Networks and Complexity

Solution 10-5

This is an example solution from the forthcoming book Networks and Complexity.
Find more exercises at https://github.com/NC-Book/NCB

Ex 10.5: Abstract Targeted Attack [2]

Consider a ER random graph of N nodes with mean degree z. An attack removes all nodes of degree 0, half the nodes of degree 1, a quarter of the nodes of degree 2 and so on. Compute the size of the network N_{a}, and the mean excess degree q_{a} after the attack.

Solution

The attack is described by

$$r_k = p_k/2^k \tag{1}$$

Substituting the ER degree distribution

$$p_k = \frac{z^k \mathrm{e}^{-z}}{k!} \tag{2}$$

we can write the attack generating function as

$$\begin{aligned} R &= \sum r_k x^k & (3) \\ &= \sum \frac{(zx/2)^k \mathrm{e}^{-z}}{k!} & (4) \\ &= \mathrm{e}^{-z} \sum \frac{(zx/2)^k}{k!} & (5) \\ &= \mathrm{e}^{-z} \mathrm{e}^{zx/2} = \mathrm{e}^{z(x/2-1)} & (6) \end{aligned}$$

Since we know R we can now compute the removed proportion of nodes

$$r = R(1) = \mathrm{e}^{-z/2} \tag{7}$$

and the surviving proportion of nodes

$$c = 1 - r = 1 - \mathrm{e}^{-z/2} \tag{8}$$

which gives us the number of nodes after the attack

$$N_{\mathrm{a}} = (1 - \mathrm{e}^{-z/2})N \tag{9}$$

for example if the network had a mean degree of zero before the attack, the size after the attack is zero, which makes sense.

To find q_{a} we recall that the mean excess degree of the ER network before the attack is $q = z$. Using a result from the lecture we can compute the reduction of the mean excess degree by the attack as

$$
\begin{aligned}
\delta &= \frac{R''(1)}{z} && (10)\\
&= \frac{1}{z}\frac{\partial^2}{\partial x^2}\mathrm{e}^{z(x/2-1)}\bigg|_{x=1} && (11)\\
&= \frac{1}{z}\frac{\partial}{\partial x}\frac{z}{2}\mathrm{e}^{z(x/2-1)}\bigg|_{x=1} && (12)\\
&= \frac{1}{z}\frac{z^2}{4}\mathrm{e}^{z(x/2-1)}\bigg|_{x=1} && (13)\\
&= \frac{z^2}{4}\mathrm{e}^{z(x/2-1)}\bigg|_{x=1} && (14)\\
&= \frac{z}{4}\mathrm{e}^{-z/2} && (15)
\end{aligned}
$$

Hence the mean excess degree after then attack is

$$
q_{\mathrm{a}} = q - \delta = z - \frac{z}{4}\mathrm{e}^{-z/2} = z(1 - \mathrm{e}^{-z/2}/4). \tag{16}
$$