# Digital Signal Processing SS 2022: 3. Assignment

## Sampling and Reconstruction

Group number 34

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## 1. Task

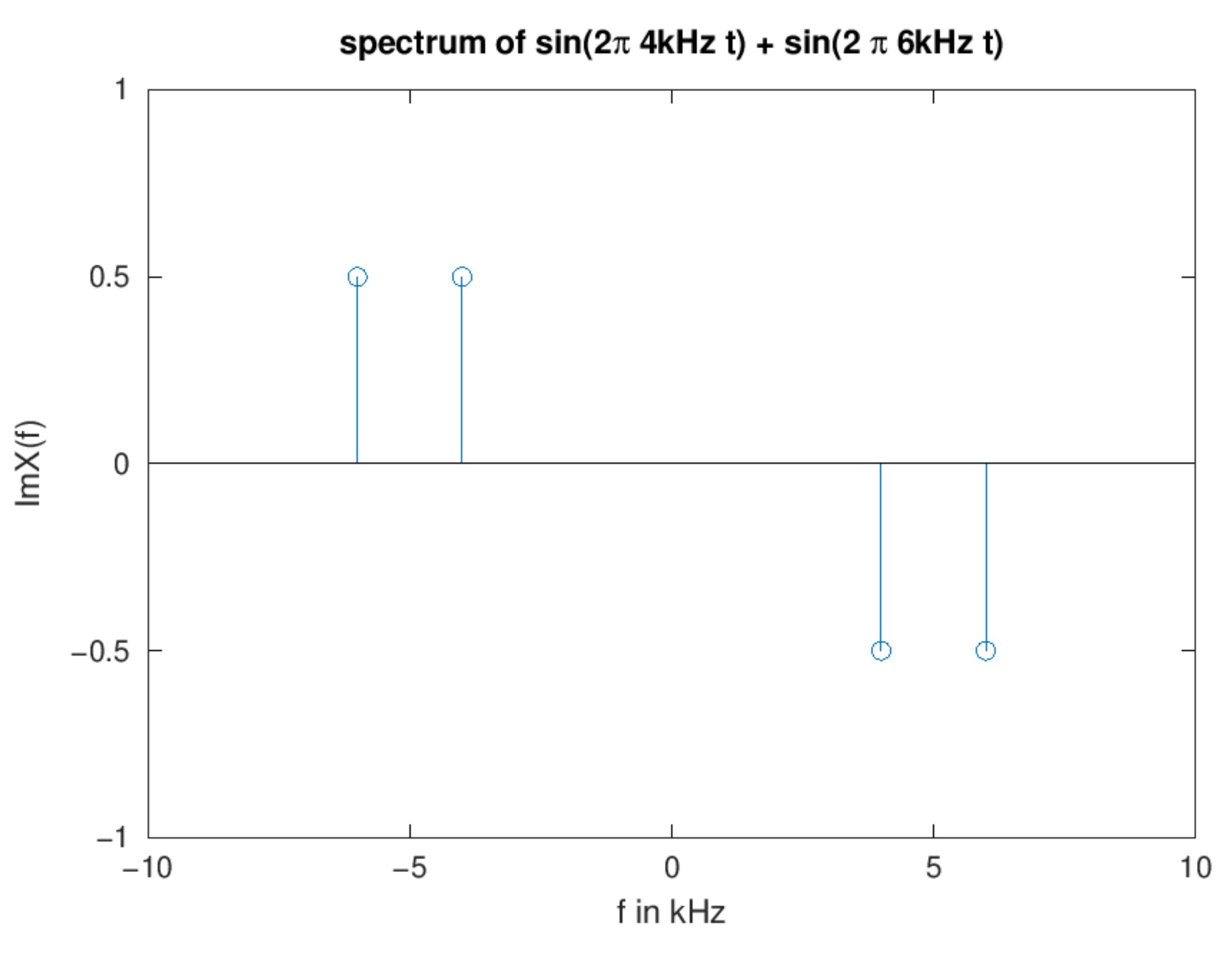
a) Plot X(f)

Given: $x(t) = x_1(t) + x_2(t) = sin(2\pi f_1 t) + sin(2\pi f_2 t)$ with $f_1 = 4kHz, f_2 = 6kHz$

From lecture 2 page 44 linarity: $ax(t) + by(t) \circ -. aX(f) + bY(f)$. Therefore, look at $x_1(t), x_2(t)$ separately.

From lecture 2 page 39 we know: $sin(2\pi f_1 t) \circ -. X_1(f) = -\frac{j}{2}\delta(f - f_1) + \frac{j}{2}\delta(f + f_1)$. From the dirac function we get that the spectrum is not 0 only at $-f_1, +f_1$. When the y-axis is only the imaginary part (real part does not exist without any cosine), then we get the heights $-1/2, +1/2$ for both peaks because for the imaginary y-axis also negative values are possible. Because of the used Fourier Series (for periodic and continuous signals) we have non-periodicity for the belonging spectrum.

Furthermore, we have $x_2(f)$. For this we have exactly the same rules with the peaks at $-f_2, +f_2$.



b) Plot spectra with shifts of $-f_s, 0, +f_s$ and of $x[n]$

From DSP_06 slide 15 we get: $X_s(f) = \frac{1}{T_s}\sum_{-\infty}^{\infty} X(f - kf_s)$ where $X(f - kf_s)$ is the spectrum of the continuous time signal.

The instructions tell us that we only need a range from $-f_s$ to $+f_s$. This range has a length of $2 \cdot f_s$.

