Introduction to Deep Learning

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Setting

Goal: Learn a mapping from input to output

Computer program

$$y(x, \mathbf{w}) = f(\phi(x)^T \mathbf{w})$$

Object label [Grumpy cat]

House price [400.000]

Regression and classification

- We transform the input x into some features $\phi(x)$
- <u>Approach 1:</u> We choose the mapping $\phi(x)$ ourselves
- <u>Approach 2:</u> We let $\phi(x)$ be a **neural network** and learn the features from the data

Workflow

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2. Choose the **loss** function.

- <u>Example:</u> $\text{Loss}(\theta) = \frac{1}{N}\sum_{i=1}^{N} \|f_\theta(\mathbf{x}_i) - \mathbf{y}_i\|_2^2$

Workflow

1. Design the neural network **architecture**

2. Choose the **loss** function.

   - Example: $\text{Loss}(\theta) = \dfrac{1}{N}\sum_{i=1}^{N} ||f_\theta(\mathbf{x}_i) - \mathbf{y}_i||_2^2$

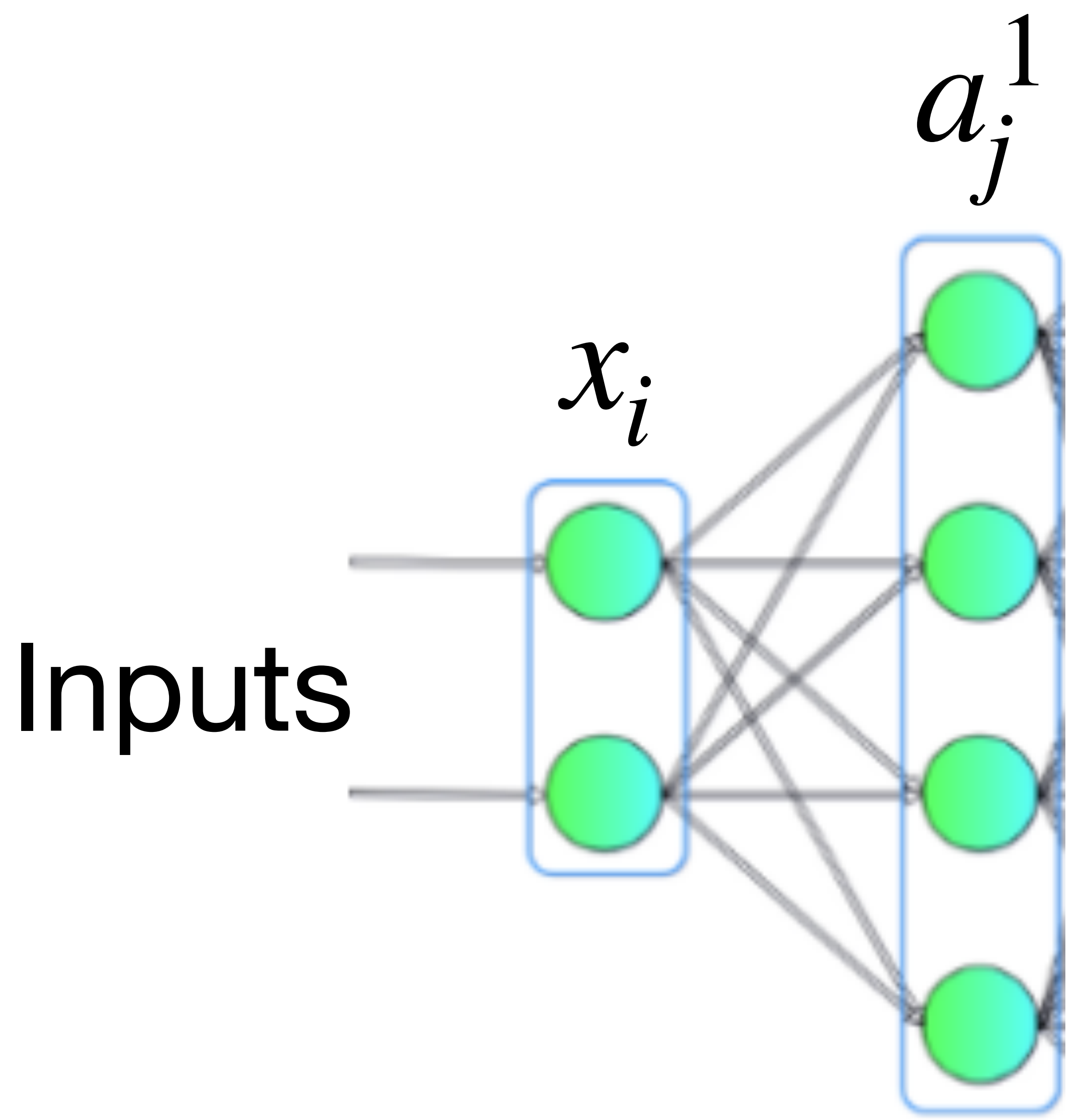
3. Train the neural network using a **gradient-based optimisation** algorithm (example: stochastic gradient descent)

Basic neural network model

- Suppose the input $\mathbf{x} = [x_1 \quad \cdots \quad x_D]^\top$

- First layer: construct M linear combinations of the input variables $x_1, \ldots, x_D$ of the form:

$$a_j^1 = \sum_{i=1}^{D} w_{ji}^1 x_i + b_j^1 \text{ for } 1 \leq j \leq M$$



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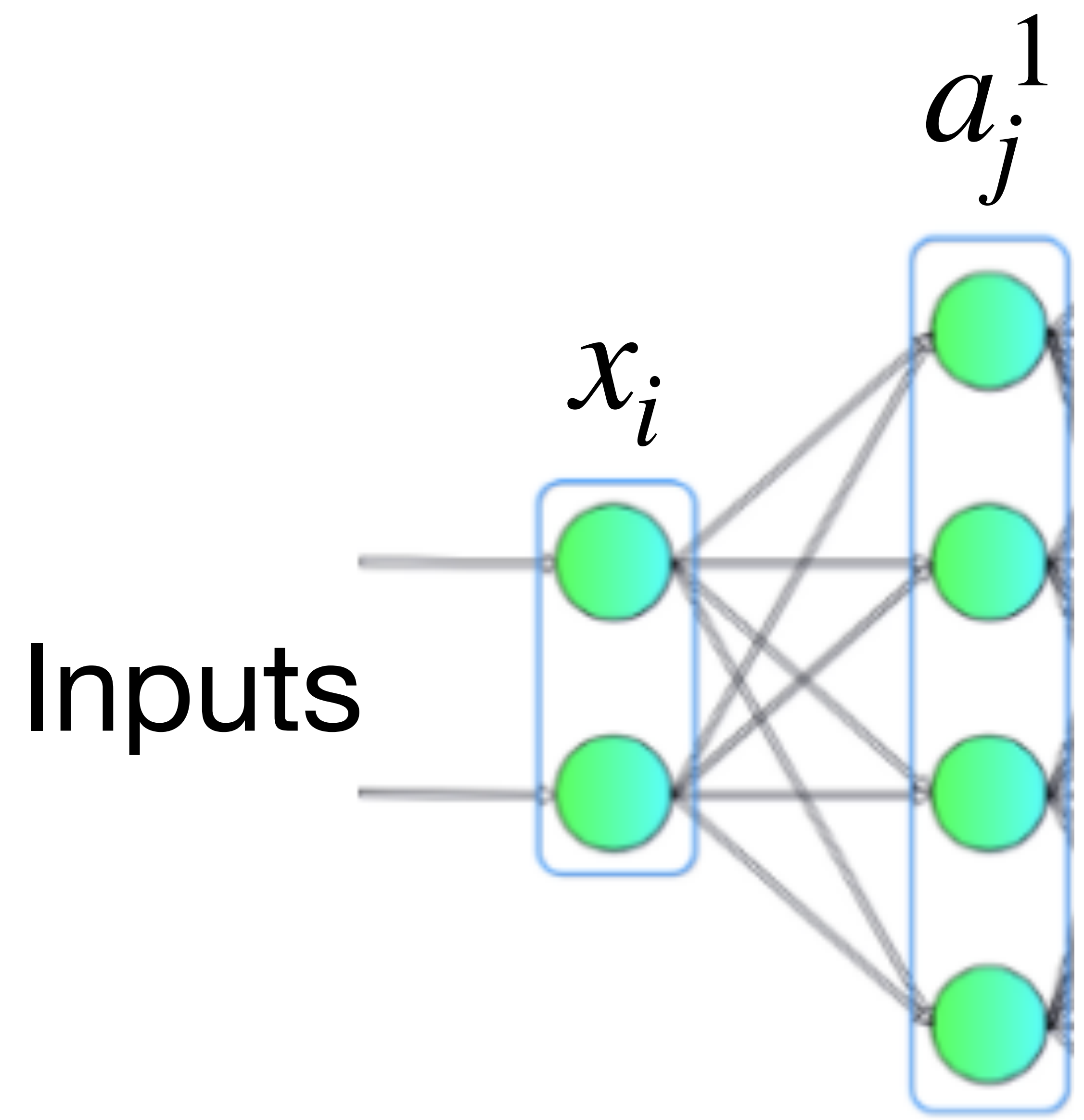
Weights



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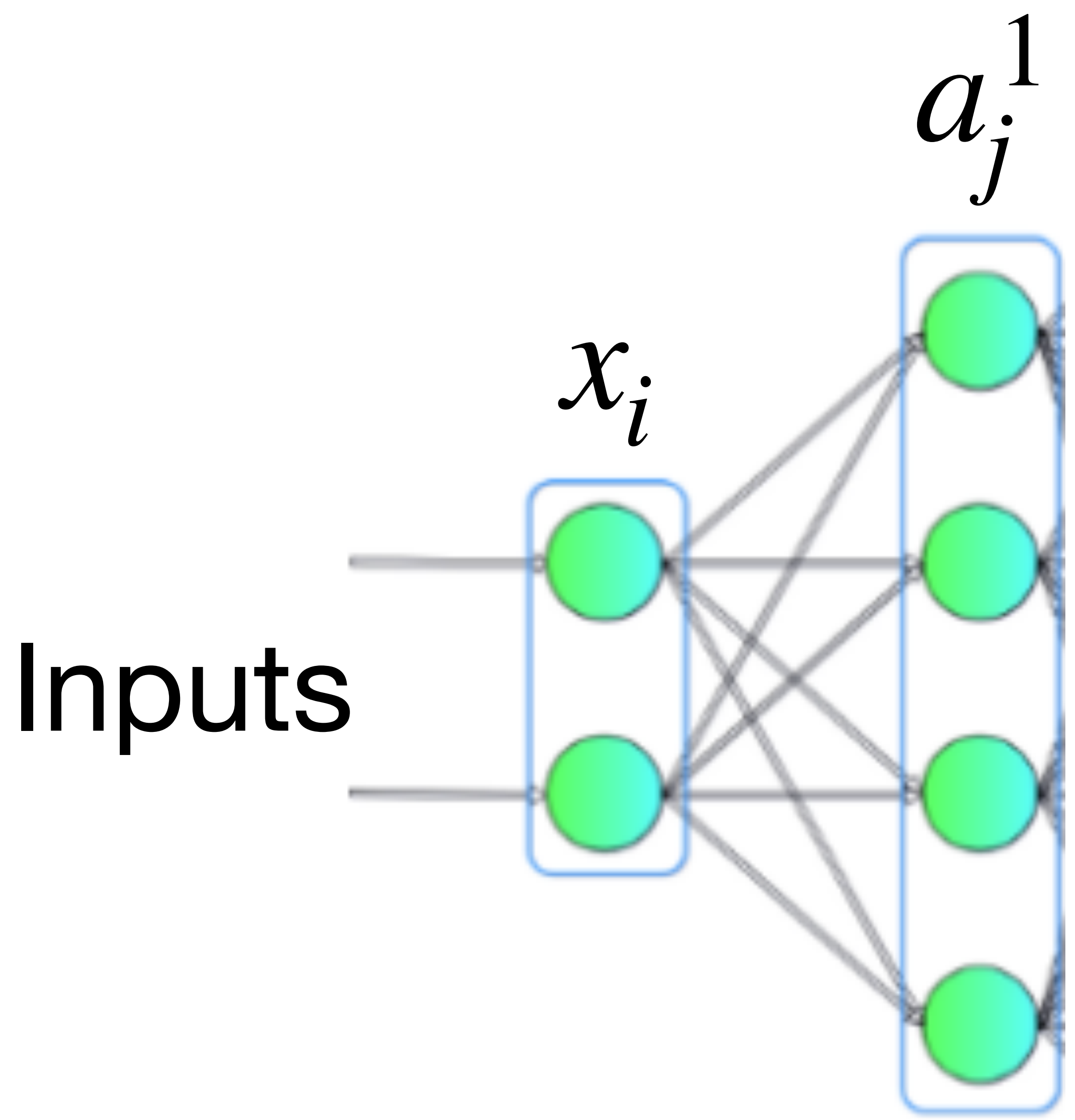
Bias



