ASEN 5050 Fall 2023 HW 2 Solution

- $Gm_{Earth} = 3.986004415 \times 10^5$ km^3/s^2
- $Gm_{moon} = 4.902799 \times 10^3$ km^3/s^2
- ER_{Earth} = 6378.1363 km
- ER_{moon} = 1738.0 km

Problem 1:

$\bar{r} = -3.02073 \cdot 10^2 \hat{X} + 2.25688 \cdot 10^1 \hat{Y} + 1.80166 \cdot 10^3 \hat{Z}$ km

$\bar{v} = 1.31575 \hat{X} - 0.95477 \hat{Y} + 0.20835 \hat{Z}$ km/s

Assume $m_{moon} >> m_{s/c}$ $\therefore$ $\mu = G(m_{moon} + m_{s/c}) \approx Gm_{moon}$

a) Calculate $a, e, i, \Omega, \omega, \theta^*$

Calculate r and v:

$r = |\bar{r}| = 1.8269 \cdot 10^3$ km

$v = |\bar{v}| = 1.6390$ km/s

Calculate a:

$\mathcal{E} = v^2/2 - \mu/r = -1.3405$ km^2/s^2

$a = -\mu/(2\mathcal{E})$

$\boxed{a = 1.8287 \cdot 10^3 \text{ km}}$

Calculate e: $\vec{h} = \vec{r} \times \vec{v} = 1.7249 \times 10^3 \hat{X} + 2.4335 \times 10^3 \hat{Y} + 0.2587 \times 10^3 \hat{Z}$ km^2/s

$h = |\bar{r} \times \bar{v}| = 2.9940 \cdot 10^3$ km^2/s

$e = \sqrt{1 + 2h^2\mathcal{E}/\mu^2}$

$\boxed{e = 0.0146}$

Calculate i:

$i = \cos^{-1}\left(\frac{h_z}{h}\right)$ Note: only 1 solution because $i = [0, 180°]$

$\boxed{i = 1.4843 \text{ rad} = 85.0428°}$

Calculate Ω:

$\bar{n} = \hat{Z} \times \bar{h} = -2.4335 \cdot 10^3 \hat{X} + 1.7249 \cdot 10^3 \hat{Y}$

$\Omega = \pm\cos^{-1}\left(\frac{\bar{n} \cdot \hat{X}}{|\bar{n}|}\right)$

*Check: $\bar{n} \cdot \hat{Y} > 0$ $\therefore$ $\Omega > 0$

$\boxed{\Omega = 2.5250 \text{ rad} = 144.6705°}$

Calculate ω:

$\bar{e} = \frac{\bar{v} \times \bar{h}}{\mu} - \frac{\bar{r}}{r} = 0.0115 \hat{X} - 0.0085 \hat{Y} + 0.0028 \hat{Z}$

$\omega = \pm\cos^{-1}((\bar{n} \cdot \bar{e})/|\bar{n}||\bar{e}|)$

*Check: $\bar{e} \cdot \hat{Z} > 0$ $\therefore$ $\omega > 0$

$\boxed{\omega = 2.9475 \text{ rad} = 168.8771°}$

Calculate θ^*:

$$\theta^* = \pm\cos^{-1}(\bar{r}\cdot\bar{e}/|\bar{r}||\bar{e}|)$$

*Check: $\bar{r}\cdot\bar{v} < 0 \therefore \theta^* < 0$

$$\boxed{\theta^* = -1.5192 \text{ rad} = -87.0428^\circ}$$

b) Write the position and velocity vectors in the rotating frame using:
i) DCM
ii) Using the 2BP analytical expressions

i) $c\Omega = \cos(\Omega)$; $s\Omega = \sin(\Omega)$, $ci = \cos(i)$; $si = \sin(i)$; $c\theta = \cos(\theta)$; $s\theta = \sin(\theta)$

$$[C] = \begin{bmatrix} c\Omega c\theta - s\Omega ci s\theta & -c\Omega s\theta - s\Omega ci c\theta & s\Omega si \\ s\Omega c\theta + c\Omega ci s\theta & -s\Omega s\theta + c\Omega ci c\theta & -c\Omega si \\ si s\theta & si c\theta & ci \end{bmatrix} = \begin{bmatrix} -0.1653 & 0.8005 & 0.5761 \\ 0.0124 & -0.5824 & 0.8128 \\ 0.9862 & 0.1415 & 0.0864 \end{bmatrix}$$

