The equation for the gravitational potential function with mu, J2, J3 in cartesian coordinates is presented:

$$U = \frac{\mu}{\sqrt{x^2+y^2+z^2}} + \frac{J_2\, a^2\, \mu\, (x^2+y^2-2\,z^2)}{2\,(x^2+y^2+z^2)^{5/2}} + \frac{J_3\, a^3\, \mu\, (3\,z\,(x^2+y^2+z^2)-5\,z^3)}{2\,(x^2+y^2+z^2)^{7/2}}$$

The gradient of the potential function yields the acceleration of a particle in that gravity field

$$\ddot{r} = \nabla U$$

Each component of the acceleration is presented below

$$a_x = \frac{J_2\, a^2\, \mu\, x}{(x^2+y^2+z^2)^{5/2}} - \frac{\mu\, x}{(x^2+y^2+z^2)^{3/2}} + \frac{3\, J_3\, a^3\, \mu\, x\, z}{(x^2+y^2+z^2)^{7/2}} - \frac{5\, J_2\, a^2\, \mu\, x\, (x^2+y^2-2\,z^2)}{2\,(x^2+y^2+z^2)^{7/2}} - \frac{7\, J_3\, a^3\, \mu\, x\, (3\,z\,(x^2+y^2+z^2)-5\,z^3)}{2\,(x^2+y^2+z^2)^{9/2}}$$

$$a_y = \frac{J_2\, a^2\, \mu\, y}{(x^2+y^2+z^2)^{5/2}} - \frac{\mu\, y}{(x^2+y^2+z^2)^{3/2}} + \frac{3\, J_3\, a^3\, \mu\, y\, z}{(x^2+y^2+z^2)^{7/2}} - \frac{5\, J_2\, a^2\, \mu\, y\, (x^2+y^2-2\,z^2)}{2\,(x^2+y^2+z^2)^{7/2}} - \frac{7\, J_3\, a^3\, \mu\, y\, (3\,z\,(x^2+y^2+z^2)-5\,z^3)}{2\,(x^2+y^2+z^2)^{9/2}}$$

$$a_z = \frac{J_3\, a^3\, \mu\, (3\,x^2+3\,y^2-6\,z^2)}{2\,(x^2+y^2+z^2)^{7/2}} - \frac{\mu\, z}{(x^2+y^2+z^2)^{3/2}} - \frac{2\, J_2\, a^2\, \mu\, z}{(x^2+y^2+z^2)^{5/2}} - \frac{5\, J_2\, a^2\, \mu\, z\, (x^2+y^2-2\,z^2)}{2\,(x^2+y^2+z^2)^{7/2}} - \frac{7\, J_3\, a^3\, \mu\, z\, (3\,z\,(x^2+y^2+z^2)-5\,z^3)}{2\,(x^2+y^2+z^2)^{9/2}}$$

Now, with a new state vector $\boldsymbol{X} = [\boldsymbol{r}, \boldsymbol{v}, \mu, J_2, J_3]$ the partials of the acceleration vector solved for above can be obtained by taking the partials with respect to this state vector as seen below.

$$\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{X}} = \begin{bmatrix} \frac{\partial a_x}{\partial x} & \frac{\partial a_x}{\partial y} & \frac{\partial a_x}{\partial z} & \frac{\partial a_x}{\partial v_x} & \frac{\partial a_x}{\partial v_y} & \frac{\partial a_x}{\partial v_z} & \frac{\partial a_x}{\partial \mu} & \frac{\partial a_x}{\partial J_2} & \frac{\partial a_x}{\partial J_3} \\ \frac{\partial a_y}{\partial x} & \frac{\partial a_y}{\partial y} & \frac{\partial a_y}{\partial z} & \frac{\partial a_y}{\partial v_x} & \frac{\partial a_y}{\partial v_y} & \frac{\partial a_y}{\partial v_z} & \frac{\partial a_y}{\partial \mu} & \frac{\partial a_y}{\partial J_2} & \frac{\partial a_y}{\partial J_3} \\ \frac{\partial a_z}{\partial x} & \frac{\partial a_z}{\partial x} & \frac{\partial a_z}{\partial x} & \frac{\partial a_z}{\partial v_x} & \frac{\partial a_z}{\partial v_x} & \frac{\partial a_z}{\partial v_x} & \frac{\partial a_z}{\partial \mu} & \frac{\partial a_z}{\partial J_2} & \frac{\partial a_z}{\partial J_3} \end{bmatrix}$$

