Networks and Complexity

Solution 22-3

This is an example solution from the forthcoming book Networks and Complexity.
Find more exercises at https://github.com/NC-Book/NCB

Ex 22.3: Product Formula [3]

Consider the Kronecker product

$$\mathbf{J} = \mathbf{A} \otimes \mathbf{B} \tag{1}$$

where $\mathbf{A}$ and $\mathbf{B}$ are matrices. Find a formula that relates the eigenvalues of $\mathbf{J}$ to the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$. (This is very similar, but simpler than the derivation in the lecture. Formulate an Ansatz for the eigenvector $\boldsymbol{v}$, then show that it works and find the eigenvalue on the way).

Solution

We use the Ansatz

$$\boldsymbol{v} = \boldsymbol{a} \otimes \boldsymbol{b} \tag{2}$$

We now show that (under suitable assumptions) this is an eigenvector of $\mathbf{J}$. We write

$$\begin{aligned}
\mathbf{J}\boldsymbol{v} &= (\mathbf{A} \otimes \mathbf{B})(\boldsymbol{a} \otimes \boldsymbol{b}) && (3)\\
&= \mathbf{A}\boldsymbol{a} \otimes \mathbf{B}\boldsymbol{b} && (4)\\
&= \alpha\boldsymbol{a} \otimes \beta\boldsymbol{b} && (5)\\
&= \alpha\beta(\boldsymbol{a} \otimes \boldsymbol{b}) && (6)\\
&= \alpha\beta\boldsymbol{v} && (7)
\end{aligned}$$

Where we had to assume that $\boldsymbol{a}$ is an eigenvector of $\mathbf{A}$ with eigenvalue α (i.e. $\boldsymbol{a}\mathbf{A} = \alpha\boldsymbol{a}$) and similarly $\boldsymbol{b}\mathbf{B} = \beta\boldsymbol{b}$.

The calculation above shows that if $\mathbf{J}$ is a Kronecker product of 2 matrices and we know the eigenvectors of these matrices, then the Kronecker product of two a pair of eigenvectors is an eigenvector of $\mathbf{J}$ and the corresponding eigenvalue is the product of the two eigenvalues.

We can see that the number of eigenvalues we can construct in this way is

$$\dim(\mathbf{A}) \cdot \dim(\mathbf{B}) = \dim(\mathbf{J}) \tag{8}$$

hence we can find all eigenvalues and eigenvectors of $\mathbf{J}$ in this way.