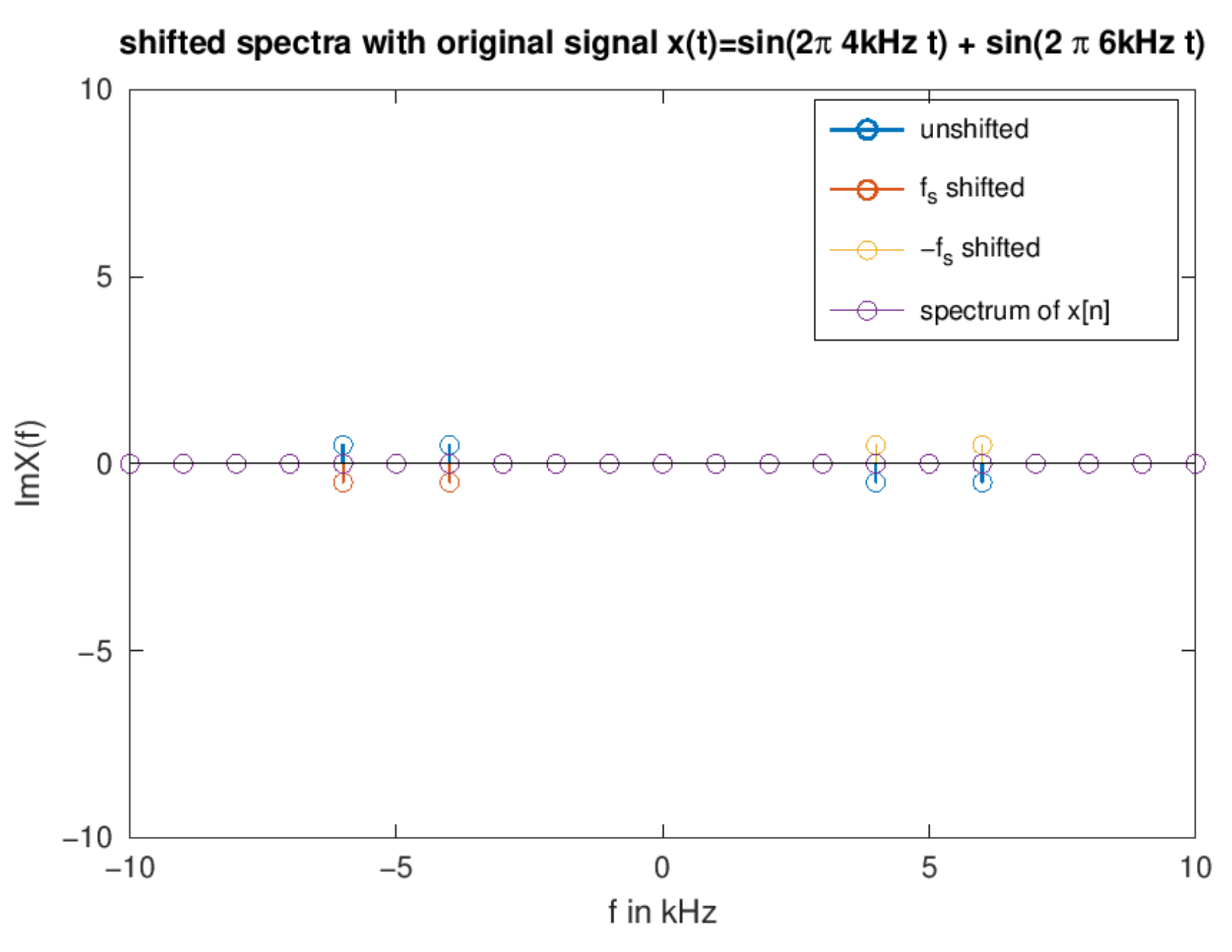
For spectrum of x[n]:
Aim:
$X_s(f) = f_s \cdot \sum_{k=-\infty}^{\infty} X_1(f - kf_s) + X_2(f - kf_s)$
$= f_s \cdot \sum_{k=-\infty}^{\infty} -\frac{j}{2}\delta(f - f_1 - kf_s) + \frac{j}{2}\delta(f + f_1 - kf_s) - \frac{j}{2}\delta(f - f_2 - kf_s) + \frac{j}{2}\delta(f + f_2 - kf_s)$
$= f_s \cdot \sum_{k=-\infty}^{\infty} a + b + c + d$