The partials for each entry of this Jacobian are presented below.

$$\frac{\partial a_x}{\partial x} = \frac{3\,\mu\,x^2}{(x^2+y^2+z^2)^{5/2}} - \frac{\mu}{(x^2+y^2+z^2)^{3/2}} + \frac{J_2\,a^2\,\mu}{(x^2+y^2+z^2)^{5/2}} - \frac{5\,J_2\,a^2\,\mu\,(x^2+y^2-2\,z^2)}{2\,(x^2+y^2+z^2)^{7/2}} - \frac{7\,J_3\,a^3\,\mu\,(3\,z\,(x^2+y^2+z^2)-5\,z^3)}{2\,(x^2+y^2+z^2)^{9/2}} + \frac{3\,J_3\,a^3\,\mu\,z}{(x^2+y^2+z^2)^{7/2}} - \frac{10\,J_2\,a^2\,\mu\,x^2}{(x^2+y^2+z^2)^{7/2}} + \frac{35\,J_2\,a^2\,\mu\,x^2\,(x^2+y^2-2\,z^2)}{2\,(x^2+y^2+z^2)^{9/2}} + \frac{63\,J_3\,a^3\,\mu\,x^2\,(3\,z\,(x^2+y^2+z^2)-5\,z^3)}{2\,(x^2+y^2+z^2)^{11/2}} - \frac{42\,J_3\,a^3\,\mu\,x^2\,z}{(x^2+y^2+z^2)^{9/2}}$$

$$\frac{\partial a_x}{\partial y} = \frac{3\,\mu\,x\,y}{(x^2+y^2+z^2)^{5/2}} - \frac{10\,J_2\,a^2\,\mu\,x\,y}{(x^2+y^2+z^2)^{7/2}} + \frac{35\,J_2\,a^2\,\mu\,x\,y\,(x^2+y^2-2\,z^2)}{2\,(x^2+y^2+z^2)^{9/2}} + \frac{63\,J_3\,a^3\,\mu\,x\,y\,(3\,z\,(x^2+y^2+z^2)-5\,z^3)}{2\,(x^2+y^2+z^2)^{11/2}} - \frac{42\,J_3\,a^3\,\mu\,x\,y\,z}{(x^2+y^2+z^2)^{9/2}}$$

$$\frac{\partial a_x}{\partial z} = \frac{3\,\mu\,x\,z}{(x^2+y^2+z^2)^{5/2}} + \frac{3\,J_3\,a^3\,\mu\,x}{(x^2+y^2+z^2)^{7/2}} + \frac{5\,J_2\,a^2\,\mu\,x\,z}{(x^2+y^2+z^2)^{7/2}} - \frac{21\,J_3\,a^3\,\mu\,x\,z^2}{(x^2+y^2+z^2)^{9/2}} - \frac{7\,J_3\,a^3\,\mu\,x\,(3\,x^2+3\,y^2-6\,z^2)}{2\,(x^2+y^2+z^2)^{9/2}} + \frac{35\,J_2\,a^2\,\mu\,x\,z\,(x^2+y^2-2\,z^2)}{2\,(x^2+y^2+z^2)^{9/2}} + \frac{63\,J_3\,a^3\,\mu\,x\,z\,(3\,z\,(x^2+y^2+z^2)-5\,z^3)}{2\,(x^2+y^2+z^2)^{11/2}}$$

Because there is no dependence on the velocity, each of those partials are zero.

