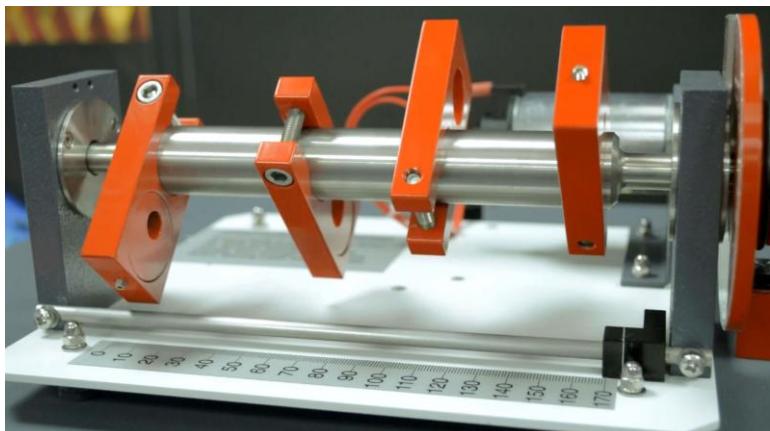


1

BALANCING OF ROTATING MASSES



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1.1 Introduction

- Often an unbalance of forces is produced in rotary or reciprocating machinery due to the inertia forces associated with the moving masses. Balancing is the process of designing or modifying machinery so that the unbalance is reduced to an acceptable level and if possible is eliminated entirely.

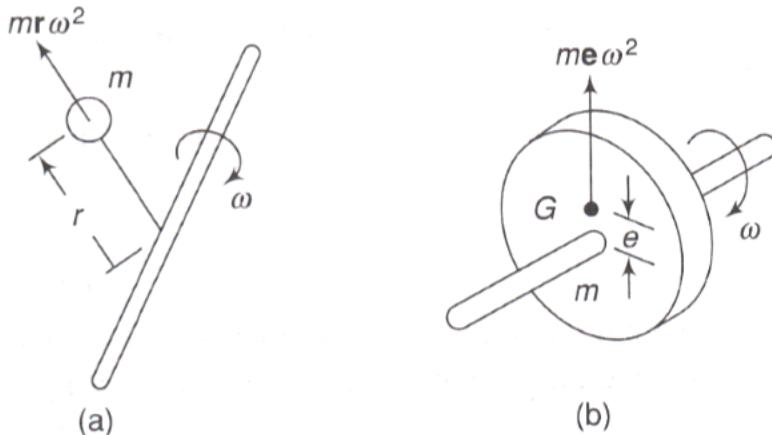


Fig. 1.1

- A particle or mass moving in a circular path experiences a centripetal acceleration and a force is required to produce it. An equal and opposite force acting radially outwards acts on the axis of rotation and is known as centrifugal force [Fig. 1.1(a)]. This is a disturbing force on the axis of rotation, the magnitude of which is constant but the direction changes with the rotation of the mass.
- In a revolving rotor, the centrifugal force remains balanced as long as the centre of the mass of the rotor lies on the axis of the shaft. When the centre of mass does not lie on the axis or there is an eccentricity, an unbalanced force is produced [Fig. 1.1(b)]. This type of unbalance is very common. For example, in steam turbine rotors, engine crankshafts, rotary compressors and centrifugal pumps.
- Most of the serious problems encountered in high-speed machinery are the direct result of unbalanced forces. These forces exerted on the frame by the moving machine members are time varying, impart vibratory motion to the frame and produce noise. Also, there are human discomfort and detrimental effects on the machine performance and the structural integrity of the machine foundation.
- The most common approach to balancing is by redistributing the mass which may be accomplished by addition or removal of mass from various machine members.
- There are two basic types of unbalance-rotating unbalance and reciprocating unbalance – which may occur separately or in combination.

1.2 Static Balancing:

- A system of rotating masses is said to be in static balance if the combined mass centre of the system lies on the axis of rotation.

1.3 Types of Balancing:

There are main two types of balancing conditions

- (i) Balancing of rotating masses
- (ii) Balancing of reciprocating masses

(i) Balancing of Rotating Masses

Whenever a certain mass is attached to a rotating shaft, it exerts some centrifugal force, whose effect is to bend the shaft and to produce vibrations in it. In order to prevent the effect of centrifugal force, another mass is attached to the opposite side of the shaft, at such a position so as to balance the effect of the centrifugal force of the first mass. This is done in such a way that the centrifugal forces of both the masses are made to be equal and opposite. The process of providing the second mass in order to counteract the effect of the centrifugal force of the first mass is called balancing of rotating masses.

The following cases are important from the subject point of view:

1. Balancing of a single rotating mass by a single mass rotating in the same plane.
2. Balancing of different masses rotating in the same plane.
3. Balancing of different masses rotating in different planes.

1.4 Balancing of Several Masses Rotating in the Same Plane

- Consider any number of masses (say four) of magnitude m_1, m_2, m_3 and m_4 at distances r_1, r_2, r_3 and r_4 from the axis of the rotating shaft. Let $\theta_1, \theta_2, \theta_3$ and θ_4 be the angles of these masses with the horizontal line OX , as shown in Fig. 1.2 (a). Let these masses rotate about an axis through O and perpendicular to the plane of paper, with a constant angular velocity of ω rad/s.

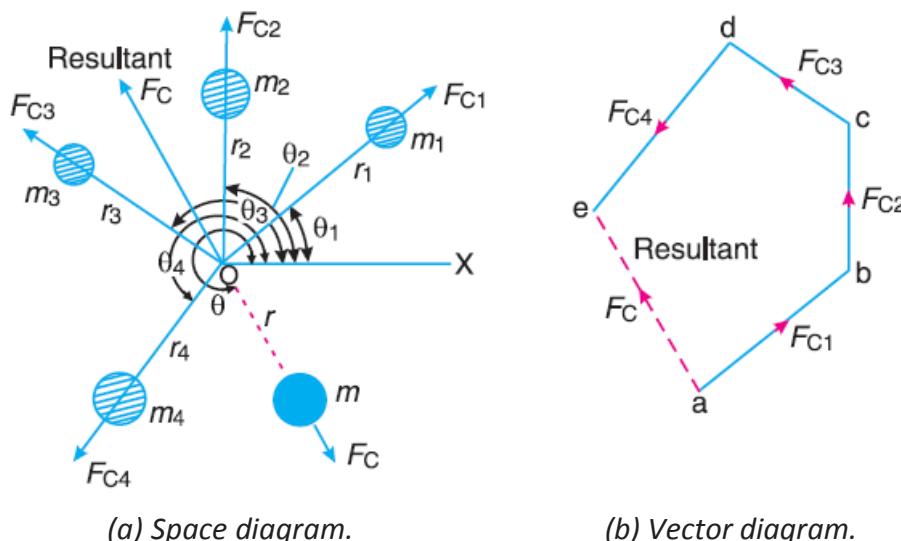


Fig. 1.2 Balancing of several masses rotating in the same plane.

- The magnitude and position of the balancing mass may be found out analytically or graphically as discussed below:

1. Analytical method

- Each mass produces a centrifugal force acting radially outwards from the axis of rotation. Let F be the vector sum of these forces.

$$F = m_1 r_1 \omega^2 + m_2 r_2 \omega^2 + m_3 r_3 \omega^2 + m_4 r_4 \omega^2$$

- The rotor is said to be statically balanced if the vector sum \mathbf{F} is zero.
 - If \mathbf{F} is not zero, i.e., the rotor is unbalanced, then produce a counterweight (balance weight) of mass m_c , at radius r_c to balance the rotor so that

$$m_1r_1\omega^2 + m_2r_2\omega^2 + m_3r_3\omega^2 + m_4r_4\omega^2 + m_cr_c\omega^2 = 0$$

$$m_1r_1 + m_2r_2 + m_3r_3 + m_4r_4 + m_cr_c = 0$$

- The magnitude of either m_c or r_c may be selected and of other can be calculated.
 - In general, if Σmr is the vector sum of $m_1.r_1$, $m_2.r_2$, $m_3.r_3$, $m_4.r_4$, etc., then

$$\sum mr + m_c r_c = 0$$

- To solve these equations by mathematics, divide each force into its x and z components, $\Sigma m_r \cos\theta + m_c r_c \cos\theta_c = 0$

$$\sum m_r \sin \theta + m_c r_c \sin \theta_c = 0$$

$$m_c r_c \cos \theta_c = -\Sigma m r \cos \theta \quad \dots \dots \dots \text{(i)}$$

$$\text{and } m_c r_c \sin \theta_c = -\sum m_r \sin \theta \quad \dots \dots \dots \text{(ii)}$$

- Squaring and adding (i) and (ii),

$$m_c r_c = \sqrt{(\sum mr \cos\theta)^2 + (\sum mr \sin\theta)^2}$$

- Dividing (ii) by (i),

$$\tan \theta_c = \frac{-\Sigma mr \sin \theta}{-\Sigma mr \cos \theta}$$

- The signs of the numerator and denominator of this function identify the quadrant of the angle.

2. Graphical method

- First of all, draw the space diagram with the positions of the several masses, as shown in Fig. 1.2 (a).
 - Find out the centrifugal force (or product of the mass and radius of rotation) exerted by each mass on the rotating shaft.
 - Now draw the vector diagram with the obtained centrifugal forces (or the product of the masses and their radii of rotation), such that ab represents the centrifugal force exerted by the mass m_1 (or $m_1 \cdot r_1$) in magnitude and direction to some suitable scale. Similarly, draw bc , cd and de to represent centrifugal forces of other masses m_2 , m_3 and m_4 (or $m_2 \cdot r_2, m_3 \cdot r_3$ and $m_4 \cdot r_4$).
 - Now, as per polygon law of forces, the closing side ae represents the resultant force in magnitude and direction, as shown in Fig. 1.2 (b).

- The balancing force is, then, equal to resultant force, but in opposite direction.
- Now find out the magnitude of the balancing mass (m) at a given radius of rotation (r), such that

$$m.r.\omega^2 = \text{Resultant centrifugal force}$$

or

$$m.r = \text{Resultant of } m_1.r_1, m_2.r_2, m_3.r_3 \text{ and } m_4.r_4$$

- (In general for graphical solution, vectors $m_1.r_1, m_2.r_2, m_3.r_3, m_4.r_4$, etc., are added. If they close in a loop, the system is balanced. Otherwise, the closing vector will be giving $m_c.r_c$. Its direction identifies the angular position of the countermass relative to the other mass.)

Example 1.1 : A circular disc mounted on a shaft carries three attached masses of 4 kg, 3 kg and 2.5 kg at radial distances of 75 mm, 85 mm and 50 mm and at the angular positions of 45° , 135° and 240° respectively. The angular positions are measured counterclockwise from the reference line along the x-axis. Determine the amount of the countermass at a radial distance of 75 mm required for the static balance.

$$m_1 = 4 \text{ kg}$$

$$r_1 = 75 \text{ mm}$$

$$\theta_1 = 45^\circ$$

$$m_2 = 3 \text{ kg}$$

$$r_2 = 85 \text{ mm}$$

$$\theta_2 = 135^\circ$$

$$m_3 = 2.5 \text{ kg}$$

$$r_3 = 50 \text{ mm}$$

$$\theta_3 = 240^\circ$$

$$m_1 r_1 = 4 \times 75 = 300 \text{ kg.mm}$$

$$m_2 r_2 = 3 \times 85 = 255 \text{ kg.mm}$$

$$m_3 r_3 = 2.5 \times 50 = 125 \text{ kg.mm}$$

Analytical Method:

$$\sum mr + m_c r_c = 0$$

$$300 \cos 45^\circ + 255 \cos 135^\circ + 125 \cos 240^\circ + m_c r_c \cos \theta_c = 0$$

and

$$300 \sin 45^\circ + 255 \sin 135^\circ + 125 \sin 240^\circ + m_c r_c \sin \theta_c = 0$$

Squaring, adding and then solving,

$$m_c r_c = \sqrt{(300 \cos 45^\circ + 255 \cos 135^\circ + 125 \cos 240^\circ)^2 + (300 \sin 45^\circ + 255 \sin 135^\circ + 125 \sin 240^\circ)^2}$$

$$m_c \times 75 = \sqrt{(-30.68)^2 + (284.2)^2}$$

$$= 285.8 \text{ kg.mm}$$

$$m_c = 3.81 \text{ kg}$$

$$\tan \theta_c = \frac{-\sum m_r \sin \theta}{-\sum m_r \cos \theta} = \frac{-284.2}{-(-30.68)} = -9.26$$

$$\theta_c = -83^\circ 50'$$

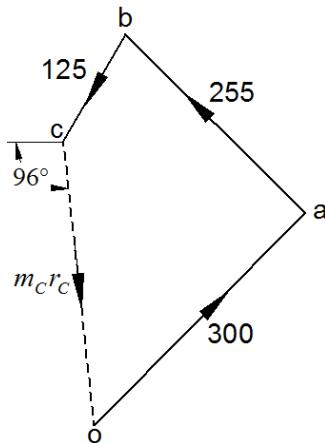
θ_c lies in the fourth quadrant (numerator is negative and denominator is positive).

$$\theta_c = 360 - 83^\circ 50'$$

$$\theta_c = 276^\circ 9'$$

Graphical Method:

- The magnitude and the position of the balancing mass may also be found graphically as discussed below :
- Now draw the vector diagram with the above values, to some suitable scale, as shown in Fig. 1.3. The closing side of the polygon co represents the resultant force. By measurement, we find that $co = 285.84 \text{ kg-mm}$.

*Fig. 1.3 Vector Diagram*

- The balancing force is equal to the resultant force. Since the balancing force is proportional to $m.r$, therefore

$$m_c \times 75 = \text{vector } co = 285.84 \text{ kg-mm} \quad \text{or} \quad m_c = 285.84 / 75$$

$$\mathbf{m_c = 3.81 \text{ kg}.}$$

- By measurement we also find that the angle of inclination of the balancing mass (m) from the horizontal or positive X-axis,

$$\theta_c = 276^\circ.$$

Example 1.2 :Four masses m_1 , m_2 , m_3 and m_4 are 200 kg, 300 kg, 240 kg and 260 kg respectively. The corresponding radii of rotation are 0.2 m, 0.15 m, 0.25 m and 0.3 m respectively and the angles between successive masses are 45°, 75° and 135°. Find the position and magnitude of the balance mass required, if its radius of rotation is 0.2 m.

$$m_1 = 200 \text{ kg}$$

$$r_1 = 0.2 \text{ m}$$

$$\theta_1 = 0^\circ$$

$$m_2 = 300 \text{ kg}$$

$$r_2 = 0.15 \text{ m}$$

$$\theta_2 = 45^\circ$$

$$m_3 = 240 \text{ kg}$$

$$r_3 = 0.25 \text{ m}$$

$$\theta_3 = 45^\circ + 75^\circ = 120^\circ$$

$$m_4 = 260 \text{ kg}$$

$$r_4 = 0.3 \text{ m}$$

$$\theta_4 = 120^\circ + 135^\circ = 255^\circ$$

$$m_1 r_1 = 200 \times 0.2 = 40$$

$$r_c = 0.2 \text{ m}$$

$$m_2 r_2 = 300 \times 0.15 = 45$$

$$m_3 r_3 = 240 \times 0.25 = 60$$

$$m_4 r_4 = 260 \times 0.3 = 78$$

$$\Sigma mr + m_c r_c = 0$$

$$40 \cos 0^\circ + 45 \cos 45^\circ + 60 \cos 120^\circ + 78 \cos 255^\circ + m_c r_c \cos \theta_c = 0 \quad \text{and}$$

$$40 \sin 0^\circ + 45 \sin 45^\circ + 60 \sin 120^\circ + 78 \sin 255^\circ + m_c r_c \sin \theta_c = 0$$

Squaring, adding and then solving,

$$m_c r_c = \sqrt{(40\cos 0^\circ + 45 \cos 45^\circ + 60 \cos 120^\circ + 78 \cos 255^\circ)^2 + (40\sin 0^\circ + 45 \sin 45^\circ + 60 \sin 120^\circ + 78 \sin 255^\circ)^2}$$

$$m_c \times 0.2 = \sqrt{(21.6)^2 + (8.5)^2}$$

$$= 23.2 \text{ kg.mm}$$

$$m_c = 116 \text{ kg}$$

$$\tan \theta_c = \frac{-\sum mr \sin \theta}{-\sum mr \cos \theta} = \frac{-8.5}{-21.6} = 0.3935$$

$$\theta_c = 21^\circ 28'$$

θ_c lies in the third quadrant (numerator is negative and denominator is negative).

$$\theta_c = 180 + 21^\circ 28'$$

$$\theta_c = 201^\circ 28'$$

Graphical Method:

- For graphical method draw the vector diagram with the above values, to some suitable scale, as shown in Fig. 1.4. The closing side of the polygon $a e$ represents the resultant force. By measurement, we find that $a e = 23 \text{ kg-m}$.

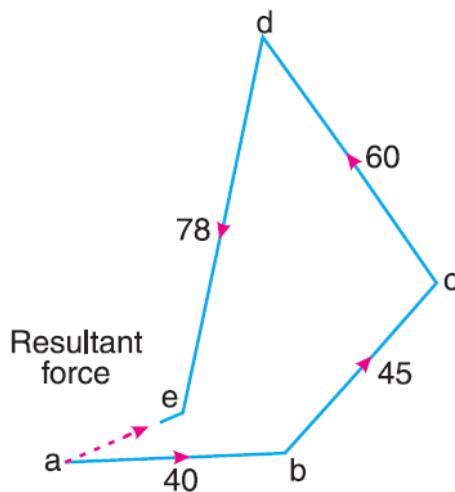


Fig. 1.4 Vector Diagram

- The balancing force is equal to the resultant force. Since the balancing force is proportional to $m.r$, therefore

$$m \times 0.2 = \text{vector } ea = 23 \text{ kg-m} \quad \text{or} \quad m_c = 23/0.2$$

$$m_c = 115 \text{ kg.}$$

- By measurement we also find that the angle of inclination of the balancing mass (m) from the horizontal or positive X-axis,

$$\theta_c = 201^\circ.$$

1.5 Dynamic Balancing

- When several masses rotate in different planes, the centrifugal forces, in addition to being out of balance, also form couples. A system of rotating masses is in dynamic balance when there does not exist any resultant centrifugal force as well as resultant couple.
- In the work that follows, the products of mr and mrl (instead of $mr\omega^2$ and $mrl\omega^2$), usually, have been referred as force and couple respectively as it is more convenient to draw force and couple polygons with these quantities.

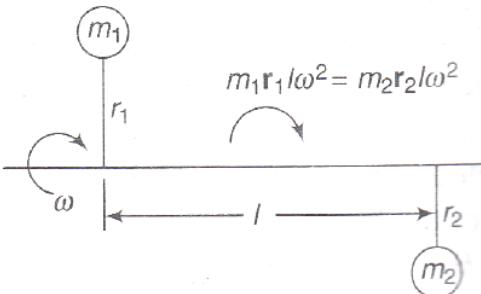


Fig. 1.5

- If m_1 , and m_2 are two masses (Fig. 1.5) revolving diametrically opposite to each other in different planes such that $m_1r_1 = m_2r_2$, the centrifugal forces are balanced, but an unbalanced couple of magnitude m_1r_1l ($= m_2r_2l$) is introduced. The couple acts in a plane that contains the axis of rotation and the two masses. Thus, the couple is of constant magnitude but variable direction.

1.6 Balancing of Several Masses Rotating in the different Planes

- Let there be a rotor revolving with a uniform angular velocity ω . m_1 , m_2 and m_3 are the masses attached to the rotor at radii r_1 , r_2 and r_3 respectively. The masses m_1 , m_2 and m_3 rotate in planes 1, 2 and 3 respectively. Choose a reference plane at O so that the distances of the planes 1, 2 and 3 from O are l_1 , l_2 and l_3 respectively.
- Transference of each unbalanced force to the reference plane introduces the like number of forces and couples.
- The unbalanced forces in the reference plane are $m_1r_1\omega^2$, $m_2r_2\omega^2$ and $m_3r_3\omega^2$ acting radially outwards.
- The unbalanced couples in the reference plane are $m_1r_1\omega^2l_1$, $m_2r_2\omega^2l_2$ and $m_3r_3\omega^2l_3$ which may be represented by vectors parallel to the respective force vectors, i.e., parallel to the respective radii of m_1 , m_2 and m_3 .
- For complete balancing of the rotor, the resultant force and resultant couple both should be zero, i.e., $m_1r_1\omega^2 + m_2r_2\omega^2 + m_3r_3\omega^2 = 0$ (a)
and $m_1r_1\omega^2l_1 + m_2r_2\omega^2l_2 + m_3r_3\omega^2l_3 = 0$ (b)
- If the Eqs (a) and (b) are not satisfied, then there are unbalanced forces and couples. A mass placed in the reference plane may satisfy the force equation but

the couple equation is satisfied only by two equal forces in different transverse planes.

- Thus in general, two planes are needed to balance a system of rotating masses.
- Therefore, in order to satisfy Eqs (a) and (b), introduce two counter-masses m_{c1} and m_{c2} at radii r_{c1} and r_{c2} respectively. Then Eq. (a) may be written as

$$\begin{aligned} m_1r_1\omega^2 + m_2r_2\omega^2 + m_3r_3\omega^2 + m_{c1}r_{c1}\omega^2 + m_{c2}r_{c2}\omega^2 &= 0 \\ m_1r_1 + m_2r_2 + m_3r_3 + m_{c1}r_{c1} + m_{c2}r_{c2} &= 0 \\ \Sigma mr + m_{c1}r_{c1} + m_{c2}r_{c2} &= 0 \end{aligned} \quad \dots \dots \dots \text{(c)}$$

- Let the two countermasses be placed in transverse planes at axial locations O and Q , i.e., the countermass m_{c1} be placed in the reference plane and the distance of the plane of m_{c2} be l_{c2} from the reference plane. Equation (b) modifies to (taking moments about O)

$$\begin{aligned} m_1r_1\omega^2 l_1 + m_2r_2\omega^2 l_2 + m_3r_3\omega^2 l_3 + m_{c2}r_{c2}\omega^2 l_{c2} &= 0 \\ m_1r_1l_1 + m_2r_2l_2 + m_3r_3l_3 + m_{c2}r_{c2}l_{c2} &= 0 \\ \Sigma mrl + m_{c2}r_{c2}l_{c2} &= 0 \end{aligned} \quad \dots \dots \dots \text{(d)}$$

- Thus, Eqs (c) and (d) are the necessary conditions for dynamic balancing of rotor. Again the equations can be solved mathematically or graphically.

Dividing Eq. (d) into component form

$$\begin{aligned} \Sigma mrl \cos \theta + m_{c2}r_{c2}l_{c2} \cos \theta_{c2} &= 0 \\ \Sigma mrl \sin \theta + m_{c2}r_{c2}l_{c2} \sin \theta_{c2} &= 0 \\ m_{c2}r_{c2}l_{c2} \cos \theta_{c2} &= -\Sigma mrl \cos \theta \quad \dots \dots \dots \text{(i)} \\ m_{c2}r_{c2}l_{c2} \sin \theta_{c2} &= -\Sigma mrl \sin \theta \quad \dots \dots \dots \text{(ii)} \end{aligned}$$

- Squaring and adding (i) and (ii)

$$m_{c2}r_{c2}l_{c2} = \sqrt{(\Sigma mrl \cos \theta)^2 + (\Sigma mrl \sin \theta)^2}$$

- Dividing (ii) by (i),

$$\tan \theta_{c2} = \frac{-\Sigma mrl \sin \theta}{-\Sigma mrl \cos \theta}$$

- After obtaining the values of m_{c2} and θ_{c2} from the above equations, solve Eq. (c) by taking its components,

$$\begin{aligned} \Sigma mrcos \theta + m_{c1}r_{c1}cos \theta_{c1} + m_{c2}r_{c2}cos \theta_{c2} &= 0 \\ \Sigma mrsin \theta + m_{c1}r_{c1}sin \theta_{c1} + m_{c2}r_{c2}sin \theta_{c2} &= 0 \\ m_{c1}r_{c1}cos \theta_{c1} &= -(\Sigma mrcos \theta + m_{c2}r_{c2}cos \theta_{c2}) \\ m_{c1}r_{c1}sin \theta_{c1} &= -(\Sigma mrsin \theta + m_{c2}r_{c2}sin \theta_{c2}) \\ m_{c1}r_{c1} &= \sqrt{(\Sigma mr cos \theta + m_{c2}r_{c2} cos \theta_{c2})^2 + (\Sigma mr sin \theta + m_{c2}r_{c2} sin \theta_{c2})^2} \\ \tan \theta_{c1} &= \frac{-(\Sigma mr sin \theta + m_{c2}r_{c2} sin \theta_{c2})}{-(\Sigma mr cos \theta + m_{c2}r_{c2} cos \theta_{c2})} \end{aligned}$$

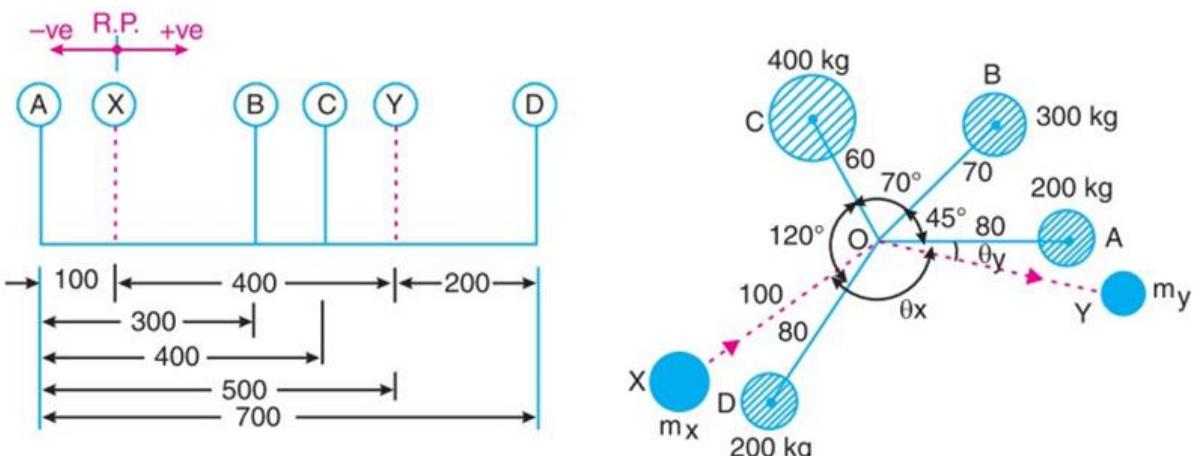
Example 1.3 : A shaft carries four masses A, B, C and D of magnitude 200 kg, 300 kg, 400 kg and 200 kg respectively and revolving at radii 80 mm, 70 mm, 60 mm and 80 mm in planes measured from A at 300 mm, 400 mm and 700 mm. The angles between the cranks measured anticlockwise are A to B 45°, B to C 70° and C to D 120°. The balancing masses are to be placed in planes X and Y. The distance between the planes A and X is 100 mm, between X and Y is 400 mm and between Y and D is 200 mm. If the balancing masses revolve at a radius of 100 mm, find their magnitudes and angular positions.

$m_A = 200 \text{ kg}$	$r_A = 80 \text{ mm}$	$\theta_A = 0^\circ$	$l_A = -100 \text{ mm}$
$m_B = 300 \text{ kg}$	$r_B = 70 \text{ mm}$	$\theta_B = 45^\circ$	$l_B = 200 \text{ mm}$
$m_C = 400 \text{ kg}$	$r_C = 60 \text{ mm}$	$\theta_C = 45^\circ + 70^\circ = 115^\circ$	$l_C = 300 \text{ mm}$
$m_D = 200 \text{ kg}$	$r_D = 80 \text{ mm}$	$\theta_D = 115^\circ + 120^\circ = 235^\circ$	$l_D = 600 \text{ mm}$
$r_X = r_Y = 100 \text{ mm}$			$l_Y = 400 \text{ mm}$

Let m_X = Balancing mass placed in plane X, and
 m_Y = Balancing mass placed in plane Y.

The position of planes and angular position of the masses (assuming the mass A as horizontal) are shown in Fig. 1.5 (a) and (b) respectively.

Assume the plane X as the reference plane (R.P.). The distances of the planes to the right of plane X are taken as +ve while the distances of the planes to the left of plane X are taken as -ve.



(a) Position of planes.

(b) Angular position of masses.

Fig. 1.6

$$m_A r_A l_A = 200 \times 0.08 \times (-0.1) = -1.6 \text{ kg.m}^2$$

$$m_B r_B l_B = 300 \times 0.07 \times 0.2 = 4.2 \text{ kg.m}^2$$

$$m_C r_C l_C = 400 \times 0.06 \times 0.3 = 7.2 \text{ kg.m}^2$$

$$m_D r_D l_D = 200 \times 0.08 \times 0.6 = 9.6 \text{ kg.m}^2$$

$$m_A r_A = 200 \times 0.08 = 16 \text{ kg.m}$$

$$m_B r_B = 300 \times 0.07 = 21 \text{ kg.m}$$

$$m_C r_C = 400 \times 0.06 = 24 \text{ kg.m}$$

$$m_D r_D = 200 \times 0.08 = 16 \text{ kg.m}$$

Analytical Method:

For unbalanced couple

$$\sum mrl + m_Y r_Y l_Y = 0$$

$$m_Y r_Y l_Y = \sqrt{(\sum mrl \cos \theta)^2 + (\sum mrl \sin \theta)^2}$$

$$m_Y r_Y l_Y = \sqrt{(-1.6 \cos 0^\circ + 4.2 \cos 45^\circ + 7.2 \cos 115^\circ + 9.6 \cos 235^\circ)^2 + (-1.6 \sin 0^\circ + 4.2 \sin 45^\circ + 7.2 \sin 115^\circ + 9.6 \sin 235^\circ)^2}$$

$$m_Y r_Y l_Y = \sqrt{(-7.179)^2 + (1.63)^2}$$

$$m_Y \times 0.1 \times 0.4 = 7.36$$

$m_Y = 184 \text{ kg.}$

$$\tan \theta_Y = \frac{-\sum mrl \sin \theta}{-\sum mrl \cos \theta} = \frac{-1.63}{-(-7.179)} = -0.227$$

$$\theta_Y = -12^\circ 47'$$

θ_Y lies in the fourth quadrant (numerator is negative and denominator is positive).

$$\theta_Y = 360 - 12^\circ 47'$$

$\theta_Y = 347^\circ 12'$

For unbalanced centrifugal force

$$\sum mr + m_X r_X + m_Y r_Y = 0$$

$$m_X r_X = \sqrt{(\sum mrc \cos \theta + m_Y r_Y \cos \theta_Y)^2 + (\sum mrs \sin \theta + m_Y r_Y \sin \theta_Y)^2}$$

$$m_X r_X = \sqrt{(16 \cos 0^\circ + 21 \cos 45^\circ + 24 \cos 115^\circ + 16 \cos 235^\circ + 18.4 \cos 347^\circ 12')^2 + (16 \sin 0^\circ + 21 \sin 45^\circ + 24 \sin 115^\circ + 16 \sin 235^\circ + 18.4 \sin 347^\circ 12')^2}$$

$$m_X r_X = \sqrt{(29.47)^2 + (19.42)^2}$$

$$m_X \times 0.1 = 35.29$$

$m_X = 353 \text{ kg.}$

$$\tan \theta_X = \frac{-\sum mrs \sin \theta}{-\sum mrc \cos \theta} = \frac{-19.42}{-29.47} = 0.6589$$

$$\theta_X = 33^\circ 22'$$

θ_X lies in the third quadrant (numerator is negative and denominator is negative).

$$\theta_X = 180 + 33^\circ 22'$$

$\theta_X = 213^\circ 22'$

Graphical Method:

The balancing masses and their angular positions may be determined graphically as discussed below :

Table 1.1

Plane	Angle	Mass (m) kg	Radius (r)m	Cent.force $\div \omega^2$ (mr) kg-m	Distance from Ref. Plane (l) m	Couple $\div \omega^2$ (mrl) kg-m ²
A	0°	200	0.08	160	- 0.1	- 1.6
X (R.P.)	θ_x	m_x	0.1	$0.1 m_x$	0	0
B	45°	300	0.07	21	0.2	4.2
C	115°	400	0.06	24	0.3	7.2
Y	θ_y	m_y	0.1	$0.1 m_y$	0.4	$0.04 m_y$
D	235°	200	0.08	16	0.6	9.6

- First of all, draw the couple polygon from the data given in Table 1.1 (column 7) as shown in Fig. 1.7 (a) to some suitable scale. The vector $d'o'$ represents the balanced couple. Since the balanced couple is proportional to $0.04 m_y$, therefore by measurement,

$$0.04 m_y = \text{vector } d'o' = 73 \text{ kg-m}^2$$

or

$$m_y = 182.5 \text{ kg}$$

- The angular position of the mass m_y is obtained by drawing Om_y in Fig. 1.6 (b), parallel to vector $d'o'$. By measurement, the angular position of m_y is $\theta_y = 12^\circ$ in the clockwise direction from mass m_A (i.e. 200 kg), so $\theta_y = 360^\circ - 12^\circ = 348^\circ$.

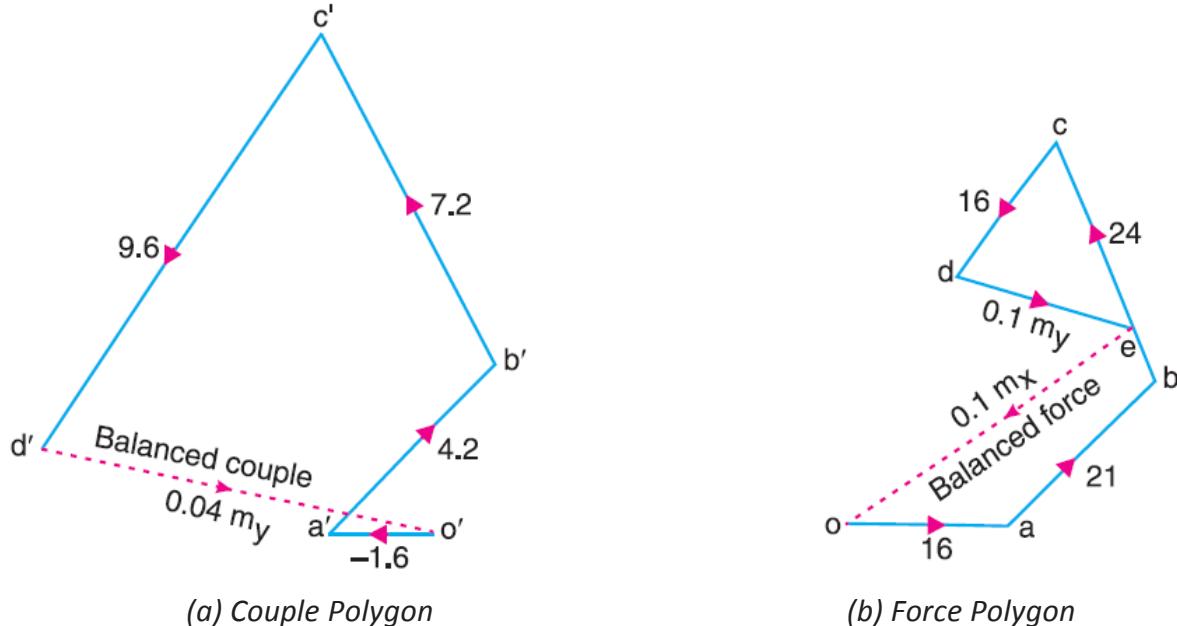


Fig. 1.7

- Now draw the force polygon from the data given in Table 1.1 (column 5) as shown in Fig. 1.7 (b). The vector eo represents the balanced force. Since the balanced force is proportional to $0.1 m_x$, therefore by measurement,

$$0.1 m_x = \text{vector } eo = 35.5 \text{ kg-m}$$

$$\text{or } m_x = 355 \text{ kg.}$$

- The angular position of the mass m_x is obtained by drawing Om_x in Fig. 1.6 (b), parallel to vector eo . By measurement, the angular position of m_x is $\theta_x = 145^\circ$ in the clockwise direction from mass m_A (i.e. 200 kg), so $\theta_x = 360^\circ - 145^\circ = 215^\circ$.

Example 1.4: Four masses A, B, C and D carried by a rotating shaft are at radii 100, 140, 210 and 160 mm respectively. The planes in which the masses revolve are spaced 600 mm apart and the masses of B, C and D are 16 kg, 10 kg and 8 kg respectively. Find the required mass A and the relative angular positions of the four masses so that shaft is in complete balance.

$$m_A = ? \quad r_A = 100\text{mm}$$

$$m_B = 16 \text{ kg} \quad r_B = 140\text{mm} \quad l_B = 600 \text{ mm}$$

$$m_C = 10 \text{ kg} \quad r_C = 210\text{mm} \quad l_C = 1200 \text{ mm}$$

$$m_D = 8 \text{ kg} \quad r_D = 160\text{mm} \quad l_D = 1800 \text{ mm}$$

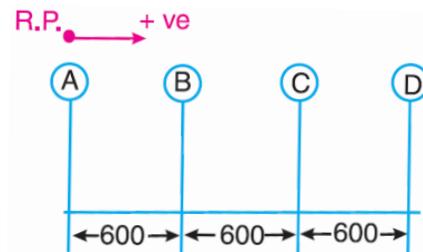
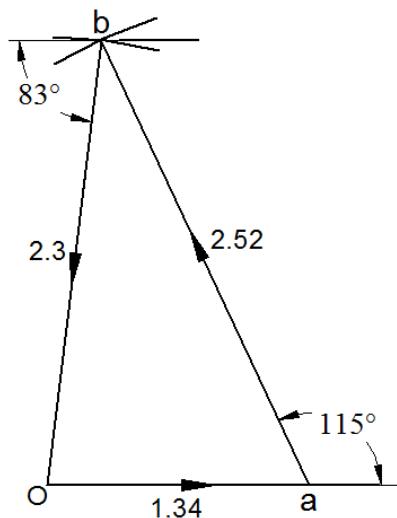


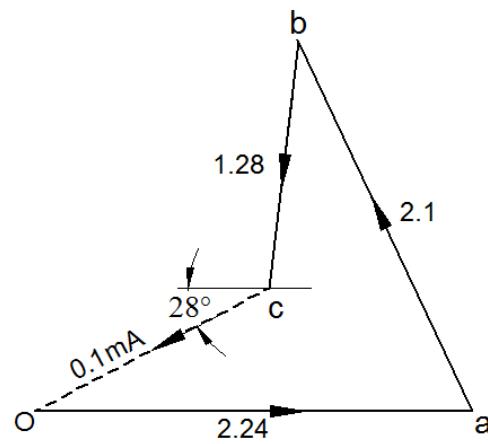
Table 1.2

Plane	Angle	Mass (m) kg	Radius (r) m	Cent.force $\div \omega^2$ (mr) kg-m	Distance from Ref. Plane (l) m	Couple $\div \omega^2$ (mrl) kg-m ²
A (R.P.)	θ_A	m_A	0.1	$0.1m_A$	0	0
B	0°	16	0.14	2.24	0.6	1.34
C	θ_C	10	0.21	2.1	1.2	2.52
D	θ_D	8	0.16	1.28	1.8	2.3

- First of all, draw the couple polygon from the data given in Table 1.2 (column 7) as shown in Fig. 1.8 (a) to some suitable scale. By measurement, the angular position of m_C is $\theta_C = 115^\circ$ in the anticlockwise direction from mass m_B and the angular position of m_D is $\theta_D = 263^\circ$ in the anticlockwise direction from mass m_B .



(a) Couple Polygon



(b) Force Polygon

Fig. 1.8

- Now draw the force polygon from the data given in Table 1.2 (column 5) as shown in Fig. 1.8 (b). The vector co represents the balanced force. Since the balanced force is proportional to $0.1 m_A$, therefore by measurement,

$$0.1m_A = \text{vector } co = 1.36 \text{ kg-m}$$

$$\text{Or } m_A = 13.6 \text{ kg.}$$

- By measurement, the angular position of m_A is $\theta_A = 208^\circ$ in the anticlockwise direction from mass m_B (i.e. 16 kg).

Example 1.5 :Four masses 150 kg, 200 kg, 100 kg and 250 kg are attached to a shaft revolving at radii 150 mm, 200 mm, 100 mm and 250 mm; in planes A, B, C and D respectively. The planes B, C and D are at distances 350 mm, 500 mm and 800 mm from plane A. The masses in planes B, C and D are at an angle 105°, 200° and 300° measured anticlockwise from mass in plane A. It is required to balance the system by placing the balancing masses in the planes P and Q which are midway between the planes A and B, and between C and D respectively. If the balancing masses revolve at radius 180 mm, find the magnitude and angular positions of the balance masses.

$$m_A = 150 \text{ kg}$$

$$r_A = 150\text{mm}$$

$$\theta_A = 0^\circ$$

$$m_B = 200 \text{ kg}$$

$$r_B = 200\text{mm}$$

$$\theta_B = 105^\circ$$

$$m_c = 100 \text{ kg}$$

$$r_c = 100\text{mm}$$

$$\theta_c = 200^\circ$$

$$m_D = 250 \text{ kg}$$

$$r_D = 250 \text{ mm}$$

$$\theta_D = 300^\circ$$

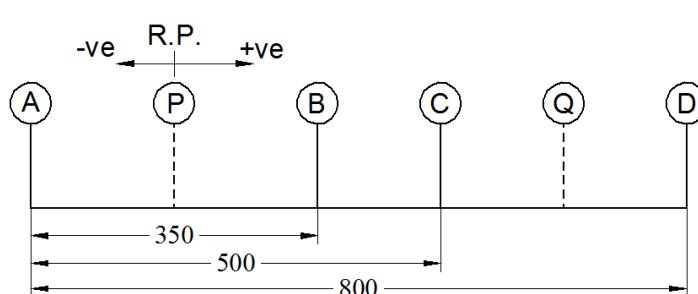


Fig. 1.9

Table 1.3

Plane	Angle	Mass (m) kg	Radius (r) m	Cent.force $\div \omega^2$ (mr) kg-m	Distance from Ref. Plane (l) m	Couple $\div \omega^2$ (mrl) kg-m ²
A (R.P.)	0°	150	0.15	22.5	-0.175	-3.94
P	θ_P	m_P	0.18	0.18 m_P	0	0
B	105°	200	0.2	40	0.175	7
C	200°	100	0.1	10	0.325	3.25
Q	θ_Q	m_Q	0.18	0.18 m_Q	0.475	0.0855 m_Q
D	300°	250	0.25	62.5	0.625	39.06

Analytical Method:

Table 1.4

$mrl\cos\theta$ (H_c)	$mrl \sin\theta$ (V_c)	$mrc\cos\theta$ (H_F)	$mr \sin\theta$ (V_F)
-3.94	0	22.5	0
0	0	$0.18 m_p \cos\theta_p$	$0.18 m_p \sin\theta_p$
-1.81	6.76	-10.35	38.64
-3.05	-1.11	-9.4	-3.42
$0.0855 m_Q \cos\theta_Q$	$0.0855 m_Q \sin\theta_Q$	$0.18 m_Q \cos\theta_Q$	$0.18 m_Q \sin\theta_Q$
19.53	-33.83	31.25	-54.13

$$\begin{aligned} \Sigma V_C &= 0 \\ 0 + 0 + 6.76 - 1.11 + 0.0855 m_Q \sin\theta_Q - 33.83 &= 0 \\ 0.0855 m_Q \sin\theta_Q &= 28.18 \\ m_Q \sin\theta_Q &= 329.59 \quad \dots \dots \dots \text{(ii)} \end{aligned}$$

$$m_Q = \sqrt{(-125.497)^2 + (329.59)^2}$$

m_Q = 352.67 kg.

$$\frac{m_o \sin \theta_o}{m_o \cos \theta_o} = \frac{329.59}{-125.497}$$

$$\tan\theta_0 = -2.626$$

$$\theta_0 = -69.15$$

$$\theta_O = 180 - 69.15$$

$$\theta_0 = 110.84^\circ$$

$$\sum H_F = 0$$

$$22.5 + 0.18 m_p \cos\theta_p - 10.35 - 9.4 + 0.18 m_Q \cos\theta_Q + 31.25 = 0$$

$$22.5 + 0.18 m_p \cos\theta_p - 10.35 - 9.4 + 0.18 (352.67) \cos 110.84^\circ + 31.25 = 0$$

$$0.18 m_p \cos\theta_p = -11.416$$

$$m_p \cos\theta_p = -63.42$$

$$\sum V_F = 0$$

$$0 + 0.18 m_p \sin\theta_p + 38.64 - 3.42 + 0.18 m_q \sin\theta_q - 54.13 = 0$$

$$0 + 0.18 m_p \sin\theta_p + 38.64 - 3.42 + 0.18 (352.67) \sin 110.84^\circ - 54.13 = 0$$

$$0.18 m_p \sin\theta_p = -40.417$$

$$m_p \sin\theta_p = -224.54$$

$$m_p = \sqrt{(-63.42)^2 + (-224.54)^2}$$

$$m_p = 233.32 \text{ kg.}$$

$$\frac{m_p \sin \theta_p}{m_p \cos \theta_p} = \frac{-224.54}{-63.42}$$

$$\tan\theta_P = 3.54$$

$$\theta_P = 74.23$$

$$\theta_P = 180 + 74.23$$

$\theta_B = 254.23^\circ$

Graphical Method :

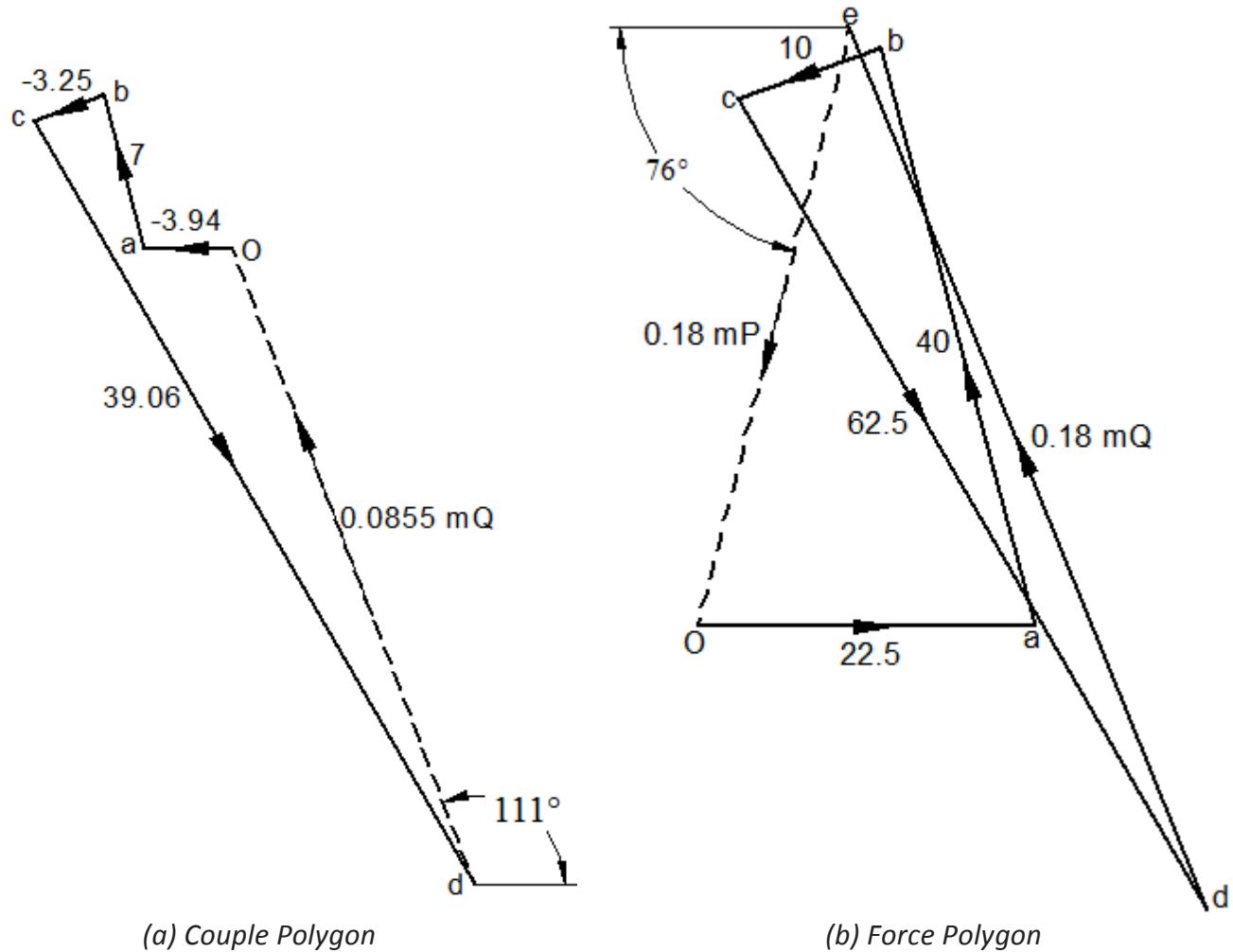


Fig. 1.10

- First of all, draw the couple polygon from the data given in Table 1.4 (column 7) as shown in Fig. 1.10 (a) to some suitable scale. The vector do represents the balanced couple. Since the balanced couple is proportional to $0.0855 m_a$, therefore by measurement,

$$0.0855 \text{ m}_Q = \text{vector } do = 30.15 \text{ kg}\cdot\text{m}^2$$

$$\text{or } m_Q = 352.63 \text{ kg.}$$

- By measurement, the angular position of m_Q is $\theta_Q = 111^\circ$ in the anticlockwise direction from mass m_A (i.e. 150 kg).
 - Now draw the force polygon from the data given in Table 1.4 (column 5) as shown in Fig. 1.10 (b). The vector eo represents the balanced force. Since the balanced force is proportional to $0.18 m_P$, therefore by measurement,

$0.18 m_P = \text{vector } eo = 41.5 \text{ kg-m}$

$$\text{Or } m_p = 230.5 \text{ kg.}$$

- By measurement, the angular position of m_p is $\theta_p = 256^\circ$ in the anticlockwise direction from mass m_A (i.e. 150kg).

Example 1.6 : A shaft carries four masses in parallel planes A, B, C and D in this order along its length. The masses at B and C are 18 kg and 12.5 kg respectively, and each has an eccentricity of 60 mm. The masses at A and D have an eccentricity of 80 mm. The angle between the masses at B and C is 100° and that between the masses at B and A is 190°, both being measured in the same direction. The axial distance between the planes A and B is 100 mm and that between B and C is 200 mm. If the shaft is in complete dynamic balance, determine:

1. The magnitude of the masses at A and D;
2. The distance between planes A and D; and
3. The angular position of the mass at D.

$$m_A = ?$$

$$r_A = 80 \text{ mm}$$

$$\theta_A = 190^\circ$$

$$m_B = 18 \text{ kg}$$

$$r_B = 60 \text{ mm}$$

$$\theta_B = 0^\circ$$

$$m_C = 12.5 \text{ kg}$$

$$r_C = 60 \text{ mm}$$

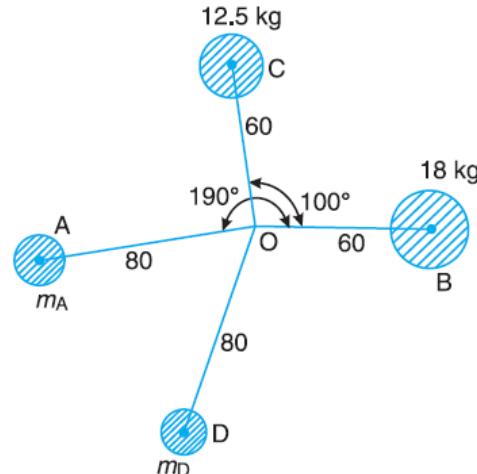
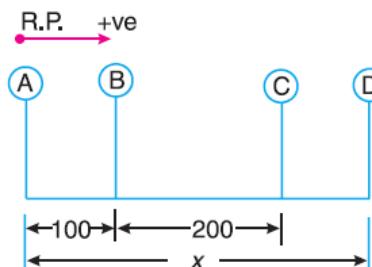
$$\theta_C = 100^\circ$$

$$m_D = ?$$

$$r_D = 80 \text{ mm}$$

$$\theta_D = ?$$

X= Distance between planes A and D.



(a) Position of planes.

(b) Angular position of masses.

Fig. 1.11

- The position of the planes and angular position of the masses is shown in Fig. 1.11 (a) and (b) respectively. The position of mass B is assumed in the horizontal direction, i.e. along OB. Taking the plane of mass A as the reference plane, the data may be tabulated as below:

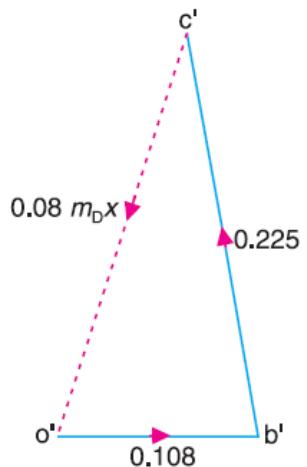
Table 1.5

Plane	Angle	Mass (m) kg	Radius (r) m	Cent.force $\div \omega^2$ (mr) kg-m	Distance from Ref. Plane (l) m	Couple $\div \omega^2$ (mrl) kg-m ²
A (R.P.)	190°	m_A	0.08	$0.08m_A$	0	0
B	0°	18	0.06	1.08	0.1	0.108
C	100°	12.5	0.06	0.75	0.3	0.225
D	θ_D	m_D	0.08	$0.08 m_D$	X	$0.08 m_D X$

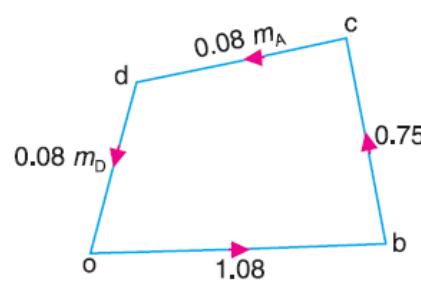
- First of all, draw the couple polygon from the data given in Table 1.5 (column 7) as shown in Fig. 1.12 (a) to some suitable scale. The closing side of the polygon (vector $c'o'$) is proportional to $0.08 m_D X$, therefore by measurement,

$$0.08 m_D X = \text{vector } c'o' = 0.235 \text{ kg-m}^2 \quad \dots \dots \dots \text{(i)}$$

- By measurement, the angular position of m_D is $\theta_D = 251^\circ$ in the anticlockwise direction from mass m_B (i.e. 18 kg).



(a) Couple Polygon



(b) Force Polygon

Fig. 1.12

- Now draw the force polygon, to some suitable scale, as shown in Fig. 1.11 (b), from the data given in Table 1.5 (column 5), as discussed below :
 - Draw vector ob parallel to OB and equal to 1.08 kg-m .
 - From point b , draw vector bc parallel to OC and equal to 0.75 kg-m .
 - For the shaft to be in complete dynamic balance, the force polygon must be a closed. Therefore from point c , draw vector cd parallel to OA and from point o draw vector od parallel to OD . The vectors cd and od intersect at d . Since the vector cd is proportional to $0.08 m_A$, therefore by measurement

$$0.08 m_A = \text{vector } cd = 0.77 \text{ kg-m}$$

$$\text{or} \quad m_A = 9.625 \text{ kg.}$$

- and vector do is proportional to $0.08 m_D$, therefore by measurement,

$$0.08 m_D = \text{vector } do = 0.65 \text{ kg-m}$$

$$\text{or} \quad m_D = 8.125 \text{ kg.}$$

- Distance between planes A and D

From equation (i),

$$0.08 m_D X = 0.235 \text{ kg-m}^2$$

$$0.08 \times 8.125 \times X = 0.235 \text{ kg-m}^2$$

$$X = 0.3615 \text{ m}$$

$$= 361.5 \text{ mm}$$

Example 1.7 : A rotating shaft carries four masses A, B, C and D which are radially attached to it. The mass centers are 30 mm, 40 mm, 35 mm and 38 mm respectively from the axis of rotation. The masses A, C and D are 7.5 kg, 5 kg and 4 kg respectively. The axial distances between the planes of rotation of A and B is 400 mm and between B and C is 500 mm. The masses A and C are at right angles to each other. Find for a complete balance,

- the angles between the masses B and D from mass A,
- the axial distance between the planes of rotation of C and D, and
- the magnitude of mass B.

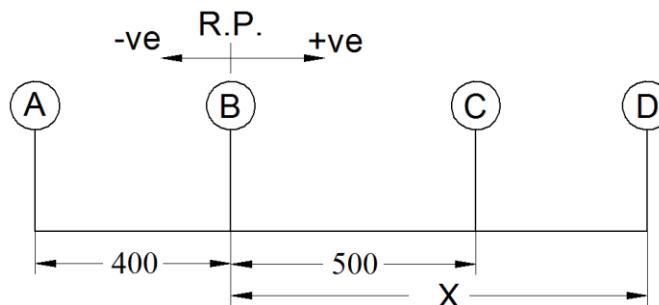
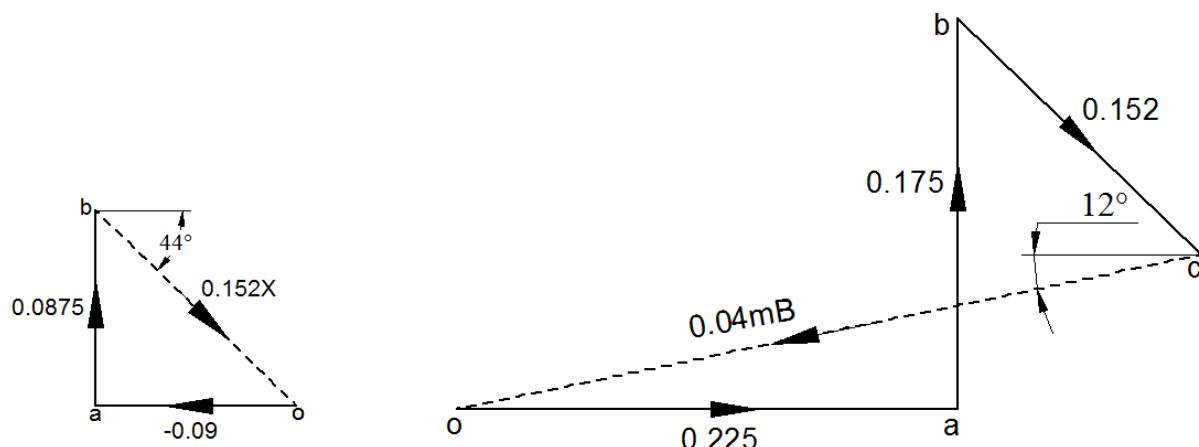


Fig. 1.13 Position of planes

Table 1.6

Plane	Angle	Mass (m) kg	Radius (r) m	Cent.force $\div \omega^2$ (mr) kg-m	Distance from Ref. Plane (l) m	Couple $\div \omega^2$ (mrl) kg-m ²
A	0°	7.5	0.03	0.225	- 0.4	-0.09
B(R.P.)	θ_B	m_B	0.04	$0.04m_B$	0	0
C	90°	5	0.035	0.175	0.5	0.0875
D	θ_D	4	0.038	0.152	X	$0.152X$



(a) Couple Polygon

(b) Force Polygon

Fig. 1.14

- First of all, draw the couple polygon from the data given in Table 1.6 (column 7) as shown in Fig. 1.14 (a) to some suitable scale. The vector bo represents the balanced couple. Since the balanced couple is proportional to $0.152X$, therefore by measurement,

$$\begin{aligned} 0.152X &= \text{vector } bo \\ &= 0.13 \text{ kg-m}^2 \\ \text{or} \quad X &= 0.855 \text{ m.} \end{aligned}$$

The axial distance between the planes of rotation of **C** and **D** = **855 – 500 = 355 mm**

- By measurement, the angular position of m_D is $\theta_D = 360^\circ - 44^\circ = 316^\circ$ in the anticlockwise direction from mass m_A (i.e. 7.5 kg).
- Now draw the force polygon from the data given in Table 1.6 (column 5) as shown in Fig. 1.14 (b). The vector co represents the balanced force. Since the balanced force is proportional to $0.04 m_B$, therefore by measurement,

$$\begin{aligned} 0.04 m_B &= \text{vector } co \\ &= 0.34 \text{ kg-m} \\ \text{or} \quad m_B &= 8.5 \text{ kg.} \end{aligned}$$

- By measurement, the angular position of m_B is $\theta_B = 180^\circ + 12^\circ = 192^\circ$ in the anticlockwise direction from mass m_A (i.e. 7.5 kg).

Example 1.8: The four masses A, B, C and D revolve at equal radii are equally spaces along the shaft. The mass B is 7 kg and radii of C and D makes an angle of 90° and 240° respectively (counterclockwise) with radius of B, which is horizontal. Find the magnitude of A, C and D and angular position of A so that the system may be completely balance. Solve problem by analytically.

Table 1.7

Plane	Angle	Mass (m) kg	Radius (r) m	Cent.force $\div \omega^2$ (mr) kg-m	Distance from Ref. Plane (l) m	Couple $\div \omega^2$ (mrl) kg-m ²
A (R.P.)	θ_A	m_A	X	m_A	0	0
B	0°	7	X	7	Y	$7Y$
C	90°	m_C	X	m_C	$2Y$	$2m_C Y$
D	240°	m_D	X	m_D	$3Y$	$3m_D Y$

$mrl\cos\theta$ (H_c)	$mrl\sin\theta$ (V_c)	$mrcos\theta$ (H_F)	$mr\sin\theta$ (V_F)
0	0	$m_A\cos\theta_A$	$m_A\sin\theta_A$
$7Y$	0	7	0
0	$2m_C Y$	0	m_C
$-1.5m_D Y$	$-2.59m_D Y$	$-0.5m_D$	$-0.866m_D$

$$\begin{aligned} \Sigma H_c &= 0 \\ 0 + 7Y + 0 - 1.5m_D Y &= 0 \\ m_D &= 7/1.5 \\ m_D &= 4.67 \text{ kg} \end{aligned}$$

$$\begin{aligned}\sum V_C &= 0 \\ 0 + 0 + 2m_C Y - 2.59m_D Y &= 0 \\ m_C &= 6.047 \text{ kg}\end{aligned}$$

$$\begin{aligned}\sum H_F &= 0 \\ m_A \cos\theta_A + 7 + 0 - 0.5m_D &= 0 \\ m_A \cos\theta_A &= -4.665\end{aligned}$$

$$\begin{aligned}\sum V_F &= 0 \\ m_A \sin\theta_A + 0 + m_C - 0.866m_D &= 0 \\ m_A \sin\theta_A &= -2.00278\end{aligned}$$

$$\begin{aligned}m_A &= \sqrt{(-4.665)^2 + (-2.00278)^2} \\ m_A &= 5.076 \text{ kg}\end{aligned}$$

$$\tan\theta_A = \frac{m_A \sin\theta_A}{m_A \cos\theta_A} = \frac{-2.00278}{-4.665} = 0.43$$

$$\theta_A = 23.23^\circ$$

$$\theta_A = 180^\circ + 23.23^\circ$$

$$\theta_A = 203.23^\circ$$

1.7 Balancing Machines

- A balancing machine is able to indicate whether a part is in balance or not and if it is not, then it measures the unbalance by indicating its magnitude and location.

1.7.1. Static Balancing Machines

- Static balancing machines are helpful for parts of small axial dimensions such as fans, gears and impellers, etc., in which the mass lies practically in a single plane.
- There are two machine which are used as static balancing machine: Pendulum type balancing machine and Cradle type balancing machine.

(i) Pendulum type balancing machine

- Pendulum type balancing machine as shown in Figure 1.15 is a simple kind of static balancing machine. The machine is of the form of a weighing machine.
- One arm of the machine has a mandrel to support the part to be balanced and the other arm supports a suspended deadweight to make the beam approximately horizontal.
- The mandrel is then rotated slowly either by hand or by a motor. As the mandrel is rotated, the beam will oscillate depending upon the unbalance of the part.
- If the unbalance is represented by a mass m at radius r , the apparent weight is greatest when m is at the position A and least when it is at B as the lengths of the arms in the two cases will be maximum and minimum.

- A calibrated scale along with the pointer can also be used to measure the amount of unbalance. Obviously, the pointer remains stationary in case the body is statically balanced.

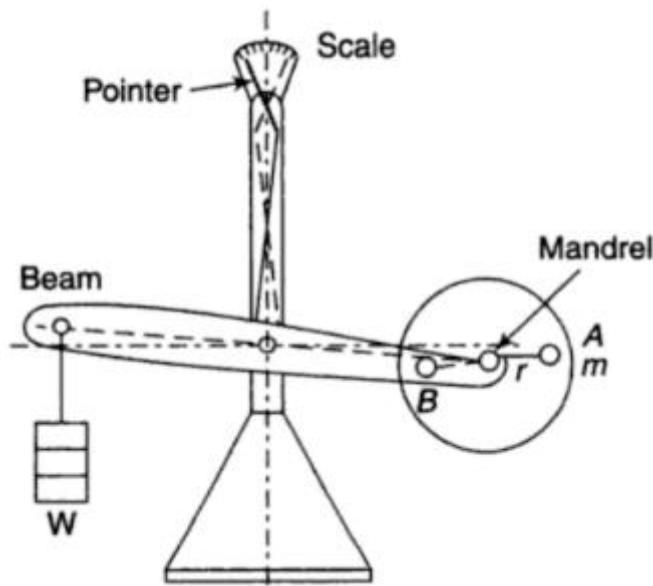


Fig. 1.15

(ii) Cradle type balancing machine

- Cradle type balancing machine as shown in fig. 1.16 is more sensitive machine than the pendulum type balancing machine.
- It consists of a cradle supported on two pivots P-P parallel to the axis of rotation of the part and held in position by two springs S-S.
- The part to be tested is mounted on the cradle and is flexibly coupled to an electric motor. The motor is started and the speed of rotation is adjusted so that it coincides with the natural frequency of the system.
- Thus, the condition of resonance is obtained under which even a small amount of unbalance generates large amplitude of the cradle.
- The moment due to unbalance = $(mr\omega^2 \cos \theta).l$ where ω is the angular velocity of rotation. Its maximum value is $mr\omega^2 l$. If the part is in static balance but dynamic unbalance, no oscillation of the cradle will be there as the pivots are parallel to the axis of rotation.

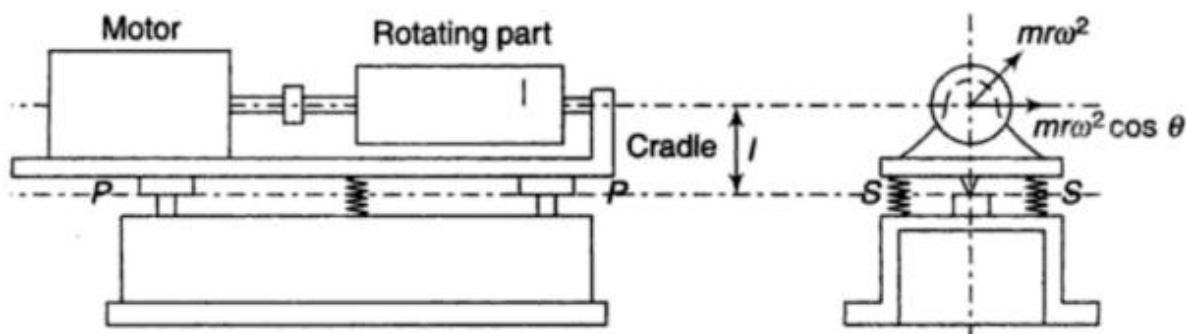


Fig. 1.16

1.7.2. Dynamic Balancing Machines

- For dynamic balancing of a rotor, two balancing or countermasses are required to be used in any two convenient planes. This implies that the complete unbalance of any rotor system can be represented by two unbalances in those two planes.
- Balancing is achieved by addition or removal of masses in these two planes, whichever is convenient. The following is a common type of dynamic balancing machine.

Pivoted-cradle Balancing Machine

- Fig 1.17 shows a pivot cradle type dynamic balancing machine. Here, part which is required to be balanced is to be mounted on cradle supported by supported rollers and it is connected to drive motor through universal coupling.
- Two planes are selected for dynamic balancing as shown in fig. 1.17 where pivots are provided about which the cradle is allowed to oscillate.
- As shown in fig 1.17, right pivot is released condition and left pivot is in locked position so as to allow the cradle and part to oscillate about the pivot.
- At the both ends of the cradle, the spring and dampers are attached such that the natural frequency can be adjusted and made equal to the motor speed. Two amplitude indicators are attached at each end of the cradle.
- The permanent magnet is mounted on the cradle which moves relative to stationary coil and generates a voltage which is directly proportional to the unbalanced couple. This voltage is amplified and read from the calibrated voltmeter and gives output in terms of kg-m.
- When left pivot is locked, the unbalance in the right correction plane will cause vibration whose amplitude is measured by the right amplitude indicator.
- After that right pivot is locked and another set of measurement is made for left hand correction plane using the amplitude indicator of the left hand side.

