aligned}$$

We observe a special kind of structure. When n shrinks the factors g grows exponentially. We take the factors in front of the x-es and combine them with dirac functions to show the shift in time. So we get:

$$h[n] = \sum_{k=0}^{k=n} (1-g) \cdot g^{n-k} \delta[n - |(k-n)|]$$

b) Plot the result with octave



We used the stem function because we have discrete n steps on the x-axis. You can observe that the memory factor g influences how strong the past responses influnce the response of the current x[n] input. With a high g value the past responses aren't 'forgotten' that much and that fast.
In the instructions no concrete n were given. However, you can see the described effects very well with n=20.

2. Task

Block diagram

a) Derive the linear difference equation from the given block diagram.

$$\begin{aligned} x[n] &= y[n] - a \cdot y[n-1] \\ y[n] &= a \cdot y[n-1] + x[n] \end{aligned}$$

b) Calculate the frequency response of the LTI system.

Using the linearity, time-shifting and convolution properties, we get

$$\begin{aligned} x[n] \longleftrightarrow X(e^{j\Omega}) &= Y(e^{j\Omega}) - a \cdot Y(e^{j\Omega}) \cdot e^{-j\Omega} \\ &= Y(e^{j\Omega}) \cdot \left(1 - a \cdot e^{-j\Omega}\right) \\ y[n] \longleftrightarrow Y(e^{j\Omega}) &= a \cdot Y(e^{j\Omega}) \cdot e^{-j\Omega} + X(e^{j\Omega}) \\ y[n] = x[n] * h[n] \longleftrightarrow Y(e^{j\Omega}) &= X(e^{j\Omega}) \cdot H(e^{j\Omega}) \,. \end{aligned}$$

It follows from the above that

$$\begin{aligned} H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} &= \frac{Y(e^{j\Omega})}{Y(e^{j\Omega}) \cdot (1 - a \cdot e^{-j\Omega})} \\ &= \frac{1}{1 - a \cdot e^{-j\Omega}} \,. \end{aligned}$$

c) Calculate the impulse response of the LTI system for $|a| < 1$.

Using the DTFT table, we finally obtain

$$h[n] = a^n \cdot u[n], \;\; |a| < 1 \longleftrightarrow H(e^{j\Omega}) = \frac{1}{1 - a \cdot e^{-j\Omega}} \,,$$

where $u[n]$ denotes the discrete-time unit-step function.

3. Task

Input and output signals

a) Calculate the frequency response of the given LTI system.

Using the linearity and time-shifting properties as well as the DTFT table, we get

$$\begin{aligned} x[n] = \left(\frac{1}{2}\right)^n \cdot u[n] - \frac{1}{4} \cdot \left(\frac{1}{2}\right)^{n-1} \cdot u[n-1] \longleftrightarrow X(e^{j\Omega}) &= \frac{1}{1 - \frac{1}{2} \cdot e^{-j\Omega}} - \frac{1}{4} \cdot \frac{1}{1 - \frac{1}{2} \cdot e^{-j\Omega}} \cdot e^{-j\Omega} \\ &= \frac{1}{1 - \frac{1}{2} \cdot e^{-j\Omega}} \cdot \left(1 - \frac{1}{4} \cdot e^{-j\Omega}\right) \\ y[n] = \left(\frac{1}{3}\right)^n \cdot u[n] \longleftrightarrow Y(e^{j\Omega}) &= \frac{1}{1 - \frac{1}{3} \cdot e^{-j\Omega}} \,. \end{aligned}$$

Using the convolution property and the partial-fraction decomposition, we then get

$$\begin{aligned} H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} &= \frac{1 - \frac{1}{2} \cdot e^{-j\Omega}}{\left(1 - \frac{1}{3} \cdot e^{-j\Omega}\right) \cdot \left(1 - \frac{1}{4} \cdot e^{-j\Omega}\right)} \\ &= \frac{A}{1 - \frac{1}{3} \cdot e^{-j\Omega}} - \frac{B}{1 - \frac{1}{4} \cdot e^{-j\Omega}} \\ &= -\frac{2}{1 - \frac{1}{3} \cdot e^{-j\Omega}} + \frac{3}{1 - \frac{1}{4} \cdot e^{-j\Omega}} \,. \end{aligned}$$