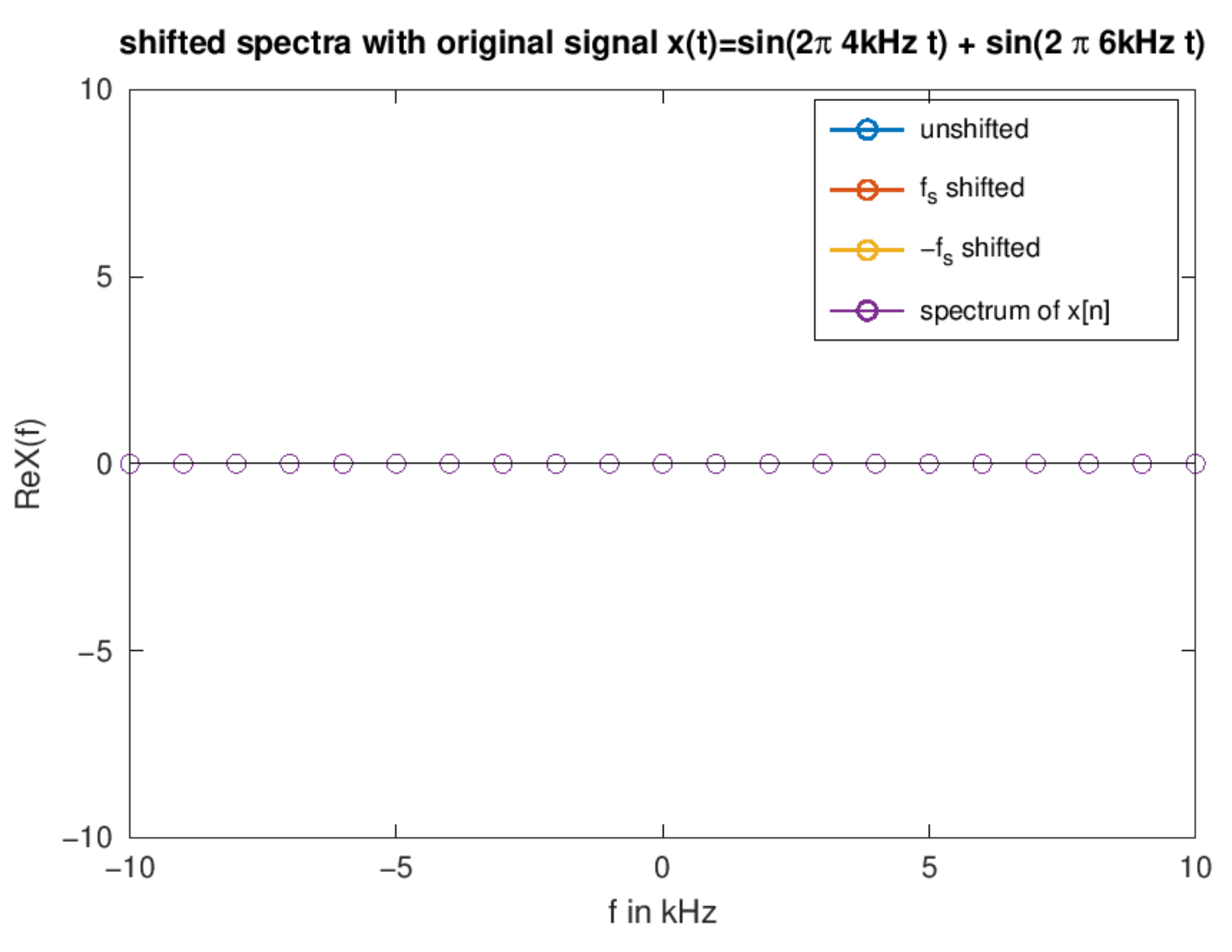
We realized that the shifts described in the instruction relate to k element of {-1,0,+1}. The remaining k-values of the formula above are not needed.

Because the signal is a sin-wave we have again here only an imaginary part for the spectrum. The real part is always 0.

Implementation: Compute in a loop for each f the a,b,c,d separately. Save them all in an array and a second array with all indices. Sum up the values with the same indices and multiply them with $1/T_s = f_s$.

We have peaks at $+f_1, -f_1, +f_2, -f_2$, else function is 0. Therefore simply shift the peaks.





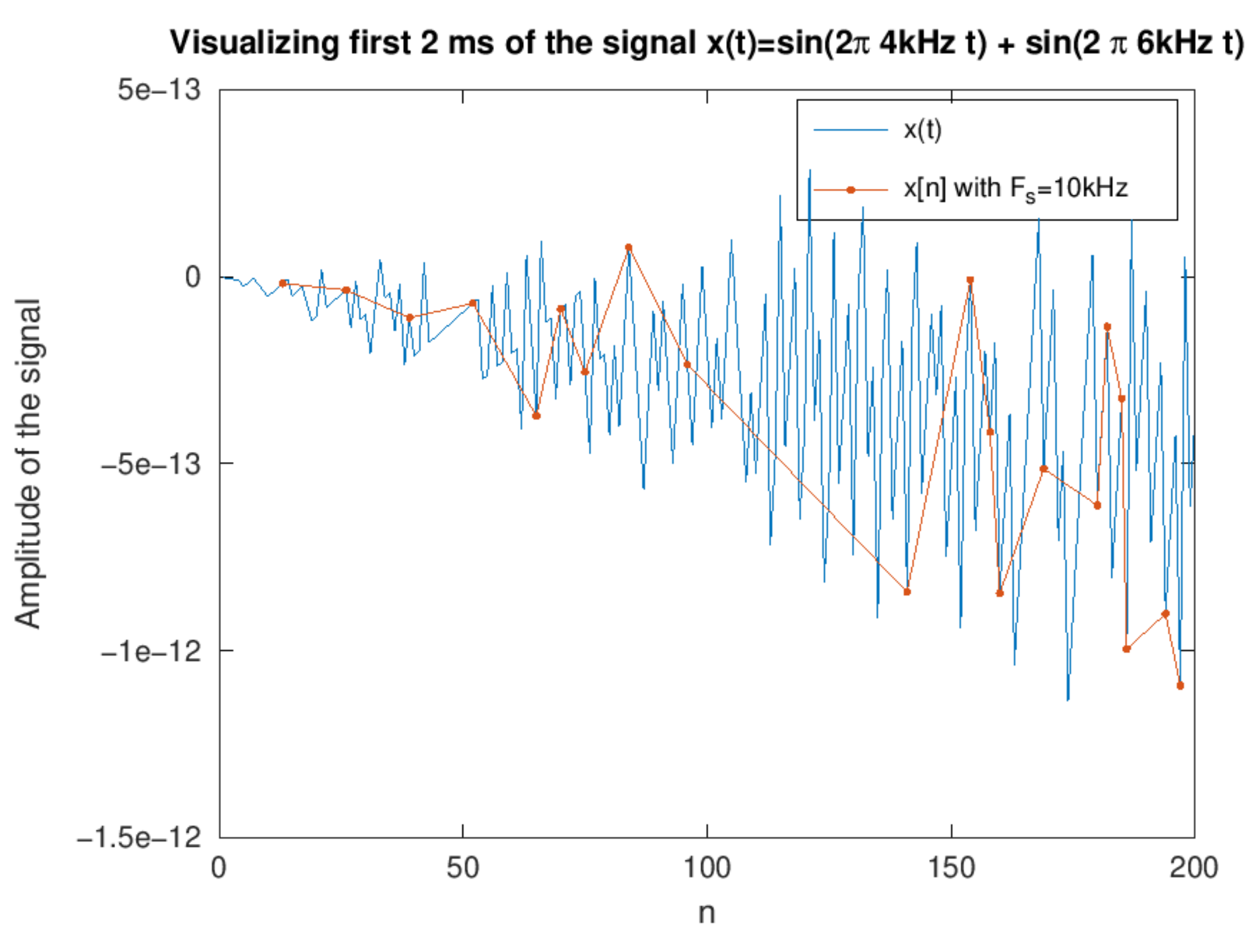
c) Plotting 0-2ms of x(t) (sampling rate) with Matlab. (We use octave because the cost free Matlab version is gone now). Then plot $x[n] = x(nT_s)$ and show that it corresponds to spectrum of b). First of all we wantt to plot 0-2ms of signal. We want that it 'looks' like the analogue signal. For this purpose the sampling rate is rather high. We make some work of thinking first:

