Digital Signal Processing SS 2022: 2. Assignment

Analog Signals and Systems

Group number 34

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1. Task

Working with discrete time signals

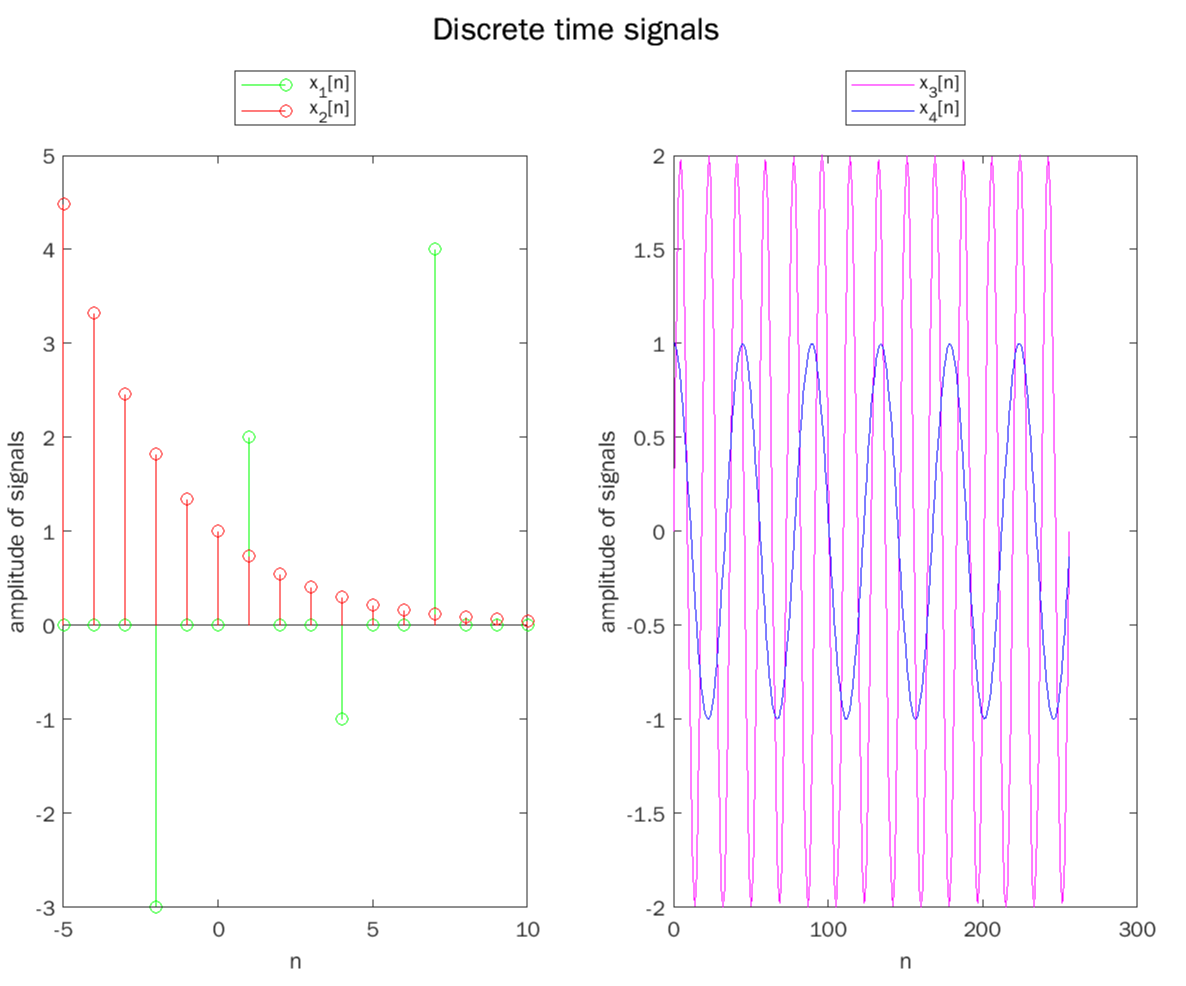
a) plotting the four different signals:

$$x_1[n] = -3\delta[n+2] + 2\delta[n-1] - \delta[n-4] + 4\delta[n-7] \qquad \text{for } -5 \le n \le 10$$

$$x_2[n] = e^{-0.3n} \qquad \text{for } -5 \le n \le 10$$

$$x_3[n] = 2\sin\left(2\pi \frac{3.5}{64} n\right) \qquad \text{for } 0 \le n \le 256$$

$$x_4[n] = \cos\left(\frac{9}{64} n\right) \qquad \text{for } 0 \le n \le 256$$



In this figure you can see two subplots each containing two of the four discrete signals.
Reasons of our design choices:
left figure: Here our range for samples is only from -5 up to 10. Therefore, we don't have a lot of samples such that it makes sense to create a plot with help of the stem function.
right figure: In this case we have a lot of samples ([0;256]). Therefore, we used the plot function to estimate how the continuous function would look like.

