HW 6 Solutions - ASEN 5050 - Fall 2023

Problem 1:

Assume:
- 2BP
- $\mu_{sun} = 1.32712428 \cdot 10^{11}$ km^3/s^2
- $m_{SK} << m_{sun}$

$\therefore \mu \simeq Gm_{sun}$

At a Julian Date of 2460012.5 TDB, the state of Earth in an SCI frame is:

$$\bar{R}_1 = -1.451526 \cdot 10^8 \hat{X} + 2.872782 \cdot 10^7 \hat{Y} + 1.245389 \cdot 10^7 \hat{Z} \text{ km}$$
$$\bar{V}_1 = -6.766088 \hat{X} - 26.810318 \hat{Y} - 11.620955 \hat{Z} \text{ km/s}$$

At a Julian Date of 2460146.5 TDB, the state of Venus in an SCI frame is:

$$\bar{R}_2 = 2.454556 \cdot 10^7 \hat{X} - 9.611903 \cdot 10^7 \hat{Y} - 4.480278 \cdot 10^7 \hat{Z} \text{ km}$$
$$\bar{V}_2 = 33.882657 \hat{X} + 7.872920 \hat{Y} + 1.598796 \hat{Z} \text{ km/s}$$

a) Describe the setup for a numerical method to solve Lambert's Equation only for arcs along elliptical orbits. List the stopping condition(s) and initial guess.

- Let the $\boxed{\text{initial condition be } a_0 = a_{min} + \Delta a}$ and TOFd is provided in seconds
- At each iteration calculate n_i, α_i, β_i using a_i
- The function to solve within some tolerance is:

$$f(a_i) = TOF_i - TOF_d = \frac{1}{n_i}\left((\alpha_i - \beta_i) - (\sin(\alpha_i) - \sin(\beta_i))\right) - TOF_d < \text{tolerance}$$

  - Utilizes some root-finding method such as fsolve( ), Newton-Raphson, Bi-section, etc.

- Stopping Conditions:
  - Tolerance: $|f(a_i)|$ is below a user-defined value (i.e. 1e-6)
  - Iterations: Max number of function evaluations (i.e. 1000)
  - Step-Tolerance: $|a_i - a_{i-1}|$ is below some user-defined value

b) For $\Delta\theta^* < 180°$, use Lambert's eq. to solve for a conic connecting $\bar{R}_1$ and $\bar{R}_2$ using the difference in Julian Dates. Calculate a and e for the transfer conic.

Step 1: Calculate $\Delta\theta^*$:

$$r_1 = |\bar{R}_1| = 1.4849 \cdot 10^8 \text{ km}$$
$$r_2 = |\bar{R}_2| = 1.0885 \cdot 10^8 \text{ km}$$
$$\Delta\theta^* = \pm \cos^{-1}\left(\frac{\bar{R}_1 \cdot \bar{R}_2}{r_1 r_2}\right) = 115.2001° = 2.0106 \text{ rad} \quad {}^*\Delta\theta^* < 180° \ ✓$$

Step 2: Calculate geometric quantities c and s:

$$c = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\Delta\theta^*)}$$
$$\therefore c = 2.1832 \cdot 10^8 \text{ km}$$
$$s = \frac{1}{2}(r_1 + r_2 + c)$$
$$\therefore s = 2.3783 \cdot 10^8 \text{ km}$$

Step 3: Determine type of transfer arc

$$TOF_d = (2460146.5 - 2460012.5)(86400 s/1 day)$$
$$\therefore TOF_d = 11577600 s \text{ or } 1.15776 \cdot 10^7 s$$
$$TOF_p = \frac{1}{3}\sqrt{\frac{2}{\mu_{sun}}}\left(s^{3/2} - (s-c)^{3/2}\right) \quad {}^* \text{for } \Delta\theta^* < 180°$$
$$\therefore TOF_p = 4.6346 \cdot 10^6 s$$

