FRESNEL TYPE INTEGRAL

Exercise 1:

Let $\beta \in \mathbb{C}$ such that $\mathrm{Re}(\beta) \in (-1, 1)$. We set:

$$I(\beta) := \int_0^{+\infty} t^\beta \mathrm{e}^{-\mathrm{i}t^2} \mathrm{d}t.$$

1. By defining $z \mapsto z^\beta \mathrm{e}^{-z^2}$ on an appropriate contour in $\mathbb{C}$, show that the integral $I(\beta)$ converges and compute its value in terms of β.

2. Deduce the values (if exist) the following limits:

$$\lim_{\beta \to \pm 1^\mp} \int_0^{+\infty} t^\beta \cos(t^2) \mathrm{d}t, \qquad \lim_{\beta \to \pm 1^\mp} \int_0^{+\infty} t^\beta \sin(t^2) \mathrm{d}t.$$

3. Using the previous question only, get the value of the Dirichlet integral:

$$I := \int_0^{+\infty} \frac{\sin(t)}{t} \mathrm{d}t.$$

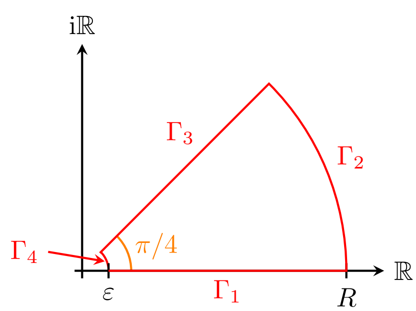
4. Generalization: For all $n \in \mathbb{N} \setminus \{0, 1\}$, show the convergence and compute the value of

$$I(\beta, n) := \int_0^{+\infty} t^\beta \mathrm{e}^{-\mathrm{i}t^n} \mathrm{d}t.$$

Solution 1:

1. Put $f(z) := z^\beta \mathrm{e}^{-z^2}$ where the complex power is defined as $z^\beta := \mathrm{e}^{\beta \mathrm{Log}(z)}$; here Log the complex logarithm defined on $\mathbb{C} \setminus A$ where A is a branch cut starting at 0 and lying in the left half plane $\{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \leq 0\}$.

   We fix $0 < \varepsilon < R$ then we define the positively oriented contour $\Gamma := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ as below:



   We integrate f over Γ. On the one hand, we have (using $\mathrm{Re}(\beta) > -1$):

$$\int_{\Gamma_1} f(z) \mathrm{d}z = \int_0^R t^\beta \mathrm{e}^{-t^2} \mathrm{d}t \underset{R \to +\infty}{\longrightarrow} \int_0^{+\infty} t^\beta \mathrm{e}^{-t^2} \mathrm{d}t = \frac{\Gamma\left(\frac{\beta+1}{2}\right)}{2}.$$

Above Γ is Euler's gamma function. On the other hand:

$$\begin{aligned}\left|\int_{\Gamma_2} f(z)\mathrm{d}z\right| &= \left|\int_0^1 \left(R\mathrm{e}^{\mathrm{i}\frac{\pi}{4}t}\right)^{\beta} \mathrm{e}^{-\left(R\mathrm{e}^{\mathrm{i}\frac{\pi}{4}t}\right)^2} \mathrm{i}\frac{\pi}{4}R\mathrm{e}^{\mathrm{i}\frac{\pi}{4}t}\mathrm{d}t\right| \\ &\leq \int_0^1 \left|\left(R\mathrm{e}^{\mathrm{i}\frac{\pi}{4}t}\right)^{\beta} \mathrm{e}^{-\left(R\mathrm{e}^{\mathrm{i}\frac{\pi}{4}t}\right)^2} \mathrm{i}\frac{\pi}{4}R\mathrm{e}^{\mathrm{i}\frac{\pi}{4}t}\right| \mathrm{d}t \\ &= \frac{\pi R^{\mathrm{Re}(\beta)+1}}{4}\int_0^1 \mathrm{e}^{-\frac{\pi \mathrm{Im}(\beta)}{4}t}\mathrm{e}^{-R^2\cos\left(\frac{\pi}{2}t\right)}\mathrm{d}t.\end{aligned}$$

