FOURIER TRANSFORM OF THE DIRAC DISTRIBUTION

In this document, the Fourier transform convention (when it is well-defined) is:

$$\mathcal{F}[u](\xi) := \int_{-\infty}^{+\infty} u(x)\mathrm{e}^{-\mathrm{i}x\xi}\mathrm{d}x.$$

We set $\mathcal{S}(\mathbb{R}) := \left\{u \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \,\middle|\, \forall(\alpha,\beta) \in \mathbb{N}^2,\, |x^\alpha u^{(\beta)}(x)| \in L^\infty(\mathbb{R},\mathbb{R})\right\}$ for the Schwartz class and we denote by $\mathcal{S}'(\mathbb{R})$ its topological dual (the space of tempered distributions). It is well-known that $\mathcal{F}[\mathcal{S}(\mathbb{R})] = \mathcal{S}(\mathbb{R})$.

For any $T \in \mathcal{S}'(\mathbb{R})$, $\mathcal{F}[T] \in \mathcal{S}'(\mathbb{R})$ is well-defined by the formula: for all $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\langle \mathcal{F}[T], \varphi\rangle := \langle T, \mathcal{F}[\varphi]\rangle.$$

Exercise 1:

The goal of this exercise is to prove the following fundamental formulæ: as elements in $\mathcal{S}'(\mathbb{R})$,

$$\mathcal{F}[\delta] = 1, \qquad \mathcal{F}[1] = 2\pi\delta.$$

1. For all $a > 0$, let $\Delta_a : \mathbb{R} \to \mathbb{R}$ be the function defined by:

$$\Delta_a(x) := \begin{cases} a & \text{if } -\frac{1}{2a} \leq x \leq \frac{1}{2a}, \\ 0 & \text{else.} \end{cases}$$

   (a) Compute $\mathcal{F}[\Delta_a](\xi)$ for all $\xi \in \mathbb{R}$.

   (b) Prove that, as elements in $\mathcal{S}'(\mathbb{R})$,

$$\lim_{a\to+\infty} \Delta_a = \delta.$$

   (c) Prove that, as elements in $\mathcal{S}'(\mathbb{R})$,

$$\lim_{a\to+\infty} \mathcal{F}[\Delta_a] = 1.$$

   (d) Deduce that, as elements in $\mathcal{S}'(\mathbb{R})$,

$$\mathcal{F}[\delta] = 1.$$

2. For all $a > 0$, let $\mathrm{Tri} : \mathbb{R} \to [0,1]$ be the function defined by:

$$\mathrm{Tri}_a(x) := \begin{cases} 1 + \frac{x}{a} & \text{if } -a \leq x \leq 0, \\ 1 - \frac{x}{a} & \text{if } 0 \leq x \leq a, \\ 0 & \text{else.} \end{cases}$$

   (a) Show that, as elements of $\mathcal{S}'(\mathbb{R})$,

$$\lim_{a\to+\infty} \mathrm{Tri}_a = 1.$$

(b) Compute the following integral:

$$I := \int_{-\infty}^{+\infty} \frac{1-\cos(\eta)}{\eta^2} \mathrm{d}\eta.$$

Hint: We can differentiate $I(t) := \int_{-\infty}^{+\infty} \frac{1-\cos(\eta)}{\eta^2} \mathrm{e}^{-t\eta^2} \mathrm{d}\eta$ or apply the residue theorem.

(c) Deduce that, as elements in $\mathcal{S}'(\mathbb{R})$,

$$\lim_{a\to+\infty} \mathcal{F}[\mathrm{Tri}_a] = 2\pi\delta.$$

(d) Deduce that, as elements in $\mathcal{S}'(\mathbb{R})$,

$$\mathcal{F}[1] = 2\pi\delta.$$

Solution 1:

1. (a) For all $\xi \in \mathbb{R}$, we compute:

$$\mathcal{F}[\Delta_a](\xi) = a \int_{-\frac{1}{2a}}^{\frac{1}{2a}} \mathrm{e}^{-\mathrm{i}x\xi} \mathrm{d}x = 2a \frac{\sin\left(\frac{\xi}{2a}\right)}{\xi}.$$

(b) Let $\varphi \in \mathcal{S}(\mathbb{R})$. We have:

$$\int_{-\infty}^{+\infty} (\Delta_a)(x)\varphi(x)\mathrm{d}x = a \int_{-\frac{1}{2a}}^{\frac{1}{2a}} \varphi(x)\mathrm{d}x.$$

Set:

$$F_\varphi(t) := \int_{-t}^{t} \varphi(x)\mathrm{d}x \qquad \forall t \in \mathbb{R}.$$