$TOF_d > TOF_p$ $\therefore$ transfer orbit is elliptical

Step 4: • Compare TOF to minimum energy transfer

$\alpha_{min} = \pi = 180°$

$\beta_{min} = 2\sin^{-1}\left(\sqrt{\frac{s-c}{s}}\right) = 0.5810 \text{ rad} = 33.2892°$

$a_{min} = s/2 = 1.1892 \cdot 10^8 \text{ km}$

$n_{min} = \sqrt{\frac{\mu}{a_{min}^3}} = 2.8093 \cdot 10^{-7} \text{ 1/s}$

$TOF_{min} = \frac{1}{n_{min}}\left((\alpha_{min} - \beta_{min}) - (\sin(\alpha_{min}) \cdot \sin(\beta_{min})\right)$ * Eqn for $\Delta\theta^* < 180°$

$\therefore TOF_{min} = 11068361.82 \text{ s}$ or $1.1068 \cdot 10^7 \text{ s}$

$TOF_d > TOF_{min}$ use: $\alpha = 2\pi - \alpha_0$ and $\beta = \beta_0$

□ $\alpha_0 = 2\sin^{-1}\left(\sqrt{s/2a}\right)$, $\beta_0 = 2\sin^{-1}\left(\sqrt{(s-c)/2a}\right)$

Step 5: • Solve Lambert's Equation for TOF_d until function tolerance is met (see previous question)

$\boxed{a = 1.1906 \cdot 10^8 \text{ km}}$

Step 6: Plug a into Lambert's equation: $|TOF(a) - TOF_d| < 1 \times 10^{-7} \text{s}$

Step 7: • Calculate e:

$p = \left[(4a(s-r_1)(s-r_2))/c^2\right] \sin^2\left(\frac{\alpha+\beta}{2}\right)$

$\therefore p = 1.03416 \cdot 10^8 \text{ km}$

$e = \sqrt{1 - p/a}$

$\therefore \boxed{e = 0.36244}$

c) What is a_{min}? (a_{min} = lowest value of a for transfer connecting these 2 $\vec{R}$)

$\boxed{a_{min} = s/2 = 1.1892 \cdot 10^8 \text{ km}}$

d) At both the beginning and end of the transfer, calculate θ_1^*, θ_2^*, $\bar{V}_1^+$, $\bar{V}_2^-$ in SCI frame. Then calculate ΔV_1 and ΔV_2.

• Calculate θ_1^* and θ_2^*

Using $r = \frac{a(1-e^2)}{1+e\cos(\theta^*)} \Rightarrow \theta^* = \pm\cos^{-1}\left(\frac{a(1-e^2)-r}{re}\right)$ ← Choose θ^* by values that satisfy $\Delta\theta^* = \theta_2^* - \theta_1^*$

$\Delta\theta^* = -244.7999° = -4.2726 \text{ rad}$

$= 115.2001°$

$\boxed{\theta_1^* = +146.8807° = +2.5636 \text{ rad}}$ $\boxed{\theta_2^* = -97.9192° = -1.7090 \text{ rad}}$

(Note: could also calculate $\vec{V}_1^+$, $\vec{V}_2^-$ first and use them in a sign check!)

• Calculate $\bar{V}_1^+$ and $\bar{V}_2^-$ using f and g functions. Then calculate ΔV_1 and ΔV_2

$f = 1 - \frac{r_2}{p}(1 - \cos(\Delta\theta^*)) = -0.5007 \text{ 1/s}$

$g = \frac{r_1 r_2}{\sqrt{p\mu_{sun}}} \sin(\Delta\theta^*) = 3.9478 \cdot 10^6$

$\dot{f} = \sqrt{\frac{\mu_{sun}}{p}} \tan\left(\frac{\Delta\theta^*}{2}\right)\left(\frac{1-\cos(\Delta\theta^*)}{p} - \frac{1}{r_1} - \frac{1}{r_2}\right) = -1.2048 \cdot 10^{-7} \text{ 1/s}$