b) Normalized angular frequencies for $x_3[n]$ and $x_4[n]$

$$\text{For } x_3[n]\text{: } \Omega = \frac{2\pi \cdot 3.5}{64}$$
$$\text{For } x_4[n]\text{: } \Omega = \frac{9}{64}$$

c) Periodicity of $x_3[n]$ and $x_4[n]$

$$\text{For } x_3[n]\text{: inner part of sin}=\frac{2\pi 3.5n}{64} = 2\pi$$
$$n = \frac{64}{3.5} = \frac{64}{\frac{7}{2}} = \frac{128}{7}$$
$$period_{fundamental} = N = n \cdot m = \frac{128}{7} \cdot 7 = 128$$
$$\text{N has to be an integer in discrete case! m is the number of cycles done by continuous function to set one period of discrete signal}$$
$$periodicity: x[n] = x[n+N]$$
$$x_3[n+N] = 2sin\left(\frac{2\pi 3.5}{64} \cdot (n+128)\right) = 2sin\left(\frac{2\pi 3.5n}{64} + 14\pi\right) = 2sin\left(\frac{2\pi 3.5n}{64} + 7 \cdot 2\pi\right) = x_3[n]$$
$$\text{because } sin(0) = sin(2\pi) = sin(2\pi \cdot 7) = 0$$

$x_3[n]$ is periodic!

For $x_4[n]$: We have $cos\left(\frac{9}{64}n\right) = cos(\omega n)$

$$\omega = \frac{9}{64}$$
$$\frac{\omega}{2\pi} = \frac{m}{N} = \frac{9}{128\pi} \text{is NOT rational and therefore not periodic!! N is also no integer}$$

d) Writing the power function

```
function P=power_(vector)
    P = sum(abs(vector).^2)/length(vector);
end
```

Fed in $x_3[n]$ with range of (0:N-1)=(0:128-1)=(0:127) because 0 is an included sample such that we feed in 128 values in total.

e) writing energy function

```
function W=energy(signal)
    W = sum(abs(signal).^2);
end
```

We feed each signal with the given range into the function because outside the range we have only zero-values.

f) table summarizing results from d) and e)

	Power	Energy
x_1[n]	NaN	30
x_2[n]	NaN	44.514
x_3[n]	2	512
x_4[n]	NaN	128.96

Energy explanation: The sample numbers given through the range from $x_1[n]$ and $x_2[n]$ are much lower than the number for the other two signals. Therefore, $x_3[n]$ and $x_4[n]$ have of course much higher energy scores.

2. Task

Discrete convolution in the style of slides from tutorial 2 page 5

a) Length y[n]

$L_y = L_x + L_h - 1 = 5 + 4 - 1 = 8$

b) Calculating y[n]

1. write down x with all its delays needed for calculations and the belonging h/impulse responses

$$x[n-0] = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid h[0] = 2$$

$$x[n-1] = \begin{bmatrix} 0 \\ 3 \\ -1 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \mid h[1] = 3$$

$$x[n-2] = \begin{bmatrix} 0 \\ 0 \\ 3 \\ -1 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \mid h[2] = 4$$

$$x[n-3] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \\ -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \mid h[3] = 1$$

2. Multiply the x-es with the belonging h-es, e.g. x[n-0].*h[0]. Then bind the x-es column wise into one matrix

$$\begin{bmatrix} 6 & 0 & 0 & 0 \\ -2 & 9 & 0 & 0 \\ 4 & -3 & 12 & 0 \\ 0 & 6 & -4 & 3 \\ 2 & 0 & 8 & -1 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. Take the sum of each row to get y[n]