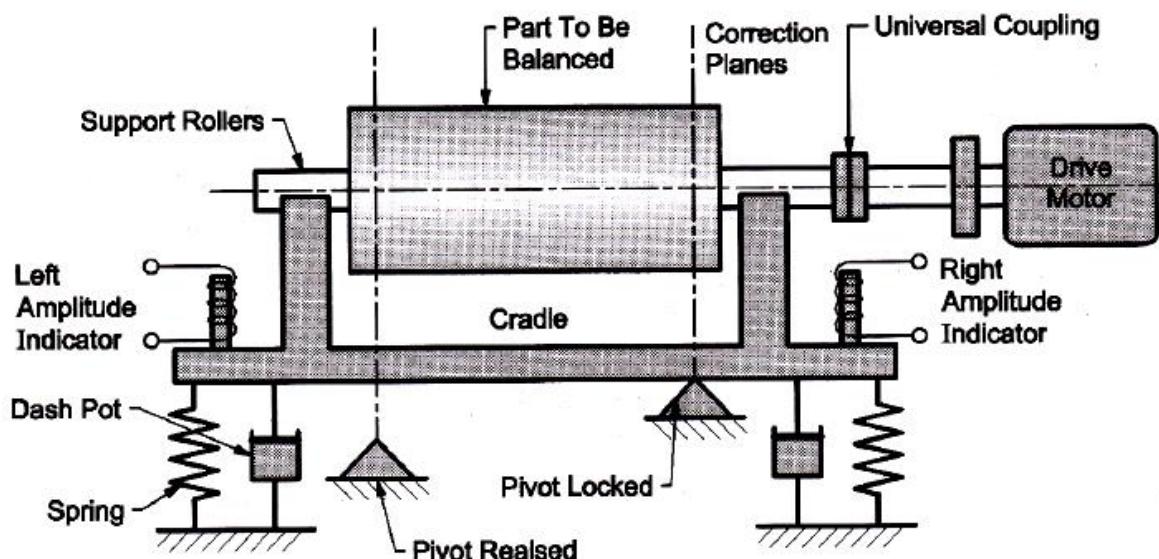
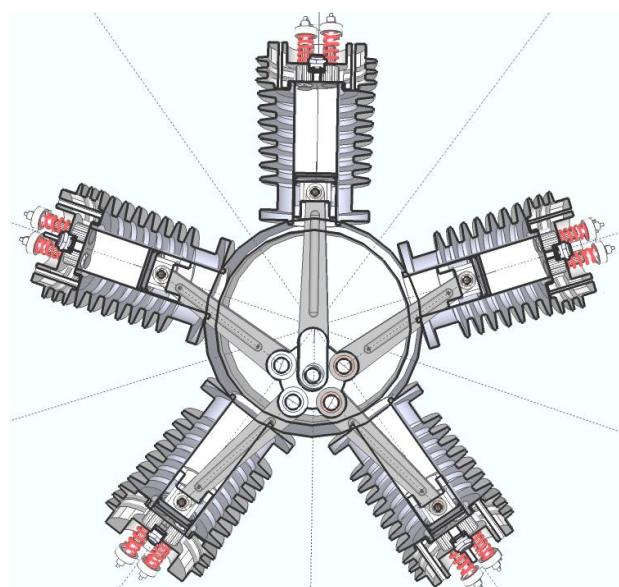
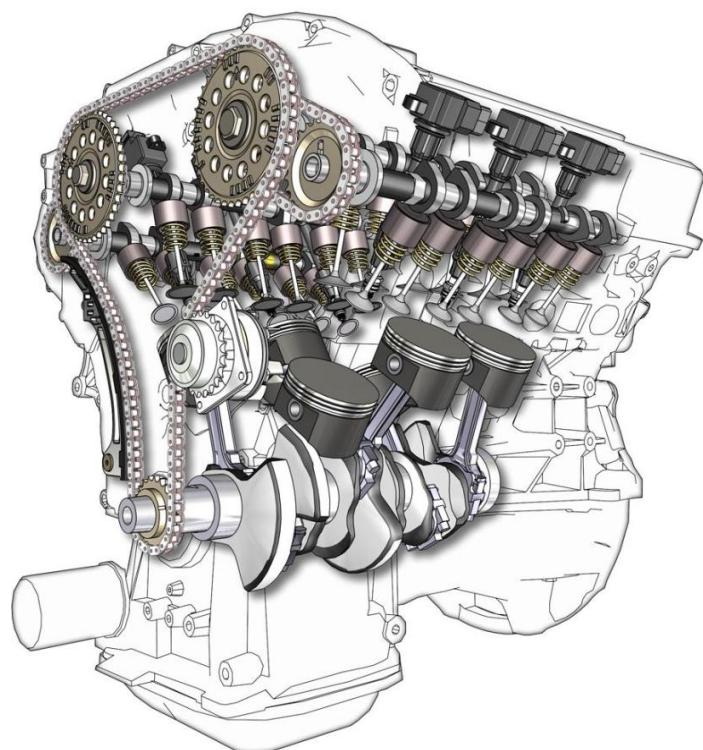


Fig. 1.17

2

DYNAMICS OF RECIPROCATING ENGINES



Course Contents

- 2.1 Slider Crank Kinematics (Analytical)
- 2.2 Gas Force and Gas Torque
- 2.3 Inertia and Shaking Forces
- 2.4 Inertia and Shaking Torques
- 2.5 Dynamically Equivalent Systems
- 2.6 Pin Forces in the Single Cylinder Engine
- 2.7 Balancing of unbalanced forces in reciprocating masses
- 2.8 Balancing of Locomotives
- 2.9 Balancing of Multi Cylinder Engine
- 2.10 Balancing of V – Engine
- 2.11 Balancing of Radial Engine

2.1 Slider Crank Kinematics (Analytical)

- There are many applications where moving parts are having reciprocating motion. For example, IC engine, shaper machine, air compressors and many more where parts are in reciprocating motion where they are subjected to continuous acceleration and retardation.
- Due to this motion, inertia force acts in the opposite direction of acceleration of the moving parts. This opposite direction force which is termed as inertia force is the unbalanced dynamic force which is acting on the reciprocating parts.
- Consider a horizontal reciprocating engine mechanism as shown in fig. 2.1. Let the crank radius be r and the connecting rod length be l . The angle of the crank is θ , and the angle that the connecting rod makes with the X axis is ϕ . For any constant crank angular velocity ω , the crank angle $\theta = \omega t$. The instantaneous piston position is x . Two right triangles rqS and lQu are constructed. Then from geometry:

$$\begin{aligned} q &= r \sin \theta = l \sin \phi \\ \theta &= \omega t \end{aligned} \quad (2.1)$$

$$\sin \phi = \frac{r}{l} \sin \omega t \quad (2.2)$$

$$\begin{aligned} s &= r \cos \omega t \\ u &= l \cos \phi \\ x &= s + u = r \cos \omega t + l \cos \phi \end{aligned} \quad (2.2)$$

$$\cos \phi = \sqrt{1 - \sin^2 \phi} = \sqrt{1 - \left(\frac{r}{l} \sin \omega t \right)^2} \quad (2.3)$$

$$x = r \cos \omega t + l \sqrt{1 - \left(\frac{r}{l} \sin \omega t \right)^2} \quad (2.4)$$

- Equ.2.4 is an exact expression for the piston position x as a function of r , l and ωt . This can be differentiated versus time to obtain exact expressions for the velocity and acceleration of the piston. For a steady-state analysis we will assume ω to be constant.

$$\dot{x} = -r\omega \left[\sin \omega t + \frac{r}{2l} \frac{\sin 2\omega t}{\sqrt{1 - \left(\frac{r}{l} \sin \omega t \right)^2}} \right] \quad (2.5)$$

$$\ddot{x} = -r\omega^2 \left\{ \cos \omega t - \frac{r \left[l^2 (1 - 2 \cos^2 \omega t) - r^2 \sin^4 \omega t \right]}{\left[l^2 - (r \sin \omega t)^2 \right]^{\frac{3}{2}}} \right\} \quad (2.6)$$

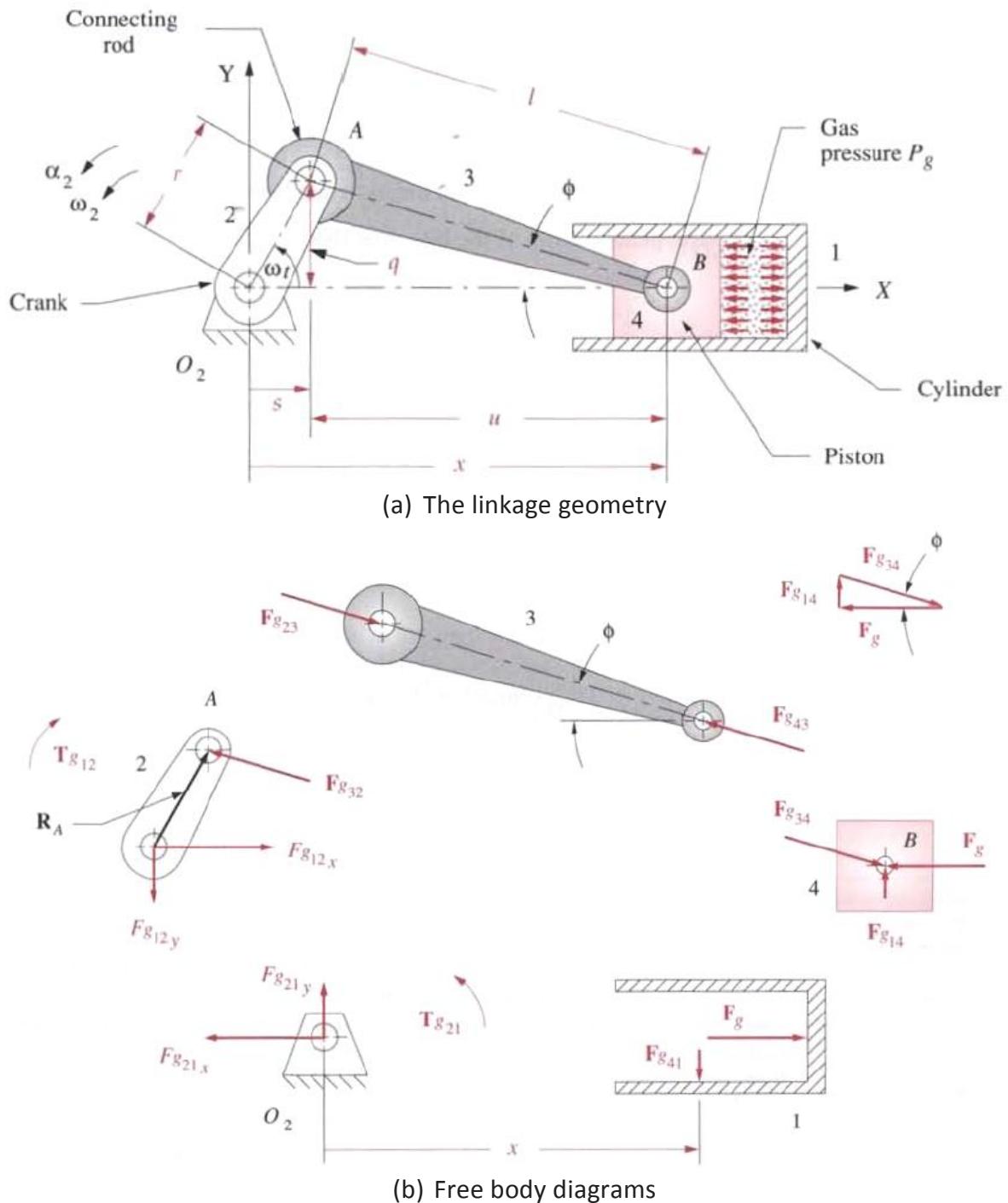


Fig. 2.1

– Applying binomial theorem and Fourier series rules for simplification, we get

$$x = l - \frac{r^2}{4l} + r \left(\cos \omega t + \frac{r}{4l} \cos 2\omega t \right) \quad (2.7)$$

– Differentiate for velocity of the piston (with constant ω):

$$\dot{x} = -r\omega \left(\sin \omega t + \frac{r}{2l} \sin 2\omega t \right) \quad (2.8)$$

– Differentiate again for acceleration (with constant ω):

$$\ddot{x} = -r\omega^2 \left(\cos \omega t + \frac{r}{l} \cos 2\omega t \right) \quad (2.9)$$

2.2 Gas Force and Gas Torque

- The gas force is due to the gas pressure from the exploding fuel-air mixture impinging on the top of the piston surface as shown in Fig. 2.1(a). Let F_g = gas force. P_g = gas pressure, A_p = area of piston and B = bore of cylinder, which is also equal to the piston diameter. Then:

$$F_g = -P_g A_p \quad \text{where, } A_p = \frac{\pi}{4} B^2$$

$$F_g = -\frac{\pi}{4} P_g B^2$$

- The negative sign is due to the choice of engine orientation in the coordinate system. The gas pressure P_g in this expression is a function of crank angle ωt and is defined by the thermodynamics of the engine.
- The gas torque is due to the gas force acting at a moment arm about the crank center in Fig. 2.1. This moment arm varies from zero to a maximum as the crank rotates.

2.3 Inertia and Shaking Forces

- The simplified lumped mass model of Fig. 2.2 can be used to develop expressions for the forces and torques due to the accelerations of the masses in the system.
- The acceleration for point B is given in equation

$$\ddot{x} = -r\omega^2 \left(\cos \omega t + \frac{r}{I} \cos 2\omega t \right)$$

- The acceleration of point A in pure rotation is obtained by differentiating the position vector R_A twice, assuming a constant crankshaft ω , which gives:

$$R_A = r \cos \omega t + r \sin \omega t \quad (2.10)$$

$$a_A = -r \omega^2 \cos \omega t - r \omega^2 \sin \omega t \quad (2.11)$$

- The total inertia force F_i is the sum of the centrifugal (inertia) force at point A and the inertia force at point B.

$$F_i = -m_A a_A - m_B a_B \quad (2.12)$$

- Breaking it into x and y components:

$$F_{ix} = -m_A (-r\omega^2 \cos \omega t) - m_B \ddot{x} \quad (2.13)$$

$$F_{iy} = -m_A (-r\omega^2 \sin \omega t) \quad (2.14)$$

- Note that only the x component is affected by the acceleration of the piston.

$$F_{ix} = -m_A (-r\omega^2 \cos \omega t) - m_B \left[-r\omega^2 \left(\cos \omega t + \frac{r}{I} \cos 2\omega t \right) \right]$$

$$F_{iy} = -m_A (-r\omega^2 \sin \omega t)$$

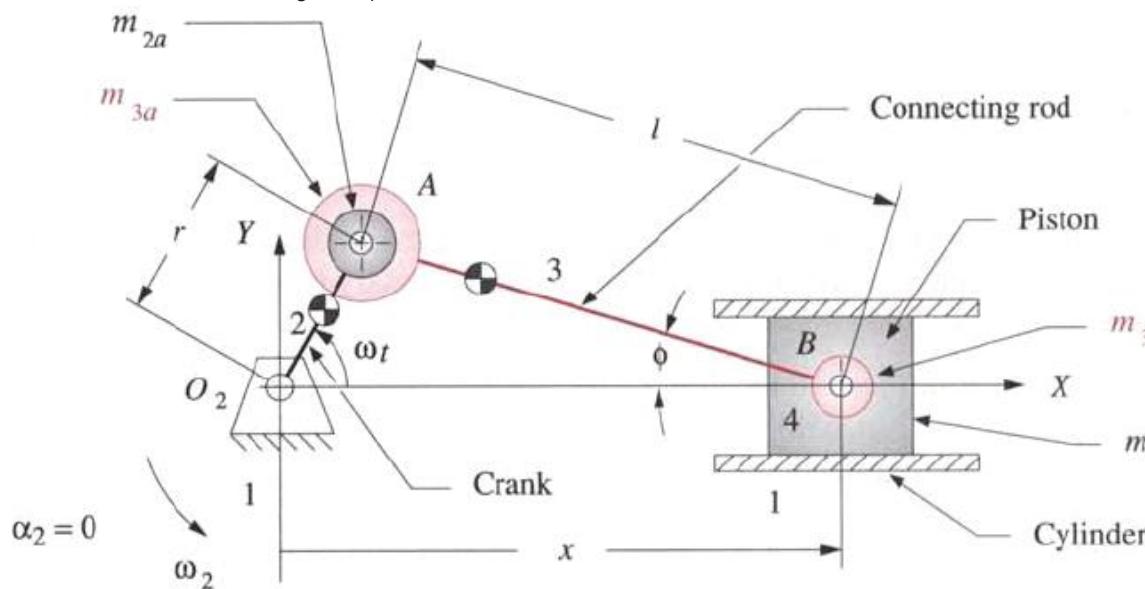
- The shaking force is defined as the sum of all forces acting on the ground plane. From the free-body diagram for link 1 in Fig. 2.2

$$\sum F_{sx} = -m_A (r\omega^2 \cos \omega t) - m_B \left[r\omega^2 \left(\cos \omega t + \frac{r}{I} \cos 2\omega t \right) \right]$$

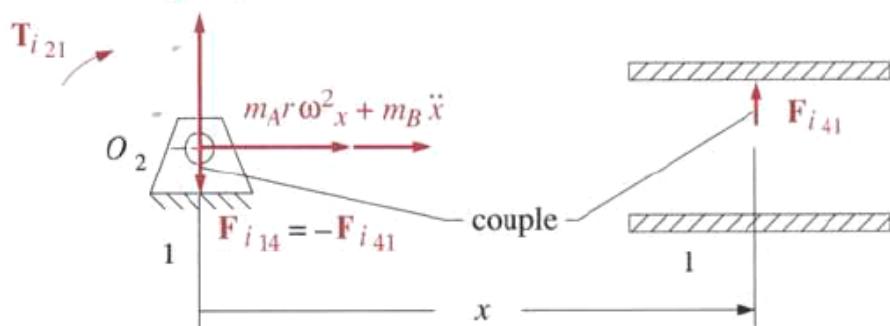
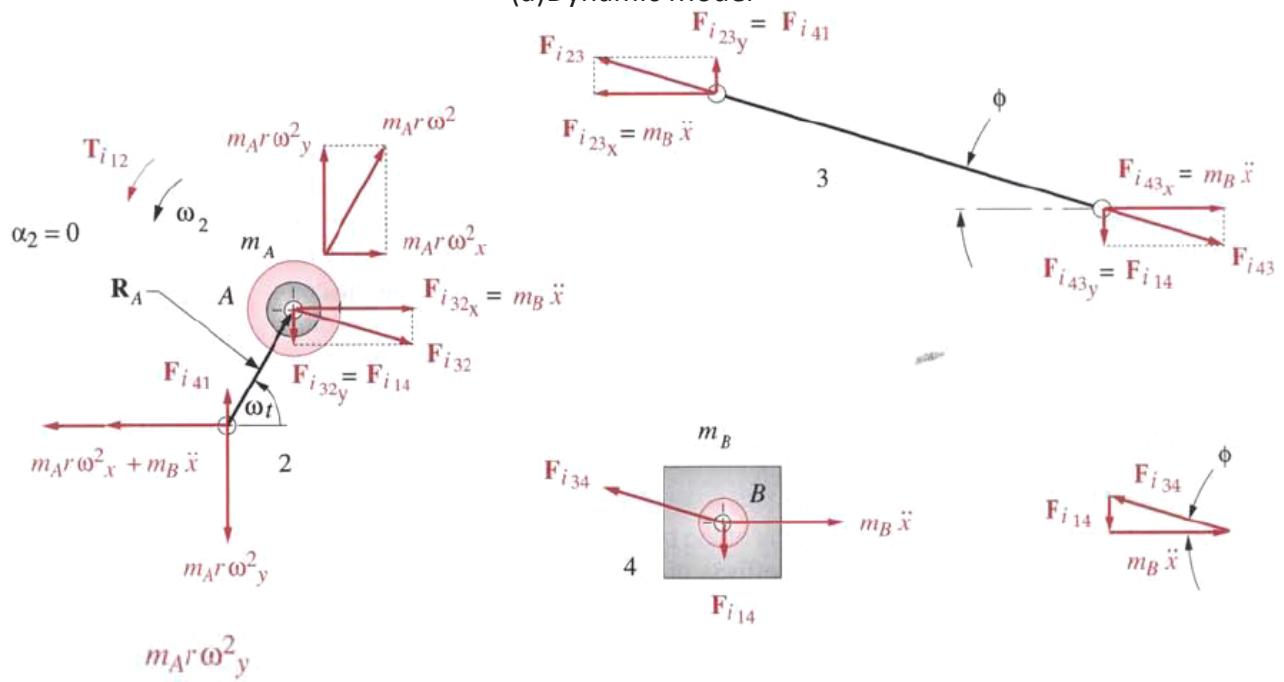
$$\sum F_{iy} = -m_A(r\omega^2 \sin\omega t)$$

- The shaking force F_S is equal to the negative of the inertia force.

$$F_S = -F_i$$



(a) Dynamic Model



(b) Free body diagrams

Fig. 2.2

2.4 Inertia and Shaking Torques

- The inertia torque results from the action of the inertia forces at a moment arm. The inertia force at point A in Fig. 2.2 has two components, radial and tangential. The radial component has no moment arm. The tangential component has a moment arm of crank radius r .
- If the crank ω is constant, the mass at A will not contribute to inertia torque. The inertia force at B has a nonzero component perpendicular to the cylinder wall except when the piston is at TDC or BDC.
- As we did for the gas torque, we can express the inertia torque in terms of the couple – F_{i41}, F_{i41} whose forces act always perpendicular to the motion of the slider (neglecting friction), and the distance x , which is their instantaneous moment arm. The inertia torque is:

$$T_{i21} = F_{i41} \cdot x = -F_{i41} \cdot x$$

- Substituting value of F_{i41} and x , we get

$$T_{i21} = -(-m_B \ddot{x} \tan \phi) \left[I - \frac{r^2}{4I} + r \left(\cos \omega t + \frac{r}{4I} \cos 2\omega t \right) \right]$$

- The shaking torque is equal to the inertia torque.

$$T_s = T_{i21}.$$

2.5 Dynamically Equivalent System

- The expression for the turning moment of the crankshaft has been obtained for the net force F on the piston. This force F may be the gas force with or without the consideration of inertia force acting on the piston.
- As the mass of the connecting rod is also significant, the inertia due to the same should also be taken into account. As neither the mass of the connecting rod is uniformly distributed nor the motion is linear, its inertia cannot be found as such. Usually, the inertia of the connecting rod is taken into account by considering a dynamically-equivalent system.
- A dynamically equivalent system means that the rigid link is replaced by a link with two point masses in such a way that it has the same motion as the rigid link when subjected to the same force, i.e., the centre of mass of the equivalent link has the same linear acceleration and the link has the same angular acceleration.

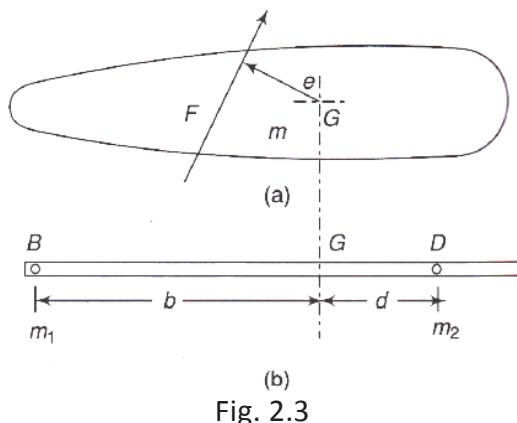


Fig. 2.3

- Fig. 2.3(a) shows a rigid body of mass m with the centre of mass at G. Let it be acted upon by a force F which produces linear acceleration a of the centre of mass as well as angular acceleration of the body as the force F does not pass through G.
- As we know,

$$F = m.a \text{ and } F.e = I. \alpha$$

Acceleration of G,

$$a = \frac{F}{m}$$

Angular acceleration of the body,

$$\alpha = \frac{F.e}{I}$$

where e = perpendicular distance of F from G

and I = moment of inertia of the body about perpendicular axis through G

- Now to have the dynamically equivalent system, let the replaced massless link [Fig. 2.3(b)] has two point masses m_1 (at B and m_2 at D) at distances b and d respectively from the centre of mass G as shown in fig. 2.3(b).
- 1. To satisfy the first condition, as the force F is to be same, the sum of the equivalent masses m_1 and m_2 has to be equal to m to have the same acceleration. Thus, $m = m_1 + m_2$.
- 2. To satisfy the second condition, the numerator $F.e$ and the denominator I must remain the same. F is already taken same, Thus, e has to be same which means that the perpendicular distance of F from G should remain same or the combined centre of mass of the equivalent system remains at G. This is Possible if

$$m_1b = m_2d$$

- To have the same moment of inertia of the equivalent system about perpendicular axis through their combined centre of mass G, we must have

$$I = m_1b^2 + m_2d^2$$

- Thus, any distributed mass can be replaced by two point masses to have the same dynamical properties if the following conditions are fulfilled:
 - The sum of the two masses is equal to the total mass.
 - The combined centre of mass coincides with that of the rod.
 - The moment of inertia of two point masses about the perpendicular axis through their combined centre of mass is equal to that of the rod.

2.6 Pin Forces in the Single Cylinder Engine

- In addition to calculating the overall effects on the ground plane of the dynamic forces present in the engine, we also need to know the magnitudes of the forces at the pin joints.
- These forces will dictate the design of the pins and the bearings at the joints. Though we were able to lump the mass due to both connecting rod and piston, or connecting rod and crank at points A and B for an overall analysis of the linkage's effects on the ground plane, we cannot do so for the pin force calculations.

- This is because the pins feel the effect of the connecting rod pulling on one "side" and the piston (or crank) pulling on the other "side" of the pin. Thus we must separate the effects of the masses of the links joined by the pins.
- We will calculate the effect of each component due to the various masses, and the gas force and then superpose them to obtain the complete pin force at each joint. The resultant bearing loads have the following components:
 - i. The gas force components.
 - ii. The inertia force due to the piston mass.
 - iii. The inertia force due to the mass of the connecting rod at the wrist pin.
 - iv. The inertia force due to the mass of the connecting rod at the crank pin.
 - v. The inertia force due to the mass of the crank at the crank pin.

2.7 Balancing of unbalanced forces in reciprocating masses

- Acceleration of reciprocating mass of a slider-crank mechanism is given by

$$a = r\omega^2 \left(\cos\theta + \frac{\cos 2\theta}{n} \right)$$

- Therefore, the force required to accelerate mass m is

$$F = mr\omega^2 \left(\cos\theta + \frac{\cos 2\theta}{n} \right)$$

$$F = mr\omega^2 \cos\theta + mr\omega^2 \frac{\cos 2\theta}{n}$$

- $mr\omega^2 \cos\theta$ is called the primary accelerating force and $mr\omega^2 \cos 2\theta/n$ is called the secondary accelerating force.

Maximum value of the primary force = $mr\omega^2$

Maximum value of the secondary force = $mr\omega^2/n$

- As n is, usually, much greater than unity, the secondary force is small, compared with the primary force and can be safely neglected for slow-speed engines.
- The inertia force due to primary accelerating force is shown in Fig. 2.4(a). In Fig. 2.4(b), the forces acting on the engine frame due to this inertia force are shown. The force exerted by the crankshaft on the main bearings has two components, F_{21}^h and F_{21}^v .
- The horizontal force F_{21}^h is an unbalanced shaking force. The vertical forces F_{21}^v and F_{41}^v balance each other, but form an unbalanced shaking couple. The magnitude and direction of this force and couple go on changing with the rotation of the crank angle θ .
- The shaking force produces linear vibration of the frame in the horizontal direction whereas the shaking couple produces an oscillating vibration.
- Thus, it is seen that the shaking force F_{21}^h is the only unbalanced force. It may hamper the smooth running of the engine and thus, effort is made to balance the same. However, it is not at all possible to balance it completely and only some modification can be made.

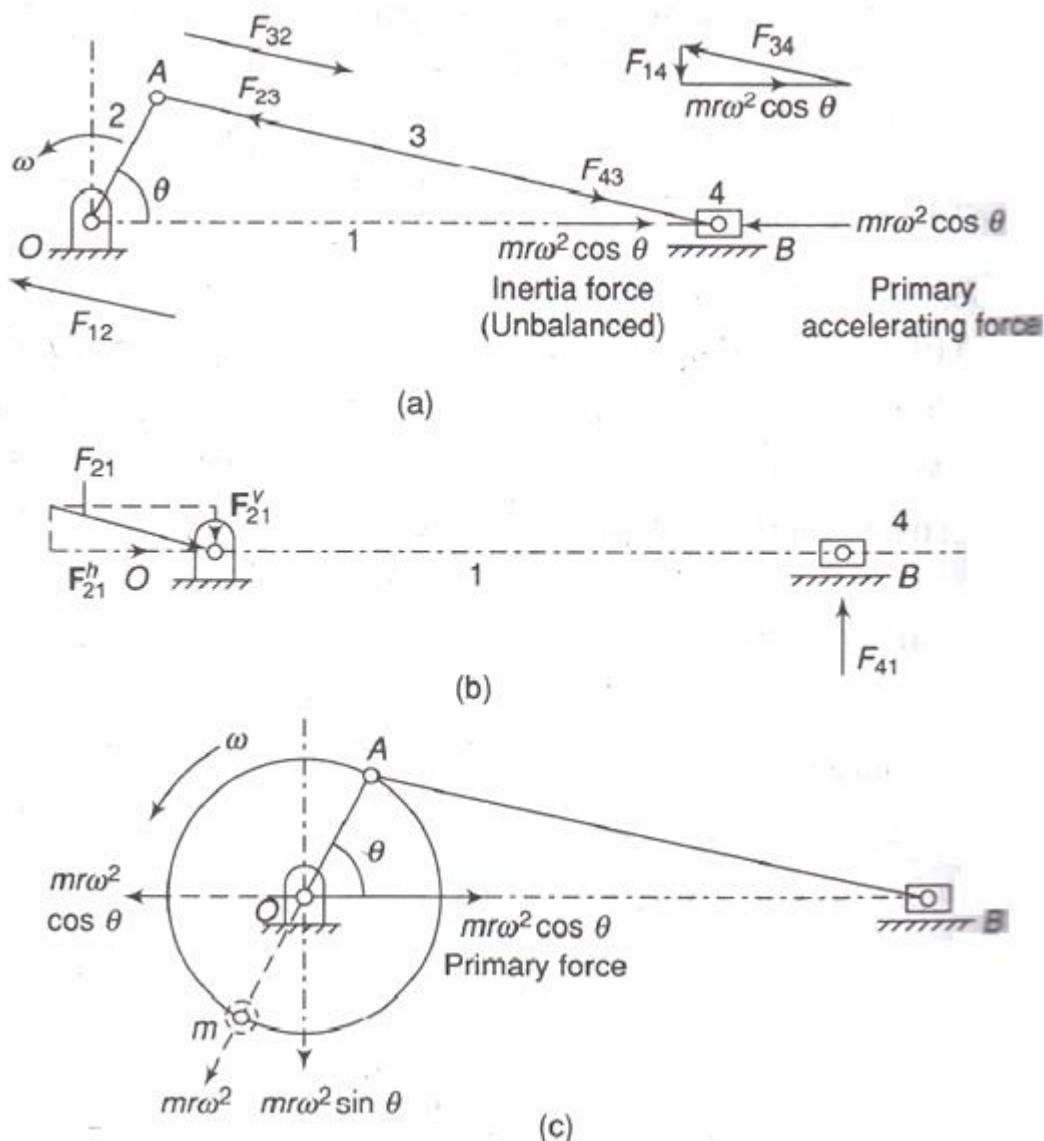


Fig. 2.4

- The usual approach of balancing the shaking force is by addition of a rotating countermass at radius r directly opposite the crank which however, provides only a partial balance. This countermass is in addition to the mass used to balance the rotating unbalance due to the mass at the crank pin.
- Fig. 2.4(c) shows the reciprocating mechanism with a countermass m at the radial distance r . The horizontal component of the centrifugal force due to the balancing mass is $mr\omega^2 \cos \theta$ in the line of stroke.
- This neutralizes the unbalanced reciprocating force. But the rotating mass also has a component $mr\omega^2 \sin \theta$ perpendicular to the line of stroke which remains unbalanced. The unbalanced force is zero at the ends of the stroke when $\theta = 0^\circ$ or 180° and maximum at the middle when $\theta = 90^\circ$.
- The magnitude or the maximum value of the unbalanced force remains the same, i.e., equal to $mr\omega^2$. Thus, instead of sliding to and fro on its mounting, the mechanism tends to jump up and down.

- To minimize the effect of the unbalanced force, a compromise is, usually, made, i.e., 2/3 of the reciprocating mass is balanced (or a value between one-half and three-quarters). If c is the fraction of the reciprocating mass Thus, balanced then

$$\text{primary force balanced by the mass} = cmr\omega^2 \cos\theta$$

$$\text{primary force unbalanced by the mass} = (1 - c)mr\omega^2 \cos\theta$$

- Vertical component of centrifugal force which remains unbalanced

$$= cmr\omega^2 \sin\theta$$

- In fact, in reciprocating engines, unbalanced forces in the direction of the line of stroke are more dangerous than the forces perpendicular to the line of stroke.
- Resultant unbalanced force at any instant

$$= \sqrt{[(1 - c)mr\omega^2 \cos\theta]^2 + [cmr\omega^2 \sin\theta]^2}$$

- The resultant unbalanced force is minimum when $c = 1/2$.
- The method just discussed above to balance the disturbing effect of a reciprocating mass is just equivalent to as if a revolving mass at the crankpin is completely balanced by providing a countermass at the same radius diametrically opposite the crank.
- Thus, if m_p is the mass at the crankpin and c is the fraction of the reciprocating mass m to be balanced, the mass at the crankpin may be considered as $(cm + m_p)$ which is to be completely balanced.

Example 2.1: The following data relate to a single - cylinder reciprocating engine:

Mass of reciprocating parts = 40 kg

Mass of revolving parts = 30 kg at crank radius

Speed = 150 rpm

Stroke = 350 mm

If 60% of the reciprocating parts and all the revolving parts are to be balanced, determine (i) balance mass required at a radius of 320 mm

(ii) unbalanced force when the crank has turned 45° from top dead centre.

$$m = 40 \text{ kg}$$

$$m_p = 30 \text{ kg}$$

$$N = 150 \text{ rpm}$$

$$r = l/2 = 175 \text{ mm}$$

$$\begin{aligned}\omega &= \frac{2\pi N}{60} = \frac{2\pi \times 150}{60} \\ &= 15.7 \text{ rad/s}\end{aligned}$$

$$\begin{aligned}\text{(i) Mass to be balanced at crank pin} &= cm + m_p \\ &= 0.6 \times 40 + 30 \\ &= 54 \text{ kg}\end{aligned}$$

$$\begin{aligned}m_C r_C &= mr \\m_C \times 320 &= 54 \times 175 \\m_C &= 29.53 \text{ kg.}\end{aligned}$$

(ii) Unbalanced force (at $\theta = 45^\circ$)

$$\begin{aligned}&= \sqrt{[(1-c)mr\omega^2 \cos\theta]^2 + [cmr\omega^2 \sin\theta]^2} \\&= \sqrt{[(1-0.6) \times 40 \times 0.175 \times (15.7)^2 \cos 45^\circ]^2 + [0.6 \times 40 \times 0.175 \times (15.7)^2 \sin 45^\circ]^2} \\&= 880.1 \text{ N.}\end{aligned}$$

Example 2.2: A single cylinder reciprocating engine has speed 240 rpm, stroke 300 mm, mass of reciprocating parts 50 kg, mass of revolving parts at 150 mm radius 30 kg. If all the mass of revolving parts and two-third of the mass of reciprocating parts are to be balanced, find the balance mass required at radius of 400 mm and the residual unbalanced force when the crank has rotated 60° from IDC.

$$N = 240 \text{ rpm}$$

$$l = 300 \text{ mm}$$

$$m = 50 \text{ kg}$$

$$m_p = 30 \text{ kg}$$

$$r = l/2 = 150 \text{ mm}$$

$$\begin{aligned}\omega &= \frac{2\pi N}{60} = \frac{2\pi \times 240}{60} \\&= 25.13 \text{ rad/s}\end{aligned}$$

(i) Mass to be balanced at crank pin = $cm + m_p$

$$\begin{aligned}&= 2/3 \times 50 + 30 \\&= 63.33 \text{ kg}\end{aligned}$$

$$m_C r_C = mr$$

$$m_C \times 400 = 63.33 \times 150$$

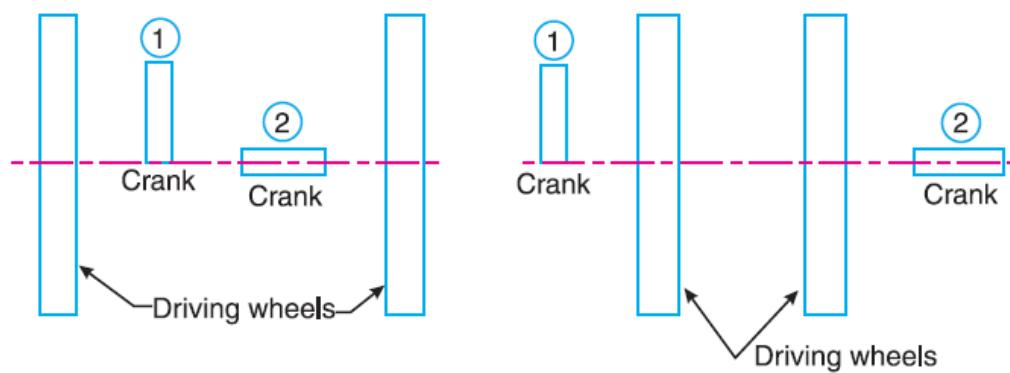
$$m_C = 23.75 \text{ kg.}$$

(ii) Unbalanced force (at $\theta = 45^\circ$)

$$\begin{aligned}&= \sqrt{[(1-c)mr\omega^2 \cos\theta]^2 + [cmr\omega^2 \sin\theta]^2} \\&= \sqrt{\left[\left(1 - \frac{2}{3}\right) \times 50 \times 0.15 \times (25.13)^2 \cos 60^\circ\right]^2 + \left[\frac{2}{3} \times 50 \times 0.15 \times (25.13)^2 \sin 60^\circ\right]^2} \\&= \sqrt{(789.36)^2 + (2734.55)^2} \\&= 2846.2 \text{ N}\end{aligned}$$

2.8 Balancing of Locomotives

- Locomotives are of two types, coupled and uncoupled. If two or more pairs of wheels are coupled together to increase the adhesive force between the wheels and the track, it is called a coupled locomotive. Otherwise, it is an uncoupled locomotive.
- Locomotives usually have two cylinders. If the cylinders are mounted between the wheels, it is called an inside cylinder locomotive and if the cylinders are outside the wheels, it is an outside cylinder locomotive. The cranks of the two cylinders are set at 90° to each other so that the engine can be started easily after stopping in any position. Balance masses are placed on the wheels in both types.
- In coupled locomotives, wheels are coupled by connecting their crankpins with coupling rods. As the coupling rod revolves with the crankpin, its proportionate mass can be considered as a revolving mass which can be completely balanced.
- Thus, whereas in uncoupled locomotives, there are four planes for consideration, two of the cylinders and two of the driving wheels, in coupled locomotives there are six planes, two of cylinders, two of coupling rods and two of the wheels. The planes which contain the coupling rod masses lie outside the planes that contain the balance (counter) masses. Also, in case of coupled locomotives, the mass required to balance the reciprocating parts is distributed among all the wheels which are coupled. This results in a reduced hammer blow.
- Locomotives have become obsolete nowadays.



(a) Inside cylinder locomotives (b) Outside cylinder locomotives

Fig. 2.5

2.8.1 Effects of Partial Balancing in Locomotives

- Reciprocating parts are only partially balanced. Due to this partial balancing of the reciprocating parts, there is an unbalanced primary force along the line of stroke and also an unbalanced primary force perpendicular to the line of stroke.

I. Hammer-blow

- Hammer-blow is the maximum vertical unbalanced force caused by the mass provided to balance the reciprocating masses. Its value is $mr\omega^2$. Thus, it varies as a square of the speed. At high speeds, the force of the hammer-blow could exceed the static load on the

wheels and the wheels can be lifted off the rail when the direction of the hammer-blow will be vertically upwards.

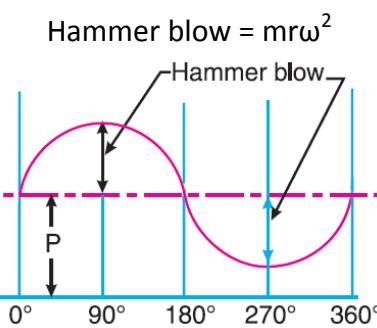


Fig. 2.6

II. Variation of Ttractive Force

- A variation in the tractive force (effort) of an engine is caused by the unbalanced portion of primary force which acts along the line of stroke of a locomotive engine.

If c is the fraction of the reciprocating mass that is balanced then

$$\text{unbalanced primary force for cylinder 1} = (1 - c) mr\omega^2 \cos\theta$$

$$\begin{aligned}\text{unbalanced primary force for cylinder 2} &= (1 - c) mr\omega^2 \cos(90^\circ + \theta) \\ &= -(1 - c) mr\omega^2 \sin\theta\end{aligned}$$

$$\begin{aligned}\text{Total unbalanced primary force or the variation in the tractive force} \\ &= (1 - c) mr\omega^2 (\cos\theta - \sin\theta)\end{aligned}$$

This is maximum when $(\cos\theta - \sin\theta)$ is maximum,

or when

$$\frac{d}{d\theta}(\cos\theta - \sin\theta) = 0$$

$$-\sin\theta - \cos\theta = 0$$

$$\sin\theta = -\cos\theta$$

$$\tan\theta = -1$$

$$\theta = 135^\circ \text{ or } 315^\circ$$

When $\theta = 135^\circ$

$$\begin{aligned}\text{Maximum variation in tractive force} &= (1 - c) mr\omega^2 (\cos 135^\circ - \sin 135^\circ) \\ &= (1 - c) mr\omega^2 \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \\ &= -\sqrt{2}(1 - c) mr\omega^2\end{aligned}$$

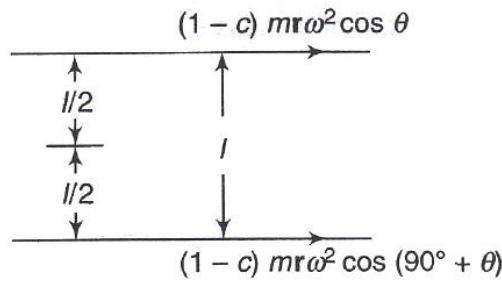
When $\theta = 315^\circ$

$$\begin{aligned}\text{Maximum variation in tractive force} &= (1 - c) mr\omega^2 (\cos 315^\circ - \sin 315^\circ) \\ &= (1 - c) mr\omega^2 \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) \\ &= \sqrt{2}(1 - c) mr\omega^2\end{aligned}$$

Thus, maximum variation = $\pm\sqrt{2}(1 - c) mr\omega^2$

III. Swaying Couple

Unbalanced primary forces along the lines of stroke are separated by a distance l apart and thus, constitute a couple. This tends to make the leading wheels sway from side to side.



- Swaying couple = moments of forces about the engine centre line

$$\begin{aligned}
 &= [(1-c)mr\omega^2 \cos \theta] \frac{l}{2} - [(1-c)mr\omega^2 \cos(90^\circ + \theta)] \frac{l}{2} \\
 &= (1-c)mr\omega^2 (\cos \theta + \sin \theta) \frac{l}{2}
 \end{aligned}$$

- This is maximum when $(\cos \theta + \sin \theta)$ is maximum.

i.e., when $\frac{d}{dt}(\cos \theta + \sin \theta) = 0$

$$-\sin \theta + \cos \theta = 0$$

$$\sin \theta = \cos \theta$$

$$\tan \theta = 1$$

$$\theta = 45^\circ \text{ or } 225^\circ$$

- When $\theta = 45^\circ$, maximum swaying couple = $\frac{1}{\sqrt{2}}(1-c)mr\omega^2 l$
- When $\theta = 225^\circ$, maximum swaying couple = $-\frac{1}{\sqrt{2}}(1-c)mr\omega^2 l$
- Thus, maximum swaying couple = $\pm \frac{1}{\sqrt{2}}(1-c)mr\omega^2 l$

Example 2.3: An inside cylinder locomotive has its cylinder centre lines 0.7 m apart and has a stroke of 0.6 m. The rotating masses per cylinder are equivalent to 150 kg at the crank pin, and the reciprocating masses per cylinder to 180 kg. The wheel centre lines are 1.5 m apart. The cranks are at right angles.

The whole of the rotating and $2/3$ of the reciprocating masses are to be balanced by masses placed at a radius of 0.6 m. Find the magnitude and direction of the balancing masses.

Find the fluctuation in rail pressure under one wheel, variation of tractive effort and the magnitude of swaying couple at a crank speed of 300 r.p.m.

$$\begin{aligned}
 I_B = I_C &= 0.6 \text{ m} & \text{or } r_B = r_C &= 0.3 \text{ m}; \\
 m_1 &= 150 \text{ kg} & r_A = r_D &= 0.6 \text{ m}; \\
 m_2 &= 180 \text{ kg}; & c &= 2/3 \\
 N &= 300 \text{ r.p.m.}
 \end{aligned}$$

$$\begin{aligned}
 \omega &= \frac{2\pi N}{60} = \frac{2\pi \times 300}{60} \\
 &= 31.42 \text{ rad/s}
 \end{aligned}$$

Equivalent mass of the rotating parts to be balanced per cylinder at the crank pin,

$$m = m_B = m_C = m_1 + c.m_2$$

$$= 150 + \frac{2}{3} \times 180 = 270 \text{ kg}$$

The magnitude and direction of balancing masses may be determined graphically as below:

- First of all, draw the space diagram to show the positions of the planes of the wheels and the cylinders, as shown in Fig. 2.7 (a). Since the cranks of the cylinders are at right angles, therefore assuming the position of crank of the cylinder *B* in the horizontal direction, draw *OC* and *OB* at right angles to each other as shown in Fig. 2.7 (b).

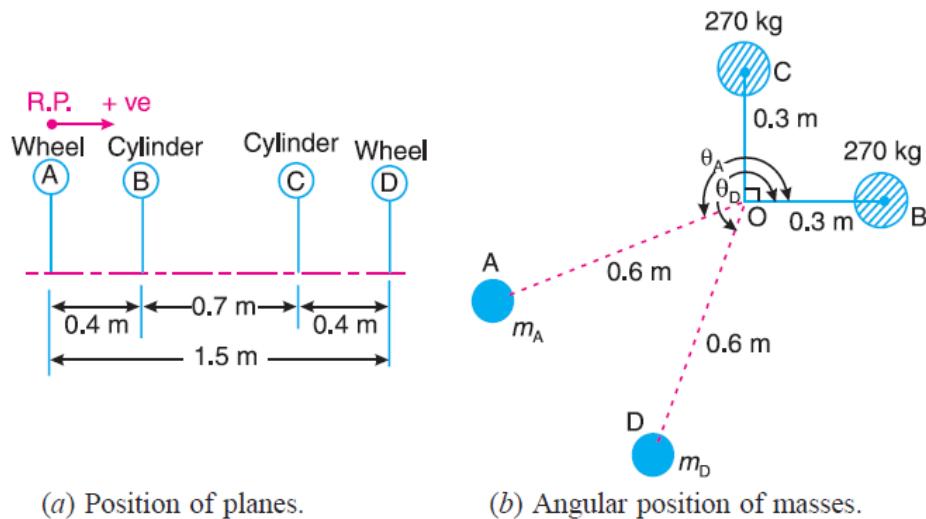


Fig. 2.7

- Tabulate the data as given in the following table. Assume the plane of wheel *A* as the reference plane.

Table 2.1

Plane	Angle	Mass (m) kg	Radius (r) m	Cent.force $\div \omega^2$ (mr) kg-m	Distance from Ref. Plane (l) m	Couple $\div \omega^2$ (mrl) kg-m ²
A (R.P.)	θ_A	m_A	0.6	$0.6 m_A$	0	0
B	0°	270	0.3	81	0.4	32.4
C	90°	270	0.3	81	1.1	89.1
D	θ_D	m_D	0.6	$0.6 m_D$	1.5	$0.9 m_D$

- Now, draw the couple polygon from the data given in Table 2.1 (column 7), to some suitable scale, as shown in Fig 2.8 (a). The closing side *c'o'* represents the balancing couple and it is proportional to $0.9 m_D$. Therefore, by measurement,

$$0.9 m_D = \text{vector } c'o' = 94.5 \text{ kg-m}^2$$

$$m_D = 105 \text{ kg}$$

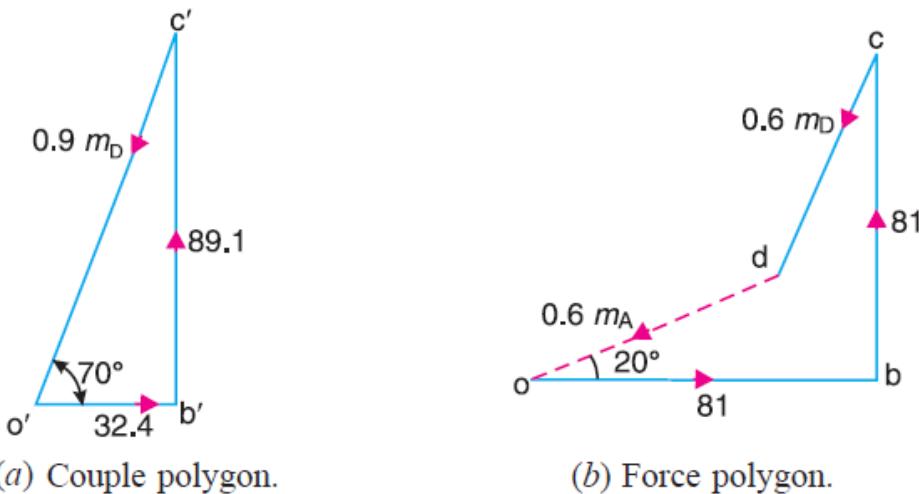


Fig. 2.8

- By measurement, the angular position of m_D is $\theta_D = 250^\circ$ in the anticlockwise direction from mass m_B .
 - In order to find the balancing mass A , draw the force polygon from the data given in Table 2.1 (column 5), to some suitable scale, as shown in Fig. 2.8 (b), The vector do represents the balancing force and it is proportional to $0.6 m_A$. Therefore by measurement,

$$0.6 m_A = \text{vector } do = 63 \text{ kg-m}$$

$$m_A = 105 \text{ kg}$$

- By measurement, the angular position of m_A is $\theta_A = 200^\circ$ in the anticlockwise direction from mass m_B .

Each balancing mass = 105 kg

$$\text{Balancing mass for rotating masses, } M = \frac{m_1}{m} \times 105 = \frac{150}{270} \times 105 \\ = 58.3 \text{ kg}$$

$$\text{Balancing mass for reciprocating masses, } M' = \frac{cm_2}{m} \times 105 = \frac{2}{3} \times \frac{180}{270} \times 105 \\ = 46.6 \text{ kg}$$

Balancing mass of 46.6 kg for reciprocating masses gives rise to centrifugal force.

$$\therefore \text{Fluctuation in rail pressure or hammer blow} = M'r\omega^2 \\ = 46.6 \times 0.6 \times (31.42)^2 \\ = 27602 \text{ N}$$

- ### **= Variation of tractive effort**

Maximum variation of tractive effort $\equiv \pm\sqrt{2}(1-c)mr\omega^2$

$$= \sqrt{2} \left(1 - \frac{2}{3}\right) \times 180 \times 0.3 \times (31.42)^2$$

= 25.123 KN

– **Swaying couple**

$$\begin{aligned}\text{Maximum swaying couple} &= \pm \frac{1}{\sqrt{2}}(1-c)mr\omega^2 l \\ &= \frac{1}{\sqrt{2}}\left(1-\frac{2}{3}\right) \times 180 \times 0.3 \times (31.42)^2 \times 0.7 \\ &= 8797 \text{ N.m}\end{aligned}$$

Example 2.4: The following data refers to two-cylinder uncoupled locomotive:

Rotating mass per cylinder	= 280 kg
Reciprocating mass per cylinder	= 300 kg
Distance between wheels	= 1400 mm
Distance between cylinder centers	= 600 mm
Diameter of treads of driving wheels	= 1800 mm
Crank radius	= 300 mm
Radius of centre of balance mass	= 620 mm
Locomotive speed	= 50 km/hr
Angle between cylinder cranks	= 90°
Dead load on each wheel	= 3.5 tonne

Determine

- Balancing mass required in planes of driving wheels if whole of the revolving and 2/3 of reciprocating mass are to be balanced
- Swaying couple
- Variation in tractive force
- Maximum and minimum pressure on the rails
- Maximum speed of locomotive without lifting the wheels from rails.

$$\begin{array}{ll} m_1 = 150 \text{ kg} & r_B = r_C = 0.3 \text{ m}; \\ m_2 = 180 \text{ kg} & r_A = r_D = 0.6 \text{ m}; \\ v = 50 \text{ km/hr} & c = 2/3 \end{array}$$

Dead load, W = 3.5tonne

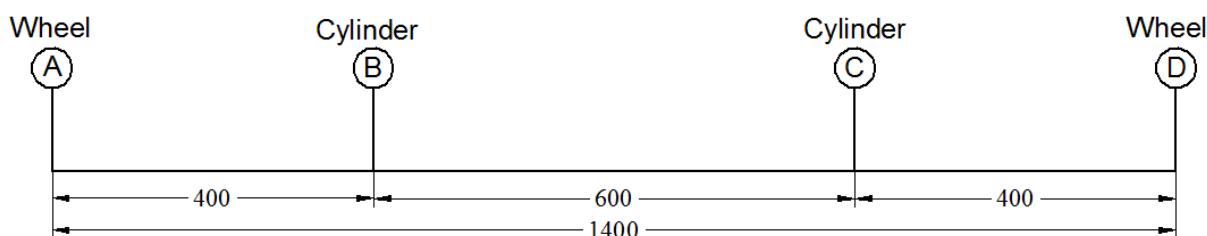


Fig. 2.9

Total mass to be balanced per cylinder, $m_B = m_C = m_p + cm$

$$\begin{aligned}&= 280 + \frac{2}{3} \times 300 \\ &= 480 \text{ kg}\end{aligned}$$

Table 2.2

Plane	Angle	Mass (m) kg	Radius (r) m	Cent.force $\div \omega^2$ (mr) kg-m	Distance from Ref. Plane (l) m	Couple $\div \omega^2$ (ml) kg-m ²
A (R.P.)	θ_A	m_A	0.62	$0.62m_A$	0	0
B	0°	480	0.3	144	0.4	57.6
C	90°	480	0.3	144	1	144
D	θ_D	m_D	0.62	$0.62m_D$	1.4	$0.868m_D$

- Draw the couple polygon from the data given in Table 2.2 (column 7), to some suitable scale, as shown in Fig 2.10 (a). The closing side *bo* represents the balancing couple and it is proportional to $0.868m_D$. Therefore, by measurement,

$$0.868m_D = \text{vector } bo = 155.1 \text{ kg-m}^2$$

$$m_D = 178.68 \text{ kg}$$

- By measurement, the angular position of m_D is $\theta_D = 248^\circ$ in the anticlockwise direction from mass m_B .
- Draw the force polygon from the data given in Table 2.2 (column 5), to some suitable scale, as shown in Fig 2.10 (b). The closing side *co* represents the balancing couple and it is proportional to $0.62m_A$. Therefore, by measurement,

$$0.62m_A = \text{vector } co = 110.78 \text{ kg-m}^2$$

$$m_A = 178.68 \text{ kg}$$

- By measurement, the angular position of m_A is $\theta_A = 202^\circ$ in the anticlockwise direction from mass m_B .

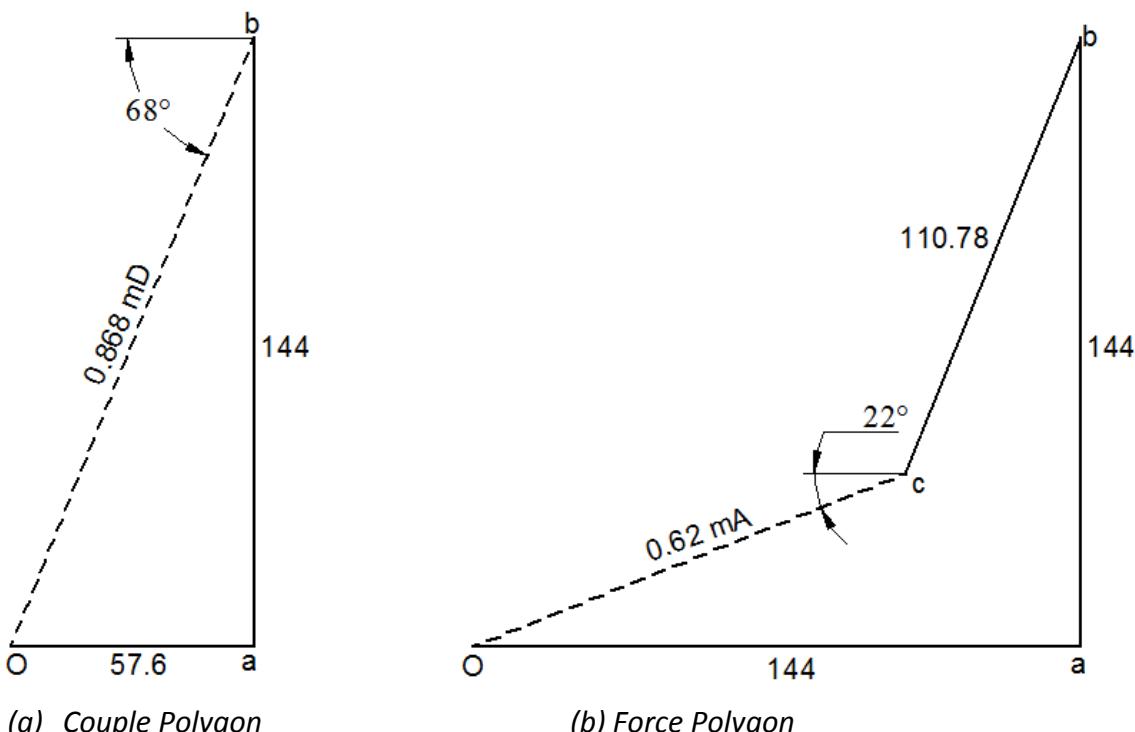


Fig. 2.10

$$v = r\omega$$

$$\omega = \frac{v}{r} = \frac{50 \times 10^6}{60 \times 60} \times \frac{1}{1800/2}$$

$$= 15.43 \text{ rad/s}$$

$$\begin{aligned}\text{Swaying couple} &= \pm \frac{1}{\sqrt{2}}(1-c)mr\omega^2 l \\ &= \frac{1}{\sqrt{2}}\left(1 - \frac{2}{3}\right) \times 300 \times 0.3 \times (15.43)^2 \times 0.6 \\ &= \mathbf{3030.3 \text{ N.m}}\end{aligned}$$

$$\begin{aligned}\text{Variation of tractive effort} &= \pm \sqrt{2}(1-c)mr\omega^2 \\ &= \sqrt{2}\left(1 - \frac{2}{3}\right) \times 300 \times 0.3 \times (15.43)^2 \\ &= \mathbf{10100 \text{ N}}\end{aligned}$$

Balance mass for reciprocating parts only

$$= 178.7 \times \frac{\frac{2}{3} \times 300}{480} = 74.46 \text{ kg}$$

$$\begin{aligned}\text{Hammer blow} &= mr\omega^2 \\ &= 74.46 \times 0.62 \times (15.43)^2 = 10991 \text{ N}\end{aligned}$$

$$\begin{aligned}\text{Dead load} &= 3.5 \times 1000 \times 9.81 \\ &= 34335 \text{ N}\end{aligned}$$

$$\begin{aligned}\text{Maximum pressure on rails} &= 34335 + 10991 \\ &= \mathbf{45326 \text{ N}}\end{aligned}$$

$$\begin{aligned}\text{Minimum pressure on rails} &= 34335 - 10991 \\ &= \mathbf{23344 \text{ N}}\end{aligned}$$

Maximum speed of the locomotive without lifting the wheels from the rails will be when the dead load becomes equal to the hammer blow

$$\begin{aligned}74.46 \times 0.62 \times \omega^2 &= 34335 \\ \omega &= 27.27 \text{ rad/s}\end{aligned}$$

Velocity of wheels = $r\omega$

$$\begin{aligned}&= \left(\frac{1.8}{2} \times 27.27 \right) \text{ m/s} \\ &= \left(27.27 \times \frac{1.8}{2} \times \frac{60 \times 60}{1000} \right) \text{ km/hr}\end{aligned}$$

$$\mathbf{V = 88.36 \text{ km / hr}}$$

Example 2.5: The three cranks of a three cylinder locomotive are all on the same axle and are set at 120° . The pitch of the cylinders is 1 meter and the stroke of each piston is 0.6 m. The reciprocating masses are 300 kg for inside cylinder and 260 kg for each outside cylinder and the planes of rotation of the balance masses are 0.8 m from the inside crank.

If 40% of the reciprocating parts are to be balanced, find :

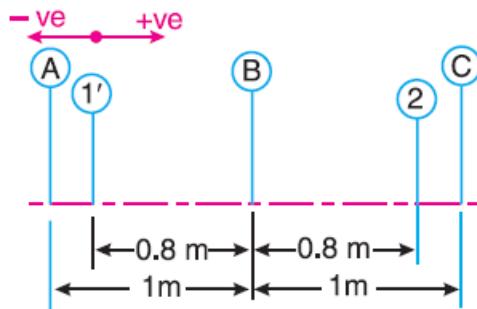
- The magnitude and the position of the balancing masses required at a radius of 0.6 m
- The hammer blow per wheel when the axle makes 6 r.p.s.

$$I_A = I_B = I_C = 0.6 \text{ m} \quad \text{or} \quad r_A = r_B = r_C = 0.3 \text{ m} ;$$

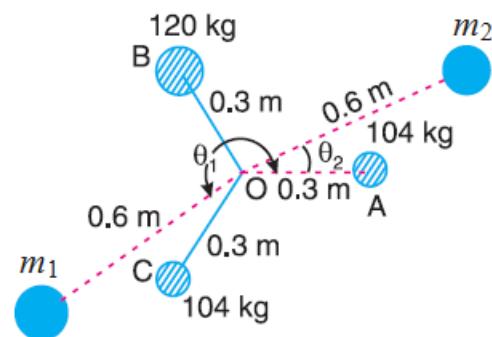
$$m_I = 300 \text{ kg} \quad r_1 = r_2 = 0.6 \text{ m} ;$$

$$m_O = 260 \text{ kg} \quad c = 40\% = 0.4 ;$$

$$N = 6 \text{ r.p.s.} = 6 \times 2 \pi = 37.7 \text{ rad/s}$$



(a) Position of planes.



(b) Position of cranks.

Fig. 2.11

Since 40% of the reciprocating masses are to be balanced, therefore mass of the reciprocating parts to be balanced for each outside cylinder,

$$\begin{aligned} m_A &= m_C = c \times m_O = 0.4 \times 260 \\ &= 104 \text{ kg} \end{aligned}$$

and mass of the reciprocating parts to be balanced for inside cylinder,

$$\begin{aligned} m_B &= c \times m_I = 0.4 \times 300 \\ &= 120 \text{ kg} \end{aligned}$$

Table 2.3

Plane	Angle	Mass (m) Kg	Radius (r) m	Cent.force $\div \omega^2$ (mr) kg-m	Distance from Ref. Plane (l) m	Couple $\div \omega^2$ (mrl) kg-m ²
A	0°	104	0.3	31.2	-0.2	-6.24
1 (R.P.)	θ_1	m_1	0.6	$0.6 m_1$	0	0
B	120°	120	0.3	36	0.8	28.8
2	θ_2	m_2	0.6	$0.6 m_2$	1.6	$0.96 m_2$
C	240°	104	0.3	31.2	1.8	56.16

- Draw the couple polygon with the data given in Table 2.3 (column 7), to some suitable scale, as shown in Fig. 2.12 (a). The closing side $c'o'$ represents the balancing couple and it is proportional to $0.96 m_2$. Therefore, by measurement,

$$0.96 m_2 = \text{vector } c'o' = 55.2 \text{ kg-m}^2$$

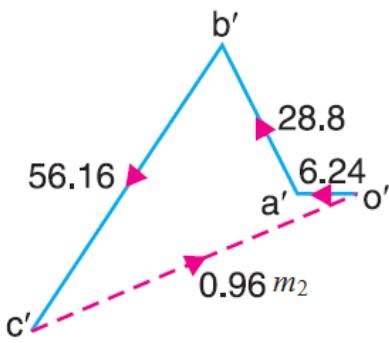
$$m_2 = 57.5 \text{ kg.}$$

- By measurement, the angular position of m_2 is $\theta_2 = 24^\circ$ in the anticlockwise direction from mass m_A .
- Draw the force polygon with the data given in Table 2.3 (column 5), to some suitable scale, as shown in Fig. 2.12 (b). The closing side co represents the balancing force and it is proportional to $0.6 m_1$. Therefore, by measurement,

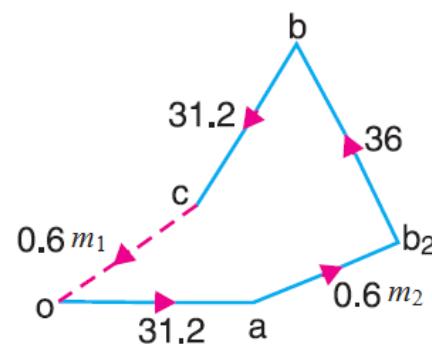
$$0.6 m_1 = \text{vector } co = 34.5 \text{ kg-m}$$

$$m_1 = 57.5 \text{ kg.}$$

- By measurement, the angular position of m_1 is $\theta_1 = 215^\circ$ in the anticlockwise direction from mass m_A .



(a) Couple polygon.



(b) Force polygon.

Fig. 2.12

$$\begin{aligned} \text{Hammer blow per wheel} &= mr\omega^2 \\ &= 57.5 \times 0.6 \times (37.7)^2 \\ &= 49035 \text{ N.} \end{aligned}$$

Example 2.6: A two cylinder locomotive has the following specifications;

Reciprocating mass per cylinder = 306 Kg

Crank radius = 300 mm

Angle between cranks = 90°

Driving wheels diameter = 1800 mm

Distance between cylinder centers = 650 mm

Distance between driving wheel planes = 1550 mm

- Determine (a) The fraction of reciprocating masses to be balanced, if the hammer blow is not to exceed 46 KN at 96.5 Km/hr.
- (b) The variation in tractive force.
- (c) The maximum swaying couple.

$$m = 300 \text{ kg}$$

$$D = 1.8 \text{ m or } R = 0.9 \text{ m}$$

$$r = 0.3 \text{ m}$$

$$\text{Hammer blow} = 46 \text{ kN}$$

$$v = 96.5 \text{ km/h} = 26.8 \text{ m/s}$$

$$\text{The mass of the reciprocating parts to be balanced} = c.m = 300c \text{ kg}$$

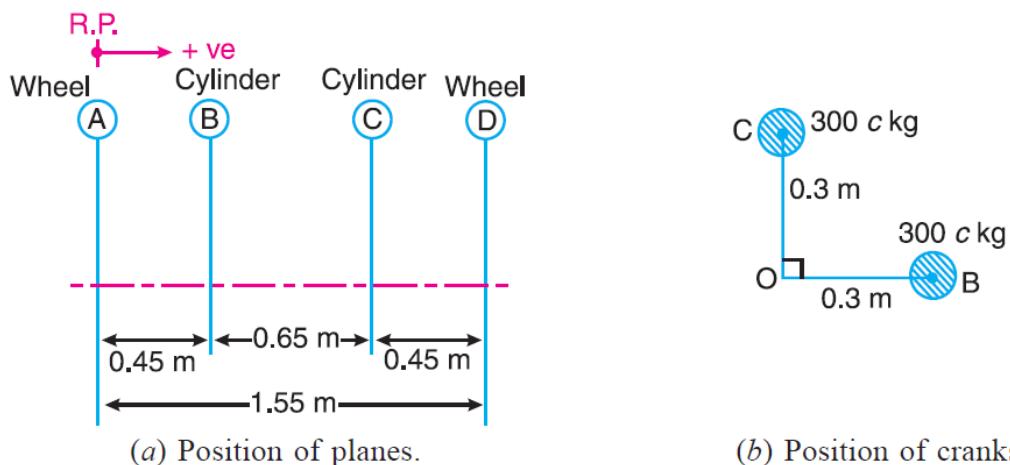


Fig. 2.13

Table 2.4

Plane	Angle	Mass (m) kg	Radius (r) m	Cent. force $\div \omega^2$ (mr) kg-m	Distance from Ref. Plane (l) m	Couple $\div \omega^2$ (mrl) kg-m ²
A (R.P.)	θ_A	m_A	r_A	$m_A r_A$	0	0
B	0°	$300c$	0.3	$90c$	0.45	$40.5c$
C	90°	$300c$	0.3	$90c$	1.1	$99c$
D	θ_D	m_D	r_D	$m_D r_D$	1.55	$1.55m_D r_D$

Now the couple polygon, to some suitable scale, may be drawn the data given in Table 2.4 (column 7), as shown in Fig. 2.14. The closing side of the polygon (vector $c'o'$) represents the balancing couple and is proportional to $1.55 B.b$.

