ASEN 5050 Fall 2023 HW 8 Solutions

Problem 1

given:

$r_p = 7500$ km
$r_a = 8500$ km
$i = 105°$
$\mathbb{P} = 110$ min

$r_{planet} = 6500$ km
$a_{planet} = 2.25$ AU
sun-synchronous orbit!

Assumptions: ① $m_{sc} \ll m_{planet}$

$G = 6.673 \times 10^{-20}\ km^3/kg \cdot s^2$

$$\mathbb{P} = 2\pi\sqrt{\frac{a^3}{\mu}} \rightarrow \mu = \frac{(2\pi)^2 a^3}{\mathbb{P}^2}$$

$$a = \frac{r_p + r_a}{2} = 8000 \text{ km}$$

$\mu_{planet} \approx GM_{planet}$ b/c $m_{sc} \ll M_{planet}$

$$M_{planet} = \mu_{planet}/G = \left(\frac{2\pi}{110min(60s)}\right)^2 (8000)^3/G = 6.01377 \times 10^{12}/G$$

$$\boxed{M_{planet} = 6.9538 \times 10^{24} \text{ kg}}$$

Assuming J_2 is the only perturbation effect.

in a sun-synch orbit, $\dot{\Omega} = 360°/\mathbb{P}$ where $\mathbb{P}$ is the orbit of the planet.

$$\dot{\Omega} = -\left[\frac{3}{2}\frac{\sqrt{\mu}\ J_2\ R^2}{(1-e^2)^2\ a^{7/2}}\right]\cos(i) \rightarrow J_2 = -\frac{2}{3}\frac{\dot{\Omega}(1-e^2)a^{7/2}}{\sqrt{\mu}\ R^2 \cos(i)}$$

we need to find e

$$e = \frac{r_a - r_p}{r_a + r_p} = 0.0625$$

find $\mathbb{P}_{planet}$

$$\mathbb{P}_{plan} = 2\pi\sqrt{\frac{a_{planet}^3}{\mu_{sun}}} = 1.0651 \times 10^8 \text{ sec}$$

Determine J_2

$$J_2 = -\frac{2}{3}\frac{(2\pi/1.0651\times10^8)(1-0.0625^2)^2(8000\text{ km})^{7/2}}{(\mu_{planet})^{1/2}\ (6500)^2\ \cos(105°)} \boxed{= 2.3989 \times 10^{-4} = J_2}$$

2a. $\bar{R}_0 = [-6402,\ -1809,\ 1065]$ km $\quad \mu_E = 3.986004415 \cdot 10^5$ km³/s²

$\bar{V}_0 = [0.999,\ -6.471,\ -4.302]$ km/s

rel tol $= 1\cdot 10^{-12}$ abs tol $= 1 \cdot 10^{-12}$

$\ddot{\bar{r}} = -\frac{GM_2}{r^3}\bar{r} \qquad \bar{r} = f(\bar{c}, t) \rightarrow$ No perturbations

$\bar{c} = [a, e, i, \omega, \Omega, \theta^*]$

state components in inertial frame

1. Write initial condions: $\vec{X} = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \vec{R}_0^T \\ \vec{V}_0^T \end{bmatrix}$

2. Write function for Equation of Motion: $\dot{\vec{X}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ -\frac{\mu}{r^3}x \\ -\frac{\mu}{r^3}y \\ -\frac{\mu}{r^3}z \end{bmatrix}$

3. Set options with wanted tolerances

4. Run ODE45: [t, y1] = ode45(@EOM, tspan, yo, options)

Scaler ← Relative tolerance: constrains error relative to magnitude of each state component

Scaler or vector ← Absolute tolerance: applies error to absolute value of each state component

For i-th state component: Error constrained to be less than $\max(\text{RelTol}\cdot|\vec{x}_i|, \text{AbsTol}_i)$

Runge-Kutta 45: at each step, two different approximations for the solution are calculated and compared. If the two answers are in agreement given a certain tolerance, the answer is accepted. The approximations are created using the Runge-Kutta method to the order of 4 and 5. using the Prince-Dormand approach

Source: MATLAB Documentation

b.