$$2ms = 2milliseconds = 2\frac{s}{1000}$$
$$\text{With sampling rate we have number of samples per seconds but we want number of samples per milliseconds (n):}$$
$$n = 2 \cdot \frac{\frac{numberSamples}{1000}}{\frac{s}{1000}} = 2 \cdot \frac{\frac{100000}{1000}}{\frac{s}{1000}} = 2 \cdot \frac{\frac{100}{1}}{ms} = \frac{200}{ms}$$

Next we have a look at the sampled signal:

$$n_2 = 2 \cdot \frac{\frac{numberSamples}{1000}}{\frac{s}{1000}} = 2 \cdot \frac{\frac{10000}{1000}}{\frac{s}{100}} = 2 \cdot \frac{\frac{10}{1}}{ms} = \frac{20}{ms}$$

We have only 20 samples in the first 2 ms of x(t). for this purpose we can simply draw randomly 20 samples out of 'n'.



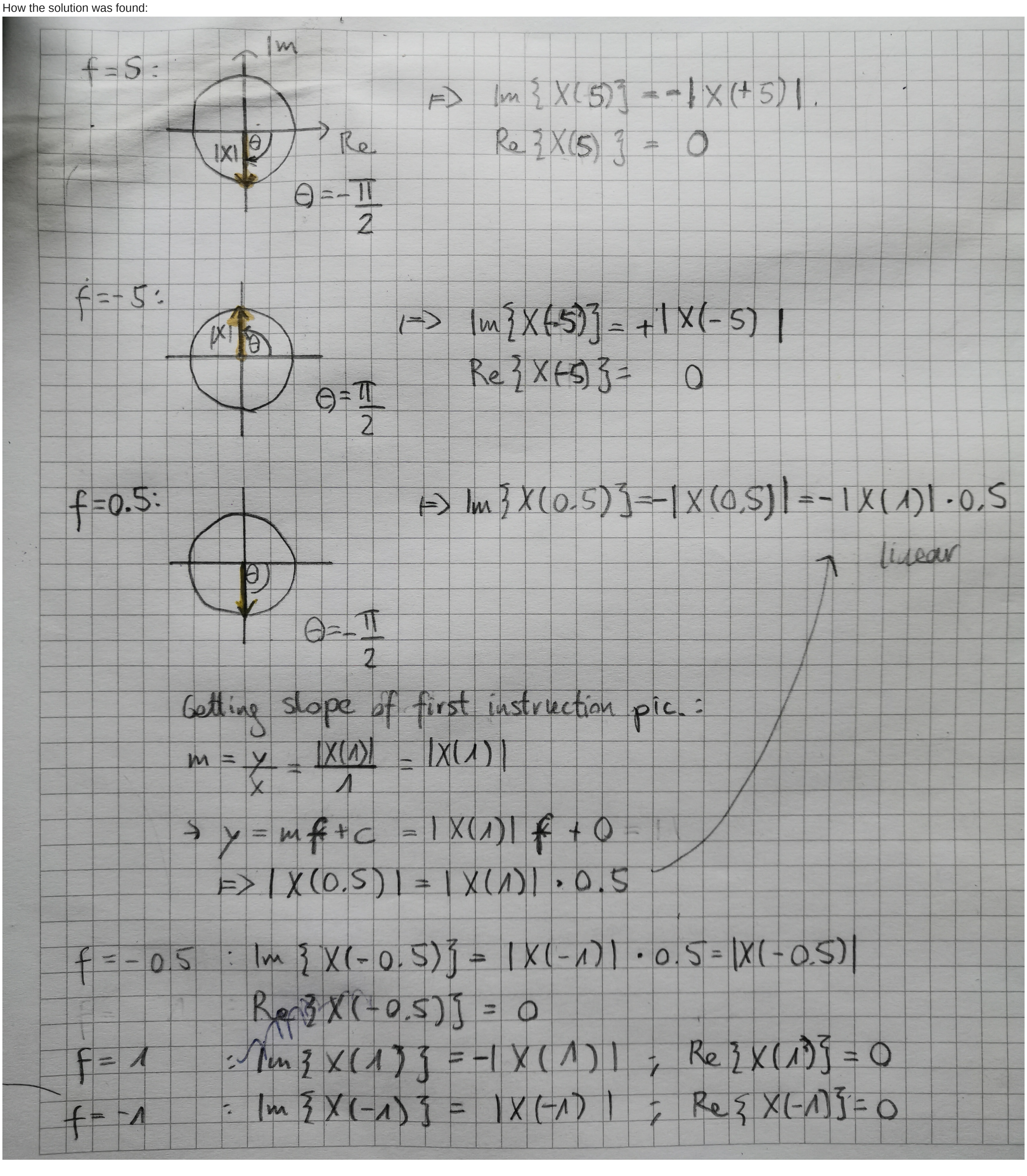
We can see that the sampled version is of course not able to imitate precisely the originial signal with a sampling rate of 10kHz.