From the couple polygon,

$$1.55m_D r_D = \sqrt{(40.5c)^2 + (99c)^2} = 107c$$

$$m_D r_D = 69c$$

$$\text{Angular speed, } \omega = v/R$$

$$= 26.8/0.9$$

$$= 29.8 \text{ rad/s}$$

$$\text{Hammer blow} = m_D r_D \omega^2$$

$$46000 = 69c(29.8)^2$$

$$c = 0.751$$

$$\text{Variation of tractive effort} = \pm \sqrt{2}(1-c)m r \omega^2$$

$$= \sqrt{2}(1-0.751) \times 300 \times 0.3 \times (29.8)^2$$

$$= 28140 \text{ N}$$

$$\text{Swaying couple} = \pm \frac{1}{\sqrt{2}}(1-c)m r \omega^2 l$$

$$= \frac{1}{\sqrt{2}}(1-0.751) \times 300 \times 0.3 \times (29.8)^2 \times 0.65 = 9148 \text{ N.m}$$

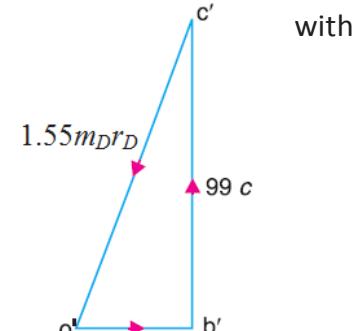


Fig. 2.14

Example 2.7: The following data apply to an outside cylinder uncoupled locomotive:

Mass of rotating parts per cylinder	= 360 kg
Mass of reciprocating parts per cylinder	= 300 kg
Angle between cranks	= 90°
Crank radius	= 0.3 m
Cylinder centres	= 1.75 m
Radius of balance masses	= 0.75 m
Wheel centres	= 1.45 m.

If whole of the rotating and two-thirds of reciprocating parts are to be balanced in planes of the driving wheels, find:

- (a) Magnitude and angular positions of balance masses,
- (b) Speed in km/hr at which the wheel will lift off the rails when the load on each driving wheel is 30 kN and the diameter of tread of driving wheels is 1.8 m, and
- (c) Swaying couple at speed arrived at in (b) above.

$$m_1 = 360 \text{ kg} \quad r_A = r_D = 0.3 \text{ m}$$

$$m_2 = 300 \text{ kg} \quad r_B = r_C = 0.75 \text{ m}$$

$$c = 2 / 3.$$

The equivalent mass of the rotating parts to be balanced per cylinder,

$$m = m_A = m_D = m_1 + c.m_2$$

$$= 360 + \frac{2}{3} \times 300$$

$$= 560 \text{ kg}$$

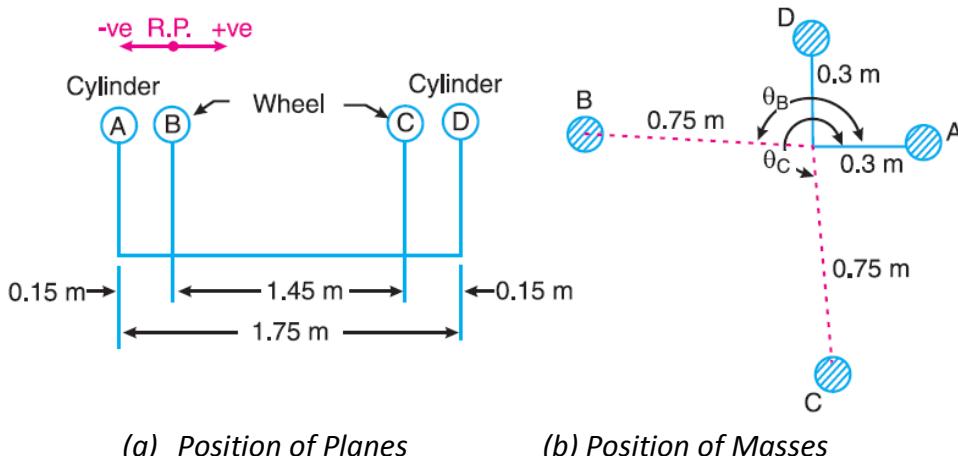


Fig. 2.15

Table 2.5

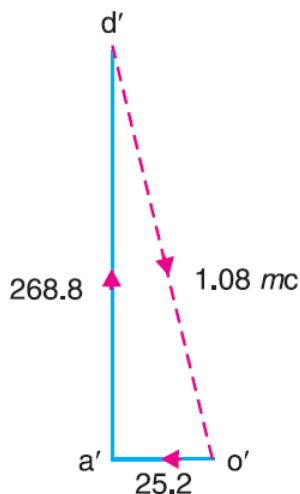
Plane	Angle	Mass (m) Kg	Radius (r) m	Cent.force $\div \omega^2$ (mr) kg-m	Distance from Ref. Plane (l) m	Couple $\div \omega^2$ (mrl) kg-m ²
A	0°	560	0.3	168	-0.15	-25.2
B (R.P.)	θ_B	m_B	0.75	$0.75 m_B$	0	0
C	θ_C	m_C	0.75	$0.75 m_C$	1.45	$1.08 m_C$
D	90°	560	0.3	168	1.6	268.8

- Draw the couple polygon with the data given in Table 2.5 column (7), to some suitable scale as shown in Fig. 2.16(a). The closing side $d'o'$ represents the balancing couple and it is proportional to $1.08 m_C$. Therefore, by measurement,

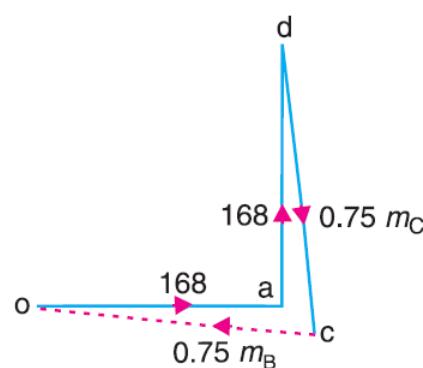
$$1.08 m_C = 269.6 \text{ kg-m}^2$$

$$m_C = 249 \text{ kg}$$

- By measurement, the angular position of m_C is $\theta_C = 275^\circ$ in the anticlockwise direction from mass m_A .



(a) Couple Polygon



(b) Force Polygon

Fig. 2.16

- Draw the force polygon with the data given in Table 2.5 column (5), to some suitable scale as shown in Fig. 2.16(b). The closing side co represents the balancing force and it is proportional to $0.75 m_B$. Therefore, by measurement,

$$0.75 m_B = 186.75 \text{ kg-m}$$

$$m_B = 249 \text{ kg}$$

- By measurement, the angular position of m_B is $\theta_B = 174.5^\circ$ in the anticlockwise direction from mass m_A .

Speed at which the wheel will lift off the rails

Given : $P = 30 \text{ kN} = 30000 \text{ N}$

$D = 1.8 \text{ m}$

ω = Angular speed at which the wheels will lift off the rails in rad/s, and

v = Corresponding linear speed in km/h.

Each balancing mass = $m_B = m_C = 249 \text{ kg}$

$$\text{Balancing mass for reciprocating parts, } M = \frac{cm_2}{m} \times 249 = \frac{2}{3} \times \frac{300}{560} \times 249 = 89 \text{ kg.}$$

$$\omega = \sqrt{\frac{P}{Mr}} = \sqrt{\frac{30 \times 10^3}{89 \times 0.75}} = 21.2 \text{ rad/s} \quad (r = r_B = r_C)$$

$$\begin{aligned} v &= \omega \times D / 2 = 21.2 \times 1.8 / 2 \\ &= 19.08 \text{ m/s} \\ &= 19.08 \times 3600 / 1000 = 68.7 \text{ km/h} \end{aligned}$$

Swaying couple at speed $\omega = 21.1 \text{ rad/s}$

$$\begin{aligned}\text{Swaying couple} &= \pm \frac{1}{\sqrt{2}}(1-c)m_2r\omega^2l \\ &= \frac{1}{\sqrt{2}}\left(1 - \frac{2}{3}\right) \times 300 \times 0.3 \times (21.2)^2 \times 1.75 \\ &= \mathbf{16687 \text{ N.m}}\end{aligned}$$

Example 2.8: The following data refers to a four-coupled wheel locomotive with two inside cylinder

Pitch of cylinders	= 600 mm
Reciprocating mass/cylinder	= 350 kg
Revolving mass/cylinder	= 260 kg
Distance between driving wheels	= 1.6 m
Distance between coupling rods	= 2 m
Diameter of driving wheels	= 1.9 m
Revolving parts for each coupling rod crank	= 130 kg
Engine crank radius	= 300 mm
Coupling rod crank radius	= 240 mm

Distance of centre of balance mass in planes of driving wheels from axle centre = 750 mm

Angle between engine cranks = 90°

Angle between coupling rod crank with adjacent engine crank = 180°

The balanced mass required for the reciprocating parts is equally divided between each pair of coupled wheels. Determine the

- (i) Magnitude and position of the balance mass required to balance two-third of reciprocating and whole of the revolving parts
- (ii) Hammer-blow and the maximum variation of tractive force when the locomotive speed is 80 km/h

$$\begin{array}{lll}m_1 = m_6 = 130 \text{ kg} & r_1 = r_6 = 0.24 \text{ m} & \theta_3 = 0^\circ \\ l_1 = -0.2 \text{ m} & r_2 = r_5 = 0.75 \text{ m} & \theta_4 = 90^\circ \\ l_3 = 0.5 \text{ m} & r_3 = r_4 = 0.3 \text{ m} & \theta_1 = 180^\circ \\ l_4 = 1.1 \text{ m} & & \theta_6 = 270^\circ \\ l_5 = 1.6 \text{ m} & & \\ l_6 = 1.8 \text{ m} & & \end{array}$$

Leading wheels: Balance mass on each leading wheel = $m_p + \frac{1}{2}cm$

$$= 260 + \frac{1}{2} \left(\frac{2}{3} \times 315 \right)$$

$$m_3 = m_4 = 365 \text{ kg}$$

$$m_1 r_1 l_1 = 130 \times 0.24 \times (-0.2) = -6.24$$

$$m_3 r_3 l_3 = 365 \times 0.3 \times 0.5 = 54.75$$

$$m_4 r_4 l_4 = 365 \times 0.3 \times 1.1 = 120.45$$

$$m_5 r_5 l_5 = m_5 \times 0.75 \times 1.6 = 1.2 m_5$$

$$m_6 r_6 l_6 = 130 \times 0.24 \times 1.8 = 56.16$$

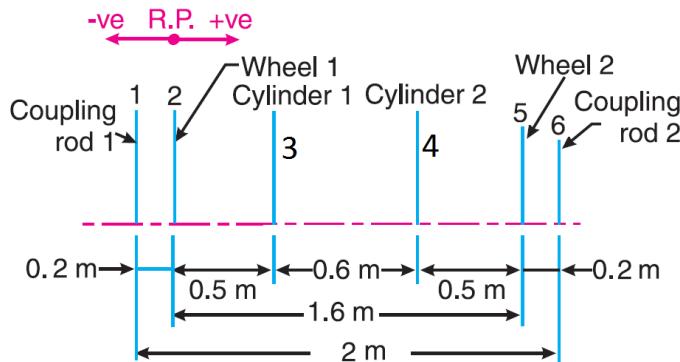


Fig. 2.17

$$\sum mrl = 0$$

$$-6.24 \cos 180^\circ + 54.75 \cos 0^\circ + 120.45 \cos 90^\circ + 56.16 \cos 270^\circ = 0$$

and

$$-6.24 \sin 180^\circ + 54.75 \sin 0^\circ + 120.45 \sin 90^\circ + 56.16 \sin 270^\circ = 0$$

Squaring, adding and then solving,

$$1.2m_5 = \sqrt{(-6.24 \cos 180^\circ + 54.75 \cos 0^\circ + 120.45 \cos 90^\circ + 56.16 \cos 270^\circ)^2 + (-6.24 \sin 180^\circ + 54.75 \sin 0^\circ + 120.45 \sin 90^\circ + 56.16 \sin 270^\circ)^2}$$

$$m_5 \times 1.2 = \sqrt{(60.99)^2 + (64.29)^2}$$

$$= 88.62 \text{ kg.mm}$$

$$m_5 = 73.85 \text{ kg}$$

$$\tan \theta_5 = \frac{-\sum mrl \sin \theta}{-\sum mrl \cos \theta} = \frac{-64.29}{-60.99} = 1.054$$

$$\theta_5 = 46^\circ 30'$$

θ_5 lies in the third quadrant (numerator is negative and denominator is negative).

$$\theta_5 = 180 + 46^\circ 30'$$

$$\theta_5 = 226^\circ 30'$$

From symmetry of the system, $m_2 = m_5 = 73.85 \text{ kg}$

$$\tan \theta_2 = \frac{-60.99}{-64.29} = 0.949$$

$$\theta_2 = 43^\circ 30'$$

θ_2 lies in the third quadrant (numerator is negative and denominator is negative).

$$\theta_2 = 180 + 43^\circ 30'$$

$$\theta_2 = 223^\circ 30'$$

Trailing wheels: The arrangement remains the same except that only half of the required reciprocating masses have to be balanced at the cranks.

$$m_3 = m_4 = \frac{1}{2} \left(\frac{2}{3} \times 315 \right) = 105 \text{ kg}$$

$$m_3 r_3 l_3 = 105 \times 0.3 \times 0.5 = 15.75$$

$$m_4 r_4 l_4 = 105 \times 0.3 \times 1.1 = 34.65$$

$$\sum mrl = 0$$

$$-6.24 \cos 180^\circ + 15.75 \cos 0^\circ + 34.65 \cos 90^\circ + 56.16 \cos 270^\circ = 0 \quad \text{and}$$

$$-6.24 \sin 180^\circ + 15.75 \sin 0^\circ + 34.65 \sin 90^\circ + 56.16 \sin 270^\circ = 0$$

Squaring, adding and then solving,

$$1.2m_5 = \sqrt{(-6.24 \cos 180^\circ + 15.75 \cos 0^\circ + 34.65 \cos 90^\circ + 56.16 \cos 270^\circ)^2 + (-6.24 \sin 180^\circ + 15.75 \sin 0^\circ + 34.65 \sin 90^\circ + 56.16 \sin 270^\circ)^2}$$

$$m_5 \times 1.2 = \sqrt{(21.99)^2 + (-21.51)^2}$$

$$= 30.76 \text{ kg.mm}$$

$$m_5 = 25.63 \text{ kg}$$

$$\tan \theta_5 = \frac{-\sum mrl \sin \theta}{-\sum mrl \cos \theta} = \frac{-(-21.51)}{-21.99} = -0.978$$

$$\theta_5 = -44^\circ 21'$$

θ_5 lies in the second quadrant (numerator is positive and denominator is negative).

$$\theta_5 = 180 - 44^\circ 21'$$

$$\theta_5 = 135^\circ 38'$$

From symmetry of the system, $m_2 = m_5 = 25.63 \text{ kg}$

$$\tan \theta_2 = \frac{-21.99}{+21.51} = -1.022$$

$$\theta_2 = -45^\circ 37'$$

θ_2 lies in the fourth quadrant (numerator is negative and denominator is positive).

$$\theta_2 = 360 - 45^\circ 37'$$

$$\theta_2 = 314^\circ 22'$$

(iii) Hammer – blow = $mr\omega^2$

where m is the balance mass for reciprocating parts only and neglecting m_1 and m_6 in the above calculations.

Thus, $m_1 r_1 l_1 = m_6 r_6 l_6 = 0$

$$1.2m_5 = \sqrt{(15.75 \cos 0^\circ + 34.65 \cos 90^\circ)^2 + (15.75 \sin 0^\circ + 34.65 \sin 90^\circ)^2}$$

$$m_5 \times 1.2 = \sqrt{(15.75)^2 + (34.65)^2}$$

$$= 38.06 \text{ kg.mm}$$

$$m_5 = 31.75 \text{ kg}$$

$$v = r\omega$$

$$\omega = \frac{80 \times 1000}{60 \times 60} \times \frac{1}{1.9/2} = 23.39 \text{ rad/s}$$

$$\text{Hammer blow} = 31.75 \times 0.75 \times (23.39)^2 = 13015 \text{ N}$$

$$\text{Variation of tractive effort} = \pm \sqrt{2(1-c)m r \omega^2}$$

$$= \sqrt{2(1-2/3)} \times 315 \times 0.3 \times (23.39)^2$$

$$= 24372 \text{ N}$$

2.9 Balancing of Multi Cylinder Engine

➤ Balancing of Primary force and couple

- The multi-cylinder engines with the cylinder centre lines in the same plane and on the same side of the centre line of the crankshaft are known as In-line engines.
- The following two conditions must be satisfied in order to give the primary balance of the reciprocating parts of a multi-cylinder engine :
 - The algebraic sum of the primary forces must be equal to zero. In other words, the primary force polygon must close and

$$\text{Primary force, } F_{PF} = \sum m r \omega^2 \cos \theta$$

- The algebraic sum of the couples about any point in the plane of the primary forces must be equal to zero. In other words, primary couple polygon must close.

$$\text{Primary couple, } F_{PC} = \sum m r l \omega^2 \cos \theta$$

- The primary unbalanced force due to the reciprocating masses is equal to the component, parallel to the line of stroke, of the centrifugal force produced by the equal mass placed at the crankpin and revolving with it. Therefore, in order to give the primary balance of the reciprocating parts of a multi-cylinder engine, it is convenient to imagine the reciprocating masses to be transferred to their respective crankpins and to treat the problem as one of revolving masses.

➤ Balancing of Secondary force and couple

- When the connecting rod is not too long (i.e. when the obliquity of the connecting rod is considered), then the secondary disturbing force due to the reciprocating mass arises.
- The following two conditions must be satisfied in order to give a complete secondary balance of an engine :
 - The algebraic sum of the secondary forces must be equal to zero. In other words, the secondary force polygon must close, and

$$\text{Secondary force, } F_{SF} = \sum m r \omega^2 \frac{\cos 2\theta}{n}$$

- The algebraic sum of the couples about any point in the plane of the secondary forces must be equal to zero. In other words, the secondary couple polygon must close.

$$\text{Secondary couple, } F_{SC} = \sum m r \omega^2 / \frac{\cos 2\theta}{n}$$

Example 2.9: A four crank engine has the two outer cranks set at 120° to each other, and their reciprocating masses are each 400 kg. The distance between the planes of rotation of adjacent cranks are 450 mm, 750 mm and 600 mm. If the engine is to be in complete primary balance, find the reciprocating mass and the relative angular position for each of the inner cranks. If the length of each crank is 300 mm, the length of each connecting rod is 1.2 m and the speed of rotation is 240 r.p.m., what is the maximum secondary unbalanced force?

$$m_1 = m_4 = 400 \text{ kg}$$

$$N = 240 \text{ r.p.m}$$

$$r = 300 \text{ mm} = 0.3 \text{ m}$$

$$\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 240}{60} = 25.14 \text{ rad/s}$$

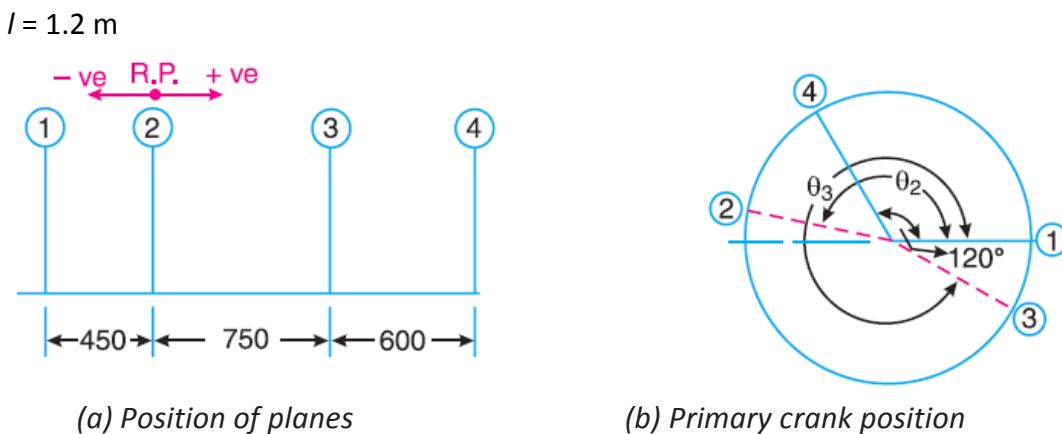


Fig. 2.18

Table 2.6

Plane	Angle		Mass (m) Kg	Radius (r)m	Cent. force $\div \omega^2$ (mr) kg-m	Distance from Ref. Plane (l) m	Couple $\div \omega^2$ (mrl) kg-m ²
	θ	2θ					
1	0°	0°	400	0.3	120	-0.45	-54
2 (R.P.)	θ_2	336°	m_2	0.3	$0.3 m_2$	0	0
3	θ_3	292°	m_3	0.3	$0.3 m_3$	0.75	$0.225 m_3$
4	120°	240°	400	0.3	120	1.35	162

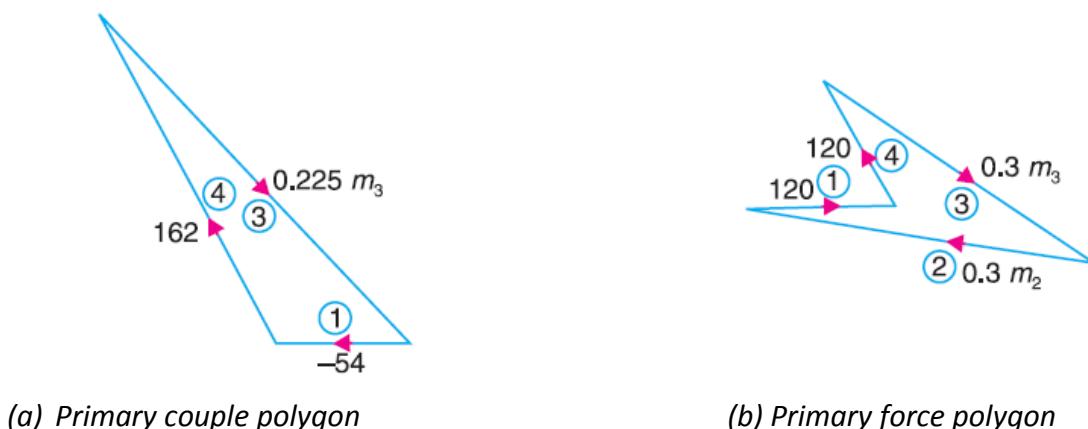


Fig. 2.19

Since the engine is to be in complete primary balance, therefore the primary couple polygon and the primary force polygon must close. First of all, the primary couple polygon, as shown in Fig. 2.19 (a), is drawn to some suitable scale from the data given in Table 2.6 (column 8), in order to find the reciprocating mass for crank 3. Now by measurement, we find that

$$0.225 m_3 = 196 \text{ kg-m}^2$$

$$m_3 = 871 \text{ kg.}$$

and its angular position with respect to crank 1 in the anticlockwise direction,

$$\theta_3 = 326^\circ.$$

Now in order to find the reciprocating mass for crank 2, draw the primary force polygon, as shown in Fig. 2.19 (b), to some suitable scale from the data given in Table 2.6 (column 6). Now by measurement, we find that

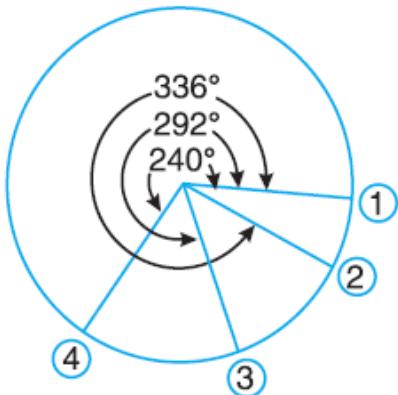
$$0.3 m_2 = 284 \text{ kg-m}$$

$$m_2 = 947 \text{ kg.}$$

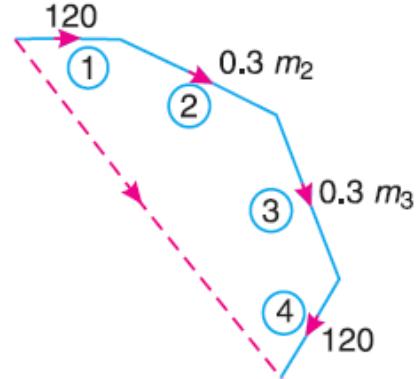
and its angular position with respect to crank 1 in the anticlockwise direction,

$$\theta_2 = 168^\circ.$$

Maximum secondary unbalanced force



(a) Secondary crank positions



(b) Secondary force polygon

Fig. 2.20

The secondary crank positions obtained by rotating the primary cranks at twice the angle, is shown in Fig. 2.20 (a). Now draw the secondary force polygon, as shown in Fig. 2.20 (b), to some suitable scale, from the data given in Table 2.6 (column 6). The closing side of the polygon shown dotted in Fig. 2.20 (b) represents the maximum secondary unbalanced force. By measurement, we find that the maximum secondary unbalanced force is proportional to 582 kg-m.

∴ Maximum Unbalanced Secondary Force,

$$\text{U.S.F.} = 582 \times \frac{\omega^2}{n}$$

$$\text{U.S.F.} = 582 \times \frac{(25.14)^2}{1.2 / 0.3}$$

$$\text{U.S.F.} = 91960 \text{ N}$$

Example 2.10: The intermediate cranks of a four cylinder symmetrical engine, which is in complete primary balance, are 90° to each other and each has a reciprocating mass of 300 kg. The centre distance between intermediate cranks is 600 mm and between extreme cranks it is 1800 mm. Lengths of the connecting rod and cranks are 900 mm and 300 mm respectively. Calculate the masses fixed to the extreme cranks with their relative angular positions. Also find the magnitudes of secondary forces and couples about the centre line of the system if the engine speed is 1500 rpm.

$$m_2 = m_3 = 300 \text{ kg}$$

$$N = 1500 \text{ r.p.m}$$

$$r = 300 \text{ mm} = 0.3 \text{ m}$$

$$\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 1500}{60} = 157.08 \text{ rad/s}$$

$$l = 0.9 \text{ m}$$

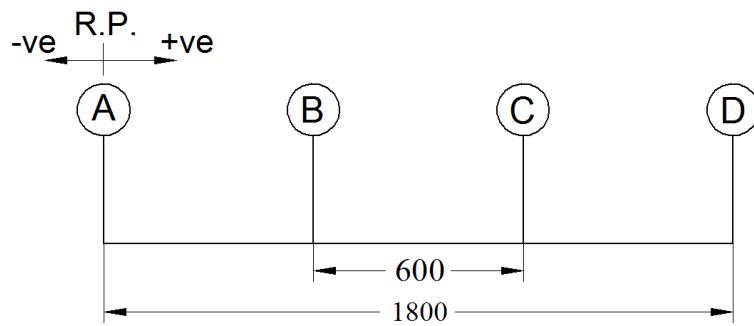
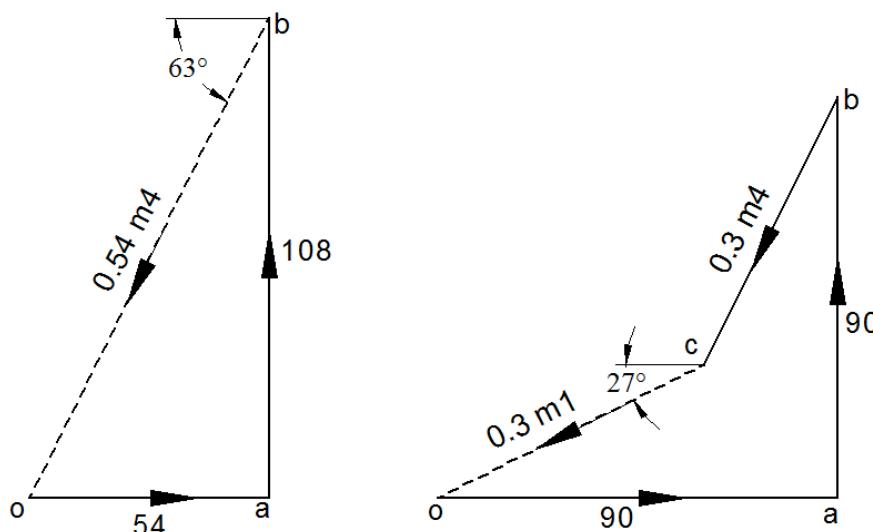


Fig. 2.21 Position of planes

Table 2.7

Plane	Angle		Mass (m) kg	Radius (r)m	Cent. force $\div \omega^2$ (mr) kg-m	Distance from Ref. Plane(l) m	Couple $\div \omega^2$ (mrl) kg-m ²
	θ	2θ					
1 (R.P.)	θ_1	54°	m_1	0.3	$0.3 m_1$	0	0
2	0°	0°	300	0.3	90	0.6	54
3	90°	180°	300	0.3	90	1.2	108
4	θ_4	126°	m_4	0.3	$0.3 m_4$	1.8	$0.54 m_4$



(a) Primary couple polygon

(b) Primary force polygon

Fig. 2.22

Since the engine is to be in complete primary balance, therefore the primary couple polygon and the primary force polygon must close. First of all, the primary couple polygon, as shown in Fig. 2.22 (a), is drawn to some suitable scale from the data given in Table 2.7 (column 8), in order to find the reciprocating mass for crank 4. Now by measurement, we find that

$$0.54 m_4 = 120.75 \text{ kg-m}^2$$

$$m_4 = 223.61 \text{ kg.}$$

and its angular position with respect to crank 2 in the anticlockwise direction,

$$\theta_4 = 180^\circ + 63^\circ = 243^\circ.$$

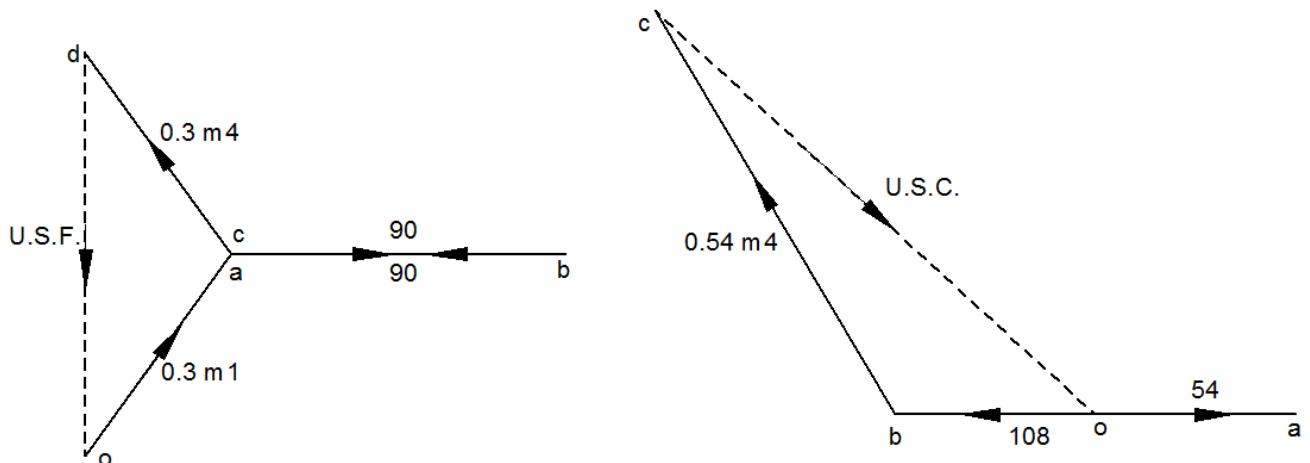
Now in order to find the reciprocating mass for crank 2, draw the primary force polygon, as shown in Fig. 2.22 (b), to some suitable scale from the data given in Table 2.7 (column 6). Now by measurement, we find that

$$0.3 m_1 = 67.08 \text{ kg-m}$$

$$m_1 = 223.6 \text{ kg.}$$

and its angular position with respect to crank 1 in the anticlockwise direction,

$$\theta_1 = 180^\circ + 27^\circ = 207^\circ.$$



(a) Secondary force polygon

(b) Secondary couple polygon

Fig. 2.23

The secondary crank positions obtained by rotating the primary cranks at twice the angle. Now draw the secondary force polygon, as shown in Fig. 2.23 (a), to some suitable scale, from the data given in Table 2.7 (column 6). The closing side of the polygon shown dotted in Fig. 2.23 (a) represents the maximum secondary unbalanced force. By measurement, we find that the maximum secondary unbalanced force is proportional to 108.54 kg-m.

\therefore Maximum Unbalanced Secondary Force,

$$\text{U.S.F.} = 108.54 \times \frac{\omega^2}{n}$$

$$\text{U.S.F.} = 108.54 \times \frac{(157.08)^2}{0.9 / 0.3}$$

$$\text{U.S.F.} = 892.71 \text{ KN}$$

Now draw the secondary couple polygon, as shown in Fig. 2.23 (b), to some suitable scale, from the data given in Table 2.7 (column 8). The closing side of the polygon shown dotted in Fig. 2.23 (b) represents the maximum secondary unbalanced couple. By measurement, we find that the maximum secondary unbalanced couple is proportional to 160.47 kg-m².

\therefore Maximum Unbalanced Secondary Couple,

$$\text{U.S.C.} = 160.47 \times \frac{\omega^2}{n}$$

$$\text{U.S.C.} = 160.47 \times \frac{(157.08)^2}{0.9 / 0.3}$$

$$\text{U.S.C.} = 1319.82 \text{ KN.m}$$

Example 2.11: The cranks and connecting rods of a 4-cylinder in-line engine running at 1800 r.p.m. are 60 mm and 240 mm each respectively and the cylinders are spaced 150 mm apart. If the cylinders are numbered 1 to 4 in sequence from one end, the cranks appear at intervals of 90° in an end view in the order 1-4-2-3. The reciprocating mass corresponding to each cylinder is 1.5 kg.

Determine: (i) Unbalanced primary and secondary forces, if any, and (ii) Unbalanced primary and secondary couples with reference to central plane of the engine.

$$r = 60 \text{ mm}$$

$$N = 1800 \text{ r.p.m}$$

$$l = 240 \text{ mm}$$

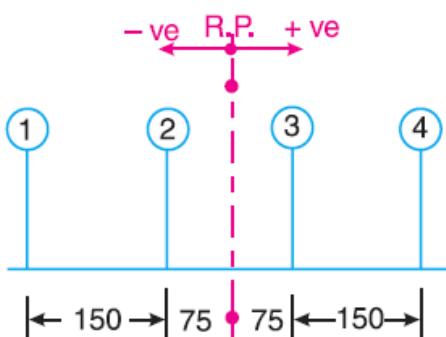
$$\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 1800}{60}$$

$$m = 1.5 \text{ kg}$$

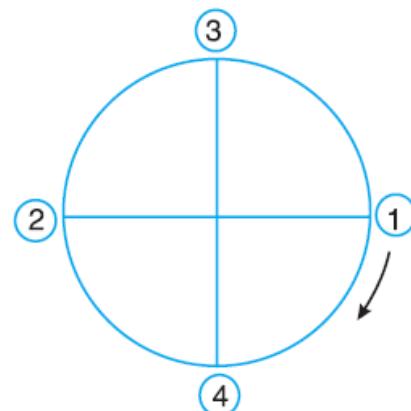
$$= 188.5 \text{ rad/s}$$

Table 2.8

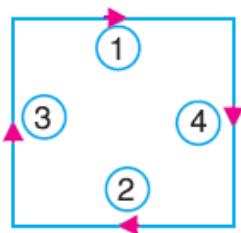
Angle 2θ	Angle θ	Plane	Mass (m) kg	Radius (r) m	Cent.force ÷ ω² (mr) kg-m	Distance from Ref. Plane (l) m	Couple ÷ ω² (mrl) kg-m²
0°	0°	1	1.5	0.06	0.09	-0.225	-0.02025
360°	180°	2	1.5	0.06	0.09	-0.075	-0.00675
540°	270°	3	1.5	0.06	0.09	0.075	0.00675
180°	90°	4	1.5	0.06	0.09	0.225	0.02025



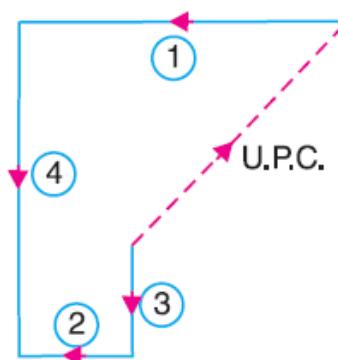
(a) Cylinder plane positions.



(b) Primary crank positions.



(c) Primary force polygon.



(d) Primary couple polygon.

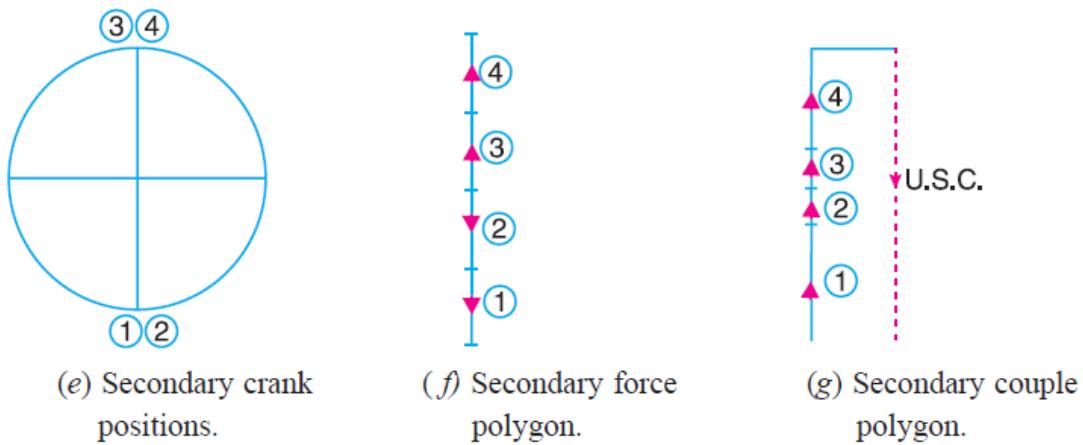


Fig. 2.24

Unbalanced primary forces and couples

The position of the cylinder planes and cranks is shown in Fig. 2.24 (a) and (b) respectively. With reference to central plane of the engine, the data may be tabulated as above:

The primary force polygon from the data given in Table 2.8 (column 6) is drawn as shown in Fig. 2.24 (c). Since the primary force polygon is a closed figure, therefore there are no unbalanced primary forces,

$$\therefore \text{Unbalanced Primary Force, U.P.F.} = 0.$$

The primary couple polygon from the data given in Table 2.8 (column 8) is drawn as shown in Fig. 2.24 (d). The closing side of the polygon, shown dotted in the figure, represents unbalanced primary couple. By measurement, the unbalanced primary couple is proportional to 0.0191 kg-m^2 .

$$\therefore \text{Unbalanced Primary Couple,}$$

$$\text{U.P.C} = 0.0191 \times \omega^2 = 0.0191 (188.52)^2$$

$$\text{U.P.C} = 678.81 \text{ N-m.}$$

Unbalanced secondary forces and couples

The secondary crank positions, taking crank 3 as the reference crank, as shown in Fig. 2.24 (e). From the secondary force polygon as shown in Fig. 2.24 (f), it is a closed figure. Therefore there are no unbalanced secondary forces.

$$\therefore \text{Unbalanced Secondary Force, U.S.F.} = 0.$$

The secondary couple polygon is shown in Fig. 2.24 (g). The unbalanced secondary couple is shown by dotted line. By measurement, we find that unbalanced secondary couple is proportional to 0.054 kg-m^2 .

$$\therefore \text{Unbalanced Secondary Couple,}$$

$$\text{U.S.C.} = 0.054 \times \frac{\omega^2}{n} = 0.054 \times \frac{(188.52)^2}{0.24 / 0.06}$$

$$\text{U.S.C.} = 479.78 \text{ N.m} \quad (n = l/r)$$

Example 2.12: The successive cranks of a five cylinder in-line engine are at 144° apart. The spacing between cylinder centre lines is 400 mm. The lengths of the crank and the connecting rod are 100 mm and 450 mm respectively and the reciprocating mass for each cylinder is 20 kg. The engine speed is 630 r.p.m. Determine the maximum values of the primary and secondary forces and couples and the position of the central crank at which these occur.

$$l = 450 \text{ mm} = 0.45 \text{ m}$$

$$r = 0.1 \text{ m}$$

$$m = 20 \text{ kg}$$

$$N = 630 \text{ r.p.m.}$$

$$n = l/r = 4.5$$

$$\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 630}{60} = 65.97 \text{ rad/s}$$

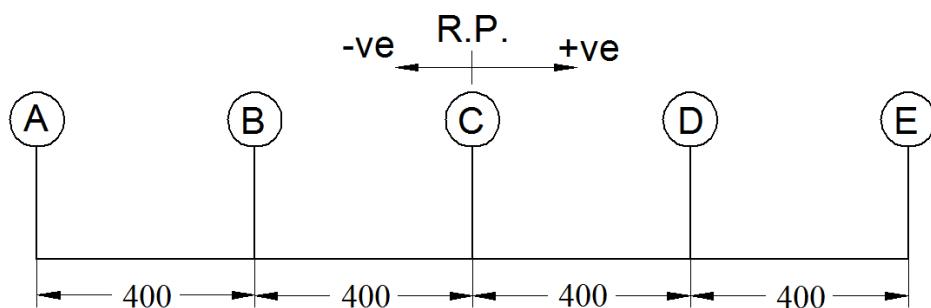


Fig. 2.25 Cylinder plane position

Table 2.9

Angle 2θ	Angle θ	Plane	Mass (m) Kg	Radius (r) m	Cent.force $\div \omega^2$ (mr) kg-m	Distance from Ref. Plane (l) m	Couple $\div \omega^2$ (mrl) kg-m 2
0°	0°	1	20	0.1	2	-0.8	-1.6
288°	144°	2	20	0.1	2	-0.4	-0.8
216°	288°	3	20	0.1	2	0	0
144°	72°	4	20	0.1	2	0.4	0.8
72°	216°	5	20	0.1	2	0.8	1.6

Unbalanced primary forces and couples

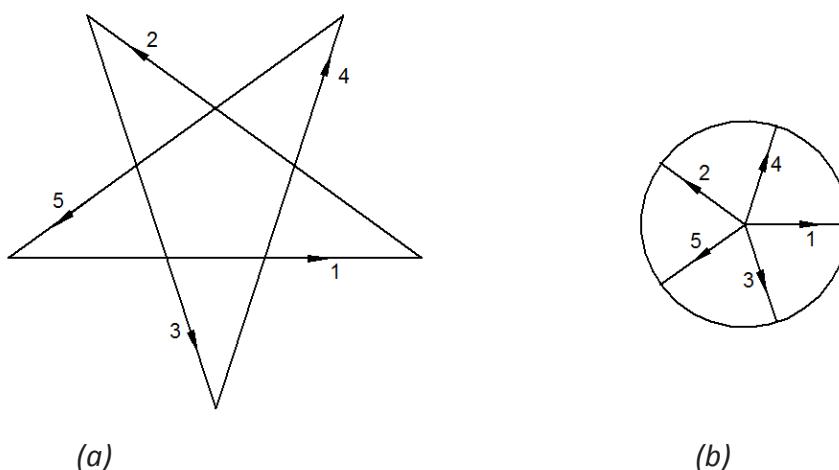


Fig. 2.26 Primary force polygon

The position of the cylinder planes and cranks is shown in Fig. 2.25 and Fig. 2.26 (b) respectively. With reference to central plane of the engine, the data may be tabulated as above:

The primary force polygon from the data given in Table 2.9 (column 6) is drawn as shown in Fig. 2.26 (a). Since the primary force polygon is a closed figure, therefore there are no unbalanced primary forces,

\therefore Unbalanced Primary Force, U.P.F. = 0.

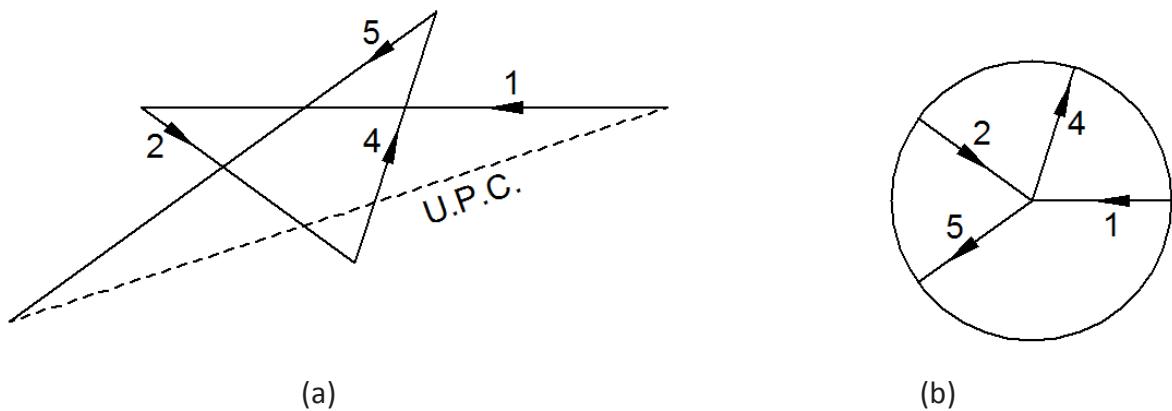


Fig. 2.27 Primary couple polygon

The primary couple polygon from the data given in Table 2.9 (column 8) is drawn as shown in Fig. 2.27 (a). The closing side of the polygon, shown dotted in the figure, represents unbalanced primary couple. By measurement, the unbalanced primary couple is proportional to 2.1 kg-m^2 .

\therefore Unbalanced Primary Couple,

$$\text{U.P.C.} = 2.1 \times \omega^2 = 2.1 (65.97)^2$$

$$\text{U.P.C.} = 9139.3 \text{ N-m.}$$

Unbalanced secondary forces and couples

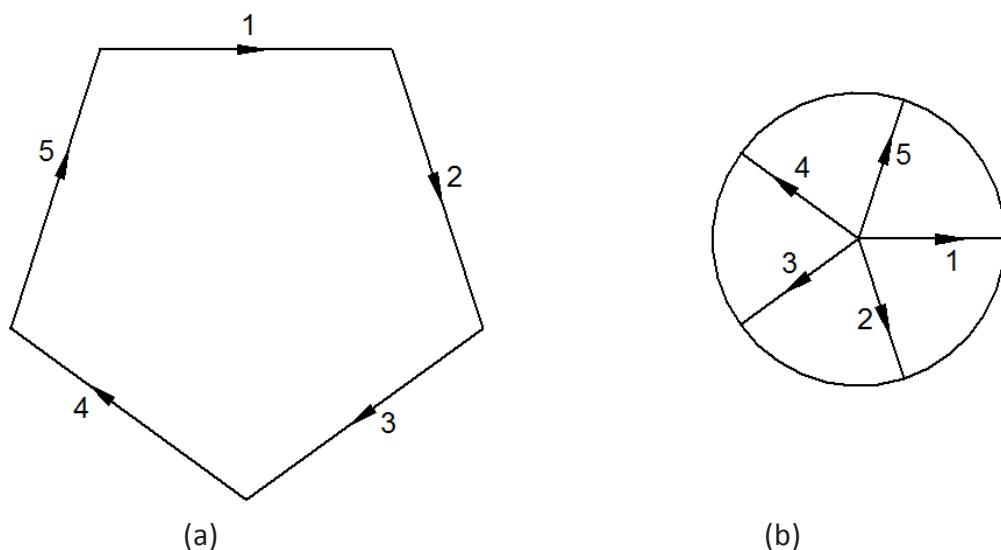


Fig. 2.28 Secondary force polygon

The secondary crank positions are shown in Fig. 2.28 (b). From the secondary force polygon as shown in Fig. 2.28 (a), it is a closed figure. Therefore there are no unbalanced secondary forces.

\therefore Unbalanced Secondary Force, $\text{U.S.F.} = 0$.

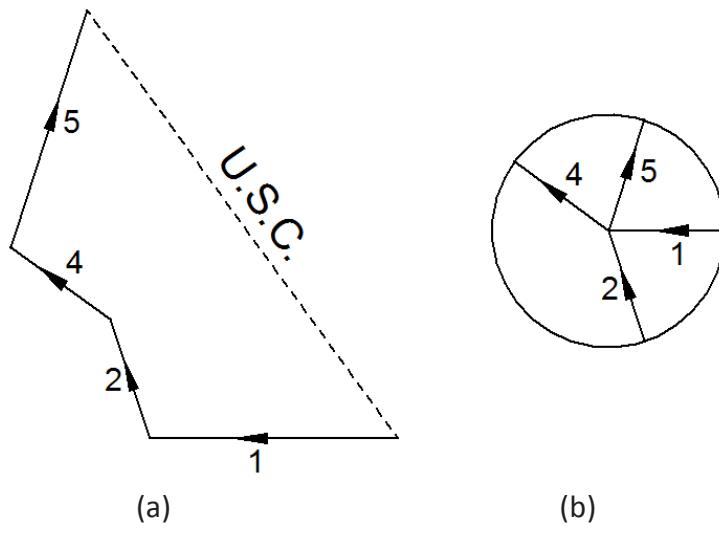


Fig. 2.29 Secondary couple polygon

The secondary couple polygon is shown in Fig. 2.29 (a). The unbalanced secondary couple is shown by dotted line. By measurement, we find that unbalanced secondary couple is proportional to 3.4 kg-m^2 .

\therefore Unbalanced Secondary Couple,

$$\text{U.S.C.} = 3.4 \times \frac{\omega^2}{n} = 3.4 \times \frac{(65.97)^2}{0.45 / 0.1}$$

$$\text{U.S.C.} = 3288.2 \text{ N.m}$$

$$(n = l/r)$$

Example 2.13: A four stroke five cylinder in-line engine has a firing order of 1-4-5-3-2-1. The centers lines of cylinders are spaced at equal intervals of 15 cm, the reciprocating parts per cylinder have a mass of 15 kg, the piston stroke is 10 cm and the connecting rods are 17.5 cm long. The engine rotates at 600 rpm. Determine the values of maximum primary and secondary unbalanced forces and couples about the central plane.

$$l = 10\text{cm} = 0.1 \text{ m} \quad \text{or}$$

$$r = 0.05 \text{ m}$$

$$m = 15 \text{ kg}$$

$$N = 600 \text{ r.p.m.}$$

$$n = l/r = 17.5/5 = 3.5$$

$$\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 600}{60} = 62.83 \text{ rad/s}$$

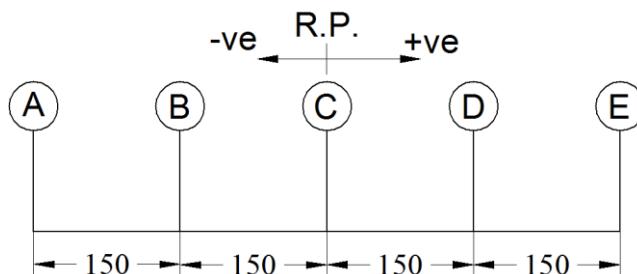


Fig. 2.30 Cylinder plane position

Table 2.10

Angle 2θ	Angle θ	Plane	Mass (m) kg	Radius (r) m	Cent.force $\div \omega^2$ (mr) kg-m	Distance from Ref. Plane (l) m	Couple $\div \omega^2$ (mrl) kg-m 2
0°	0°	1	15	0.05	0.75	-0.3	-0.225
216°	288°	2	15	0.05	0.75	-0.15	-0.1125
72°	216°	3	15	0.05	0.75	0	0
144°	72°	4	15	0.05	0.75	0.15	0.1125
288°	144°	5	15	0.05	0.75	0.3	0.225

Unbalanced primary forces and couples

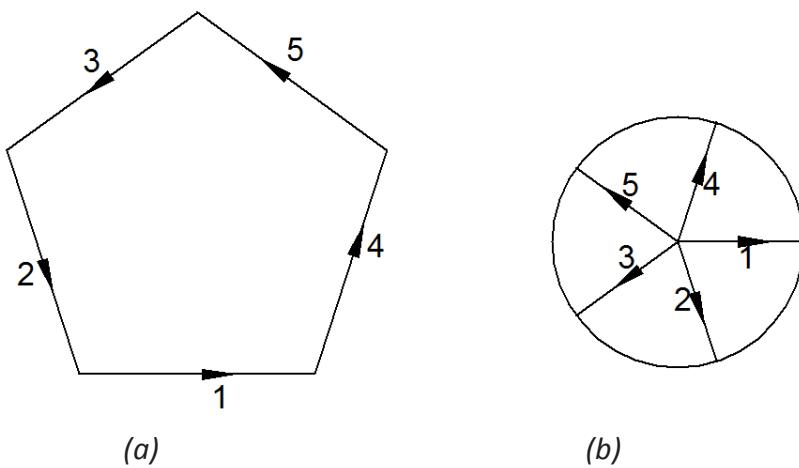


Fig. 2.31 Primary force polygon

The position of the cylinder planes and cranks is shown in Fig. 2.30 and Fig. 2.31(b) respectively. With reference to central plane of the engine, the data may be tabulated as above:

The primary force polygon from the data given in Table 2.10 (column 6) is drawn as shown in Fig. 2.31 (a). Since the primary force polygon is a closed figure, therefore there are no unbalanced primary forces,

∴ Unbalanced Primary Force, **U.P.F. = 0.**

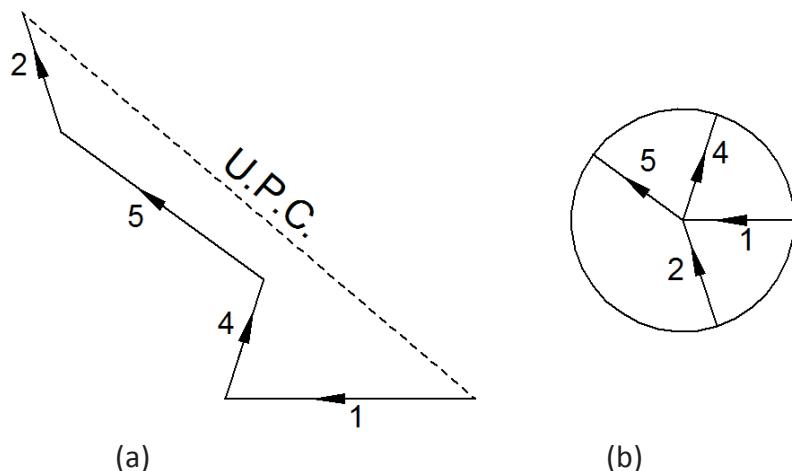


Fig. 2.32 Primary couple polygon

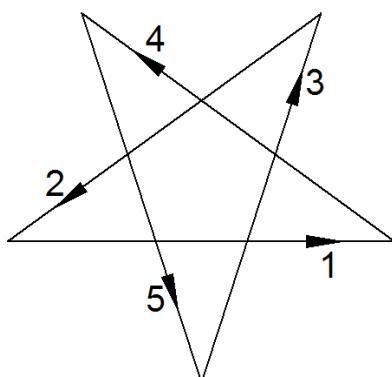
The primary couple polygon from the data given in Table 2.10 (column 8) is drawn as shown in Fig. 2.32 (a). The closing side of the polygon, shown dotted in the figure, represents unbalanced primary couple. By measurement, the unbalanced primary couple is proportional to 0.53 kg-m^2 .

\therefore Unbalanced Primary Couple,

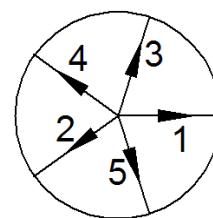
$$\text{U.P.C.} = 0.53 \times \omega^2 = 0.53 (62.83)^2$$

$$\text{U.P.C.} = 2092.23 \text{ N-m.}$$

Unbalanced secondary forces and couples



(a)

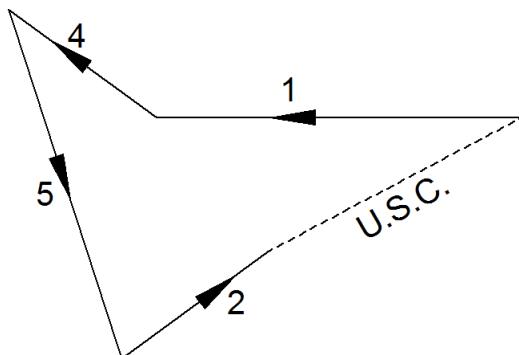


(b)

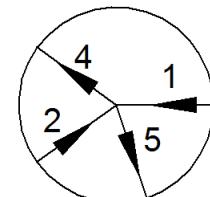
Fig. 2.33 Secondary force polygon

The secondary crank positions are shown in Fig. 2.33 (b). From the secondary force polygon as shown in Fig. 2.33 (a), it is a closed figure. Therefore there are no unbalanced secondary forces.

\therefore Unbalanced Secondary Force, $\text{U.S.F.} = 0$.



(a)



(b)

Fig. 2.34 Secondary couple polygon

The secondary couple polygon is shown in Fig. 2.34 (a). The unbalanced secondary couple is shown by dotted line. By measurement, we find that unbalanced secondary couple is proportional to 0.18 kg-m^2 .

\therefore Unbalanced Secondary Couple,

$$\text{U.S.C.} = 0.18 \times \frac{\omega^2}{n}$$

$$\text{U.S.C.} = 0.18 \times \frac{(62.83)^2}{3.5}$$

$$\text{U.S.C.} = 203 \text{ N.m}$$

$$(n = l/r)$$

Example 2.14: In an in-line six cylinder engine working on two stroke cycle, the cylinder centre lines are spaced at 600 mm. In the end view, the cranks are 60° apart and in the order 1-4-5-2-3-6. The stroke of each piston is 400 mm and the connecting rod length is 1 m. The mass of the reciprocating parts is 200 kg per cylinder and that of rotating parts 100 kg per crank. The engine rotates at 300 r.p.m. Examine the engine for the balance of primary and secondary forces and couples. Find the maximum unbalanced forces and couples.

$$L = 400 \text{ mm}$$

$$\text{or } r = L / 2 = 200 \text{ mm} = 0.2 \text{ m}$$

$$l = 1 \text{ m}$$

$$N = 300 \text{ r.p.m.}$$

$$m_1 = 200 \text{ kg}$$

$$\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 300}{60} = 31.42 \text{ rad/s}$$

$$m_2 = 100 \text{ kg}$$

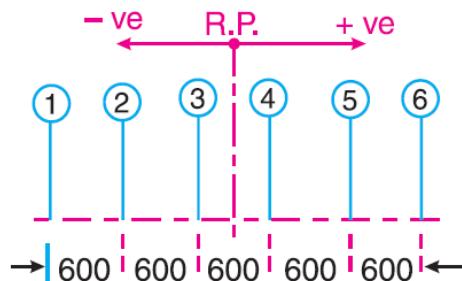


Fig. 2.35 Positions of planes of cylinders

Table 2.11

Angle 2θ	Angle θ	Plane	Mass (m) kg	Radius (r) m	Cent. force $\div \omega^2$ (mr) kg-m	Distance from Ref. Plane (l) m	Couple $\div \omega^2$ (mrl) kg-m 2
0°	0°	1	300	0.2	60	-1.5	-90
360°	180°	2	300	0.2	60	-0.9	-54
120°	240°	3	300	0.2	60	-0.3	-18
120°	60°	4	300	0.2	60	0.3	18
240°	120°	5	300	0.2	60	0.9	54
240°	300°	6	300	0.2	60	1.5	90

Unbalanced primary forces and couples

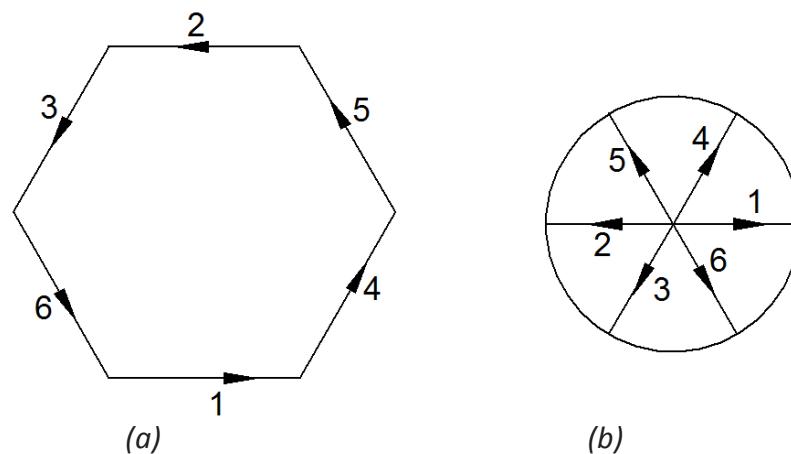


Fig. 2.36 Primary force polygon

The position of the cylinder planes and cranks is shown in Fig. 2.35 and Fig. 2.36(b) respectively. With reference to central plane of the engine, the data may be tabulated as above:

The primary force polygon from the data given in Table 2.11 (column 6) is drawn as shown in Fig. 2.36 (a). Since the primary force polygon is a closed figure, therefore there are no unbalanced primary forces,

$$\therefore \text{Unbalanced Primary Force, U.P.F.} = 0.$$

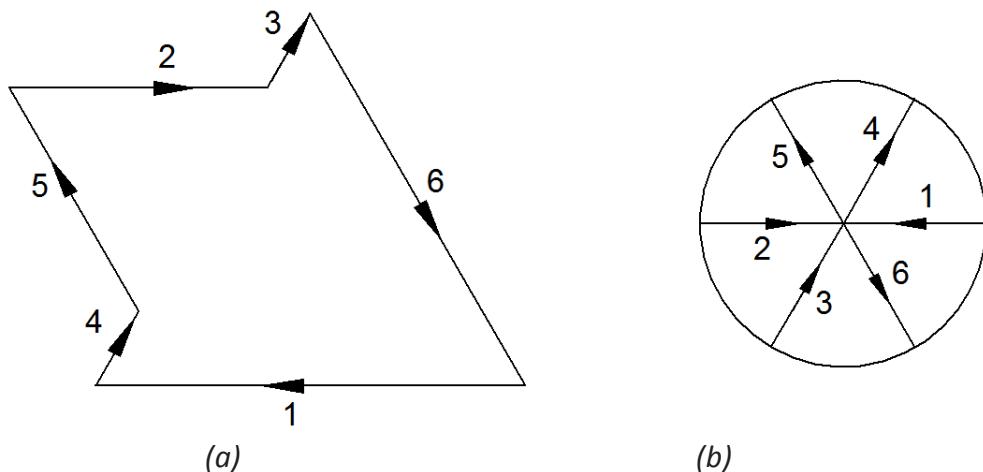


Fig. 2.37 Primary couple polygon

The primary couple polygon from the data given in Table 2.11 (column 8) is drawn as shown in Fig. 2.37 (a). Since the primary couple polygon is a closed figure, therefore there are no unbalanced primary couples,

$$\therefore \text{Unbalanced Primary Couple, U.P.C.} = 0$$

Unbalanced secondary forces and couples

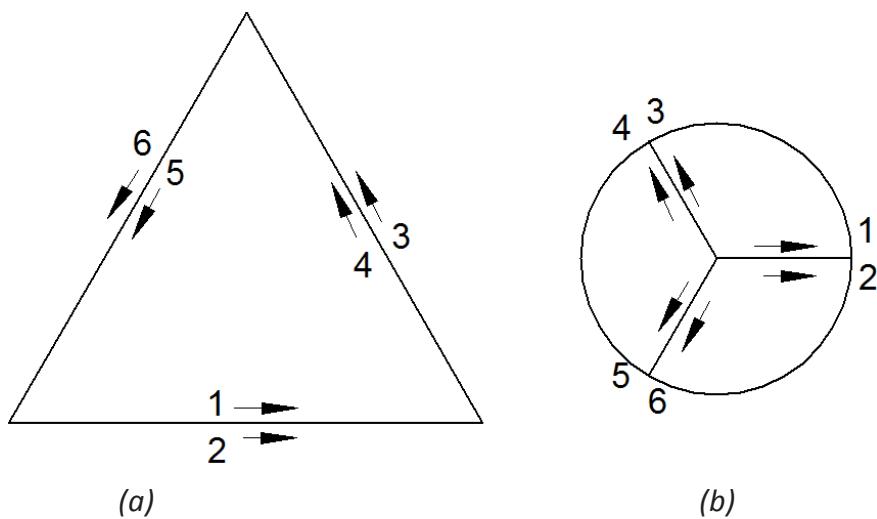


Fig. 2.38 Secondary force polygon

The secondary crank positions are shown in Fig. 2.38 (b). From the secondary force polygon as shown in Fig. 2.38 (a), it is a closed figure. Therefore there are no unbalanced secondary forces.

$$\therefore \text{Unbalanced Secondary Force, U.S.F.} = 0.$$

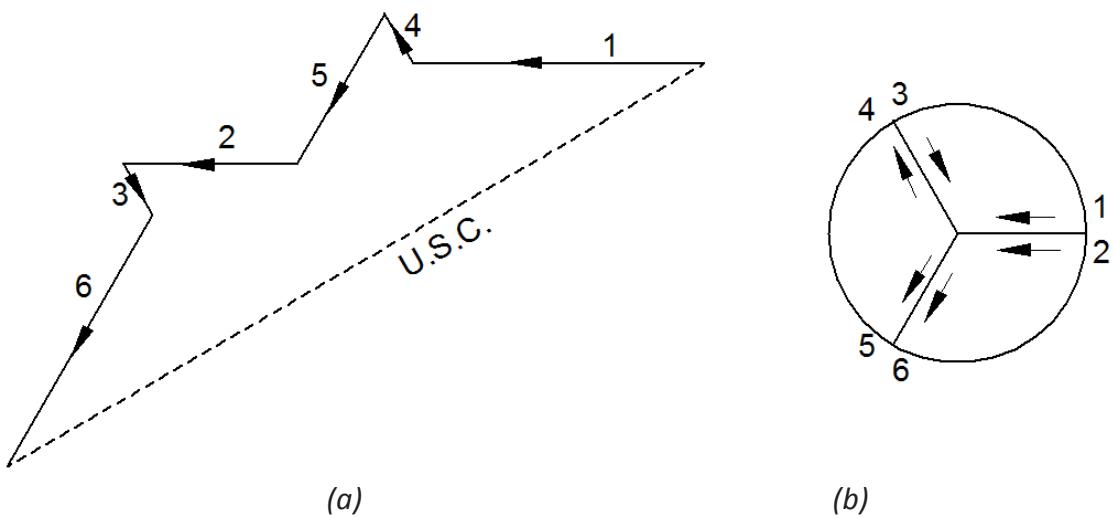


Fig. 2.39 Secondary couple polygon

The secondary couple polygon is shown in Fig. 2.39 (a). The unbalanced secondary couple is shown by dotted line. By measurement, we find that unbalanced secondary couple is proportional to 249 kg-m^2 .

\therefore Unbalanced Secondary Couple,

$$U.S.C. = 249 \times \frac{\omega^2}{n}$$

$$U.S.C. = 249 \times \frac{(31.42)^2}{1/0.2}$$

$$\mathbf{U.S.C. = 49163.37 \text{ N.m}}$$

$$(n = l/r)$$

Example 2.15: The firing order in a 6 cylinder vertical four stroke in-line engine is 1-4-2-6-3-5. The piston stroke is 100 mm and the length of each connecting rod is 200 mm. The pitch distances between the cylinder centre lines are 100 mm, 100 mm, 150 mm, 100 mm, and 100 mm respectively. The reciprocating mass per cylinder is 1 kg and the engine runs at 3000 r.p.m. Determine the out-of-balance primary and secondary forces and couples on this engine, taking a plane midway between the cylinder 3 and 4 as the reference plane.

$$L = 100 \text{ mm} \quad \text{or} \quad r = L/2 = 50 \text{ mm} = 0.05 \text{ m}$$

$$l = 200 \text{ mm} \quad N = 3000 \text{ r.p.m.}$$

$$m = 1 \text{ kg} \quad \omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 3000}{60} = 314.16 \text{ rad/s}$$

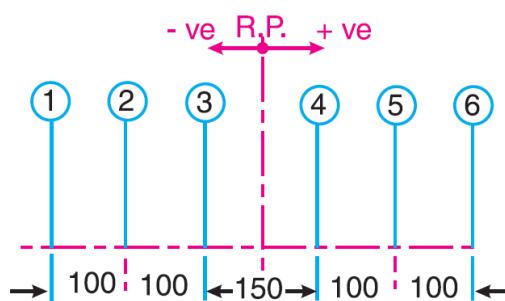


Fig. 2.40 Positions of planes

Table 2.12

Angle 2θ	Angle θ	Plane	Mass (m) kg	Radius (r) m	Cent. force $\div \omega^2$ (mr) kg-m	Distance from Ref. Plane (l) m	Couple $\div \omega^2$ (mrl) kg-m 2
0°	0°	1	1	0.05	0.05	-0.275	-0.01375
240°	120°	2	1	0.05	0.05	-0.175	-0.00875
120°	240°	3	1	0.05	0.05	-0.075	-0.00375
120°	60°	4	1	0.05	0.05	0.075	0.00375
240°	300°	5	1	0.05	0.05	0.175	0.00875
360°	180°	6	1	0.05	0.05	0.275	0.01375

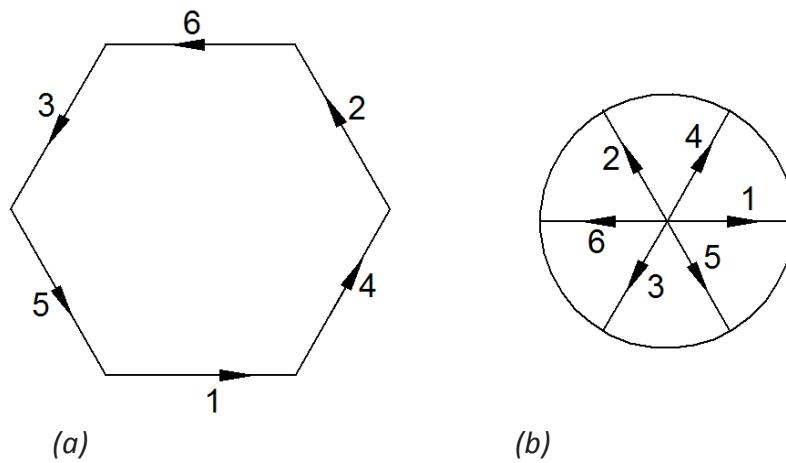
Unbalanced primary forces and couples

Fig. 2.41 Primary force polygon

The position of the cylinder planes and cranks is shown in Fig. 2.40 and Fig. 2.41(b) respectively. With reference to central plane of the engine, the data may be tabulated as above:

The primary force polygon from the data given in Table 2.12 (column 6) is drawn as shown in Fig. 2.41 (a). Since the primary force polygon is a closed figure, therefore there are no unbalanced primary forces,

\therefore Unbalanced Primary Force, U.P.F. = 0.