As $[0,1] \ni t \mapsto \cos\left(\frac{\pi}{2}t\right)$ is concave, we have $\cos\left(\frac{\pi}{2}t\right) \leq \frac{\pi}{2}(1-t)$ and thus (using $\mathrm{Re}(\beta) < 1$ and taking $R \gg |\mathrm{Im}(\beta)|$):

$$\left|\int_{\Gamma_2} f(z)\mathrm{d}z\right| \leq \frac{\pi R^{\mathrm{Re}(\beta)+1}}{4}\int_0^1 \mathrm{e}^{-\frac{\pi \mathrm{Im}(\beta)}{4}t}\mathrm{e}^{-R^2\frac{\pi}{2}(1-t)}\mathrm{d}t = R^{\mathrm{Re}(\beta)+1}\mathrm{e}^{-\frac{\pi}{2}R^2}\frac{\mathrm{e}^{\frac{\pi}{4}(2R^2-\mathrm{Im}(\beta))}-1}{2R^2-\mathrm{Im}(\beta)} \underset{R\to+\infty}{\longrightarrow} 0.$$

Next, we have:

$$\begin{aligned}\int_{\Gamma_3} f(z)\mathrm{d}z &= \int_0^R \left(\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}(R-t)\right)^{\beta}\mathrm{e}^{-\left(\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}(R-t)\right)^2}\left(-\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right)\mathrm{d}t \\ &= -\mathrm{e}^{\mathrm{i}\frac{\pi(\beta+1)}{4}}\int_0^R (R-t)^{\beta}\mathrm{e}^{-\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}(R-t)^2}\mathrm{d}t \\ &= -\mathrm{e}^{\mathrm{i}\frac{\pi(\beta+1)}{4}}\int_0^R t^{\beta}\mathrm{e}^{-\mathrm{i}t^2}\mathrm{d}t.\end{aligned}$$

We eventually compute:

$$\begin{aligned}\left|\int_{\Gamma_4} f(z)\mathrm{d}z\right| &= \left|\int_0^1 \left(\varepsilon\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}t}\right)^{\beta} \mathrm{e}^{-\left(\varepsilon\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}t}\right)^2} \mathrm{i}\frac{\pi}{4}\varepsilon\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}t}\mathrm{d}t\right| \\ &\leq \int_0^1 \left|\left(\varepsilon\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}t}\right)^{\beta} \mathrm{e}^{-\left(\varepsilon\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}t}\right)^2} \mathrm{i}\frac{\pi}{4}\varepsilon\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}t}\right| \mathrm{d}t \\ &= \frac{\pi \varepsilon^{\mathrm{Re}(\beta)+1}}{4}\int_0^1 \mathrm{e}^{\frac{\pi \mathrm{Im}(\beta)}{4}t}\mathrm{e}^{-\varepsilon^2\cos\left(\frac{\pi}{2}t\right)}\mathrm{d}t \\ &\leq \frac{\pi\varepsilon^{\mathrm{Re}(\beta)+1}\mathrm{e}^{\frac{\pi\max\{\mathrm{Im}(\beta),0\}}{4}}}{4} \underset{\varepsilon\to0^+}{\longrightarrow} 0.\end{aligned}$$

Since f is holomorphic in $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$, Cauchy's theorem entails:

$$\int_{\Gamma} f(z)\mathrm{d}z = 0.$$

Therefore,

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}\frac{\pi(\beta+1)}{4}} I(\beta) &= \lim_{\substack{\varepsilon\to 0^+\\ R\to+\infty}} \mathrm{e}^{\mathrm{i}\frac{\pi(\beta+1)}{4}} \int_0^R \mathrm{e}^{-\mathrm{i}t^2}\mathrm{d}t \\
&= -\lim_{\substack{\varepsilon\to 0^+\\ R\to+\infty}} \int_{\Gamma_1} f(z)\mathrm{d}z \\
&= \lim_{\substack{\varepsilon\to 0^+\\ R\to+\infty}} \left(\int_{\Gamma_2} f(z)\mathrm{d}z + \int_{\Gamma_3} f(z)\mathrm{d}z + \int_{\Gamma_4} f(z)\mathrm{d}z\right) = \frac{\Gamma\left(\frac{\beta+1}{2}\right)}{2}
\end{aligned}$$

whence:

$$\boxed{I(\beta) = \mathrm{e}^{-\mathrm{i}\frac{\pi(\beta+1)}{4}} \frac{\Gamma\left(\frac{\beta+1}{2}\right)}{2}.}$$