Last but not least, we have to show that x[n] corresponds to the spectrum in b). This was already done in b). Here again: We have for our sinus signal four peaks. The sampled spectrum of a continuous time signal can be described by $X_s(f) = f_s \cdot \sum_{k=-\infty}^{\infty} X_1(f - kf_s) + X_2(f - kf_s)$ (lecture 6 slide 15). At b) we only had a look at the x_axis range from $-f_s$ to $f_s$ in which are all four peaks. The formula tells us that we can shift them to get the sampled spectrum. We can only use k element of {-1,0,1} as shift values. Else the shift would be so large that we would leaf the x-axis range of $[-f_s, +f_s]$. Therefore we end up with the shifts of $-(-1) \cdot f_s, -0 \cdot f_s, -(+1) \cdot f_s$ which are precisely the shifts from b). Summing them up to get the spectrum of x[n] fulfills the formula. So there is the connection.

## 2. Task

a) Draw the imaginary and real parts of a spectrum.

How the solution was found:



f=5: $\Theta = -\frac{\pi}{2}$ ⇒ $Im\{X(5)\} = -|X(+5)|$, $Re\{X(5)\} = 0$

f=-5: $\Theta = \frac{\pi}{2}$ ⇒ $Im\{X(-5)\} = +|X(-5)|$, $Re\{X(-5)\} = 0$

f=0.5: $\Theta = -\frac{\pi}{2}$ ⇒ $Im\{X(0.5)\} = -|X(0.5)| = -|X(1)| \cdot 0.5$ linear

Getting slope of first instruction pic.:

$m = \frac{y}{x} = \frac{|X(1)|}{1} = |X(1)|$

$\rightarrow y = mf + c = |X(1)| f + 0$

$\Rightarrow |X(0.5)| = |X(1)| \cdot 0.5$

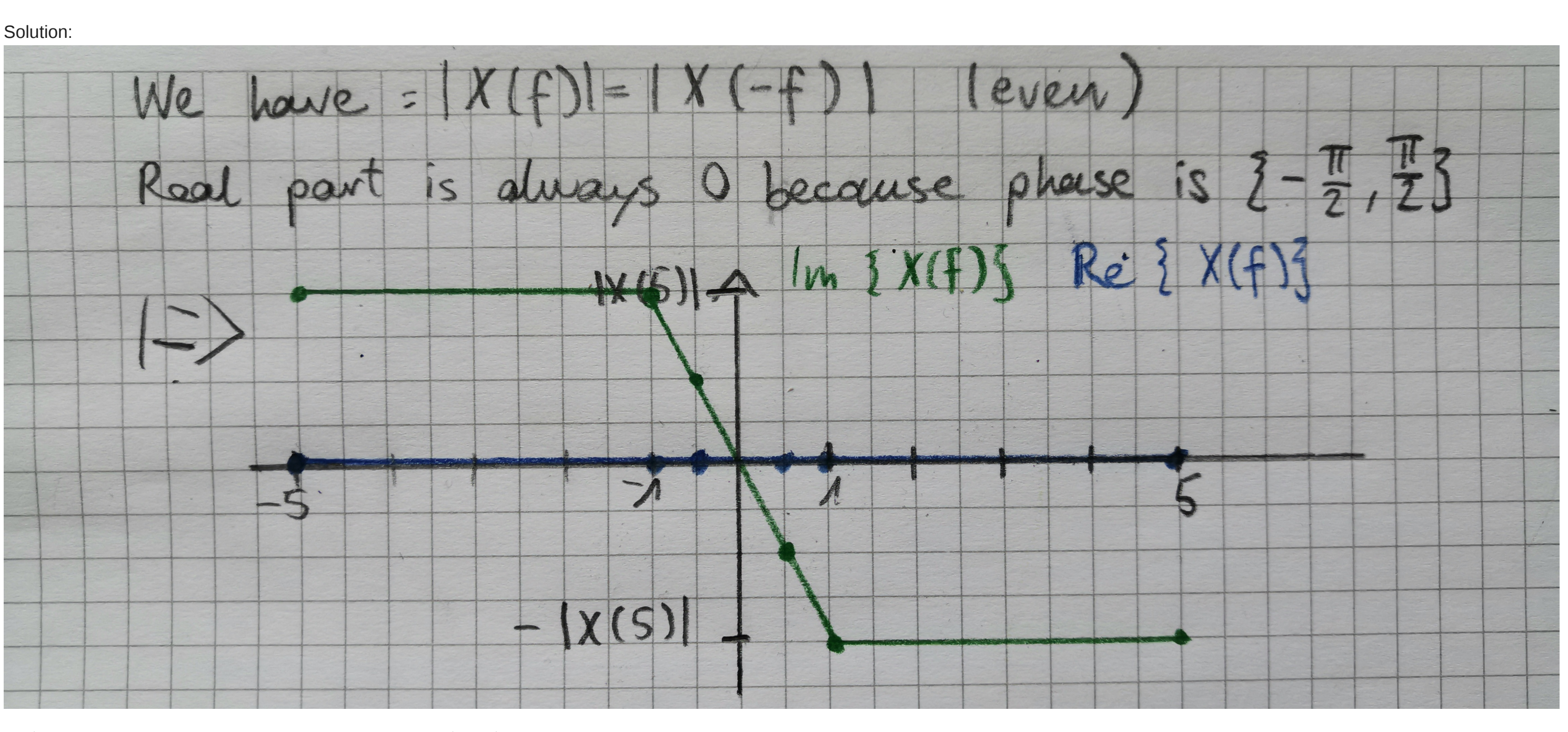
f=-0.5 : $Im\{X(-0.5)\} = |X(-1)| \cdot 0.5 = |X(-0.5)|$

