ASEN 5050 Fall 2023 HW 9 Solutions

Problem 1

Constants:

GM_{SUN}: $1.32712428 \cdot 10^{11}$ km^3/s^2
GM_{Earth}: $3.986004415 \cdot 10^{5}$ km^3/s^2
a_{Earth}: 1.00000010178 AU
1 AU: 149,597,870.7 km

Assumptions:

1) $M_{Earth} + M_{S/C} \approx M_{Earth} \Rightarrow GM_{Earth} + GM_{S/C} \approx GM_{Earth}$
$M_{Earth} \gg M_{S/C}$

2) The planets are travelling along circular orbits around the sun.

Known CubeSat relative state vector components after deployment:

$x_0 = 5$ m
$\dot{x}_0 = 0$ m/s

Known orbit condition:

- CubeSat and S/C1 remain in the same orbit plane
  ↳ null cross-track components
- Bounded, oscillatory motion
  ↳ $x(t)$ & $y(t)$ function of cos(t) and sin(t)
- Max along-track deviation 15 m, −15 m (absolute value constrained, 2 options)
  ↳ $\max|y(t)| = 15$ m

(a) Given the conditions above, we can find the remaining components by subtituting the known terms in the CW Eqs.

$$x(t) = 4x_0 + \frac{2}{n}\dot{y}_0 + \cancel{\frac{\dot{x}_0}{n}}\sin(nt) - \left(3x_0 + \frac{2}{n}\dot{y}_0\right)\cos(nt)$$

$$= 4x_0 + \frac{2}{n}\dot{y}_0 - \left(3x_0 + \frac{2}{n}\dot{y}_0\right)\cos(nt)$$

$$x(t) = 4x_0 + \frac{2}{n}\dot{y}_0 + \frac{\cancel{\dot{x}_0}}{\cancel{n}}\sin(nt) - (3x_0 + \frac{2}{n}\dot{y}_0)\cos(nt)$$

$$\overset{!}{=} \quad 4x_0 + \frac{2}{n}\dot{y}_0 - (3x_0 + \frac{2}{n}\dot{y}_0)\cos(nt)$$

$$y(t) = y_0 - \cancel{\frac{2}{n}\dot{x}_0} - 3(2nx_0 + \dot{y}_0)t + 2(3x_0 + \frac{2}{n}\dot{y}_0)\sin(t) + \cancel{\frac{2}{n}\dot{x}_0}\cos(nt)$$

$$\overset{!}{=} \quad y_0 - 3(2nx_0 + \dot{y}_0)t + 2(3x_0 + \frac{2}{n}\dot{y}_0)\sin(t)$$

$z(t) = 0$ and $\dot{z}(t) = 0$ $\forall t$ b/c of the 1^{st} condition above

$\therefore \boxed{z_0 = \dot{z}_0 = 0}$

Compute the mean motion n

$$n = \sqrt{\frac{\mu_E}{R^3}} = 6.3135 \cdot 10^{-4} \, \frac{1}{s}$$

To obtain the bounded oscillatory motion, we need to suppress the drift term in $y(t)$

$$2nx_0 + \dot{y}_0 = 0 \quad \Rightarrow \quad \boxed{\dot{y}_0 = -2nx_0 = -0.006313 \, \frac{m}{s}}$$

Use the max deviation in along track direction to find y_0

$$\max\left(\left|y_0 + 2(3x_0 + \frac{2}{n}y_0)\right|\right) = \max\left(\left|y_0 - 2x_0\right|\right) = 15\,m$$