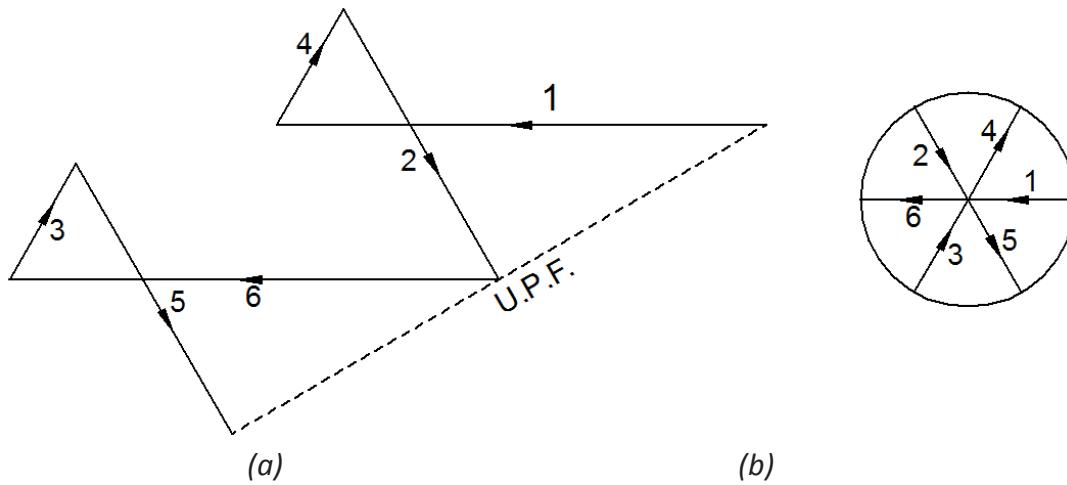


Fig. 2.42 Primary couple polygon

The primary couple polygon from the data given in Table 2.12 (column 8) is drawn as shown in Fig. 2.42 (a). The closing side of the polygon, shown dotted in the figure, represents unbalanced primary couple. By measurement, the unbalanced primary couple is proportional to 0.01732 kg-m².

\therefore Unbalanced Primary Couple,

$$\text{U.P.C.} = 0.01732 \times \omega^2 = 0.01732 (314.16)^2$$

$$\text{U.P.C.} = 1709.42 \text{ N}\cdot\text{m}.$$

Unbalanced secondary forces and couples

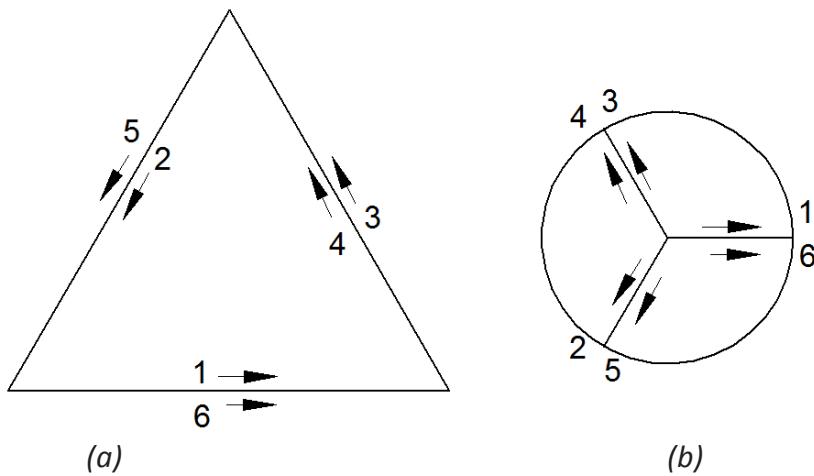


Fig. 2.43 Secondary force polygon

The secondary crank positions are shown in Fig. 2.43 (b). From the secondary force polygon as shown in Fig. 2.43 (a), it is a closed figure. Therefore there are no unbalanced secondary forces.

\therefore Unbalanced Secondary Force, $\text{U.S.F.} = 0$.

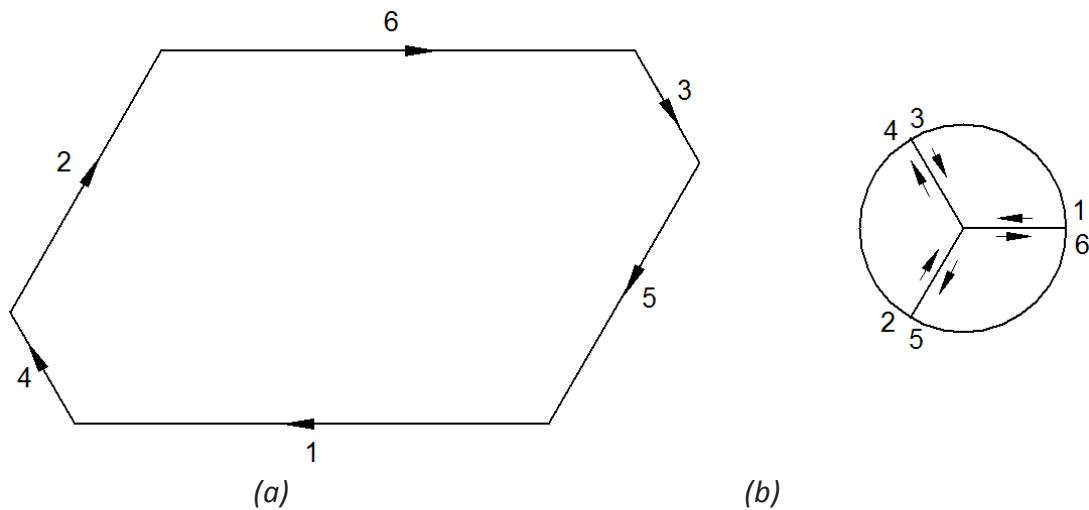


Fig. 2.44 Secondary couple polygon

The secondary couple polygon is drawn as shown in Fig. 2.44 (a). Since the secondary couple polygon is a closed figure, therefore there are no unbalanced secondary couples,

\therefore Unbalanced Secondary Couple, $\text{U.S.C.} = 0$

2.10 Balancing of V – Engine

In V-engines, a common crank OA is operated by two connecting rods AB₁, and AB₂. Fig. 2.45 shows a symmetrical two cylinder V-cylinder, the centre lines of which are inclined at an angle α to the x-axis.

Let θ be the angle moved by the crank from the x-axis.

I. Primary force

Primary force of 1 along line of stroke

$$OB_1 = mr\omega^2 \cos(\theta - \alpha)$$

Primary force of 1 along x-axis

$$= mr\omega^2 \cos(\theta - \alpha) \cos \alpha$$

Primary force of 2 along line of stroke OB₂ = $mr\omega^2 \cos(\theta + \alpha)$

Primary force of 2 along x-axis = $mr\omega^2 \cos(\theta + \alpha) \cos \alpha$

Total primary force along x-axis

$$\begin{aligned} &= mr\omega^2 \cos \alpha [\cos(\theta - \alpha) + \cos(\theta + \alpha)] \\ &= mr\omega^2 \cos \alpha [(\cos \theta \cos \alpha + \sin \theta \sin \alpha) + (\cos \theta \cos \alpha - \sin \theta \sin \alpha)] \\ &= 2mr\omega^2 \cos \alpha \cos \theta \cos \alpha \\ &= 2mr\omega^2 \cos^2 \alpha \cos \theta \end{aligned}$$

Similarly, total primary force along z-axis

$$\begin{aligned} &= mr\omega^2 \sin \alpha [\cos(\theta - \alpha) - \cos(\theta + \alpha)] \\ &= mr\omega^2 \sin \alpha [(\cos \theta \cos \alpha + \sin \theta \sin \alpha) - (\cos \theta \cos \alpha - \sin \theta \sin \alpha)] \\ &= 2mr\omega^2 \sin \alpha \sin \theta \sin \alpha \\ &= 2mr\omega^2 \sin^2 \alpha \sin \theta \end{aligned}$$

$$\begin{aligned} \text{Resultant primary force} &= \sqrt{(2mr\omega^2 \cos^2 \alpha \cos \theta)^2 + (2mr\omega^2 \sin^2 \alpha \sin \theta)^2} \\ &= 2mr\omega^2 \sqrt{(\cos^2 \alpha \cos^2 \theta + \sin^2 \alpha \sin^2 \theta)} \end{aligned}$$

It will be at an angle β with the x-axis given by

$$\tan \beta = \frac{\sin^2 \alpha \sin \theta}{\cos^2 \alpha \cos \theta}$$

$$\begin{aligned} \text{If } 2\alpha = 90^\circ, \text{ resultant force} &= 2mr\omega^2 \sqrt{(\cos^2 45^\circ \cos \theta)^2 + (\sin^2 45^\circ \sin \theta)^2} \\ &= mr\omega^2 \end{aligned}$$

$$\tan \beta = \frac{\sin^2 45^\circ \sin \theta}{\cos^2 45^\circ \cos \theta} = \tan \theta$$

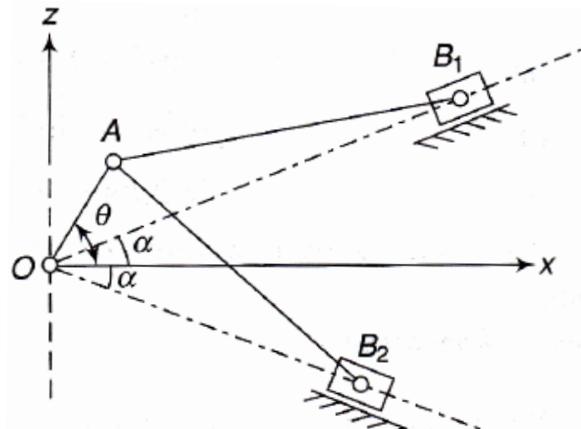


Fig. 2.45

i.e., $\beta = \theta$ or it acts along the crank and, therefore, can be completely balanced by a mass at a suitable radius diametrically opposite to the crank such that $m_r r_r = mr$.

For a given value of α , the resultant primary force is maximum when

$$(\cos^2 \alpha \cos \theta)^2 + (\sin^2 \alpha \sin \theta)^2 \text{ is maximum}$$

$$(\cos^4 \alpha \cos^2 \theta + \sin^4 \alpha \sin^2 \theta) \text{ is maximum}$$

$$\frac{d}{d\theta} (\cos^4 \alpha \cos^2 \theta + \sin^4 \alpha \sin^2 \theta) = 0$$

$$-\cos^4 \alpha \cdot 2 \cos \theta \sin \theta + \sin^4 \alpha \cdot 2 \sin \theta \cos \theta = 0$$

$$-\cos^4 \alpha \cdot \sin 2\theta + \sin^4 \alpha \cdot \sin 2\theta = 0$$

$$\sin 2\theta (\sin^4 \alpha - \cos^4 \alpha) = 0$$

As α is not zero, therefore, for a given value of α , the resultant primary force is maximum when θ is zero degree.

II. Secondary force

$$\text{Secondary force of 1 along OB}_1 = \frac{mr\omega^2}{n} \cos 2(\theta - \alpha)$$

$$\text{Secondary force of 1 along x-axis} = \frac{mr\omega^2}{n} \cos 2(\theta - \alpha) \cos \alpha$$

$$\text{Secondary force of 2 along OB}_2 = \frac{mr\omega^2}{n} \cos 2(\theta + \alpha)$$

$$\text{Secondary force of 2 along x-axis} = \frac{mr\omega^2}{n} \cos 2(\theta + \alpha) \cos \alpha$$

Total secondary force along x-axis

$$= \frac{mr\omega^2}{n} \cos \alpha [\cos 2(\theta - \alpha) + \cos 2(\theta + \alpha)]$$

$$= \frac{mr\omega^2}{n} \cos \alpha [(\cos 2\theta \cos 2\alpha + \sin 2\theta \sin 2\alpha) + (\cos 2\theta \cos 2\alpha - \sin 2\theta \sin 2\alpha)]$$

$$= \frac{2mr\omega^2}{n} \cos \alpha \cos 2\theta \cos 2\alpha$$

$$\text{Similarly, secondary force along z-axis} = \frac{2mr\omega^2}{n} \sin \alpha \sin 2\theta \sin 2\alpha$$

$$\text{Resultant secondary force} = \frac{2mr\omega^2}{n} \sqrt{(\cos \alpha \cos 2\theta \cos 2\alpha)^2 + (\sin \alpha \sin 2\theta \sin 2\alpha)^2}$$

$$\tan \beta' = \frac{\sin \alpha \sin 2\theta \sin 2\alpha}{\cos \alpha \cos 2\theta \cos 2\alpha}$$

If $2\alpha = 90^\circ$ or $\alpha = 45^\circ$

$$\begin{aligned}\text{Secondary force} &= \frac{2mr\omega^2}{n} \sqrt{\left(\frac{\sin 2\theta}{\sqrt{2}}\right)^2} \\ &= \frac{\sqrt{2}mr\omega^2}{n} \sin 2\theta \\ \tan \beta' &= \infty, \beta' = 90^\circ\end{aligned}$$

This means that the force acts along z-axis and is a harmonic force and special methods are needed to balance it.

Example 2.16: Reciprocating mass per cylinder in 60° V-twin engine is 1.5 kg. The stroke and connecting rod length are 100 mm and 250 mm respectively. If the engine runs at 2500 rpm, determine the maximum and minimum values of primary and secondary forces.

$$\begin{array}{ll}2\alpha = 60^\circ & L = 250 \text{ mm} \\ l = 100 \text{ mm} & \text{or} \quad r = 50 \text{ mm} \\ m = 1.5 \text{ kg} & N = 2500 \text{ rpm} \\ \omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 2500}{60} & = 261.8 \text{ rad/s}\end{array}$$

$$\begin{aligned}\text{Resultant primary force, } F_P &= 2mr\omega^2 \sqrt{(\cos^2 \alpha \cos \theta)^2 + (\sin^2 \alpha \sin \theta)^2} \\ &= 2mr\omega^2 \sqrt{(\cos^2 30^\circ \cos \theta)^2 + (\sin^2 30^\circ \sin \theta)^2} \\ &= 2mr\omega^2 \sqrt{\left(\frac{3}{4} \cos \theta\right)^2 + \left(\frac{1}{4} \sin \theta\right)^2} \\ &= \frac{mr\omega^2}{2} \sqrt{(9 \cos^2 \theta) + (\sin^2 \theta)} \quad \dots \dots \dots \text{(i)}\end{aligned}$$

The primary force is maximum, when $\theta = 0^\circ$. Therefore substituting $\theta = 0^\circ$ in equation (i), maximum primary force,

$$\begin{aligned}F_{P(\max)} &= \frac{mr\omega^2}{2} \times 3 = \frac{1.5 \times 0.05 \times (261.8)^2}{2} \times 3 \\ &= 7710.7 \text{ N}\end{aligned}$$

The primary force is minimum, when $\theta = 90^\circ$. Therefore substituting $\theta = 90^\circ$ in equation (i), minimum primary force,

$$\begin{aligned}F_{P(\min)} &= \frac{mr\omega^2}{2} = \frac{1.5 \times 0.05 \times (261.8)^2}{2} \\ &= 2570.2 \text{ N}\end{aligned}$$

$$\begin{aligned}\text{Resultant secondary force, } F_S &= \frac{2mr\omega^2}{n} \sqrt{(\cos \alpha \cos 2\theta \cos 2\alpha)^2 + (\sin \alpha \sin 2\theta \sin 2\alpha)^2} \\ &= \frac{2mr\omega^2}{n} \sqrt{(\cos 30^\circ \cos 2\theta \cos 60^\circ)^2 + (\sin 30^\circ \sin 2\theta \sin 60^\circ)^2}\end{aligned}$$

$$\begin{aligned}
 &= \frac{2mr\omega^2}{n} \sqrt{\left(\frac{\sqrt{3}}{2} \times \frac{1}{2} \cos 2\theta\right)^2 + \left(\frac{1}{2} \times \frac{\sqrt{3}}{2} \sin 2\theta\right)^2} \\
 &= \frac{\sqrt{3}}{2} \times \frac{mr\omega^2}{n} \\
 &= \frac{\sqrt{3}}{2} \times \frac{1.5 \times 0.05 \times (261.8)^2}{0.25 / 0.05} \quad (n = l/r)
 \end{aligned}$$

Resultant secondary force, $F_S = 890.3 \text{ N}$

Example 2.17: A V-twin engine has the cylinder axes at right angles and the connecting rods operate a common crank. The reciprocating mass per cylinder is 11.5 kg and the crank radius is 75 mm. The length of the connecting rod is 0.3 m. Show that the engine may be balanced for primary forces by means of a revolving balance mass. If the engine speed is 500 rpm, what is the value of maximum resultant secondary force?

$$2\alpha = 90^\circ$$

$$m = 11.5 \text{ kg}$$

$$r = 75 \text{ mm} = 0.075 \text{ m}$$

$$l = 0.3 \text{ m}$$

$$N = 500 \text{ r.p.m.}$$

$$\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 500}{60} = 52.37 \text{ rad/s}$$

$$\begin{aligned}
 \text{Primary force, } F_P &= 2mr\omega^2 \sqrt{(\cos^2 \alpha \cos \theta)^2 + (\sin^2 \alpha \sin \theta)^2} \\
 &= 2mr\omega^2 \sqrt{(\cos^2 45^\circ \cos \theta)^2 + (\sin^2 45^\circ \sin \theta)^2} \\
 &= 2mr\omega^2 \sqrt{\left(\frac{\cos \theta}{2}\right)^2 + \left(\frac{\sin \theta}{2}\right)^2} \\
 &= mr\omega^2
 \end{aligned}$$

Since the resultant primary force $mr\omega^2$ is the centrifugal force of a mass m at the crank radius r when rotating at ω rad/s, therefore, the engine may be balanced by a rotating balance mass.

$$\text{Secondary force, } F_S = \frac{\sqrt{2}mr\omega^2}{n} \sin 2\theta$$

This is maximum, when $\sin 2\theta$ is maximum i.e. when $\sin 2\theta = \pm 1$ or $\theta = 45^\circ$ or 135° . \therefore Maximum resultant secondary force,

$$\begin{aligned}
 F_{S(\max)} &= \frac{\sqrt{2}mr\omega^2}{n} \quad (\text{Substituting } \theta = 45^\circ) \\
 F_{S(\max)} &= \frac{\sqrt{2} \times 11.5 \times 0.075 \times (52.37)^2}{0.3 / 0.075} \quad (n = l/r) \\
 &= 836 \text{ N}
 \end{aligned}$$

2.11 Balancing of Radial Engine

A radial engine is a multi-cylinder engine in which all the connecting rods are connected to a common crank. The analysis of forces in such type of engines is much simplified by using the method of direct and reverse cranks. As all the forces are in the same plane, no unbalance couples exist.

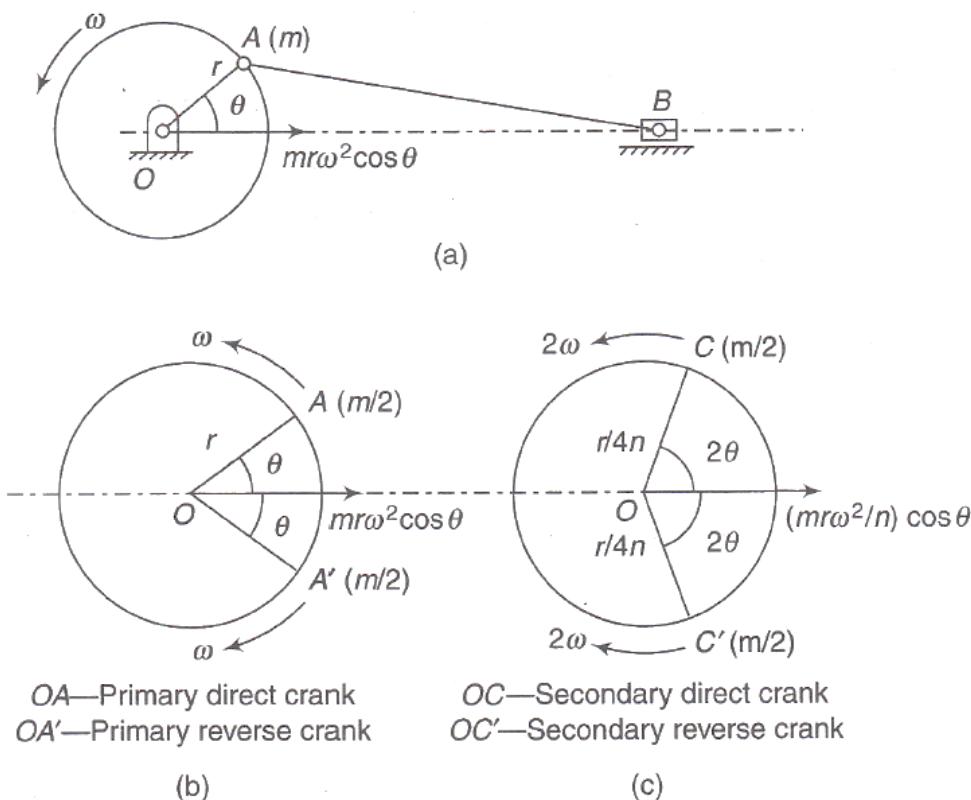


Fig. 2.46

In a reciprocating engine [Fig. 2.46(a)],

Primary force: $mr\omega^2 \cos \theta$ (along line of stroke)

In the method of direct and reverse cranks, a force identical to this force is generated by two masses in the following way:

- A mass $m/2$, placed at the crank pin A and rotating at an angular velocity ω in the given direction [Fig. 2.46(b)].
- A mass $m/2$, placed at the crank pin of an imaginary crank OA' at the same angular position as the real crank but in the opposite direction of the line of stroke. This imaginary crank is assumed to rotate at the same angular velocity ω in the opposite direction to that of the real crank. Thus, while rotating; the two masses coincide only on the cylinder centre line. Now, the components of centrifugal force due to rotating masses along line of stroke are

$$\text{Due to mass at A} = \frac{m}{2} r \omega^2 \cos \theta$$

$$\text{Due to mass at A'} = \frac{m}{2} r \omega^2 \cos \theta$$

Thus, total force along line of stroke = $mr\omega^2 \cos \theta$ which is equal to the primary force. At any instant, the components of the centrifugal forces of these two masses normal to the line of stroke will be equal and opposite.

The crank rotating in the direction of engine rotation is known as the direct crank and the imaginary crank rotating in the opposite direction is known as the reverse crank.

$$\begin{aligned}\text{Secondary accelerating force} &= mr\omega^2 \frac{\cos 2\theta}{n} = mr(2\omega)^2 \frac{\cos 2\theta}{4n} \\ &= m \frac{r}{4n} (2\omega)^2 \cos 2\theta \quad (\text{along line of stroke})\end{aligned}$$

This force can also be generated by two masses in a similar way as follows:

- A mass $m/2$, placed at the end of direct secondary crank of length $r/(4n)$ at angle 2θ and rotating at an angular velocity 2ω in the given direction [Fig. 6.1(c)].
- A mass $m/2$, placed at the end of reverse secondary crank of length $r/(4n)$ at angle -2θ rotating at an angular velocity 2ω in the opposite direction. Now, the components of centrifugal force due to rotating masses along line of stroke are

$$\text{Due to mass at } C = \frac{m}{2} \frac{r}{4n} (2\omega)^2 \cos 2\theta = \frac{mr\omega^2}{2n} \cos 2\theta$$

$$\text{Due to mass at } C' = \frac{m}{2} \frac{r}{4n} (2\omega)^2 \cos 2\theta = \frac{mr\omega^2}{2n} \cos 2\theta$$

$$\text{Thus total force along line of stroke} = 2 \times \frac{m}{2} \frac{r}{4n} (2\omega)^2 \cos 2\theta = \frac{mr\omega^2}{n} \cos 2\theta$$

Which is equal to the secondary force.

3

Mechanical Vibration



Course Contents

- 3.1 Introduction to Mechanical Vibrations
- 3.2 Single Degrees of Freedom System (Linear and Torsional)
- 3.3 Two Degrees of Freedom System
- 3.4 Multi degree freedom systems and analysis (Free vibrations)
- 3.5 Vibrations of Continuous Systems (Free Vibrations)
- 3.7 Rotating unbalance
- 3.8 Vibration Measurement

3.1 Introduction to Mechanical Vibrations

Introduction: When an elastic body such as, a spring, a beam and a shaft are displaced from the equilibrium position by the application of external forces, and then released, they execute a vibratory motion, due to the elastic or strain energy present in the body. When the body reaches the equilibrium position, the whole of the elastic or stain energy is converted into kinetic energy due to which the body continues to move in the opposite direction. The entire KE is again converted into strain energy due to which the body again returns to the equilibrium position. Hence the vibratory motion is repeated indefinitely.

Oscillatory motion is any pattern of motion where the system under observation moves back and forth across some equilibrium position, but dose not necessarily have any particular repeating pattern.

Periodic motion is a specific form of oscillatory motion where the motion pattern repeats itself with a uniform time interval. This uniform time interval is referred to as the *period* and has units of seconds per cycle. The reciprocal of the period is referred to as the *frequency* and has units of cycles per second. This unit of combination has been given a special unit symbol and is referred to as *Hertz* (Hz)

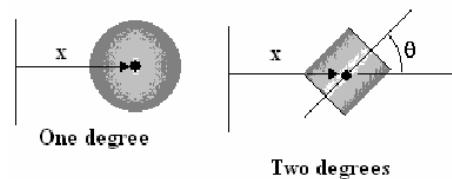
Harmonic motion is a specific form of periodic motion where the motion pattern can be describe by either a sine or cosine. This motion is also sometimes referred to as simple harmonic motion. Because the sine or cosine technically used angles in *radians*, the frequency term expressed in the units radians per seconds (*rad/sec*). This is sometimes referred to as the *circular frequency*. The relationship between the frequency in Hz (cps) and the frequency in *rad/sec* is simply the relationship. $2\pi \text{ rad/sec}$.

Natural frequency is the frequency at which an undamped system will tend to oscillate due to initial conditions in the absence of any external excitation. Because there is no damping, the system will oscillate indefinitely.

Damped natural frequency is frequency that a damped system will tend to oscillate due to initial conditions in the absence of any external excitation. Because there is damping in the system, the system response will eventually decay to rest.

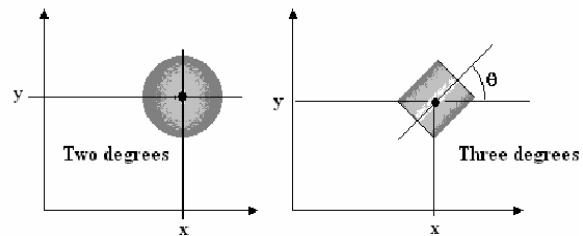
Resonance is the condition of having an external excitation at the natural frequency of the system. In general, this is undesirable, potentially producing extremely large system response.

Degrees of freedom: The numbers of degrees of freedom that a body possesses are those necessary to completely define its position and orientation in space. This is useful in several fields of study such as robotics and vibrations. Consider a spherical object that can only be positioned somewhere on the *x* axis.



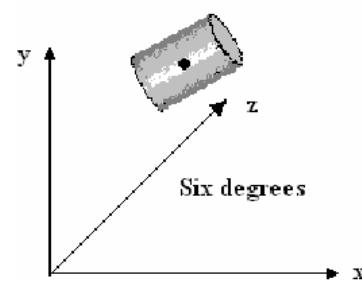
This needs only one dimension, ‘ x ’ to define the position to the centre of gravity so it has one degree of freedom. If the object was a cylinder, we also need an angle ‘ θ ’ to define the orientation so it has two degrees of freedom.

Now consider a sphere that can be positioned in Cartesian coordinates anywhere on the z plane. This needs two coordinates ‘ x ’ and ‘ y ’ to define the position of the centre of gravity so it has two degrees of freedom. A cylinder, however, needs the angle ‘ θ ’ also to define its orientation in that plane so it has three degrees of freedom.



In order to completely specify the position and orientation of a cylinder in Cartesian space, we would need three coordinates x , y and z and three angles relative to each angle. This makes six degrees of freedom. A rigid body in space has $(x, y, z, \theta_x, \theta_y, \theta_z)$.

In the study of free vibrations, we will be constrained to one degree of freedom.



Types of Vibrations:

Free or natural vibrations: A free vibration is one that occurs naturally with no energy being added to the vibrating system. The vibration is started by some input of energy but the vibrations die away with time as the energy is dissipated. In each case, when the body is moved away from the rest position, there is a natural force that tries to return it to its rest position. Free or natural vibrations occur in an elastic system when only the internal restoring forces of the system act upon a body. Since these forces are proportional to the displacement of the body from the equilibrium position, the acceleration of the body is also proportional to the displacement and is always directed towards the equilibrium position, so that the body moves with SHM.

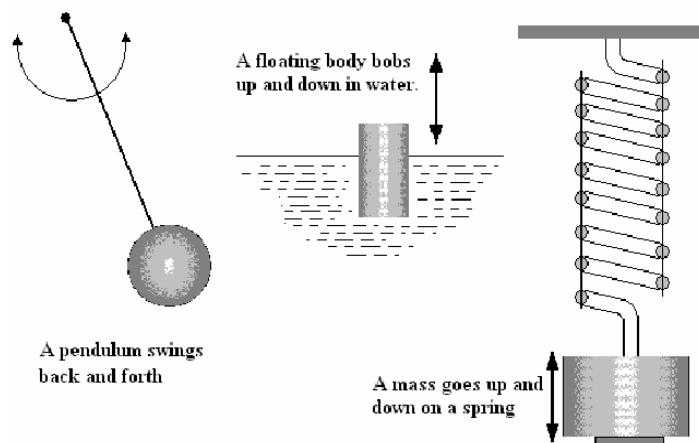


Figure 1. Examples of vibrations with single degree of freedom.

Note that the mass on the spring could be made to swing like a pendulum as well as bouncing up and down and this would be a vibration with two degrees of freedom. The number of degrees of freedom of the system is the number of different modes of vibration which the system may possess.

The motion that all these examples perform is called SIMPLE HARMONIC MOTION (S.H.M.). This motion is characterized by the fact that when the displacement is plotted against time, the resulting graph is basically sinusoidal. Displacement can be linear (e.g. the distance moved by the mass on the spring) or angular (e.g. the angle moved by the simple pendulum). Although we are studying natural vibrations, it will help us understand S.H.M. if we study a forced vibration produced by a mechanism such as the Scotch Yoke.

Simple Harmonic Motion

The wheel revolves at ω radians/sec and the pin forces the yoke to move up and down. The pin slides in the slot and Point P on the yoke oscillates up and down as it is constrained to move only in the vertical direction by the hole through which it slides. The motion of point P is simple harmonic motion. Point P moves up and down so at any moment it has a displacement x , velocity v and an acceleration a .

The pin is located at radius R from the centre of the wheel. The vertical displacement of the pin from the horizontal centre line at any time is x . This is also the displacement of point P . The yoke reaches a maximum displacement equal to R when the pin is at the top and $-R$ when the pin is at the bottom.

This is the amplitude of the oscillation. If the wheel rotates at ω radian/sec then after time t seconds the angle rotated is $\theta = \omega t$ radians. From the right angle triangle we find $x = R \sin(\omega t)$ and the graph of $x - \theta$ is shown on figure 3a.

Velocity is the rate of change of distance with time. The plot is also shown on figure 3a.

$$v = dx/dt = \omega R \cos(\omega t).$$

The maximum velocity or amplitude is ωR and this occurs as the pin passes through the horizontal position and is plus on the way up and minus on the way down. This makes sense since the tangential velocity of a point moving in a circle is $v = \omega R$ and at the horizontal point they are the same thing.

Acceleration is the rate of change of velocity with time. The plot is also shown on figure 3a.

$$a = dv/dt = -\omega^2 R \sin(\omega t)$$

The amplitude is $\omega^2 R$ and this is positive at the bottom and minus at the top (when the yoke is about to change direction)

$$\text{Since } R \sin(\omega t) = x; \text{ then substituting } x \text{ we find } a = -\omega^2 x$$

This is the usual definition of S.H.M. The equation tells us that any body that performs sinusoidal motion must have an acceleration that is directly proportional to the displacement and is always directed to the point of zero displacement. The constant of proportionality is ω^2 . Any vibrating body that has a motion that can be described in this way must vibrate with S.H.M. and have the same equations for displacement, velocity and acceleration.

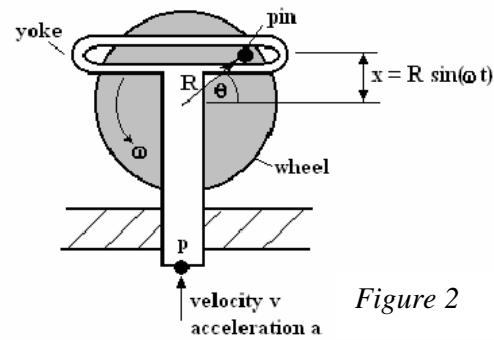
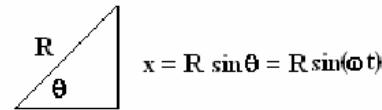


Figure 2



$$x = R \sin \theta = R \sin(\omega t)$$

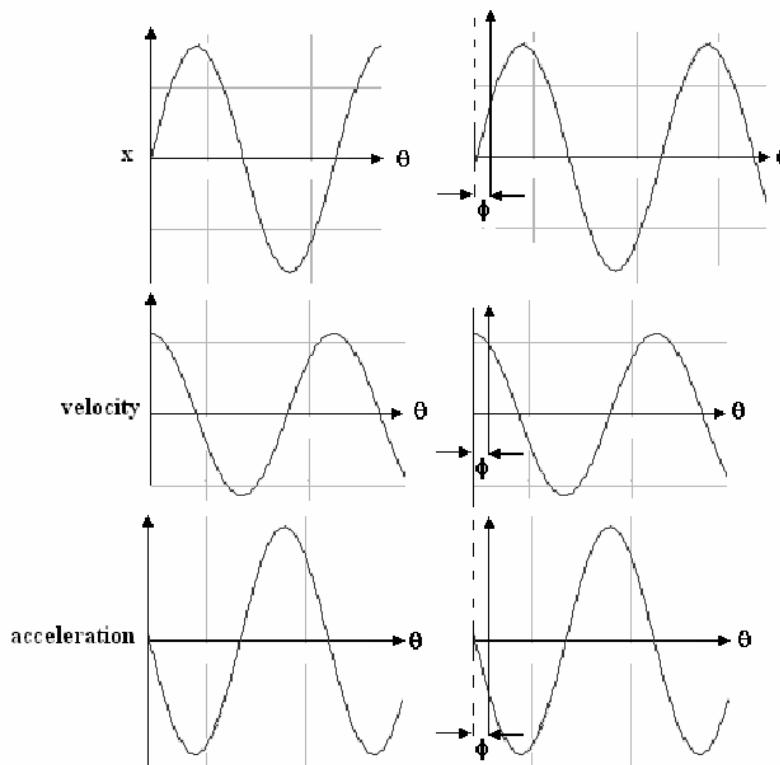


FIGURE 3a

FIGURE 3b

Angular Frequency, Frequency and Periodic time

ω is the angular velocity of the wheel but in any vibration such as the mass on the spring, it is called the angular frequency as no physical wheel exists.

The frequency of the wheel in revolutions/second is equivalent to the frequency of the vibration. If the wheel rotates at 2 rev/s the time of one revolution is $1/2$ seconds. If the wheel rotates at 5 rev/s the time of one revolution is $1/5$ second. If it rotates at f rev/s the time of one revolution is $1/f$. This formula is important and gives the periodic time.

Periodic Time T = time needed to perform one cycle.

f is the frequency or number of cycles per second.

It follows that: $T = 1/f$ and $f = 1/T$

Each cycle of an oscillation is equivalent to one rotation of the wheel and 1 revolution is an angle of 2π radians.

When $\theta = 2\pi$ and $t = T$.

It follows that since $\theta = \omega t$; then $2\pi = \omega T$

Rearrange and $\theta = 2\pi/T$. Substituting $T = 1/f$, then $\omega = 2\pi f$

Equations of S.H.M.

Consider the three equations derived earlier.

Displacement $x = R \sin(\omega t)$.

Velocity $v = dx/dt = \omega R \cos(\omega t)$ and Acceleration $a = dv/dt = -\omega^2 R \sin(\omega t)$

The plots of x , v and a against angle θ are shown on figure 3a. In the analysis so far made, we measured angle θ from the horizontal position and arbitrarily decided that the time was zero at this point.

Suppose we start the timing after the angle has reached a value of ϕ from this point. In these cases, ϕ is called the phase angle. The resulting equations for displacement, velocity and acceleration are then as follows.

Displacement $x = R \sin(\omega t + \phi)$.

Velocity $v = dx/dt = \omega R \cos(\omega t + \phi)$.

Acceleration $a = dv/dt = -\omega^2 R \sin((\omega t + \phi))$.

The plots of x , v and a are the same but the vertical axis is displaced by ϕ as shown on figure 3b. A point to note on figure 3a and 3b is that the velocity graph is shifted $\frac{1}{4}$ cycle (90°) to the left and the acceleration graph is shifted a further $\frac{1}{4}$ cycle making it $\frac{1}{2}$ cycle out of phase with x .

Forced vibrations: When the body vibrates under the influence of external force, then the body is said to be under forced vibrations. The external force, applied to the body is a periodic disturbing force created by unbalance. The vibrations have the same frequency as the applied force.

(Note: When the frequency of external force is same as that of the natural vibrations, resonance takes place)

Damped vibrations: When there is a reduction in amplitude over every cycle of vibration, the motion is said to be damped vibration. This is due to the fact that a certain amount of energy possessed by the vibrating system is always dissipated in overcoming frictional resistance to the motion.

Types of free vibrations:

Linear / Longitudinal vibrations: When the disc is displaced vertically downwards by an external force and released as shown in the figure 4, all the particles of the rod and disc move parallel to the axis of shaft. The rod is elongated and shortened alternately and thus the tensile and compressive stresses are induced alternately in the rod. The vibration occurs is known as *Linear/Longitudinal vibrations*.

Transverse vibrations: When the rod is displaced in the transverse direction by an external force and released as shown in the figure 5, all the particles of rod and disc move approximately perpendicular to the axis of the rod. The shaft is straight and bends alternately inducing bending stresses in the rod. The vibration occurs is known as *transverse vibrations*.

Torsional vibrations: When the rod is twisted about its axis by an external force and released as shown in the figure 6, all the particles of the rod and disc move in a circle about the axis of the rod. The rod is subjected to twist and torsional shear stress is induced. The vibration occurs is known as *torsional vibrations*.

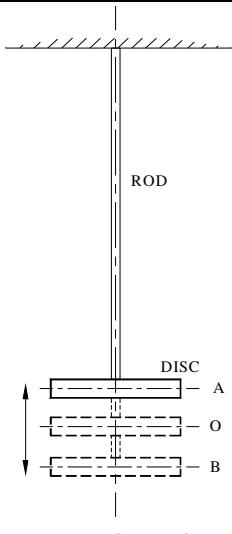


FIGURE 4

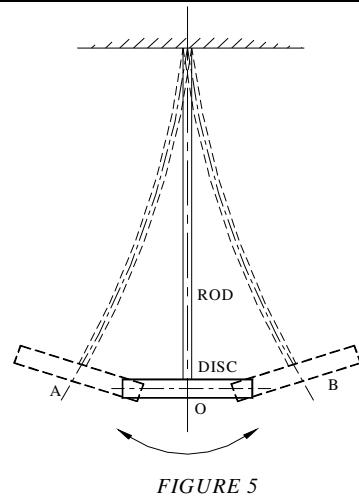


FIGURE 5

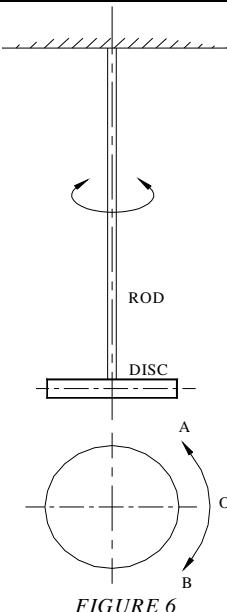
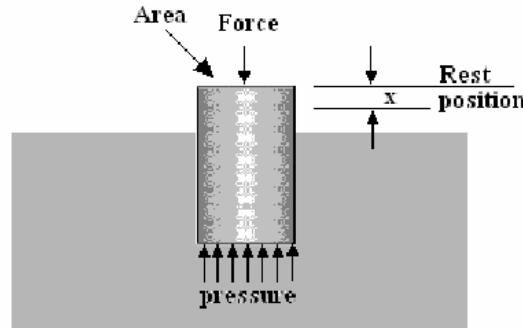


FIGURE 6

Oscillation of a floating body:

You may have observed that some bodies floating in water bob up and down. This is another example of simple harmonic motion and the restoring force in this case is **buoyancy**.

Consider a floating body of mass M kg. Initially it is at rest and all the forces acting on it add up to zero. Suppose a force F is applied to the top to push it down a distance x . The applied force F must overcome this buoyancy force and also overcome the inertia of the body.



Buoyancy force:

The pressure on the bottom increases by $\Delta p = \rho g x$.

The buoyancy force pushing it up on the bottom is F_b and this increases by $\Delta p A$.

Substitute for Δp and $F_b = \rho g x A$

Inertia force:

The inertia force acting on the body is $F_i = M a$

Balance of forces:

The applied force must be $F = F_i + F_b$ -this must be zero if the vibration is free.

$$0 = Ma + \rho g x A$$

$$a = -\frac{\rho g}{M} x$$

This shows that the acceleration is directly proportional to displacement and is always directed towards the rest position so the motion must be simple harmonic and the constant of proportionality must be the angular frequency squared.

$$\omega^2 = \frac{\rho A g}{M}$$

$$\omega = \sqrt{\frac{\rho A g}{M}}$$

$$f_n = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{\rho A g}{M}}$$

Example: A cylindrical rod is 80 mm diameter and has a mass of 5 kg. It floats vertically in water of density 1036 kg/m³. Calculate the frequency at which it bobs up and down. (Ans. 0.508 Hz)

Principal of super position:

The principle of superposition means that, when TWO or more waves meet, the wave can be added or subtracted.

Two waveforms combine in a manner, which simply adds their respective Amplitudes linearly at every point in time. Thus, a complex SPECTRUM can be built by mixing together different Waves of various amplitudes.

The principle of superposition may be applied to waves whenever two (or more) waves traveling through the same medium at the same time. The waves pass through each other without being disturbed. The net displacement of the medium at any point in space or time, is simply the sum of the individual wave dispacements.

General equation of physical systems is:

$m\ddot{x} + c\dot{x} + kx = F(t)$ - This equation is for a linear system, the inertia, damping and spring force are linear function \ddot{x} , \dot{x} and x respectively. This is not true case of non-linear systems.

$m\ddot{x} + \phi(\dot{x}) + f(x) = F(t)$ - Damping and spring force are not linear functions of \dot{x} and x

Mathematically for linear systems, if x_1 is a solution of;

$$m\ddot{x}_1 + c\dot{x}_1 + kx_1 = F_1(t)$$

and x_2 is a solution of;

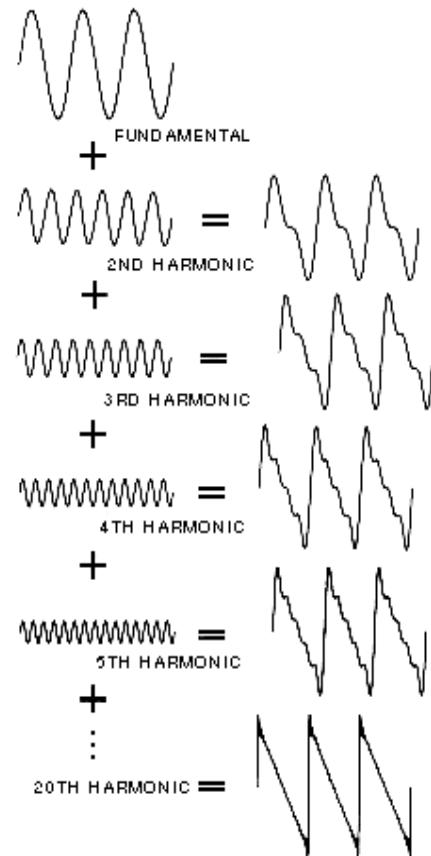
$$m\ddot{x}_2 + c\dot{x}_2 + kx_2 = F_2(t)$$

then $(x_1 + x_2)$ is a solution of;

$$m\ddot{x} + c\dot{x} + kx = F_1(t) + F_2(t)$$

Law of superposition does not hold good for non-linear systems.

If more than one wave is traveling through the medium: The resulting net wave is given by the *Superposition Principle given by the sum of the individual waveforms*"



3.2 (A) Undamped Free Vibrations

Undamped Free Vibrations:

NATURAL FREQUENCY OF FREE LONGITUDINAL VIBRATION

Equilibrium Method: Consider a body of mass 'm' suspended from a spring of negligible mass as shown in the figure 4.

Let m = Mass of the body

W = Weight of the body = mg

K = Stiffness of the spring

δ = Static deflection of the spring due to ' W '

By applying an external force, assume the body is displaced vertically by a distance ' x ', from the equilibrium position. On the release of external force, the unbalanced forces and acceleration imparted to the body are related by Newton Second Law of motion.

∴ The restoring force = $F = -k \times x$

(-ve sign indicates, the restoring force ' $k.x$ ' is opposite to the direction of the displacement ' x ')

By Newton's Law; $F = m \times a$

$$\therefore F = -kx = m \frac{d^2x}{dt^2}$$

∴ The differential equation of motion, if a body of mass ' m ' is acted upon by a restoring force ' k ' per unit displacement from the equilibrium position is;

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \quad - \text{This equation represents SHM}$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \omega^2 = \frac{k}{m} \quad - \text{for SHM}$$

The natural period of vibration is $T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}} \text{ Sec}$

The natural frequency of vibration is $f_n = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \text{ cycles/sec}$

From the figure 7; when the spring is strained by an amount of ' δ ' due to the weight $W = mg$
 $\delta k = mg$

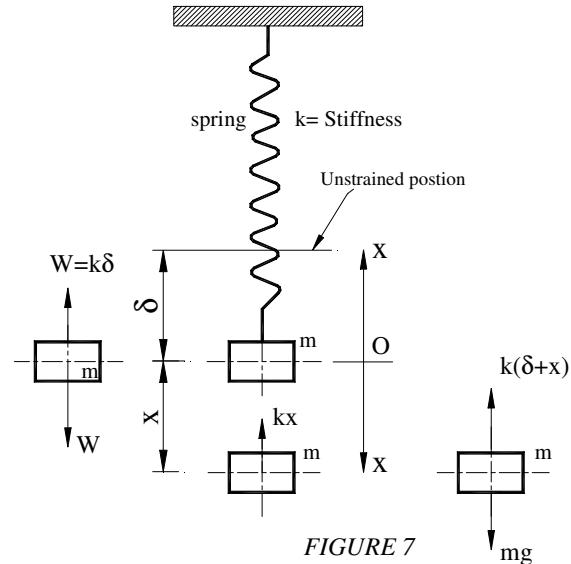


FIGURE 7

Hence

$$\frac{k}{m} = \frac{g}{\delta}$$

$$\therefore f_n = \frac{1}{2\pi} \sqrt{\frac{g}{\delta}} \quad Hz \text{ or cps}$$

Energy method: The equation of motion of a conservative system may be established from energy considerations. If a conservative system set in motion, the mechanical energy in the system is partially kinetic and partially potential. The *KE* is due to the velocity of mass and the *PE* is due to the strain energy of the spring by virtue of its deformation.

Since the system is conservative; and no energy is transmitted to the system and from the system in the free vibrations, the sum of *PE* and *KE* is constant. Both velocity of the mass and deformation of spring are cyclic. Thus, therefore be constant interchange of energy between the mass and the spring.

(*KE* is maximum, when *PE* is minimum and *PE* is maximum, when *KE* is minimum - so system goes through cyclic motion)

$$KE + PE = Constant$$

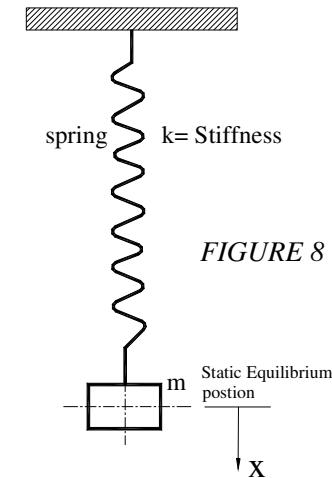
$$\frac{d}{dt}[KE+PE]=0 \quad -(1)$$

$$\text{We have } KE = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 \quad -(2)$$

Potential energy due to the displacement is equal to the strain energy in the spring, minus the *PE* change in the elevation of the mass.

$$\begin{aligned} \therefore PE &= \int_0^x (\text{Total spring force})dx - mg dx \\ &= \int_0^x (mg + kx - mg)dx = \frac{1}{2}kx^2 \quad -(3) \end{aligned}$$

Equation (1) becomes



$$\left[PE = \left(\frac{0+kx}{2} \right) x = \frac{1}{2}kx^2 \right]$$

$$\frac{d}{dt} \left[\left(\frac{dx}{dt} \right)^2 + \frac{1}{2}kx^2 \right] = 0$$

$$\left(m \frac{d^2x}{dt^2} + kx \right) \frac{dx}{dt} = 0$$

$$\text{Either } \left(m \frac{d^2x}{dt^2} + kx \right) = 0 \quad OR \quad \frac{dx}{dt} = 0$$

But velocity $\frac{dx}{dt}$ can be zero for all values of time.

$$\therefore m \frac{d^2x}{dt^2} + kx = 0 \quad [m\ddot{x} + kx = 0]$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \quad - \text{Equation represents SHM}$$

$$\therefore \text{Time period} = T = 2\pi \sqrt{\frac{m}{k}} \text{ sec} \quad \text{and}$$

$$\text{Natural frequency of vibration} = f_n = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \text{ cycles/sec}$$

(The natural frequency is inherent in the system. It is the function of the system parameters 'k' and 'm' and it is independent of the amplitude of oscillation or the manner in which the system is set into motion.)

Rayleigh's Method: The concept is an extension of energy method. We know, there is a constant interchange of energy between the PE of the spring and KE of the mass, when the system executes cyclic motion. At the static equilibrium position, the KE is maximum and PE is zero; similarly when the mass reached maximum displacement (amplitude of oscillation), the PE is maximum and KE is zero (velocity is zero). But due to conservation of energy total energy remains constant.

Assuming the motion executed by the vibration to be simple harmonic, then;

$$x = A \sin \omega t$$

x = displacement of the body from the mean position after time 't' sec and

A = Maximum displacement from the mean position

$$\dot{x} = A \omega \cos \omega t$$

At mean position, $t = 0$; Velocity is maximum

$$\therefore v_{\max} = \left(\frac{dx}{dt} \right)_{\max} = \dot{x}_{\max} = \omega A$$

$$\therefore \text{Maximum K. E} = \frac{1}{2} m \omega^2 A^2$$

$$\text{Maximum P.E} = \frac{1}{2} k x_{\max}^2 \quad x_{\max} = A$$

$$\therefore \text{Maximum P.E} = \frac{1}{2} k A^2$$

$$\text{We know} \quad (KE)_{\max} = (PE)_{\max}$$

$$m \omega^2 = k$$

$$\omega = \left(\frac{k}{m} \right)^{1/2}$$

$$f_n = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad \begin{aligned} & \left[\because \delta k = m g \right] \\ & \Rightarrow \frac{k}{m} = \frac{g}{\delta} \end{aligned}$$

1. Determine the natural frequency of the spring-mass system, taking mass of the spring into account.

Let l = Length of the spring under equilibrium condition

ρ = Mass/unit length of the spring
 m_s = Mass of the spring = $\rho \times l$

Consider an elemental length of 'dy' of the

spring at a distance 'y' from support.

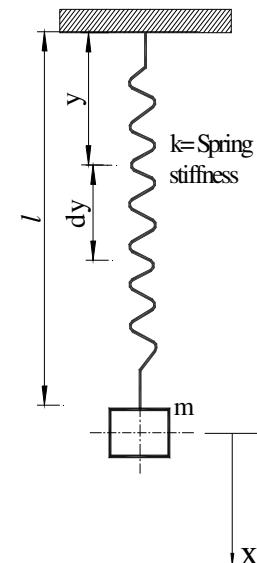
\therefore Mass of the element = ρdy

At any instant, the mass 'm' is displaced by a

$$\therefore PE = \frac{1}{2} k x^2$$

K E of the system at this instant,

is the sum of (KE) mass and (KE) spring



$$\begin{aligned}\therefore KE &= \frac{1}{2} m \dot{x}^2 + \int_0^l \frac{1}{2} (\rho dy) \left(\frac{y \dot{x}}{l} \right)^2 \\ &= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \rho \frac{\dot{x}^2}{l^2} \int_0^l y^2 dy \\ &= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \rho \frac{\dot{x}^2}{l^2} \cdot \frac{l^3}{3} \\ &= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \dot{x}^2 \cdot \frac{m_s}{3} = \frac{1}{2} \left[m + \frac{m_s}{3} \right] \dot{x}^2\end{aligned}$$

$$\frac{1}{2} k x^2 + \frac{1}{2} \left(m + \frac{m_s}{3} \right) \dot{x}^2 = 0$$

Differentiating with respect to 't'; $\frac{d}{dt}(PE + KE) = 0$

$$k x \dot{x} + \left(m + \frac{m_s}{3} \right) \dot{x} \ddot{x} = 0 \quad -\text{Differential equation}$$

$$\ddot{x} + \frac{k}{\left(m + \frac{m_s}{3} \right)} x = 0$$

$$\therefore f_n = \frac{1}{2\pi} \sqrt{\frac{k}{\left(m + \frac{m_s}{3} \right)}} \text{ cps} \quad [\rho l = m_s]$$

OR

$$\therefore f_n = \sqrt{\frac{k}{\left(m + \frac{m_s}{3} \right)}} \text{ radians/sec}$$

We know that $PE + KE = \text{Constant}$

2. Determine the natural frequency of the system shown in figure by Energy and Newton's method.

When mass 'm' moves down a distance 'x' from its equilibrium position, the center of the disc if mass m_1 moves down by $|x|$ and rotates through an angle θ .

$$\therefore \frac{x}{2} = r \theta$$

$$\Rightarrow \dot{\theta} = \frac{\dot{x}}{2r}$$

$$KE = (KE)_{Tr} + (KE)_{rot}$$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m_1 \frac{\dot{x}^2}{4} + \frac{1}{2} I_o \dot{\theta}^2$$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{8} m_1 \dot{x}^2 + \frac{1}{4} m_1 r^2 \frac{\dot{x}^2}{4r^2}$$

$$= \frac{1}{2} m \dot{x}^2 + \frac{3}{16} m_1 \dot{x}^2$$

$$PE = \frac{1}{2} k \left(\frac{x}{2} \right)^2$$

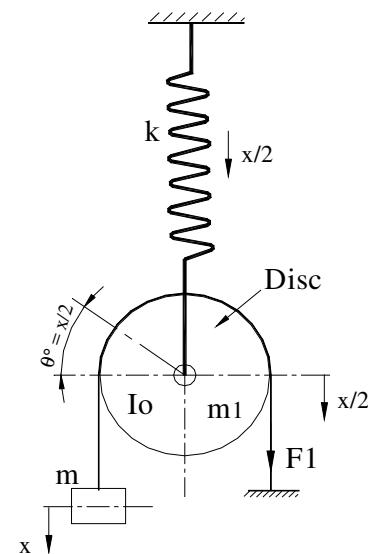
$$\frac{d}{dt} (KE + PE) = 0$$

$$\Rightarrow m \ddot{x} + \frac{3}{8} m_1 \ddot{x} + \frac{1}{4} kx = 0$$

$$\therefore \ddot{x} \left(\frac{8m + 3m_1}{8} \right) + \frac{1}{4} kx = 0$$

$$\therefore f_n = \frac{1}{2\pi} \sqrt{\frac{2k}{8m + 3m_1}} \text{ cps}$$

$$\text{or } f_n = \sqrt{\frac{2k}{8m + 3m_1}} \text{ rad/sec}$$



Newton's Method: Use x, as co-ordinate

$$\text{or mass } m; m \ddot{x} = -F \quad -1$$

$$\text{for disc } m_1; m_1 \frac{\ddot{x}}{2} = F + F_1 - \frac{kx}{2} \quad -2$$

$$I_o \ddot{\theta} = Fr - F_1 r \quad -3$$

$$\text{Substituting (1) in (2) and (3) and replace } \ddot{\theta} \text{ by } \frac{\ddot{x}}{2r}$$

$$m_1 \frac{\ddot{x}}{2} = -m \ddot{x} + F_1 - \frac{kx}{2} \quad -4$$

$$I_o \frac{\ddot{x}}{2r} = -m \ddot{x} r - F_1 r \quad -5$$

$$I_o \frac{\ddot{x}}{2r^2} = -m \ddot{x} - F_1 \quad -6$$

Adding equations (4) and (6)

$$m_1 \frac{\ddot{x}}{2} + I_o \frac{\ddot{x}}{2r^2} = -2m \ddot{x} - \frac{kx}{2} \quad \{I_o = \frac{1}{2} m_1 r^2\}$$

$$\ddot{x} \left(m_1 + \frac{m_1}{2} + 4m \right) + kx = 0$$

$$f_n = \sqrt{\frac{2k}{3m_1 + 8m}} \text{ rad/sec}$$

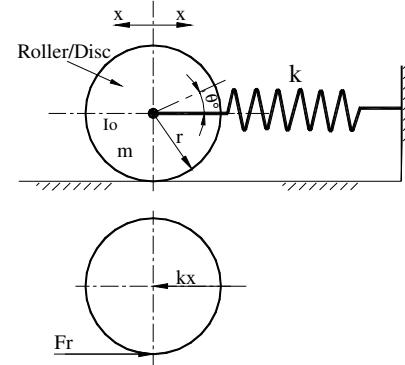
$$f_n = \frac{1}{2\pi} \sqrt{\frac{2k}{3m_1 + 8m}} \text{ cps or Hz}$$

3. Determine the natural frequency of the system shown in figure by Energy and Newton's method. Assume the cylinder rolls on the surface without slipping.

a) Energy method:

When mass 'm' rotates through an angle θ , the center of the roller move distance 'x'

$$\begin{aligned} KE &= (KE)_{Tr} + (KE)_{rot} \\ &= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I_o \dot{\theta}^2 \\ &= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \left(\frac{1}{2} m r^2 \right) \frac{\dot{x}^2}{r^2} \\ \therefore KE &= \frac{3}{4} m \dot{x}^2 \\ PE &= \frac{1}{2} k x^2 \\ \frac{d}{dt} (KE + PE) &= 0 \\ \Rightarrow \frac{3}{2} m \ddot{x} + k x &= 0 \\ \text{Natural frequency } f_n &= \frac{1}{2\pi} \sqrt{\frac{2k}{3m}} \end{aligned}$$



Newton's method:

$$ma = \sum F$$

$$m\ddot{x} = -kx + F_r$$

using torque equation $I_o \ddot{\theta} = -F_r r$

$$F_r = -\frac{m\ddot{x}}{2}$$

$$m\ddot{x} = -kx - \frac{1}{2}m\ddot{x}$$

$$\frac{3}{2}m\ddot{x} + kx = 0$$

$$\therefore f_n = \frac{1}{2\pi} \sqrt{\frac{2k}{3m}} \text{ cps} \quad \text{or} \quad f_n = \sqrt{\frac{2k}{3m}} \text{ rad/sec}$$

4. Determine the natural frequency of the system shown in figure by Energy and Newton's method.

Energy Method: Use θ or x as coordinate

$$KE = (KE)_{Tr} + (KE)_{rot}$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_o\dot{\theta}^2$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}m_1r^2\right)\frac{\dot{x}^2}{r^2}$$

$$\therefore KE = \left(\frac{1}{2}m\dot{x}^2 + \frac{1}{4}m_1\dot{x}^2\right)$$

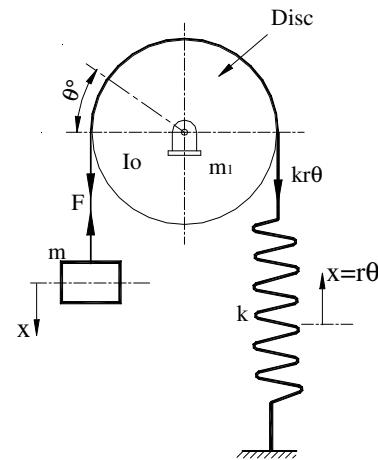
$$PE = \frac{1}{2}kx^2$$

$$\frac{d}{dt}(KE + PE) = 0$$

$$\Rightarrow kx\ddot{x} + \left(\frac{1}{2}m + \frac{1}{4}m_1\right)2\dot{x}\ddot{x} = 0$$

$$kx + \left(m + \frac{1}{2}m_1\right)\dot{x} = 0$$

$$\text{Natural frequency } f_n = \sqrt{\frac{2k}{2m+m_1}} \text{ rad/sec} = \frac{1}{2\pi} \sqrt{\frac{2k}{2m+m_1}} \text{ cps OR } \frac{1}{2\pi} \sqrt{\frac{kr^2}{I_o + mr^2}} \text{ cps or Hz}$$



Newton's Method: Consider motion of the disc with ' θ ' as coordinate.

For the mass 'm' : $m\ddot{x} = -F_r$ -(1)

$$\text{For the disc} : I_o\ddot{\theta} = -F_r r - kr^2\theta \quad \text{-(2)}$$

Substitute (1) in (2):

$$I_o\ddot{\theta} = -mr\ddot{x} - kr^2\theta$$

$$I_o\ddot{\theta} = -mr^2\ddot{\theta} - kr^2\theta$$

$$(I_o + mr^2)\ddot{\theta} + kr^2\theta = 0$$

$$\therefore f_n = \sqrt{\frac{kr^2}{I_o + mr^2}} \text{ rad/sec}$$

$$= \frac{1}{2\pi} \sqrt{\frac{kr^2}{I_o + mr^2}} \text{ cps or Hz}$$

$$= \frac{1}{2\pi} \sqrt{\frac{2k}{m_1 + 2m}} \text{ cps} \quad \left(I_o = \frac{1}{2}m_1r^2 \right)$$

5. Determine the natural frequency of the system shown in figure 5 by Energy and Newton's method. Assume the cylinder rolls on the surface without slipping.

Energy Method:

$$KE = (KE)_{Tr} + (KE)_{rot}$$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I_o \dot{\theta}^2$$

$$= \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2} \left(\frac{1}{2} m r^2 \right) \dot{\theta}^2$$

$$\therefore KE = \frac{3}{4} m r^2 \dot{\theta}^2$$

$$PE = 2 \left(\frac{1}{2} k x^2 \right) = k (r+a)^2 \theta^2$$

$$\frac{d}{dt} (KE + PE) = 0$$

$$\Rightarrow \frac{3}{2} m r^2 \ddot{\theta} \dot{\theta} + 2 k (r+a)^2 \theta \dot{\theta} = 0$$

$$3 m r^2 \ddot{\theta} + 4 k (r+a)^2 \theta = 0$$

$$\text{Natural frequency } f_n = \sqrt{\frac{4k(r+a)^2}{3mr^2}} \text{ rad/sec} = \frac{1}{2\pi} \sqrt{\frac{4k(r+a)^2}{3mr^2}} \text{ cps or Hz}$$

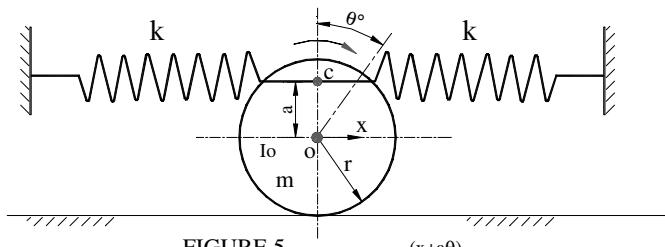


FIGURE 5

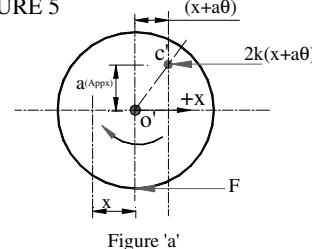


Figure 'a'

Newton's Method: Considering combined translation and rotational motion as shown in Figure 'a'.

Hence it must satisfy:

$$m \ddot{x} = \sum \text{Force in } x \text{ direction}$$

$$m \ddot{x} = -F - 2k(x + a\theta)$$

$$\text{and } I_o \ddot{\theta} = \sum \text{Torque about } \theta$$

$$= Fr - 2k(x + a\theta)a$$

$$mr \ddot{\theta} = -F - 2k(r+a)\theta \quad -(1)$$

$$\text{and } \frac{1}{2} mr^2 \ddot{\theta} = Fr - 2k(r+a)a\theta \quad -(2)$$

Multiply equation (1) by $2r$ and (2) by 2. Then add

$$3mr^2 \ddot{\theta} = -4kr(r+a)\theta - 4k(r+a)a\theta$$

$$3mr^2 \ddot{\theta} + 4k[r(r+a) + (r+a)a]\theta = 0$$

$$3mr^2 \ddot{\theta} + 4k(r+a)^2 = 0$$

$$\text{Natural frequency } f_n = \sqrt{\frac{4k(r+a)^2}{3mr^2}} \text{ rad/sec} = \frac{1}{2\pi} \sqrt{\frac{4k(r+a)^2}{3mr^2}} \text{ cps or Hz}$$

Refer PPT – For more problems

3.2 (B) Damped Freee Vibrations Single DoF

Introduction:

Damping – dissipation of energy.

For a system to vibrate, it requires energy. During vibration of the system, there will be continuous transformation of energy. Energy will be transformed from potential/strain to kinetic and vice versa.

In case of undamped vibrations, there will not be any dissipation of energy and the system vibrates at constant amplitude. Ie, once excited, the system vibrates at constant amplitude for infinite period of time. But this is a purely hypothetical case. But in an actual vibrating system, energy gets dissipated from the system in different forms and hence the amplitude of vibration gradually dies down. Fig.1 shows typical response curves of undamped and damped free vibrations.

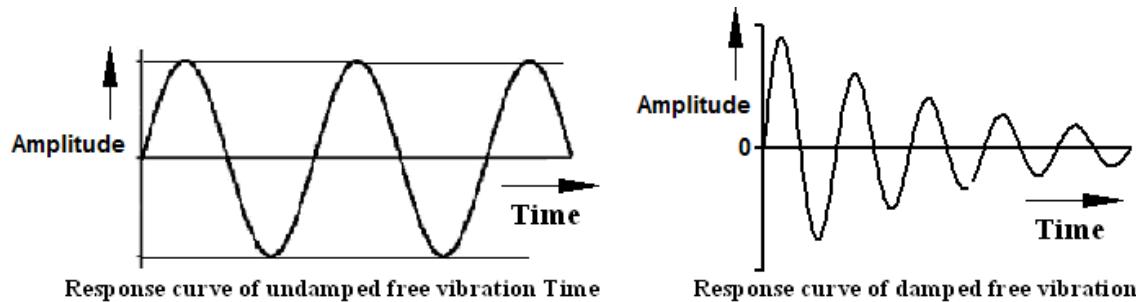


Fig 1

Types o damping:

(i) Viscous damping

In this type of damping, the damping resistance is proportional to the relative velocity between the vibrating system and the surroundings. For this type of damping, the differential equation of the system becomes linear and hence the analysis becomes easier. A schematic representation of viscous damper is shown in Fig.2.

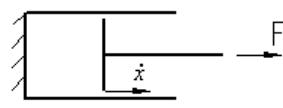


Fig 2

Here, $F \propto \dot{x}$ or $F = c\dot{x}$, where, F is damping resistance, \dot{x} is relative velocity and c is the damping coefficient.

(ii) Dry friction or Coulomb damping

In this type of damping, the damping resistance is independent of rubbing velocity and is practically constant.

(iii) Structural damping

This type of damping is due to the internal friction within the structure of the material, when it is deformed.