The coefficients A and B are obtained by solving the equation

$$\begin{aligned} 1 - \frac{1}{2} \cdot e^{-j\Omega} &= A \cdot \left(1 - \frac{1}{4} \cdot e^{-j\Omega}\right) - B \cdot \left(1 - \frac{1}{3} \cdot e^{-j\Omega}\right) \\ &= A - \frac{1}{4} \cdot A \cdot e^{-j\Omega} - B + \frac{1}{3} \cdot B \cdot e^{-j\Omega} \\ &= A - B - \left(\frac{1}{4} \cdot A - \frac{1}{3} \cdot B\right) \cdot e^{-j\Omega} \,, \end{aligned}$$

which is equivalent to solving the system of equations

$$\begin{aligned} \text{I.} \qquad\qquad A - B = 1 &\longleftrightarrow A = B + 1 \\ &\overset{*}{\longleftrightarrow} A = -3 + 1 \quad {}^*\text{inserting } B \\ &\longleftrightarrow A = -2 \\ \text{II.} \; \frac{1}{4} \cdot A - \frac{1}{3} \cdot B = \frac{1}{2} &\longleftrightarrow \frac{1}{4} \cdot (B+1) - \frac{1}{3} \cdot B = \frac{1}{2} \\ &\longleftrightarrow 3 \cdot B + 3 - 4 \cdot B = 6 \\ &\longleftrightarrow B = -3 \,. \end{aligned}$$

This yields the corresponding coefficients.

b) Calculate the impulse response of the given LTI system.

Using the linearity property and the DTFT table, we get

$$h[n] = -2 \cdot \left(\frac{1}{3}\right)^n \cdot u[n] + 3 \cdot \left(\frac{1}{4}\right)^n \cdot u[n] \longleftrightarrow H(e^{j\Omega}) = -\frac{2}{1 - \frac{1}{3} \cdot e^{-j\Omega}} + \frac{3}{1 - \frac{1}{4} \cdot e^{-j\Omega}} \,.$$

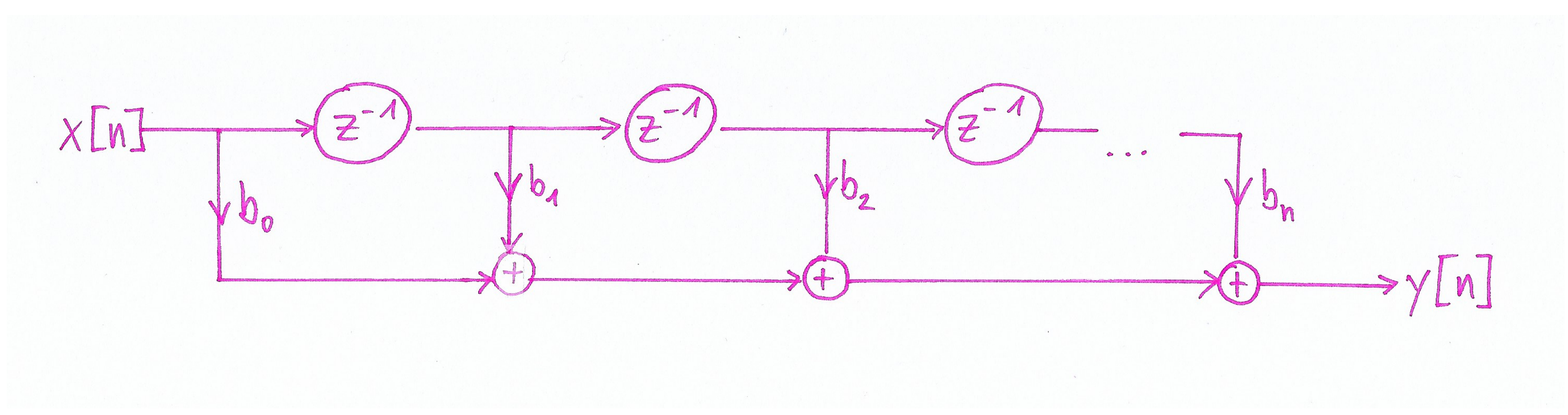
c) Draw a block diagram of the given LTI system.

The output signal $y[n]$ is given by

$$\begin{aligned} y[n] = h[n] * x[n] &= \sum_{i=0}^{n} h[i] \cdot x[n-i] \\ &= h[0] \cdot x[n] + h[1] \cdot x[n-1] + \cdots + h[n-1] \cdot x[1] + h[n] \cdot x[0] \\ &= b_0 \cdot x[n] + b_1 \cdot x[n-1] + \cdots + b_{n-1} \cdot x[1] + b_n \cdot x[0] \,, \end{aligned}$$

since $h[i] \cdot x[n-i]$ is non-zero for $0 \le i \le n$.