Note that F_φ is differentiable at finite $t \in \mathbb{R}$ and $F_\varphi(0) = 0$ and $F'_\varphi(0) = 2\varphi(0)$. Letting $h := \frac{1}{2a}$, we see that:

$$\lim_{a\to+\infty} \int_{-\infty}^{+\infty} (\Delta_a)(x)\varphi(x)\mathrm{d}x = \frac{1}{2} \lim_{h\to 0^+} \frac{F_\varphi(h)}{h} = \frac{1}{2} \lim_{h\to 0^+} \frac{F_\varphi(h) - F_\varphi(0)}{h} = F'_\varphi(0) = \varphi(0) = \langle \delta, \varphi \rangle.$$

This proves that, as tempered distributions:

$$\lim_{a\to+\infty} \Delta_a = \delta.$$

(c) Let $\varphi \in \mathcal{S}(\mathbb{R})$. We have:

$$\int_{-\infty}^{+\infty} \mathcal{F}[\Delta_a](\xi)\varphi(\xi)\mathrm{d}\xi = 2a \int_{-\infty}^{+\infty} \frac{\sin\left(\frac{\xi}{2a}\right)}{\xi} \varphi(\xi)\mathrm{d}\xi = \int_{-\infty}^{+\infty} \mathrm{sinc}\left(\frac{\xi}{2a}\right) \varphi(\xi)\mathrm{d}\xi.$$

The function $f_a : \xi \mapsto \mathrm{sinc}\left(\frac{\xi}{2a}\right) \varphi(\xi)$ is integrable (as sinc is uniformly bounded over $\mathbb{R}$) and tends pointwisely to $\varphi(\xi)$ as $a \to +\infty$; since $|f_a| \leq |\varphi|$ for all $a \in \mathbb{R}$ and $|\varphi|$ is integrable, Lebesgue's dominated convergence theorem ensures that the limit $a \to +\infty$ can be taken through the integral:

$$\lim_{a\to+\infty} \int_{-\infty}^{+\infty} \mathcal{F}[\Delta_a](\xi)\varphi(\xi)\mathrm{d}\xi = \int_{-\infty}^{+\infty} \mathrm{sinc}(0)\varphi(\xi)\mathrm{d}\xi = \int_{-\infty}^{+\infty} 1\cdot\varphi(\xi)\mathrm{d}\xi = \langle 1, \varphi\rangle.$$

This proves that, as tempered distributions:

$$\lim_{a\to+\infty} \mathcal{F}[\Delta_a] = 1.$$

(d) Let $\varphi \in \mathcal{S}(\mathbb{R})$. As $\mathcal{F}[\mathcal{S}(\mathbb{R})] = \mathcal{S}(\mathbb{R})$, we also have $\mathcal{F}[\varphi] \in \mathcal{S}(\mathbb{R})$. Then, using the above results, we can write:

$$\langle 1, \varphi\rangle = \lim_{a\to+\infty} \langle \mathcal{F}[\Delta_a], \varphi\rangle = \lim_{a\to+\infty} \langle \Delta_a, \mathcal{F}[\varphi]\rangle = \langle \delta, \mathcal{F}[\varphi]\rangle = \langle \mathcal{F}[\delta], \varphi\rangle.$$

This completes the proof.

2. (a) Let $\varphi \in \mathcal{S}(\mathbb{R})$. The function $f_a : x \mapsto \mathrm{Tri}(x/a)\varphi(\xi)$ is integrable (as Tri is uniformly bounded over $\mathbb{R}$) and tends pointwisely to φ as $a \to +\infty$; since $|f_a(\xi)| \leq |\varphi|$ for all $a \in \mathbb{R}$ and $|\varphi|$ is integrable, Lebesgue's dominated convergence theorem ensures that the limit $a \to +\infty$ can be taken through the integral:

$$\lim_{a\to+\infty} \int_{-\infty}^{+\infty} \mathrm{Tri}(x/a)\varphi(x)\mathrm{d}x = \int_{-\infty}^{+\infty} \mathrm{Tri}(0)\varphi(x)\mathrm{d}x = \int_{-\infty}^{+\infty} 1\cdot\varphi(x)\mathrm{d}x = \langle 1, \varphi\rangle.$$

Hence, as tempered distributions,

$$\lim_{a\to+\infty} \mathrm{Tri}_a = 1.$$

(b) *Feynman's trick.* For all $t > 0$, we define:

$$I(t) := \int_{-\infty}^{+\infty} \frac{1-\cos(\eta)}{\eta^2}\mathrm{e}^{-t\eta^2}\mathrm{d}\eta.$$

The decay of the integrand allows us to differentiate through the integral:

$$I'(t) = -\int_{-\infty}^{+\infty} (1-\cos(\eta))\mathrm{e}^{-t\eta^2}\mathrm{d}\eta = -\sqrt{\frac{\pi}{t}}(1-\mathrm{e}^{-1/4t}).$$

Solving for the first order equation with boundary condition $I(t) \to 0$ as $t \to +\infty$ (by dominated convergence), we obtain:

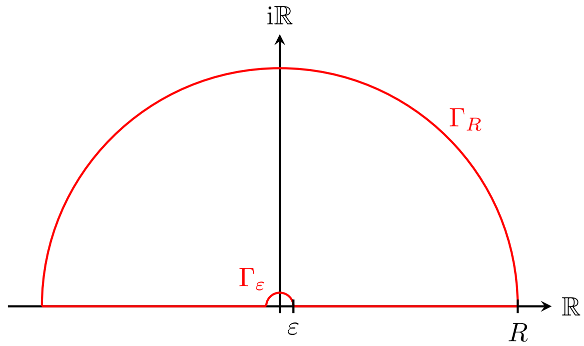
$$I(t) = -2\sqrt{\frac{\pi}{t}}(1-\mathrm{e}^{-1/4t}) + \pi\mathrm{erf}(1/2\sqrt{t}).$$

Hence:

$$\int_{-\infty}^{+\infty} \frac{1-\cos(\eta)}{\eta^2}\mathrm{d}\eta = \lim_{t\to 0^+} I(t) = \pi.$$

Residue theorem. We integrate the function $f(z) := \frac{1-\mathrm{e}^{\mathrm{i}z}}{z^2}$ over the positively oriented contour

Γ depicted below:



We have:

$$\left|\int_{\Gamma_R} f(z)\mathrm{d}z\right| \leq \int_0^{\pi} \frac{1+\mathrm{e}^{-R\sin(\theta)}}{R^2} R\mathrm{d}\theta \leq \frac{2\pi}{R} \underset{R\to+\infty}{\longrightarrow} 0;$$
$$\int_{\Gamma_\varepsilon} f(z)\mathrm{d}z = -\pi\mathrm{i}(\mathrm{i}) = \pi.$$

Hence:

$$I = \pi.$$

(c) Let $\varphi \in \mathcal{S}(\mathbb{R})$. We compute:

$$\mathcal{F}[\mathrm{Tri}_a](\xi) = \int_{-a}^{0} \left(1+\frac{x}{a}\right)\mathrm{e}^{-\mathrm{i}x\xi}\mathrm{d}x + \int_0^a \left(1-\frac{x}{a}\right)\mathrm{e}^{-\mathrm{i}x\xi}\mathrm{d}x = 2\frac{1-\cos(a\xi)}{a\xi^2}.$$

By dominated convergence:

$$\lim_{a\to+\infty}\int_{-\infty}^{+\infty} 2\frac{1-\cos(a\xi)}{a\xi^2}\varphi(\xi)\mathrm{d}\xi = \lim_{a\to+\infty}\int_{-\infty}^{+\infty} 2\frac{1-\cos(\eta)}{\eta^2}\varphi(\eta/a)\mathrm{d}\eta = 2\varphi(0)\int_{-\infty}^{+\infty}\frac{1-\cos(\eta)}{\eta^2}\mathrm{d}\eta.$$

From the above computation, we deduce that:

$$\lim_{a\to+\infty}\int_{-\infty}^{+\infty} 2\frac{1-\cos(a\xi)}{a\xi^2}\varphi(\xi)\mathrm{d}\xi = 2\pi\varphi(0) = 2\pi\langle\delta,\varphi\rangle.$$

This proves that, as tempered distributions:

$$\lim_{a\to+\infty}\mathcal{F}[T_a] = 2\pi\delta.$$

(d) Let $\varphi \in \mathcal{S}(\mathbb{R})$. As $\mathcal{F}[\mathcal{S}(\mathbb{R})] = \mathcal{S}(\mathbb{R})$, we also have $\mathcal{F}[\varphi] \in \mathcal{S}(\mathbb{R})$. Then, using the above results, we can write:

$$\langle 2\pi\delta,\varphi\rangle = \lim_{a\to+\infty}\langle\mathcal{F}[T_a],\varphi\rangle = \lim_{a\to+\infty}\langle T_a,\mathcal{F}[\varphi]\rangle = \langle 1,\mathcal{F}[\varphi]\rangle = \langle\mathcal{F}[1],\varphi\rangle.$$

This completes the proof.

Exercise 2:

The goal of this exercise is to prove the following formula: as elements in $\mathcal{S}'(\mathbb{R})$,

$$\mathcal{F}[\delta] = 1.$$

Let $t > 0$. For all $x \in \mathbb{R}$, we define:

$$G_t(x) := \frac{\mathrm{e}^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}.$$

1. For all $\xi \in \mathbb{R}$, compute $\mathcal{F}[G_t](\xi)$. *Hint:* We can differentiate $\mathcal{F}[G_t](\xi)$ with respect to ξ.
2. Prove that, as elements in $\mathcal{S}'(\mathbb{R})$,

$$\lim_{t \to 0^+} \mathcal{F}[G_t] = 1.$$

3. Prove that, as elements in $\mathcal{S}'(\mathbb{R})$,

$$\lim_{t \to 0^+} G_t = \delta.$$

4. Deduce that, as elements in $\mathcal{S}'(\mathbb{R})$,

$$\mathcal{F}[\delta] = 1.$$

Solution 2:

1. We compute:

$$\mathcal{F}[G_t](\xi) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}x\xi} \mathrm{e}^{-\frac{x^2}{4t}} \,\mathrm{d}x.$$

The decay of the integrand above allows to differentiate through the integral:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{F}[G_t](\xi) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} (-\mathrm{i}x) \mathrm{e}^{-\mathrm{i}x\xi} \mathrm{e}^{-\frac{x^2}{4t}} \,\mathrm{d}x \\
&= \mathrm{i}\sqrt{\frac{t}{\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}x\xi} \left(-\frac{x}{2t}\right) \mathrm{e}^{-\frac{x^2}{4t}} \,\mathrm{d}x \\
&= \mathrm{i}\sqrt{\frac{t}{\pi}} \left(\left[\mathrm{e}^{-\mathrm{i}x\xi} \mathrm{e}^{-\frac{\xi^2}{4t}} \right]_{-\infty}^{+\infty} + \mathrm{i}\xi \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}x\xi} \mathrm{e}^{-\frac{x^2}{4t}} \,\mathrm{d}x \right) \\
&= -\sqrt{\frac{t}{\pi}} \xi \mathcal{F}[G_t](\xi).
\end{aligned}$$

Solving this ordinary first order equation yields:

$$\mathcal{F}[G_t](\xi) = \mathcal{F}[G_t](0) \mathrm{e}^{-\sqrt{\frac{t}{\pi}}\xi^2}.$$

We then use the Gauss integral:

$$\mathcal{F}[G_t](0) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{x^2}{4t}} \mathrm{d}x = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\eta^2} \mathrm{d}\eta = 1.$$

Whence:

$$\mathcal{F}[G_t](\xi) = \mathrm{e}^{-\sqrt{\frac{t}{\pi}}\xi^2}.$$

2. Let $\varphi \in \mathcal{S}(\mathbb{R})$. We can exchange limit and integration by dominated convergence:

$$\lim_{t\to 0^+} \int_{-\infty}^{+\infty} \mathcal{F}[G_t](\xi)\varphi(\xi)\mathrm{d}\xi = \int_{-\infty}^{+\infty} 1 \cdot \varphi(\xi)\mathrm{d}\xi = \langle 1, \varphi \rangle.$$

This proves that, as tempered distributions:

$$\lim_{t\to 0^+} \mathcal{F}[G_t] = 1.$$

3. Let $\varphi \in \mathcal{S}(\mathbb{R})$. By dominated convergence:

$$\lim_{t\to 0^+} \int_{-\infty}^{+\infty} G_t(\xi)\varphi(\xi)\mathrm{d}\xi = \lim_{t\to 0^+} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\eta^2} \varphi(2\sqrt{t}\eta)\mathrm{d}\eta = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\eta^2} \varphi(0)\mathrm{d}\eta = \varphi(0).$$

4. Let $\varphi \in \mathcal{S}(\mathbb{R})$. As $\mathcal{F}[\mathcal{S}(\mathbb{R})] = \mathcal{S}(\mathbb{R})$, we also have $\mathcal{F}[\varphi] \in \mathcal{S}(\mathbb{R})$. Then, using the above results, we can write:

$$\langle 1, \varphi \rangle = \lim_{t\to 0^+} \langle \mathcal{F}[G_t], \varphi \rangle = \lim_{t\to 0^+} \langle G_t, \mathcal{F}[\varphi] \rangle = \langle \delta, \mathcal{F}[\varphi] \rangle = \langle \mathcal{F}[\delta], \varphi \rangle.$$

This completes the proof.

Exercise 3:
For all $a > 0$, we define:

$$T_a(x) = a^2 \begin{cases} -1 & \text{if } -\frac{1}{a} \leq x \leq 0, \\ 1 & \text{if } 0 \leq x \leq \frac{1}{a}, \\ 0 & \text{else.} \end{cases}$$

1. Prove that, as elements in $\mathcal{S}'(\mathbb{R})$,

$$\lim_{a\to+\infty} T_a = \delta'.$$

2. Prove that, as elements in $\mathcal{S}'(\mathbb{R})$,

$$\lim_{a\to+\infty} \mathcal{F}[T_a] = -\mathrm{i}\xi.$$

3. Deduce (without using Fourier's derivative formula) that, as elements in $\mathcal{S}'(\mathbb{R})$,

$$\mathcal{F}[\delta'] = -\mathrm{i}\xi.$$

Solution 3:

1. Let $\varphi \in \mathcal{S}(\mathbb{R})$. Set $h := \frac{1}{a}$. We compute:

$$\int_{-\infty}^{+\infty} T_a(x)\varphi(x)\mathrm{d}x = a^2\left(-\int_{-\frac{1}{a}}^{0}\varphi(x)\mathrm{d}x + \int_0^{\frac{1}{a}}\varphi(x)\mathrm{d}x\right) = \frac{1}{h^2}\left(-\int_{-h}^{0}\varphi(x)\mathrm{d}x + \int_0^{h}\varphi(x)\mathrm{d}x\right).$$