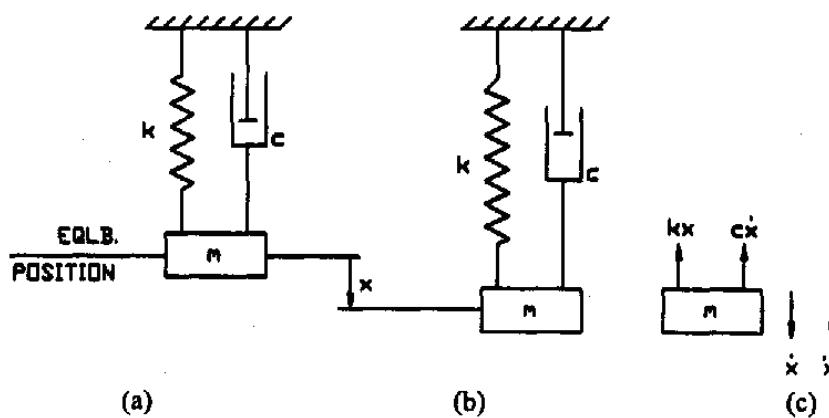
Spring-mass-damper system:

Fig 3

Fig.3 shows the schematic of a simple spring-mass-damper system, where, m is the mass of the system, k is the stiffness of the system and c is the damping coefficient.

If x is the displacement of the system, from Newton's second law of motion, it can be written

$$m\ddot{x} = -c\dot{x} - kx$$

$$\text{Ie } m\ddot{x} + c\dot{x} + kx = 0 \quad (1)$$

This is a linear differential equation of the second order and its solution can be written as

$$x = e^{st} \quad (2)$$

Differentiating (2),

$$\frac{dx}{dt} = \dot{x} = se^{st}$$

$$\frac{d^2x}{dt^2} = \ddot{x} = s^2 e^{st}$$

Substituting in (1),

$$ms^2 e^{st} + cse^{st} + ke^{st} = 0$$

$$(ms^2 + cs + k)e^{st} = 0$$

$$\text{Or } ms^2 + cs + k = 0 \quad (3)$$

Equation (3) is called the characteristic equation of the system, which is quadratic in s .

The two values of s are given by

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \quad (4)$$

The general solution for (1) may be written as

$$x = C_1 e^{s_1 t} + C_2 e^{s_2 t} \quad (5)$$

Where, C_1 and C_2 are arbitrary constants, which can be determined from the initial conditions.

In equation (4), the values of $s_1 = s_2$, when $\left(\frac{c}{2m}\right)^2 = \frac{k}{m}$

$$\text{Or, } \left(\frac{c}{2m}\right) = \sqrt{\frac{k}{m}} = \omega_n \quad (6)$$

Or $c = 2m\omega_n$, which is the property of the system and is called critical damping coefficient and is represented by c_c .

$$\text{Ie, critical damping coefficient} = c_c = 2m\omega_n$$

The ratio of actual damping coefficient c and critical damping coefficient c_c is called damping factor or damping ratio and is represented by ζ .

$$\text{Ie, } \zeta = \frac{c}{c_c} \quad (7)$$

In equation (4), $\frac{c}{2m}$ can be written as $\frac{c}{2m} = \frac{c}{c_c} \times \frac{c_c}{2m} = \zeta \cdot \omega_n$

$$\text{Therefore, } s_{1,2} = -\zeta \cdot \omega_n \pm \sqrt{(\zeta \cdot \omega_n)^2 - \omega_n^2} = [-\zeta \pm \sqrt{\zeta^2 - 1}] \omega_n \quad (8)$$

The system can be analyzed for three conditions.

- (i) $\zeta > 1$, ie, $c > c_c$, which is called over damped system.
- (ii) $\zeta = 1$. ie, $c = c_c$, which is called critically damped system.
- (iii) $\zeta < 1$, ie, $c < c_c$, which is called under damped system.

Depending upon the value of ζ , value of s in equation (8), will be real and unequal, real and equal and complex conjugate respectively.

(i) Analysis of over-damped system ($\zeta > 1$).

In this case, values of s are real and are given by

$$s_1 = [-\zeta + \sqrt{\zeta^2 - 1}] \omega_n \text{ and } s_2 = [-\zeta - \sqrt{\zeta^2 - 1}] \omega_n$$

Then, the solution of the differential equation becomes

$$x = C_1 e^{[-\zeta + \sqrt{\zeta^2 - 1}] \omega_n t} + C_2 e^{[-\zeta - \sqrt{\zeta^2 - 1}] \omega_n t} \quad (9)$$

This is the final solution for an over damped system and the constants C_1 and C_2 are obtained by applying initial conditions. Typical response curve of an over damped system is shown in fig.4. The amplitude decreases exponentially with time and becomes zero at $t = \infty$.

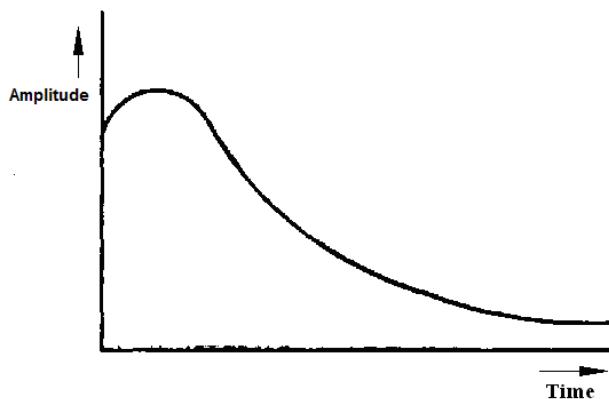


Fig 4 Typical response curve of an over damped system

(ii) Analysis of critically damped system ($\zeta = 1$).

In this case, based on equation (8), $s_1 = s_2 = -\omega_n$

The solution of the differential equation becomes

$$\begin{aligned}x &= C_1 e^{s_1 t} + C_2 t e^{s_2 t} \\ \text{Ie, } x &= C_1 e^{-\omega_n t} + C_2 t e^{-\omega_n t} \\ \text{Or, } x &= (C_1 + C_2 t) e^{-\omega_n t}\end{aligned}\quad (10)$$

This is the final solution for the critically damped system and the constants C_1 and C_2 are obtained by applying initial conditions. Typical response curve of the critically damped system is shown in fig.5. In this case, the amplitude decreases at much faster rate compared to over damped system.

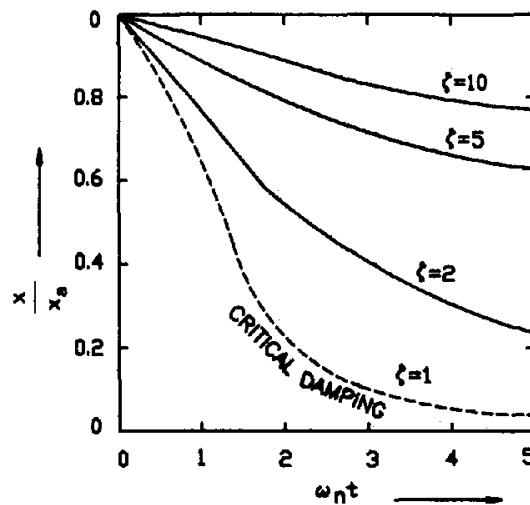


Fig 5 Displacement-time plots of over-damped and critically damped systems with zero starting velocity

(iii) Analysis of under damped system ($\zeta < 1$).

In this case, the roots are complex conjugates and are given by

$$s_1 = -\zeta + j\sqrt{1-\zeta^2} \omega_n$$

$$s_2 = -\zeta - j\sqrt{1-\zeta^2} \omega_n$$

The solution of the differential equation becomes

$$x = C_1 e^{-\zeta \omega_n t} e^{j\sqrt{1-\zeta^2} \omega_n t} + C_2 e^{-\zeta \omega_n t} e^{-j\sqrt{1-\zeta^2} \omega_n t}$$

This equation can be rewritten as

$$x = e^{-\zeta \omega_n t} \left[C_1 e^{j\sqrt{1-\zeta^2} \omega_n t} + C_2 e^{(-j\sqrt{1-\zeta^2}) \omega_n t} \right] \quad (11)$$

Using the relationships

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{-i\theta} = \cos \theta - i \sin \theta$$

Equation (11) can be written as

$$x = e^{-\zeta \omega_n t} \left[C_1 \left\{ \cos \sqrt{1-\zeta^2} \omega_n t + j \sin \sqrt{1-\zeta^2} \omega_n t \right\} + C_2 \left\{ \cos \sqrt{1-\zeta^2} \omega_n t - j \sin \sqrt{1-\zeta^2} \omega_n t \right\} \right]$$

$$\text{Or } x = e^{-\zeta \omega_n t} \left[(C_1 + C_2) \left\{ \cos \sqrt{1-\zeta^2} \omega_n t \right\} + j(C_1 - C_2) \left\{ \sin \sqrt{1-\zeta^2} \omega_n t \right\} \right] \quad (12)$$

In equation (12), constants (C_1+C_2) and $j(C_1-C_2)$ are real quantities and hence, the equation can also be written as

$$x = e^{-\zeta \omega_n t} \left[A \left\{ \cos \sqrt{1-\zeta^2} \omega_n t \right\} + B \left\{ \sin \sqrt{1-\zeta^2} \omega_n t \right\} \right]$$

$$\text{Or, } x = A_1 e^{-\zeta \omega_n t} \left[\sin \left(\sqrt{1-\zeta^2} \omega_n t + \phi_1 \right) \right] \quad (13)$$

The above equations represent oscillatory motion and the frequency of this motion is represented by $\omega_d = \sqrt{1-\zeta^2} \omega_n$ (14)

Where, ω_d is the damped natural frequency of the system. Constants A_1 and Φ_1 are determined by applying initial conditions. The typical response curve of an under damped system is shown in Fig.6.

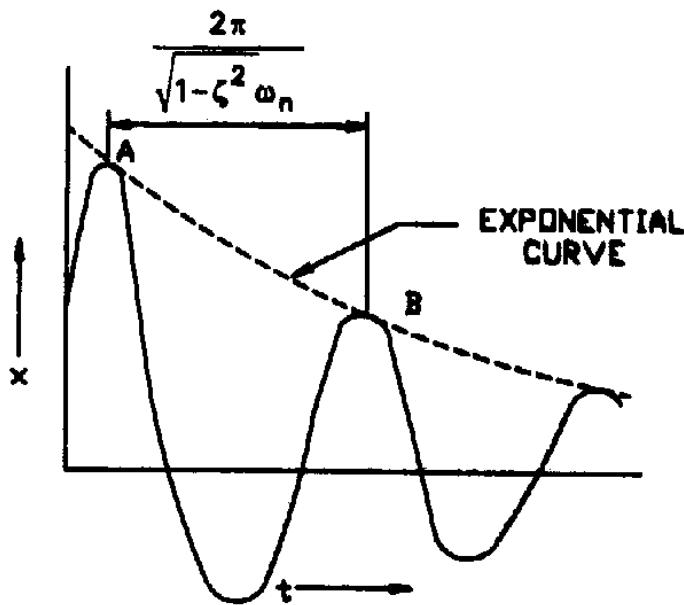


Fig 6 Typical response curve of an under damped system

Applying initial conditions,

$x = X_0$ at $t = 0$; and $\dot{x} = 0$ at $t = 0$, and finding constants A_1 and Φ_1 , equation (13) becomes

$$x = \frac{X_o}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \left[\left\{ \sin \sqrt{1-\zeta^2} \omega_n t + \tan^{-1} \left(\frac{\sqrt{1-\zeta^2}}{\zeta} \right) \right\} \right] \quad (15)$$

The term $\frac{X_o}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}$ represents the amplitude of vibration, which is observed to decay exponentially with time.

LOGARITHMIC DECREMENT

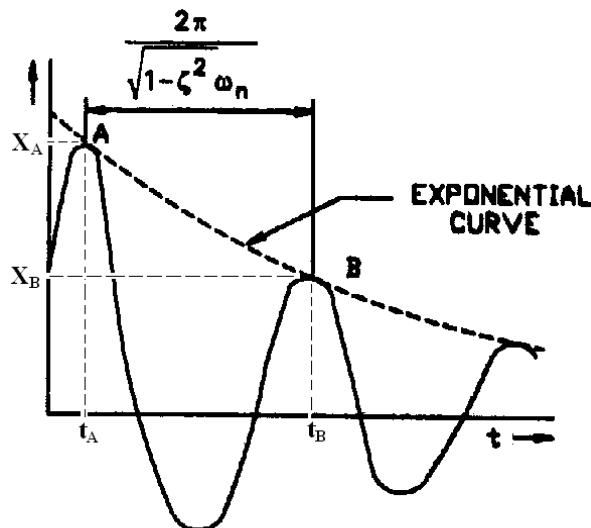


Fig.7 Logarithmic decrement

Referring to Fig.7, points A & B represent two successive peak points on the response curve of an under damped system. X_A and X_B represent the amplitude corresponding to points A & B and t_A & t_B represents the corresponding time.

We know that the natural frequency of damped vibration = $\omega_d = \sqrt{1-\zeta^2} \omega_n$ rad/sec.

$$\text{Therefore, } f_d = \frac{\omega_d}{2\pi} \text{ cycles/sec}$$

$$\text{Hence, time period of oscillation} = t_B - t_A = \frac{1}{f_d} = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{1-\zeta^2} \omega_n} \text{ sec} \quad (16)$$

From equation (15), amplitude of vibration

$$X_A = \frac{X_o}{\sqrt{1-\zeta^2}} e^{-\zeta \omega_n t_A}$$

$$X_B = \frac{X_o}{\sqrt{1-\zeta^2}} e^{-\zeta \omega_n t_B}$$

$$\text{Or, } \frac{X_A}{X_B} = e^{-\zeta \omega_n (t_A - t_B)} = e^{\zeta \omega_n (t_B - t_A)}$$

$$\text{Using eqn. (16), } \frac{X_A}{X_B} = e^{\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$\text{Or, } \log_e \frac{X_A}{X_B} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

This is called logarithmic decrement. It is defined as the logarithmic value of the ratio of two successive amplitudes of an under damped oscillation. It is normally denoted by δ .

$$\text{Therefore, } \delta = \log_e \frac{X_A}{X_B} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \quad (17)$$

This indicates that the ratio of any two successive amplitudes of an under damped system is constant and is a function of damping ratio of the system.

For small values of ζ , $\delta \approx 2\pi\zeta$

If X_0 represents the amplitude at a particular peak and X_n represents the amplitude after 'n' cycles, then, logarithmic decrement $= \delta = \log_e \frac{X_0}{X_1} = \log_e \frac{X_1}{X_2} = \dots = \log_e \frac{X_{n-1}}{X_n}$

Adding all the terms, $n\delta = \log_e \frac{X_0}{X_1} \times \frac{X_1}{X_2} \times \dots \times \frac{X_{n-1}}{X_n}$

$$\text{Or, } \delta = \frac{1}{n} \log_e \frac{X_0}{X_n} \quad (18)$$

Solved problems

1) The mass of a spring-mass-dashpot system is given an initial velocity $5\omega_n$, where ω_n is the undamped natural frequency of the system. Find the equation of motion for the system, when (i) $\zeta = 2.0$, (ii) $\zeta = 1.0$, (iii) $\zeta = 0.2$.

Solution:

Case (i) For $\zeta = 2.0$ – Over damped system

For over damped system, the response equation is given by

$$x = C_1 e^{\left[-\zeta + \sqrt{\zeta^2 - 1}\right]\omega_n t} + C_2 e^{\left[-\zeta - \sqrt{\zeta^2 - 1}\right]\omega_n t}$$

$$\text{Substituting } \zeta = 2.0, \quad x = C_1 e^{-0.27\omega_n t} + C_2 e^{-3.73\omega_n t} \quad (\text{a})$$

$$\text{Differentiating,} \quad \dot{x} = -0.27\omega_n C_1 e^{-0.27\omega_n t} - 3.73\omega_n C_2 e^{-3.73\omega_n t} \quad (\text{b})$$

Substituting the initial conditions

$$x = 0 \text{ at } t = 0; \text{ and } \dot{x} = 5\omega_n \text{ at } t = 0 \text{ in (a) \& (b),}$$

$$0 = C_1 + C_2 \quad (\text{c})$$

$$5\omega_n = -0.27\omega_n C_1 - 3.73\omega_n C_2 \quad (\text{d})$$

$$\text{Solving (c) \& (d),} \quad C_1 = 1.44 \text{ and } C_2 = -1.44.$$

Therefore, the response equation becomes

$$x = 1.44 \left(e^{-0.27\omega_n t} - e^{-3.73\omega_n t} \right) \quad (\text{e})$$

Case (ii) For $\zeta = 1.0$ – Critically damped system

For critically damped system, the response equation is given by

$$x = (C_1 + C_2 t) e^{-\omega_n t} \quad (\text{f})$$

$$\text{Differentiating,} \quad \dot{x} = -(C_1 + C_2 t)\omega_n e^{-\omega_n t} + C_2 e^{-\omega_n t} \quad (\text{g})$$

Substituting the initial conditions

$$x = 0 \text{ at } t = 0; \text{ and } \dot{x} = 5\omega_n \text{ at } t = 0 \text{ in (f) \& (g),}$$

$$C_1 = 0 \text{ and } C_2 = 5\omega_n$$

Substituting in (f), the response equation becomes

$$x = (5\omega_n t) e^{-\omega_n t} \quad (\text{h})$$

Case (iii) For $\zeta = 0.2$ – under damped system

For under damped system, the response equation is given by

$$x = A_1 e^{-\zeta\omega_n t} \left[\sin \sqrt{1 - \zeta^2} \omega_n t + \phi_1 \right]$$

$$\text{Substituting } \zeta = 0.2, \quad x = A_1 e^{-0.2\omega_n t} \left[\sin(0.98\omega_n t + \phi_1) \right] \quad (\text{p})$$

Differentiating,

$$\dot{x} = -0.2\omega_n A_1 e^{-0.2\omega_n t} [\sin(0.98\omega_n t + \phi_1)] + 0.98\omega_n A_1 e^{-0.2\omega_n t} \cos(0.98\omega_n t + \phi_1) \quad (\text{q})$$

Substituting the initial conditions

$$x = 0 \text{ at } t = 0; \text{ and } \dot{x} = 5\omega_n \text{ at } t = 0 \text{ in (p) \& (q),}$$

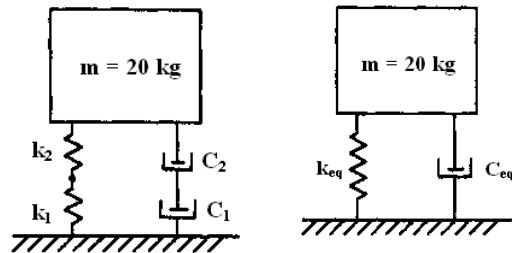
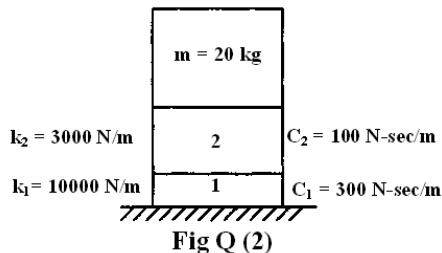
$$A_1 \sin \Phi_1 = 0 \text{ and } A_1 \cos \Phi_1 = 5.1$$

$$\text{Solving, } A_1 = 5.1 \text{ and } \Phi_1 = 0$$

Substituting in (p), the response equation becomes

$$x = 5.1 e^{-0.2\omega_n t} [\sin(0.98\omega_n t)] \quad (\text{r})$$

2) A mass of 20kg is supported on two isolators as shown in fig.Q.2. Determine the undamped and damped natural frequencies of the system, neglecting the mass of the isolators.



Solution:

Equivalent stiffness and equivalent damping coefficient are calculated as

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} = \frac{1}{10000} + \frac{1}{3000} = \frac{13}{30000}$$

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{300} + \frac{1}{100} = \frac{4}{300}$$

$$\text{Undamped natural frequency} = \omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{30000/13}{20}} = 10.74 \text{ rad/sec}$$

$$f_n = \frac{10.74}{2\pi} = 1.71 \text{ cps}$$

$$\text{Damped natural frequency} = \omega_d = \sqrt{1 - \zeta^2} \omega_n$$

$$\zeta = \frac{C_{eq}}{2\sqrt{k_{eq}m}} = \frac{300/4}{2 \times \sqrt{30000/13 \times 20}} = 0.1745$$

$$\therefore \omega_d = \sqrt{1 - 0.1745^2} \times 10.74 = 10.57 \text{ rad/sec}$$

Or,

$$f_d = \frac{10.57}{2 \times \pi} = 1.68 \text{ cps}$$

3) A gun barrel of mass 500kg has a recoil spring of stiffness 3,00,000 N/m. If the barrel recoils 1.2 meters on firing, determine,

- (a) initial velocity of the barrel
- (b) critical damping coefficient of the dashpot which is engaged at the end of the recoil stroke
- (c) time required for the barrel to return to a position 50mm from the initial position.

Solution:

- (a) Strain energy stored in the spring at the end of recoil:

$$P = \frac{1}{2} kx^2 = \frac{1}{2} \times 300000 \times 1.2^2 = 216000 \text{ N-m}$$

Kinetic energy lost by the gun barrel:

$$T = \frac{1}{2} mv^2 = \frac{1}{2} \times 500 \times v^2 = 250v^2, \text{ where } v = \text{initial velocity of the barrel}$$

Equating kinetic energy lost to strain energy gained, ie $T = P$,

$$250v^2 = 216000$$

$$v = 29.39 \text{ m/s}$$

(b) Critical damping coefficient = $C_c = 2\sqrt{km} = 2\sqrt{300000 \times 500} = 24495 \text{ N-sec/m}$

(c) Time for recoiling of the gun (undamped motion):

$$\text{Undamped natural frequency} = \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{300000}{500}} = 24.49 \text{ r/s}$$

$$\text{Time period} = \tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{24.49} = 0.259 \text{ sec}$$

$$\text{Time of recoil} = \frac{\tau}{4} = \frac{0.259}{4} = 0.065 \text{ sec}$$

Time taken during return stroke:

Response equation for critically damped system = $x = (C_1 + C_2 t)e^{-\omega_n t}$

$$\text{Differentiating, } \dot{x} = C_2 e^{-\omega_n t} - (C_1 + C_2 t)\omega_n e^{-\omega_n t}$$

Applying initial conditions, $x = 1.2$, at $t = 0$ and $\dot{x} = 0$ at $t = 0$,

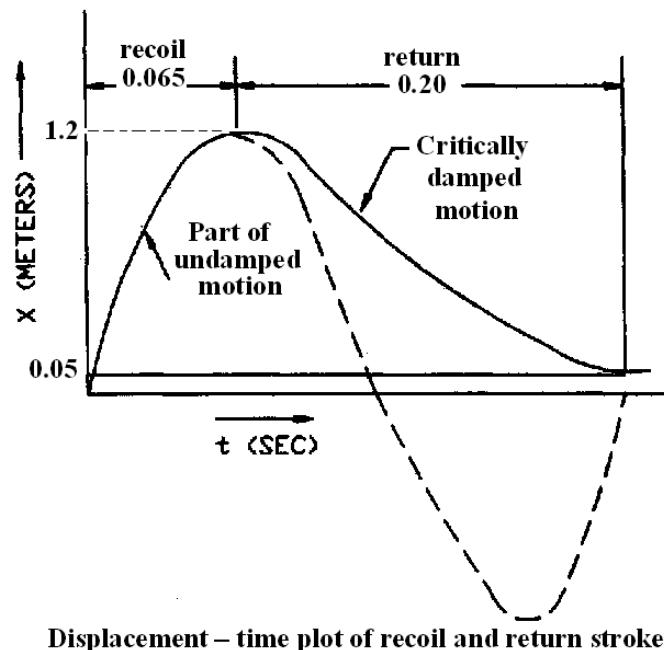
$$C_1 = 1.2, \text{ & } C_2 = 29.39$$

Therefore, the response equation = $x = (1.2 + 29.39t)e^{-24.49t}$

When $x = 0.05 \text{ m}$, by trial and error, $t = 0.20 \text{ sec}$

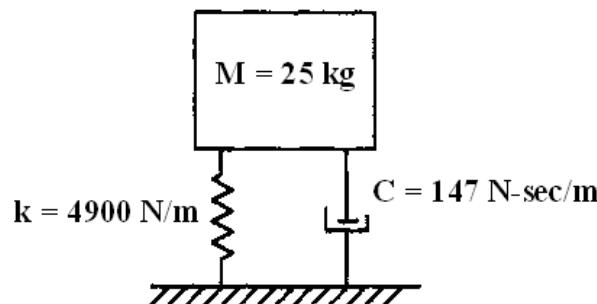
Therefore, total time taken = time for recoil + time for return = $0.065 + 0.20 = 0.265 \text{ sec}$

The displacement – time plot is shown in the following figure.



- 4) A 25 kg mass is resting on a spring of 4900 N/m and dashpot of 147 N-sec/m in parallel. If a velocity of 0.10 m/sec is applied to the mass at the rest position, what will be its displacement from the equilibrium position at the end of first second?

Solution:



The above figure shows the arrangement of the system.

$$\text{Critical damping coefficient} = c_c = 2m\omega_n$$

$$\text{Where } \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4900}{25}} = 14 \text{ r/s}$$

$$\text{Therefore, } c_c = 2 \times 25 \times 14 = 700 \text{ N-sec/m}$$

$$\text{Since } C < C_c, \text{ the system is under damped and } \zeta = \frac{c}{c_c} = \frac{147}{700} = 0.21$$

Hence, the response equation is $x = A_1 e^{-\zeta \omega_n t} [\sin(\sqrt{1 - \zeta^2} \omega_n t + \phi_1)]$

$$\text{Substituting } \zeta \text{ and } \omega_n, x = A_1 e^{-0.21 \times 14t} [\sin(\sqrt{1 - 0.21^2} 14t + \phi_1)] \\ x = A_1 e^{-2.94t} [\sin(13.7t + \phi_1)]$$

$$\text{Differentiating, } \dot{x} = -2.94 A_1 e^{-2.94t} [\sin(13.7t + \phi_1)] + 13.7 A_1 e^{-2.94t} \cos(13.7t + \phi_1)$$

Applying the initial conditions, $x = 0$, at $t = 0$ and $\dot{x} = 0.10 \text{ m/s}$ at $t = 0$

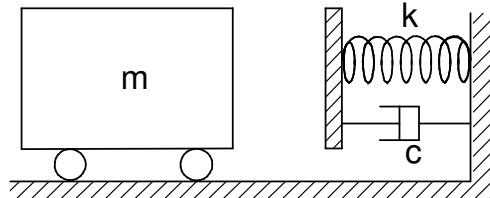
$$\Phi_1 = 0$$

$$0.10 = -2.94 A_1 [\sin(\phi_1)] + 13.7 A_1 \cos(\phi_1)$$

Since, $\Phi_1 = 0$, $0.10 = 13.7 A_1$; $A_1 = 0.0073$

$$\text{Displacement at the end of 1 second} = x = 0.0073 e^{-2.94} [\sin(13.7)] = 3.5 \times 10^{-4} \text{ m}$$

5) A rail road bumper is designed as a spring in parallel with a viscous damper. What is the bumper's damping coefficient such that the system has a damping ratio of 1.25, when the bumper is engaged by a rail car of 20000 kg mass. The stiffness of the spring is 2E5 N/m. If the rail car engages the bumper, while traveling at a speed of 20m/s, what is the maximum deflection of the bumper?



Solution: Data = $m = 20000 \text{ kg}$; $k = 200000 \text{ N/m}$; $\zeta = 1.25$

Critical damping coefficient =

$$c_c = 2 \times \sqrt{m \times k} = 2 \times \sqrt{20000 \times 200000} = 1.24 \times 10^5 \text{ N-sec/m}$$

$$\text{Damping coefficient } C = \zeta \times C_c = 1.25 \times 1.24 \times 10^5 = 1.58 \times 10^5 \text{ N-sec/m}$$

$$\text{Undamped natural frequency} = \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{200000}{20000}} = 3.16 \text{ rad/s}$$

Since $\zeta = 1.25$, the system is over damped.

For over damped system, the response equation is given by

$$x = C_1 e^{[-\zeta + \sqrt{\zeta^2 - 1}] \omega_n t} + C_2 e^{[-\zeta - \sqrt{\zeta^2 - 1}] \omega_n t}$$

$$\text{Substituting } \zeta = 1.25, x = C_1 e^{[-0.5] \omega_n t} + C_2 e^{[-2.0] \omega_n t} \quad (\text{a})$$

Differentiating, $\dot{x} = -0.5\omega_n C_1 e^{-0.5\omega_n t} - 2.0\omega_n C_2 e^{-2.0\omega_n t}$ (b)

Substituting the initial conditions

$x = 0$ at $t = 0$; and $\dot{x} = 20 \text{ m/s}$ at $t = 0$ in (a) & (b),

$$0 = C_1 + C_2 \quad (\text{c})$$

$$20 = -0.5\omega_n C_1 - 2.0\omega_n C_2 \quad (\text{d})$$

Solving (c) & (d), $C_1 = 4.21$ and $C_2 = -4.21$

Therefore, the response equation becomes

$$x = 4.21(e^{-1.58\omega_n t} - e^{-6.32\omega_n t})m \quad (\text{e})$$

The time at which, maximum deflection occurs is obtained by equating velocity equation to zero.

$$\text{Ie, } \dot{x} = -0.5\omega_n C_1 e^{-0.5\omega_n t} - 2.0\omega_n C_2 e^{-2.0\omega_n t} = 0$$

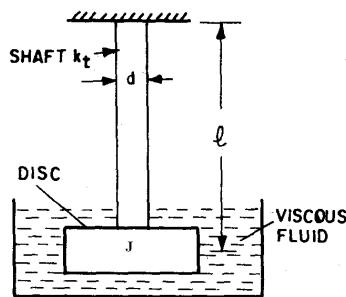
$$\text{Ie, } -6.65e^{-1.58t} + 26.61e^{-6.32t} = 0$$

Solving the above equation, $t = 0.292 \text{ secs.}$

Therefore, maximum deflection at $t = 0.292 \text{ secs.}$

$$\text{Substituting in (e), } x = 4.21(e^{-1.58 \times 0.292} - e^{-6.32 \times 0.292})m, = 1.99 \text{ m.}$$

6) A disc of a torsional pendulum has a moment of inertia of $6 \times 10^{-2} \text{ kg-m}^2$ and is immersed in a viscous fluid. The shaft attached to it is 0.4 m long and 0.1 m in diameter. When the pendulum is oscillating, the observed amplitudes on the same side of the mean position for successive cycles are 9° , 6° and 4° . Determine (i) logarithmic decrement (ii) damping torque per unit velocity and (iii) the periodic time of vibration. Assume $G = 4.4 \times 10^10 \text{ N/m}^2$, for the shaft material.



Shaft dia. = $d = 0.1 \text{ m}$

Shaft length = $l = 0.4 \text{ m}$

Moment of inertia of disc = $J = 0.06 \text{ kg-m}^2$.

Modulus of rigidity = $G = 4.4 \times 10^{10} \text{ N/m}^2$

Solution:

The above figure shows the arrangement of the system.

$$(i) \text{ Logarithmic decrement} = \delta = \log_e \frac{9}{6} = \log_e \frac{6}{4} = 0.405$$

(ii) The damping torque per unit velocity = damping coefficient of the system 'C'.

We know that logarithmic decrement = $\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$, rearranging which, we get

$$\text{Damping factor } \zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{0.405}{\sqrt{4\pi^2 + 0.405^2}} = 0.0645$$

Also, $\zeta = \frac{C}{C_c}$, where, critical damping coefficient = $C_c = 2\sqrt{k_t J}$

$$\text{Torsional stiffness} = k_t = \frac{GI_p}{l} = \frac{G}{l} \times \frac{\pi d^4}{32} = \frac{4.4 \times 10^{10}}{0.4} \times \frac{\pi \times 0.1^4}{32} = 1.08 \times 10^6 \text{ N-m/rad}$$

$$\text{Critical damping coefficient} = C_c = 2\sqrt{k_t J} = 2\sqrt{1.08 \times 10^6 \times 0.06} = 509 \text{ N-m/rad}$$

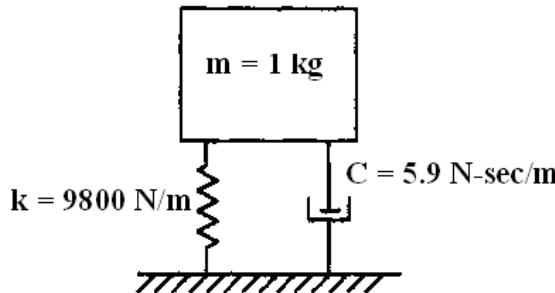
$$\text{Damping coefficient of the system} = C = C_c \times \zeta = 509 \times 0.0645 = 32.8 \text{ N-m/rad}$$

$$(iii) \text{ Periodic time of vibration} = \tau = \frac{1}{f_d} = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1-\zeta^2}}$$

$$\text{Where, undamped natural frequency} = \omega_n = \sqrt{\frac{k_t}{J}} = \sqrt{\frac{1.08 \times 10^6}{0.06}} = 4242.6 \text{ rad/sec}$$

$$\text{Therefore, } \tau = \frac{2\pi}{4242.6 \times \sqrt{1-0.0645^2}} = 0.00148 \text{ sec}$$

7) A mass of 1 kg is to be supported on a spring having a stiffness of 9800 N/m. The damping coefficient is 5.9 N-sec/m. Determine the natural frequency of the system. Find also the logarithmic decrement and the amplitude after three cycles if the initial displacement is 0.003m.



Solution:

$$\text{Undamped natural frequency} = \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{9800}{1}} = 99 \text{ rad/s}$$

$$\text{Damped natural frequency} = \omega_d = \sqrt{1-\zeta^2} \omega_n$$

$$\text{Critical damping coefficient} = c_c = 2 \times m \times \omega_n = 2 \times 1 \times 99 = 198 \text{ N-sec/m}$$

$$\text{Damping factor} = \zeta = \frac{c}{c_c} = \frac{5.9}{198} = 0.03$$

$$\text{Hence damped natural frequency} = \omega_d = \sqrt{1-\zeta^2} \omega_n = \sqrt{1-0.013^2} \times 99 = 98.99 \text{ rad/sec}$$

$$\text{Logarithmic decrement } \delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi \times 0.03}{\sqrt{1-0.03^2}} = 0.188$$

$$\text{Also, } \delta = \frac{1}{n} \log_e \frac{X_0}{X_n}; \text{ if } x_0 = 0.003,$$

$$\text{then, after 3 cycles, } \delta = \frac{1}{n} \log_e \frac{X_0}{X_n} \text{ ie, } 0.188 = \frac{1}{3} \times \log_e \frac{0.003}{X_3}$$

$$\text{ie, } X_3 = \frac{0.003}{e^{3 \times 0.188}} = 1.71 \times 10^{-3} \text{ m}$$

8) The damped vibration record of a spring-mass-dashpot system shows the following data.

**Amplitude on second cycle = 0.012m; Amplitude on third cycle = 0.0105m;
Spring constant k = 7840 N/m; Mass m = 2kg. Determine the damping constant, assuming it to be viscous.**

Solution:

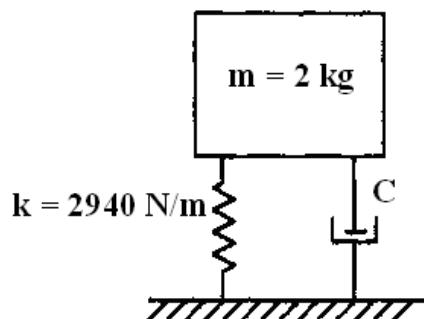
$$\text{Here, } \delta = \log_e \frac{X_2}{X_3} = \log_e \frac{0.012}{0.0105} = 0.133$$

$$\text{Also, } \delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}, \text{ rearranging, } \zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{0.133}{\sqrt{4\pi^2 + 0.133^2}} = 0.021$$

$$\text{Critical damping coefficient } c_c = 2 \times \sqrt{m \times k} = 2 \times \sqrt{2 \times 7840} = 250.4 \text{ N - sec/m}$$

$$\text{Damping coefficient } C = \zeta \times C_c = 0.021 \times 250.4 = 5.26 \text{ N - sec/m}$$

9) A mass of 2kg is supported on an isolator having a spring scale of 2940 N/m and viscous damping. If the amplitude of free vibration of the mass falls to one half its original value in 1.5 seconds, determine the damping coefficient of the isolator.



Solution:

$$\text{Undamped natural frequency } \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2940}{2}} = 38.34 \text{ rad/s}$$

Critical damping coefficient = $c_c = 2 \times m \times \omega_n = 2 \times 2 \times 38.34 = 153.4 \text{ N} - \text{sec/m}$

Response equation of under damped system = $x = A_1 e^{-\zeta \omega_n t} \left[\sin \left(\sqrt{1 - \zeta^2} \omega_n t + \phi_1 \right) \right]$

Here, amplitude of vibration = $A_1 e^{-\zeta \omega_n t}$

If amplitude = X_0 at $t = 0$, then, at $t = 1.5$ sec, amplitude = $\frac{X_0}{2}$

Ie, $A_1 e^{-\zeta \omega_n \times 0} = X_0$ or $A_1 = X_0$

Also, $A_1 e^{-\zeta \omega_n \times 1.5} = \frac{X_0}{2}$ or $X_0 \times e^{-\zeta \times 38.34 \times 1.5} = \frac{X_0}{2}$ or $e^{-\zeta \times 38.34 \times 1.5} = \frac{1}{2}$

Ie, $e^{\zeta \times 38.34 \times 1.5} = 2$, taking log, $\zeta \times 38.34 \times 1.5 = 0.69 \therefore \zeta = 0.012$

Damping coefficient $C = \zeta \times C_c = 0.012 \times 153.4 = 1.84 \text{ N} - \text{sec/m}$

3.2 (C) Forced Vibrations

Introduction:

In free un-damped vibrations a system once disturbed from its initial position executes vibrations because of its elastic properties. There is no damping in these systems and hence no dissipation of energy and hence it executes vibrations which do not die down. These systems give natural frequency of the system.

In free damped vibrations a system once disturbed from its position will execute vibrations which will ultimately die down due to presence of damping. That is there is dissipation of energy through damping. Here one can find the damped natural frequency of the system.

In forced vibration there is an external force acts on the system. This external force which acts on the system executes the vibration of the system. The external force may be harmonic and periodic, non-harmonic and periodic or non periodic. In this chapter only external harmonic forces acting on the system are considered. Analysis of non harmonic forcing functions is just an extension of harmonic forcing functions.

Examples of forced vibrations are air compressors, I.C. engines, turbines, machine tools etc.,

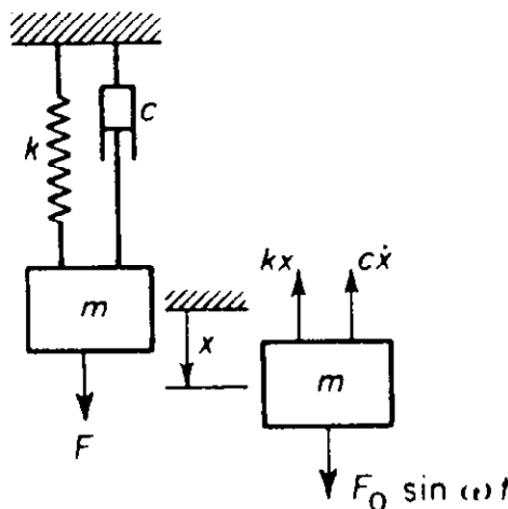
Analysis of forced vibrations can be divided into following categories as per the syllabus.

1. Forced vibration with constant harmonic excitation
 2. Forced vibration with rotating and reciprocating unbalance
 3. Forced vibration due to excitation of the support
- A: Absolute amplitude

B: Relative amplitude
 4. Force and motion transmissibility

For the above first a differential equation of motion is written. Assume a suitable solution to the differential equation. On obtaining the suitable response to the differential equation the next step is to non-dimensionalise the response. Then the frequency response and phase angle plots are drawn.

1. Forced vibration with constant harmonic excitation



From the figure it is evident that spring force and damping force oppose the motion of the mass. An external excitation force of constant magnitude acts on the mass with a frequency ω . Using Newton's second law of motion an equation can be written in the following manner.

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t \quad \text{--- --- 1}$$

Equation 1 is a linear non homogeneous second order differential equation. The solution to eq. 1 consists of complimentary function part and particular

integral. The complimentary function part of eq. 1 is obtained by setting the equation to zero. This derivation for complementary function part was done in damped free vibration chapter.

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p \quad \text{--- --- 2}$$

The complementary function solution is given by the following equation.

$$\mathbf{X}_c = A_2 e^{-\zeta \omega_n t} \sin \left[\sqrt{1 - \xi^2} \omega_n t + \varphi_2 \right] \quad \text{--- 3}$$

Equation 3 has two constants which will have to be determined from the initial conditions. But initial conditions cannot be applied to part of the solution of eq. 1 as given by eq. 3. The complete response must be determined before applying the initial conditions. For complete response the particular integral of eq. 1 must be determined. This particular solution will be determined by vector method as this will give more insight into the analysis.

Assume the particular solution to be

$$\mathbf{X}_p = X \sin(\omega t - \varphi) \quad \text{--- 4}$$

Differentiating the above assumed solution and substituting it in eq. 1

$$\dot{\mathbf{X}}_p = \omega X \sin \left(\omega t - \varphi + \frac{\pi}{2} \right)$$

$$\ddot{\mathbf{X}}_p = \omega^2 X \sin(\omega t - \varphi + \pi)$$

$$\begin{aligned} F_o \sin \omega t - k X \sin(\omega t - \varphi) - c \omega X \left(\omega t - \varphi + \frac{\pi}{2} \right) \\ - m \omega^2 X \sin(\omega t - \varphi + \pi) = 0 \quad \text{--- --- 5} \end{aligned}$$

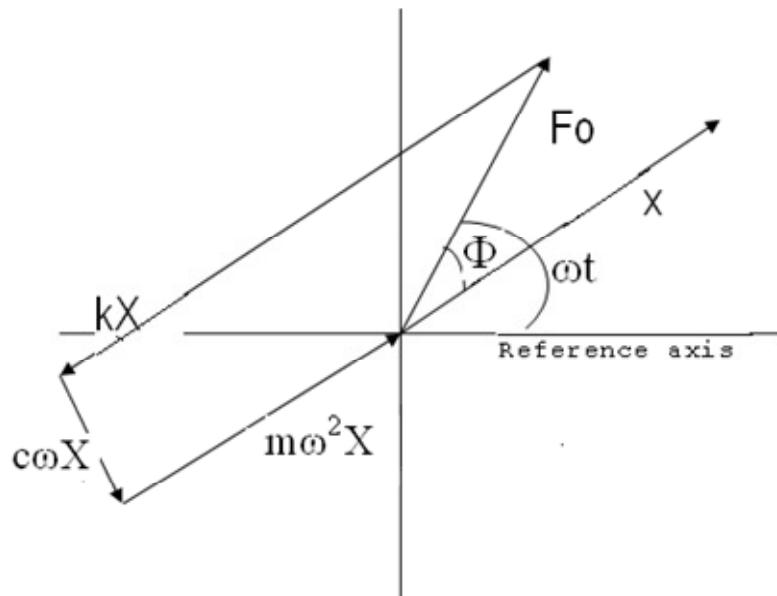


Fig. Vector representation of forces on system subjected to forced vibration

Following points are observed from the vector diagram

1. The displacement lags behind the impressed force by an angle Φ .
2. Spring force is always opposite in direction to displacement.
3. The damping force always lags the displacement by 90° . Damping force is always opposite in direction to velocity.
4. Inertia force is in phase with the displacement.

The relative positions of vectors and their magnitudes do not change with time.

From the vector diagram one can obtain the steady state amplitude and phase angle as follows

$$X = F_o / \sqrt{[(k - m\omega^2)^2 + (c\omega)^2]} \quad \dots \dots 6$$

$$\varphi = \tan^{-1} [c\omega / (k - m\omega^2)] \quad \dots \dots 7$$

The above equations are made non-dimensional by dividing the numerator and denominator by K .

$$X = \frac{X_{st}}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\xi(\omega/\omega_n)]^2}} \quad \dots \dots 8$$

$$\varphi = \tan^{-1} \left[\frac{2\xi(\omega/\omega_n)}{1 - (\omega/\omega_n)^2} \right] \quad \dots \dots 9$$

where, $X_{st} = F_o/k$ is zero frequency deflection

Therefore the complete solution is given by

$$X = X_c + X_p$$

$$x = A_2 e^{-\zeta \omega_n t} \sin \left[\sqrt{1 - \xi^2} \omega_n t + \varphi_2 \right] + \frac{X_{st} \sin(\omega t - \varphi)}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\xi(\omega/\omega_n)]^2}} \quad \dots \dots 10$$

The two constants A_2 and ϕ_2 have to be determined from the initial conditions.

The first part of the complete solution that is the complementary function decays with time and vanishes completely. This part is called transient vibrations. The second part of the complete solution that is the particular integral is seen to be sinusoidal vibration with constant amplitude and is called as steady state vibrations. Transient vibrations take place at damped natural frequency of the system, whereas steady state vibrations take place at frequency of excitation. After transients die out the complete solution consists of only steady state vibrations.

In case of forced vibrations without damping equation 10 changes to

$$x = A_2 \sin[\omega_n t + \phi_2] + \frac{X_{st} \sin(\omega t)}{\left| 1 - (\omega/\omega_n)^2 \right|} \quad \dots \dots 11$$

ϕ_2 is either 0° or 180° depending on whether $\omega < \omega_n$ or $\omega > \omega_n$

Steady state Vibrations: The transients die out within a short period of time leaving only the steady state vibrations. Thus it is important to know the steady state behavior of the system,

Thus Magnification Factor (M.F.) is defined as the ratio of steady state amplitude to the zero frequency deflection.

$$M.F. = \frac{X}{X_{st}} = \frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + [2\xi(\omega/\omega_n)]^2}} \quad \dots \dots 12$$

$$\phi = \tan^{-1} \left[\frac{2\xi(\omega/\omega_n)}{1 - (\omega/\omega_n)^2} \right] \quad \dots \dots 13$$

Equations 12 and 13 give the magnification factor and phase angle. The steady state amplitude always lags behind the impressed force by an angle Φ . The above equations are used to draw frequency response and phase angle plots.

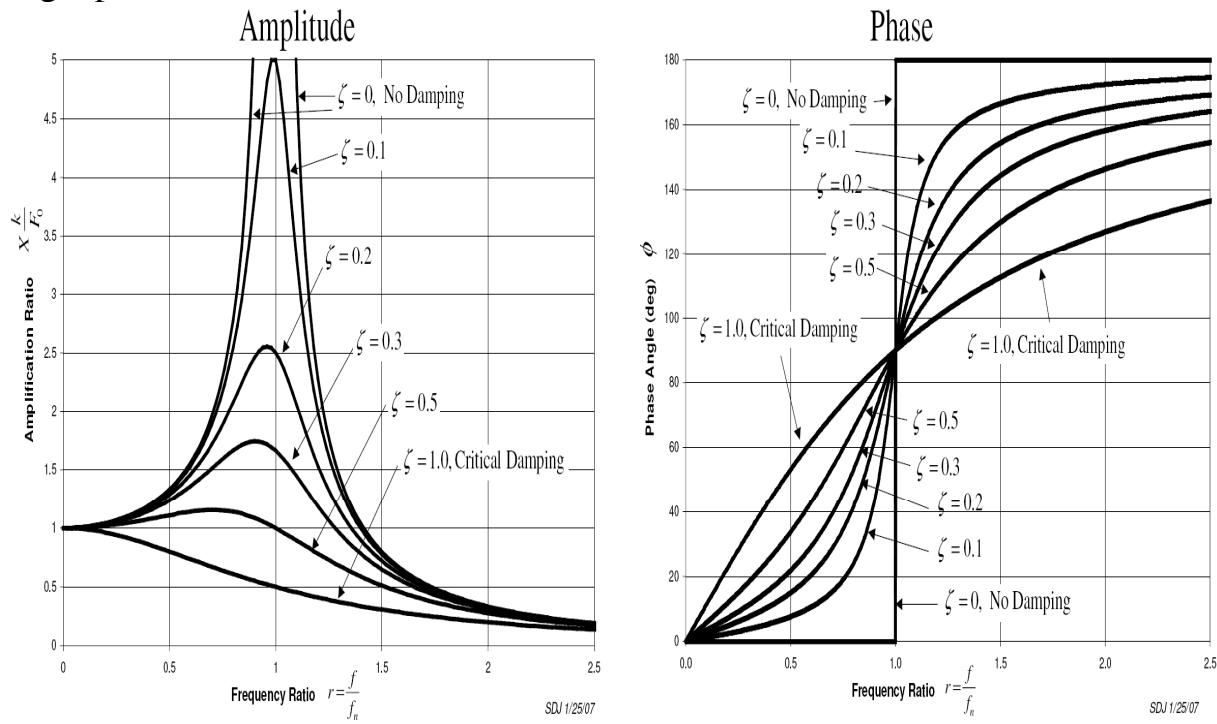


Fig. Frequency response and phase angle plots for system subjected forced vibrations.

Frequency response plot: The curves start from unity at frequency ratio of zero and tend to zero as frequency ratio tends to infinity. The magnification factor increases with the increase in frequency ratio up to 1 and then decreases as frequency ratio is further increased. Near resonance the amplitudes are very high and decrease with the increase in the damping ratio. The peak of magnification factor lies slightly to the left of the resonance line. This tilt to the left increases with the increase in the damping ratio. Also the sharpness of the peak of the curve decreases with the increase in the damping.

Phase angle plot: At very low frequency the phase angle is zero. At resonance the phase angle is 90° . At very high frequencies the phase angle tends to 180° . For low values of damping there is a steep change in the phase angle near resonance. This decreases with the increase in the damping. The sharper the change in the phase angle the sharper is the peak in the frequency response plot.

The amplitude at resonance is given by equation 14

$$\begin{aligned} c\omega X_r &= F_o \\ X_r &= X_{st}/2\xi \quad \text{--- 14} \end{aligned}$$

The frequency at which maximum amplitude occurs is obtained by differentiating the magnification factor equation with respect to frequency ratio and equating it to zero.

$$\left(\frac{\omega_p}{\omega_n} \right) = \sqrt{1 - 2\xi^2} \quad \text{--- 15}$$

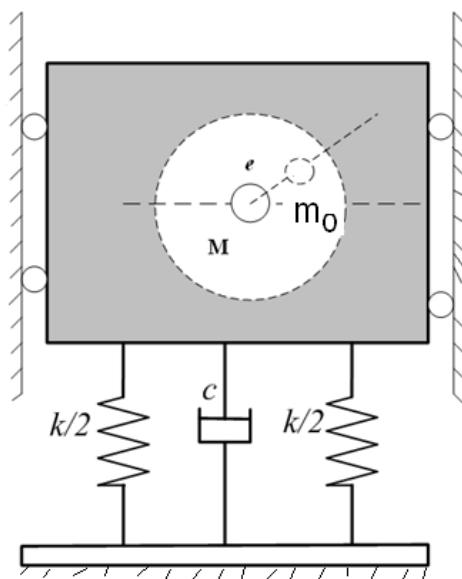
Also no maxima will occur for $\xi > \sqrt{\frac{1}{2}}$ or $\xi > 0.707$

2. Rotating and Reciprocating Unbalance

Machines like electric motors, pumps, fans, clothes dryers, compressors have rotating elements with unbalanced mass. This generates centrifugal type harmonic excitation on the machine.

The final unbalance is measured in terms of an equivalent mass m_o rotating with its c.g. at a distance e from the axis of rotation. The centrifugal force is proportional to the square of frequency of rotation. It varies with the speed

of rotation and is different from the harmonic excitation in which the maximum force is independent of the frequency.



Let m_o = Unbalanced mass
 e = eccentricity of the unbalanced mass
 M = Total mass of machine including unbalanced m_o
 m_o makes an angle ωt with ref. axis.
 $M_o e \omega^2$ is the centrifugal force that acts radially outwards.

Equation of motion is

$$(M - m_o) \frac{d^2 x}{dt^2} + m_o \frac{d^2}{dt^2} (x + esin\omega t) = -kx - c \frac{dx}{dt}$$

$$M\ddot{x} + c\dot{x} + kx = m_o e \omega^2 \sin \omega t \quad \dots \dots 1$$

The solution of following equation 2 is given by

$$m\ddot{x} + c\dot{x} + kx = F_o \sin \omega t \quad \dots \dots 2$$

$$x = A_2 e^{-\zeta \omega_n t} \sin \left[\sqrt{1 - \xi^2} \omega_n t + \varphi_2 \right]$$

$$+ \frac{X_{st} \sin(\omega t - \varphi)}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\xi(\omega/\omega_n)]^2}}$$

Compare eq. 1 with eq.2 the only change is F_0 is replaced by $m_o e\omega^2$

The transient part of the solution remains the same. The only change is in the steady state part of the solution.

Therefore the steady state solution of eq.1 can be written as

$$x = X \sin(\omega t - \varphi)$$

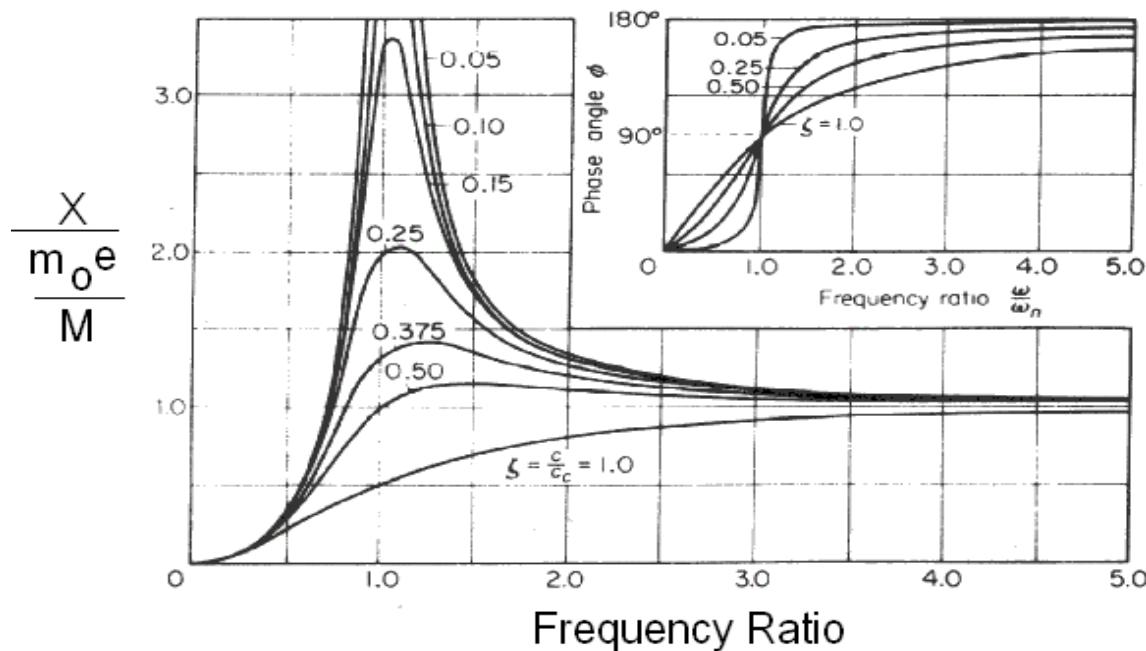
$$\text{where...} X = \frac{m_o e\omega^2 / k}{\sqrt{\left(1 - \frac{M\omega^2}{k}\right)^2 + \left(\frac{C\omega}{k}\right)^2}}$$

The above equation reduces to dimensionless form as

$$\frac{X}{\left(\frac{m_o e\omega^2}{M}\right)} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi \frac{\omega}{\omega_n}\right]^2}}$$

The phase angle equation and its plot remains the same as shown below

$$\varphi = \tan^{-1} \left[\frac{2\xi \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \right]$$



Frequency response and phase angle plots (Unbalance)

At low speeds $m_o e \omega^2$ is small, hence all response curves start from zero.

At resonance $\omega / \omega_n = 1$, therefore
$$\frac{X}{(m_o e) / M} = \frac{1}{2\xi}$$

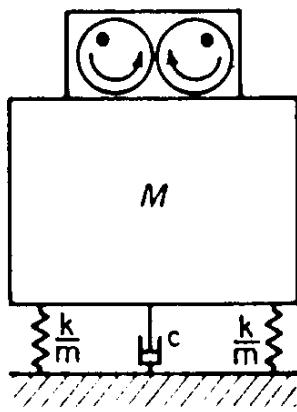
And amplitude is limited due to damping present in the system. Under these conditions the motion of main mass ($M-m_o$) lags that of the mass m_o by 90°.

When ω / ω_n is very large the ratio $X/(moe/M)$ tends to unity and the main mass ($M-m_o$) has an amplitude of $X = moe/M$. This motion is 180° out of phase with the exciting force. That is when unbalanced mass moves up, the main mass moves down and vice versa.

Problem 1

A counter rotating eccentric weight exciter is used to produce the forced oscillation of a spring-supported mass as shown in Fig. By varying the speed

of rotation, a resonant amplitude of 0.60 cm was recorded. When the speed of rotation was increase considerably beyond the resonant frequency, the amplitude appeared to approach a fixed value of 0.08 cm. Determine the damping factor of the system.



$$X = \frac{\frac{m_o e}{M}}{2\xi} = 0.6 \text{ cm}$$

$$X = \frac{m_o e}{M} = 0.08 \text{ cm}$$

Dividing one by the other

$$\xi = 0.0667$$

(Answer)

Problem 2

A system of beam supports a mass of 1200 kg. The motor has an unbalanced mass of 1 kg located at 6 cm radius. It is known that the resonance occurs at 2210 rpm. What amplitude of vibration can be expected at the motors operating speed of 1440 rpm if the damping factor is assumed to be less than 0.1

Solution:

Given: $M = 1200 \text{ kg}$, $m_o = 1 \text{ kg}$, eccentricity $e = 0.06 \text{ m}$, Resonance at 2210 rpm, Operating speed = 1440 rpm, $\xi = 0.1$, $X = ?$.

$$\omega_n = \frac{2\pi N}{60} = 231.43 \text{ rad/s} \quad \omega = \frac{2\pi N_{op}}{60} = 150.79 \text{ rad/s}$$

$$\frac{\omega}{\omega_n} = r = 0.652$$

$$\frac{X}{\left(\frac{m_o e \omega^2}{M}\right)} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi \frac{\omega}{\omega_n}\right]^2}}$$

Substituting the appropriate values in the above eq.

$$X = 0.036 \text{ mm} \text{ (Answer)}$$

However, if ζ is made zero, the amplitude $X = 0.037 \text{ mm}$ (Answer)

This means if the damping is less than 0.1, the amplitude of vibration will be between 0.036 mm and 0.037 mm. (Answer)

Problem 3

An eccentric mass exciter is used to determine the vibratory characteristics of a structure of mass 200 kg. At a speed of 1000 rpm a stroboscope showed the eccentric mass to be at the bottom position at the instant the structure was moving downward through its static equilibrium position and the corresponding amplitude was 20 mm. If the unbalance of the eccentric is 0.05 kg-m, determine, (a) un damped natural frequency of the system (b) the damping factor of the structure (c) the angular position of the eccentric at 1300 rpm at the instant when the structure is moving downward through its equilibrium position.

Solution:

Given: $M = 200 \text{ kg}$, Amplitude at 1000 rpm = 20 mm, $m_o e = 0.05 \text{ kg-m}$
At 1000 rpm the eccentric mass is at the bottom when the structure was moving downward – This means there is phase lag of 90° (i.e., at resonance). At resonance $\omega = \omega_n$.

$$\omega = \omega_n = \frac{2\pi N}{60} = 104.72 \text{ rad/s}$$

$$\frac{X}{\left(\frac{m_o e}{M}\right)} = \frac{1}{2\xi}$$

$$\xi = 0.00625 \text{ (Answer)}$$

$$\Phi = \tan^{-1} \left[\frac{2\xi \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n} \right)^2} \right]$$

$$\phi = 176.89^\circ \text{ (Answer)}$$

Problem 4

A 40 kg machine is supported by four springs each of stiffness 250 N/m. The rotor is unbalanced such that the unbalance effect is equivalent to a mass of 5 kg located at 50mm from the axis of rotation. Find the amplitude of vibration when the rotor rotates at 1000 rpm and 60 rpm. Assume damping coefficient to be 0.15

Solution:

Given: $M = 40 \text{ kg}$, $m_o = 5 \text{ kg}$, $e = .05 \text{ m}$, $\xi = 0.15$, $N = 1000 \text{ rpm}$ and 60 rpm .
When $N = 1000 \text{ rpm}$

$$\omega = \frac{2\pi N}{60} = 104.67 \text{ rad/s}$$

$$\omega_n = \sqrt{\frac{k}{M}} = 5 \text{ rad/s} \quad \frac{\omega}{\omega_n} = 20.934$$

$$\frac{X}{\left(\frac{m_o e \omega^2}{M}\right)} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi \frac{\omega}{\omega_n}\right]^2}}$$

X at 1000 rpm = 6.26 mm (Solution)

When $N = 60 \text{ rpm}$

$$\omega = \frac{2\pi N}{60} = 6.28 \text{ rad/s}$$

$$\frac{\omega}{\omega_n} = 1.256$$

Using the same eq. X at 60 rpm = 14.29 mm (Solution)

Problem 5

A vertical single stage air compressor having a mass of 500 kg is mounted on springs having a stiffness of $1.96 \times 10^5 \text{ N/m}$ and a damping coefficient of 0.2. The rotating parts are completely balanced and the equivalent reciprocating parts have a mass of 20 kg. The stroke is 0.2 m. Determine the

dynamic amplitude of vertical motion and the phase difference between the motion and excitation force if the compressor is operated at 200 rpm.

Solution

Given: $M = 500 \text{ kg}$, $k = 1.96 \times 10^5 \text{ N/m}$, $\zeta = 0.2$, $m_o = 20 \text{ kg}$, stroke = 0.2 m, $N = 200 \text{ rpm}$, $X = ?$.

Stroke = 0.2 m, i.e. eccentricity $e = \text{stroke}/2 = 0.1 \text{ m}$

Using the equations $X = 10.2 \text{ mm}$ and $\phi = 105.9^\circ$ (Solution)

(a)

$$c = 500/4 = 125 \text{ N-s/m}$$

$$c_e = 2\sqrt{kM} = 2\sqrt{6400 \times 20} = 715.54 \text{ N-s/m}$$

$$\xi = \frac{c}{c_e} = \frac{125}{715.5} = 0.175 \text{ (Solution)}$$

(b)

$$\frac{X}{\left(\frac{m_o e \omega^2}{M}\right)} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi \frac{\omega}{\omega_n}\right]^2}} \quad \Phi = \tan^{-1} \left[\frac{2\xi \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \right]$$

$X = 0.15 \text{ cm}$ and $\phi = 169^\circ$ (Solution)

(c)

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{6400}{20}} = 17.88 \text{ rad/s}$$

$$N_r = \frac{60 \times \omega_n}{2 \times \pi} = 170.74 \text{ rpm}$$

$$X_r = \frac{m_o e}{2\xi M} = 0.357 \text{ cm}$$

(d)

$$c\omega X = 125 \times \frac{2 \times \pi \times 400}{60} \times 0.15 \times 10^{-2} = 7.85 \text{ N}$$

$$kX = 6400 \times 0.15 \times 10^{-2} = 9.6 \text{ N}$$

$$F = \sqrt{(c\omega X)^2 + (kX)^2} = 12.4 \text{ N}$$

Conclusions on rotating and reciprocating unbalance

- Unbalance in machines cannot be made zero. Even small unbalanced mass can produce high centrifugal force. This depends on the speed of operation.
- Steady state amplitude is determined for a machine subjected unbalanced force excitation.
- For reciprocating machines, the eccentricity can be taken as half the crank radius.
- Frequency response plot starts from zero at frequency ratio zero and tends to end at unity at very high frequency ratios.

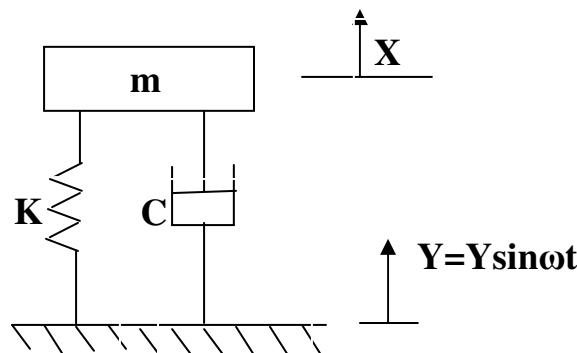
2. Response of a damped system under the harmonic motion of the base

In many cases the excitation of the system is through the support or the base instead of being applied through the mass. In such cases the support will be considered to be excited by a regular sinusoidal motion.

Example of such base excitation is an automobile suspension system excited by a road surface, the suspension system can be modeled by a linear spring in parallel with a viscous damper. such model is depicted in Figure 1.

There are two cases: (a) Absolute Amplitude of mass m
 (b) Relative amplitude of mass m

(a) Absolute Amplitude of mass m



$$\begin{aligned} m\ddot{x} &= -k(x - y) - c(\dot{x} - \dot{y}) \\ m\ddot{x} + c\dot{x} + kx &= c\dot{y} + ky \end{aligned}$$

It is assumed that the base moves harmonically, that is

$$y(t) = Y \sin \omega t$$

where Y denotes the amplitude of the base motion and ω represents the frequency of the base excitation

Substituting Eq.2 in Eq. 1

$$m\ddot{x} + c\dot{x} + kx = c\omega Y \cos \omega t + kY \sin \omega t$$

The above equation can be expressed as

$$m\ddot{x} + c\dot{x} + kx = Y \sqrt{k^2 + (c\omega)^2} \sin (\omega t + \alpha)$$

where

$$\alpha = \tan^{-1} \left(\frac{c\omega}{k} \right)$$

where $Y[k^2 + (c\omega)^2]^{1/2}$ is the amplitude of excitation force.

Examination of equation 3 reveals that it is identical to an Equation developed during derivation for M.F. The solution is:

$$\begin{aligned} m\ddot{x} + c\dot{x} + kx &= F_o \sin \omega t \\ x &= X \sin(\omega t - \varphi) \end{aligned}$$

$$X = \frac{F_o/k}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\xi(\omega/\omega_n)]^2}}$$

$$m\ddot{x} + c\dot{x} + kx = Y \sqrt{k^2 + (c\omega)^2} \sin(\omega t + \alpha)$$

where

$$\alpha = \tan^{-1} \left(\frac{c\omega}{k} \right)$$

$$x = X \sin(\omega t + \alpha - \phi)$$

Therefore the steady state amplitude and phase angle to eq. 3 is

$$X = \frac{Y \sqrt{k^2 + (c\omega)^2}}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \quad \tan \phi = \frac{c\omega}{k - m\omega^2}$$

The above equations can be written in dimensionless form as follows

$$\frac{X}{Y} = \frac{\sqrt{1 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi \frac{\omega}{\omega_n}\right]^2}}$$

$$\phi = \tan^{-1} \left[\frac{2\xi \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \right]$$

$$\varphi - \alpha = \tan^{-1} \left[\frac{2\xi \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n} \right)^2} \right] - \tan^{-1} \left[2\xi \frac{\omega}{\omega_n} \right]$$

The motion of the mass m lags that of the support by an angle $(\varphi - \alpha)$ as shown by equation 6.

Equation 5 which gives the ratio of (X/Y) is also known as motion transmissibility or displacement transmissibility.

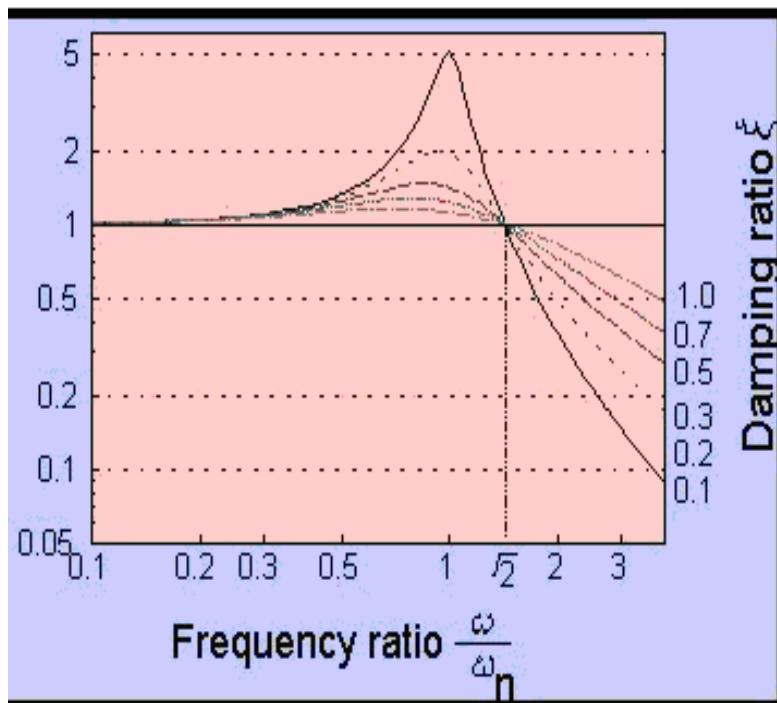


Fig. gives the frequency response curve for motion transmissibility.

1. For low frequency ratios the system moves as a rigid body and $X/Y \approx 1$.
 2. At resonance the amplitudes are large
 3. For very high frequency ratios the body is almost stationary ($X/Y \approx 0$)
- It will be seen later that the same response curve is also used for Force Transmissibility.