$$\frac{\partial a_x}{\partial v_x} = \frac{\partial a_x}{\partial v_y} = \frac{\partial a_x}{\partial v_z} = 0$$

$$\frac{\partial a_x}{\partial \mu} = \frac{J_2\,a^2\,x}{(x^2+y^2+z^2)^{5/2}} - \frac{x}{(x^2+y^2+z^2)^{3/2}} - \frac{5\,J_2\,a^2\,x\,(x^2+y^2-2\,z^2)}{2\,(x^2+y^2+z^2)^{7/2}} - \frac{7\,J_3\,a^3\,x\,(3\,z\,(x^2+y^2+z^2)-5\,z^3)}{2\,(x^2+y^2+z^2)^{9/2}} + \frac{3\,J_3\,a^3\,x\,z}{(x^2+y^2+z^2)^{7/2}}$$

$$\frac{\partial a_x}{\partial J_2} = \frac{a^2\,\mu\,x}{(x^2+y^2+z^2)^{5/2}} - \frac{5\,a^2\,\mu\,x\,(x^2+y^2-2\,z^2)}{2\,(x^2+y^2+z^2)^{7/2}}$$

$$\frac{\partial a_x}{\partial J_3} = \frac{3\,a^3\,\mu\,x\,z}{(x^2+y^2+z^2)^{7/2}} - \frac{7\,a^3\,\mu\,x\,(3\,z\,(x^2+y^2+z^2)-5\,z^3)}{2\,(x^2+y^2+z^2)^{9/2}}$$

Now, the partials for the y component of the acceleration with respect to the state vector are presented.

$$\frac{\partial a_y}{\partial x} = \frac{3\,\mu\,x\,y}{(x^2+y^2+z^2)^{5/2}} - \frac{10\,J_2\,a^2\,\mu\,x\,y}{(x^2+y^2+z^2)^{7/2}} + \frac{35\,J_2\,a^2\,\mu\,x\,y\,(x^2+y^2-2\,z^2)}{2\,(x^2+y^2+z^2)^{9/2}} + \frac{63\,J_3\,a^3\,\mu\,x\,y\,(3\,z\,(x^2+y^2+z^2)-5\,z^3)}{2\,(x^2+y^2+z^2)^{11/2}} - \frac{42\,J_3\,a^3\,\mu\,x\,y\,z}{(x^2+y^2+z^2)^{9/2}}$$

$$\frac{\partial a_y}{\partial y} = \frac{3\,\mu\,y^2}{(x^2+y^2+z^2)^{5/2}} - \frac{\mu}{(x^2+y^2+z^2)^{3/2}} + \frac{J_2\,a^2\,\mu}{(x^2+y^2+z^2)^{5/2}} - \frac{5\,J_2\,a^2\,\mu\,(x^2+y^2-2\,z^2)}{2\,(x^2+y^2+z^2)^{7/2}} - \frac{7\,J_3\,a^3\,\mu\,(3\,z\,(x^2+y^2+z^2)-5\,z^3)}{2\,(x^2+y^2+z^2)^{9/2}} + \frac{3\,J_3\,a^3\,\mu\,z}{(x^2+y^2+z^2)^{7/2}} - \frac{10\,J_2\,a^2\,\mu\,y^2}{(x^2+y^2+z^2)^{7/2}} + \frac{35\,J_2\,a^2\,\mu\,y^2\,(x^2+y^2-2\,z^2)}{2\,(x^2+y^2+z^2)^{9/2}} + \frac{63\,J_3\,a^3\,\mu\,y^2\,(3\,z\,(x^2+y^2+z^2)-5\,z^3)}{2\,(x^2+y^2+z^2)^{11/2}} - \frac{42\,J_3\,a^3\,\mu\,y^2\,z}{(x^2+y^2+z^2)^{9/2}}$$

$$\frac{\partial a_y}{\partial z} = \frac{3\,\mu\,y\,z}{(x^2+y^2+z^2)^{5/2}} + \frac{3\,J_3\,a^3\,\mu\,y}{(x^2+y^2+z^2)^{7/2}} + \frac{5\,J_2\,a^2\,\mu\,y\,z}{(x^2+y^2+z^2)^{7/2}} - \frac{21\,J_3\,a^3\,\mu\,y\,z^2}{(x^2+y^2+z^2)^{9/2}} - \frac{7\,J_3\,a^3\,\mu\,y\,(3\,x^2+3\,y^2-6\,z^2)}{2\,(x^2+y^2+z^2)^{9/2}} + \frac{35\,J_2\,a^2\,\mu\,y\,z\,(x^2+y^2-2\,z^2)}{2\,(x^2+y^2+z^2)^{9/2}} + \frac{63\,J_3\,a^3\,\mu\,y\,z\,(3\,z\,(x^2+y^2+z^2)-5\,z^3)}{2\,(x^2+y^2+z^2)^{11/2}}$$