$\bar{r}_{\hat{r}\hat{\theta}\hat{h}} = [C]^T \bar{r}_{\hat{x}\hat{y}\hat{z}}$ ← Position vector given in problem

Note: $\theta = \theta^* + \omega = 1.4283 \text{ rad} = 81.8343^\circ$

$$\boxed{\vec{r} = 1826.9472\hat{r} \text{ km}}$$

Note: $\vec{r}\cdot\hat{\theta}$, $\vec{r}\cdot\hat{h}$, $\vec{v}\cdot\hat{h}$ should all be negligibly small through these computations.
↳ $< O(10^{-10})$

$\bar{v}_{\hat{r}\hat{\theta}\hat{h}} = [C]^T \bar{v}_{\hat{x}\hat{y}\hat{z}}$

$$\boxed{\vec{v} = -0.0239\hat{r} + 1.6388\hat{\theta} \text{ km/s}}$$

ii) $r = (h^2/\mu)/(1 + e\cos(\theta^*))$

$$\boxed{\bar{r} = r\hat{r} = 1826.9472\hat{r} \text{ km}}$$

$V_r = (\mu/h)e\sin(\theta^*) = -0.0239$ km/s

$V_\theta = (\mu/h)(1 + e\cos(\theta^*)) = 1.6388$ km/s

$$\boxed{\bar{v} = V_r\hat{r} + V_\theta\hat{\theta} = -0.0239\hat{r} + 1.6388\hat{\theta} \text{ km/s}}$$

c) Calculate $\bar{r}$ and $\bar{v}$ in the inertial frame at periapsis

→ First, calculate $\bar{r}, \bar{v}$ in rotating frame. Then, update the DCM w/ new θ. Finally, perform transformation

$$[C] = \begin{bmatrix} c\Omega c\theta - s\Omega ci s\theta & -c\Omega s\theta - s\Omega ci c\theta & s\Omega si \\ s\Omega c\theta + c\Omega ci s\theta & -s\Omega s\theta + c\Omega ci c\theta & -c\Omega si \\ & & ci \end{bmatrix}$$

$r_p = a(1-e) = 1802.01$ km

$\bar{r}_p = r_p\hat{r} = 1802.01\hat{r}$ km since the s/c is at periapsis

$\theta_p^* = 0$ at periapsis → ∴ New value of $\theta_p = \theta_p^* + \omega = \omega$

$V_r = (\mu/h)e\sin(\theta_p^*) = 0$ km/s

$V_\theta = (\mu/h)(1 + e\cos(\theta_p^*)) = 1.6615$ km/s

$\bar{v}_p = V_r\hat{r} + V_\theta\hat{\theta} = 0\hat{r} + 1.6615\hat{\theta}$ km/s

$$[C] = \begin{bmatrix} 0.7909 & 0.2064 & 0.5761 \\ -0.5810 & -0.0424 & 0.8128 \\ 0.1922 & -0.9775 & 0.0864 \end{bmatrix}$$

Transforming from rotating frame to inertial frame

$\bar{r}_{\hat{x}\hat{y}\hat{z}} = [C]\bar{r}_{\hat{r}\hat{\theta}\hat{h}} \Rightarrow \boxed{\bar{r}_p = 1425.17\hat{X} - 1047.00\hat{Y} + 346.33\hat{Z} \text{ km}}$