$\dot{g} = 1 - \left(\frac{r_1}{p}\right)(1 - \cos(\Delta\theta^*)) = -1.0472$

$\bar{V}_1^+ = \frac{1}{g}(\bar{R}_2 - f\bar{R}_1) \Rightarrow \boxed{\bar{V}_1^+ = -12.1931\hat{x} - 20.7040\hat{y} - 9.7693\hat{z} \text{ km/s}}$

$\bar{V}_2^- = \dot{f}\bar{R}_1 + \dot{g}\bar{V}_1^- \Rightarrow \boxed{\bar{V}_2^- = 30.2571\hat{x} + 18.2207\hat{y} + 8.7302\hat{z} \text{ km/s}}$

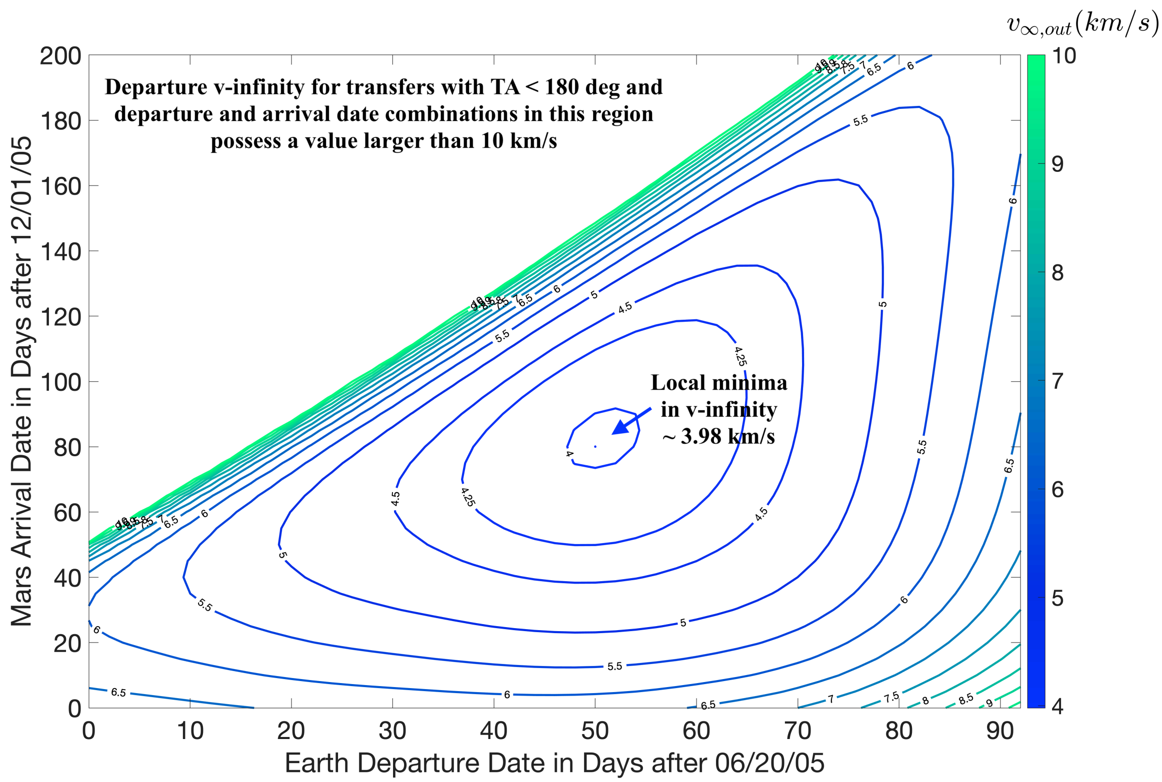
$\Delta V_1 = |\bar{V}_1^+ - \bar{V}_1| = |-5.4270\hat{x} + 6.1063\hat{y} + 1.8517\hat{z}| \text{ km/s} \therefore \boxed{\Delta V_1 = 8.3766 \text{ km/s}}$