(b) Relative Amplitude of mass m

Here amplitude of mass m relative to the base motion is considered. The equations are basically made use in the Seismic instruments. If z represents the relative motion of the mass w.r.t. support,

$$\mathbf{z} = \mathbf{x} - \mathbf{y}$$

$$\mathbf{x} = \mathbf{y} + \mathbf{z}$$

Substituting the value of x in eq.

$$m\ddot{x} + c\dot{x} + kx = c\omega Y \cos \omega t + kY \sin \omega t$$

$$m\ddot{z} + c\dot{z} + kz = m\omega^2 Y \sin \omega t$$

The above equation is similar to the equation developed for rotating and reciprocating unbalances. Thus the relative steady state amplitude can be written as

$$\frac{z}{Y} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

$$\phi = \tan^{-1} \left[\frac{2\xi\left(\omega/\omega_n\right)}{1 - \left(\omega/\omega_n\right)^2} \right]$$

Thus eq. 7 and eq.8 are similar to the one developed during the study of rotating an reciprocating unbalances. Frequency and phase response plots will also remain same.

Problem 1

The support of a spring mass system is vibrating with an amplitude of 5 mm and a frequency of 1150 cpm. If the mass is 0.9 kg and the stiffness of springs is 1960 N/m, Determine the amplitude of vibration of mass. What amplitude will result if a damping factor of 0.2 is included in the system.

Solution:

Given: $Y = 5 \text{ mm}$, $f = 1150 \text{ cpm}$, $m = 0.9 \text{ kg}$, $k = 1960 \text{ N/m}$, $X = ?$

$\xi = 0.2$, then $X = ?$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1960}{0.9}} = 46.67 \text{ rad/s}$$

$$\omega = 2\pi f = 2\pi \times 1150/60 = 120.43 \text{ rad/s}$$

$$\frac{\omega}{\omega_n} = r = 2.58$$

$$\frac{X}{Y} = \frac{\sqrt{1 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi \frac{\omega}{\omega_n}\right]^2}}$$

When $\xi = 0$, $X = 0.886 \text{ mm}$ (Solution)

When $\xi = 0.2$, $X = 1.25 \text{ mm}$ (Solution)

Observe even when damping has increased the amplitude has not decreased but it has increased.

Problem 2

The springs of an automobile trailer are compressed 0.1 m under its own weight. Find the critical speed when the trailer is travelling over a road with a profile approximated by a sine wave of amplitude 0.08 m and a wavelength of 14 m. What will be the amplitude of vibration at 60 km/hr.

Solution:

Given: Static deflection = $d_{st} = 0.1$ m, $Y = 0.08$ m, $\gamma = 14$ m, Critical Speed = ?, $V_{60} = ?$.

Critical speed can be found by finding natural frequency.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{d_{st}}} = \sqrt{\frac{9.81}{.1}} = 9.9 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = 1.576 \text{ cps}$$

$$V = \text{wavelength} \times f_n = 14 \times 1.576 = 22.06 \text{ m/s}$$

Corresponding $V = 22.06 \text{ m/s} = 79.4 \text{ km/hr}$

Amplitude X at 60 km/hr

$V_{60} = 16.67 \text{ m/s}$

$$\frac{\omega}{\omega_n} = r = .756 \quad f = \text{velocity} / \text{wavelength} = \frac{16.67}{14} = 1.19 \text{ cps}$$

$$\omega = 2\pi f = 7.48 \text{ rad/s}$$

$$\frac{X}{Y} = \frac{\sqrt{1 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi \frac{\omega}{\omega_n}\right]^2}} \text{ km/hr} = 0.186 \text{ m (Solution)}$$

Problem 3

A heavy machine of 3000 N, is supported on a resilient foundation. The static deflection of the foundation due to the weight of the machine is found to be 7.5 cm. It is observed that the machine vibrates with an amplitude of 1 cm when the base of the machine is subjected to harmonic oscillations at the undamped natural frequency of the system with an amplitude of 0.25 cm. Find (a) the damping constant of the foundation (b) the dynamic force amplitude on the base (c) the amplitude of the displacement of the machine relative to the base.

Solution

Given: $mg = 3000 \text{ N}$, Static deflection = $dst = 7.5 \text{ cm}$, $X = 1 \text{ cm}$, $Y = 0.25 \text{ cm}$, $\omega = \omega_n$, $\xi = ?$, $F_{\text{base}} = ?$, $Z = ?$

(a)

$$\frac{\omega}{\omega_n} = \frac{X}{Y} = \frac{0.010}{0.0025} = 4 = \sqrt{\frac{1 + (2\xi)^2}{(2\xi)^2}}$$

Solving for $\xi = 0.1291$

$$C = \xi C_c = \xi \times 2\sqrt{km} = 903.05 \text{ N-s/m}$$

(c)

$$\frac{Z}{Y} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi \frac{\omega}{\omega_n}\right]^2}}$$

Note: $Z = 0.00968 \text{ m}$, $X = 0.001 \text{ m}$, $Y = 0.0025 \text{ m}$, Z is not equal to $X-Y$ due phase difference between x, y, z.

Using the above eq. when $\omega = \omega_n$, the

relative amplitude is $Z = 0.00968$ m (Solution)

(b)

$$F_d = \sqrt{(c\omega Z)^2 + (kZ)^2}$$

$$\omega = \omega_n = 3.65 \text{ rad/s}$$

$$F_d = 388.5 \text{ N} \quad (\text{Solution})$$

Problem 4

The time of free vibration of a mass hung from the end of a helical spring is 0.8 s. When the mass is stationary, the upper end is made to move upwards with displacement y mm given by $y = 18 \sin 2\pi t$, where t is time in seconds measured from the beginning of the motion. Neglecting the mass of spring and damping effect, determine the vertical distance through which the mass is moved in the first 0.3 seconds.

Solution:

Given: Time period of free vibration = 0.8 s., $y = 18 \sin 2\pi t$,

$\zeta = 0$, x at the end of first 0.3 s. = ?

$$m\ddot{x} + kx = ky$$

$$m\ddot{x} + kx = kY \sin \omega t \quad \text{where, } Y = 18 \text{ mm, and } \omega = 2\pi \text{ rad/s.}$$

The complete solution consists of Complementary function and Particular integral part.

$$x = x_c + x_p$$

$$x_c = A \cos \omega_n t + B \sin \omega_n t$$

$$x_p = X \sin(\omega t + \alpha - \varphi)$$

$$\text{where, } X = \frac{Y}{\sqrt{1 - \left(\frac{\omega}{\omega_n}\right)^2}}$$

$$\omega_n = \frac{2\pi}{T} = \frac{2\pi}{0.8}$$

$$\omega = 2\pi$$

$$(\varphi - \alpha) = 0, \text{ if } \frac{\omega}{\omega_n} < 1$$

$$\frac{\omega}{\omega_n} = 0.8$$

$$(\varphi - \alpha) = 180^\circ, \text{ if } \left(\frac{\omega}{\omega_n}\right) > 1$$

$$\text{and } \varphi - \alpha = 0$$

$$\omega_n = \frac{2\pi}{T} = \frac{2\pi}{0.8}$$

$$\omega = 2\pi$$

$$\frac{\omega}{\omega_n} = 0.8$$

$$\text{and } \varphi - \alpha = 0$$

Hence, $x_p = \frac{Y}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]} \sin \omega t$

The complete solution is given by

$$x = A \cos \omega_n t + B \sin \omega_n t + \frac{Y}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]} \sin \omega t$$

Substituting the initial conditions in the above eq. constants A and B can be obtained

$$x = 0; \text{ at } t = 0 \text{ gives } A = 0$$

$$\dot{x} = 0, \text{ at } t = 0 \text{ and } B = -\frac{Y \frac{\omega}{\omega_n}}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]}$$

Thus the complete solution after substituting the values of A and B

$$x = \frac{Y}{\left[1 - \left(\frac{\omega}{\omega_n} \right)^2 \right]} \left[\sin \omega t - \left(\frac{\omega}{\omega_n} \right) \sin \omega_n t \right]$$

when $t = 0.3$ s, the value of x from the above eq.
is $x = 19.2$ mm (Solution)

Conclusions on Response of a damped system under the harmonic motion of the base

- Review of forced vibration (constant excitation force and rotating and reciprocating unbalance).
- Steady state amplitude and phase angle is determined when the base is excited sinusoidally. Derivations were made for both absolute and relative amplitudes of the mass.

4. Vibration Isolation and Force Transmissibility

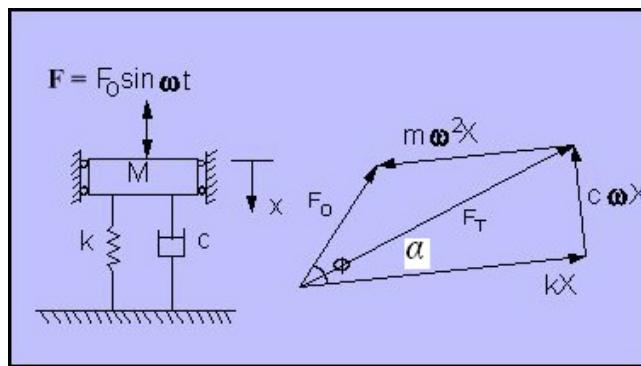
- Vibrations developed in machines should be isolated from the foundation so that adjoining structure is not set into vibrations. (Force isolation)
- Delicate instruments must be isolated from their supports which may be subjected to vibrations. (Motion Isolation)
- To achieve the above objectives it is necessary to choose proper isolation materials which may be cork, rubber, metallic springs or other suitable materials.

- Thus in this study, derivations are made for force isolation and motion isolation which give insight into response of the system and help in choosing proper isolation materials.

Transmissibility is defined as the ratio of the force transmitted to the foundation to that impressed upon the system.

$$m\ddot{x} + c\dot{x} + kx = F_o \sin \omega t \quad ---1$$

$$x_p = X \sin(\omega t - \phi)$$



The force transmitted to the base is the sum of the spring force and damper force. Hence, the amplitude of the transmitted force is:

$$\begin{aligned}
 F_T &= \sqrt{(kX)^2 + (c\omega X)^2} \\
 &= kX \sqrt{1 + \left(\frac{c\omega}{k}\right)^2} \\
 &= kX \sqrt{1 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}
 \end{aligned}$$

$$\frac{Xk}{F_o} = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}}$$

Substituting the value of X from Eq. 2 in Eq. 3 yields

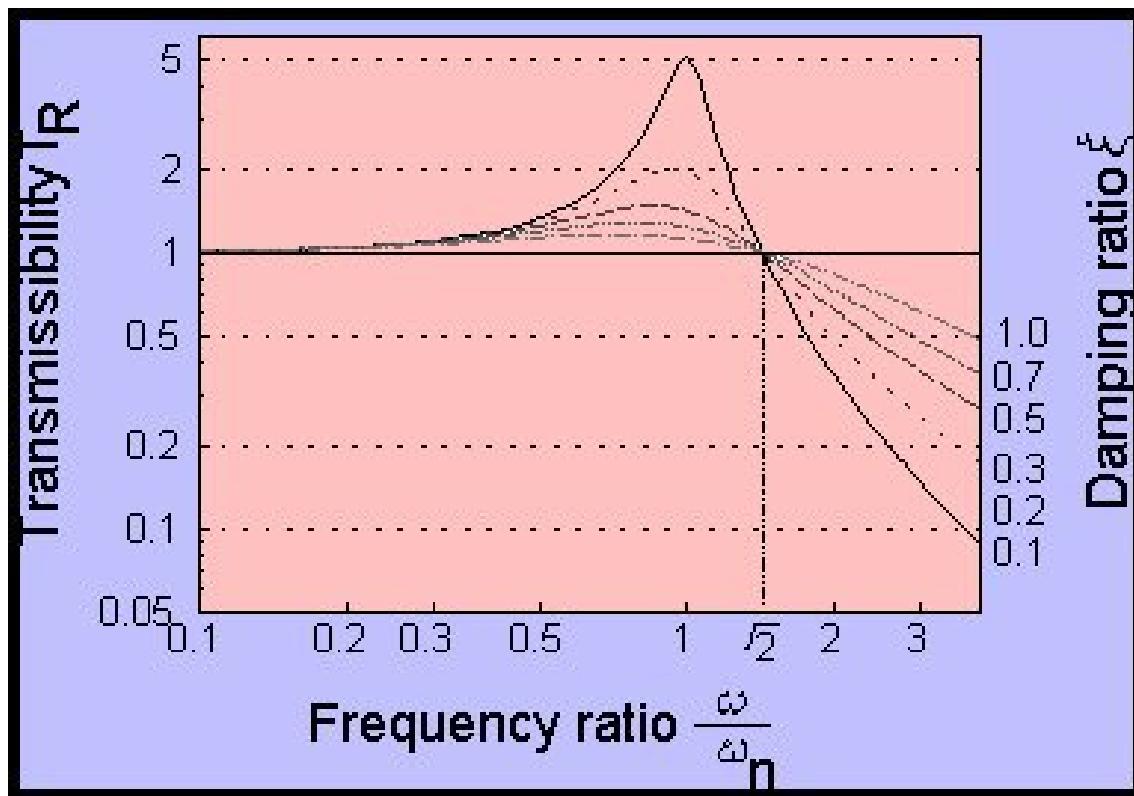
$$F_T = \frac{F_o \sqrt{1 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}}$$

Hence, the force transmission ratio or transmissibility, TR is given by

$$TR = \frac{F_T}{F_0} = \frac{\sqrt{1 + \left[2\zeta \left(\frac{\omega}{\omega_n} \right) \right]^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n} \right)^2 \right]^2 + \left[2\zeta \left(\frac{\omega}{\omega_n} \right) \right]^2}}$$

$$\phi - \alpha = \tan^{-1} \left[\frac{2\xi \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n} \right)^2} \right] - \tan^{-1} \left[2\xi \frac{\omega}{\omega_n} \right]$$

Eq. 6 gives phase angle relationship between Impressed force and transmitted force.



$$TR = \frac{F_T}{F_0} = \frac{\sqrt{1 + \left[2\xi \left(\frac{\omega}{\omega_n} \right) \right]^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n} \right)^2 \right]^2 + \left[2\xi \left(\frac{\omega}{\omega_n} \right) \right]^2}}$$

- Curves start from unity value of transmissibility pass through unit value of transmissibility at $(\omega / \omega_n) = 1$ and after that they Tend to zero as $(\omega / \omega_n) \rightarrow \infty$.

- Response plot can be divided into three regions depending on the control of spring, damper and mass.
- When (ω / ω_n) is large, it is mass controlled region. Damping in this region deteriorates the performance of machine.
- When (ω / ω_n) is very small, it is spring controlled region.
- When (ω / ω_n) ranges from 0.6 to 1.0, it is damping controlled region.
- For effective isolation (ω / ω_n) should be large. It means it will have spring with low stiffness (hence large static deflections).

Motion Transmissibility

Motion transmissibility is the ratio of steady state amplitude of mass m (X) to the steady amplitude (Y) of the supporting base.

$$\frac{X}{Y} = \frac{\sqrt{1 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi \frac{\omega}{\omega_n}\right]^2}}$$

$$\varphi - \alpha = \tan^{-1} \left[\frac{2\xi \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \right] - \tan^{-1} \left[2\xi \frac{\omega}{\omega_n} \right]$$

The equations are same as that of force transmissibility. Thus the frequency response and phase angle plots are also the same.

Problem 1

A 75 kg machine is mounted on springs of stiffness $k=11.76 \times 10^6$ N/m with a damping factor of 0.2. A 2 kg piston within the machine has a reciprocating motion with a stroke of 0.08 m and a speed of 3000 cpm. Assuming the motion of the piston to be harmonic, determine the amplitude of vibration of machine and the vibratory force transmitted to the foundation.

Solution:

Given: $M = 75$ kg, $\xi = 0.2$, $m_0 = 2$ kg, stroke = 0.08 m, $N = 3000$ cpm, $X = ?$, $FT = ?$.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{11.76 \times 10^5}{75}} = 125 \text{ rad/s}$$

$$\omega = \frac{2\pi \times 3000}{60} = 314.16 \text{ rad/s}$$

$$\frac{\omega}{\omega_n} = 2.51 \quad e = \frac{0.08}{2} = 0.04 \text{ m}$$

$$\frac{X}{\left(\frac{m_o e}{M}\right)} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

Using the above eq. $X = 1.25 \text{ mm}$ (Solution)

$$F_o = m_o e \omega^2 = 7900 \text{ N}$$

$$TR = \frac{F_T}{F_o} = \frac{\sqrt{1 + \left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}}$$

Using the above eq. $F_T = 2078 \text{ N}$ (Solution)

3.3 Two Degree of Freedom System

Introduction

A two degree of freedom system is one that requires two coordinates to completely describe its equation of motion. These coordinates are called generalized coordinates when they are independent of each other. Thus system with two degrees of freedom will have two equation of motion and hence has two frequencies.

A two degree freedom system differs from a single degree of freedom system in that it has two natural frequencies and for each of these natural frequencies there correspond a natural state of vibration with a displacement configuration known as **NORMAL MODE**. Mathematical terms related to these quantities are known as Eigen values and Eigen vectors. These are established from the two simultaneous equation of motion of the system and posses certain dynamic properties associated.

A system having two degrees of freedom are important in as far as they introduce to the coupling phenomenon where the motion of any of the two independent coordinates depends also on the motion of the other coordinate through the coupling spring and damper. The free vibration of two degrees of freedom system at any point is a combination of two harmonics of these two natural frequencies.

Under certain condition, during free vibrations any point in a system may execute harmonic vibration at any of the two natural frequencies and the amplitude are related in a specific manner and the configuration is known as **NORMAL MODE or PRINCIPAL MODE** of vibration. Thus system with two degrees of freedom has two normal modes of vibration corresponding two natural frequencies.

Free vibrations of two degrees of freedom system:

Consider an un-damped system with two degrees of freedom as shown in Figure 6.1a, where the masses are constrained to move in the direction of the spring axis and executing free vibrations. The displacements are measured from the un-stretched positions of the springs. Let x_1 and x_2 be the displacement of the masses m_1 and m_2 respectively at any given instant of time measured from the

equilibrium position with $x_2 > x_1$. Then the spring forces acting on the masses are as shown in free body diagram in Figure 6.1b

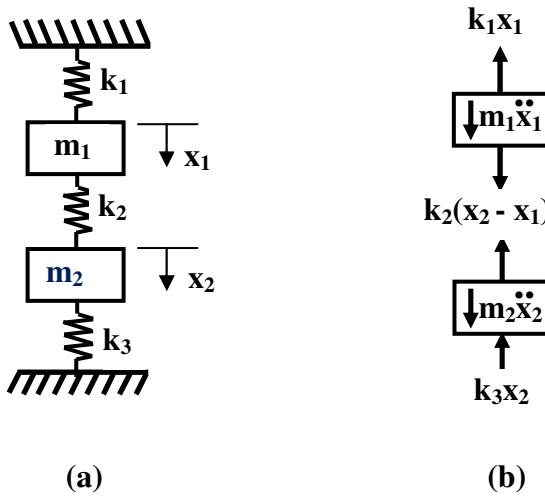


Figure 6.1

Based on Newton's second law of motion $\sum f = m \ddot{x}$

For mass m_1

$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1)$$

$$m_1 \ddot{x}_1 + k_1 x_1 - k_2 x_2 + k_2 x_1 = 0$$

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 = k_2 x_2 \quad \text{----- (1)}$$

for mass (2)

$$m_2 \ddot{x}_2 = -k_3 x_2 - k_2(x_2 - x_1)$$

$$m_2 \ddot{x}_2 + k_3 x_2 + k_2 x_2 - k_2 x_1 = 0$$

$$m_2 \ddot{x}_2 + (k_2 + k_3) x_2 = k_2 x_1 \quad \text{----- (2)}$$

The solution for x_1 and x_2 are obtained by considering that they can have harmonic vibration under steady state condition. Then considering the case when the mass m_1 execute harmonic vibration at frequency ω_1 and the mass m_2 execute harmonic vibration at frequency ω_2 then we have

$$x_1 = X_1 \sin \omega_1 t, \quad \text{and} \quad x_2 = X_2 \sin \omega_2 t \quad \text{----- (3)}$$

Where X_1 and X_2 are the amplitudes of vibrations of the two masses under steady state conditions.

Substituting equation (3) into equation (1) we have

$$-m_1 \omega_1^2 X_1 \sin \omega_1 t + (k_1 + k_2) X_1 \sin \omega_1 t = k_2 X_2 \sin \omega_2 t$$

$$\text{Therefore } \frac{X_1}{X_2} = \frac{k_2}{(k_1 + k_2) - m\omega_1^2} \frac{\sin \omega_2 t}{\sin \omega_1 t}$$

Since X_1 and X_2 are the amplitude of two harmonic motions, their ratio must be constant and independent of time. Therefore $\sin \omega_2 t / \sin \omega_1 t = C$ a constant.

Consider if $C > 1$. Then at time $t = \pi/2\omega_1$, $\sin\omega_1 t$ will be $\sin\omega_1 \cdot \pi/2\omega_1 = \sin \pi/2 = 1$

Therefore $\sin\omega_2 t / \sin\omega_1 t > 1$ or $\sin\omega_2 t > 1$ which is impossible. Hence $C > 1$ is not possible.

Similarly it can be shown that $C < 1$ is also not possible. Thus the only possibility is that $C = 1$

Hence $\sin\omega_2 t / \sin\omega_1 t = 1$ which is only possible if $\omega_2 = \omega_1 = \omega$. Hence the two harmonic motion have to be of the same frequency. Thus the solution of equation (1) and (2) can be

$$x_1 = X_1 \sin \omega t, \text{ and } x_2 = X_2 \sin \omega t \quad \dots \quad (4)$$

$$\ddot{x}_1 = -\omega^2 X_1 \sin \omega t \quad \ddot{x}_2 = -\omega^2 X_2 \sin \omega t \quad \dots \quad (5)$$

Substitute equation (4) and (5) into the equation (1) and (2)

$$-m_1\omega^2 X_1 \sin \omega t + (k_1 + k_2) X_1 \sin \omega t = k_2 X_2 \sin \omega t \quad \dots \quad (6)$$

$$-m_2\omega^2 X_2 \sin \omega t + (k_2 + k_3) X_2 \sin \omega t = k_2 X_1 \sin \omega t. \quad \dots \quad (7)$$

Canceling the common term $\sin \omega t$ on both the sides and re arranging the terms we have from equation (6)

$$X_1/X_2 = k_2 / (k_1 + k_2 - m_1\omega^2) \quad \dots \quad (8)$$

$$X_1/X_2 = [(k_2 + k_3) - m_2\omega^2] / k_2 \quad \dots \quad (9)$$

Thus equating equation (8) and (9) we have

$$X_1/X_2 = k_2 / (k_1 + k_2 - m_1\omega^2) = [(k_2 + k_3) - m_2\omega^2] / k_2 \quad \dots \quad (10)$$

Cross multiplying in equation (10) we have

$$(k_1 + k_2 - m_1\omega^2)(k_2 + k_3 - m_2\omega^2) = k_2^2 \quad \text{on simplification we get}$$

$$m_1 m_2 \omega^4 - [m_1(k_2 + k_3) + m_2(k_1 + k_2)] \omega^2 + [k_1 k_2 + k_2 k_3 + k_3 k_1] = 0 \quad \dots \quad (11)$$

The above equation (11) is quadratic in ω^2 and gives two values of ω^2 and therefore the two positive values of ω correspond to the two natural frequencies ω_{n1} and ω_{n2} of the system. The above equation is called frequency equation since the roots of the above equation give the natural frequencies of the system.

Now considering $m_1 = m_2 = m$ and $k_1 = k_3 = k$

Then the frequency equation (11) becomes

$$m^2 \omega^4 - 2m(k + k_2)\omega^2 + (k^2 + 2kk_2) = 0 \quad \dots \quad (12)$$

$$\text{Let: } \omega^2 = \lambda \quad \therefore \lambda^2 = \omega^4, \quad \therefore m^2 \lambda^2 - 2m(k + k_2)\lambda + (k^2 + 2kk_2) = 0 \quad \dots \quad (13)$$

The roots of the above equation (13) are as follows: Let $a = m^2$, $b = -2m(k + k_2)$; $c = (k^2 + 2kk_2)$

$$\therefore \lambda_{1,2} = [-b \pm \sqrt{(b^2 - 4ac)}] / 2a$$

$$\lambda_{1,2} = [-(-2m)(k + k_2) \pm \sqrt{[-2m(k + k_2)]^2 - 4(m^2)(k^2 + 2kk_2)}] / 2m^2$$

$$= [+2m(k + k_2)] / 2m^2 \pm [\sqrt{4m^2[(k^2 + k_2^2 + 2kk_2) - (k^2 + 2kk_2)] / 4m^4}] = (k + k_2) / m \pm \sqrt{(k_2^2 / m^2)}$$

$$= (k + k_2) / m \pm k_2 / m$$

Thus $\lambda_1 = (k + k_2)/m - k_2/m = k/m$. Then $\omega_{n1}^2 = K/m \therefore \omega_{n1} = \sqrt{(k/m)}$ ----- (14)

and $\therefore \lambda_2 = (k + 2k_2)/m$ Thus $\omega_{n2}^2 = (k + 2k_2)/m$. Then $\therefore \omega_{n2} = \sqrt{[(k + 2k_2)/m]}$ --- (15)

ω_{n1} is called the first or fundamental frequency or 1st mode frequency, ω_{n2} is called the second or 2nd mode frequency. Thus the number of natural frequencies of a system is equal to the number of degrees of freedom of system.

Modes Shapes: From equation (10) we have $X_1/X_2 = k_2/(k+k_2) - m\omega^2 = (k_2 + k) - m\omega^2/k_2$ ---(16)

Substitute $\omega_{n1} = \sqrt{(k/m)}$ in any one of the above equation (16).

$$(X_1/X_2)_{\omega n1} = k_2 / (k + k_2 - m(k/m)) \text{ or } ((k_2 + k) - m(k/m))/k_2 = k_2/k_2 = 1$$

$$(X_1/X_2)_{\omega n1} = 1 \text{ ----- (17)}$$

Similarly substituting $\omega_{n2} = \sqrt{[(k + 2k_2)/m]}$ in any one of the above equation (16).

$$(X_1/X_2)_{\omega n2} = k_2 / (k + k_2 - m(k+ 2k_2)/m) \text{ or } ((k_2 + k) - m(k+ 2k_2)/m)/k_2 = -k_2/k_2 = -1$$

$$(X_1/X_2)_{\omega n2} = -1 \text{ ----- (18)}$$

The displacements X_1 and X_2 corresponding to the two natural frequency of the system can be plotted as shown in Figure 6.2, which describe the mode in which the masses vibrate. When the system vibrates in principal mode the masses oscillate in such a manner that they reach maximum displacements simultaneously and pass through their equilibrium points simultaneously or all moving parts of the system oscillate in phase with one frequency. Since the ratio X_1/X_2 is important rather than the amplitudes themselves, it is customary to assign a unit value of amplitude to either X_1 or X_2 . When this is done, the principal mode is referred as normal mode of the system.

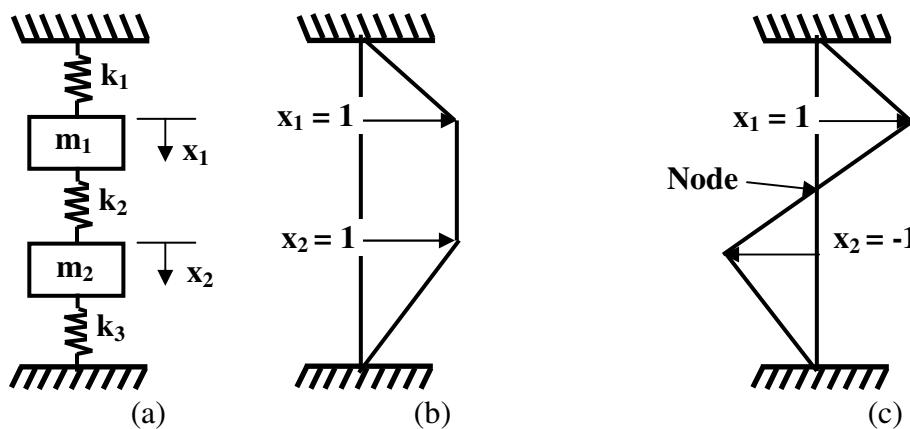


Figure – 6.2

$$\omega_{n1} = \sqrt{(k/m)}$$

1st Mode

$$\omega_{n2} = \sqrt{[(k + 2k_2)/m]}$$

2nd Mode

It can be observed from the figure – 6.2b when the system vibrates at the first frequency, the amplitude of two masses remain same. The motion of both the masses are in phase i.e., both the masses move up or down together, the length of the middle spring remains constant, this spring (coupling spring) is neither stretched nor compressed. It moves rigid bodily with both the masses and hence totally ineffective as shown in Figure 6.3a. Even if the coupling spring is removed the two masses will vibrate as two single degree of freedom systems with $\omega_n = \sqrt{(K/m)}$.

When the system vibrates at the second frequency the displacement of the two masses have the same magnitude but with opposite signs. Thus the motions of m_1 and m_2 are 180° out of phase, the midpoint of the middle spring remains stationary for all the time. Such a point which experiences no vibratory motion is called a node, as shown in Figure 6.3b which is as if the middle of the coupling spring is fixed

When the two masses are given equal initial displacements in the same direction and released, they will vibrate at first frequency. When they are given equal initial displacements in opposite direction and released they will vibrate at the second frequency as shown in Figures 6.3a and 6.3b

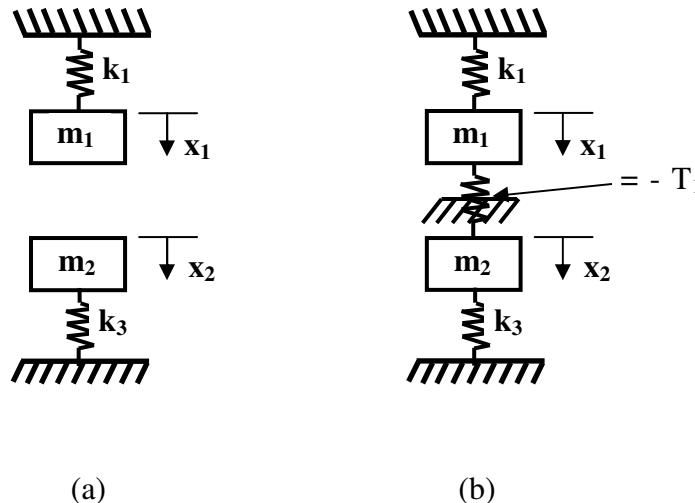


Figure – 6.3

$$\omega_{n1} = \sqrt{(k/m)}$$

1st Mode

$$\omega_{n2} = \sqrt{[(k + 2k_2)/m]}$$

2nd Mode

If unequal displacements are given to the masses in any direction, the motion will be superposition of two harmonic motions corresponding to the two natural frequencies.

Problems

1. Obtain the frequency equation for the system shown in Figure – 6.4. Also determine the natural frequencies and mode shapes when $k_1 = 2k$, $k_2 = k$, $m_1 = m$ and $m_2 = 2m$.

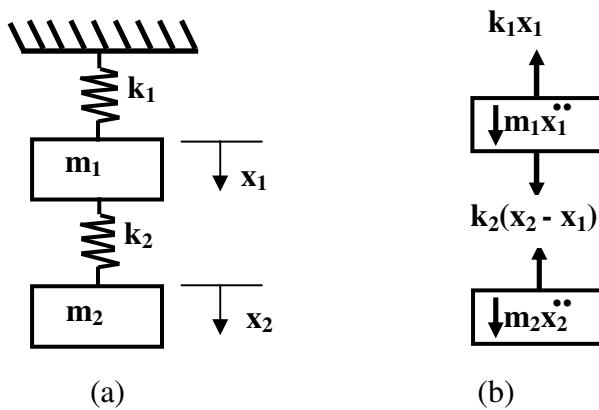


Figure – 6.4.

Solution

Consider two degrees of freedom system shown in Figure 6.4a, where the masses are constrained to move in the direction of the spring axis and executing free vibrations. The displacements are measured from the un-stretched positions of the springs. Let x_1 and x_2 be the displacement of the masses m_1 and m_2 respectively at any given instant of time measured from the equilibrium position with $x_2 > x_1$. Then the spring forces acting on the masses are as shown in free body diagram in Figure 6.4b

Based on Newton's second law of motion $\sum f = m \ddot{x}$

For mass m_1

$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1)$$

$$m_1 \ddot{x}_1 + k_1 x_1 - k_2 x_2 + k_2 x_1 = 0$$

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 = k_2 x_2 \quad \text{----- (1)}$$

for mass (2)

$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1)$$

$$m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0$$

$$m_2 \ddot{x}_2 + k_2 x_2 = k_2 x_1 \quad \text{----- (2)}$$

The solution for x_1 and x_2 are obtained by considering that they can have harmonic vibration under steady state condition. Then considering the case when the masses execute harmonic vibration at frequency ω . Thus if $x_1 = X_1 \sin \omega t$, and $x_2 = X_2 \sin \omega t$ ----- (3)

Then we have $x_1 = -\omega^2 X_1 \sin \omega t$, $x_2 = -\omega^2 X_2 \sin \omega t$ ----- (4)

Substitute equation (3) and (4) into the equation (1) and (2) we get

$$-m_1\omega^2 X_1 \sin \omega t + (k_1 + k_2) X_1 \sin \omega t = k_2 X_2 \sin \omega t \text{ ----- (5)}$$

$$-m_2\omega^2 X_2 \sin \omega t + k_2 X_2 \sin \omega t = k_2 X_1 \sin \omega t \text{ ----- (6)}$$

From equation (5) we have $X_1/X_2 = k_2/[(k_1 + k_2) - m_1\omega^2]$ ----- (7)

From equation (6) we have $X_1/X_2 = [k_2 - m_2\omega^2]/k_2$ ----- (8)

Equating (7) and (8)

$$k_2/(k_1 + k_2 - m_1\omega^2) = [k_2 - m_2\omega^2]/k_2$$

$$k_2^2 = (k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2)$$

$$k_2^2 = (k_1 + k_2)k_2 - m_1\omega^2 k_2 - m_2\omega^2(k_1 + k_2) + m_1 m_2\omega^4$$

$$m_1 m_2 \omega^4 - \omega^2 [m_1 k_2 + m_2 (k_1 + k_2)] + k_1 k_2 = 0 \text{ ----- (9)}$$

$$\text{letting } \omega^2 = \lambda \quad m_1 m_2 \lambda^2 - \lambda [m_1 k_2 + m_2 (k_1 + k_2)] + k_1 k_2 = 0 \text{ ----- (10)}$$

Equation (10) is the frequency equation of the system which is quadratic in λ and hence the solution is

$$\lambda = [[m_1 k_2 + m_2 (k_1 + k_2)] \pm \sqrt{[(m_1 k_2 + m_2 (k_1 + k_2))^2] - 4 m_1 m_2 k_1 k_2}] / 2m_1 m_2$$

To determine the natural frequencies Given $k_1 = 2 k$, $k_2 = k$ and $m_1 = m$, $m_2 = 2m$

$$\lambda = [mk + 2m(2k + k) \pm \sqrt{[mk + 6mk]^2 - 4m^2 2mk^2 k}] / 2m \cdot 2m$$

$$= [7mk \pm \sqrt{(7mk)^2 - 4(4m^2 k^2)}] / 4m^2$$

$$= [7mk \pm \sqrt{(49m^2 k^2 - 16m^2 k^2)}] / 4m^2$$

$$\lambda = [7mk \pm 5.744 mk] / 4m^2 \quad \text{Thus } \lambda_1 = [7mk - 5.744 mk] / 4m^2 \text{ and } \lambda_2 = [7mk + 5.744 mk] / 4m^2$$

$$\lambda_1 = \omega_{n1}^2 = [7mk - 5.744 mk] / 4m^2 = 1.255 mk / 4m^2 = 0.3138 \text{ k/m} \quad \text{Thus } \omega_{n1} = 0.56 \sqrt{\text{k/m}}$$

$$\lambda_2 = \omega_{n2}^2 = [7mk + 5.744 mk] / 4m^2 = 3.186 \text{ k/m. Thus } \omega_{n2} = 1.784 \sqrt{\text{k/m}}$$

Substituting the values of frequencies into the amplitude ratio equation as given by equation (7) and (8) one can determine the mode shapes:

FOR THE FIRST MODE:

Substituting $\omega_{n1}^2 = 0.3138 \text{ K/m}$ into either of the equation (7) or (8) we get first mode shape:

$$\text{i.e. } X_1/X_2 = k_2/[(k_1 + k_2) - m_1\omega^2] \text{ ----- (7)} \quad X_1/X_2 = [k_2 - m_2\omega^2]/k_2 \text{ ----- (8)}$$

$$X_1/X_2 = k / [(2k + k - m\omega^2)] = k / [3k - m\omega_{n1}^2]$$

$$X_1/X_2 = k / [3k - 2m \cdot 0.3138 \text{ k/m}] = 1 / (3 - 0.3138) = 1 / 2.6862 = 0.3724$$

Thus we have $X_1/X_2 = 0.3724$. Then If $X_1 = 1$, $X_2 = 2.6852$

FOR THE SECOND MODE:

Substituting $\omega_{n2}^2 = 3.186 \text{ K/m}$ into either of the equation (7) or (8) we get first mode shape:

$$\text{i.e. } X_1/X_2 = k_2 / [(k_1 + k_2) - m_1 \omega^2] \quad \dots \dots \dots \quad (7) \quad X_1/X_2 = [k_2 - m_2 \omega^2] / k_2 \quad \dots \dots \dots \quad (8)$$

$$X_1/X_2 = k / [(2k + k - m \omega^2)] = k / [3k - m \cdot \omega_{n1}^2]$$

$$X_1/X_2 = k / [3k - 2m \cdot 3.186 \text{ K/m}] = 1 / (3 - 3.186) = 1 / (-0.186) = -5.37 \text{ or}$$

$$X_1/X_2 = (k - 2m \cdot 3.186 \text{ K/m})/k = [1 - 2(3.186)] = -5.37$$

Thus we have $X_1/X_2 = -5.37$. Then If $X_1 = 1$, $X_2 = -0.186$

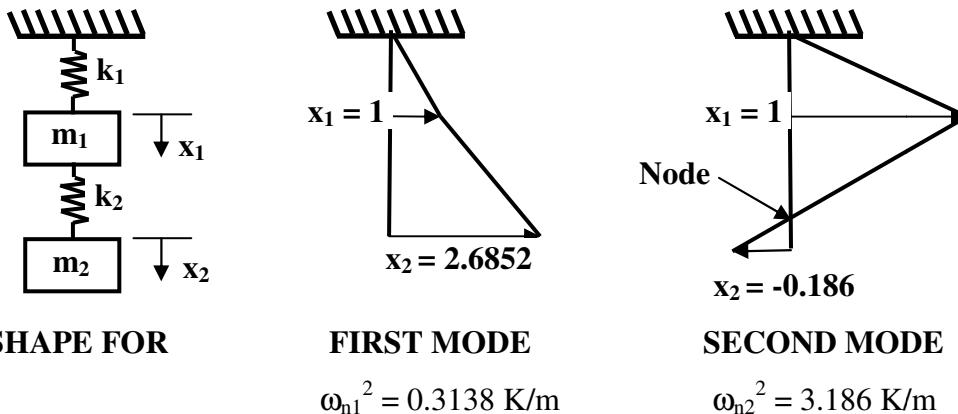


Figure – 6.5.

Derive the frequency equation for a double pendulum shown in figure 6.6. Determine the natural frequency and mode shapes of the double pendulum when $m_1 = m_2 = m$ $l_1 = l_2 = l$

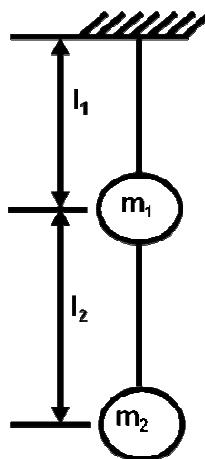


Figure 6.6

Consider two masses m_1 and m_2 suspended by string of length l_1 and l_2 as shown in the figure 6.6. Assume the system vibrates in vertical plane with small amplitude under which it only has the oscillation.

Let θ_1 and θ_2 be the angle at any given instant of time with the vertical and x_1 and x_2 be the horizontal displacement of the masses m_1 and m_2 from the initial vertical position respectively.

For small angular displacement we have $\sin\theta_1 = x_1 / l_1$ and $\sin\theta_2 = (x_2 - x_1) / l_2$ --- (1)

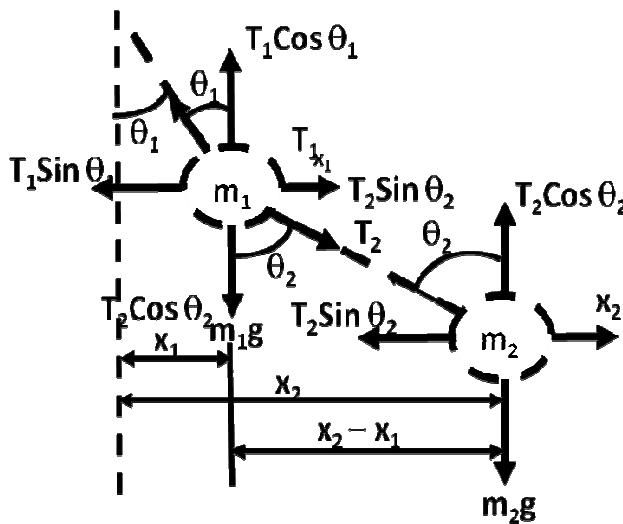


Figure 6.7 Free body diagram

Figure 6.7 shows the free body diagram for the two masses. For equilibrium under static condition the summation of the vertical forces should be equal to zero. Thus we have

$$\text{At mass } m_1 \quad T_1 \cos\theta_1 = mg + T_2 \cos\theta_2 \quad \dots\dots (2)$$

$$\text{At mass } m_2 \quad T_2 \cos\theta_2 = mg \quad \dots\dots (3)$$

For smaller values of θ we have $\cos\theta = 1$. Then the above equations can be written as

$$T_2 = m_2 g \quad \dots\dots (4) \quad \text{and} \quad T_1 = m_1 g + m_2 g \quad T_1 = (m_1 + m_2)g \quad \dots\dots (5)$$

When the system is in motion, the differential equation of motion in the horizontal direction can be derived by applying Newton Second Law of motion.

Then we have for mass m_1

$$\begin{aligned} m_1 \ddot{x}_1 &= -T_1 \sin\theta_1 + T_2 \sin\theta_2 \\ m_1 \ddot{x}_1 + T_1 \sin\theta_1 &= T_2 \sin\theta_2 \quad \dots\dots (6) \end{aligned}$$

For mass m_2

$$\begin{aligned} m_2 \ddot{x}_2 &= -T_2 \sin\theta_2 \\ m_2 \ddot{x}_2 + T_2 \sin\theta_2 &= 0 \quad \dots\dots (7) \end{aligned}$$

Substituting the expression for T_2 and T_1 from equation (4) and (5) and for $\sin\theta_1$ and $\sin\theta_2$ from equation (1) into the above equation (6) and (7) we have

Equation (6) becomes

$$m_1 \ddot{x}_1 + [(m_1 + m_2)g](x_1/l_1) = m_2 g[(x_2 - x_1)/l_2]$$

$$m_1 \ddot{x}_1 + \{(m_1 + m_2)/l_1\} + m_2/l_2 g x_1 = (m_2 g/l_2) x_2 \quad \dots\dots (8)$$

Equation (7) becomes

$$m_2 \ddot{x}_2 + m_2 g(x_2 - x_1)/l_2$$

$$m_2 \ddot{x}_2 + (m_2 g/l_2) x_1 = (m_2 g/l_2) x_2 \quad \dots\dots (9)$$

Equations (8) and (9) represent the governing differential equation of motion. Thus assuming the solution for the principal mode as

$$\ddot{x}_1 = -\omega^2 A \sin \omega t \quad \text{and} \quad \ddot{x}_2 = -\omega^2 B \sin \omega t \quad \dots\dots (10)$$

Substitute in (10) into equation (8) and (9) and cancelling the common term $\sin \omega t$ we have

$$[-m_1 \omega^2 + \{(m_1+m_2)/l_1 + m_2/l_2\}g]A = (m_2 g/l_2)B \quad \dots\dots (11)$$

$$[-m_2 \omega^2 + (m_2 g/l_2)]B = (m_2 g/l_2)A \quad \dots\dots (12)$$

From equation (11) we have

$$A/B = (m_2 g/l_2) / [\{(m_1+m_2)/l_1 + m_2/l_2\}g] - m_1 \omega^2 \quad \dots\dots (13)$$

From equation (12) we have

$$A/B = [(m_2 g/l_2) - m_2 \omega^2] / (m_2 g/l_2) \quad \dots\dots (14)$$

Equating equation (13) and (14) we have

$$A/B = (m_2 g/l_2) / [\{(m_1+m_2)/l_1 + m_2/l_2\}g - m_1 \omega^2] = [(m_2 g/l_2) - m_2 \omega^2] / (m_2 g/l_2)$$

$$[\{(m_1+m_2)/l_1 + m_2/l_2\}g - m_1 \omega^2][(m_2 g/l_2) - m_2 \omega^2] = (m_2 g/l_2)^2 \quad \dots\dots (15)$$

Equation (15) is a quadratic equation in ω^2 which is known as the frequency equation.

Solving for ω^2 we get the natural frequency of the system.

Particular Case:

When $m_1 = m_2 = m$ and $l_1 = l_2 = l$

Then equation (13) will be written as $A/B = (mg/l)/[(3mg/l) - m\omega^2] = (g/l)/[(3g/l) - \omega^2]$

$$A/B = 1/[3 - (\omega^2 l/g)] \quad \dots\dots (16)$$

$$\text{and equation (14) will be written as } A/B = [1 - (\omega^2 l/g)] \quad \dots\dots (17)$$

$$\text{Equating equation (16) and (17) we get } A/B = 1/[3 - (\omega^2 l/g)] = [1 - (\omega^2 l/g)]$$

$$[3 - (\omega^2 l/g)] * [1 - (\omega^2 l/g)] = 1 \quad \text{or} \quad (3g - \omega^2 l) * (g - \omega^2 l) = g^2$$

$$3g^2 - 3gl\omega^2 - gl\omega^2 + l^2\omega^4 = g^2 \quad \text{or} \quad l^2\omega^4 - 4gl\omega^2 + 2g^2 = 0 \quad \text{or}$$

$$\omega^4 - (4g/l)\omega^2 + (2g^2/l^2) = 0 \quad \dots\dots (18)$$

$$\text{letting } \lambda = \omega^2 \text{ in equation (18) we get } \lambda^2 - (4g/l)\lambda + (2g^2/l^2) = 0 \quad \dots\dots (19)$$

Which is a quadratic equation in l and the solution for the equation (19) is

$$\lambda_{1,2} = (2g/l) \pm \sqrt{[(4g^2/l^2) - (2g^2/l^2)]} \quad \text{or} \quad \lambda_{1,2} = (g/l)(2 \pm \sqrt{2}) \quad \dots\dots\dots (20)$$

$$\lambda_1 = (g/l)(2 - \sqrt{2}) = 0.5858(g/l) \quad \dots\dots (21) \quad \text{and} \quad \lambda_2 = (g/l)(2 + \sqrt{2}) = 3.4142(g/l) \quad \dots\dots (22)$$

Since $\lambda = \omega^2$ then the natural frequency $\omega_{n1} = \sqrt{l_1} = 0.7654\sqrt{g/l}$ thus $\omega_{n1} = 0.7654\sqrt{g/l}$ --- (23)

and $\omega_{n2} = \sqrt{l_2} = 1.8478\sqrt{g/l}$ thus $\omega_{n2} = 1.8478\sqrt{g/l}$ --- (24)

Substituting ω_{n1} and ω_{n2} from equation (23) and (24) into either of the equation (16) or (17) we get the mode shape

FOR THE FIRST MODE:

Mode shapes for the first natural frequency $\omega_{n1} = 0.7654\sqrt{g/l}$ or $\omega_{n1}^2 = (g/l)(2 - \sqrt{2})$

I mode from equation (16) $A/B = 1/[3 - (\omega^2 l/g)]$

$$(A/B)_1 = 1/[3 - \omega_{n1}^2 l/g] = 1/[(3 - \{(g/l)(2 - \sqrt{2})\}l/g)] = 1/(3 - 2 + \sqrt{2}) = 1/(1 + \sqrt{2}) = 1/2.4142 = 0.4142$$

Thus when $A = 1$ $B = 2.4142$

Also from equation (17) $A/B = [1 - (\omega^2 l/g)]$

For $\omega_{n1} = 0.7654\sqrt{g/l}$ or $\omega_{n1}^2 = (g/l)(2 - \sqrt{2})$

$$(A/B)_1 = (1 - \omega_{n1}^2 l/g) = [1 - \{(g/l)(2 - \sqrt{2})\}l/g]$$

$$\text{or } (A/B)_1 = (1 - 2 + \sqrt{2}) = \sqrt{2} - 1 = 0.4142$$

Thus when $A = 1$ $B = 2.4142$

Modes shape is shown in figure-6.8

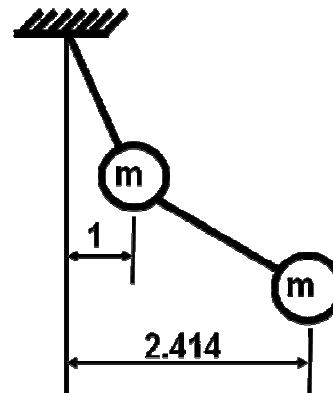


Figure- 6.8

FOR THE SECOND MODE:

Mode shapes for second natural frequency $\omega_{n2} = 1.8478\sqrt{g/l}$ or $\omega_{n2}^2 = (g/l)(2 + \sqrt{2})$

II mode from equation (16) is given by $A/B = 1/[3 - (\omega^2 l/g)]$

$$(A/B)_2 = 1/[3 - \omega_{n1}^2 l/g] = 1/[(3 - \{(g/l)(2 + \sqrt{2})\}l/g)] = 1/(3 - 2 - \sqrt{2}) = 1/(1 - \sqrt{2})$$

$$(A/B)_2 = 1/(-0.4142) = -2.4142 \quad \text{or} \quad \text{Thus when } A = 1 \text{ } B = -0.4142$$

Also from equation (17) $A/B = [1 - (\omega^2 l/g)]$

$\omega_{n2} = 1.8478\sqrt{g/l}$ or $\omega_{n2}^2 = (g/l)(2 + \sqrt{2})$

$$(A/B)_2 = (1 - \omega_{n1}^2 l/g) = [1 - \{(g/l)(2 + \sqrt{2})\}l/g]$$

$$(A/B)_2 = (1 - 2 - \sqrt{2}) = -(1 + \sqrt{2}) = -2.4142$$

Thus when $A = 1$ $B = -0.4142$

Modes shape is shown in figure-6.9

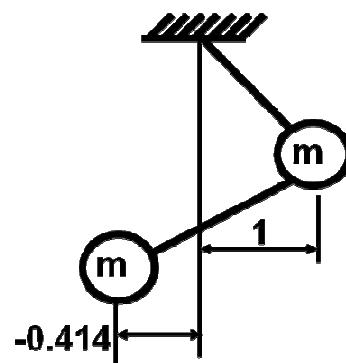


Figure- 6.9

Determine the natural frequencies of the coupled pendulum shown in the figure – 6.10. Assume that the light spring of stiffness 'k' is un-stretched and the pendulums are vertical in the equilibrium position.

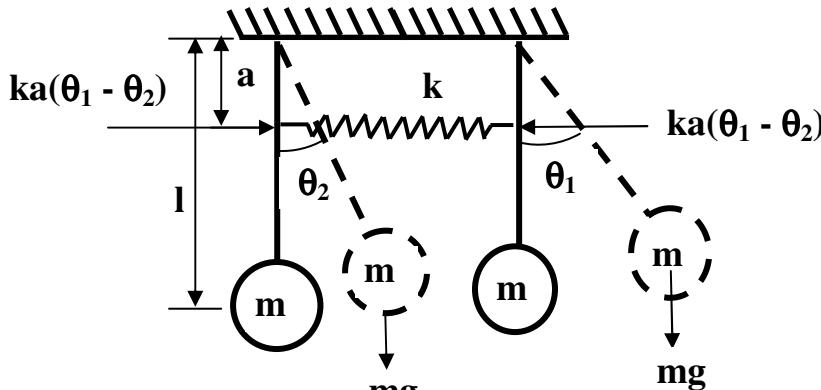


Figure – 6.10.

Solution:

Considering counter clockwise angular displacement to be positive and taking the moments about the pivotal point of suspension by D'Alembert's principle we have

$$ml^2\ddot{\theta}_1 = -mgl\theta_1 - ka(\theta_1 - \theta_2) \quad \dots \quad (1)$$

$$ml^2\ddot{\theta}_2 = -mgl\theta_2 + ka(\theta_1 - \theta_2) \quad \dots \quad (2)$$

Equation (1) and (2) can also be written as

$$ml^2\ddot{\theta}_1 + (mgl + ka)\theta_1 = ka\theta_2 \quad \dots \quad (3)$$

$$ml^2\ddot{\theta}_2 + (mgl + ka)\theta_2 = ka\theta_1 \quad \dots \quad (4)$$

Equation (3) and (4) are the second order differential equation and the solution for θ_1 and θ_2 are obtained by considering that they can have harmonic vibration under steady state condition. Then considering the case when the masses execute harmonic vibration at frequency ω

$$\text{Thus if } \theta_1 = A \sin \omega t, \text{ and } \theta_2 = B \sin \omega t \quad \dots \quad (5)$$

Substitute equation (5) into the equation (3) and (4) and canceling the common terms we get

$$(-ml^2\omega^2 + mgl + ka^2)A = ka^2B \quad \dots \quad (6)$$

$$(-ml^2\omega^2 + mgl + ka^2)B = ka^2A \quad \dots \quad (7)$$

$$\text{From equation (6) we have } A/B = ka^2 / [mgl + ka^2 - ml^2\omega^2] \quad \dots \quad (8)$$

$$\text{From equation (7) we have } A/B = [mgl + ka^2 - ml^2\omega^2] / ka^2 \quad \dots \quad (9)$$

Equating (8) and (9)

$$A/B = ka^2 / [mgl + ka^2 - ml^2\omega^2] = [mgl + ka^2 - ml^2\omega^2] / ka^2$$

$$[mgl + ka^2 - ml^2\omega^2]^2 = [ka^2]^2 \quad \dots \quad (10) \quad \text{or}$$

$$mgl + ka^2 - ml^2\omega^2 = \pm ka^2 \quad \omega^2 = (mgl + ka^2 \pm ka^2) / ml^2 \quad \dots \quad (11)$$

$$\omega_{1,2} = \sqrt{[(mgl + ka^2 \pm ka^2) / ml^2]} \quad \dots\dots \quad (12)$$

$$\omega_1 = \sqrt{[(mgl + ka^2 - ka^2) / ml^2]} = \sqrt{g/l} \quad \dots\dots \quad (13)$$

$$\omega_2 = \sqrt{[(mgl + ka^2 + ka^2) / ml^2]} = \sqrt{[(g/l) + (2ka^2/ml^2)]} \quad \dots\dots \quad (14)$$

Substituting the values of frequencies into the amplitude ratio equation as given by equation (8) and (9) one can determine the mode shapes:

FOR THE FIRST MODE:

Substituting $\omega_{n1}^2 = g/l$ into either of the equation (8) or (9) we get first mode shape:

$$A/B = ka^2 / [mgl + ka^2 - ml^2\omega^2] = ka^2 / [mgl + ka^2 - ml^2g/l] = ka^2 / [mgl + ka^2 - mlg] = ka^2 / ka^2$$

$$A/B = 1$$

FOR THE SECOND MODE:

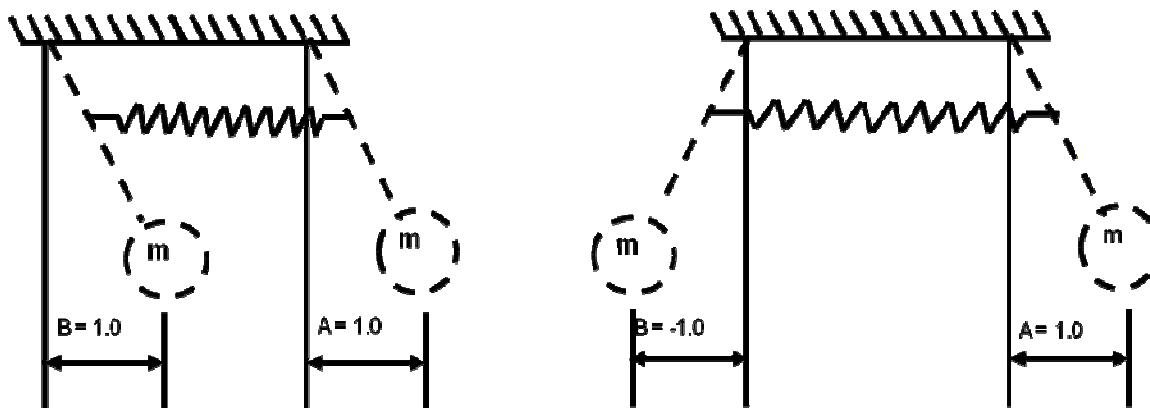
Substituting $\omega_{n2}^2 = [(g/l) + (2ka^2/ml^2)]$ into either of the equation (8) or (9) we get second mode shape:

$$A/B = ka^2 / [mgl + ka^2 - ml^2\omega^2] = ka^2 / [mgl + ka^2 - ml^2[(g/l) + (2ka^2/ml^2)]]$$

$$= ka^2 / [mgl + ka^2 - mlg - 2ka^2] = (ka^2 / -ka^2) = -1 \text{ Thus } A/B = -1$$

Mode shapes at these two natural frequencies are as shown in figure- 6.10

MODE SHAPES AT TWO DIFFERENT FREQUENCIES



FIRST MODE

$$\omega_{n1}^2 = g/l \quad A/B = 1$$

Figure-6.10 Mode Shapes at first frequency

SECOND MODE

$$\omega_{n2}^2 = [(g/l) + (2ka^2/ml^2)] \quad A/B = -1$$

Figure-6.11 Mode Shapes at second frequency

Derive the equation of motion of the system shown in figure 6.12. Assume that the initial tension 'T' in the string is too large and remains constant for small amplitudes. Determine the natural frequencies, the ratio of amplitudes and locate the nodes for each mode of vibrations when $m_1 = m_2 = m$ and $l_1 = l_2 = l_3 = l$.

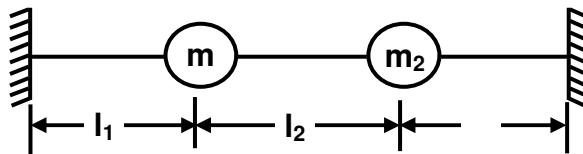
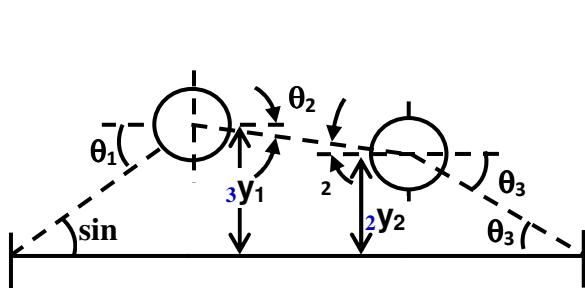


Figure 6.12.

At any given instant of time let y_1 and y_2 be the displacement of the two masses m_1 and m_2 respectively. The configuration is as shown in the figure 6.13.



(a)

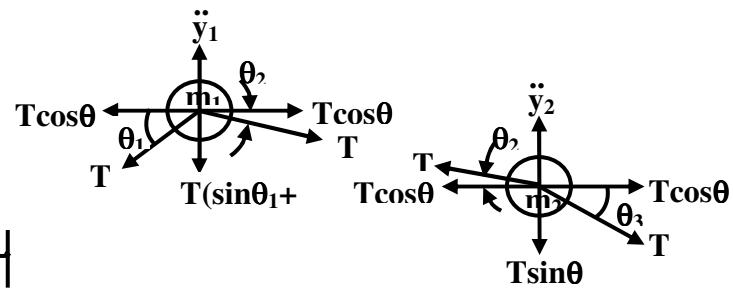


Figure 6.14

Figure 6.13.

The forces acting on the two masses are shown in the free body diagram in figure 6.14(a) and (b)

From figure 6.13 we have $\sin\theta_1 = (y_1/l_1)$, $\sin\theta_2 = [(y_1 - y_2)/l_2]$ and $\sin\theta_3 = (y_2/l_3)$

For small angle we have $\sin\theta_1 = \theta_1 = (y_1/l_1)$, $\sin\theta_2 = \theta_2 = [(y_1 - y_2)/l_2]$ and $\sin\theta_3 = \theta_3 = (y_2/l_3)$

and $\cos\theta_1 = \cos\theta_2 = \cos\theta_3 = 1.0$ Thus the equation of motion for lateral movement of the masses

For the mass m_1

$$m_1 \ddot{y}_1 = - (T\sin\theta_1 + T\sin\theta_2) = - T(\theta_1 + \theta_2)$$

$$m_1 \ddot{y}_1 = - T[(y_1/l_1) + (y_1 - y_2)/l_2] \quad \text{or} \quad m_1 \ddot{y}_1 + [(T/l_1) + (T/l_2)]y_1 = (T/l_2)y_2 \quad \dots (1)$$

For the mass m_2

$$m_2 \ddot{y}_2 = (T\sin\theta_2 - T\sin\theta_3)$$

$$m_2 y_2 = T[(y_1 - y_2)/l_2 - (y_2/l_3)] \quad \text{or} \quad y_2 + \ddot{y}_2 + [(T/l_2) + (T/l_3)]y_2 = (T/l_2)y_1 \quad \dots (2)$$

Assuming harmonic motion as $y_1 = A \sin \omega t$ and $y_2 = B \sin \omega t$ ----- (3) and substituting this into equation (1) and (2) we have $[-m_1 \omega^2 + (T/l_1) + (T/l_2)] A = (T/l_2) B$ ----- (4)

$$[-m_2 \omega^2 + (T/l_2) + (T/l_3)] B = (T/l_2) A ----- (5)$$

Thus from equation (4) we have $A/B = (T/l_2) / [(T/l_1) + (T/l_2) - m_1 \omega^2]$ ----- (6)

and from equation (5) we have $A/B = [(T/l_2) + (T/l_3) - m_2 \omega^2] / (T/l_2)$ ----- (7)

Equating equation (6) and (7) we have $A/B = (T/l_2) / [(T/l_1) + (T/l_2) - m_1 \omega^2] / [(T/l_2) + (T/l_3) - m_2 \omega^2] / (T/l_2)$

$$\text{Thus we have } [(T/l_1) + (T/l_2) - m_1 \omega^2][(T/l_2) + (T/l_3) - m_2 \omega^2] = (T^2/l_2^2) ----- (8)$$

Equation (8) is the equation on motion which is also known as frequency equation. Solving this equation gives the natural frequencies of the system.

Particular Case: When $m_1 = m_2 = m$ and $l_1 = l_2 = l_3 = l$ then equation (6) can be written as

$$A/B = (T/l) / [(T/l) + (T/l) - m\omega^2] = (T/l) / [(2T/l) - m\omega^2] ----- (9)$$

$$\text{and equation (7) can be written as } A/B = [(T/l) + (T/l) - m\omega^2] / (T/l) = [(2T/l) - m\omega^2] / (T/l) ----- (10)$$

$$\text{Equating equation (9) and (10) we have } [(2T/l) - m\omega^2]^2 = (T/l)^2 ----- (11)$$

$$\text{Thus } 2T/l - m\omega^2 = \pm (T/l) ----- (12) \quad \text{Therefore we have } \omega^2 = [(2T \pm T)/ml] ----- (13)$$

$$\omega_{n1} = \sqrt{(2T-T)/ml} = \sqrt{T/ml} ----- (14) \quad \text{and} \quad \omega_{n2} = \sqrt{(2T+T)/ml} = \sqrt{3T/ml} ----- (15)$$

Substituting equation (14) and (15) into either of the equation (9) or (10) we have the ratio of amplitudes for the two natural frequencies. For the first natural frequency $\omega_{n1} = \sqrt{T/ml}$ then from equation (9) we have $(A/B)_{\omega_{n1}} = (T/l) / [(2T/l) - m\omega^2] = (T/l) / [(2T/l) - m(2T/ml)] = (T/l) / (T/l) = +1$ or from equation (10) we have $(A/B)_{\omega_{n1}} = [(2T/l) - m\omega^2] / (T/l) = [(2T/l) - m(2T/ml)] / (T/l)$

$$\text{Thus } (A/B)_{\omega_{n1}} = (T/l) / (T/l) = +1$$

For the second natural frequency $\omega_{n2} = \sqrt{3T/ml}$ then from equation (9) we have

$$(A/B)_{\omega_{n2}} = (T/l) / [(2T/l) - m\omega^2] = (T/l) / [(2T/l) - m(3T/ml)] = (T/l) / (-T/l) = -1$$

Thus $(A/B)_{\omega_{n2}} = (T/l) / (-T/l) = -1$ Then the mode shape will be as shown in figure 6.15(a) and (b)

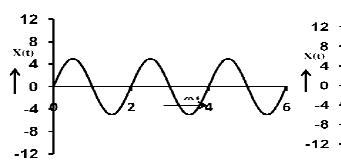
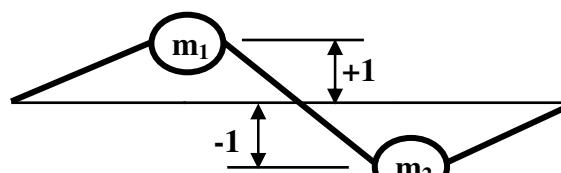
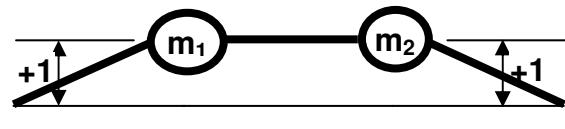


Figure 6.15(a)

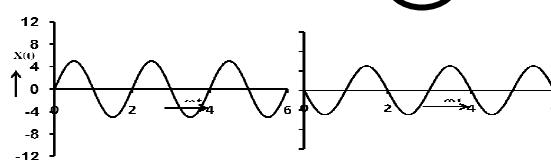


Figure 6.15(b)

First Mode $\omega_{n1} = \sqrt{T/ml}$, $(A/B)_{\omega_{n1}} = +1$

Second Mode $\omega_{n2} = \sqrt{3T/ml}$, $(A/B)_{\omega_{n2}} = -1$

Torsional Vibratory systems

Derive the equation of motion of a torsional system shown in figure 6.16. Let J_1 and J_2 be the mass moment of inertia of the two rotors which are coupled by shafts having torsional stiffness of K_{t1} and K_{t2} .

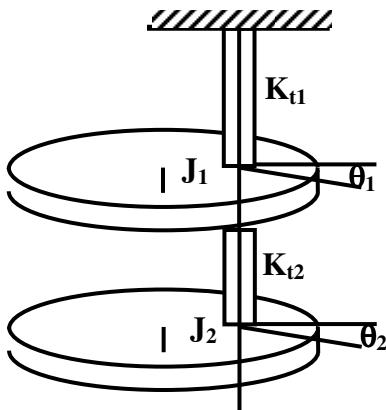


Figure 6.16 Two Degree of Freedom
torsional system

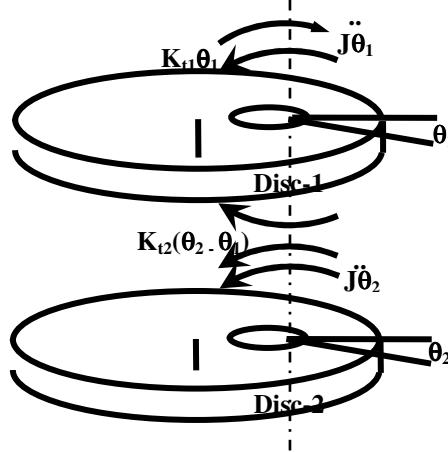


Figure 6.17 Free Body Diagram

If θ_1 and θ_2 are the angular displacement of the two rotors at any given instant of time, then the shaft with the torsional stiffness K_{t1} exerts a torque of $K_{t1}\theta_1$ and the shaft with the torsional stiffness K_{t2} exerts a torque of $K_{t2}(\theta_2 - \theta_1)$ as shown in the free body diagram figure 6.17

Then by Newton second law of motion we have for the mass m_1

$$J_1\ddot{\theta}_1 = -K_{t1}\theta_1 + K_{t2}(\theta_2 - \theta_1) \quad \text{or} \quad J_1\ddot{\theta}_1 + (K_{t1} + K_{t2})\theta_1 = K_{t2}\theta_2 \quad (1)$$

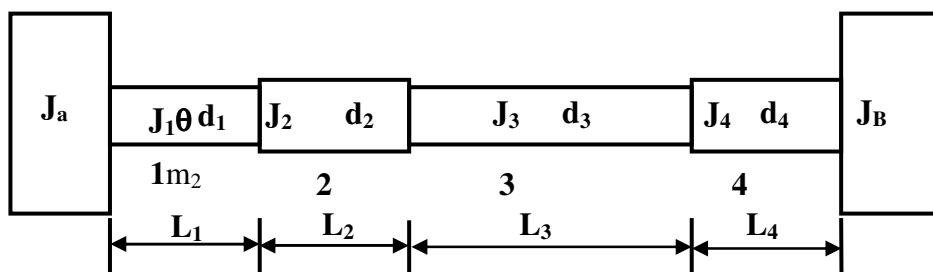
for the mass m_2

$$J_2\ddot{\theta}_2 = -K_{t2}(\theta_2 - \theta_1) \quad \text{or} \quad J_2\ddot{\theta}_2 + K_{t2}\theta_2 = K_{t1}\theta_1 \quad (2)$$

Equation (1) and (2) are the governing Equations of motion of the system.