$\bar{v}_{\hat{x}\hat{y}\hat{z}} = [C]\bar{v}_{\hat{r}\hat{\theta}\hat{h}} \Rightarrow \boxed{\bar{v}_p = 0.3430\hat{X} - 0.0704\hat{Y} - 1.6242\hat{Z} \text{ km/s}}$

Problem 2:

Known: $\hat{n} = 0.3746\hat{X} + 0.9272\hat{Y} + 0\hat{Z}$
$\hat{h} = 0.7595\hat{X} - 0.3069\hat{Y} + 0.5736\hat{Z}$
$\bar{e} = 0.0999\hat{X} - 0.2515\hat{Y} - 0.2668\hat{Z}$

These vectors describe the orientation and shape of the orbit

Assume $m_{Earth} >> m_{sc}$ $\therefore \mu = G(m_{Earth} + m_{sc}) \approx G m_{Earth}$

a) Calculate e, i, Ω, ω:

Calculate e:

$$\boxed{e = |\bar{e}| = 0.3800}$$

Calculate i:

$i = \cos^{-1}\left(\frac{h_z}{|\bar{h}|}\right)$ * $i \geq 0$, $\therefore$ no quadrant check needed

$$\boxed{i = 0.9599 \text{ rad} = 54.9993^\circ}$$

Calculate Ω:

$\Omega = \pm\left(\frac{\hat{n}\cdot\hat{X}}{|\bar{n}|}\right)$

* Check: $\hat{n}\cdot\hat{Y} > 0$ $\therefore \Omega > 0$

$$\boxed{\Omega = 1.1868 \text{ rad or } 68.0007^\circ}$$

Calculate ω:

$\omega = \pm\cos^{-1}\left(\frac{\hat{n}\cdot\bar{e}}{|\bar{n}||\bar{e}|}\right)$

* Check: $\bar{e}\cdot\hat{Z} < 0$ $\therefore \omega < 0$

$$\boxed{\omega = -2.1120 \text{ rad or } -121.0073^\circ}$$

b) When the s/c is located at the ascending node, $r = 19148$ km. Calculate a:

At asc. node: $\theta^*_a = -\omega = 121.0073^\circ$

From conic eq: $r = p/(1 + e\cos(\theta^*_a)) = a(1-e^2)/(1 + e\cos(\theta^*_a))$

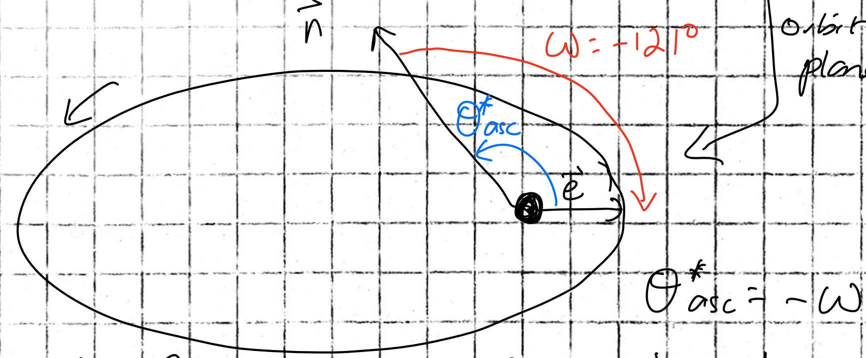
Rearranging: $a = r(1 + e\cos(\theta^*_a))/(1-e^2)$