Let us set:

$$F_\varphi(t) := -\int_{-h}^{0}\varphi(x)\mathrm{d}x + \int_0^{h}\varphi(x)\mathrm{d}x.$$

Note that F_φ is two times differentiable at finite $t \in \mathbb{R}$ and $F_\varphi(0) = F'_\varphi(0) = 0$, $F''_\varphi(0) = 2\varphi'(0)$. Then:

$$\begin{aligned}
\int_{-\infty}^{+\infty} T_a(x)\varphi(x)\mathrm{d}x &= \frac{F_\varphi(h)}{h^2} \\
&= \frac{F_\varphi(0) + hF'_\varphi(0) + \frac{h^2}{2}F''_\varphi(0) + o_{h\to 0^+}(h^2)}{h^2} \\
&= \frac{1}{2}F''_\varphi(0) + o_{h\to 0^+}(1) \underset{h\to 0^+}{\longrightarrow} \varphi'(0).
\end{aligned}$$

It follows that, as distributions of order 1:

$$\lim_{a\to+\infty} T_a = \delta'.$$

2. Let $\varphi \in \mathcal{S}(\mathbb{R})$. We compute:

$$\begin{aligned}
\mathcal{F}[T_a](\xi) = \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}x\xi}T_a(x)\mathrm{d}x &= a^2\left(-\int_{-\frac{1}{a}}^{0}\mathrm{e}^{-\mathrm{i}x\xi}\mathrm{d}x + \int_0^{\frac{1}{a}}\mathrm{e}^{-\mathrm{i}x\xi}\mathrm{d}x\right) \\
&= a^2\left(-\frac{1-\mathrm{e}^{\mathrm{i}\frac{\xi}{a}}}{-\mathrm{i}\xi} + \frac{\mathrm{e}^{-\mathrm{i}\frac{\xi}{a}}-1}{-\mathrm{i}\xi}\right) \\
&= \frac{2\mathrm{i}a^2}{\xi}\left(\cos\left(\frac{\xi}{a}\right) - 1\right),
\end{aligned}$$

$$\lim_{a\to+\infty}\int_{-\infty}^{+\infty}\mathcal{F}[T_a](\xi)\varphi(\xi)\mathrm{d}\xi = \lim_{a\to+\infty}\int_{-\infty}^{+\infty}\frac{2\mathrm{i}a^2}{\xi}\left(\cos\left(\frac{\xi}{a}\right) - 1\right)\varphi(\xi)\mathrm{d}\xi.$$

The function $f_a : \xi \mapsto \frac{2\mathrm{i}a^2}{\xi}\left(\cos\left(\frac{\xi}{a}\right) - 1\right)\varphi(\xi)$ is integrable (as $\frac{\cos\left(\frac{\xi}{a}\right)-1}{\xi}$ is uniformly bounded over $\mathbb{R}$) and tends pointwisely to $-\mathrm{i}\xi\varphi(\xi)$ as $a \to +\infty$; since $|f_a(\xi)| \leq 4|\xi\varphi(\xi)|$ for all $a \in \mathbb{R}$ and $|\xi \mapsto \xi\varphi(\xi)|$ is integrable, Lebesgue's dominated convergence theorem ensures that the limit

$a \to +\infty$ can be taken through the integral:

$$\lim_{a\to+\infty} \int_{-\infty}^{+\infty} \mathcal{F}[T_a](\xi)\varphi(\xi)\mathrm{d}\xi = \lim_{a\to+\infty} \int_{-\infty}^{+\infty} (-\mathrm{i}\xi)\varphi(\xi)\mathrm{d}\xi = \langle -\mathrm{i}\xi, \varphi\rangle.$$

This proves that, as tempered distributions:

$$\lim_{a\to+\infty} \mathcal{F}[T_a] = -\mathrm{i}\xi.$$

3. Let $\varphi \in \mathcal{S}(\mathbb{R})$. As $\mathcal{F}[\mathcal{S}(\mathbb{R})] = \mathcal{S}(\mathbb{R})$, we also have $\mathcal{F}[\varphi] \in \mathcal{S}(\mathbb{R})$. Then, using the above results, we can write:

$$\langle -\mathrm{i}\xi, \varphi\rangle = \lim_{t\to0^+} \langle \mathcal{F}[T_a], \varphi\rangle = \lim_{t\to0^+} \langle T_a, \mathcal{F}[\varphi]\rangle = \langle \delta', \mathcal{F}[\varphi]\rangle = \langle \mathcal{F}[\delta'], \varphi\rangle.$$

This completes the proof.