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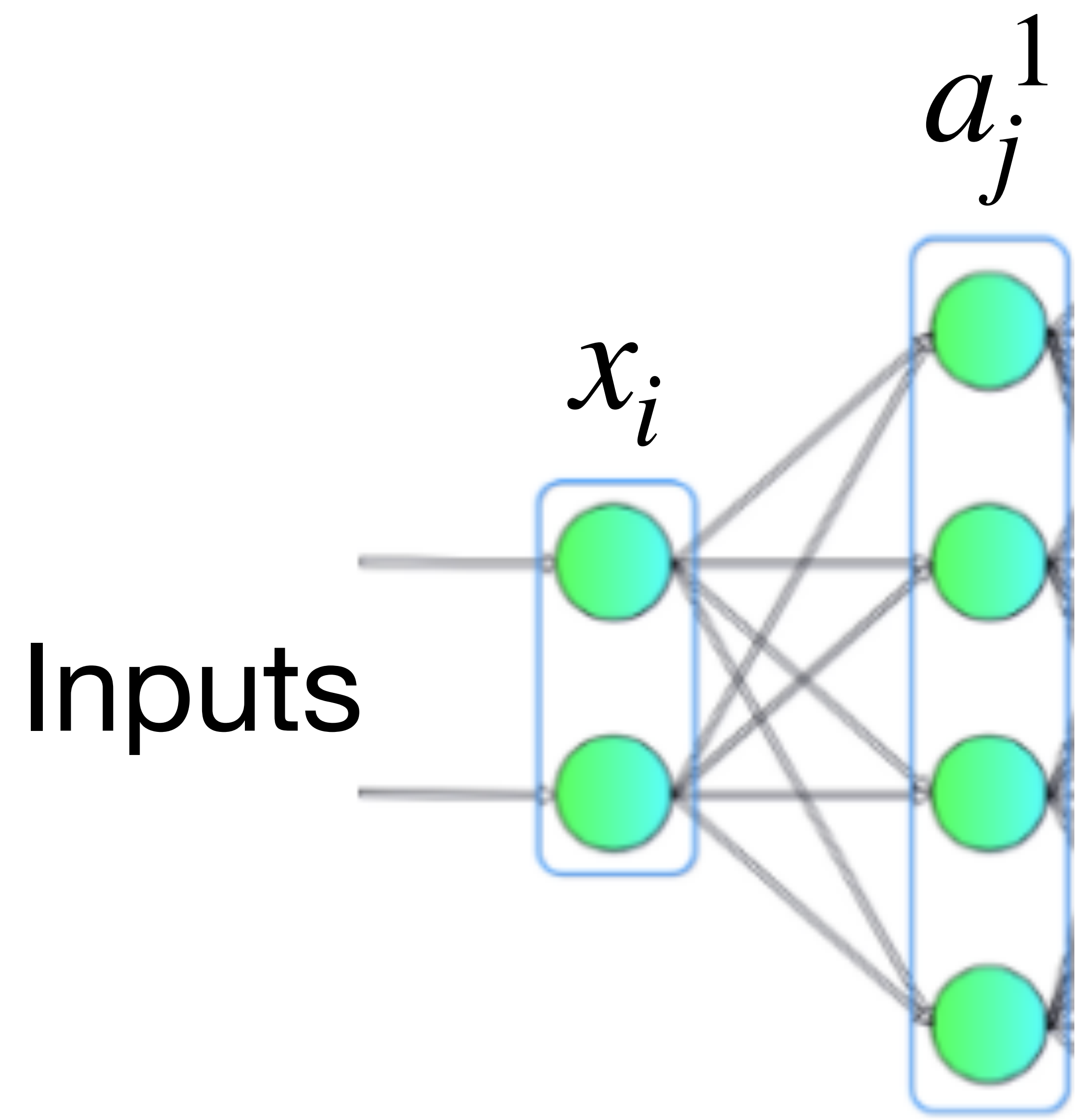
Pre-activations



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- Activation: We transform the a's using a **nonlinear activation function** σ

$$\boxed{h_j^1} = \sigma(a_j^1)$$

Hidden units

Basic neural network model

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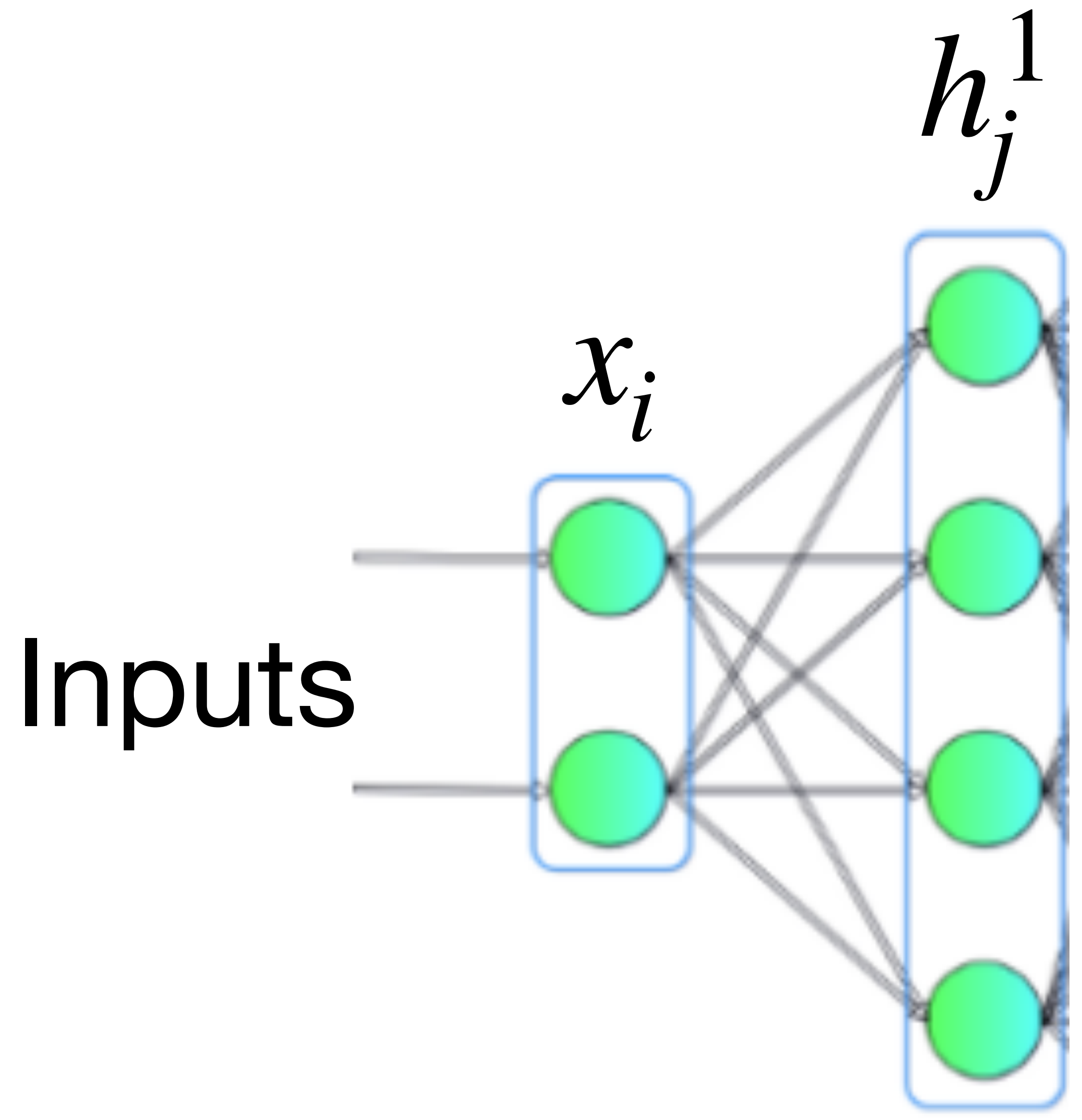