The figure below shows the resulting block diagram.



4. Task

Working with a LTI system and its frequency response $H(e^{j\Omega}) = 1 + \alpha e^{-j\Omega} + \beta e^{-j\Omega 2}$

a)Finding out the singla values α and β

From DSP 5 slide 15 we know: $y[n] = x[n] * h[n]$ can be converted into $Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega})$

We know: y[n] = 0; therefore from the DFTF table (exercise) we get: $Y(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} 0 \cdot e^{-j\Omega n} = 0$

From lecture 5 slide 20 we know: $X(e^{j\Omega}) = \pi \sum_{k=-\infty}^{\infty} (\delta[\Omega - \Omega_0 + 2\pi k] + \delta[\Omega + \Omega_0 + 2\pi k])$

On the same slides we see that the $X(e^{j\Omega})$ is only NOT 0 at $\Omega = +/- \Omega_0$ when we only consider the period around the y-axis. That becomes also clear when we have a look at the formula itself. The Dirac function describes a shift in time where a function is 1 (multiplied when a factor stands in front of it in the formula, here π).When we consider k=0 because we only want to have a look at that one representative period (more is not needed for this task in which $X(e^{j\Omega})$ follows always the same pattern (periodicity)), we have:

$\pi \cdot (\delta[\Omega - \Omega_0] + \delta[\Omega + \Omega_0])$

and we know that a dirac function per definition: $\delta[0] = 1$ and else $\delta[i \neq 0] = 0$ for digital signals.

Therefore, we get $\Omega = +/- \Omega_0$ as markers where the function is non zero.

For this markers we need that $H(e^{j\Omega}) = 0$ because we want $0 = Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega})$ and $X(e^{j\Omega}) \neq 0$

$$\begin{gathered} \text{1. Case } \Omega = +\Omega_0\text{:} \\ H(e^{j\Omega_0}) = 1 + \alpha e^{-j\Omega_0} + \beta e^{-j\Omega_0 2} = 0 \\ \alpha = (-1 - \beta e^{-j\Omega_0 2}) / e^{-j\Omega_0} \\ \alpha = (-1 - \beta e^{-j\Omega_0 2}) \cdot e^{j\Omega_0} \end{gathered}$$

$$\begin{gathered} \text{2. Case } \Omega = -\Omega_0\text{:} \\ H(e^{j-\Omega_0}) = 1 + \alpha e^{j\Omega_0} + \beta e^{j\Omega_0 2} = 0 \end{gathered}$$

$$\begin{gathered} \text{Put result 1. case into formula of 2. case. Solve by } \beta\text{:} \\ 1 + (-1 - \beta e^{-j\Omega_0 2}) \cdot e^{j\Omega_0} \cdot e^{j\Omega_0} + \beta e^{j\Omega_0 2} = 0 \\ 1 + (-1 - \beta e^{-j\Omega_0 2}) \cdot e^{2j\Omega_0} + \beta e^{j\Omega_0 2} = 0 \\ 1 - e^{2j\Omega_0} - \beta + \beta e^{2j\Omega_0} = 0 \\ \beta(-1 + e^{2j\Omega_0}) = -1 + e^{2j\Omega_0} \\ \beta = 1 \end{gathered}$$