2. In particular, if $\mathrm{Re}(\beta) \in (-1, 1)$, then we deduce:

$$\begin{aligned}
\int_0^{+\infty} t^\beta \cos(t^2)\mathrm{d}t &= \mathrm{Re}(I(\beta)) = \cos\left(\frac{\pi(\beta+1)}{4}\right)\frac{\Gamma\left(\frac{\beta+1}{2}\right)}{2}, \\
\int_0^{+\infty} t^\beta \sin(t^2)\mathrm{d}t &= -\mathrm{Re}(I(\beta)) = \sin\left(\frac{\pi(\beta+1)}{4}\right)\frac{\Gamma\left(\frac{\beta+1}{2}\right)}{2}.
\end{aligned}$$

When $\beta = 0$, we get back Fresnel's integral:

$$\int_0^{+\infty} \cos(t^2)\mathrm{d}t = \int_0^{+\infty} \sin(t^2)\mathrm{d}t = \sqrt{\frac{\pi}{8}}.$$

Using that $\Gamma(z) = \frac{1}{z} + \mathcal{O}_{z\to 0}(1)$, we compute:

$$\begin{aligned}
\lim_{\beta\to -1^+} \int_0^{+\infty} t^\beta \cos(t^2)\mathrm{d}t &= \lim_{\beta\to -1^+} \cos\left(\frac{\pi(\beta+1)}{4}\right)\frac{\Gamma\left(\frac{\beta+1}{2}\right)}{2} \\
&= \lim_{\beta\to -1^+} \left(\mathcal{O}_{\beta\to -1^+}(1)\frac{\frac{2}{\beta+1} + \mathcal{O}_{\beta\to -1^+}(1)}{2}\right) = +\infty, \\
\lim_{\beta\to -1^+} \int_0^{+\infty} t^\beta \sin(t^2)\mathrm{d}t &= \lim_{\beta\to -1^+} \sin\left(\frac{\pi(\beta+1)}{4}\right)\frac{\Gamma\left(\frac{\beta+1}{2}\right)}{2} \\
&= \lim_{\beta\to -1^+} \left(\left(\frac{\pi(\beta+1)}{4} + \mathcal{O}_{\beta\to -1^+}\big((\beta+1)^3\big)\right)\frac{\frac{2}{\beta+1} + \mathcal{O}_{\beta\to -1^+}(1)}{2}\right) \\
&= \frac{\pi}{4}.
\end{aligned}$$

On the other hand:

$$\begin{aligned}
\lim_{\beta\to 1^-} \int_0^{+\infty} t^\beta \cos(t^2)\mathrm{d}t &= \lim_{\beta\to 1^-} \cos\left(\frac{\pi(\beta+1)}{4}\right)\frac{\Gamma\left(\frac{\beta+1}{2}\right)}{2} = \cos\left(\frac{\pi}{2}\right)\cdot\frac{1}{2} = 0, \\
\lim_{\beta\to 1^-} \int_0^{+\infty} t^\beta \sin(t^2)\mathrm{d}t &= \lim_{\beta\to 1^-} \sin\left(\frac{\pi(\beta+1)}{4}\right)\frac{\Gamma\left(\frac{\beta+1}{2}\right)}{2} = \sin\left(\frac{\pi}{2}\right)\cdot\frac{1}{2} = \frac{1}{2}.
\end{aligned}$$

3. Using the change of variables $u := t^2$, we get:

$$\int_0^{+\infty} t^{2\beta+1}\sin(t^2)\mathrm{d}t = \frac{1}{2}\int_0^{+\infty} u^\beta \sin(u)\mathrm{d}u$$