Basic neural network model

- Suppose the input $\mathbf{x} = [x_1 \quad \cdots \quad x_D]^\top$
- Second layer: construct K linear combinations of the hidden variables $h_1^1, \ldots, h_M^1$ of the form:

Number of outputs

$$a_j^2 = \sum_{i=1}^{M} w_{ji}^2 h_i^1 + b_j^2 \text{ for } 1 \leq j \leq K$$

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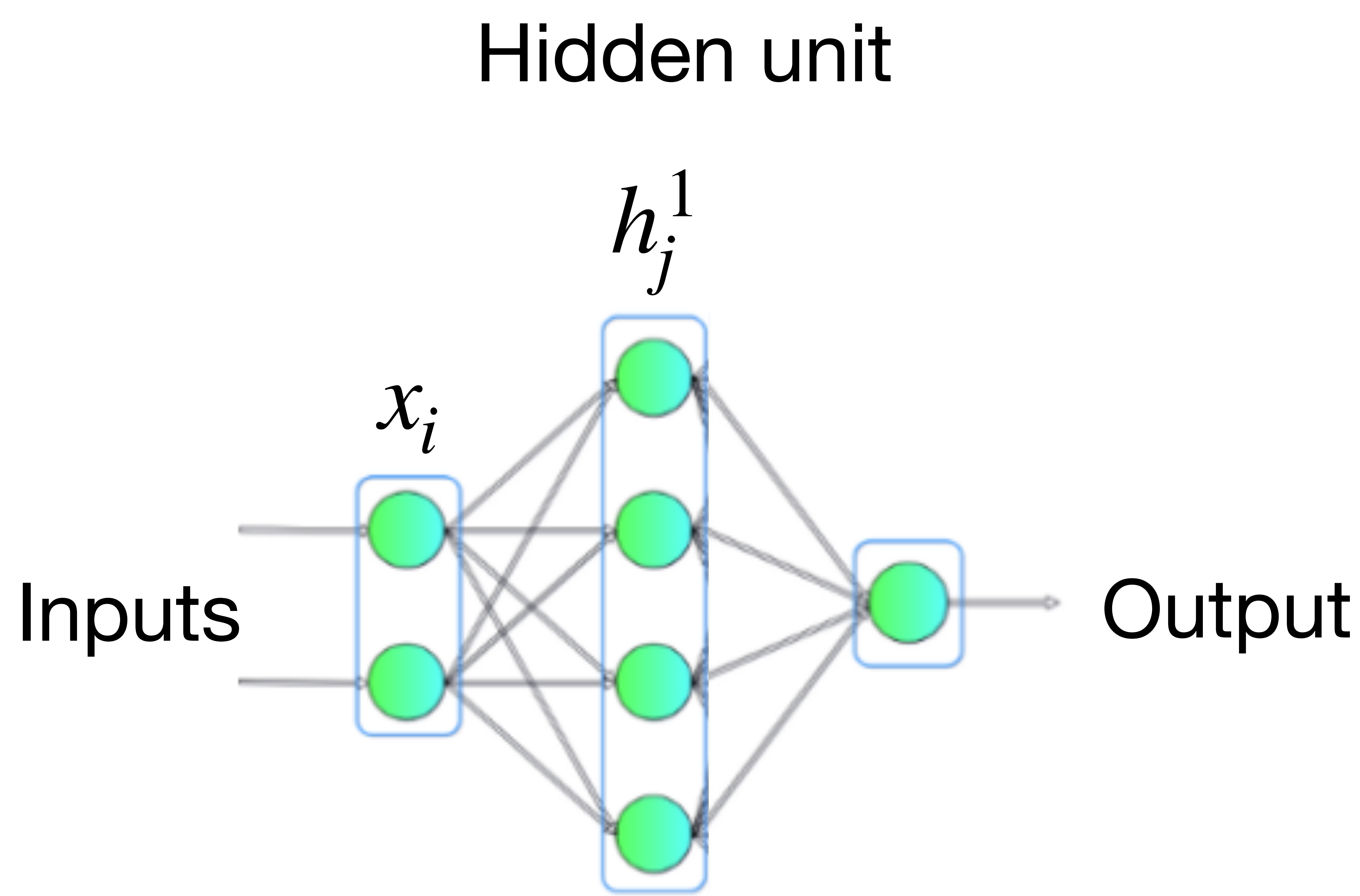
- <u>Final activation:</u> We transform the a's using a **nonlinear activation function** σ

