Networks and Complexity

Solution 13-9

This is an example solution from the forthcoming book Networks and Complexity. Find more exercises at https://github.com/NC-Book/NCB

Ex 13.9: The Zombie Apocalypse ... will be weird [3]

During the zombie apocalypse each zombie apocalypse each zombie is killed within a day, but not before infecting 3 people with the zombie virus. Within a day the infected people turn to zombies.

a) Model the system as a discrete-time map where the Z_t and I_t are the numbers of zombies and infected on day t.

Solution

Using the information from the question, we can model the system by the discrete time map

$$\begin{aligned} Z_{t+1} &= I_t && (1) \\ I_{t+1} &= 3Z_t && (2) \end{aligned}$$

b) Use the method from the chapter to compute Z_t, I_t for all $t > 0$.

Solution

We define the vector

$$\boldsymbol{X_t} = \begin{pmatrix} Z_t \\ I_t \end{pmatrix}. \qquad (3)$$

We can now write the system in matrix

$$\boldsymbol{X_{t+1} = \mathbf{J}X_t,} \qquad (4)$$

where

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}. \qquad (5)$$

To compute the eigenvectors of $\mathbf{J}$ we solve

$$\lambda \begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} \qquad (6)$$

which we can also write as

$$\begin{aligned} \lambda &= x && (7) \\ \lambda x &= 3 && (8) \end{aligned}$$

Substituting the first line into the second one gives us

$$x^2 = 3 \qquad (9)$$

and hence

$$x = \pm\sqrt{3}. \tag{10}$$

Using $\lambda = x$ we have now two eigenvalues

$$\lambda_1 = \sqrt{3}, \qquad \lambda_2 = -\sqrt{3} \tag{11}$$

and the corresponding eigenvectors

$$\boldsymbol{v_1} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}, \qquad \boldsymbol{v_1} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}. \tag{12}$$

Therefore the general solution is

$$\begin{pmatrix} Z_t \\ I_t \end{pmatrix} = c_1(\sqrt{3})^t \begin{pmatrix} 1 \\ sqrt3 \end{pmatrix} + c_2(-\sqrt{3})^t \begin{pmatrix} 1 \\ sqrt3 \end{pmatrix} \begin{pmatrix} 1 \\ -sqrt3 \end{pmatrix} \tag{13}$$

c) Using your solution, consider the case where the outbreak starts with one infected on day $t = 0$ and compute the number of zombies on days 1,2,20, and 21.

<u>*Solution*</u>
To find the particular solution we need to determine c_1 and c_2 by solving the system on day 0,

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ sqrt3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ \sqrt{-3} \end{pmatrix}. \tag{14}$$

The upper line reads

$$0 = c_1 + c_2 \tag{15}$$

hence

$$c_1 = -c_2. \tag{16}$$

Substituting this into the lower line yields

$$1 = c_1\sqrt{3} + c_1\sqrt{3} = 2c_1\sqrt{3} \tag{17}$$

hence

$$c_1 = \frac{1}{2\sqrt{3}} \qquad c_2 = -\frac{1}{2\sqrt{3}} \tag{18}$$

We now have the particular solution

$$\begin{pmatrix} Z_t \\ I_t \end{pmatrix} = \frac{1}{2\sqrt{3}}(\sqrt{3})^t \begin{pmatrix} 1 \\ sqrt3 \end{pmatrix} - \frac{1}{2\sqrt{3}}(-\sqrt{3})^t \begin{pmatrix} 1 \\ sqrt3 \end{pmatrix} \tag{19}$$

We can write the line for the Zombies as

$$Z_t = \frac{1}{2\sqrt{3}}(\sqrt{3})^t - \frac{1}{2\sqrt{3}}(-\sqrt{3})^t \tag{20}$$

by pulling a factor $(-1)^t$ from the second term we can further simplify this to

$$Z_t = \frac{1}{2\sqrt{3}}\left((\sqrt{3})^t - (-1)^t(\sqrt{3})^t\right) \tag{21}$$

and by cancelling a factor of $\sqrt{3}$ and collecting the terms we arrive at

$$Z_t = \frac{1-(-1)^t}{2}\sqrt{3})^{t-1}. \tag{22}$$

Form this form it is now apparent that the number of zombies will be 0 on all even days. On odd days first factor becomes 1 and we have

$$Z_t = \sqrt{3})^{t-1} \qquad \text{[On odd days]} \tag{23}$$

Hence the number of zombies is zero on day 2 and day 20. On day 1 it is $\sqrt{3}^{0} = 1$ and on day 21 it is $\sqrt{3}^{2}0 = 3^{1}0$ which is 59049. Of course it will be zero again on day 22.

So why is the Zombie apocalypse so weird? We will return to this question below.

d) Bonus: Make a model of the Zombie Apocalypse using a differential equation system. Let $Z(t)$ and $I(t)$ be the number of zombies and infected at time t. Use the initial condition $Z(0) = 0$, $I(0) = 1$. Parameterize your model such one unit of time is: the average life expectancy of zombies, the average time it takes an infected person to turn into a zombie, and the average time in which one zombie infects 3 people (assuming the zombie doesn't die). This may require careful tuning of the parameters. Then solve the model.