Once again, there is no dependence on velocity and those partials evaluate to zero.

$$\frac{\partial a_y}{\partial v_x} = \frac{\partial a_y}{\partial v_y} = \frac{\partial a_y}{\partial v_z} = 0$$

$$\frac{\partial a_y}{\partial \mu} = \frac{J_2\,a^2\,y}{(x^2+y^2+z^2)^{5/2}} - \frac{y}{(x^2+y^2+z^2)^{3/2}} - \frac{5\,J_2\,a^2\,y\,(x^2+y^2-2\,z^2)}{2\,(x^2+y^2+z^2)^{7/2}} - \frac{7\,J_3\,a^3\,y\,(3\,z\,(x^2+y^2+z^2)-5\,z^3)}{2\,(x^2+y^2+z^2)^{9/2}} + \frac{3\,J_3\,a^3\,y\,z}{(x^2+y^2+z^2)^{7/2}}$$

$$\frac{\partial a_y}{\partial J_2} = \frac{a^2\,\mu\,y}{(x^2+y^2+z^2)^{5/2}} - \frac{5\,a^2\,\mu\,y\,(x^2+y^2-2\,z^2)}{2\,(x^2+y^2+z^2)^{7/2}}$$

$$\frac{\partial a_y}{\partial J_3} = \frac{3\,a^3\,\mu\,y\,z}{(x^2+y^2+z^2)^{7/2}} - \frac{7\,a^3\,\mu\,y\,(3\,z\,(x^2+y^2+z^2)-5\,z^3)}{2\,(x^2+y^2+z^2)^{9/2}}$$

Lastly, the partials for the x component of acceleration are presented.

$$\begin{aligned}\frac{\partial a_z}{\partial x} = {} & \frac{3\,\mu\,x\,z}{(x^2+y^2+z^2)^{5/2}} + \frac{3\,J_3\,a^3\,\mu\,x}{(x^2+y^2+z^2)^{7/2}} + \frac{5\,J_2\,a^2\,\mu\,x\,z}{(x^2+y^2+z^2)^{7/2}} - \frac{21\,J_3\,a^3\,\mu\,x\,z^2}{(x^2+y^2+z^2)^{9/2}} \\ & - \frac{7\,J_3\,a^3\,\mu\,x\,(3\,x^2+3\,y^2-6\,z^2)}{2\,(x^2+y^2+z^2)^{9/2}} + \frac{35\,J_2\,a^2\,\mu\,x\,z\,(x^2+y^2-2\,z^2)}{2\,(x^2+y^2+z^2)^{9/2}} \\ & + \frac{63\,J_3\,a^3\,\mu\,x\,z\,(3\,z\,(x^2+y^2+z^2)-5\,z^3)}{2\,(x^2+y^2+z^2)^{11/2}}\end{aligned}$$

$$\begin{aligned}\frac{\partial a_z}{\partial y} = {} & \frac{3\,\mu\,y\,z}{(x^2+y^2+z^2)^{5/2}} + \frac{3\,J_3\,a^3\,\mu\,y}{(x^2+y^2+z^2)^{7/2}} + \frac{5\,J_2\,a^2\,\mu\,y\,z}{(x^2+y^2+z^2)^{7/2}} - \frac{21\,J_3\,a^3\,\mu\,y\,z^2}{(x^2+y^2+z^2)^{9/2}} \\ & - \frac{7\,J_3\,a^3\,\mu\,y\,(3\,x^2+3\,y^2-6\,z^2)}{2\,(x^2+y^2+z^2)^{9/2}} + \frac{35\,J_2\,a^2\,\mu\,y\,z\,(x^2+y^2-2\,z^2)}{2\,(x^2+y^2+z^2)^{9/2}} \\ & + \frac{63\,J_3\,a^3\,\mu\,y\,z\,(3\,z\,(x^2+y^2+z^2)-5\,z^3)}{2\,(x^2+y^2+z^2)^{11/2}}\end{aligned}$$