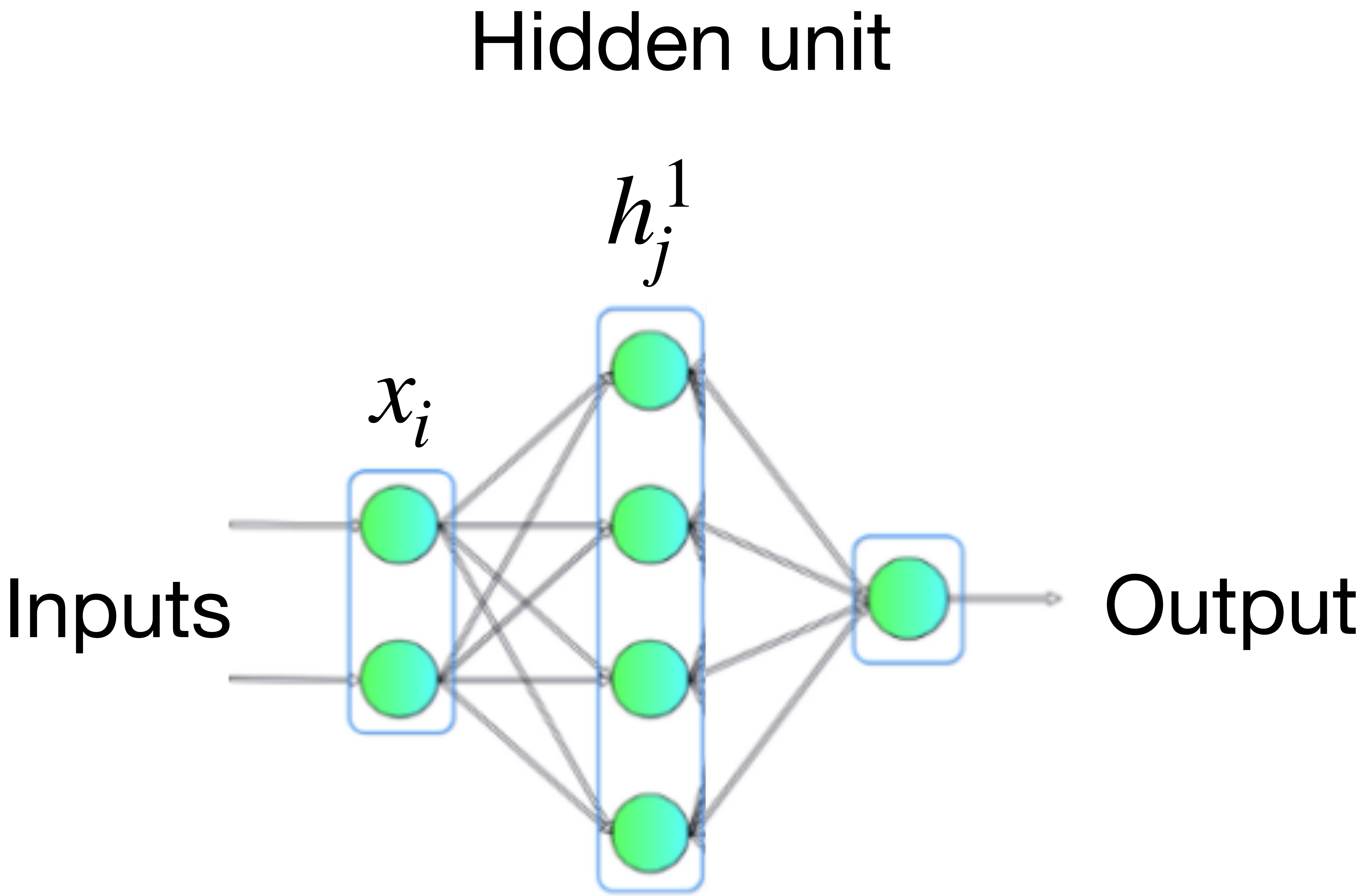
$$y_j^1 = \sigma(a_j^1)$$

Basic neural network model

Visualisation



Basic neural network model
Visualisation



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Basic neural network model

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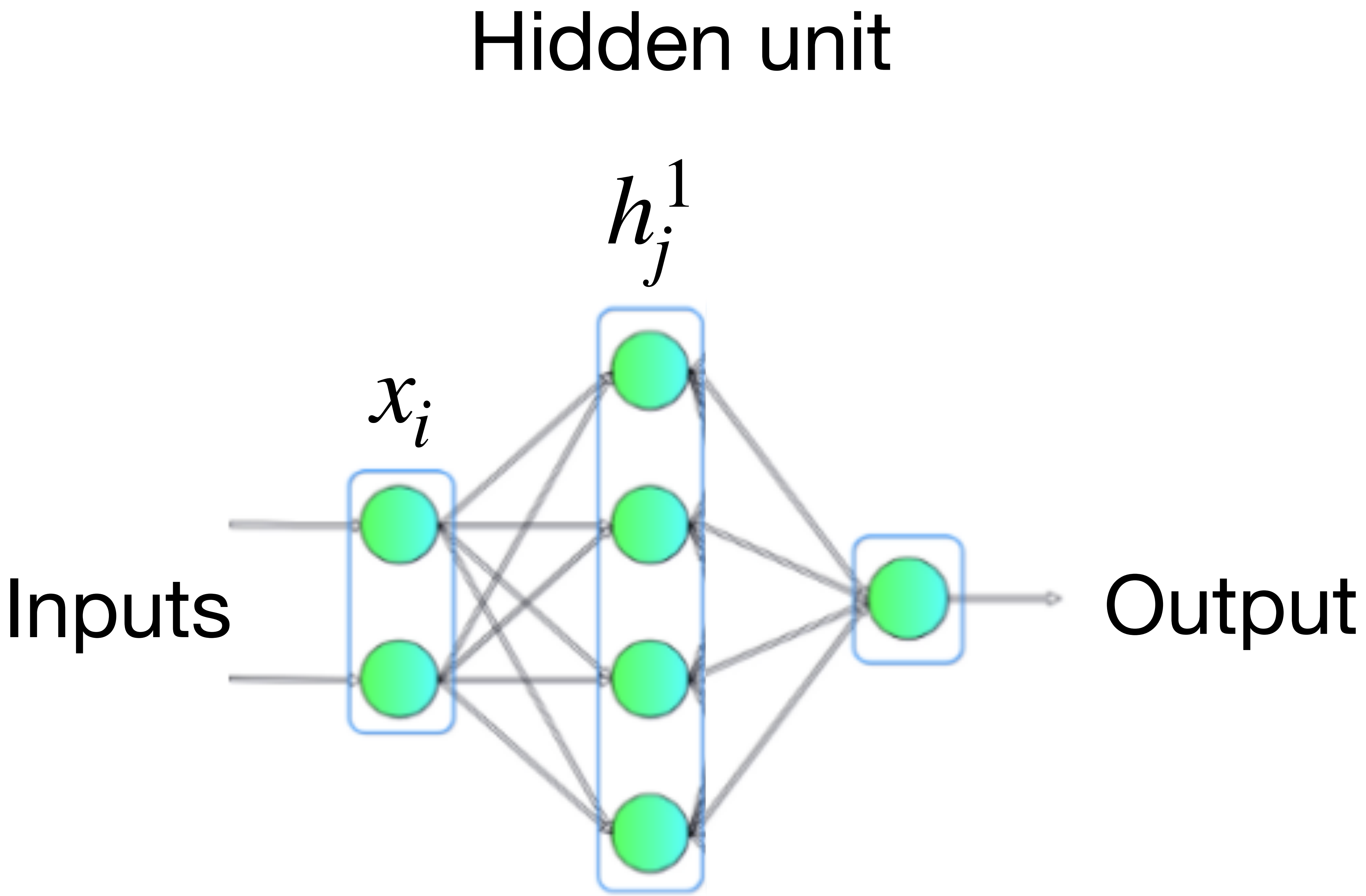

