ASEN 5050 Fall 2023 HW 3 Solution

Problem 1 $\vec{r}_1 = -720{,}000\hat{x} + 670{,}000\hat{y} + 310{,}000\hat{z}$ km

(a)

given: $GM_{saturn} = 3.794\times10^7$ km^3/s^2 , $R_{sat} = 60268$ km $\vec{v}_1 = 2.160\hat{x} - 3.360\hat{y} + 0.620\hat{z}$ km/s

assume: 2BP, saturn sphere , $Gm_{s/c} << Gm_{sat}$ $\therefore \mu \approx Gm_{sat}$

at t_2 $r_2 = R_{saturn}$

in $\hat{r}\hat{\theta}\hat{h}$ $\vec{r}_2 = r_2\hat{r} = 60{,}268\,\hat{r}$ km

use DCM to get into $\hat{x}\hat{y}\hat{z}$ saturn-centered frame - get orbital elements

2 BP conserves h, e, $\mathcal{E}$ so we can find those using $\vec{r}_1, \vec{v}_1$

$|\vec{h}| = |\vec{r}_1\times\vec{v}_1| = 2.077\times10^6$ km^2/s and $\vec{h} = 1{,}457{,}000\hat{x} + 1{,}116{,}000\hat{y} + 972{,}000\hat{z}$ km^2/s

$\mathcal{E} = \frac{v_1^2}{2} - \frac{\mu}{r_1} = -28.62$ km^2/s^2

$a = -\frac{\mu}{2\mathcal{E}} = 6.6278\times10^5$ km

$e = \sqrt{1 + \frac{2h^2\mathcal{E}}{\mu^2}} = 0.9102$

$\therefore$ we can find θ_2^* whenever $r = R_{saturn}$ by the conic equation

$r_2 = \frac{p}{1+e\cos\theta_2^*} \rightarrow \theta_2^* = \pm\cos^{-1}\left(\frac{p-r_2}{r_2 e}\right)$

b/c the s/c is hitting the surface select $\theta_2^* < 0$ $(r_1 > r_2)$

$\therefore$ $\theta_2^* = -13.17°$

$i = \cos^{-1}\left(\frac{h_3}{h}\right) = 62.1°$

$\vec{n} = \hat{z}\times\vec{h} = -1{,}116{,}000\hat{x} + 1{,}457{,}000\hat{y}$

$\Omega = \pm\cos^{-1}\left(\frac{\vec{n}\cdot\hat{x}}{n}\right) = \pm127.45° \Rightarrow$ check $\vec{n}\cdot\hat{y} > 0$, $\Omega = 127.45°$

$\omega = \pm\cos^{-1}\left(\frac{\vec{n}\cdot\vec{e}}{ne}\right) = \pm172.28° \Rightarrow$ check $\vec{e}\cdot\hat{z} < 0$, $\omega = -172.28°$

note that i, Ω, ω does not change from t_1 to t_2

$v_{r_{1,2}} = \frac{\mu}{h}e\sin\theta_2^* = -3.788$ km/s

$v_{\theta_{1,2}} = \frac{h}{r_2} = 34.46$ km/s

in rotating frame then

$\vec{v}_2 = -3.788\,\hat{r} + 34.46\,\hat{\theta}$ km/s

To rewrite $\vec{r}_2, \vec{v}_2$ in Saturn-centered inertial system with frame $\hat{x}\hat{y}\hat{z}$, use DCM w/ $\theta = \omega + \theta_2^*$

$$[C] = \begin{bmatrix} c_\Omega c_\theta - s_\Omega c_i s_\theta & -c_\Omega s_\theta - s_\Omega c_i c_\theta & s_\Omega s_i \\ s_\Omega c_\theta + c_\Omega c_i s_\theta & -s_\Omega s_\theta + c_\Omega c_i c_\theta & -c_\Omega s_i \\ s_i s_\theta & s_i c_\theta & c_i \end{bmatrix}$$

at t_2
$$C = \begin{bmatrix} 0.5700 & 0.4276 & 0.7016 \\ -0.8173 & 0.2079 & 0.5374 \\ 0.0839 & -0.8797 & 0.4680 \end{bmatrix}$$

Recall $\vec{r}_{xyz} = [C]\,\vec{r}_{r\theta h}$ $\quad \vec{v}_{xyz} = [C]\,\vec{v}_{r\theta h}$

$$\therefore \boxed{\begin{aligned} \vec{r}_2 &= 3.4356\times10^4\hat{x} - 4.9258\times10^4\hat{y} + 5.0576\times10^3\hat{z} \text{ km} \\ \vec{v}_2 &= 12.5761\hat{x} + 10.2611\hat{y} - 30.6325\hat{z} \text{ km/s} \end{aligned}}$$

b) Our prediction used a simplified scenario modeled via the 2BP. However, there are some limitations that would influence the accuracy of our prediction:

- Saturn is not actually a sphere w/ a radius equal to the equatorial radius
- Near Saturn, a point mass assumption for the gravitational environment is not accurate
- There are other bodies in the Saturnian system, atmospheric drag.