Networks and Complexity

Solution 12-10

This is an example solution from the forthcoming book Networks and Complexity. Find more exercises at https://github.com/NC-Book/NCB

Ex 12.10: Change of variables [3]

Consider a system of differential equations where

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x \end{aligned}$$

We want to transform these equations into polar coordinates, where

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \phi &= \arctan(y/x) \end{aligned}$$

Find (closed) differential equation for ϕ and r. Simplify where possible. Then solve the differential equations.

Solution

Computing the time derivative of r we find

$$\begin{aligned} \dot{r} &= \frac{1}{2}\frac{1}{\sqrt{x^2+y^2}}(2x\dot{x} + 2y\dot{y}) \quad (1) \\ &= \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2+y^2}} \quad (2) \end{aligned}$$

Now substituting the equations for x and y we find

$$\dot{r} = \frac{xy - yx}{\sqrt{x^2+y^2}} = 0 \quad (3)$$

That is nice. Now consider ϕ.

$$\begin{aligned} \dot{\phi} &= \frac{\mathrm{d}}{\mathrm{d}t}\arctan(y/x) \quad (4) \\ &= \frac{1}{1 + y/x^2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{y}{x}\right) \quad (5) \\ &= \frac{1}{1 + y/x^2}\left(\frac{\dot{y}}{x} - \frac{\dot{x}y}{x^2}\right) \quad (6) \\ &= \frac{1}{1 + y^2/x^2}\left(-1 - \frac{y^2}{x^2}\right) \quad (7) \\ &= -\frac{1 + y^2/x^2}{1 + y^2/x^2} \quad (8) \\ &= -1 \quad (9) \end{aligned}$$

So it turns out that r is actually a conserved quantity. It is thus a parameter of our system and not a variable. We only need to integrate the equation for ϕ which we can do by direct integration

$$\phi(t) = \int -1 \mathrm{d}t = -t + C \tag{10}$$

To translate back into our original coordinates we need the inverse coordinate transformation, which is

$$\begin{aligned} x &= r \sin\phi \qquad (11) \\ y &= r \cos\phi. \qquad (12) \end{aligned}$$

Substituting our solution for ϕ yields

$$\begin{aligned} x(t) &= r \sin(C - t) \qquad (13) \\ y(t) &= r \cos(C - t) \qquad (14) \end{aligned}$$

so the system is going in a circle. We have found a periodic solution.