$Re\{X(-0.5)\} = 0$

f=1 : $Im\{X(1)\} = -|X(1)|$ ; $Re\{X(1)\} = 0$

f=-1 : $Im\{X(-1)\} = |X(-1)|$ ; $Re\{X(-1)\} = 0$

Solution:



We have : $|X(f)| = |X(-f)|$ (even)

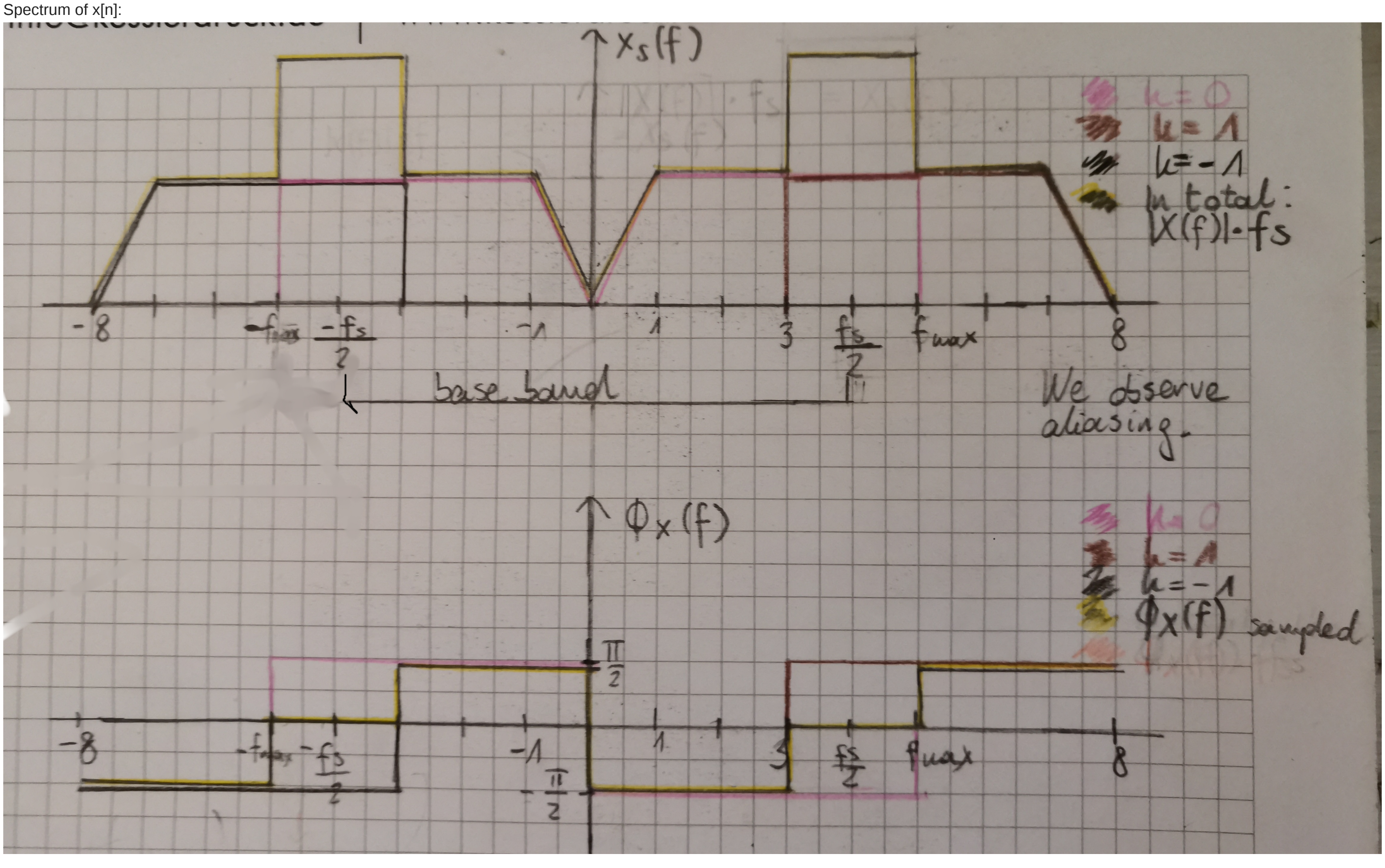
Real part is always 0 because phase is $\{-\frac{\pi}{2}, \frac{\pi}{2}\}$

b) $f_s = 8kHz$ to create x[n]. Draw its spectrum from $-f_s$ to $f_s$ and baseband.

As in task 1 we can make use of following formula (DSP*06, slide 15): $X*{sampled}(f) = fs *cdot* *sum*{k=-\infty}^{\infty} X_{ofContinuousSignal}(f-kf_s)

< br > kisdescribingbyhowfartheshiftsgo. Inthiscasekiselementof−1, 0, 1, elsethethespectrumwouldovergothex − axislimitof + /−f_s. < br > Basebandisdefinedby -\frac{f_s}{2}<f<\frac{f_s}{2}$ (DSP_06, slide 10)

Spectrum of x[n]:



Yellow lines as final solution

## 3. Task

**Reconstruction**

a) Up to $20\,\mathrm{kHz}$, list all positive frequencies occurring in $x(t)$. Calculating the DTFT yields

$$x[n] = \sqrt{2} \cdot \sin\left(2\pi \frac{1}{8} n\right) = \sqrt{2} \cdot \sin(\Omega_0 n) = x[n] = \frac{\sqrt{2}}{2j} \cdot \left(e^{j\Omega_0 n} - e^{-j\Omega_0 n}\right)$$
$$\longleftrightarrow$$
$$X(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} \frac{\sqrt{2}}{2j} \cdot \left(e^{j\Omega_0 k} - e^{-j\Omega_0 k}\right) \cdot e^{-j\Omega k} = \frac{\sqrt{2}}{2j} \cdot \left(\sum_{k=-\infty}^{\infty} e^{-j(\Omega - \Omega_0)k} - \sum_{k=-\infty}^{\infty} e^{-j(\Omega + \Omega_0)k}\right)$$
$$= \frac{\sqrt{2}}{2j} \cdot \left(2\pi \cdot \delta(\Omega - \Omega_0) - 2\pi \cdot \delta(\Omega + \Omega_0)\right) = \sqrt{2} \cdot j\pi \cdot \left(\delta(\Omega + \Omega_0) - \delta(\Omega - \Omega_0)\right)$$
$$= \sqrt{2} \cdot j\pi \cdot \left(\sum_{k=-\infty}^{\infty} \delta(\Omega + \Omega_0 - 2\pi k) - \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k)\right).$$

For positive frequencies, we need

$$\begin{aligned} \delta(\Omega - \Omega_0 - 2\pi k) = \infty &\longleftrightarrow \Omega - \Omega_0 - 2\pi k = 0 \\ &\longleftrightarrow \Omega = \Omega_0 + 2\pi k \\ &\longleftrightarrow 2\pi f \frac{1}{8000} = 2\pi \frac{1}{8} + 2\pi k \\ &\longleftrightarrow f = 1000 + 8000k\,. \end{aligned}$$

From this, we obtain $1\,\mathrm{kHz}$ for $k = 0$, $9\,\mathrm{kHz}$ for $k = 1$ and $17\,\mathrm{kHz}$ for $k = 2$.

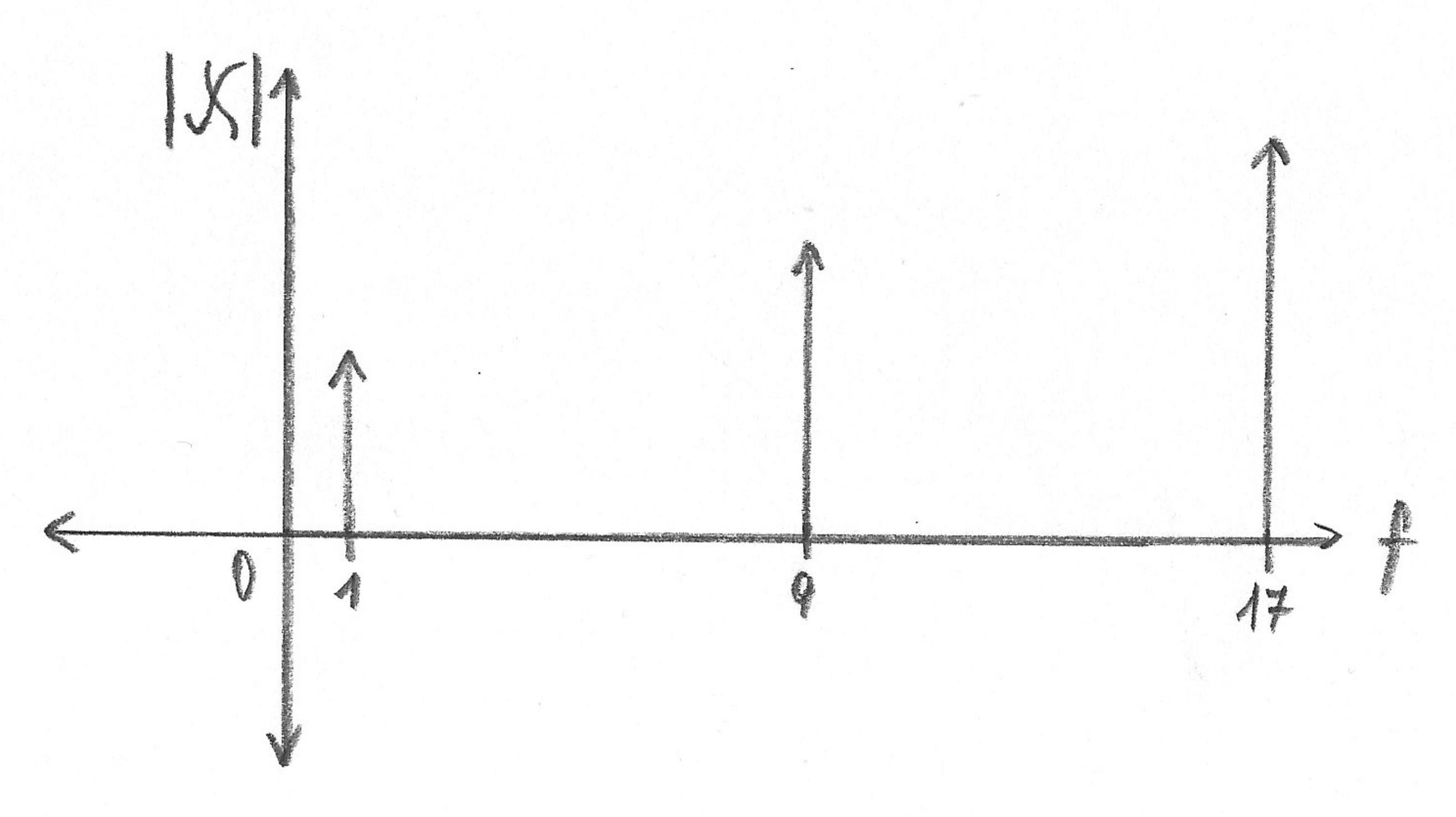
b) Calculate the power (in dB) of the baseband sine wave and the first out-of-band sine wave. The sine-modulated Dirac comb is given by

$$Z(j\omega) = \sqrt{2} \cdot j\pi \cdot \left(\sum_{k=-\infty}^{\infty} 8000 \cdot \delta(\omega + \omega_0 - 2\pi \cdot 8000k) - \sum_{k=-\infty}^{\infty} 8000 \cdot \delta(\omega - \omega_0 - 2\pi \cdot 8000k)\right).$$

The power is thus given by

$$P = \frac{\left(\sqrt{2}\right)^2}{2} = 1\,.$$

c) Draw the spectrum of $x(t)$ up to $20\,\mathrm{kHz}$. Dirac delta pulses should be indicated by an arrow, where the height of the arrow corresponds to the weight.



