ASEN 5050 Fall 2023 HW 4 Solutions

Problem 1

given:

$\vec{r}_1 = -720000\hat{x} + 670000\hat{y} + 310000\hat{z}$ km

$\vec{v}_1 = 2.160\hat{x} - 3.360\hat{y} + 0.620\hat{z}$ km/s

$GM_{saturn} = 3.794\times10^{7}$ km^3/s^2

$r_{saturn} = 60268$ km

from HW3:

$\mathcal{E} = -28.62$ km^2/s

$a = 6.6278\times10^5$ km

$e = 0.9102$

$i = 62.1°$

$\Omega = 127.45°$

$\omega = -172.28°$

$h = 2.077\times10^6$

$\vec{h} = 1.457\times10^6\hat{x} + 1.116\times10^6\hat{y} + 9.72\times10^5\hat{z}$ km^2/s

$\theta_2^* = -13.17°$

$\theta_1^* = -167.83°$

note: E always in radians, $[-\pi, \pi]$

$\tan\left(\frac{E}{2}\right) = \sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta^*}{2}\right)$ sign check: if $\theta^* > 0$, $E > 0$

$M = n(t-t_p) = E - e\sin E$

$n = \sqrt{\mu/a^3}$

assume: 2BP, Saturn is a sphere, $GM_{s/c} \ll GM_{sat} \therefore \mu \approx GM_{saturn}$

$$E = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta^*}{2}\right)\right)$$

$E_1 = -2.2278$ rads $= 4.0554$ rad

$E_2 = -0.05$ rads $= 6.2331$ rad

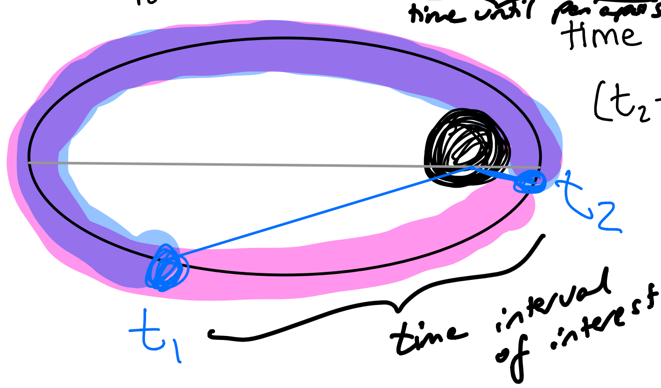
n = mean motion $= \sqrt{\frac{\mu_{sat}}{a^3}} = 1.1415\times10^{-5}$ rads/s

$(t_1 - t_p) = \frac{1}{n}(E_1 - e\sin E_1) = -1.3202\times10^5$ sec $= \overbrace{4.1839\times10^5\,\text{s}}^{\text{time past periapsis}}$

$(t_2 - t_p) = \frac{1}{n}(E_2 - e\sin E_2) = \underbrace{-395.322\text{ sec}}_{\text{time until periapsis}} = 5.5001\times10^5$ s

time for s/c to travel from 1 to 2:

$(t_2 - t_1) = (t_2 - t_p) - (t_1 - t_p)$



$t_2 - t_1 = 1.3162\times10^5$ sec $= 36.56$ hours

Problem 2:

At t_1: $\bar{R}_1 = 1.89737 \cdot 10^6 \hat{x} - 4.08985 \cdot 10^6 \hat{y} - 4.59509 \cdot 10^6 \hat{z}$

$\bar{V}_1 = 0.133694 \hat{x} - 0.546366 \hat{y} + 0.369530 \hat{z}$ km/s

$m_{sc} << m_{Jupiter}$

Assume Jupiter - S/C 2BP $\therefore \mu = G(m_{Jupiter} + \cancel{m_{S/C}}) \approx G(m_{Jupiter}) = 1.268 \cdot 10^8 \ km^3/s^2$

a) Calculate θ_1^* and E_1 at t_1

Calculate $\mathcal{E}$ and a:

$r_1 = |\bar{R}_1| = 6.4375 \cdot 10^5$ km

$v_1 = |\bar{V}_1| = 0.6730$ km/s

$\mathcal{E} = v_1^2/2 - \mu/r_1$

$\therefore \underline{\mathcal{E} = -19.4706} \ km^2/s^2$

$a = -\mu/2\mathcal{E}$

$\therefore \underline{a = 3.2562 \cdot 10^6 \text{ km}}$

Calculate h:

$\bar{h} = \bar{R}_1 \times \bar{V}_1$

$\therefore \underline{\bar{h} = -4.02 \cdot 10^6 \hat{x} - 1.32 \cdot 10^6 \hat{y} - 0.49 \hat{z} \ km^2/s}$

$h = |\bar{h}|$

$\therefore \underline{h = 4.2597 \cdot 10^6 \ km^2/s}$

Calculate e

$e = \sqrt{1 + 2h^2 \mathcal{E}/\mu^2}$

$\therefore \underline{e = 0.9778}$

Calculate θ_1^*

$p = a(1-e^2) = 1.4310 \cdot 10^5$ km

$\theta_1^* = \pm\cos^{-1}(p/(r_1 e) - 1/e) \rightarrow \vec{R}_1 \cdot \vec{V}_1 = 7.9 \times 10^5 > 0$

$\therefore \theta_1^* = [0°, 180°]$

$\boxed{\therefore \theta_1^* = 3.1374 \text{ rad} = 179.7584°}$

Calculate E_1

$r = a(1 - e\cos(E))$

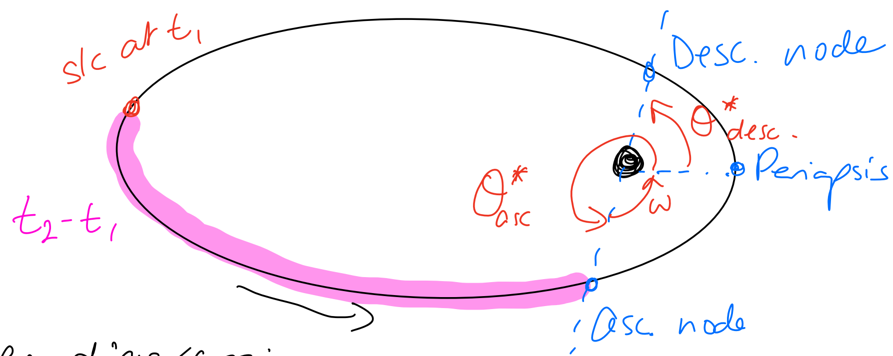
$E = \pm\cos^{-1}((1 - r/a)/e) \rightarrow$ Because $\theta_1^* = [0°, 180°]$, $E_1 = [0, \pi]$ rad

$\boxed{\therefore E = 3.1018 \text{ rad}}$

b) Calculate θ_2^* and E_2 at t_2 when the s/c crosses the eq. plane after t_1

Note: s/c crosses equatorial plane (in this case, xy plane) at <u>both</u> the ascending and descending nodes.
→ Need to calculate their θ^*

Consider a conceptual diagram:



From the diagram:

$\theta^*_{asc} = -\omega \qquad \theta^*_{desc} = 180° - \omega$

Calculate argument of periapsis:

$\bar{n} = \hat{Z} \times \bar{h} = (1.3155\,\hat{x} - 4.0218\,\hat{y}) \cdot 10^6 \text{ km}^2/\text{s}$

$\bar{e} = \bar{V}_1 \times \bar{h}/\mu - \bar{R}_1/r_1 = -0.2888\,\hat{x} + 0.6241\,\hat{y} + 0.6951\,\hat{z}$

$\omega = \cos^{-1}(\bar{n} \cdot \bar{e})/(|\bar{n}||\bar{e}|)$ * Check $e_z > 0 \therefore \omega > 0$