$$\boxed{a = 17998.75 \text{ km}}$$

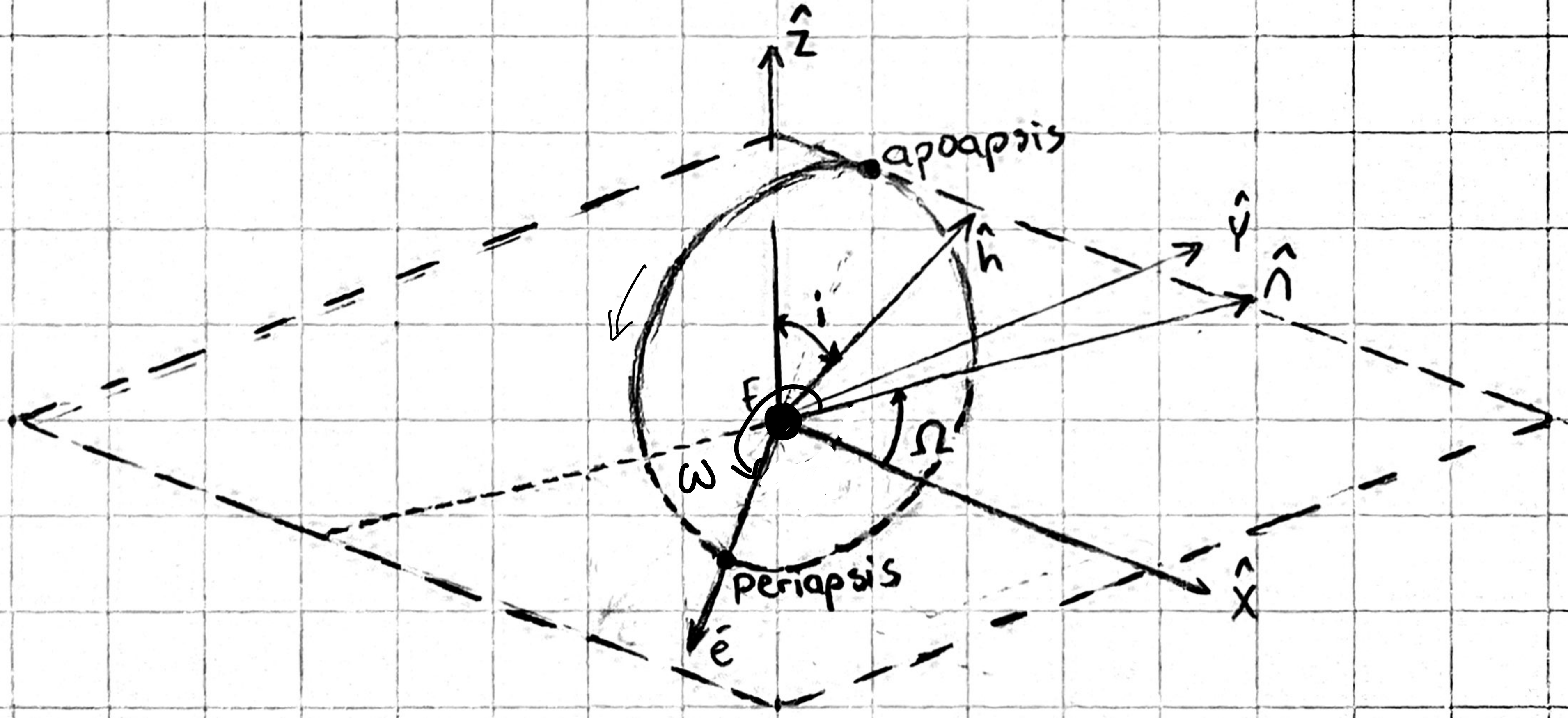
To see this, draw a diagram looking down on the orbit plane.

c) Sketch the orbit and include $\hat{X}, \hat{Y}, \hat{Z}$, F, periapsis, apoapsis, $\bar{e}$, $\hat{n}$, $\hat{h}$, i, Ω, ω

Next page



Or conceptually, θ is measured from $\vec{n}$ to location of interest. $\therefore$ $\theta = 0^\circ$ at asc. node $\Rightarrow 0^\circ = \theta^* + \omega$



$\hat{z}$
apoapsis
$\hat{y}$
$\hat{h}$
$\hat{n}$
i
E
Ω
ω
periapsis
$\hat{x}$
$\hat{e}$