$$\Rightarrow \quad y_0 \pm 10m = 15\,m$$

$$\Rightarrow \quad y_0 = \pm 5\,m$$ We select $\boxed{y_0 = -5m}$

but we could also select $y_0 = 5m \rightarrow$ different solution

Therefore the relative state vector after deployment is

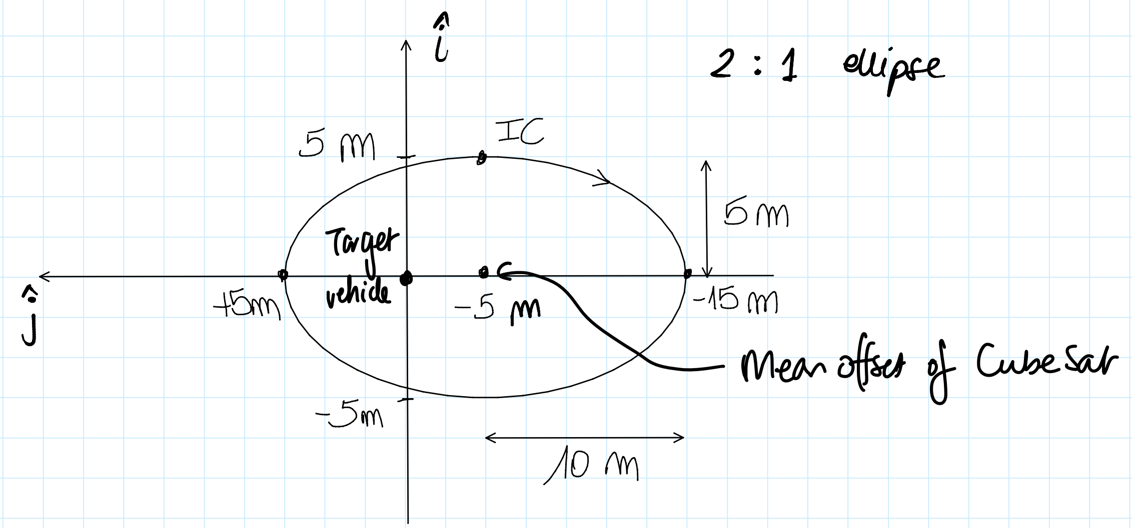
$$\boxed{[x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0] = [5m, -5m, 0, 0, -0.006313 \frac{m}{s}, 0]}$$

one possible solution

b)

$$x(t) = 5\cos(nt)$$

$$y(t) = -5 - 10\sin(nt)$$



Problem 2

- Transfer with 2 impulsive maneuvers

State components in radial, along-track and cross-track direction $(\hat{i}, \hat{j}, \hat{k})$

@ $t_0 = 0$ $\quad [x_0, y_0, z_0] = [2, 2, 0]$ m

$[\dot{x}_0, \dot{y}_0, \dot{z}_0] = [-0.03, +0.01, 0.05]$ m/s

@ $t_1 = \frac{P}{2}$ $\quad$ GOAL $\quad [x_0, y_0, z_0] = [-2, 2, 0]$ m

a) Relative velocity required at to after maneuver

$$P = 2\pi\sqrt{\frac{R^3}{\mu}} = \frac{2\pi}{n} = 9.9520 \cdot 10^3 \text{ s}$$

$$\Rightarrow t_1 = 4.9760 \cdot 10^3 \text{ s}$$

This is a design choice as any $\dot{z}_0$ will satisfy this req given $z(t) = \frac{\dot{z}_0}{n}\sin(nt)$

Let's use planar transfer because $z_0 = z_f = 0$

$$\bar{r}(t_1) = \phi_{rr}\,\bar{r}(t_0) + \phi_{rv}\,\bar{v}(t_0)$$

$$\phi_{rr} = \begin{bmatrix} 4 - 3\cos(nt) & 0 \\ 6(\sin(nt) - nt) & 1 \end{bmatrix} \overset{@ t_1}{=} \begin{bmatrix} 7 & 0 \\ \ldots & 1 \end{bmatrix}$$

$$\Phi_{rr} = \begin{bmatrix} 4 - 3\cos(nt) & 0 \\ 6(\sin(nt) - nt) & 1 \end{bmatrix} \overset{@t_1}{=} \begin{bmatrix} 7 & 0 \\ -18.85 & 1 \end{bmatrix}$$

$$\Phi_{rv} = \begin{bmatrix} \frac{1}{n}\sin(nt) & \frac{2}{n}(1 - \cos(nt)) \\ \frac{2}{n}(\cos(nt) - 1) & \frac{4}{n}\sin(nt) - 3t \end{bmatrix} \overset{@t_1}{=} 10^4 \cdot \begin{bmatrix} 0 & 0.6336 \\ -0.6336 & -1.4928 \end{bmatrix}$$

$$\bar{v}^+(t_0) = -\Phi_{rv}^{-1}(t_1)\,\Phi_{rr}(t_1)\,\bar{r}(t_0) + \Phi_{rv}^{-1}(t_1)\,\bar{r}(t_1)$$