$$y[n] = \begin{bmatrix} 6 \\ 7 \\ 13 \\ 5 \\ 9 \\ 5 \\ 4 \\ 1 \end{bmatrix}$$

3. Task

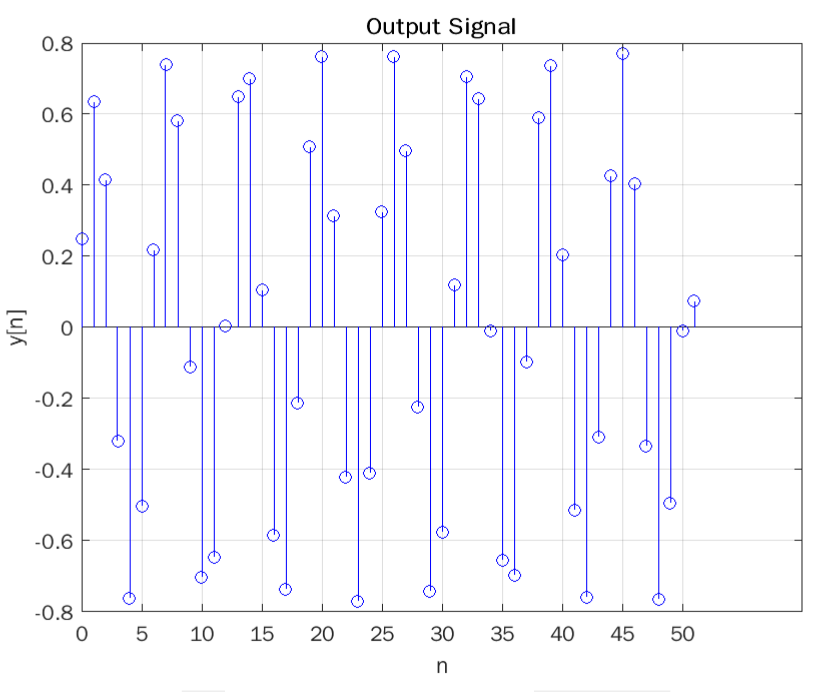
Convolution

a) Calculate the length of the output signal $y[n]$.

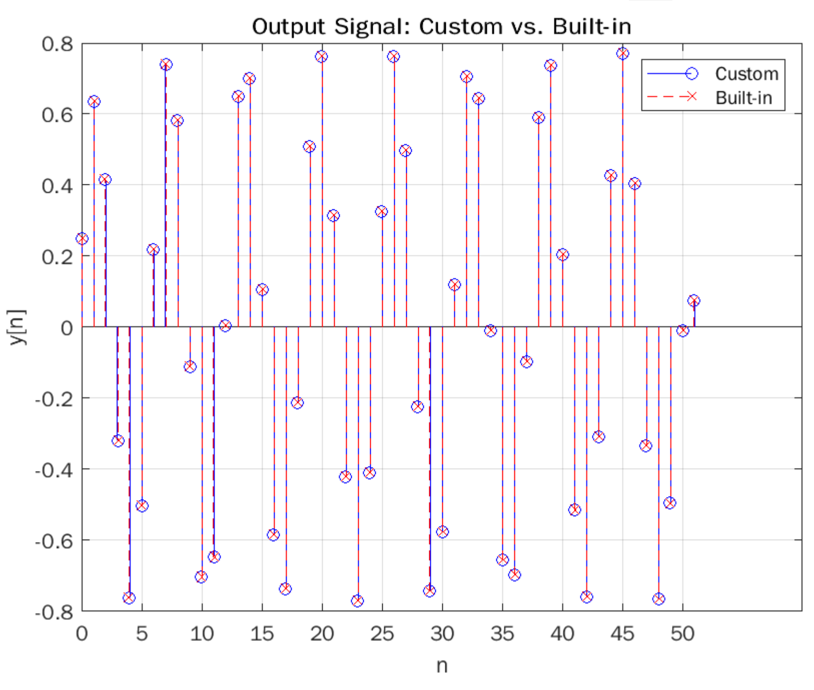
$$h[n] = (0.25, 0.5, 0.25) \text{ for } 0 \le n \le 2 \longrightarrow L_h = 3$$
$$x[n] = \cos\left(\frac{\pi}{10} \cdot n\right) \text{ for } 0 \le n \le 49 \longrightarrow L_x = 50$$
$$y[n] = \sum_{i=0}^{L_h - 1} h[i] \cdot x[n-i] \text{ for } (0+0) \le n \le (2+49) \longrightarrow 0 \le n \le 51 \longrightarrow L_y = L_x + L_h - 1 = 49 + 3 - 1 = 52$$

b) Implement the above-defined convolution operation using two nested for-loops. Please refer to `assignment2_3.m` for details. The output signal $y[n]$ is computed and plotted when executing the script.

c) Plot the computed output signal. The figure below shows the output signal $y[n]$ on the given interval of samples.



b) Apply the built-in function `conv` and plot the result for verification. Please refer to `assignment2_3.m` for details. The output signal $y[n]$ is computed and plotted when executing the script. The figure below shows the previously computed output signal $y[n]$ in comparsion to the built-in function `conv`. As can be observed, all values are precisely the same.