Equivalent Shaft for a Torsional system

In many engineering applications we find shaft of different diameters as shown in Figure 6.18 are in use.



For vibration analysis it is required to have an equivalent system. In this section we will study how to obtain the torsionally equivalent shaft. Let θ be the total angle of twist in the shaft by application of torque T , and $\theta_1, \theta_2, \theta_3$ and θ_4 be twists in section 1, 2, 3 and 4 respectively. Then we have

$$\theta = \theta_1 + \theta_2 + \theta_3 + \theta_4$$

From torsion theory we have,

$$T = G\theta \quad \text{Where } J = pd^4/32 \text{ Polar moment of inertia of shaft.}$$

$$J = L$$

Thus $\theta = \theta_1 + \theta_2 + \theta_3 + \theta_4$ will be

$$\theta = \frac{TL_1}{J_1G_1} + \frac{TL_2}{J_2G_2} + \frac{TL_3}{J_3G_3} + \frac{TL_4}{J_4G_4}$$

If material of shaft is same, then the above equation can be written as

$$\theta = \frac{32T}{\pi G} \left[\frac{L_1}{d_1^4} + \frac{L_2}{d_2^4} + \frac{L_3}{d_3^4} + \frac{L_4}{d_4^4} \right]$$

If d_e and L_e are equivalent diameter and lengths of the shaft, then:

$$\frac{L_e}{d_e} = \left[\frac{L_1}{d_1^4} + \frac{L_2}{d_2^4} + \frac{L_3}{d_3^4} + \frac{L_4}{d_4^4} \right]$$

$$L_e = L_1 \left[\frac{d_e}{d_1^4} \right] + L_2 \left[\frac{d_e}{d_2^4} \right] + L_3 \left[\frac{d_e}{d_3^4} \right] + L_4 \left[\frac{d_e}{d_4^4} \right]$$

Equivalent shaft of the system shown in Figure- 6.19

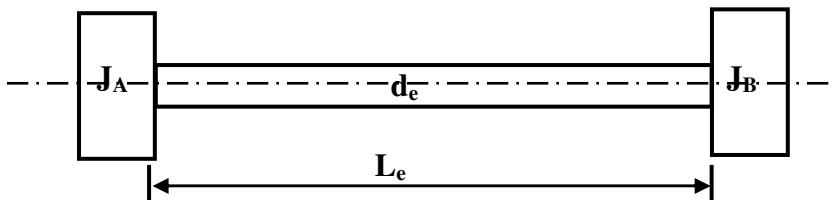


Figure – 6.19 Equivalent shaft of the system shown in figure – 6.18

Definite and Semi-Definite Systems

Definite Systems

A system, which is fixed from one end or both the ends is referred as definite system. A definite system has nonzero lower natural frequency. A system, which is free from both the ends, is referred as semi-definite system. For semi-definite systems, the first natural frequency is zero.

Various definite linear and a torsional systems are shown in figure-6.19

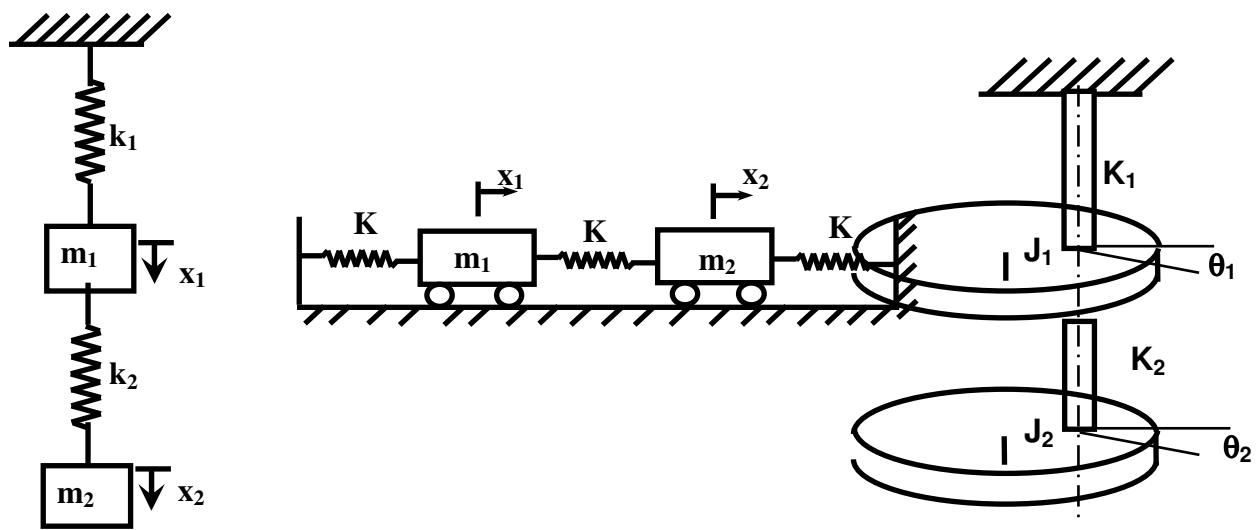


Figure-6.19 Various definite systems

Semi Definite or Degenerate System Systems

Systems for which one of the natural frequencies is equal to zero are called semi definite systems.

Various definite linear and a torsional systems are shown in figure-6.20

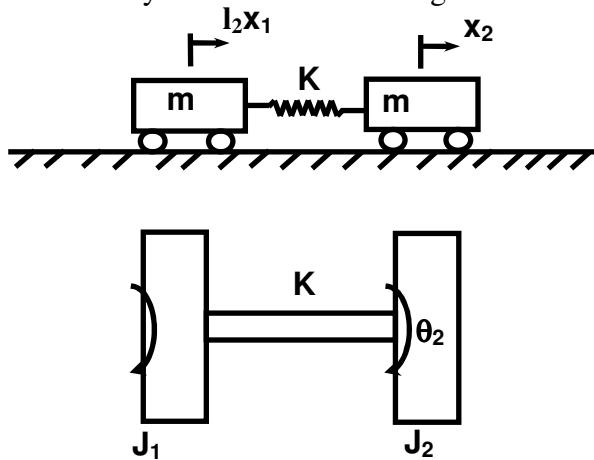


Figure-6.20 Semi-Definite systems

Problem to solve

Derive the equation of motion of a torsional system shown in figure 6.21.

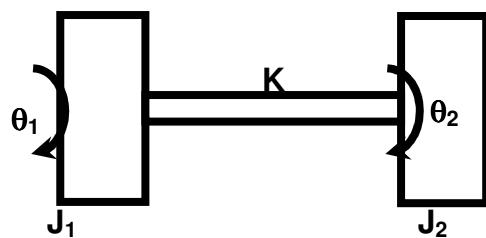


Figure-6.21 Two Rotor System

Vibration of Geared Systems

Consider a Turbo-generator geared system is shown in the figure 6.22.

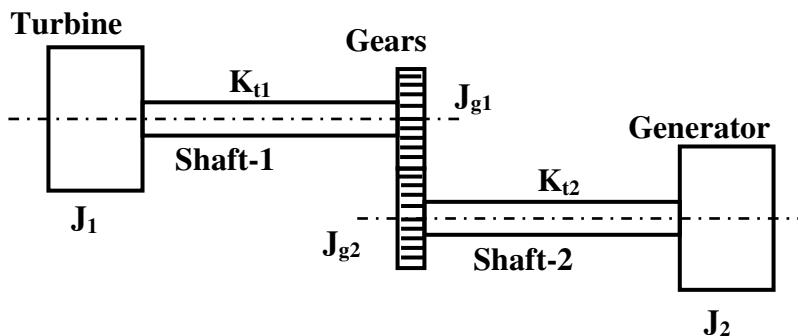


Figure-6.22: Turbo-Generator Geared System.

The analysis of this system is complex due to the presence of gears. Let 'i' be the speed ratio of the system given by

$$i = \frac{\text{Speed of Turbine}}{\text{Speed of Generator}}$$

First step in the analysis of this system is to convert the original geared system into an equivalent rotor system. Which is done with respect to either of the shafts.

When the Inertia of Gears is Neglected

The basis for this conversion is to consider the energies i.e. the kinetic and potential energy for the equivalent system should be same as that of the original system. Thus if θ_1 and θ_2 are the angular displacement of the rotors of moment of inertia J_1 and J_2 respectively then neglecting the inertia of the gears the **Kinetic and Potential energy of the original system are given by**

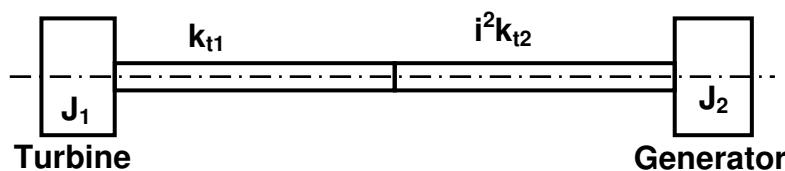
$$T = 1/2 J_1 \dot{\theta}_1^2 + 1/2 J_2 \dot{\theta}_2^2 \quad U = 1/2 k_{t1} \theta_1^2 + 1/2 k_{t2} \theta_2^2$$

Since $\theta_2 = i\theta_1$ Then the above equations can be written as

$$T = 1/2 J_1 \dot{\theta}_1^2 + 1/2 J_2 (i\dot{\theta}_1)^2 = 1/2 J_1 \dot{\theta}_1^2 + 1/2 (i^2 J_2) \dot{\theta}_1^2$$

$$U = 1/2 k_{t1} \theta_1^2 + 1/2 k_{t2} (i\theta_1)^2 = 1/2 k_{t1} \theta_1^2 + 1/2 (i^2 k_{t2}) \theta_1^2$$

Thus the above equation shows that the original system can be converted into equivalent system with respect to the first shaft as shown in figure- 6.23



Which is obtained by multiplying the inertia and stiffness of the second shaft by i^2 and keeping this part of the system in series with the first part. Thus the stiffness of this equivalent two rotor system is

$$k_{te} = i^2 k_{t1} k_{t2} / (k_{t1} + i^2 k_{t2})$$

Thus the frequency of the system is given by $\omega_n = \sqrt{k_{te}(J_1 + i^2 J_2) / i^2 J_1 J_2}$ rad/sec

When the Inertia of Gears is Considered

If the inertia of the gears is not negligible then the equivalent system with respect to the first shaft can be obtained in the same manner and finally we have the three rotor system as shown in figure-6.24

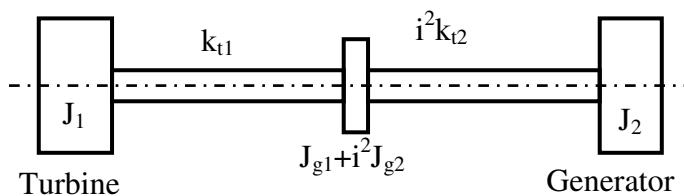


Figure-6.24 Considering the inertia of gears

3.4 Multidegree Freedom Systems and Analysis

Approximate methods

- (i) Dunkerley's method
- (ii) Rayleigh's method
- Influence co-efficients

Numerical methods

- (i) Matrix iteration method
- (ii) Stodola's method
- (iii) Holzar's method

1. Influence co-efficients

It is the influence of unit displacement at one point on the forces at various points of a multi-DOF system.

OR

It is the influence of unit Force at one point on the displacements at various points of a multi-DOF system.

The equations of motion of a multi-degree freedom system can be written in terms of influence co-efficients. A set of influence co-efficients can be associated with each of matrices involved in the equations of motion.

$$[M]\{\ddot{x}\} + [K]\{x\} = [0]$$

For a simple linear spring the force necessary to cause unit elongation is referred as stiffness of spring. For a multi-DOF system one can express the relationship between displacement at a point and forces acting at various other points of the system by using influence co-efficients referred as stiffness influence coefficients

The equations of motion of a multi-degree freedom system can be written in terms of inverse of stiffness matrix referred as flexibility influence co-efficients.

Matrix of flexibility influence co-efficients = $[K]^{-1}$

The elements corresponds to inverse mass matrix are referred as flexibility mass/inertia co-efficients.

Matrix of flexibility mass/inertia co-efficients = $[M]^{-1}$

The flexibility influence co-efficients are popular as these coefficents give elements of inverse of stiffness matrix. The flexibility mass/inertia co-efficients give elements of inverse of mass matrix

Stiffness influence co-efficients.

For a multi-DOF system one can express the relationship between displacement at a point and forces acting at various other points of the system by using influence coefficients referred as stiffness influence coefficients.

$$\{F\} = [K]\{x\}$$

$$[K] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{32} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

where, k_{11}, \dots, k_{33} are referred as stiffness influence coefficients

k_{11} -stiffness influence coefficient at point 1 due to a unit deflection at point 1

k_{21} - stiffness influence coefficient at point 2 due to a unit deflection at point 1

k_{31} - stiffness influence coefficient at point 3 due to a unit deflection at point 1

Example-1.

Obtain the stiffness coefficients of the system shown in Fig.1.

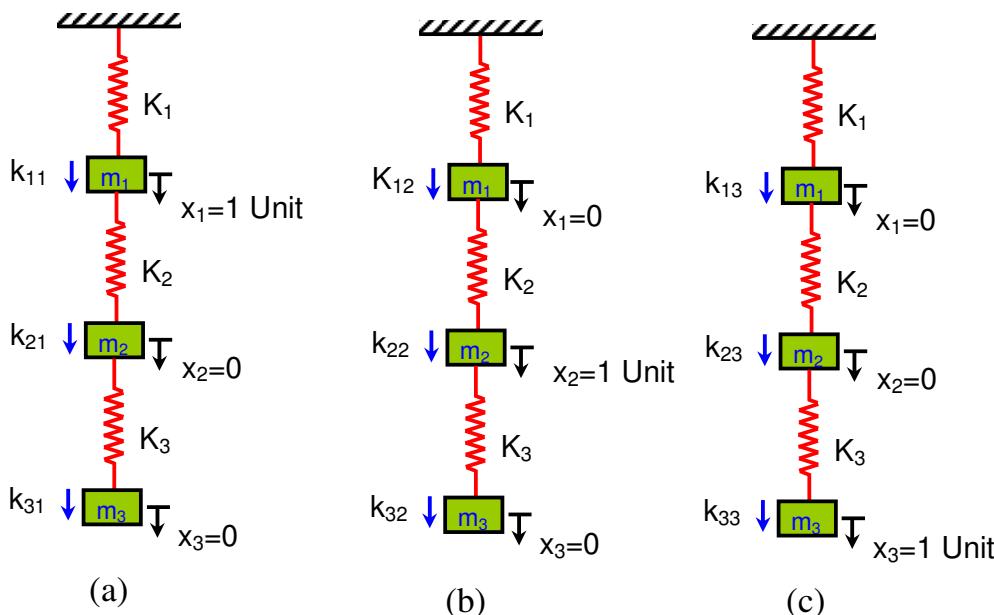


Fig.1 Stiffness influence coefficients of the system

I-step:

Apply 1 unit deflection at point 1 as shown in Fig.1(a) and write the force equilibrium equations. We get,

$$k_{11} = K_1 + K_2$$

$$k_{21} = -K_2$$

$$k_{31} = 0$$

II-step:

Apply 1 unit deflection at point 2 as shown in Fig.1(b) and write the force equilibrium equations. We get,

$$k_{12} = -K_2$$

$$k_{22} = K_2 + K_3$$

$$k_{31} = -K_3$$

III-step:

Apply 1 unit deflection at point 3 as shown in Fig.1(c) and write the force equilibrium equations. We get,

$$k_{13} = 0$$

$$k_{23} = -K_3$$

$$k_{33} = K_3$$

$$[K] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

$$[K] = \begin{bmatrix} (K_1 + K_2) & -K_2 & 0 \\ -K_2 & (K_2 + K_3) & -K_3 \\ 0 & -K_3 & K_3 \end{bmatrix}$$

From stiffness coefficients K matrix can be obtained without writing Eqns. of motion.

Flexibility influence co-efficients.

$$\{F\} = [K]\{x\}$$

$$\{x\} = [K]^{-1}\{F\}$$

$$\{x\} = [\alpha]\{F\}$$

where, $[\alpha]$ - Matrix of Flexibility influence co-efficients given by

$$[\alpha] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}$$

where, $\alpha_{11}, \dots, \alpha_{33}$ are referred as stiffness influence coefficients

α_{11} -flexibility influence coefficient at point 1 due to a unit force at point 1

α_{21} - flexibility influence coefficient at point 2 due to a unit force at point 1

α_{31} - flexibility influence coefficient at point 3 due to a unit force at point 1

Example-2.

Obtain the flexibility coefficients of the system shown in Fig.2.

I-step:

Apply 1 unit Force at point 1 as shown in Fig.2(a) and write the force equilibrium equations. We get,

$$\alpha_{11} = \alpha_{21} = \alpha_{31} = \frac{1}{K_1}$$

II-step:

Apply 1 unit Force at point 2 as shown in Fig.2(b) and write the force equilibrium equations. We get,

$$\alpha_{22} = \alpha_{32} = \frac{1}{K_1} + \frac{1}{K_2}$$

III-step:

Apply 1 unit Force at point 3 as shown in Fig.2(c) and write the force equilibrium equations. We get,

$$\alpha_{33} = \frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3}$$

$$\alpha_{23} = \frac{1}{K_1} + \frac{1}{K_2}$$

Therefore,

$$\alpha_{11} = \alpha_{21} = \alpha_{12} = \alpha_{31} = \alpha_{13} = \frac{1}{K_1}$$

$$\alpha_{22} = \alpha_{32} = \alpha_{23} = \frac{1}{K_1} + \frac{1}{K_2}$$

$$\alpha_{33} = \frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3}$$

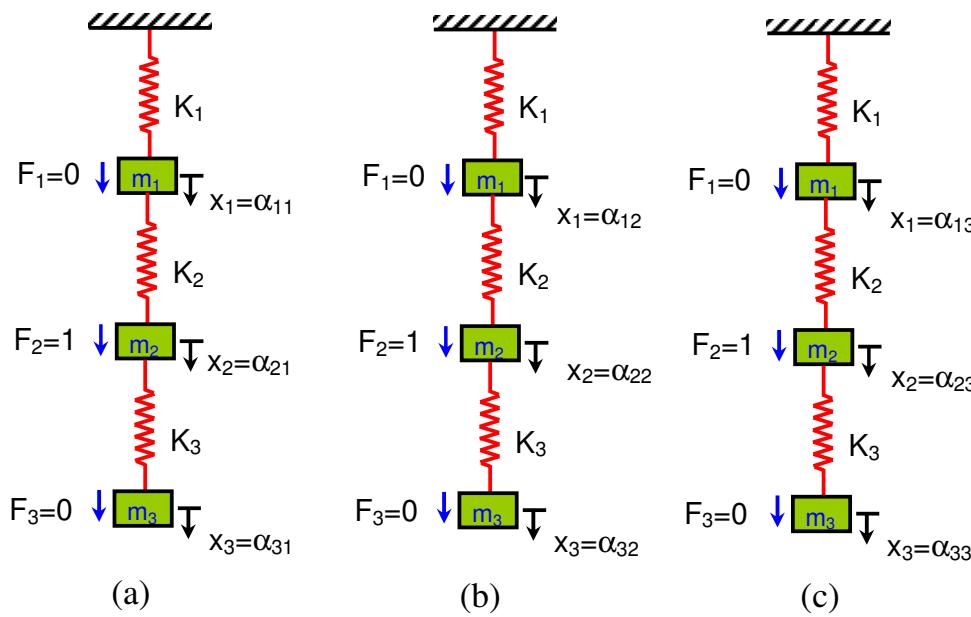


Fig.2 Flexibility influence coefficients of the system

For simplification, let us consider : $K_1 = K_2 = K_3 = K$

$$\alpha_{11} = \alpha_{21} = \alpha_{12} = \alpha_{31} = \alpha_{13} = \frac{1}{K_1} = \frac{1}{K}$$

$$\alpha_{22} = \alpha_{32} = \alpha_{23} = \frac{1}{K} + \frac{1}{K} = \frac{2}{K}$$

$$\alpha_{33} = \frac{1}{K} + \frac{1}{K} + \frac{1}{K} = \frac{3}{K}$$

$$[\alpha] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}$$

$$[\alpha] = \frac{1}{K} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$[\alpha] = [K]^{-1}$$

In Vibration analysis if there is need of $[K]^{-1}$ one can use flexibility co-efficient matrix.

Example-3

Obtain of the Flexibility influence co-efficients of the pendulum system shown in the Fig.3.

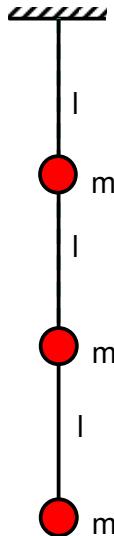


Fig.3 Pendulum system

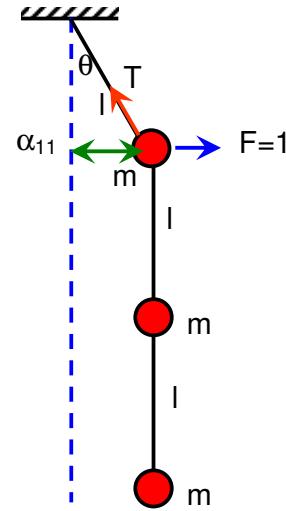


Fig.4 Flexibility influence co-efficients

I-step:

Apply 1 unit Force at point 1 as shown in Fig.4 and write the force equilibrium equations. We get,

$$T \sin \theta = l$$

$$T \cos \theta = g(m + m + m) = 3mg$$

$$\tan \theta = \frac{1}{3mg}$$

θ is small, $\tan \theta = \sin \theta$

$$\sin \theta = \frac{\alpha_{11}}{l}$$

$$\alpha_{11} = l \sin \theta$$

$$\alpha_{11} = \frac{l}{3mg}$$

Similarly apply 1 unit force at point 2 and next at point 3 to obtain,

$$\alpha_{22} = \frac{l}{5mg}$$

the influence coefficients are:

$$\alpha_{11} = \alpha_{21} = \alpha_{12} = \alpha_{31} = \alpha_{13} = \frac{l}{5mg}$$

$$\alpha_{22} = \alpha_{32} = \alpha_{23} = \frac{11l}{6mg}$$

$$\alpha_{33} = \frac{11l}{6mg}$$

Approximate methods

In many engineering problems it is required to quickly estimate the first (fundamental) natural frequency. Approximate methods like Dunkerley's method, Rayleigh's method are used in such cases.

(i) Dunkerley's method

Dunkerley's formula can be determined by frequency equation,

$$[-\omega^2[M] + [K]] = [0]$$

$$[-[K] + \omega^2[M]] = [0]$$

$$-\frac{1}{\omega^2}[I] + [K]^{-1}[M] = [0]$$

$$-\frac{1}{\omega^2}[I] + [\alpha][M] = [0]$$

For n DOF systems,

$$\left[-\frac{1}{\omega^2} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & 1 \end{bmatrix} + \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & & \ddots & \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{bmatrix} \begin{bmatrix} m_1 & 0 & \dots & 0 \\ 0 & m_2 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & m_n \end{bmatrix} \right] = [0]$$

$$\left[\begin{pmatrix} -\frac{1}{\omega^2} + \alpha_{11}m_1 \\ \alpha_{12}m_2 \\ \vdots \\ \alpha_{n1}m_1 \end{pmatrix} \quad \begin{matrix} \alpha_{12}m_2 & \dots & \alpha_{1n}m_n \\ 0 & \begin{pmatrix} -\frac{1}{\omega^2} + \alpha_{22}m_2 \\ \alpha_{23}m_3 \\ \vdots \\ \alpha_{n2}m_2 \end{pmatrix} & \dots & \alpha_{2n}m_n \\ \vdots & \ddots & \ddots & \vdots \\ \alpha_{n2}m_2 & \dots & \begin{pmatrix} -\frac{1}{\omega^2} + \alpha_{nn}m_n \\ \alpha_{n(n-1)}m_{n-1} \\ \vdots \\ \alpha_{n1}m_1 \end{pmatrix} \end{matrix} \right] = [0]$$

Solve the determinant

$$\left(\frac{1}{\omega^2} \right)^n - (\alpha_{11}m_1 + \alpha_{22}m_2 + \dots + \alpha_{nn}m_n) \left(\frac{1}{\omega^2} \right)^{n-1} \\ (\alpha_{11}\alpha_{22}m_1m_2 + \alpha_{11}\alpha_{33}m_1m_3 + \dots + \alpha_{nn}m_n + \dots) = [0] \quad (1)$$

It is the polynomial equation of nth degree in $(1/\omega^2)$. Let the roots of above Eqn. are:

$$\frac{1}{\omega_1^2}, \frac{1}{\omega_2^2}, \dots, \frac{1}{\omega_n^2} \\ \left(\frac{1}{\omega^2} - \frac{1}{\omega_1^2} \right), \left(\frac{1}{\omega^2} - \frac{1}{\omega_2^2} \right), \dots, \left(\frac{1}{\omega^2} - \frac{1}{\omega_n^2} \right) \\ = \left(\frac{1}{\omega^2} \right)^n - \left(\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \dots + \frac{1}{\omega_n^2} \right) \left(\frac{1}{\omega^2} \right)^{n-1} - \dots = 0 \quad (2)$$

Comparing Eqn.(1) and Eqn. (2), we get,

$$\left(\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \dots + \frac{1}{\omega_n^2} \right) = (\alpha_{11}m_1 + \alpha_{22}m_2 + \dots + \alpha_{nn}m_n)$$

In mechanical systems higher natural frequencies are much larger than the fundamental (first) natural frequencies. Approximately, the first natural frequency is:

$$\left(\frac{1}{\omega_1^2} \right) \approx (\alpha_{11}m_1 + \alpha_{22}m_2 + \dots + \alpha_{nn}m_n)$$

The above formula is referred as Dunkerley's formula, which can be used to estimate first natural frequency of a system approximately.

The natural frequency of the system considering only mass m_1 is:

$$\omega_{1n} = \sqrt{\frac{1}{\alpha_{11}m_1}} = \sqrt{\frac{K_1}{m_1}}$$

The Dunkerley's formula can be written as:

$$\frac{1}{\omega_1^2} \approx \frac{1}{\omega_{1n}^2} + \frac{1}{\omega_{2n}^2} + \dots + \frac{1}{\omega_{nn}^2} \quad (3)$$

where, $\omega_{1n}, \omega_{2n}, \dots$ are natural frequency of single degree of freedom system considering each mass separately.

The above formula given by Eqn. (3) can be used for any mechanical/structural system to obtain first natural frequency

Examples: 1

Obtain the approximate fundamental natural frequency of the system shown in Fig.5 using Dunkerley's method.

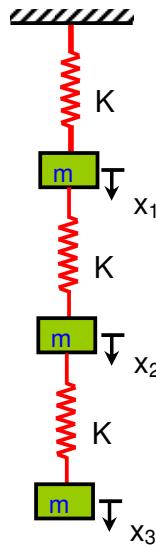


Fig.5 Linear vibratory system

Dunkerley's formula is:

$$\left(\frac{1}{\omega_1^2} \right) \cong (\alpha_{11}m_1 + \alpha_{22}m_2 + \dots + \alpha_{nn}m_n) \text{ OR}$$

$$\frac{1}{\omega_1^2} \cong \frac{1}{\omega_{1n}^2} + \frac{1}{\omega_{2n}^2} + \dots + \frac{1}{\omega_{nn}^2}$$

Any one of the above formula can be used to find fundamental natural frequency approximately.

Find influence flexibility coefficients.

$$\alpha_{11} = \alpha_{21} = \alpha_{12} = \alpha_{31} = \alpha_{13} = \frac{1}{K}$$

$$\alpha_{22} = \alpha_{32} = \alpha_{23} = \frac{2}{K}$$

$$\alpha_{33} = \frac{3}{K}$$

Substitute all influence coefficients in the Dunkerley's formula.

$$\left(\frac{1}{\omega_1^2} \right) \cong (\alpha_{11}m_1 + \alpha_{22}m_2 + \dots + \alpha_{nn}m_n)$$

$$\left(\frac{1}{\omega_1^2} \right) \cong \left(\frac{m}{K} + \frac{2m}{K} + \frac{3m}{K} \right) = \frac{6m}{K}$$

$$\omega_1 = 0.40\sqrt{K/m} \text{ rad/s}$$

Examples: 2

Find the lowest natural frequency of the system shown in Figure by Dunkerley's method. Take $m_1=100 \text{ kg}$, $m_2=50 \text{ kg}$

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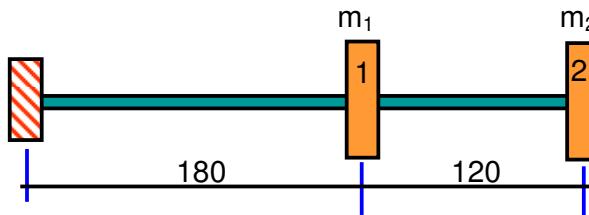


Fig.6 A cantilever rotor system.

Obtain the influence co-efficients:

$$\alpha_{11} = \frac{1.944 \times 10^{-3}}{EI}$$

$$\alpha_{22} = \frac{9 \times 10^{-3}}{EI}$$

$$\left(\frac{1}{\omega_n^2} \right) \cong (\alpha_{11}m_1 + \alpha_{22}m_2)$$

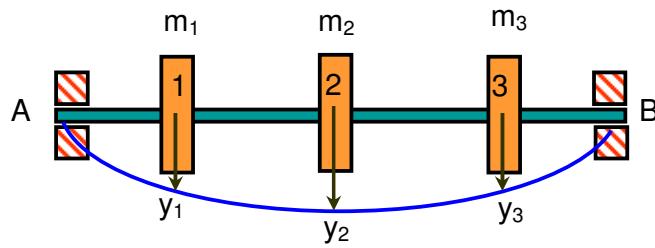
$$\omega_n = 1.245 \text{ rad/s}$$

(ii) Rayleigh's method

It is an approximate method of finding fundamental natural frequency of a system using energy principle. This principle is largely used for structural applications.

Principle of Rayleigh's method

Consider a rotor system as shown in Fig.7. Let, m_1 , m_2 and m_3 are masses of rotors on shaft supported by two bearings at A and B and y_1 , y_2 and y_3 are static deflection of shaft at points 1, 2 and 3.

**Fig.7** A rotor system.

For the given system maximum potential energy and kinetic energies are:

$$V_{\max} = \frac{1}{2} \sum_{i=1}^n m_i g y_i \quad (4)$$

$$T_{\max} = \frac{1}{2} \sum_{i=1}^n m_i \dot{y}_i^2$$

where, m_i - masses of the system, y_i -displacements at mass points.

Considering the system vibrates with SHM,

$$\ddot{y}_i = \omega^2 y_i$$

From above equations

$$T_{\max} = \frac{\omega^2}{2} \sum_{i=1}^n m_i y_i^2 \quad (5)$$

According to Rayleigh's method,

$$V_{\max} = T_{\max} \quad (6)$$

substitute Eqn. (4) and (5) in (6)

$$\omega^2 = \frac{\sum_{i=1}^n m_i g y_i}{\sum_{i=1}^n m_i y_i^2} \quad (7)$$

The deflections at point 1, 2 and 3 can be found by.

$$y_1 = \alpha_{11} m_1 g + \alpha_{12} m_2 g + \alpha_{13} m_3 g$$

$$y_2 = \alpha_{21} m_1 g + \alpha_{22} m_2 g + \alpha_{23} m_3 g$$

$$y_3 = \alpha_{31} m_1 g + \alpha_{32} m_2 g + \alpha_{33} m_3 g$$

Eqn.(7) is the Rayleigh's formula, which is used to estimate frequency of transverse vibrations of a vibratory systems.

Examples: 1

Estimate the approximate fundamental natural frequency of the system shown in Fig.8 using Rayleigh's method. Take: $m=1\text{kg}$ and $K=1000 \text{ N/m}$.

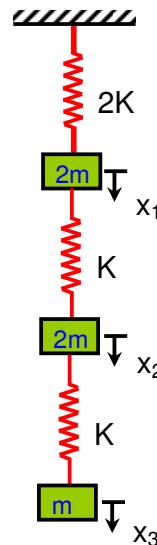


Fig.8 Linear vibratory system

Obtain influence coefficients,

$$\alpha_{11} = \alpha_{21} = \alpha_{12} = \alpha_{31} = \alpha_{13} = \frac{1}{2K}$$

$$\alpha_{22} = \alpha_{32} = \alpha_{23} = \frac{3}{2K}$$

$$\alpha_{33} = \frac{5}{2K}$$

Deflection at point 1 is:

$$y_1 = \alpha_{11}m_1g + \alpha_{12}m_2g + \alpha_{13}m_3g$$

$$y_1 = \frac{mg}{2K}(2+2+1) = \frac{5mg}{2K} = \frac{5g}{2000}$$

Deflection at point 2 is:

$$y_2 = \alpha_{21}m_1g + \alpha_{22}m_2g + \alpha_{23}m_3g$$

$$y_2 = \frac{mg}{2K}(2+6+3) = \frac{11mg}{2K} = \frac{11g}{2000}$$

Deflection at point 3 is:

$$y_3 = \alpha_{31}m_1g + \alpha_{32}m_2g + \alpha_{33}m_3g$$

$$y_3 = \frac{mg}{2K}(2+6+5) = \frac{13mg}{2K} = \frac{13g}{2000}$$

Rayleigh's formula is:

$$\omega^2 = \frac{\sum_{i=1}^n m_i g y_i}{\sum_{i=1}^n m_i y_i^2}$$

$$\omega^2 = \frac{\left(2x \frac{5}{2000} + 2x \frac{11}{2000} + 2x \frac{13}{2000}\right)g^2}{2\left(\frac{5}{2000}\right)^2 + 2\left(\frac{11}{2000}\right)^2 + 2\left(\frac{13}{2000}\right)^2 g^2}$$

$$\omega = 12.41 \text{ rad/s}$$

Examples: 2

Find the lowest natural frequency of transverse vibrations of the system shown in Fig.9 by Rayleigh's method.

E=196 GPa, I=10.6 m⁴, m₁=40 kg, m₂=20 kg

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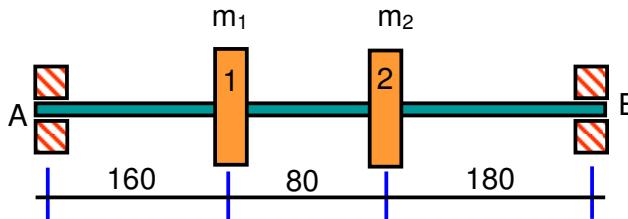


Fig.9 A rotor system.

Step-1:

Find deflections at point of loading from strength of materials principle.

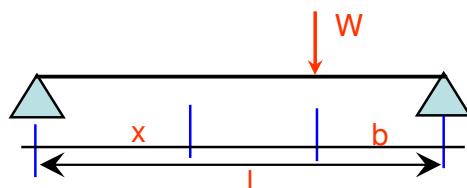


Fig.10 A simply supported beam

For a simply supported beam shown in Fig.10, the deflection of beam at distance x from left is given by:

$$y = \frac{Wbx}{6EI} (l^2 - x^2 - b^2) \text{ for } x \leq (l - b)$$

For the given problem deflection at loads can be obtained by superposition of deflections due to each load acting separately.

Deflections due to 20 kg mass

$$y_1' = \frac{(9.81 \times 20) \times 0.18 \times 0.16}{6EI \times 0.42} (0.42^2 - 0.16^2 - 0.18^2) = \frac{0.265}{EI}$$

$$y_2' = \frac{(9.81 \times 20) \times 0.18 \times 0.24}{6EI \times 0.42} (0.42^2 - 0.24^2 - 0.18^2) = \frac{0.29}{EI}$$

Deflections due to 40 kg mass

$$y_1'' = \frac{(9.81 \times 40) \times 0.16 \times 0.26}{6EI \times 0.42} (0.42^2 - 0.26^2 - 0.16^2) = \frac{0.538}{EI}$$

$$y_2'' = \frac{(9.81 \times 40) \times 0.16 \times 0.18}{6EI \times 0.42} (0.42^2 - 0.18^2 - 0.16^2) = \frac{0.53}{EI}$$

The deflection at point 1 is:

$$y_1 = y_1' + y_1'' = \frac{0.803}{EI}$$

The deflection at point 2 is:

$$y_2 = y_2' + y_2'' = \frac{0.82}{EI}$$

$$\omega^2 = \frac{\sum_{i=1}^n m_i g y_i}{\sum_{i=1}^n m_i y_i^2}$$

$$\omega^2 = \frac{9.81(40 \times 0.803 + 20 \times 0.82)}{(40 \times 0.803^2) + (20 \times 0.82^2)}$$

$$\omega_n = 1541.9 \text{ rad/s}$$

Numerical methods

(i) Matrix iteration method

Using this method one can obtain natural frequencies and modal vectors of a vibratory system having multi-degree freedom.

It is required to have $\omega_1 < \omega_2 < \dots < \omega_n$

Eqns. of motion of a vibratory system (having n DOF) in matrix form can be written as:

$$[M]\{\ddot{x}\} + [K]\{x\} = [0]$$

where,

$$\{x\} = \{A\} \sin(\omega t + \phi) \quad (8)$$

substitute Eqn.(8) in (9)

$$-\omega^2[M]\{A\} + [K]\{A\} = [0] \quad (9)$$

For principal modes of oscillations, for r^{th} mode,

$$-\omega_r^2[M]\{A\}_r + [K]\{A\}_r = [0]$$

$$[K]^{-1}[M]\{A\}_r = \frac{1}{\omega_r^2}\{A\}_r$$

$$[D]\{A\}_r = \frac{1}{\omega_r^2}\{A\}_r \quad (10)$$

where, $[D]$ is referred as Dynamic matrix.

Eqn.(10) converges to first natural frequency and first modal vector.

The Equation,

$$[M]^{-1}[K]\{A\}_r = \omega_r^2\{A\}_r$$

$$[D_1]\{A\}_r = \omega_r^2\{A\}_r \quad (11)$$

where, $[D_1]$ is referred as inverse dynamic matrix.

Eqn.(11) converges to last natural frequency and last modal vector.

In above Eqns (10) and (11) by assuming trial modal vector and iterating till the Eqn is satisfied, one can estimate natural frequency of a system.

Examples: 1

Find first natural frequency and modal vector of the system shown in the Fig.10 using matrix iteration method. Use flexibility influence co-efficients.

Find influence coefficients.

$$\alpha_{11} = \alpha_{21} = \alpha_{12} = \alpha_{31} = \alpha_{13} = \frac{1}{2K}$$

$$\alpha_{22} = \alpha_{32} = \alpha_{23} = \frac{3}{2K}$$

$$\alpha_{33} = \frac{5}{2K}$$

$$[\alpha] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}$$

$$[\alpha] = [K]^{-1} = \frac{1}{2K} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 5 \end{bmatrix}$$

First natural frequency and modal vector

$$[K]^{-1}[M]\{A\}_r = \frac{1}{\omega_r^2} \{A\}_r$$

$$[D]\{A\}_r = \frac{1}{\omega_r^2} \{A\}_r$$

Obtain Dynamic matrix $[D] = [K]^{-1}[M]$

$$[D] = \frac{m}{2K} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{m}{2K} \begin{bmatrix} 2 & 2 & 1 \\ 2 & 6 & 3 \\ 2 & 6 & 5 \end{bmatrix}$$

Use basic Eqn to obtain first frequency

$$[D]\{A\}_1 = \frac{1}{\omega_r^2} \{A\}_1$$

Assume trial vector and substitute in the above Eqn.

$$\text{Assumed vector is: } \{u\}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

First Iteration

$$[D]\{u\}_1 = \frac{m}{2K} \begin{bmatrix} 2 & 2 & 1 \\ 2 & 6 & 3 \\ 2 & 6 & 5 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{5m}{2K} \begin{Bmatrix} 1 \\ 2.2 \\ 2.6 \end{Bmatrix}$$

As the new vector is not matching with the assumed one, iterate again using the new vector as assumed vector in next iteration.

Second Iteration

$$[D]\{u\}_2 = \frac{m}{2K} \begin{bmatrix} 2 & 2 & 1 \\ 2 & 6 & 3 \\ 2 & 6 & 5 \end{bmatrix} \begin{Bmatrix} 1 \\ 2.2 \\ 2.6 \end{Bmatrix} = \frac{4.5m}{K} \begin{Bmatrix} 1 \\ 2.55 \\ 3.13 \end{Bmatrix}$$

Third Iteration

$$[D]\{u\}_3 = \frac{m}{2K} \begin{bmatrix} 2 & 2 & 1 \\ 2 & 6 & 3 \\ 2 & 6 & 5 \end{bmatrix} \begin{Bmatrix} 1 \\ 2.555 \\ 3.133 \end{Bmatrix} = \frac{5.12m}{K} \begin{Bmatrix} 1 \\ 2.61 \\ 3.22 \end{Bmatrix}$$

Fourth Iteration

$$[D]\{u\}_4 = \frac{m}{2K} \begin{bmatrix} 2 & 2 & 1 \\ 2 & 6 & 3 \\ 2 & 6 & 5 \end{bmatrix} \begin{Bmatrix} 1 \\ 2.61 \\ 3.22 \end{Bmatrix} = \frac{5.22m}{K} \begin{Bmatrix} 1 \\ 2.61 \\ 3.23 \end{Bmatrix}$$

As the vectors are matching stop iterating. The new vector is the modal vector.

To obtain the natural frequency,

$$[D] \begin{Bmatrix} 1 \\ 2.61 \\ 3.22 \end{Bmatrix} = \frac{5.22m}{K} \begin{Bmatrix} 1 \\ 2.61 \\ 3.23 \end{Bmatrix}$$

Compare above Eqn with basic Eqn.

$$[D]\{A\}_1 = \frac{1}{\omega_1^2} \{A\}_1$$

$$\frac{1}{\omega_1^2} = \frac{5.22m}{K}$$

$$\omega_1^2 = \frac{1}{5.22} \frac{K}{m}$$

$$\omega_1 = 0.437 \sqrt{\frac{K}{m}} \text{ Rad/s}$$

Modal vector is:

$$\{A\}_1 = \begin{Bmatrix} 1 \\ 2.61 \\ 3.23 \end{Bmatrix}$$

**Method of obtaining natural frequencies in between first and last one
(Sweeping Technique)**

For understanding it is required to clearly understand Orthogonality principle of modal vectors.

Orthogonality principle of modal vectors

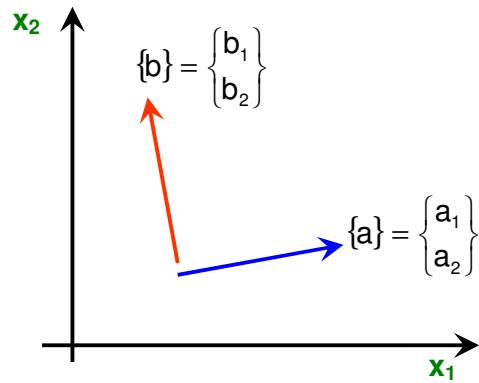


Fig.11 Vector representation graphically

Consider two vectors shown in Fig.11. Vectors $\{a\}$ and $\{b\}$ are orthogonal to each other if and only if

$$\begin{aligned}\{a\}^T \{b\} &= 0 \\ \{a_1 \quad a_2\} \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} &= 0 \\ \{a_1 \quad a_2\} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} &= 0 \\ \{a\}^T [I] \{b\} &= 0\end{aligned}\tag{12}$$

where, $[I]$ is Identity matrix.

From Eqn.(12), Vectors $\{a\}$ and $\{b\}$ are orthogonal to each other with respect to identity matrix.

Application of orthogonality principle in vibration analysis

Eqns. of motion of a vibratory system (having n DOF) in matrix form can be written as:

$$\begin{aligned}[M]\ddot{x}\} + [K]\{x\} &= [0] \\ \{x\} &= \{A\} \sin(\omega t + \phi) \\ -\omega^2[M]\{A\}_l + [K]\{A\}_l &= [0]\end{aligned}$$

$$\omega^2 [M]\{A\}_1 = [K]\{A\}_1$$

If system has two frequencies ω_1 and ω_2

$$\omega_1^2 [M]\{A\}_1 = [K]\{A\}_1 \quad (13)$$

$$\omega_2^2 [M]\{A\}_2 = [K]\{A\}_2 \quad (14)$$

Multiply Eqn.(13) by $\{A\}_2^T$ and Eqn.(14) by $\{A\}_1^T$

$$\omega_1^2 \{A\}_2^T [M]\{A\}_1 = \{A\}_2^T [K]\{A\}_1 \quad (15)$$

$$\omega_2^2 \{A\}_1^T [M]\{A\}_2 = \{A\}_1^T [K]\{A\}_2 \quad (16)$$

Eqn.(15)-(16)

$$\{A\}_1^T [M]\{A\}_2 = 0$$

Above equation is a condition for mass orthogonality.

$$\{A\}_1^T [K]\{A\}_2 = 0$$

Above equation is a condition for stiffness orthogonality.

By knowing the first modal vector one can easily obtain the second modal vector based on either mass or stiffness orthogonality. This principle is used in the matrix iteration method to obtain the second modal vector and second natural frequency.

This technique is referred as **Sweeping technique**

Sweeping technique

After obtaining $\{A\}_1$ and ω_1 to obtain $\{A\}_2$ and ω_2 choose a trial vector $\{V\}_1$ orthogonal to $\{A\}_1$, which gives constraint Eqn.:

$$\{V\}_1^T [M]\{A\}_1 = 0$$

$$\{V_1 \ V_2 \ V_3\} \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = 0$$

$$\{(V_1 m_1 A_1) + (V_2 m_2 A_2) + (V_3 m_3 A_3)\} = 0$$

$$\{(m_1 A_1) V_1 + (m_2 A_2) V_2 + (m_3 A_3) V_3\} = 0$$

$$V_1 = \alpha V_2 + \beta V_3$$

where α and β are constants

$$\alpha = -\left(\frac{m_2 A_2}{m_1 A_1}\right)$$

$$\beta = -\left(\frac{m_3 A_3}{m_1 A_1}\right)$$

Therefore the trial vector is:

$$\begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} (\alpha V_2 + \beta V_3) \\ V_2 \\ V_3 \end{Bmatrix}$$

$$= \begin{bmatrix} 0 & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix}$$

$$= [S]\{V\}_1$$

where $[S]$ is referred as Sweeping matrix and $\{V\}_1$ is the trial vector.

New dynamics matrix is:

$$[D_s] + [D][S]$$

$$[D_s]\{V\}_1 = \frac{1}{\omega_n^2} \{A\}_2$$

The above Eqn. Converges to second natural frequency and second modal vector.

This method of obtaining frequency and modal vectors between first and the last one is referred as sweeping technique.

Examples: 2

For the Example problem 1, Find second natural frequency and modal vector of the system shown in the Fig.10 using matrix iteration method and Sweeping technique. Use flexibility influence co-efficients.

For this example already the first frequency and modal vectors are obtained by matrix iteration method in Example 1. In this stage only how to obtain second frequency is demonstrated.

First Modal vector obtained in Example 1 is:

$$\{A\}_1 = \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2.61 \\ 3.23 \end{Bmatrix}$$

$$[M] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the mass matrix

Find sweeping matrix

$$[S] = \begin{bmatrix} 0 & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = -\left(\frac{m_2 A_2}{m_1 A_1}\right) = -\left(\frac{2(2.61)}{2(1)}\right) = -2.61$$

$$\beta = -\left(\frac{m_3 A_3}{m_1 A_1}\right) = -\left(\frac{1(3.23)}{2(1)}\right) = -1.615$$

Sweeping matrix is:

$$[S] = \begin{bmatrix} 0 & -2.61 & -1.615 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

New Dynamics matrix is:

$$[D_s] + [D][S]$$

$$[D_s] = \frac{m}{2K} \begin{bmatrix} 2 & 2 & 1 \\ 2 & 6 & 3 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0 & -2.61 & -1.615 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{m}{K} \begin{bmatrix} 0 & -1.61 & -1.11 \\ 0 & 0.39 & -0.11 \\ 0 & 0.39 & 1.89 \end{bmatrix}$$

First Iteration

$$[D_s]\{V\}_1 = \frac{1}{\omega_n^2} \{A\}_2$$

$$\frac{m}{K} \begin{bmatrix} 0 & -1.61 & -1.11 \\ 0 & 0.39 & -0.11 \\ 0 & 0.39 & 1.89 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{m}{K} \begin{Bmatrix} -2.27 \\ 0.28 \\ 2.28 \end{Bmatrix} = \frac{0.28m}{K} \begin{Bmatrix} -9.71 \\ 1 \\ 8.14 \end{Bmatrix}$$

Second Iteration

$$\frac{m}{K} \begin{bmatrix} 0 & -1.61 & -1.11 \\ 0 & 0.39 & -0.11 \\ 0 & 0.39 & 1.89 \end{bmatrix} \begin{Bmatrix} -9.71 \\ 1 \\ 8.14 \end{Bmatrix} = \frac{m}{K} \begin{Bmatrix} -10.64 \\ -0.50 \\ 15.77 \end{Bmatrix} = \frac{0.5m}{K} \begin{Bmatrix} -21.28 \\ -1 \\ 31.54 \end{Bmatrix}$$

Third Iteration

$$\frac{m}{K} \begin{bmatrix} 0 & -1.61 & -1.11 \\ 0 & 0.39 & -0.11 \\ 0 & 0.39 & 1.89 \end{bmatrix} \begin{Bmatrix} -21.28 \\ -1 \\ 31.54 \end{Bmatrix} = \frac{m}{K} \begin{Bmatrix} -33.39 \\ -3.85 \\ 59.52 \end{Bmatrix} = \frac{3.85m}{K} \begin{Bmatrix} -8.67 \\ -1 \\ 15.38 \end{Bmatrix}$$

Fourth Iteration

$$\frac{m}{K} \begin{bmatrix} 0 & -1.61 & -1.11 \\ 0 & 0.39 & -0.11 \\ 0 & 0.39 & 1.89 \end{bmatrix} \begin{Bmatrix} -8.67 \\ -1 \\ 15.38 \end{Bmatrix} = \frac{m}{K} \begin{Bmatrix} -18.68 \\ -2.08 \\ 28.67 \end{Bmatrix} = \frac{2.08m}{K} \begin{Bmatrix} -8.98 \\ -1 \\ 13.78 \end{Bmatrix}$$

Fifth Iteration

$$\frac{m}{K} \begin{bmatrix} 0 & -1.61 & -1.11 \\ 0 & 0.39 & -0.11 \\ 0 & 0.39 & 1.89 \end{bmatrix} \begin{Bmatrix} -8.98 \\ -1 \\ 13.78 \end{Bmatrix} = \frac{m}{K} \begin{Bmatrix} -13.68 \\ -1.90 \\ 25.65 \end{Bmatrix} = \frac{1.90m}{K} \begin{Bmatrix} -7.2 \\ -1 \\ 13.5 \end{Bmatrix}$$

Sixth Iteration

$$\frac{m}{K} \begin{bmatrix} 0 & -1.61 & -1.11 \\ 0 & 0.39 & -0.11 \\ 0 & 0.39 & 1.89 \end{bmatrix} \begin{Bmatrix} -7.2 \\ -1 \\ 13.5 \end{Bmatrix} = \frac{m}{K} \begin{Bmatrix} -13.24 \\ -1.87 \\ 25.12 \end{Bmatrix} = \frac{1.87m}{K} \begin{Bmatrix} -7.08 \\ -1 \\ 13.43 \end{Bmatrix}$$

$$\frac{1}{\omega_2^2} = \frac{1087m}{K}$$

$$\omega_1^2 = \frac{1}{1.87} \frac{K}{m}$$

$$\omega_1 = 0.73 \sqrt{\frac{K}{m}}$$

Modal vector

$$\{A\}_2 = \begin{Bmatrix} -1 \\ -0.14 \\ 1.89 \end{Bmatrix}$$

Similar manner the next frequency and modal vectors can be obtained.

(ii) Stodola's method

It is a numerical method, which is used to find the fundamental natural frequency and modal vector of a vibratory system having multi-degree freedom. The method is based on finding inertia forces and deflections at various points of interest using flexibility influence coefficients.

Principle / steps

1. Assume a modal vector of system. For example for 3 dof systems:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

2. Find out inertia forces of system at each mass point,

$$F_1 = m_1 \omega^2 x_1 \text{ for Mass 1}$$

$$F_2 = m_2 \omega^2 x_2 \text{ for Mass 2}$$

$$F_3 = m_3 \omega^2 x_3 \text{ for Mass 3}$$

3. Find new deflection vector using flexibility influence coefficients, using the formula,

$$\begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \alpha_{11} + F_2 \alpha_{12} + F_3 \alpha_{13} \\ F_1 \alpha_{21} + F_2 \alpha_{22} + F_3 \alpha_{23} \\ F_1 \alpha_{31} + F_2 \alpha_{32} + F_3 \alpha_{33} \end{Bmatrix}$$

4. If assumed modal vector is equal to modal vector obtained in step 3, then solution is converged. Natural frequency can be obtained from above equation, i.e

$$\text{If } \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \cong \begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} \text{ Stop iterating.}$$

Find natural frequency by first equation,

$$x'_1 = 1 = F_1 \alpha_{11} + F_2 \alpha_{12} + F_3 \alpha_{13}$$

5. If assumed modal vector is not equal to modal vector obtained in step 3, then consider obtained deflection vector as new vector and iterate till convergence.

Example-1

Find the fundamental natural frequency and modal vector of a vibratory system shown in Fig.10 using Stodola's method.

First iteration

1. Assume a modal vector of system $\{u\}_1 = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$

2. Find out inertia forces of system at each mass point

$$F_1 = m_1 \omega^2 x_1 = 2m\omega^2$$

$$F_2 = m_2 \omega^2 x_2 = 2m\omega^2$$

$$F_3 = m_3 \omega^2 x_3 = m\omega^2$$

3. Find new deflection vector using flexibility influence coefficients

Obtain flexibility influence coefficients of the system:

$$\alpha_{11} = \alpha_{21} = \alpha_{12} = \alpha_{31} = \alpha_{13} = \frac{1}{2K}$$

$$\alpha_{22} = \alpha_{32} = \alpha_{23} = \frac{3}{2K}$$

$$\alpha_{33} = \frac{5}{2K}$$

$$x'_1 = F_1 \alpha_{11} + F_2 \alpha_{12} + F_3 \alpha_{13}$$

Substitute for F's and α 's

$$x'_1 = \frac{m\omega^2}{K} + \frac{m\omega^2}{K} + \frac{m\omega^2}{2K} = \frac{5m\omega^2}{2K}$$

$$x'_2 = F_1 \alpha_{21} + F_2 \alpha_{22} + F_3 \alpha_{23}$$

Substitute for F's and α 's

$$x'_2 = \frac{m\omega^2}{K} + \frac{6m\omega^2}{2K} + \frac{3m\omega^2}{2K} = \frac{11m\omega^2}{2K}$$

$$x'_3 = F_1 \alpha_{31} + F_2 \alpha_{32} + F_3 \alpha_{33}$$

Substitute for F's and α 's

$$x'_3 = \frac{m\omega^2}{K} + \frac{6m\omega^2}{2K} + \frac{5m\omega^2}{2K} = \frac{13m\omega^2}{2K}$$

4. New deflection vector is:

$$\begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} = \frac{m\omega^2}{2K} \begin{Bmatrix} 5 \\ 11 \\ 13 \end{Bmatrix}$$

$$\begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} = \frac{5m\omega^2}{2K} \begin{Bmatrix} 1 \\ 2.2 \\ 2.6 \end{Bmatrix} = \{u\}_2$$

The new deflection vector $\{u\}_2 \neq \{u\}_1$. Iterate again using new deflection vector $\{u\}_2$

Second iteration

1. Initial vector of system $\{u\}_2 = \begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2.2 \\ 2.6 \end{Bmatrix}$

2. Find out inertia forces of system at each mass point

$$F'_1 = m_1 \omega^2 x'_1 = 2m\omega^2$$

$$F'_2 = m_2 \omega^2 x'_2 = 4.4m\omega^2$$

$$F'_3 = m_3 \omega^2 x'_3 = 2.6m\omega^2$$

3. New deflection vector,

$$x''_1 = F'_1 \alpha_{11} + F'_2 \alpha_{12} + F'_3 \alpha_{13}$$

Substitute for F's and α 's

$$x''_1 = \frac{m\omega^2}{K} + \frac{4.4m\omega^2}{2K} + \frac{2.6m\omega^2}{2K} = \frac{9m\omega^2}{2K}$$

$$x''_2 = F'_1 \alpha_{21} + F'_2 \alpha_{22} + F'_3 \alpha_{23}$$

Substitute for F's and α 's

$$x''_2 = \frac{m\omega^2}{K} + \frac{13.2m\omega^2}{2K} + \frac{7.8m\omega^2}{2K} = \frac{23m\omega^2}{2K}$$

$$x''_3 = F'_1 \alpha_{31} + F'_2 \alpha_{32} + F'_3 \alpha_{33}$$

Substitute for F's and α 's

$$x''_3 = \frac{m\omega^2}{K} + \frac{13.2m\omega^2}{2K} + \frac{13m\omega^2}{2K} = \frac{28.2m\omega^2}{2K}$$

4. New deflection vector is:

$$\begin{Bmatrix} x''_1 \\ x''_2 \\ x''_3 \end{Bmatrix} = \frac{m\omega^2}{2K} \begin{Bmatrix} 9 \\ 23 \\ 28.2 \end{Bmatrix}$$

$$\begin{Bmatrix} x''_1 \\ x''_2 \\ x''_3 \end{Bmatrix} = \frac{9m\omega^2}{2K} \begin{Bmatrix} 1 \\ 2.55 \\ 3.13 \end{Bmatrix} = \{u\}_3$$

The new deflection vector $\{u\}_3 \neq \{u\}_2$. Iterate again using new deflection vector $\{u\}_3$

Third iteration

1. Initial vector of system $\{u\}_3 = \begin{Bmatrix} x''_1 \\ x''_2 \\ x''_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2.55 \\ 3.13 \end{Bmatrix}$

2. Find out inertia forces of system at each mass point

$$F''_1 = m_1 \omega^2 x''_1 = 2m\omega^2$$

$$F''_2 = m_2 \omega^2 x''_2 = 5.1m\omega^2$$

$$F''_3 = m_3 \omega^2 x''_3 = 3.13m\omega^2$$

3. new deflection vector,

$$x''_1 = F''_1 \alpha_{11} + F''_2 \alpha_{12} + F''_3 \alpha_{13}$$

Substitute for F's and α 's

$$x''_1 = \frac{m\omega^2}{K} + \frac{5.1m\omega^2}{2K} + \frac{3.13m\omega^2}{2K} = \frac{10.23m\omega^2}{2K}$$

$$x''_2 = F''_1 \alpha_{21} + F''_2 \alpha_{22} + F''_3 \alpha_{23}$$

Substitute for F's and α 's

$$x''_2 = \frac{m\omega^2}{K} + \frac{15.3m\omega^2}{2K} + \frac{9.39m\omega^2}{2K} = \frac{26.69m\omega^2}{2K}$$

$$x''_3 = F''_1 \alpha_{31} + F''_2 \alpha_{32} + F''_3 \alpha_{33}$$

Substitute for F's and α 's

$$x''_3 = \frac{m\omega^2}{K} + \frac{15.3m\omega^2}{2K} + \frac{16.5m\omega^2}{2K} = \frac{28.2m\omega^2}{2K}$$

4. New deflection vector is:

$$\begin{Bmatrix} x''_1 \\ x''_2 \\ x''_3 \end{Bmatrix} = \frac{m\omega^2}{2K} \begin{Bmatrix} 10.23 \\ 26.69 \\ 33.8 \end{Bmatrix}$$

$$\begin{Bmatrix} x''_1 \\ x''_2 \\ x''_3 \end{Bmatrix} = \frac{10.23m\omega^2}{2K} \begin{Bmatrix} 1 \\ 2.60 \\ 3.30 \end{Bmatrix} = \{u\}_4$$

The new deflection vector $\{u\}_4 \approx \{u\}_3$ stop Iterating

Fundamental natural frequency can be obtained by.

$$\frac{10.23m\omega^2}{2K} = 1$$

$$\omega = 0.44 \sqrt{\frac{K}{m}} \text{ rad/s}$$

Modal vector is:

$$\{A\}_1 = \begin{Bmatrix} 1 \\ 2.60 \\ 3.30 \end{Bmatrix}$$

Example-2

For the system shown in Fig.12 find the lowest natural frequency by Stodola's method
(carryout two iterations)

July/Aug 2005 VTU for 10 marks

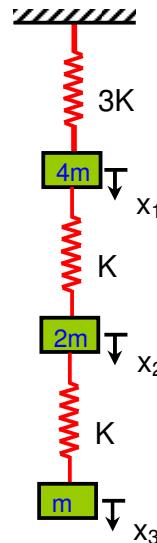


Fig.12 Linear vibratory system

Obtain flexibility influence coefficients,

$$\alpha_{11} = \alpha_{21} = \alpha_{12} = \alpha_{31} = \alpha_{13} = \frac{1}{3K}$$

$$\alpha_{22} = \alpha_{32} = \alpha_{23} = \frac{4}{3K}$$

$$\alpha_{33} = \frac{7}{3K}$$

First iteration

1. Assume a modal vector of system $\{u\}_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

2. Find out inertia forces of system at each mass point

$$F_1 = m_1 \omega^2 x_1 = 4m \omega^2$$

$$F_2 = m_2 \omega^2 x_2 = 2m \omega^2$$

$$F_3 = m_3 \omega^2 x_3 = m \omega^2$$

3. New deflection vector using flexibility influence coefficients,

$$x'_1 = F_1 \alpha_{11} + F_2 \alpha_{12} + F_3 \alpha_{13}$$

$$x'_1 = \frac{4m \omega^2}{3K} + \frac{2m \omega^2}{3K} + \frac{m \omega^2}{3K} = \frac{7m \omega^2}{3K}$$

$$x'_2 = F_1\alpha_{21} + F_2\alpha_{22} + F_3\alpha_{23}$$

$$x'_2 = \frac{4m\omega^2}{3K} + \frac{8m\omega^2}{3K} + \frac{4m\omega^2}{3K} = \frac{16m\omega^2}{3K}$$

$$x'_3 = F_1\alpha_{31} + F_2\alpha_{32} + F_3\alpha_{33}$$

$$x'_3 = \frac{4m\omega^2}{3K} + \frac{8m\omega^2}{3K} + \frac{7m\omega^2}{3K} = \frac{19m\omega^2}{3K}$$

4. New deflection vector is:

$$\begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} = \frac{m\omega^2}{3K} \begin{Bmatrix} 7 \\ 16 \\ 19 \end{Bmatrix}$$

$$\begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} = \frac{7m\omega^2}{3K} \begin{Bmatrix} 1 \\ 2.28 \\ 2.71 \end{Bmatrix} = \{u\}_2$$

The new deflection vector $\{u\}_2 \neq \{u\}_1$. Iterate again using new deflection vector $\{u\}_2$

Second iteration

$$1. \text{ Initial vector of system } \{u\}_2 = \begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2.28 \\ 2.71 \end{Bmatrix}$$

2. Find out inertia forces of system at each mass point

$$F'_1 = m_1\omega^2 x'_1 = 4m\omega^2$$

$$F'_2 = m_2\omega^2 x'_2 = 4.56m\omega^2$$

$$F'_3 = m_3\omega^2 x'_3 = 2.71m\omega^2$$

3. New deflection vector

$$x''_1 = F'_1\alpha_{11} + F'_2\alpha_{12} + F'_3\alpha_{13}$$

$$x''_1 = \frac{4m\omega^2}{3K} + \frac{4.56m\omega^2}{3K} + \frac{2.71m\omega^2}{3K} = \frac{11.27m\omega^2}{3K}$$

$$x''_2 = F'_1\alpha_{21} + F'_2\alpha_{22} + F'_3\alpha_{23}$$

$$x''_2 = \frac{4m\omega^2}{3K} + \frac{18.24m\omega^2}{3K} + \frac{10.84m\omega^2}{3K} = \frac{33.08m\omega^2}{3K}$$

$$x''_3 = F'_1\alpha_{31} + F'_2\alpha_{32} + F'_3\alpha_{33}$$

$$x''_3 = \frac{4m\omega^2}{3K} + \frac{18.24m\omega^2}{3K} + \frac{18.97m\omega^2}{3K} = \frac{41.21m\omega^2}{3K}$$

4. New deflection vector is:

$$\begin{Bmatrix} x_1'' \\ x_2'' \\ x_3'' \end{Bmatrix} = \frac{m\omega^2}{3K} \begin{Bmatrix} 11.27 \\ 33.08 \\ 41.21 \end{Bmatrix}$$

$$\begin{Bmatrix} x_1'' \\ x_2'' \\ x_3'' \end{Bmatrix} = \frac{3.75m\omega^2}{K} \begin{Bmatrix} 1 \\ 2.93 \\ 3.65 \end{Bmatrix} = \{u\}_3$$

Stop Iterating as it is asked to carry only two iterations. The Fundamental natural frequency can be calculated by,

$$\frac{3.75m\omega^2}{2K} = 1$$

$$\omega = 0.52\sqrt{\frac{K}{m}}$$

Modal vector,

$$\{A\}_1 = \begin{Bmatrix} 1 \\ 2.93 \\ 3.65 \end{Bmatrix}$$

Disadvantage of Stodola's method

Main drawback of Stodola's method is that the method can be used to find only fundamental natural frequency and modal vector of vibratory systems. This method is not popular because of this reason.

(iii) Holz's method

It is an iterative method, used to find the natural frequencies and modal vector of a vibratory system having multi-degree freedom.

Principle

Consider a multi dof semi-definite torsional semi-definite system as shown in Fig.13.

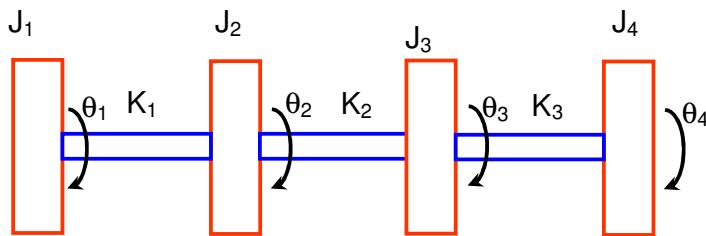


Fig.13 A torsional semi-definite system

The Eqns. of motions of the system are:

$$\begin{aligned} J_1 \ddot{\theta}_1 + K_1(\theta_1 - \theta_2) &= 0 \\ J_2 \ddot{\theta}_2 + K_1(\theta_2 - \theta_1) + K_2(\theta_2 - \theta_3) &= 0 \\ J_3 \ddot{\theta}_3 + K_2(\theta_3 - \theta_2) + K_3(\theta_3 - \theta_4) &= 0 \\ J_4 \ddot{\theta}_4 + K_3(\theta_4 - \theta_3) &= 0 \end{aligned}$$

The Motion is harmonic,

$$\theta_i = \phi_i \sin(\omega t) \quad (17)$$

where $i=1,2,3,4$

Substitute above Eqn.(17) in Eqns. of motion, we get,

$$\omega^2 J_1 \phi_1 = K_1(\phi_1 - \phi_2) \quad (18)$$

$$\omega^2 J_2 \phi_2 = K_1(\phi_2 - \phi_1) + K_2(\phi_2 - \phi_3)$$

$$\omega^2 J_3 \phi_3 = K_2(\phi_3 - \phi_2) + K_3(\phi_3 - \phi_4)$$

$$\omega^2 J_4 \phi_4 + K_3(\phi_4 - \phi_3) \quad (19)$$

Add above Eqns. (18) to (19), we get

$$\sum_{i=1}^4 \omega^2 J_i \phi_i = 0$$

For n dof system the above Eqn changes to,

$$\sum_{i=1}^n \omega^2 J_i \phi_i = 0 \quad (20)$$

The above equation indicates that sum of inertia torques (torsional systems) or inertia forces (linear systems) is equal to zero for semi-definite systems.

In Eqn. (20) ω and ϕ_i both are unknowns. Using this Eqn. one can obtain natural frequencies and modal vectors by assuming a trial frequency ω and amplitude ϕ_1 so that the above Eqn is satisfied.

Steps involved

1. Assume magnitude of a trial frequency ω
2. Assume amplitude of first disc/mass (for simplicity assume $\phi_1=1$)
3. Calculate the amplitude of second disc/mass ϕ_2 from first Eqn. of motion

$$\omega^2 J_1 \phi_1 = K_1(\phi_1 - \phi_2) = 0$$

$$\phi_2 = \phi_1 - \frac{\omega^2 J_1 \phi_1}{K_1}$$

4. Similarly calculate the amplitude of third disc/mass ϕ_3 from second Eqn. of motion.

$$\omega^2 J_2 \phi_2 = K_1(\phi_2 - \phi_1) + K_2(\phi_2 - \phi_3) = 0$$

$$\omega^2 J_2 \phi_2 = K_1(\phi_1 - \frac{\omega^2 J_1 \phi_1}{K_1} - \phi_1) + K_2(\phi_2 - \phi_3) = 0$$

$$\omega^2 J_2 \phi_2 = -\omega^2 J_1 \phi_1 + K_2(\phi_2 - \phi_3) = 0$$

$$K_2(\phi_2 - \phi_3) = \omega^2 J_1 \phi_1 + \omega^2 J_2 \phi_2$$

$$\phi_3 = \phi_2 - \frac{\omega^2 J_1 \phi_1 + \omega^2 J_2 \phi_2}{K_2} \quad (21)$$

The Eqn (21) can be written as:

$$\phi_3 = \phi_2 - \frac{\sum_{i=1}^2 J_i \phi_i \omega^2}{K_2}$$

5. Similarly calculate the amplitude of nth disc/mass ϕ_n from (n-1)th Eqn. of motion is:

$$\phi_n = \phi_{n-1} - \frac{\sum_{i=1}^{n-1} J_i \phi_i \omega^2}{K_n}$$

6. Substitute all computed ϕ_i values in basic constraint Eqn.

$$\sum_{i=1}^n \omega^2 J_i \phi_i = 0$$

7. If the above Eqn. is satisfied, then assumed ω is the natural frequency, if the Eqn is not satisfied, then assume another magnitude of ω and follow the same steps.