We have:

$$\begin{aligned}
\lim_{\beta\to-1^+}\int_0^{+\infty} t^\beta \sin(t)\mathrm{d}t &= \lim_{\beta\to-1^+}\lim_{R\to+\infty}\int_0^R t^\beta\sin(t)\mathrm{d}t \\
&= \lim_{\beta\to-1^+}\left(\int_0^{\pi/2} t^\beta \sin(t)\mathrm{d}t + \beta\lim_{R\to+\infty}\int_{\pi/2}^R t^{\beta-1}\cos(t)\mathrm{d}t\right) \\
&= \lim_{\beta\to-1^+}\left(\int_0^{\pi/2} \frac{\sin(t)}{t^\beta}\mathrm{d}t + \beta\int_{\pi/2}^{+\infty} t^{\beta-1}\cos(t)\mathrm{d}t\right).
\end{aligned}$$

By applying Lebesgue's dominated convergence theorem on the integral involving cos for the limit $\beta\to-1^+$, we discover:

$$\begin{aligned}
\lim_{\beta\to-1^+}\int_0^{+\infty} t^\beta\sin(t)\mathrm{d}t &= \int_0^{\pi/2}\frac{\sin(t)}{t}\mathrm{d}t - \int_{\pi/2}^{+\infty}\frac{\cos(t)}{t^2}\mathrm{d}t \\
&= \lim_{R\to+\infty}\left(\int_0^{\pi/2}\frac{\sin(t)}{t}\mathrm{d}t - \int_{\pi/2}^{R}\frac{\cos(t)}{t^2}\mathrm{d}t\right) \\
&= \lim_{R\to+\infty}\left(\int_0^{\pi/2}\frac{\sin(t)}{t}\mathrm{d}t + \int_{\pi/2}^{R}\frac{\sin(t)}{t}\mathrm{d}t\right) \\
&= \lim_{R\to+\infty}\int_0^R\frac{\sin(t)}{t}\mathrm{d}t \\
&= \int_0^{+\infty}\frac{\sin(t)}{t}\mathrm{d}t\ ;
\end{aligned}$$

Whence:

$$\int_0^{+\infty}\frac{\sin(t)}{t}\mathrm{d}t = \lim_{\beta\to-1^+}\int_0^{+\infty}t^\beta\sin(t)\mathrm{d}t = 2\lim_{\beta\to-1^+}\int_0^{+\infty}t^{2\beta+1}\sin(t^2)\mathrm{d}t = \frac{\pi}{2}.$$

4. Integrating $f_n(z) := z^\beta \mathrm{e}^{-z^n}$ over Γ_n defined as Γ but with an angle of $\frac{\pi}{2n}$, the same computations as for question 1 above lead to the following formula: for all $\beta\in\{z\in\mathbb{C}\mid \mathrm{Re}(\beta)\in(-1,n-1)\}$,

$$\boxed{I(\beta,n) = \mathrm{e}^{-\mathrm{i}\frac{\pi(\beta+1)}{2n}}\frac{\Gamma\left(\frac{\beta+1}{2n}\right)}{2n}.}$$

Notice that, as $\beta\mapsto\Gamma\left(\frac{\beta+1}{2n}\right)$ has an order one pole at the points $\beta\in-1-2n\mathbb{N}$, and as

$$\int_0^{+\infty}\frac{\sin(t^n)}{t}\mathrm{d}t = -\lim_{\beta\to-1^+}\mathrm{Im}(I(\beta,n)) = \lim_{\beta\to-1^+}\sin\left(\frac{\pi(\beta+1)}{2n}\right)\frac{\Gamma\left(\frac{\beta+1}{2n}\right)}{2n} = \frac{\pi}{2n},$$

the formula

$$\int_0^{+\infty} t^\beta \sin(t^n)\mathrm{d}t = \sin\left(\frac{\pi(\beta+1)}{2n}\right)\frac{\Gamma\left(\frac{\beta+1}{2n}\right)}{2n}$$

holds for all $\beta \in (-n-1, n-1)$ (since $-n-1 > -1-2kn$ when $k \in \mathbb{N} \setminus \{0\}$). The above integral form on the left-hand side is well-define as showed by the change of variables $u := t^n$ as well as an integration by parts.