Exercise Sheet 21: The Vibe of a City

This is an exercise sheet from the forthcoming book Networks and Complexity.
Find more exercises and solutions at https://github.com/NC-Book/NCB

Ex 21.1: Adjacency and Laplacian [1]

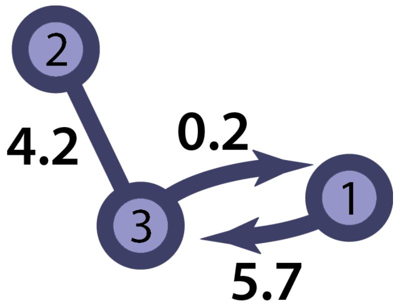
Construct the adjacency and Laplacian matrices for a three node chain (o-o-o) and a 4-cycle (a ring of four nodes).

Ex 21.2: Spectral methods in the wild [3]

A supermarket chain approaches you. Their goal is to identify different groups in their customer base. They have a large dataset that containing 20 million customers, for each customer the dataset describes which products they have bought in the past. Describe in bullet points how you would approach this question.

Ex 21.3: Weighted Laplacian [3]

Consider a number of walkers diffusion on the following network:



where the arrows are directed links. The numbers indicate the rates at which walkers at the respective source use the link.

a) Write the weighted adjacency and Laplacian matrices for this system. (Recall that A_{ij} is the weight of the link to i from j)
b) Write a system of differential equations that governs x_i, the proportion of walkers in node i. Compute the Jacobian matrix $\mathbf{J}$ and verify $\mathbf{J} = -\mathbf{L}$.
c) Find the steady state distribution of walkers with a method of your choice.

Ex 21.4: Discrete-time diffusion [4]

In the lecture we introduced the Laplacian matrix that describes continuous-time diffusion. However, the diffusion map typically uses a normalized Laplacian matrix that describes discrete-time diffusion. In this exercise we derive the normalized Laplacian from a discrete-time diffusion model. Throughout let $x_i(t)$ be the number of walkers that are at node i in timestep t.

a) To get an intuition consider the a three node chain (1)-(2)-(3). Where $n \gg 1$ walkers start in node 1. That is $x_1(0) = n$, $x_2(0) = x_3(0) = 0$. In each timestep every walker uses one of the links available to them and makes exactly one step. Use this information to work out x_1, x_2, x_3 for the first few steps. Then spot the pattern and state how the walkers will be distributed at time $t = 1001$.

b) Now consider a network consisting of three nodes that form a triangle. Write a general formulas for $x_1(t+1)$, $x_2(t+1)$, $x_3(t+1)$, given $x_1(t)$, $x_2(t)$, and $x_3(t)$.

c) Next we consider a network with N nodes and a topology described by an adjacency matrix $\mathbf{A}$. We denote the degree of a node i as k_i. For this general case write an equation for $x_i(t+1)$ given $x_1(t), \ldots, x_N(t)$.

d) We now define $\boldsymbol{x}(t) = (x_1(t), \ldots, x_N(t))^{\mathrm{T}}$ and write the dynamics in matrix form

$$\boldsymbol{x}(t+1) = \mathbf{M}\boldsymbol{x}(t) \tag{25}$$

where $\mathbf{L}$ is the discrete-time Laplacian matrix. From this equation, identify a formula for the elements L_{ij}. (Bonus: also write $\mathbf{M}$ as a function of $\mathbf{A}$ and the degree matrix $\mathbf{D}$ as a matrix equation.)

e) Consider again the triangle of three nodes from (b). Write the matrix $\mathbf{M}$ for this network and compute its eigenvalues and eigenvectors. Then, use an eigenvector decomposition to find the general solution for the discrete time diffusion problem on this network.

f) Also solve the dynamics for the three-node-chain from (a) and show that we find the same result.