4. Task

Modulation

a) Prove (or disprove) linearity and time invariance of a given system $\mathcal{S}$. The definitions for linearity (p. 52 of DSP 2) and time invariance (p. 53 of DSP 2) are used. The obtained outputs $y'(t)$ are compared with the assumptions $y''(t)$.

$$y(t) = \mathcal{S}\{x(t)\} = x(t) \cdot \cos(\omega_0 t)$$

For linearity:

$$\begin{aligned} y_1(t) &= \mathcal{S}\{x_1(t)\} = x_1(t) \cdot \cos(\omega_0 t) \\ y_2(t) &= \mathcal{S}\{x_2(t)\} = x_2(t) \cdot \cos(\omega_0 t) \\ x'(t) &= \alpha \cdot x_1(t) + \beta \cdot x_2(t) \\ y'(t) &= \mathcal{S}\{x'(t)\} = \big(\alpha \cdot x_1(t) + \beta \cdot x_2(t)\big) \cdot \cos(\omega_0 t) \\ &= \alpha \cdot x_1(t) \cdot \cos(\omega_0 t) + \beta \cdot x_2(t) \cdot \cos(\omega_0 t) \\ y''(t) &= \alpha \cdot y_1(t) + \beta \cdot y_2(t) = \alpha \cdot x_1(t) \cdot \cos(\omega_0 t) + \beta \cdot x_2(t) \cdot \cos(\omega_0 t) \end{aligned}$$

Since $y'(t) = y''(t)$, system $\mathcal{S}$ is linear.

For time invariance:

$$\begin{aligned} x'(t) &= x(t-\tau) \\ y'(t) &= \mathcal{S}\{x'(t)\} = x'(t) \cdot \cos(\omega_0 t) = x(t-\tau) \cdot \cos(\omega_0 t) \\ y''(t) &= y(t-\tau) = x(t-\tau) \cdot \cos\big(\omega_0 \cdot (t-\tau)\big) = x(t-\tau) \cdot \cos(\omega_0 t - \omega_0 \tau) \end{aligned}$$

Since $y'(t) \ne y''(t)$, system $\mathcal{S}$ is time-variant.

b) Calculate the spectrum $Y(f)$ of a given signal $y(t)$. The spectrum of the cosine (p. 39 of DSP 2), the spectrum of the rectangular function (p. 35 of DSP 2), the properties of the FT (p. 44 of DSP 2) and the convolution of a function with the Dirac delta function (p. 66 of DSP 2) are used.

$$\begin{aligned} f_0 &= \frac{1}{T_0} \\ y_1(t) &= \cos(2\pi f_0 t) \longleftrightarrow Y_1(f) = \frac{1}{2} \cdot \delta(f - f_0) + \frac{1}{2} \cdot \delta(f + f_0) \\ y_2(t) &= rect\left(\frac{t}{T_0}\right) \longleftrightarrow Y_2(f) = T_0 \cdot sinc(\pi f T_0) = T_0 \cdot \frac{\sin(\pi f T_0)}{\pi f T_0} \\ y(t) &= \cos(2\pi f_0 t) \cdot rect\left(\frac{t}{T_0}\right) = y_1(t) \cdot y_2(t) \longleftrightarrow Y(f) = Y_1(f) * Y_2(f) \\ Y(f) &= \int_{-\infty}^{\infty} Y_2(\varphi) \cdot Y_1(f - \varphi) \cdot d\varphi \\ &= \int_{-\infty}^{\infty} T_0 \cdot sinc(\pi \varphi T_0) \cdot \left(\frac{1}{2} \cdot \delta\big((f-\varphi) - f_0\big) + \frac{1}{2} \cdot \delta\big((f-\varphi) + f_0\big)\right) \cdot d\varphi \\ &= \int_{-\infty}^{\infty} \left(\frac{T_0}{2} \cdot sinc(\pi \varphi T_0) \cdot \delta(f - \varphi - f_0) + \frac{T_0}{2} \cdot sinc(\pi \varphi T_0) \cdot \delta(f - \varphi + f_0)\right) \cdot d\varphi \\ &= \frac{T_0}{2} \cdot \left(\int_{-\infty}^{\infty} sinc(\pi \varphi T_0) \cdot \delta(f - \varphi - f_0) \cdot d\varphi + \int_{-\infty}^{\infty} sinc(\pi \varphi T_0) \cdot \delta(f - \varphi + f_0) \cdot d\varphi\right) \\ &= \frac{T_0}{2} \cdot \Big(sinc\big(\pi \cdot (f - f_0) \cdot T_0\big) + sinc\big(\pi \cdot (f + f_0) \cdot T_0\big)\Big) \\ &= \frac{T_0}{2} \cdot \Big(sinc\big(\pi \cdot (f T_0 - 1)\big) + sinc\big(\pi \cdot (f T_0 + 1)\big)\Big) \\ &= \frac{T_0}{2} \cdot \big(sinc(\pi f T_0 - \pi) + sinc(\pi f T_0 + \pi)\big) \end{aligned}$$

The zeros are given by:

$$Y(f) = 0 \longrightarrow f \in \left\{\frac{k}{T_0} : k \in \mathbb{Z}, |k| \ne 1\right\}$$

The spectrum $Y(f)$ is computed and plotted when executing the script `assignment2_4.m`. The figure below shows the spectrum $Y(f)$ for $T_0 = 5$.