## 4. Task

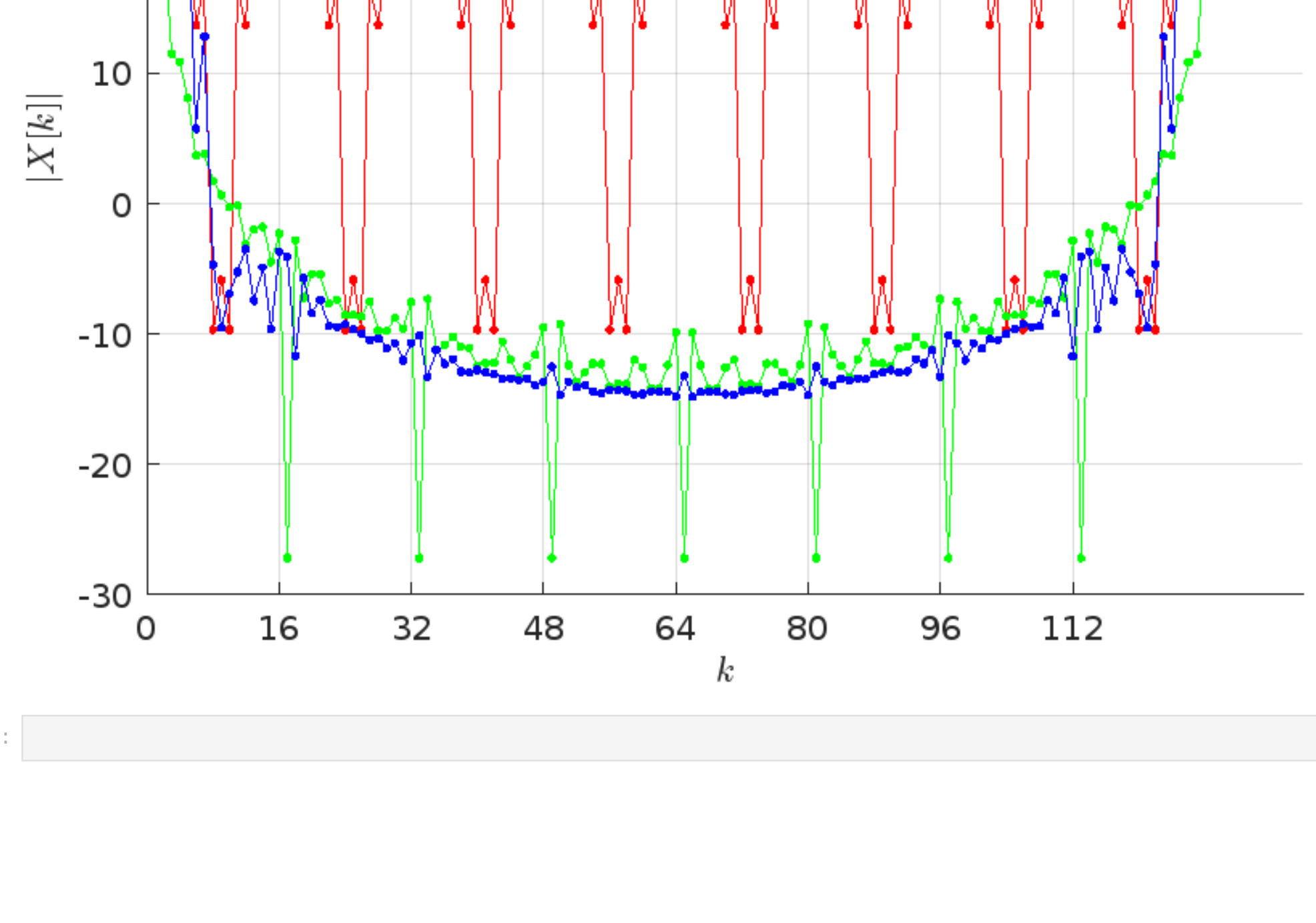
**Upsampling**

a) Generate a signal $x_1[n]$ with increased sampling rate by inserting $L - 1$ zeros after each sample of $x[n]$. Please refer to `assignment4_4.m` for the implementation in MATLAB. Contrary to the instruction, we increased the sampling rate of $x[n]$ by a factor of $R$ by inserting $R - 1$ zeros per sample. Using $L$ leads to astronomical numbers and high computation times.

b) Pass $x_1[n]$ through a moving average filter to generate the zero-order hold interpolated signal $x_2[n]$. Please refer to `assignment4_4.m` for the implementation in MATLAB.

c) Generate the signal $x_3[n]$ by using the given time-limited version of the "ideal" brick-wall filter impulse response. Please refer to `assignment4_4.m` for the implementation in MATLAB.

d) Compute the DFT spectra of the previously generated signals. Please refer to `assignment4_4.m` for the implementation in MATLAB. Also, plot the magnitude spectra in dB. For better visibility, we chose $N = \frac{L}{R} = \frac{1024}{8} = 128$ to compute the discrete Fourier transforms. Hence, the figure below shows $|X_1[k]|$, $|X_2[k]|$ and $|X_3[k]|$ for $0 \le k \le 127$.



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