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Basic neural network model

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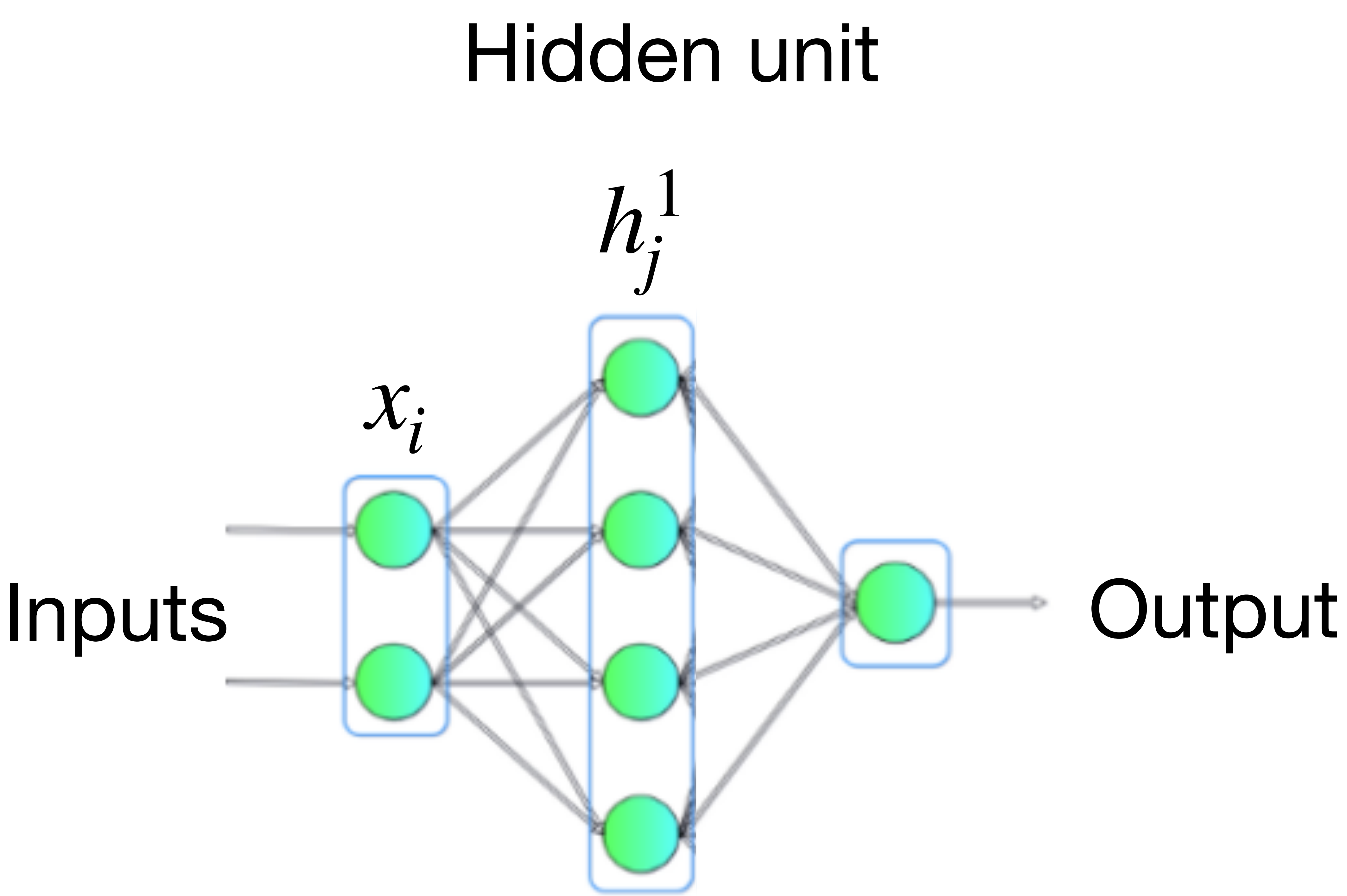