$$= [\,0,\ -0.002525\,]^T \text{ m/s} \quad \text{in } \hat{i}, \hat{j} \quad \text{with } \dot{z}_0 = 0 \text{ m/s}$$

$$\Delta\bar{v}(t_0) = \bar{v}^+(t_0) - \bar{v}^-(t_0) = [0.03, -0.012525, -0.5]^T_{\hat{i}\hat{j}\hat{k}} \ \frac{m}{s}$$

$$|\Delta\bar{v}(t_0)| = 0.0596 \text{ m/s}$$

b) Relative velocity at t_1 before maneuver

$$\bar{v}(t) = \Phi_{vr}\,\bar{r}(t_0) + \Phi_{vv}\,\bar{v}(t_0)$$

$$\Phi_{vr} = \begin{bmatrix} 3n\sin(nt) & 0 \\ 6n\cos(nt) - 6n & 0 \end{bmatrix} \overset{@t_1}{=} \begin{bmatrix} 0 & 0 \\ -0.00757 & 0 \end{bmatrix}$$

$$\Phi_{vv} = \begin{bmatrix} \cos(nt) & 2\sin(nt) \\ -2\sin(nt) & 4\cos(nt) - 3 \end{bmatrix} \overset{@t_1}{=} \begin{bmatrix} -1 & 0 \\ 0 & -7 \end{bmatrix}$$

$$\bar{v}^-(t_1) = \Phi_{vr}(t_1)\,\bar{r}(t_0) + \Phi_{vv}(t_1)\,\bar{v}^+(t_0)$$

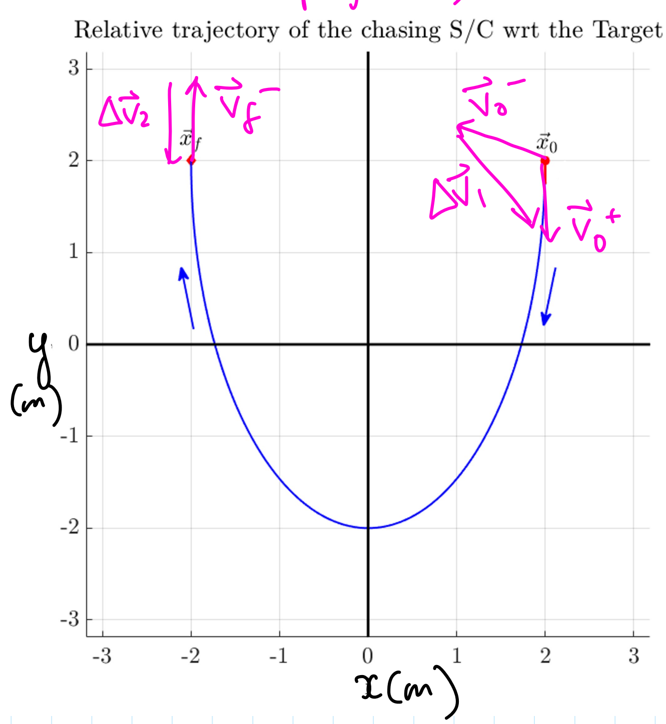
$$= [\,0,\ 0.002525\,]^T \ldots$$

$= [0,\ 0.002525]^T_{\hat{i}\hat{j}}$ m/s with $\dot{z}_f = 0$ m/s

$$\boxed{\begin{aligned} \Delta\bar{v}(t_1) &= \vec{0} - \bar{v}^-(t_1) = -\bar{v}^-(t_1) \\ |\Delta\bar{v}(t_1)| &= 0.00252 \text{ m/s} \end{aligned}}$$

c) Relative trajectory

(Planar projection)



Looking at the transfer curve, we can notice that the relative distance between the S/C and the CubeSat is not constant and it's symmetric with respect to the along track direction. From the knowledge of the primary S/C orbit (circular with R = 10.000 km around the Earth) we can conclude that this type of motion with respect to the primary S/C corresponds to an arc of ellipse around the Earth.

When $x > 0$, the CubeSat is in a higher orbit than the primary s/c and drifts behind it. After passing $x = 0$, the CubeSat is in a lower orbit and drifts ahead. Both s/c orbit planes are aligned.