$\therefore \omega = 2.3441 \text{ rad} = 134.3056°$

$\theta^*_{asc} = -\omega = -2.3441 \text{ rad} = -134.3056°$

$\theta^*_{desc} = \theta^*_{asc} + 180° = 0.7975 \text{ rad} = 45.6944°$

From the diagram, the s/c will first pass through the ascending node after t_1
(Note: if ω had different value, we might need to redraw a more accurate diagram)

Since $\theta_1^* > \theta_{desc}^*$ the s/c will cross the eq. plane at the ascending node

$$\boxed{\therefore \theta_2^* = \theta_{asc}^* = -2.3441 \text{ rad} = -134.3056°}$$

$$E_2 = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta_2^*}{2}\right)\right)$$

$$\boxed{\therefore E_2 = -0.4929 \text{ rad} = 5.7903 \text{ rad}}$$

c) Calculate the time between t_1 and t_2 in days

Calculate Mean Motion (n):

$$n = \sqrt{\frac{\mu}{a^3}}$$

$$\therefore n = 1.9164 \cdot 10^{-6} \ \frac{rad}{s}$$

Calculate time difference

*Be sure to wrap E_2 to 2π to get appropriate answer

$$(t_2 - t_p) = \frac{1}{n}(E_2 - e\sin(E_2)) \qquad (t_1 - t_p) = \frac{1}{n}(E_1 - e\sin(E_1))$$

$$\therefore (t_2 - t_p) = 3.2628 \cdot 10^6 \text{ s} \qquad \therefore (t_1 - t_p) = 1.5982 \cdot 10^6 \text{ s}$$

$$t_2 - t_1 = (t_2 - t_p) - (t_1 - t_p)$$

$$\therefore \Delta t_{1\to 2} = 1.6646 \cdot 10^6 \text{ s} \Rightarrow \boxed{\Delta t_{1\to 2} = 19.2658 \text{ days}}$$

d) Possible Method: Use Newton's method with update equation

$$E_{i+1} = E_i - (E_i - e\sin(E_i) - M)/(1 - e\cos(E_i))$$

• where M is an initial guess input for E_0 as well

*what tolerances did you choose for this problem?

Possible Stopping conditions: $|g(E_i)| = |E_i - e\sin E_i - M| < tol_g$

$|E_{i+1} - E_i| < tol_E$ (but check $g(E_i)$ after)

$i <$ maxiterations

e) t_3 occurs 25 days after t_1. Calculate E_3 (to 4 decimal places). Calculate altitude at t_3 and determine if s/c is moving toward or away from periapsis

$$(t_3 - t_p) = (t_1 - t_p) + (25 \text{ days})\left(\frac{86400 \text{ s}}{\text{day}}\right) \Rightarrow (t_3 - t_p) = 3.7582 \cdot 10^6 \text{ s or } 4.7966 \cdot 10^5 \text{ s}$$

$*3.7582 \cdot 10^6 > \mathbb{P}$

so $4.7966 \cdot 10^5 = 3.7582 \cdot 10^6 - \mathbb{P}$

$$M_3 = n(t_3 - t_p) = 7.2024 \text{ rad}$$

$$= 0.9192 \text{ rad}$$

e (cont.))

E_3 is then found numerically from the method implemented in part d)

Using $E_{3,0} = M_3$ and $tol_E = 10^{-5}$ rad (E accurate to 4 dec places)

$\boxed{\therefore E_3 = 1.8572 \text{ rad}}$ (Check: $g(E_3) = 10^{-6}$ rad)

Since E_3 is $[0,\pi]$ rad range $\boxed{\text{the s/c is moving away from periapsis}}$

Calculate altitude at t_3

$r_3 = a(1 - e\cos(E_3)) = 4.1556 \cdot 10^6$ km

$alt_3 = r_3 - ER_{JUPITER}$

$\boxed{\therefore alt_3 = 4.0841 \cdot 10^6 \text{ km}}$

*$ER_{JUPITER} = 71492$ km ← Given

Assume Jupiter is a sphere