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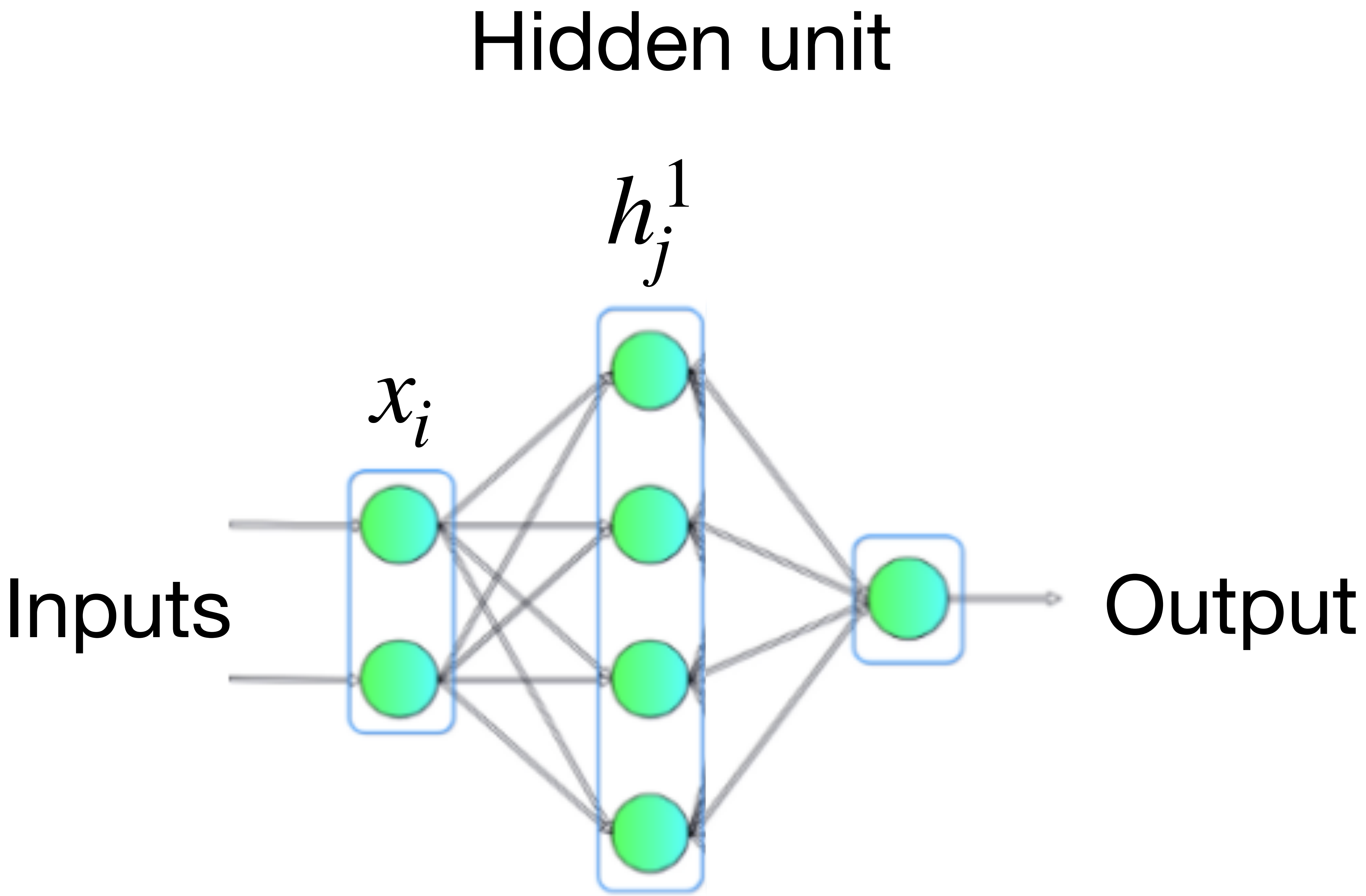
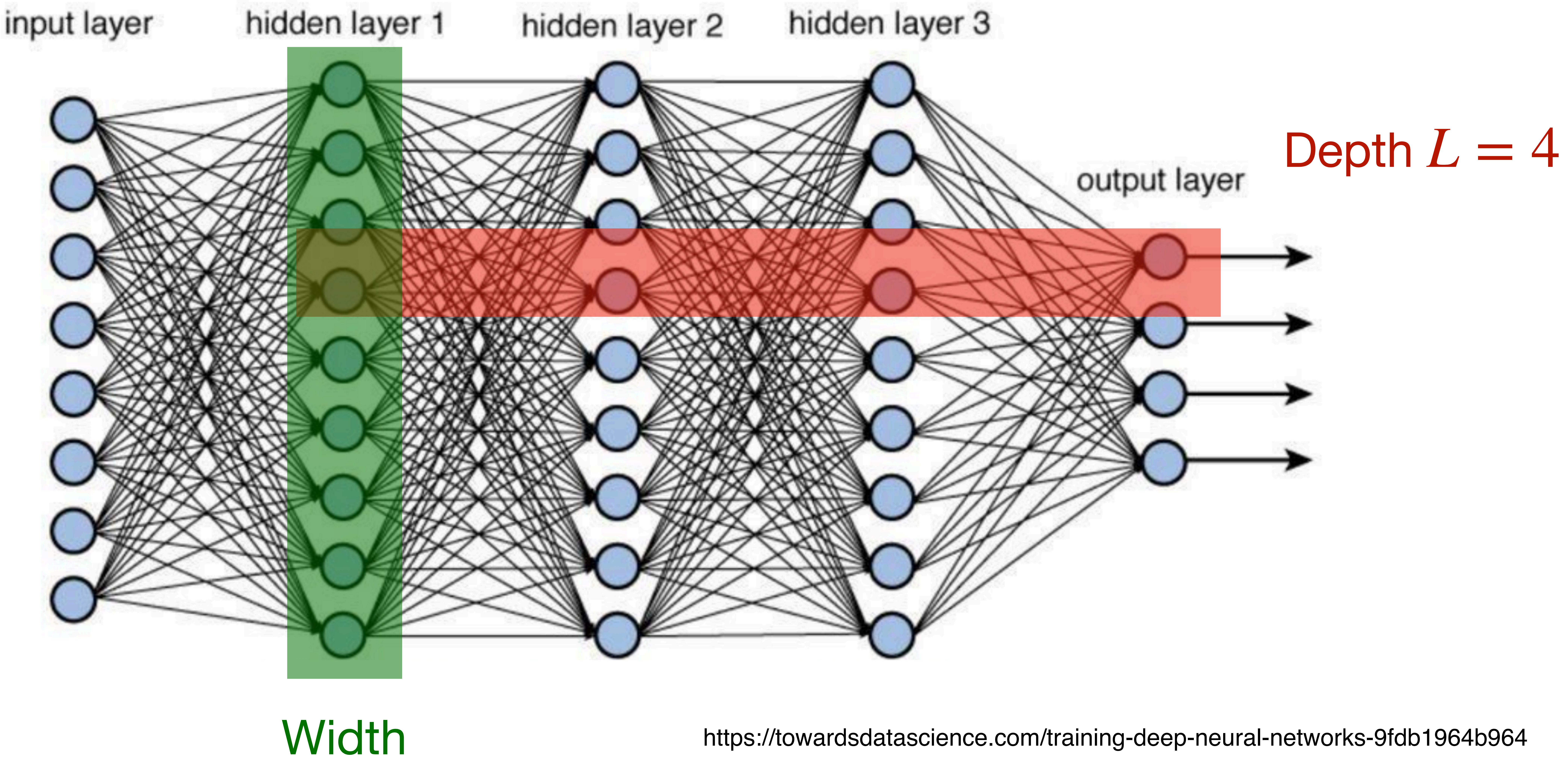


1. Linear transformation by pre-multiplying with a weight matrix W^1 and adding a bias vector b^1
2. Pass through nonlinear activation function
3. Linear transformation with W^2 and b^2
4. Pass through nonlinear activation function (if needed)

Deep neural network



https://towardsdatascience.com/training-deep-neural-networks-9fdb1964b964

PyTorch

- Machine learning library with Python interface to implement neural networks

- Open source software developed by Meta https://pytorch.org/

- All operations are performed on objects torch.Tensor that are multidimensional matrices

A neural network in PyTorch

- A network with L hidden layers in torch:

```
def forward(self, input):
    m = torch.nn.Linear(dim, nr_hidden)
    x = torch.nn.flatten(input)
    x = m(x)
    for layer in range(L):
        m = torch.nn.Linear(nr_hidden, nr_hidden)
        x = m(x)
        x = torch.nn.functional.relu(x)
    m = torch.nn.linear(nr_hidden, nr_output)
    output = m(x)
    return output
```

Training neural networks

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- We train the neural networks by optimising a certain loss or cost function, such as the mean-squared error:

$$\min_{\theta \in \mathbb{R}^d} L(\theta) = \min_{\theta \in \mathbb{R}^d} \frac{1}{m} \sum_{i=1}^{m} \|\mathbf{y}_i - f(\mathbf{x}_i; \theta)\|_2^2$$

Optimisation algorithm

- We use a gradient descent algorithm to minimise the cost function:

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- Update the parameters of the networks as