$\bar{h}_0 = \bar{r}_0 \times \bar{V}_0 = [14674,\ -26477,\ 43235]$ km²/s

$\mathcal{E}_0 = \frac{1}{2}\left(V_0^2 - \frac{\mu}{r}\right) = -28.4729$ km²/s²

$p_0 = \frac{h^2}{\mu} = 6.9885 \cdot 10^3$ km

$e_0 = \sqrt{1 + \frac{2h^2\mathcal{E}}{\mu}} = 0.0399$

$a_0 = \frac{p}{1-e^2} = 6.9996 \cdot 10^3$ km

$\theta_0^* = \pm\cos^{-1}\left(\frac{p}{re} - \frac{1}{e}\right) = \pm 21.0233°$ Because $\vec{V}_0 \cdot \vec{R}_0 > 0$, $\theta_0^* = 21.02°$

$E_0 = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta^*}{2}\right)\right) = 0.3529$ rad

$n = \sqrt{\frac{\mu}{a^3}} = 0.0011$ 1/s

$t_1 - t_0 = 100$ hr $= 360{,}000$ s $= \frac{1}{n}[(E_1 - e\sin E_1) - (E_0 - e\sin E_0)]$

$E_1 = 5.1426$ rad $\rightarrow$ Slight differences in accuracy of E will impact results.

$\Delta E_1 = E_1 - E_0 = 4.7897$ rad

$f = 1 - \frac{a}{r_0}(1-\cos\Delta E) = 0.0413$

$g = (t-t_0) - \sqrt{\frac{a^3}{\mu}}(\Delta E - \sin\Delta E) = -878.3351$ s

$\dot{f} = \frac{-\sin\Delta E\sqrt{\mu a}}{r_0 r} = 0.0011$ s^{-1}

$\dot{g} = 1 - \frac{a}{r}(1-\cos\Delta E) = 0.0616$

$\bar{R}_1 = f\bar{R}_0 + g\bar{V}_0 = [-1142, 5609, 3823]$ km

$\bar{V}_1 = \dot{f}\bar{R}_0 + \dot{g}\bar{V}_0 = [-7.209, -2.453, 0.944]$ km/s

* Note numbers rounded for answer here, but more digits retained in working!

$h_1 = \bar{R}_1 \times \bar{V}_1 = [14674, -26477, 43235] = \boxed{5\,2779 \cdot 10^4 \text{ km}^2/\text{s}}$

$\mathcal{E}_1 = \frac{1}{2}\left(v_1^2 - \frac{\mu}{r_1}\right) = \boxed{-28.473 \text{ km}^2/\text{s}^2}$

c. $\bar{R}_1 = [-1142, 5609, 3823]$ km

$\bar{V}_1 = [-7.209, -2.453, 0.0944]$ km/s

* Note: rounded numbers here, but more digits retained in working.

$\bar{h}_1 = [14674, -26477, 43235] = \boxed{5\,2779 \cdot 10^4 \text{ km}^2/\text{s}}$

$\mathcal{E}_1 = \boxed{-28.473 \text{ km}^2/\text{s}^2}$

- Position + velocity are slightly different than the analytical sln. For position vector, the error is $\sim 5\times10^{-5}$ km, velocity vector error $\sim 6\times10^{-8}$ km/s
- Do not expect exact match for two reasons:
  1) Accuracy of E calculation
  2) Numerical integration induces error at each step

d.

	ΔR (km)	ΔV (km/s)	$\Delta \mathcal{E}$ (km^2/s^2)	Δh (km^2/s)
Tol $= 1 \cdot 10^{-4}$	$1.3596 \cdot 10^{4}$	15.2317	0.8884	794.806
Tol $= 1 \cdot 10^{-6}$	9.5245	0.0105	$1.0698 \cdot 10^{-4}$	0.0913
Tol $= 1 \cdot 10^{-8}$	0.4905	$5.3765 \cdot 10^{-4}$	$6.817 \cdot 10^{-6}$	0.0063
Tol $= 1 \cdot 10^{-10}$	0.0055	$5.9770 \cdot 10^{-6}$	$7.5180 \cdot 10^{-8}$	$6.9771 \cdot 10^{-5}$
Tol $= 1 \cdot 10^{-12}$	$5.4552 \cdot 10^{-5}$	$5.9818 \cdot 10^{-8}$	$7.5046 \cdot 10^{-10}$	$6.9659 \cdot 10^{-7}$

*Numbers may differ based on reference state in part b).

e.

As expected, the difference between the analytical solution and numeric solution decreases noticeably as the tolerance is selected as a smaller value. The tradeoff would include balancing accuracy with increased computational time as the step size is lowered to meet this tolerance.