Exercise 4:

Let $\rho \in L^1(\mathbb{R}, \mathbb{C})$ be such that:

$$\int_{-\infty}^{+\infty} \rho(x)\mathrm{d}x = 1.$$

Show that, as distribution of order 0 (*i.e.* as element in the topological dual of $\mathcal{C}_\mathrm{c}^0(\mathbb{R}, C)$, the space of compactly supported continuous functions),

$$\lim_{\varepsilon\to0^+} \rho_\varepsilon = \delta$$

where $\rho_\varepsilon(x) := \varepsilon^{-1}\rho(\varepsilon^{-1}x)$.

Solution 4:

For all $\varphi \in \mathcal{C}_\mathrm{c}^0(\mathbb{R}, \mathbb{C})$, we have:

$$\begin{aligned}
|\langle \rho_\varepsilon - \delta, \varphi\rangle| &= \left| \int_{-\infty}^{+\infty} \varepsilon^{-1}\rho(\varepsilon^{-1}x)\varphi(x)\mathrm{d}x - \varphi(0) \right| \\
&= \left| \int_{-\infty}^{+\infty} \rho(x)\varphi(\varepsilon x)\mathrm{d}x - \varphi(0) \int_{-\infty}^{+\infty} \rho(x)\mathrm{d}x \right| \\
&= \left| \int_{-\infty}^{+\infty} \rho(x)(\varphi(\varepsilon x) - \varphi(0))\mathrm{d}x \right| \\
&\le \int_{-R}^{R} |\rho(x)||\varphi(\varepsilon x) - \varphi(0)|\mathrm{d}x + \int_{|x|>R} |\rho(x)||\varphi(\varepsilon x) - \varphi(0)|\mathrm{d}x \\
&\le \omega_\varphi(\varepsilon R) \int_{-R}^{R} |\rho(x)|\mathrm{d}x + 2\|\varphi\|_{L^\infty(\mathbb{R},\mathbb{C})} \int_{|x|>R} |\rho(x)|\mathrm{d}x.
\end{aligned}$$

Above ω_φ denotes the continuity modulus of φ. It follows:

$$\lim_{\varepsilon\to0^+} |\langle \rho_\varepsilon - \delta, \varphi\rangle| \le 2\|\varphi\|_{L^\infty(\mathbb{R},\mathbb{C})} \int_{|x|>R} |\rho(x)|\mathrm{d}x.$$

Since $\rho \in L^1(\mathbb{R}, \mathbb{C})$, the right-hand side above get be made arbitrarily small as $R \to +\infty$. Hence, in the sense of distributions of order 0:

$$\lim_{\varepsilon \to 0^+} \rho_\varepsilon = \delta.$$

This completes the proof.

Exercise 5:

Let $\rho \in L^1(\mathbb{R}, \mathbb{C})$ be such that:

$$\int_{-\infty}^{+\infty} \rho(x)\mathrm{d}x = 1.$$

For all $\varepsilon > 0$, let us define $\rho_\varepsilon(x) := \varepsilon^{-1}\rho(\varepsilon^{-1}x)$. We also set $H_\pm(x) := H(\pm x)$ for all $x \in \mathbb{R}$, where $H : \mathbb{R} \to [0,1]$ is the Heaviside step function ($H(x) = 1$ is $x \geq 0$, $H(x) = 0$ else). Finaly, let us define:

$$\delta_+ := \lim_{\varepsilon \to 0} H_+(\rho_{|\varepsilon|} - \rho_{-|\varepsilon|}), \qquad \delta_- := -\lim_{\varepsilon \to 0} H_-(\rho_{|\varepsilon|} - \rho_{-|\varepsilon|}).$$

1. Prove that, as elements in the topological dual of $\mathcal{C}^0_\mathrm{c}(\mathbb{R} \setminus \{0\}, \mathbb{C})$,

$$\frac{\delta_+ + \delta_-}{2} = \mathrm{P.V.}_0$$

   where $\mathrm{P.V.}_0$ is the Cauchy principal value at 0: for all $\varphi \in \mathcal{C}^0_\mathrm{c}(\mathbb{R} \setminus \{0\}, \mathbb{C})$,

$$\mathrm{P.V.}_0[\varphi] := \frac{\varphi(0^+) + \varphi(0^-)}{2}, \qquad \varphi(0^\pm) := \lim_{x \to 0^\pm} \varphi(x).$$

2. Prove that, as distributions of order 0,

$$\frac{\delta_+ + \delta_-}{2} = \delta.$$

Solution 5:

1. Let $\varphi \in \mathcal{C}^0_\mathrm{c}(\mathbb{R} \setminus \{0\}, \mathbb{C})$. The same argument as in Exercise 4 above yields:

$$\begin{aligned}
\lim_{\varepsilon \to 0} \langle H_+\rho_{|\varepsilon|}, \varphi \rangle &= \varphi(0^+) \int_{-\infty}^{0} \rho(x)\mathrm{d}x, \\
\lim_{\varepsilon \to 0} \langle H_-\rho_{|\varepsilon|}, \varphi \rangle &= \varphi(0^-) \int_{0}^{+\infty} \rho(x)\mathrm{d}x, \\
\lim_{\varepsilon \to 0} \langle H_+\rho_{-|\varepsilon|}, \varphi \rangle &= -\varphi(0^+) \int_{-\infty}^{0} \rho(x)\mathrm{d}x, \\
\lim_{\varepsilon \to 0} \langle H_-\rho_{-|\varepsilon|}, \varphi \rangle &= -\varphi(0^-) \int_{0}^{+\infty} \rho(x)\mathrm{d}x.
\end{aligned}$$

Then:

$$\langle \delta_+, \varphi \rangle = \varphi(0^+) \int_{-\infty}^{+\infty} \rho(x)\mathrm{d}x = \varphi(0^+),$$
$$\langle \delta_-, \varphi \rangle = \varphi(0^-) \int_{-\infty}^{+\infty} \rho(x)\mathrm{d}x = \varphi(0^-).$$

In particular,

$$\left\langle \frac{\delta_+ + \delta_-}{2}, \varphi \right\rangle = \frac{\varphi(0^+) + \varphi(0^-)}{2} = \mathrm{P.V}_0[\varphi]$$

2. If $\varphi \in \mathcal{C}^0_{\mathrm{c}}(\mathbb{R}, \mathbb{C})$, then $\varphi(0^+) = \varphi(0^-) = \varphi(0)$ and thus:

$$\left\langle \frac{\delta_+ + \delta_-}{2}, \varphi \right\rangle = \frac{\varphi(0^+) + \varphi(0^-)}{2} = \varphi(0) = \langle \delta, \varphi \rangle.$$

This completes the proof.

Exercise 6:

The goal of this exercise is to prove the following fundamental formula: as elements in $\mathcal{S}'(\mathbb{R})$,

$$\mathcal{F}[H] = \pi\delta - \mathrm{i}\,\mathrm{P.V.}\frac{1}{\xi} \tag{1}$$

where:

$$H(x) := \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{else,} \end{cases}$$
$$\mathrm{P.V.}\frac{1}{\xi}[\varphi] := \lim_{\varepsilon \to 0^+} \int_{|\xi| \geq \varepsilon} \frac{\varphi(\xi)}{\xi}\mathrm{d}\xi = \int_0^{+\infty} \frac{\varphi(\xi) - \varphi(-\xi)}{\xi}\mathrm{d}\xi \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}).$$

Let $a > 0$ and set:

$$T_a(x) := \begin{cases} 1 & \text{if } 0 \leq x \leq a, \\ 0 & \text{else.} \end{cases}$$

1. Prove that, as elements in $\mathcal{S}'(\mathbb{R})$,

$$\lim_{a \to +\infty} \mathcal{F}[T_a] = \mathcal{F}[H].$$

2. Prove that, for all $a > 0$ and all $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\lim_{a \to +\infty} \int_0^R \frac{\mathrm{e}^{-\mathrm{i}\eta}\varphi(\eta/a) - \mathrm{e}^{\mathrm{i}\eta}\varphi(-\eta/a)}{\mathrm{i}\eta}\mathrm{d}\eta = \pi\varphi(0).$$

3. Prove that, as elements in $\mathcal{S}'(\mathbb{R})$,

$$\lim_{a \to +\infty} \mathcal{F}[T_a] = -\mathrm{i}\,\mathrm{P.V.}\frac{1}{\xi} + \pi\delta.$$

4. Deduce the formula (1).

Solution 6:

1. Clearly, $T_a \to H$ pointwisely so that, for all $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\lim_{a\to+\infty}\langle \mathcal{F}[T_a],\varphi\rangle = \lim_{a\to+\infty}\langle T_a,\mathcal{F}[\varphi]\rangle = \langle H,\mathcal{F}[\varphi]\rangle = \langle \mathcal{F}[H],\varphi\rangle.$$

2. Pick $R > 0$. On the one hand, Lebesgue's dominated convergence implies:

$$\begin{aligned}\lim_{a\to+\infty}\int_0^R \frac{\mathrm{e}^{-\mathrm{i}\eta}\varphi(\eta/a)-\mathrm{e}^{\mathrm{i}\eta}\varphi(-\eta/a)}{\mathrm{i}\eta}\mathrm{d}\eta &= -2\varphi(0)\int_0^R \frac{\sin(\eta)}{\eta}\mathrm{d}\eta\\ &= -2\varphi(0)\left(\frac{\pi}{2}-\int_R^{+\infty}\frac{\sin(\eta)}{\eta}\mathrm{d}\eta\right)\\ &= -2\varphi(0)\left(\frac{\pi}{2}+\left[\frac{\cos(\eta)}{\eta}\right]_R^{+\infty}+\int_R^{+\infty}\frac{\cos(\eta)}{\eta^2}\mathrm{d}\eta\right)\\ &= -2\varphi(0)\left(\frac{\pi}{2}-\frac{\cos(R)}{R}+\underbrace{\int_R^{+\infty}\frac{\cos(\eta)}{\eta^2}\mathrm{d}\eta}_{\leq\frac{1}{R}}\right).\end{aligned}$$