$$\begin{aligned}\frac{\partial a_z}{\partial z} = {} & \frac{3\,\mu\,z^2}{(x^2+y^2+z^2)^{5/2}} - \frac{\mu}{(x^2+y^2+z^2)^{3/2}} - \frac{2\,J_2\,a^2\,\mu}{(x^2+y^2+z^2)^{5/2}} - \frac{5\,J_2\,a^2\,\mu\,(x^2+y^2-2\,z^2)}{2\,(x^2+y^2+z^2)^{7/2}} \\ & - \frac{7\,J_3\,a^3\,\mu\,(3\,z\,(x^2+y^2+z^2)-5\,z^3)}{2\,(x^2+y^2+z^2)^{9/2}} - \frac{6\,J_3\,a^3\,\mu\,z}{(x^2+y^2+z^2)^{7/2}} + \frac{20\,J_2\,a^2\,\mu\,z^2}{(x^2+y^2+z^2)^{7/2}} \\ & + \frac{35\,J_2\,a^2\,\mu\,z^2\,(x^2+y^2-2\,z^2)}{2\,(x^2+y^2+z^2)^{9/2}} + \frac{63\,J_3\,a^3\,\mu\,z^2\,(3\,z\,(x^2+y^2+z^2)-5\,z^3)}{2\,(x^2+y^2+z^2)^{11/2}} \\ & - \frac{7\,J_3\,a^3\,\mu\,z\,(3\,x^2+3\,y^2-6\,z^2)}{(x^2+y^2+z^2)^{9/2}}\end{aligned}$$

Once again, there is no dependence on velocity and those partials evaluate to zero.

$$\frac{\partial a_z}{\partial v_x} = \frac{\partial a_z}{\partial v_y} = \frac{\partial a_z}{\partial v_z} = 0$$

$$\begin{aligned}\frac{\partial a_z}{\partial \mu} = {} & \frac{J_3\,a^3\,(3\,x^2+3\,y^2-6\,z^2)}{2\,(x^2+y^2+z^2)^{7/2}} - \frac{2\,J_2\,a^2\,z}{(x^2+y^2+z^2)^{5/2}} - \frac{z}{(x^2+y^2+z^2)^{3/2}} \\ & - \frac{5\,J_2\,a^2\,z\,(x^2+y^2-2\,z^2)}{2\,(x^2+y^2+z^2)^{7/2}} - \frac{7\,J_3\,a^3\,z\,(3\,z\,(x^2+y^2+z^2)-5\,z^3)}{2\,(x^2+y^2+z^2)^{9/2}}\end{aligned}$$

$$\frac{\partial a_z}{\partial J_2} = -\frac{2\,a^2\,\mu\,z}{(x^2+y^2+z^2)^{5/2}} - \frac{5\,a^2\,\mu\,z\,(x^2+y^2-2\,z^2)}{2\,(x^2+y^2+z^2)^{7/2}}$$

$$\frac{\partial a_z}{\partial J_3} = \frac{a^3\,\mu\,(3\,x^2+3\,y^2-6\,z^2)}{2\,(x^2+y^2+z^2)^{7/2}} - \frac{7\,a^3\,\mu\,z\,(3\,z\,(x^2+y^2+z^2)-5\,z^3)}{2\,(x^2+y^2+z^2)^{9/2}}$$

This Jacobian that has been presented is only some of the entire $\frac{\partial \dot{\mathbf{X}}}{\partial \boldsymbol{X}}$ that needs to be constructed. For clarity, this entire Jacobian is the partial derivative of the derivative of the state vector with respect to the state vector. It will describe how the state will transition in time and is of size 9x9.

The first 3 rows of this matrix will be a direct mapping of the velocity states to each other and is presented below

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{X}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The next 3 rows of this Jacobian will be the results previously presented with the partials of the acceleration with respect to the state vector $\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{X}}$

The final 3 rows of this Jacobian will be all zeros because there is no rate of change in $\mu,\ J_2,\ J_3$.

A function was created to take in a state vector and produce this Jacobian. The results were compared with the values provided and the difference is shown below.

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0
4	1.2517e-05	2.6703e-05	-6.9618e-05	0	0	0	2.2411e-05	1.8044e-09	-7.0572e-05
5	2.6703e-05	-2.3842e-05	1.3161e-04	0	0	0	-4.2915e-05	-3.4925e-09	1.2207e-04
6	-6.9618e-05	1.3161e-04	1.6212e-05	0	0	0	5.8174e-05	1.9558e-08	-5.3406e-05
7	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0