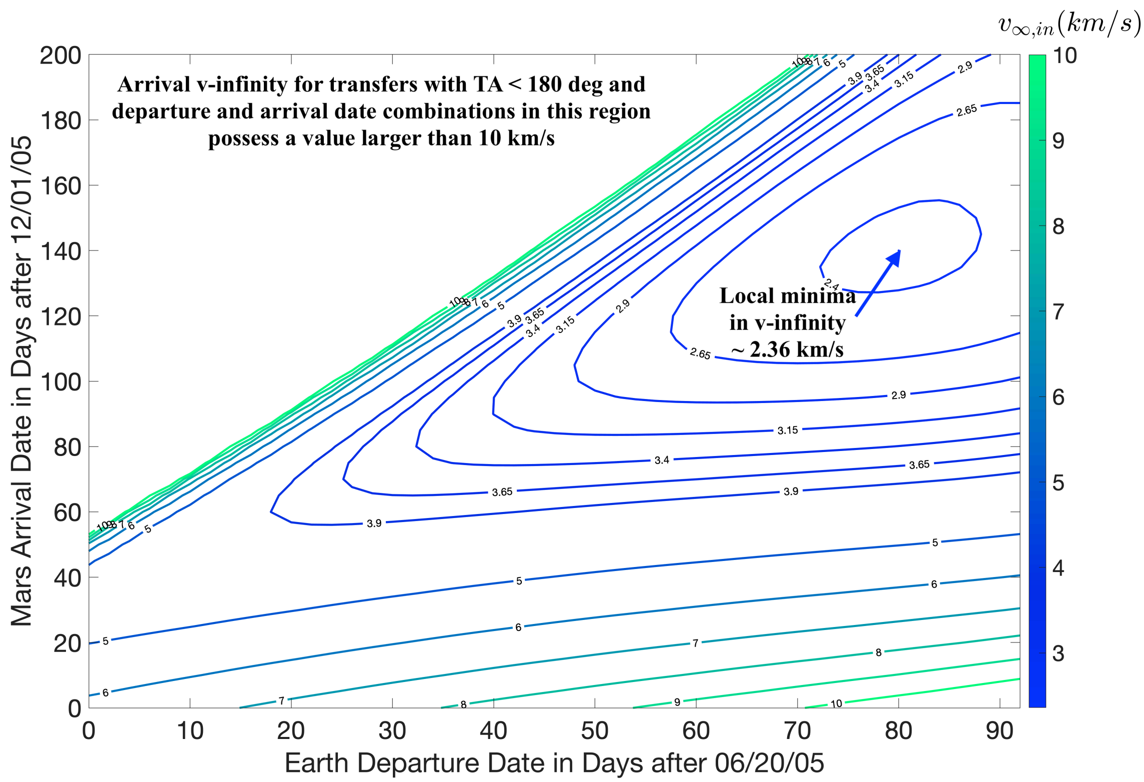
$\Delta V_2 = |\bar{V}_2 - \bar{V}_2^-| = |3.6256\hat{x} - 10.3477\hat{y} - 7.3314\hat{z}| \text{ km/s} \therefore \boxed{\Delta V_2 = 13.1898 \text{ km/s}}$ → This is large!!

PROBLEM 2

(a) code problem 1.

(b) use code from part a.
- ensure the correct true anomalies are used to obtain the correct transfer angle that is < 180°
- be sure to only use arrival dates that come <u>after</u> the departure date.





(c) You should have an indepth discussion here. Some points that folks might miss:

- The local minima value for $V_{\infty,out} \approx 3.98$ km/s
- The local minima value for $V_{\infty,in} \approx 2.36$ km/s
- These minima do not occur at the same time, so tradeoffs may be required when selecting a transfer.

In the future, this work could be expanded by:

1) Including transfers with $\Delta\theta^* > 180°$ (See lecture notes for how this will appear)
2) Increase complexity of dynamical model
3) Model local gravity fields to study any necessary $\Delta\vec{v}$ to leave/arrive into specific orbits relative to Earth/Mars.
4) Consider continuous thrust prop. systems

And more!