On the other hand:

$$\left|\int_R^{+\infty}\frac{\mathrm{e}^{-\mathrm{i}a\xi}\varphi(\xi)-\mathrm{e}^{\mathrm{i}a\xi}\varphi(-\xi)}{\mathrm{i}\xi}\mathrm{d}\xi\right| \leq \int_R^{+\infty}\frac{|\xi\varphi(\xi)|+|\xi\varphi(-\xi)|}{\xi^2}\mathrm{d}\xi \leq \frac{\|\xi\varphi(\xi)\|_{L^\infty(\mathbb{R})}+\|\xi\varphi(-\xi)\|_{L^\infty(\mathbb{R})}}{R}.$$

It therefore follows:

$$\lim_{a\to+\infty}\int_0^R \frac{\mathrm{e}^{-\mathrm{i}\eta}\varphi(\eta/a)-\mathrm{e}^{\mathrm{i}\eta}\varphi(-\eta/a)}{\mathrm{i}\eta}\mathrm{d}\eta = -\pi\varphi(0)+\mathcal{O}_{R\to+\infty}(1/R).$$

As R is arbitrary, we deduce the announced equality.

3. We compute:

$$\mathcal{F}[T_a](\xi) = \int_0^a \mathrm{e}^{-\mathrm{i}x\xi}\mathrm{d}x = \frac{1-\mathrm{e}^{-\mathrm{i}a\xi}}{\mathrm{i}\xi}.$$

For all $\varphi \in \mathcal{S}(\mathbb{R})$, we thus have:

$$\begin{aligned}\langle \mathcal{F}[T_a],\varphi\rangle &= \int_{-\infty}^{+\infty}\frac{1-\mathrm{e}^{-\mathrm{i}a\xi}}{\mathrm{i}\xi}\varphi(\xi)\mathrm{d}\xi\\ &= \mathrm{P.V.}\int_{-\infty}^{+\infty}\frac{1-\mathrm{e}^{-\mathrm{i}a\xi}}{\mathrm{i}\xi}\varphi(\xi)\mathrm{d}\xi\\ &= \mathrm{P.V.}\int_{-\infty}^{+\infty}\frac{1}{\mathrm{i}\xi}\varphi(\xi)\mathrm{d}\xi - \mathrm{P.V.}\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{-\mathrm{i}a\xi}}{\mathrm{i}\xi}\varphi(\xi)\mathrm{d}\xi\\ &= -\mathrm{i}\,\mathrm{P.V.}\frac{1}{\xi}[\varphi] - \int_0^{+\infty}\frac{\mathrm{e}^{-\mathrm{i}a\xi}\varphi(\xi)-\mathrm{e}^{\mathrm{i}a\xi}\varphi(-\xi)}{\mathrm{i}\xi}\mathrm{d}\xi\end{aligned}$$

We finally take the limit as $a \to +\infty$ using the result from the previous question:

$$\begin{aligned}
\langle \mathcal{F}[T_a], \varphi \rangle &= -\mathrm{i}\,\mathrm{P.V.}\frac{1}{\xi}[\varphi] - \lim_{a\to+\infty} \int_0^{+\infty} \frac{\mathrm{e}^{-\mathrm{i}a\xi}\varphi(\xi) - \mathrm{e}^{\mathrm{i}a\xi}\varphi(-\xi)}{\mathrm{i}\xi}\mathrm{d}\xi \\
&= -\mathrm{i}\,\mathrm{P.V.}\frac{1}{\xi}[\varphi] - \lim_{a\to+\infty} \int_0^{+\infty} \frac{\mathrm{e}^{-\mathrm{i}\eta}\varphi(\eta/a) - \mathrm{e}^{\mathrm{i}\eta}\varphi(-\eta/a)}{\mathrm{i}\eta}\mathrm{d}\eta \\
&= -\mathrm{i}\,\mathrm{P.V.}\frac{1}{\xi}[\varphi] + \pi\varphi(0) \\
&= \Big\langle -\mathrm{i}\,\mathrm{P.V.}\frac{1}{\xi} + \pi\delta, \varphi \Big\rangle.
\end{aligned}$$

4. Let $\varphi \in \mathcal{S}(\mathbb{R})$. Then, using the above results, we can write:

$$\langle \mathcal{F}[H], \varphi \rangle = \lim_{t\to 0^+} \langle \mathcal{F}[T_a], \varphi \rangle = \Big\langle -\mathrm{i}\,\mathrm{P.V.}\frac{1}{\xi} + \pi\delta, \varphi \Big\rangle.$$

This completes the proof.