For ease of computations, Prepare the following table, this facilitates the calculations.

Table-1. Holzar's Table

1	2	3	4	5	6	7	8
ω	S No	J	ϕ	$J\omega^2\phi$	$\sum J\omega^2\phi$	K	$\frac{1}{K} \sum J\omega^2\phi$

Example-1

For the system shown in the Fig.16, obtain natural frequencies using Holzar's method.

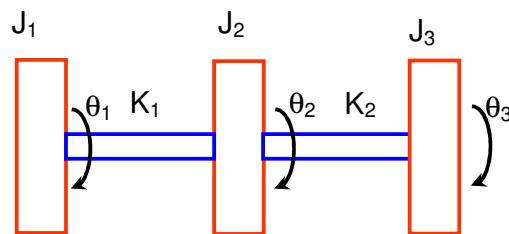


Fig.14 A torsional semi-definite system

Make a table as given by Table-1, for iterations, follow the steps discussed earlier.
Assume ω from lower value to a higher value in proper steps.

Table-2. Holzar's Table for Example-1

1	2	3	4	5	6	7	8
ω	S No	J	ϕ	$J\omega^2\phi$	$\sum J\omega^2\phi$	K	$\frac{1}{K} \sum J\omega^2\phi$
I-iteration							
0.25	1	1	1	0.0625	0.0625	1	0.0625
	2	1	0.9375	0.0585	0.121	1	0.121
	3	1	0.816	0.051	0.172	—	—
II-iteration							
0.50	1	1	1	0.25	0.25	1	0.25
	2	1	0.75	0.19	0.44	1	0.44
	3	1	0.31	0.07	0.51	—	—
III-iteration							
0.75	1	1	1	0.56	0.56	1	0.56
	2	1	0.44	0.24	0.80	1	0.80
	3	1	-0.36	-0.20	0.60	—	—
IV-iteration							
1.00	1	1	1	1	1	1	1
	2	1	0	0	1	1	1

	3	1	-1	-1	0	—	—
V-iteration							
1.25	1	1	1	1.56	1.56	1	1.56
	2	1	-0.56	-0.87	0.69	1	0.69
	3	1	-1.25	-1.95	-1.26	—	—
VI-iteration							
1.50	1	1	1	2.25	2.25	1	2.25
	2	1	-1.25	-2.82	-0.57	1	-0.57
	3	1	-0.68	-1.53	-2.10	—	—
VII-iteration							
1.75	1	1	1	3.06	3.06	1	3.06
	2	1	-2.06	-6.30	-3.24	1	-3.24
	3	1	1.18	3.60	0.36	—	—

Table.3 Iteration summary table

ω	$\sum J\omega^2\phi$
0	0
0.25	0.17
0.5	0.51
0.75	0.6
1	0
1.25	-1.26
1.5	-2.1
1.75	0.36

The values in above table are plotted in Fig.15.

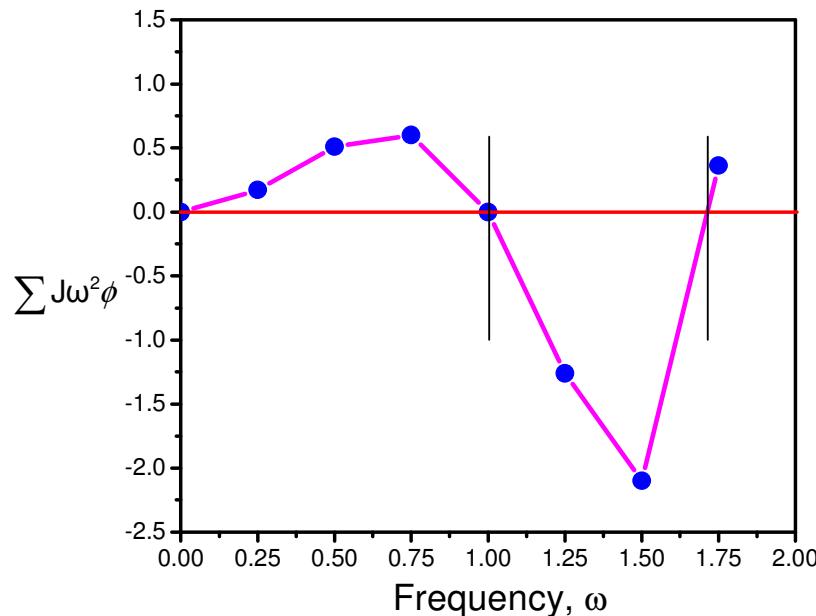


Fig.15. Holzr's plot of Table-3

From the above Graph, the values of natural frequencies are:

$$\omega_1 = 0 \text{ rad/s}$$

$$\omega_2 = 1 \text{ rad/s}$$

$$\omega_3 = 1.71 \text{ rad/s}$$

Definite systems

The procedure discussed earlier is valid for semi-definite systems. If a system is definite the basic equation Eqn. (20) is not valid. It is well-known that for definite systems, deflection at fixed point is always ZERO. This principle is used to obtain the natural frequencies of the system by iterative process. The Example-2 demonstrates the method.

Example-2

For the system shown in the figure estimate natural frequencies using Holzr's method.

July/Aug 2005 VTU for 20 marks

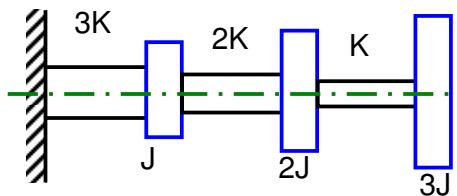


Fig.16 A torsional system

Make a table as given by Table-1, for iterations, follow the steps discussed earlier. Assume ω from lower value to a higher value in proper steps.

Table-4. Holzr's Table for Example-2

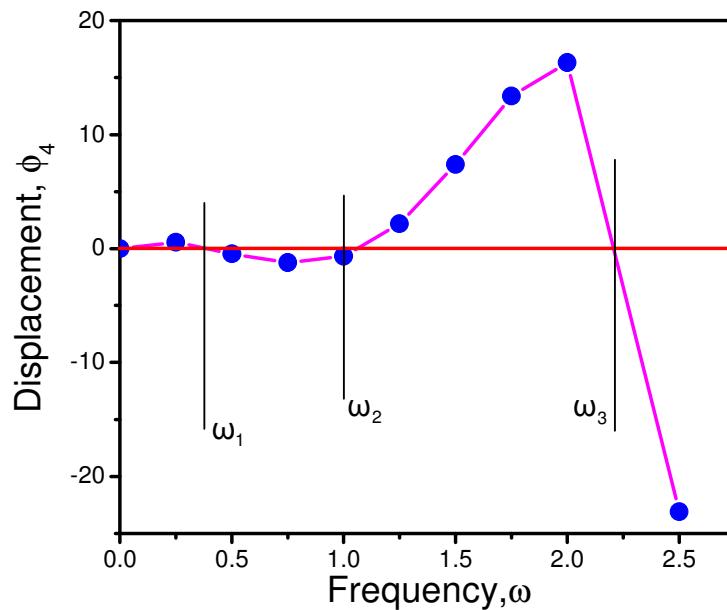
1	2	3	4	5	6	7	8
ω	S No	J	ϕ	$J\omega^2\phi$	$\sum J\omega^2\phi$	K	$\frac{1}{K} \sum J\omega^2\phi$
I-iteration							
0.25	1	3	1	0.1875	0.1875	1	0.1875
	2	2	0.8125	0.1015	0.289	2	0.1445
	3	1	0.6679	0.0417	0.330	3	0.110
	4	—	0.557	—	—	—	—
II-iteration							
0.50	1	3	1	0.75	0.75	1	0.75
	2	2	0.25	0.125	0.875	2	0.437
	3	1	-0.187	-0.046	0.828	3	0.27
	4	—	-0.463	—	—	—	—
III-iteration							
0.75	1	3	1	1.687	1.687	1	1.687
	2	2	-0.687	-0.772	0.914	2	0.457
	3	1	-1.144	-0.643	0.270	3	0.090
	4	—	-1.234	—	—	—	—
IV-iteration							
1.00	1	3	1	3	3	1	3
	2	2	-2	-4	-1	2	-0.5

	3	1	-1.5	-1.5	-2.5	3	-0.833
	4	—	-0.667	—	—	—	—
V-iteration							
1.25	1	3	1	4.687	4.687	1	4.687
	2	2	-3.687	-11.521	-6.825	2	-3.412
	3	1	-0.274	-0.154	-6.979	3	-2.326
	4	—	2.172	—	—	—	—
VI-iteration							
1.50	1	3	1	6.75	6.75	1	6.75
	2	2	-5.75	-25.875	-19.125	2	-9.562
	3	1	3.31	8.572	-10.552	3	-3.517
	4	—	7.327	—	—	—	—
	1	2	3	4	5	7	8
ω	S No	J	ϕ	$J\omega^2\phi$	$\sum J\omega^2\phi$	K	$\frac{1}{K} \sum J\omega^2\phi$
VII-iteration							
1.75	1	3	1	9.18	9.18	1	9.18
	2	2	-8.18	-50.06	-40.88	2	-20.44
	3	1	12.260	37.515	-3.364	3	-1.121
	4	—	13.38	—	—	—	—
VIII-iteration							
2.0	1	3	1	12	12	1	12
	2	2	-11	-88	-76	2	-38
	3	1	-27	108	32	3	10.66
	4	—	16.33	—	—	—	—
IX-iteration							
2.5	1	3	1	18.75	18.75	1	18.75
	2	2	-17.75	-221.87	-203.12	2	-101.56
	3	1	83.81	523.82	320.70	3	106.90
	4	—	-23.09	—	—	—	—

Table.5 Iteration summary table

ω	ϕ_4
0	0
0.25	0.557
0.5	-0.463
0.75	-1.234
1	-0.667
1.25	2.172
1.5	7.372
1.75	13.38
2	16.33
2.5	-23.09

The values in above table are plotted in Fig.17.

**Fig.17.** Holzar's plot of Table-5

From the above Graph, the values of natural frequencies are:

$$\omega_1 = 0.35 \text{ rad/s}$$

$$\omega_2 = 1.15 \text{ rad/s}$$

$$\omega_3 = 2.30 \text{ rad/s}$$

3.5 Vibrations of Continuous Systems

Continuous systems are those which have continuously distributed mass and Elasticity. These continuous systems are assumed to be homogeneous and isotropic obeying hook law within the elastic limit. Since to specify the position of every point in the continuous systems an infinite number of coordinates is required therefore a continuous systems is considered to have infinite number of degrees of freedom. Thus there will be infinite natural frequencies

Free vibration of continuous system is sum of the principal or normal modes. For the normal mode vibration, every particle of the body performs simple harmonic motion at the frequency corresponding to the particular root of the frequency equation, each particles passing simultaneously through its respective equilibrium position. If the elastic curve resulting due to vibration motion starts coinciding exactly with one of the normal modes, only that normal mode will be produced. If the elastic curve of the system under which the vibration is started, is identical to any one of the principal mode shapes, then the system will vibrate only in that principal mode.

Vibration of String.

A string stretched between two support is shown in figure-1 is an infinite degree of freedom system.

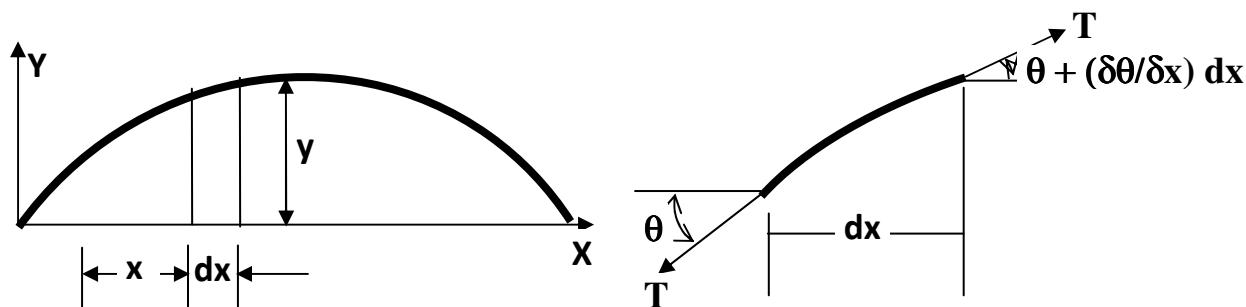


Figure-1

Let the tension T in the string be large so that for small displacement or amplitude the tension in the string remains constant throughout the string. Consider an element of length dx at a distance 'x' from the left end. At any instant of time the element of the string be displaced through a distance 'y' from the equilibrium position. Then the tension at both the ends of this element is 'T'. If θ the angle at the left end of the element makes with the horizontal x-axis then the angle at the right end of the element is $\theta + (\partial\theta/\partial x)dx$

The components of these two tensions at the ends of the element along x-axis balance each others.

The components along Y – axis $T \sin \left(\theta + \frac{\partial\theta}{\partial x} dx \right) = T \sin \theta$ Since θ is small

$$= T \left(\theta + \frac{\partial\theta}{\partial x} dx \right) - T\theta = T \frac{\partial\theta}{\partial x} dx \quad \dots (1)$$

If ' ρ ' is mass per unit length, then the mass of the element is ' ρdx ' and the differential equation of motion along Y-axis according to Newton's second law of motion is,

Mass x Acceleration = Sum of all the forces.

$$\rho dx \frac{\partial^2 y}{\partial t^2} = T \frac{\partial\theta}{\partial x} dx \quad \text{Therefore} \quad \rho \frac{\partial^2 y}{\partial t^2} = T \frac{\partial\theta}{\partial x}$$

But we know slope $\theta = \frac{\partial y}{\partial x}$ then we have

$$\rho \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 y}{\partial t^2} = \frac{\rho}{T} \frac{\partial^2 y}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 y}{\partial x^2} = \frac{1}{T/\rho} \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} \quad \dots (2)$$

Where $C = \sqrt{T/\rho}$ ----- (3) is the velocity of wave propagation along the string. Solution of equation (2) can be obtained by assuming Y through separation of variables to be a product of two functions in the form. $y(x, t) = Y(x) G(t) \dots (4)$

$$\frac{\partial^2 y}{\partial x^2} = G \frac{\partial^2 y}{\partial x^2} \quad \text{and} \quad \frac{\partial^2 y}{\partial t^2} = Y \frac{\partial^2 y}{\partial t^2} \quad \dots (5)$$

Substituting equation (5) in to equation (2) we have

$$G \frac{\partial^2 y}{\partial x^2} = \frac{Y}{c^2} \frac{\partial^2 G}{\partial t^2} \quad \text{or} \quad \frac{1}{Y} \frac{\partial^2 y}{\partial x^2} = \frac{1}{Gc^2} \frac{\partial^2 G}{\partial t^2} \quad \frac{C^2}{Y} \frac{\partial^2 y}{\partial x^2} = \frac{1}{G} \frac{\partial^2 G}{\partial t^2} \quad \dots \quad (6)$$

Left hand side of equation (6) is a function of x alone and the right hand side is a function of t alone. These two can only be equal if each one of these expression is a constant. This constant may be positive, zero or negative. If it is a positive constant or zero then there is no vibratory motion. Hence this constant has to be negative constant and equal to $-\square^2$

$$\text{Then } \frac{C^2}{Y} \frac{\partial^2 y}{\partial x^2} = \frac{1}{G} \frac{\partial^2 G}{\partial t^2} = -\square^2 \quad \dots \quad (7)$$

$$\frac{C^2}{Y} \frac{\partial^2 y}{\partial x^2} + \omega^2 = 0 \quad \text{and} \quad \frac{1}{G} \frac{\partial^2 G}{\partial t^2} + \omega^2 = 0$$

$$\frac{\partial^2 y}{\partial x^2} + \left(\frac{\omega}{c}\right)^2 Y = 0 \quad \text{and} \quad \frac{\partial^2 G}{\partial t^2} + \omega^2 = 0 \quad \dots \quad (8)$$

$$\text{With the general solution } Y = A \sin \frac{\omega}{c}x + B \cos \frac{\omega}{c}x \quad \dots \quad (9)$$

$$G = D \sin t + E \cos t \quad \dots \quad (10)$$

Substituting equation (9) and (10) into equation (4) we have

$$y(x, t) = \left(A \sin \frac{\omega}{c}x + B \cos \frac{\omega}{c}x \right) (D \sin t + E \cos t) \quad \dots \quad (11)$$

Since ω can have infinite values, being an infinite degree of freedom system the general solution can be written as

$$y(x, t) = \sum_{i=1}^{\infty} \left(A_i \sin \frac{\omega_i}{c}x + B_i \cos \frac{\omega_i}{c}x \right) (D_i \sin i t + E_i \cos i t) \quad \dots \quad (11)$$

The arbitrary constant A, B, D and E have to be determined from the boundary condition and the initial conditions.

For the string of length 'L' stretched between two fixed points. The boundary conditions are $y(0, t) = y(L, t) = 0$

The condition $y(0, t) = 0$ will require $B = 0$ so the solution will be

$$y(x, t) = \left(A \sin \frac{\omega}{c} x \right) (D \sin t + E \cos t) \quad \dots \dots \dots (12)$$

The condition $y(L, t) = 0$ them fields the equation $0 = \left(A \sin \frac{\omega}{c} L \right) (D \sin t + E \cos t)$

Since $D \sin \omega t + E \cos \omega t \neq 0$, then we have $\left(A \sin \frac{\omega}{c} L \right) = 0$

$$\text{or } \left(\frac{\omega}{c} L \right) = n\pi \text{ for } n = 1, 2, 3, \dots \dots \dots \dots \dots (13)$$

$$\text{Since } f = c/\lambda \quad \dots \dots \dots (14) \quad \text{and}$$

$$= 2\pi f = 2\pi c/\lambda \quad \dots \dots \dots (15). \quad \text{Where } \lambda \text{ is the wave length of oscillation.}$$

Substituting equation (15) into equation (13) we have

$$\left(\frac{2\pi c/\lambda}{c} \right) L = n\pi \text{ or } 2L/\lambda = n \quad \dots \dots \dots (16)$$

Each 'n' represents normal mode vibration with natural frequency determined from the equation

$$2Lf/c = n \text{ therefore } f = nc/2L = \frac{n}{2L} \sqrt{T/\rho} \text{ for } n = 1, 2, 3, \dots \dots \dots \dots \dots (17)$$

$$\text{distribution } Y = \left(\sin \frac{n\pi}{L} x \right) \quad \dots \dots \dots (18)$$

Thus the general solution is given by

$$y(x, t) = \sum_{n=1}^{\infty} (D_n \sin n t + E_n \cos n t) \sin \frac{n\pi}{L} x \quad \dots \dots \dots (19)$$

$$\text{With } \omega_n = \frac{n\pi c}{L} \quad \dots \dots \dots (20)$$

With the initial condition of $Y(x, 0)$ and $Y'(x, 0)$ the constants D_n and E_n can be evaluated.

Longitudinal vibration of Rod/Bar

The bar/rod considered for the analysis is assumed to be thin and a uniform cross sectional area throughout its length. Due to axial force the displacement 'u' along the bar will be a function of both position 'x' and time 't'. The bar/rod has infinite number of natural frequencies and modes of vibration

Consider an element of length dx at a distance 'x' from the left end of the bar as shown in the figure -2.

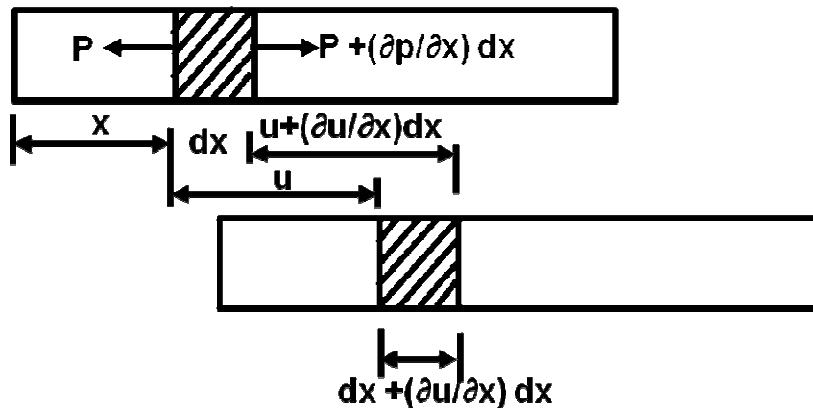


Figure -2 Displacement in rod/bar member.

At any instant of time during vibration let 'P' be the axial force at the left end of the element. Then the axial force at the right end of the element is $P + (\partial p / \partial x) dx$ and If 'u' is the displacement at x , then the displacement at $x + dx$ will be $u + (\partial u / \partial x) dx$. Then the element dx in the new position has changed in length by an amount of $(\partial u / \partial x) dx$ and hence the unit strain is $\partial u / \partial x$.

But from Hook's law the ratio of unit stress to the unit strain is equal to the Modulus of Elasticity 'E',
Hence we have $\partial u / \partial x = P / AE$ ----- (1)

In which 'A' is the cross sectional area of the bar/rod. Differentiating the equation (1) with respect to x we have $AE \partial^2 u / \partial x^2 = \partial P / \partial x$ ----- (2) But $(\partial P / \partial x) dx$ is the unbalanced force

Considering the Dynamic Equilibrium of the element from Newton's second law of motions we have.

(Mass) X (Acceleration) of the Element = (Unbalanced Resultant External force)

$$(\partial P / \partial x) dx = Ar dx \partial^2 u / \partial t^2 = ----- (3)$$

Where ρ is the density of rod/bar mass per unit volume, substituting equation (2) into (3) for $\partial P / \partial x$ we have

$$EAdx (\partial^2 u / \partial x^2) = rAdx \partial^2 u / \partial t^2$$

$$\partial^2 u / \partial x^2 = (r/E) \partial^2 u / \partial t^2$$

$$\frac{\partial^2 u}{\partial x^2} = \left[\frac{1}{(E/r)} \right] \frac{\partial^2 u}{\partial t^2} \quad \dots \dots \dots (4)$$

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{1}{c^2} \right) \frac{\partial^2 u}{\partial t^2} \quad \dots \dots \dots (5)$$

Where $c = \sqrt{(E/r)}$ ----- (6) is the velocity of propagation of the displacement or stress wave in the rod/bar.

Solution of equation (5) can be obtained by assuming u through separation of variables to be a product of two functions in the form $u(x, t) = U(x) G(t)$ ----- (7)

$$\frac{\partial^2 u}{\partial x^2} = G \left(\frac{\partial^2 U}{\partial x^2} \right) \text{ and } \frac{\partial^2 u}{\partial t^2} = U \left(\frac{\partial^2 G}{\partial t^2} \right) \dots \dots \dots (8)$$

Substituting equation (8) in to equation (5) we have $G \left(\frac{\partial^2 U}{\partial x^2} \right) = \left(\frac{U}{c^2} \right) \left(\frac{\partial^2 G}{\partial t^2} \right)$

$$\text{Then } \left(\frac{c^2}{U} \right) \left(\frac{\partial^2 U}{\partial x^2} \right) = \left(\frac{1}{G} \right) \left(\frac{\partial^2 G}{\partial t^2} \right) \dots \dots \dots (9)$$

Left hand side of equation (9) is a function of x alone and the right hand side is a function of t alone. These two can only be equal if each one of these expression is a constant.

This constant may be positive, zero or negative. If it is a positive constant or zero then there is no vibratory motion. Hence this constant have to be negative constant and equal to $-w^2$. Then we have

$$\left(\frac{c^2}{U} \right) \left(\frac{\partial^2 U}{\partial x^2} \right) = \left(\frac{1}{G} \right) \left(\frac{\partial^2 G}{\partial t^2} \right) = -\omega^2 \dots \dots \dots (10)$$

$$\left(\frac{c^2}{U} \right) \left(\frac{\partial^2 U}{\partial x^2} \right) = -w^2 \text{ and } \left(\frac{1}{G} \right) \left(\frac{\partial^2 G}{\partial t^2} \right) = -\omega^2$$

$$\frac{\partial^2 U}{\partial x^2} + (\omega/c)^2 U = 0 \text{ and } \frac{\partial^2 G}{\partial t^2} + \omega^2 G = 0 \dots \dots \dots (11)$$

With the general solution $U = A \sin(\omega/c)x + B \cos(\omega/c)x$ ----- (12) $G = D \sin\omega t + E \cos\omega t$ ----- (13)

Substituting equation (12) and (13) into equation (7) we have

$$u(x, t) = [A \sin(\omega/c)x + B \cos(\omega/c)x] * [D \sin\omega t + E \cos\omega t] \dots \dots \dots (14)$$

Since ω can have infinite values, as the system being an infinite degree of freedom the general solution can be written as

$$u(x, t) = \sum_{i=1}^{\infty} [A_i \sin(\omega_i/c)x + B_i \cos(\omega_i/c)x] [D_i \sin\omega_i t + E_i \cos\omega_i t] \dots \dots \dots (15)$$

The arbitrary constant A , B , D and E have to be determined from the boundary condition and the initial conditions.

Example:

Determine the natural frequencies and mode shape of a bar/rod when both the ends are free.

Solution: When the bar/rod with both the ends being free then the stress and strain at the ends are zero

The boundary conditions are $\partial u / \partial x = 0$ at $x=0$ and $x=L$ ----- (1)

$$u(x, t) = [A\sin(\omega/c)x + B\cos(\omega/c)x] [D\sin\omega t + E\cos\omega t] \quad \text{----- (2)}$$

$$\partial u / \partial x = [(A\omega/c)\cos(\omega/c)x + (B\omega/c)\sin(\omega/c)x] [D\sin\omega t + E\cos\omega t] \quad \text{----- (3)}$$

$$(\partial u / \partial x)_{x=0} = (A\omega/c)[D\sin\omega t + E\cos\omega t] = 0 \quad \text{--- (4)}$$

$$(\partial u / \partial x)_{x=L} = (\omega/c)[A\cos(\omega L/c) + B\sin(\omega L/c)] [D\sin\omega t + E\cos\omega t] = 0 \quad \text{----- (5)}$$

Since the equation (4) and (5) must be true for any time 't'

From equation (4) A must be equal to zero. Since B must be finite value in order to have vibration then equation (5) is satisfied only when $\sin(\omega_n L/c) = 0$ or $\omega_n L/c = \sin^{-1}(0) = n\pi$

Since $c = \sqrt{E/\rho}$ then we have $\omega_n L \sqrt{\rho/E} = n\pi$ Thus $\omega_n = (n\pi/L) \sqrt{E/\rho}$ ----- (6)

since the natural frequency is $f_n = \omega_n / 2\pi = [(n\pi/L) \sqrt{E/\rho}] / 2\pi$

$$f_n = (n/2L) \sqrt{E/\rho} \quad \text{----- (7)}$$

Each 'n' represents normal mode vibration with natural frequency ' f_n ' determined from the equation (7).

Torsional vibration of circular shaft.

Consider an element of length dx at a distance x from one end of the shaft as shown in the figure -3

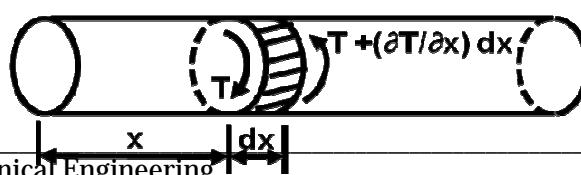


Figure -3 Torque acting on the element of the circular shaft

At any given instant of time during vibration let T be the torque at left end of the element. Then the torque at the right end of the element is given by $T + (\partial T / \partial x)dx$.

If θ is the angular twist of the shaft at the distance x , then $\theta + (\partial \theta / \partial x)dx$ is the angular twist of the shaft at the distance $x+dx$. Therefore the angular twist in the element of length dx is $(\partial \theta / \partial x)dx$.

If 'J' is the polar moment of inertia of the shaft and 'G' the modulus of rigidity then the angular twist in the element of length dx is given by torsion formula

$$d\theta = (T/GJ) dx \quad \text{--- (1)} \quad \text{or} \quad GJ(d\theta/dx) = T \quad \text{--- (2)}$$

$$\text{Differentiating with respect to } x \text{ we have } GJ(d^2\theta/dx^2) = dT/dx \quad \text{--- (3)}$$

In which GJ is the torsional stiffness of the shaft. Since the torque on the two face of the element being T and $T + (\partial T / \partial x)dx$ the net torque on the element will be $(\partial T / \partial x)dx$ --- (4)

Substituting equation (3) for $\partial T / \partial x$ in to equation (4) we have $(\partial T / \partial x)dx = GJ(\partial^2\theta / \partial x^2)dx$ ----- (5)

Considering the dynamic equilibrium of the element one can obtain the equation of motion by equating the product of mass moment of inertia ' $J\rho dx$ ' and the acceleration ' $\partial^2\theta / \partial t^2$ ' to the net torque acting on the element we have.

$$(J\rho dx) \partial^2\theta / \partial t^2 = GJ(\partial^2\theta / \partial x^2)dx$$

$$\partial^2\theta / \partial t^2 = (G/\rho)(\partial^2\theta / \partial x^2) \text{ or}$$

$$\partial^2\theta / \partial x^2 = (\rho/G) \partial^2\theta / \partial t^2$$

$$\partial^2\theta / \partial x^2 = [1/(G/\rho)] \partial^2\theta / \partial t^2$$

$$\partial^2\theta / \partial x^2 = (1/c^2) \partial^2\theta / \partial t^2 \quad \text{----- (6)}$$

Where $c = \sqrt{(G/\rho)}$ ----- (7) is the velocity of wave propagation in which ρ is the density of the shaft in mass per unit volume.

Solution of equation (6) can be obtained by assuming θ through separation of variables to be a product of two functions in the form. $\theta(x, t) = \Theta(x) G(t)$ ----- (7)

$$\frac{\partial^2 \theta}{\partial x^2} = G \left(\frac{\partial^2 \Theta}{\partial x^2} \right) \text{ and } \frac{\partial^2 \theta}{\partial t^2} = \Theta \left(\frac{\partial^2 G}{\partial t^2} \right) \text{---(8)}$$

Substituting equation (8) in to equation (6) we have $G \left(\frac{\partial^2 \Theta}{\partial x^2} \right) = \left(\Theta / c^2 \right) \left(\frac{\partial^2 G}{\partial t^2} \right)$

$$\text{Then } \left(c^2 / \Theta \right) \left(\frac{\partial^2 \Theta}{\partial x^2} \right) = \left(1 / G \right) \left(\frac{\partial^2 G}{\partial t^2} \right) \text{---- (9)}$$

Left hand side of equation (9) is a function of ' Θ ' alone and the right hand side is a function of ' t ' alone. These two can only be equal if each one of these expression is a constant. This constant may be positive, zero or negative. If it is a positive constant or zero then there is no vibratory motion. Hence this constant has to be negative constant and equal to $- \omega^2$

$$\left(c^2 / \Theta \right) \left(\frac{\partial^2 \Theta}{\partial x^2} \right) = \left(1 / G \right) \left(\frac{\partial^2 G}{\partial t^2} \right) = - \omega^2 \text{--- (10)}$$

$$\left(c^2 / \Theta \right) \left(\frac{\partial^2 \Theta}{\partial x^2} \right) = - \omega^2 \text{ and } \left(1 / G \right) \left(\frac{\partial^2 G}{\partial t^2} \right) = - \omega^2$$

$$\frac{\partial^2 \Theta}{\partial x^2} + (\omega/c)^2 \Theta = 0 \text{ and } \frac{\partial^2 G}{\partial t^2} + \omega^2 G = 0 \text{---(11)}$$

$$\text{With the general solution } \Theta = A \sin(\omega/c)x + B \cos(\omega/c)x \text{ ---- (12)} \quad G = D \sin \omega t + E \cos \omega t \text{ ---- (13)}$$

Substituting equation (12) and (13) into equation (7) we have

$$\theta(x, t) = [A \sin(\omega/c)x + B \cos(\omega/c)x] [D \sin \omega t + E \cos \omega t] \text{----- (14)}$$

Since ω can have infinite values, as the system being an infinite degree of freedom the general solution can be written as

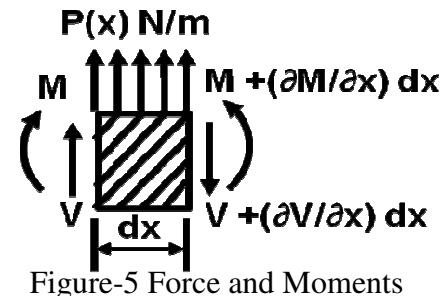
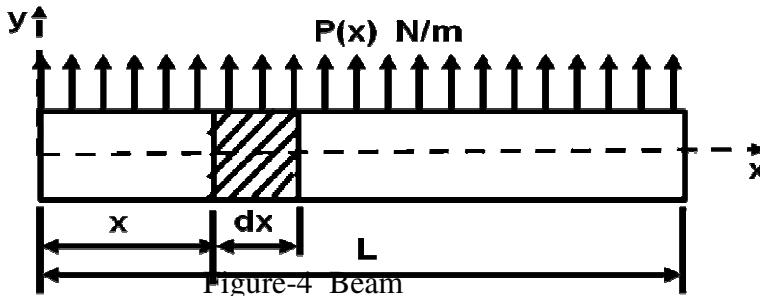
$$\theta(x, t) = \sum_{i=1}^{\infty} [A_i \sin(\omega_i/c)x + B_i \cos(\omega_i/c)x] [D_i \sin \omega_i t + E_i \cos \omega_i t] \text{----- (15)}$$

The arbitrary constant A, B, D and E have to be determined from the boundary condition and the initial conditions.

Lateral Vibration of beams

To derive the differential equation of motion for lateral vibration of beams one has to consider the forces and moments acting on the beam. Let V and M are the shear forces and bending moments respectively acting on the beam with $p(x)$ represents the intensity of lateral loading per unit length of the beam.

Consider a beam of length 'L' and moment of inerter 'I' subjected to distributed lateral load of $p(x)$ N per unit length. At any section $x-x$ at a distance x from one end consider an element of length dx as shown in the figure-4. The force and the moments acting on the element of length dx are as shown in the figure-5



For static equilibrium the summation of the forces and moment should be equal to zero. Thus summing the vertical force in the Y – direction and equating to zero we have

$$V + p(x) dx - [V + (\partial V / \partial x) dx] = 0$$

$$[p(x) - (\partial V / \partial x)] dx = 0 \text{ since } dx \neq 0$$

$$p(x) - (\partial V / \partial x) = 0 \text{ or } \partial V / \partial x = p(x) \text{ ---- (1)}$$

Which state that the rate of change of shear force along the length of the beam is equal to the load per unit length.

Summation of the moment about any point should be equal to zero. Thus summing the moments about the right bottom corner of the element considering the clockwise moment as positive we have

$$M + p(x) dx^2/2 + V dx - [M + (\partial M / \partial x) dx] = 0 \text{ or } p(x) dx^2/2 + V dx - (\partial M / \partial x) dx = 0$$

Since dx is small neglecting the higher order terms we have

$$[V - (\partial M / \partial x)] dx = 0 \text{ Since } dx \neq 0 \text{ we have}$$

$$V - (\partial M / \partial x) = 0 \quad \partial M / \partial x = V \text{ ---- (2)}$$

Which state that the rate of change of bending moment along the length of the beam is equal to the shear force.

From equation (1) and (2) we have $\partial^2 M / \partial x^2 = \partial V / \partial x = p(x)$ ---- (3)

But we known that the bending moment is related to curvature by flexure formula and for the coordinates indicated we have $EI(d^2y/dx^2) = M$ ----- (4)

Substituting equation (4) into equation (3) we have $\partial^2[EI(d^2y/dx^2)]/\partial x^2 = p(x)$

$$EI(d^4y/dx^4) = p(x) ----- (5)$$

For the dynamic equilibrium when the beam is having transverse vibrations about its static equilibrium position under its own weight and due to the distributed load per unit length must be equal to the inertia load due its mass and acceleration. By assuming harmonic motion the inertia force $\rho\omega^2y$ is in the same direction as that of the distributed load $p(x)$.

$$\text{Thus we have } p(x) = \rho\omega^2y ----- (6)$$

In which ρ is the mass density per unit length of the beam. Thus substituting equation (6) into equation (5) for $p(x)$ we have $EI(d^4y/dx^4) = \rho\omega^2y$ or $EI(d^4y/dx^4) - \rho\omega^2y = 0$ ----- (7)

If the flexural rigidity EI is constant then equation (7) can be written as $d^4y/dx^4 - (\rho\omega^2/EI)y = 0$ ---- (8)

$$\text{Letting } \beta^4 = \rho\omega^2/EI ----- (9) \quad d^4y/dx^4 - \beta^4y = 0 ----- (10)$$

The above equation (10) is a fourth order differential equation for the lateral vibration of a uniform cross section beam. The solution of the above equation (10) can be obtained by assuming the displacement 'y' of the form $y = e^{ax}$ ---- (11)

Which will satisfy the fourth order differential equation (10) when $a = \pm \beta$ and $a = \pm i\beta$ ---- (12)

$$\text{Since } e^{\pm\beta x} = \cosh\beta x \pm \sinh\beta x ----- (13) \quad \text{and} \quad e^{\pm i\beta x} = \cos\beta x \pm \sin\beta x ----- (14)$$

Then the solution of the above equation (10) will be in the form

$$Y = A\cosh\beta x + B\sinh\beta x + C\cos\beta x + D\sin\beta x -- (15) \quad \text{Which is readily established}$$

The natural frequencies of lateral vibration of beams are found from equation (9) given by

$$\beta_n^4 = \rho\omega_n^2/EI \quad \text{Thus} \quad \omega_n^2 = \beta_n^4 EI/\rho \quad \text{or} \quad \omega_n = \beta_n^2 \sqrt{EI/\rho}$$

$$\omega_n = (\beta_n L)^2 \sqrt{(EI/\rho L^4)} ----- (16) \quad \text{Where the number } \beta_n \text{ depends on the boundary conditions of the beam.}$$

The following table lists the numerical values of $(\beta_n L)^2$ for typical end boundary conditions of the beam.

Beam Condition	First Mode	Second Mode	Third Mode
Simply supported	9.87	39.5	88.9
Cantilever	3.52	22.0	61.7
Free-Free	22.4	61.7	121.0
Clemped -Clemped	22.4	61.7	121.0
Champed hinged	15.4	50.0	104.0
Hinged -Fre	0	15.4	50.0

Example-1: Determine the first three natural frequencies of a Rectangular Cantilever Beam for the following data Length L = 1 m Breath B = 0.03 m Depth D = 0.04 m Young's Modulus E = 2×10^{11} N/m² Density r = 7850.0 Kg/m³

Solution: For a cantilever beam the boundary conditions are at x= 0 displacement and slope are zero

i.e. y = 0 and dy/dx = 0 ----- (1) and at x= L the shear force 'V' and the bending moment 'M' is zero
i.e. d²y/dx² = 0 and d³y/dx³ = 0 ----- (2)

Substituting these boundary conditions into the general solution

$$Y = A\cosh\beta x + B\sinh\beta x + C\cos\beta x + D\sin\beta x \text{ --- (3)}$$

$$\text{we obtain } (y)_{x=0} = A + C = 0 \text{ then } A = -C \text{ --- (4)}$$

$$(dy/dx)_{x=0} = \beta(A\sinh\beta x + B\cosh\beta x - C\sin\beta x + D\cos\beta x)_{x=0} = 0$$

$$\beta(B + D) = 0 \text{ since } \beta \neq 0 \text{ then } B = -D \text{ --- (5)}$$

$$(d^2y/dx^2)_{x=L} = \beta^2(A\cosh\beta L + B\sinh\beta L - C\cos\beta L - D\sin\beta L) = 0$$

$$\text{Since } \beta^2 \neq 0 \text{ and } A = -C \text{ and } B = -D \text{ and we have } A(\cosh\beta L + \cos\beta L) + B(\sinh\beta L + \sin\beta L) = 0 \text{ --- (6)}$$

$$A/B = -(\sinh\beta L + \sin\beta L)/(\cosh\beta L + \cos\beta L) \text{ ---- (7)}$$

$$(d^3y/dx^3)_{x=L} = \beta^3(A\sinh\beta L + B\cosh\beta L + C\sin\beta L - D\cos\beta L) = 0$$

$$\text{Again Since } \beta^2 \neq 0 \text{ and } A = -C \text{ and } B = -D \text{ and we have } A(\sinh\beta L - \sin\beta L) + B(\cosh\beta L + \cos\beta L) = 0 \text{ ---- (8)}$$

$$A/B = -(\cosh\beta L + \cos\beta L)/(\sinh\beta L - \sin\beta L) \text{ ---- (9)}$$

From equation (7) and (9) we have $A(\cosh\beta L + \cos\beta L) + B(\sinh\beta L + \sin\beta L) = 0$ --- (10)

$$A/B = \frac{(\sinh\beta L + \sin\beta L)}{(\cosh\beta L + \cos\beta L)} = \frac{(\cosh\beta L + \cos\beta L)}{(\sinh\beta L - \sin\beta L)}$$

Which reduces to $\cosh\beta L \cos\beta L + 1 = 0$ ----- (11)

Equation (11) is satisfied by a number of values of βL , corresponding to each normal mode of vibration, which for the first three modes these values are tabulated in the table as

$$(\beta_1 l)^2 = 3.52, (\beta_2 l)^2 = 22.00, (\beta_3 l)^2 = 61.70$$

$$\text{Area } A = 0.0012 \text{ m}^2 \quad \text{Moment of Inertia } I = 1.600E-07 \text{ m}^4$$

$$\text{Density per unit length } (\rho * A) = \rho_L = 7850 \times 0.0012 = 9.4200 \text{ Kg/m}$$

Circular Frequency

$$\omega_{n1} = (\beta_1 l)^2 \sqrt{(EI)/(\rho l^4)} = 2.052E+02 \text{ rad/sec}$$

$$\omega_{n2} = (\beta_2 l)^2 \sqrt{(EI)/(\rho l^4)} = 1.282E+03 \text{ rad/sec}$$

$$\omega_{n3} = (\beta_3 l)^2 \sqrt{(EI)/(\rho l^4)} = 3.596E+03 \text{ rad/sec}$$

Natural Frequencies

$$f_{n1} = \omega_{n1}/2\pi = 32.639 \text{ Hz}$$

$$f_{n2} = \omega_{n2}/2\pi = 203.994 \text{ Hz}$$

$$f_{n3} = \omega_{n3}/2\pi = 572.111 \text{ Hz}$$

Example-2: Determine the first three natural frequencies of a Rectangular Cantilever Beam for the following data Length L = 1 m Breath B = 0.03 m Depth D = 0.04 m Young's Modulus E = 2.00E+11 N/m² Density r = 7850.0 Kg/m³

Solution: For a cantilever beam the boundary conditions are at x= 0 displacement and slope are zero

i.e. $y = 0$ and $dy/dx = 0$ ----- (1)

and at $x = L$ the shear force 'V' and the bending moment 'M' is zero

i.e. $d^2y/dx^2 = 0$ and $d^3y/dx^3 = 0$ ----- (2)

Substituting these boundary conditions into the general solution

$$Y = A\cosh\beta x + B\sinh\beta x + C\cos\beta x + D\sin\beta x \quad \dots (3)$$

we obtain $(y)_{x=0} = A + C = 0$ then $A = -C$ ----- (4)

$$(dy/dx)_{x=0} = \beta(A\sinh\beta x + B\cosh\beta x - C\sin\beta x + D\cos\beta x)_{x=0} = 0$$

$$\beta(B + D) = 0 \text{ since } \beta \neq 0 \text{ then } B = -D \quad \dots (5)$$

$$(d^2y/dx^2)_{x=L} = \beta^2(A\cosh\beta L + B\sinh\beta L - C\cos\beta L - D\sin\beta L) = 0$$

Since $\beta^2 \neq 0$ and $A = -C$ and $B = -D$ and we have

$$A(\cosh\beta L + \cos\beta L) + B(\sinh\beta L + \sin\beta L) = 0 \quad \dots (6)$$

$$A/B = -(\sinh\beta L + \sin\beta L)/(\cosh\beta L + \cos\beta L) \quad \dots (7)$$

$$(d^3y/dx^3)_{x=L} = \beta^3(A\sinh\beta L + B\cosh\beta L + C\sin\beta L - D\cos\beta L) = 0$$

Again Since $\beta^3 \neq 0$ and $A = -C$ and $B = -D$ and we have

$$A(\sinh\beta L - \sin\beta L) + B(\cosh\beta L + \cos\beta L) = 0 \quad \dots (8)$$

$$A/B = -(\cosh\beta L + \cos\beta L)/(\sinh\beta L - \sin\beta L) \quad \dots (9)$$

From equation (7) and (9) we have

$$A(\cosh\beta L + \cos\beta L) + B(\sinh\beta L + \sin\beta L) = 0 \quad \dots (10)$$

$$A/B = \frac{(\sinh\beta L + \sin\beta L)}{(\cosh\beta L + \cos\beta L)} = \frac{(\cosh\beta L + \cos\beta L)}{(\sinh\beta L - \sin\beta L)}$$

Which reduces to $\cosh\beta L \cos\beta L + 1 = 0$ ----- (11)

Equation (11) is satisfied by a number of values of βL , corresponding to each normal mode of vibration, which for the first three modes these values are tabulated in the table as

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Circular Frequency

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Natural Frequencies

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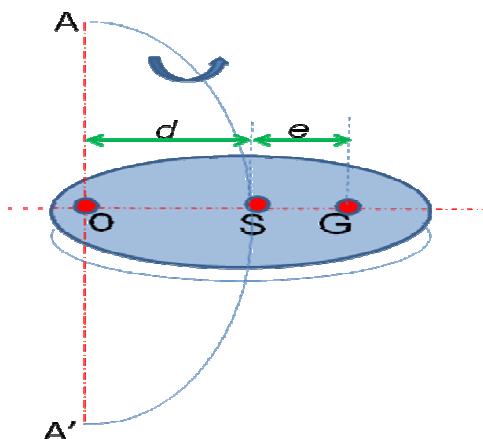
$$f_{n3} = \omega_{n3}/2\pi = 572.111 \text{ Hz}$$

3.6 Rotating Unbalance

- There are many engineering applications in which shafts carry disks (turbines, compressors, electric motors, pumps etc.,)
- These shafts vibrate violently in transverse directions at certain speed of operation known as critical speed of shaft.
- Among the various causes that create critical speeds, the mass unbalance is the most important.
- The unbalance cannot be made zero. There is always some unbalance left in rotors or disks.

- Whirling is defined as the rotation of the plane containing the bent shaft about the bearing axis.
- The whirling of the shaft can take place in the same or opposite direction as that of the rotation of the shaft.
- The whirling speed may or may not be equal to the rotation speed.

Critical speed of a light shaft having a single disc – without damping



Consider a light shaft carrying a single disc at the centre in deflected position. 'S' is the geometric centre through which centre line of shaft passes. 'G' is centre of gravity of disc. 'O' intersection of bearing centre line with the disc 'e' is distance between c.g. 'G' and the geometric centre 'S'. 'd' is displacement of the geometric centre 'S' from the un deflected position 'O'. 'k' is the stiffness of the shaft in the lateral direction.

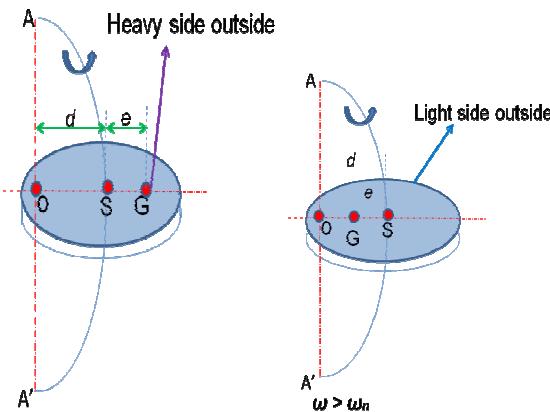
The forces acting on the disc are

- Centrifugal force ' $m\omega^2(d+e)$ ' acts radially outwards at 'G'
- Restoring force ' kd ' acts radially inwards at 'S'
- For equilibrium the two forces must be equal and act along the same line

$$m\omega^2(d+e) = kr \quad \dots \dots 1$$

$$d = \frac{m\omega^2 e}{k - m\omega^2} = \frac{\left(\frac{\omega}{\omega_n}\right)^2 e}{1 - \left(\frac{\omega}{\omega_n}\right)^2} = \frac{r^2 e}{(1 - r^2)} \quad \dots \dots 2$$

- Deflection 'd' tends to infinity when $\omega = \omega_n$
- 'd' is positive below the critical speed, the disc rotates with heavy side outside when $\omega < \omega_n$
- 'd' is negative above the critical speed, the disc rotates with light side outside when $\omega > \omega_n$
- When $\omega >> \omega_n$, ' $d \rightarrow -e$ ', which means point 'G' approaches point 'O' and the disc rotates about its centre of gravity.



Problem 1

A rotor having a mass of 5 kg is mounted midway on a 1 cm shaft supported at the ends by two bearings. The bearing span is 40 cm. Because of certain manufacturing inaccuracies, the centre of gravity of the disc is 0.02 mm away from the geometric centre of the rotor. If the system rotates at 3000 rpm find the amplitude of steady state vibrations and the dynamic force transmitted to the bearings. Assume the rotor to be simply supported. Take $E = 1.96 \times 10^{11}$ N/m².

Solution:

Given: $m = 5$ kg, $d = 1$ cm, $l = 40$ cm, $e = 0.02$ mm,
 $N = 3000$ rpm, $E = 1.96 \times 10^{11}$ N/m², $d = ?$, Simply supported

$$\frac{d}{e} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]}$$

In the above eq. e and ω are known, ω_n has to be found out in order to find d . To find ω_n , stiffness has to be determined.

For a simply supported shaft the deflection at the mid point is given by the following equation.

$$\delta = \frac{mgl^3}{48EI}, \quad k = \frac{mg}{\delta} = \frac{48EI}{l^3} = \frac{48 \times 1.96 \times 10^{11} \times \pi \times 0.01^4}{64 \times 0.4^3} = 72000 \text{ N/m}$$

$$\omega = \frac{2\pi N}{60} = 314.16 \text{ rad/s}, \omega_n = \sqrt{\frac{k}{m}} = 120 \text{ rad/s}, \frac{\omega}{\omega_n} = 2.168$$

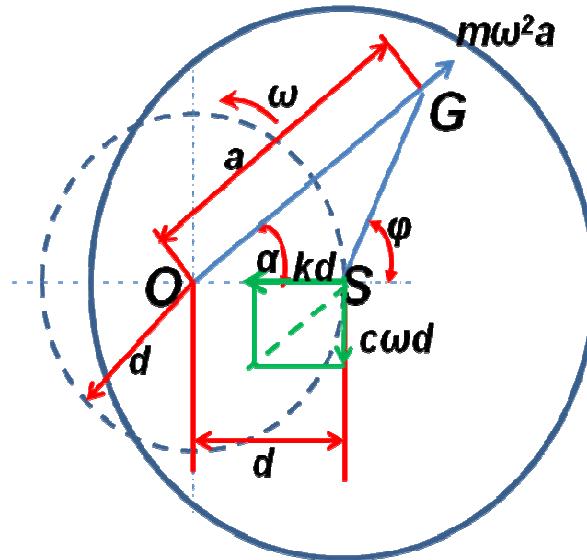
$$\frac{d}{e} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{1 - \left(\frac{\omega}{\omega_n}\right)^2} = \frac{2.168^2}{(1 - 2.168^2)} = -1.1558, d = -1.1558 \times 0.02 = -0.023 \text{ mm}$$

- Sign implies displacement is out of phase with centrifugal force

$$\text{Dynamic on bearing} = kd = 1.656 \text{ N}$$

$$\text{load on each bearing} = 0.828 \text{ N}$$

Critical speed of light shaft having a single disc – with damping



O' is intersection of bearing centre line with the disc

'S' is geometric centre of the disk

'G' is centre of gravity of disc

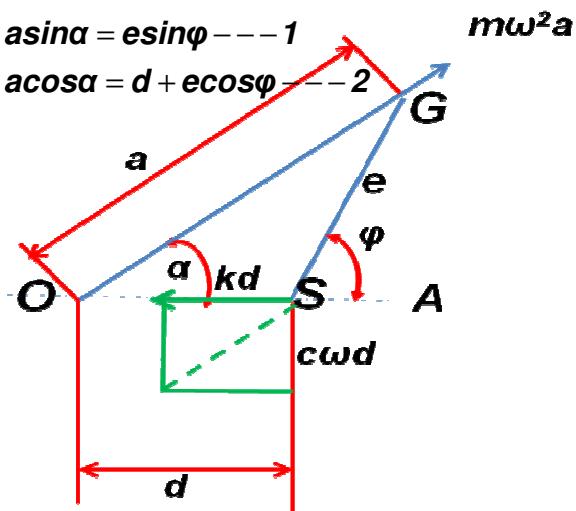
The forces acting on the disc are

1. Centrifugal force $m\omega^2 a$ at G along OG produced
2. Restoring force kd at S along SO
3. Damping force $c\omega d$ at S in a direction opposite to velocity at S
4. The points O, S and G no longer lie on straight line

Let,

$$OG = a, SG = e, OS = d, \angle GOS = \alpha, \angle GSA = \varphi$$

From the geometry



$$\sum X = 0,$$

$$-kd + m\omega^2 a = 0 \quad \dots \dots 3$$

$$\sum Y = 0,$$

$$-c\omega d + m\omega^2 asina = 0 \quad \dots \dots 4$$

Eliminating a and α from equations 3 and 4 with the help of equations 1 and 2

$$-kd + m\omega^2(d + ecos\varphi) = 0 \quad \dots \dots 5$$

$$-c\omega d + m\omega^2(esin\varphi) = 0 \quad \dots \dots 6$$

$$\text{From eq.5 } \cos\varphi = \frac{kd - m\omega^2 d}{m\omega^2 e} \quad \dots \dots 7$$

$$\text{From eq.6 } \sin\varphi = \frac{cw d}{m\omega^2 e} \quad \dots \dots 8$$

Squaring and adding the equations 7 and 8

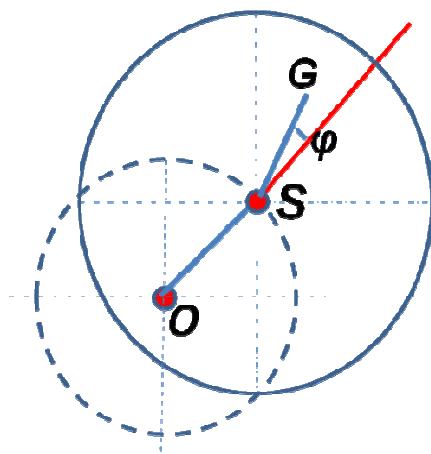
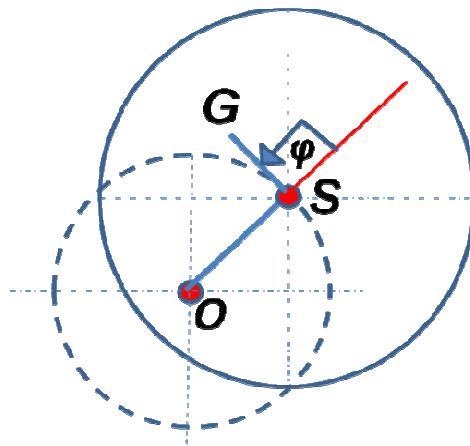
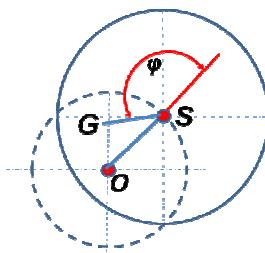
$$\frac{d}{e} = \frac{m\omega^2}{\sqrt{(k - m\omega^2)^2 + (cw)^2}}$$

$$\frac{d}{e} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{1 - \left(\frac{\omega}{\omega_n}\right)^2 + \left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}} = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} \quad \dots \dots 9$$

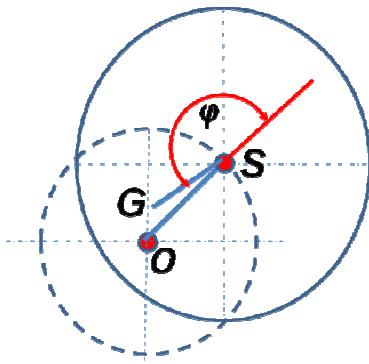
$$\varphi = \tan^{-1} \frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_n}\right)^2} = \tan^{-1} \frac{2\xi r}{1 - r^2} \quad \dots \dots 10$$

From the curves or the equation it can be seen that (Discussions of speeds above and below critical speed)

- (a) $\Phi \sim 0$ when $\omega \ll \omega_n$ (Heavy side out)
- (b) $0 < \varphi < 90^\circ$ when $\omega < \omega_n$ (Heavy side out)

(c) $\Phi = 90^\circ$ when $\omega = \omega_n$ (d) $90^\circ < \Phi < 180^\circ$ when $\omega > \omega_n$ (Light side out)

(e) $\Phi \sim 180^\circ$ when $\omega >> \omega_n$, (Light side out, disc rotates about its centre of gravity)



Critical speed of shaft may be placed above or below the operating speed.

1. If the unit is to operate at high speeds, that do not vary widely, the critical speed may be below the operating speed, and the shaft is then said to be flexible.
- 2 In bringing the shaft up to the operating speed, the critical speed must be passed through. If this is done rapidly, resonance conditions do not have chance to build up.
3. If the operating speed is low or if speeds must vary through wide ranges, the critical speed is placed over the operating speed and the shaft is said to be rigid or stiff.
4. Generally the running speed must at least 20% away from the critical speed.

Problem 1

A disk of mass 4 kg is mounted midway between the bearings

Which may be assumed to be simply supported. The bearing span is 48 cm. The steel shaft is 9 mm in diameter. The c.g. of the disc is displaced 3 mm from the geometric centre. The equivalent viscous damping at the centre of

the disc-shaft may be taken as 49 N-s/m. If the shaft rotates at 760 rpm, find the maximum stress in the shaft and compare it with the dead load stress in the shaft when the shaft is horizontal. Also find the power required to drive the shaft at this speed. Take $E = 1.96 \times 10^{11}$ N/m².

Solution:

Given: $m = 4\text{kg}$, $l = 48\text{ cm}$, $e = 3\text{ mm}$, $c = 49\text{ N-s/m}$, $N = 760\text{ rpm}$
 $E = 1.96 \times 10^{11}\text{ N/m}^2$, $s_{max} = ?$. Dia = 9 mm, $E = 1.96 \times 10^{11}\text{ N/m}^2$.

$$k = \frac{48EI}{l^3} = 27400\text{N/m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = 82.8\text{rad/s}$$

$$\omega = \frac{2\pi\pi}{60} = 79.5\text{rad/s}$$

$$\xi = \frac{c}{2\sqrt{km}} = 0.074$$

$$\frac{d}{e} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + \left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}}$$

$$d = 0.017\text{ m (solution)}$$

The dynamic load on the bearings is equal to centrifugal force of the disc which is equal to the vector sum of spring force and damping force.

$$F_{dy} = \sqrt{(kd)^2 + (cw d)^2} = d\sqrt{k^2 + (cw)^2} = 470\text{N}$$

The total maximum load on the shaft under dynamic conditions is the sum of above load and the dead load.

$$F_{max} = 470 + (4 \times 9.81) = 509.2\text{N}$$

The load under static conditions is

$$F_s = 4 \times 9.81 = 39.2\text{N}$$

The maximum stress, due to load acting at the centre of a simply supported shaft is

$$s = \frac{M \times dia}{I \times 2} = \frac{Fl \times dia \times 64}{4 \times 2 \times \pi \times dia^4} = 168 \times 10^4 F$$

The total maximum stress under dynamic conditions

$$s_{max} = 168 \times 10^4 \times F_{max} = 8.55 \times 10^8 N/m^2$$

Maximum stress under dead load

$$s_{max} = 168 \times 10^4 \times 39.2 = 6.59 \times 10^7 N/m^2$$

$$\text{Damping torque} = T = (c\omega d)d = 1.125 N-m$$

$$\text{Power} = \frac{2\pi NT}{60} = 90W$$

3.7 Vibration Measurement

Theory of Vibration measuring instruments

It is well known that the dynamic forces in a vibratory system depend on the displacement, velocity and acceleration components of a system:

Spring force \propto displacement

Damping force \propto velocity

Inertia force \propto acceleration

Therefore, in vibration analysis of a mechanical system, it is required to measure the displacement, velocity and acceleration components of a system. An instrument, which is used to measure these parameters, is referred as vibration measuring instrument or seismic instrument. A simple model of seismic instrument is shown in Fig.5.1. The major requirement of a seismic instrument is to indicate an output, which represents an input such as the displacement amplitude, velocity or acceleration of a vibrating system as close as possible.

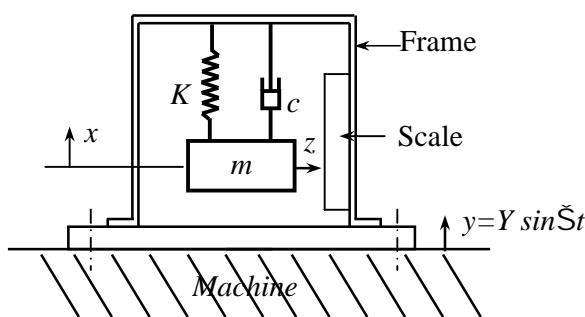


Fig.5.1 Seismic instrument

m-seismic mass

c-damping coefficient of seismic unit

K-stiffness of spring used in seismic unit

x-absolute displacement of seismic mass

y-base excitation (assume SHM)

$z = (x - y)$ displacement of seismic mass relative to frame

To study the response of the system shown in Fig.5.1, we shall obtain the equation of motion of seismic mass:

$$m\ddot{x} + c(\dot{x} - \dot{y}) + K(x - y) = 0 \quad (1)$$

$$m\ddot{z} + c\dot{z} + Kz = -m\ddot{y} \quad (2)$$

Considering base excitation to be SHM:

$$y(t) = Y \sin \tilde{\omega}t \quad (3)$$

$$m\ddot{z} + c\dot{z} + Kz = m\tilde{\omega}^2 Y \sin \tilde{\omega}t \quad (4)$$

The above equation represents a equation of motion of a forced vibration with

$$m\tilde{\omega}^2 Y = F$$

Solution of governing differential equation is:

$$z(t) = z_c(t) + z_p(t)$$

z_c is the complimentary solution, which nullifies after some time. The total solution is thus, only steady state solution z_p

Let, the steady state solution of Eqn.(4) is:

$$z(t) = Z \sin(\tilde{\omega}t - \phi) \quad (5)$$

Eqn.(5) has to satisfy Eqn.(4). Substitute Eqn.(5) in (4) and draw force polygon as already studied in forced vibration. The amplitude of steady state vibration is:

$$Z = \frac{m\tilde{\omega}^2 Y}{\sqrt{(K - m\tilde{\omega}^2)^2 + (c\tilde{\omega})^2}} \quad (6)$$

devide above equation by K

$$Z = \frac{r^2 Y}{\sqrt{(1 - r^2)^2 + (2' r)^2}} \quad (7)$$

substitute eqn.(5.7) in eqn.(5.5)

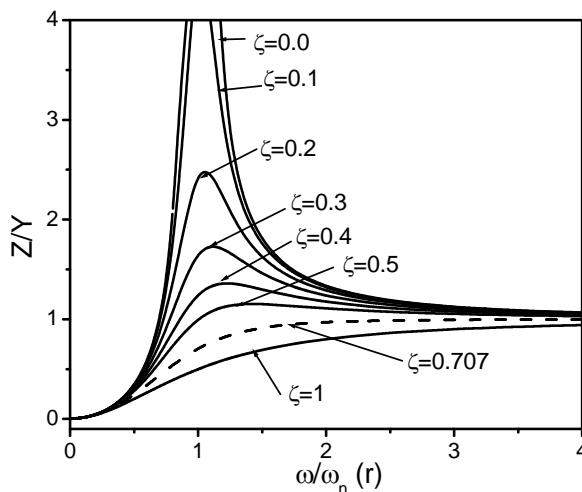
$$z(t) = \frac{r^2 Y}{\sqrt{(1 - r^2)^2 + (2' r)^2}} \sin(\tilde{\omega}t - \phi) \quad (8)$$

the phase angle is:

$$\phi = \tan^{-1} \left(\frac{c\tilde{\omega}}{K - m\tilde{\omega}^2} \right) \quad (9)$$

$$\phi = \tan^{-1} \left(\frac{2' r}{1 - r^2} \right) \quad (10)$$

The variation of non-dimensional amplitude (Z/Y) with respect to frequency ratio (r) is shown in Fig.5.2

**Fig.5.2** Plot of equation 7**Displacement measuring instrument (Vibrometer)**

It is an instrument used to measure the displacement of a vibrating system.

In Eqn.(8) if,

$$\frac{r^2}{\sqrt{(1-r^2)^2 + (2'r)^2}} \approx 1 \quad (11)$$

then,

$$z(t) = Y \sin(\tilde{\omega}t - \phi) \quad (12)$$

Eqn.(11) is the condition for vibrometer.

Acceleration measuring instrument (Accelerometer)

It is an instrument used to measure the acceleration of a vibrating system. The response of the seismic mass is given by Eqn.(8). Double differentiating the Eqn.(8), we get.

$$-z(t)\tilde{\omega}^2 = \frac{r^2}{\sqrt{(1-r^2)^2 + (2'r)^2}} (-Y\tilde{\omega}^2 \sin(\tilde{\omega}t - \phi)) \quad (13)$$

$$-z(t)\tilde{\omega}_n^2 = \frac{1}{\sqrt{(1-r^2)^2 + (2'r)^2}} (-Y\tilde{\omega}^2 \sin(\tilde{\omega}t - \phi)) \quad (14)$$

In above equation if

$$\frac{1}{\sqrt{(1-r^2)^2 + (2'r)^2}} \approx 1 \quad (15)$$

Then,

$$-z(t)\tilde{\omega}_n^2 = -Y\tilde{\omega}^2 \sin(\tilde{\omega}t - \phi) \quad (16)$$

we have acceleration component of base excitation:

Eqn.(15) is the condition for accelerometer.

Numerical problems**Problem-1**

A seismic instrument is mounted on a machine running at 1000 rpm. The natural frequency of the seismic instrument is 20 rad/sec. the instrument records relative amplitude of 0.5 mm. Compute the displacement, velocity and acceleration of the machine. Neglect the damping in seismic instrument.

Given data

$$\omega_n = 20 \text{ rad/s}, \quad \omega = ?$$

Speed of the machine (N) = 1000 rpm

$$\begin{aligned} \omega &= \frac{2\pi N}{60} = \frac{2\pi (1000)}{60} \\ &= 104.72 \text{ rad/s} \end{aligned}$$

Frequency ratio

$$r = \frac{\omega}{\omega_n} = \frac{104.72}{20} = 5.23$$

For seismic instrument

$$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

For the given system damping is neglected

$$Y = \frac{Z}{1.042} = \frac{0.5}{1.042} = 0.48 \text{ mm}$$

$$\frac{Z}{Y} = \frac{5.23^2}{1 - 5.23^2} = 1.042$$

Displacement of the machine:

$$Y = \frac{Z}{1.042} = \frac{0.5}{1.042} = 0.48 \text{ mm}$$

Velocity of the machine:

$$\dot{Y} = (104.72) 0.48 = 50.26 \text{ mm/s}$$

Acceleration of the machine:

$$\ddot{Y} = (104.72)^2 0.48 = 5263.81 \text{ mm/s}^2$$

Problem-2

A seismic instrument has natural frequency of 6 Hz. What is the lowest frequency beyond which the amplitude can be measured within 2% error. Neglect damping

Given data

$$\omega_n = 6 \text{ Hz}, \zeta = 0 \text{ and error} = 2\%$$

Damping is neglected for given system

$$\frac{Z}{Y} = \frac{r^2}{1 - r^2}$$

$$\text{Error} = \frac{Z - Y}{Y} = 0.02$$

$$Z = Y + 0.02 \quad Y = 1.02 \quad Y$$

$$\frac{Z}{Y} = 1.02 = \frac{r^2}{1 - r^2}$$

$$1.02 - 1.02r^2 = r^2$$

$$r = 0.7034$$

The lowest frequency beyond which the amplitude can be measured within 2% error is:

$$\ddot{S} = r \cdot \ddot{S}_n$$

$$\ddot{S} = (0.7034) 6$$

$$\ddot{S} = 4.22 \text{ Hz}$$

Summary

Seismic instruments are used to measure the displacement, velocity and acceleration components of a vibratory system. Basic theory of Seismic instruments is based on forced vibration considering the vibratory system under base excitation. A single Seismic instrument can be sued as vibrometer, velometrre and accelerometer using suitable calibration.