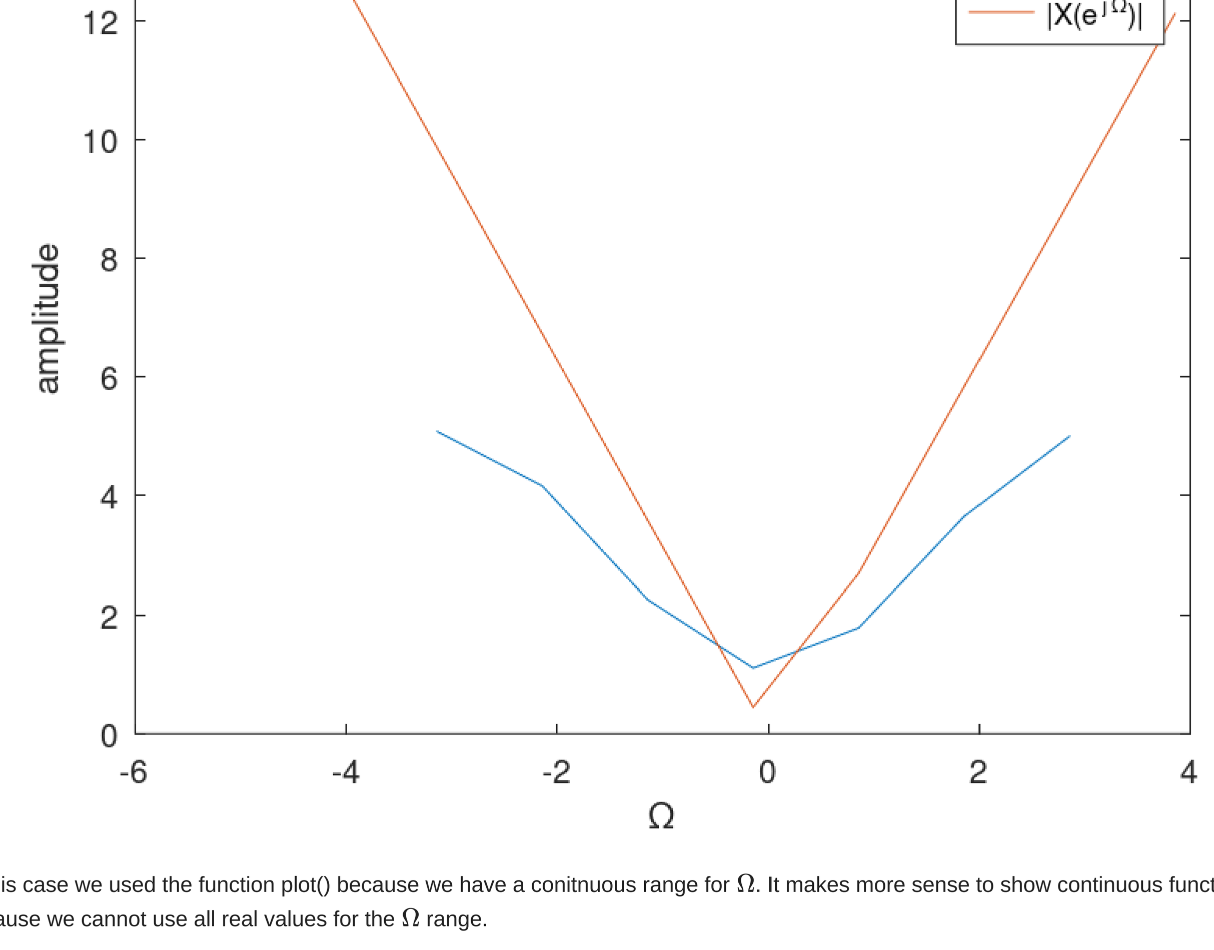
$$\begin{aligned} &\text{Put in } \beta \text{ into the 1. case's result to solve by } \alpha\text{:} \\ \alpha &= (-1 + e^{-2j\Omega_0}) e^{j\Omega_0} \\ &= -e^{j\Omega_0} - \frac{e^{j\Omega_0}}{e^{2j\Omega_0}} \\ &= -e^{j\Omega_0} - \frac{1}{e^{j\Omega_0}} \\ &= -2cos(j\Omega_0) \end{aligned}$$

b) Choose some value Ω_0 from $[0, \pi]$. Plot $|H(e^{j\Omega})|$ and $|X(e^{j\Omega})|$ in range $-\pi <= \Omega <= \pi$ by using Octave.

We know from lecture 4 slide 41 that $|e^{-j\Omega n}| = 1$. Furthermore, it is known that $|e^{Re + jIm}| = |e^{Re}| \cdot |e^{jIm}| = |e^{Re}| \cdot 1$

For $X(e^{j\Omega})$ we get from lecture 5 slide 20 that we already have the absolute form.

Because we only talked about the two non-zero values of $X(e^{j\Omega_0})$ we will use $\pi \cdot (\delta[\Omega - \Omega_0] + \delta[\Omega + \Omega_0])$ which only contains the information for one period where k=0 and also has only these two non-zero values within the given range for Ω.



In this case we used the function plot() because we have a conitnuous range for Ω. It makes more sense to show continuous function with help of a line. Furhtermore, we only estimated the functions because we cannot use all real values for the Ω range.