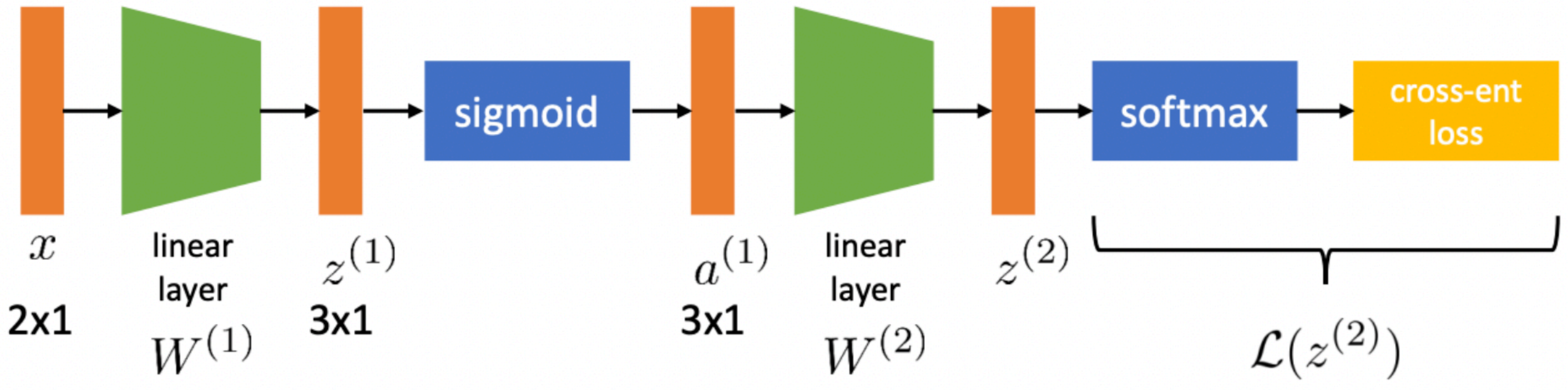
$$\theta^{(k+1)} = \theta^{(k)} - \eta \nabla_\theta L(\theta^{(k)})$$

$\eta > 0$ is called the learning rate

Backpropagation algorithm

- How do we compute the gradient $\nabla_\theta L(\theta)$?

- We use the chain rule (or backpropagation algorithm):



$$L(\theta) = L([W^1, W^2])$$

$$\frac{d\mathscr{L}}{dW^1} = \frac{d\mathscr{L}}{dz^2}\frac{dz^2}{da^1}\frac{da^1}{dz^1}\frac{dz^1}{dW^1}$$

$$\frac{d\mathscr{L}}{dW^2} = \frac{d\mathscr{L}}{dz^2}\frac{dz^2}{dW^2}$$

Backpropagation algorithm

In PyTorch:

```
# Set gradients to zero
optimizer.zero_grad()
```

Build the model $f(\mathbf{x};\theta)$ and remember operations (similar to pyadjoint)

```
# Forward pass
output = model(x)
```

Compute the cost function $L(\theta)$

```
# Calculate loss
loss = criterion(output, y)
```

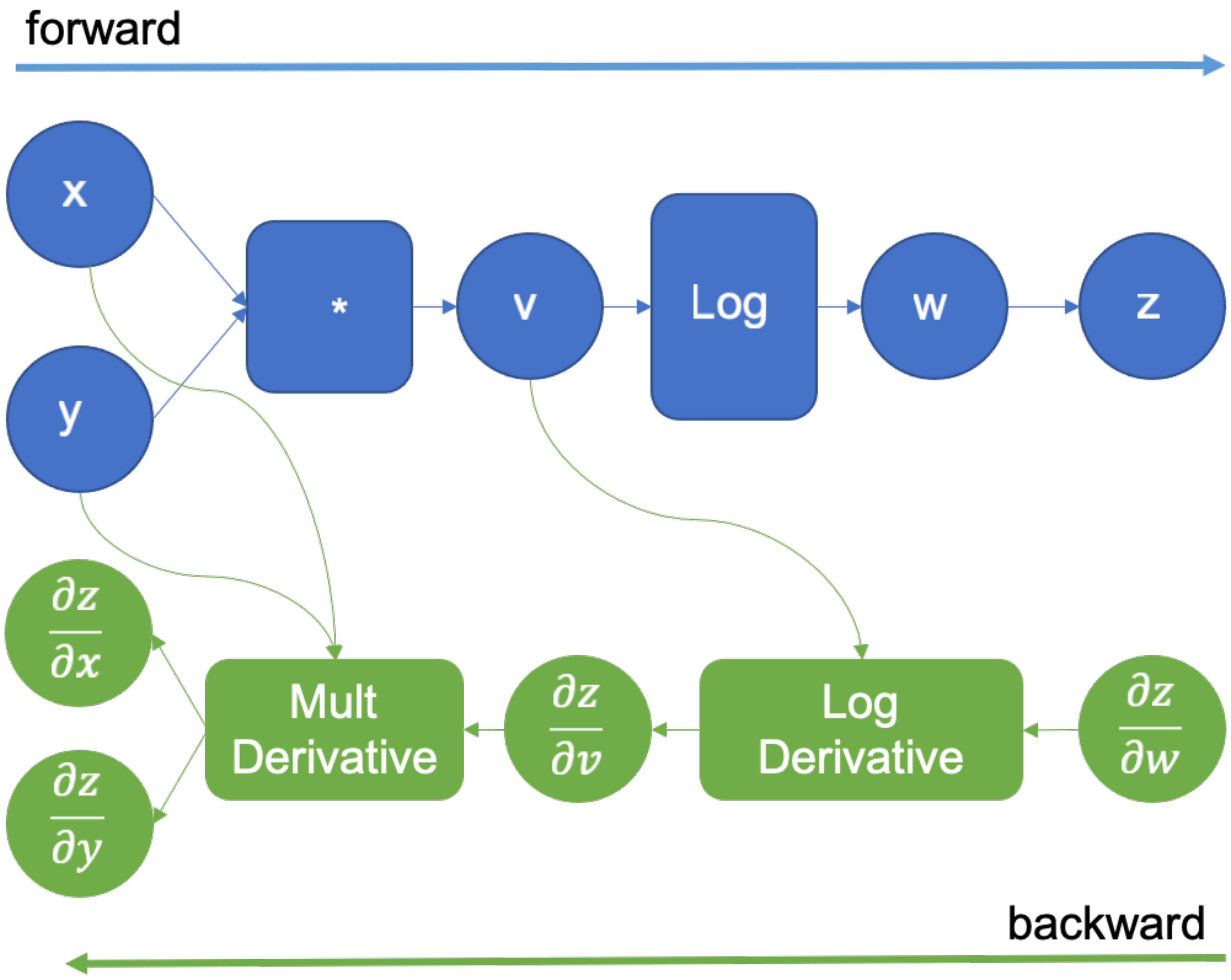
Calculate the gradient $\nabla_\theta L(\theta)$

```
# Backward pass
loss.backward()
```

Update the parameters:
$\theta \leftarrow \theta - \eta \nabla_\theta L(\theta)$

```
# Update weights
optimizer.step()
```

Computational graphs



$$z = \log(x \times y)$$

z.backward()

$$\text{x.grad} = \frac{\partial z}{\partial x}$$

$$\text{y.grad} = \frac{\partial z}{\partial y}$$

https://pytorch.org/blog/overview-of-pytorch-autograd-engine/