Networks and Complexity

Exercise Sheet 9: Massively Parallel Maths

This is an exercise sheet from the forthcoming book Networks and Complexity. Find more exercises and solutions at https://github.com/NC-Book/NCB

Ex 9.1: Simple Taylor expansion [2]

Taylor expand the function

$$f(x) = \frac{1}{x+1}$$

up to linear order around $x = 0$.

Ex 9.2: Coordinate shift [2]

Consider the function $f(x) = \sqrt{x}$. We are going to approximate the function around the point $x^* = 1$.

a) Introduce a new variable y, such that the point of interest ($x^* = 1$) is at $y^* = 0$.
b) Write f in terms of y, call the resulting function $h(y)$.
c) Approximate $h(y)$ by a function of the form $g(y) = c_0 + c_1 y$ around $y^* = 0$.
d) Write the approximation in terms of the original variable x.
e) Apply the same steps to approximate f up to quadratic order around $x^* = 2$.

Ex 9.3: Degree generating functions [2]

Derive the generating functions G for the degree distribution of the following networks:

a) A regular graph where every node has degree 3.
b) A network where half the nodes have degree 10 and the other half has degree 20.
c) An Erdős-Rényi random graph with mean degree z (find a nice form for the result).
d) A network with the degree distribution

$$p_k = \frac{\mathrm{e}^{-5}5^k}{2k!} + \frac{1}{2}\delta_{5,k}$$

Ex 9.4: Ways and means [2]

For the networks from the previous exercise. Use generating functions to...

a) show that the degree distributions are properly normalized.
b) compute the mean degree.
c) construct the generating function for the excess degree distribution and use it to compute the mean excess degree.

Ex 9.5: A quick test [2]

Consider a network where every node has degree 1. In this network it is not very hard to guess what the distribution of component sizes looks like.

a) Construct the generating function G of the degree distribution and use it to compute the generating function Q of the excess degree distribution.
b) Use the equation from the lecture, $Y = xQ(Y)$, to determine the generating function Y that generates the number of nodes in a branch.

c) Use $C = xG(Y)$ to find the function C that generates the component size distribution and explain the the result.

Ex 9.6: Tangled cables [3]

I have a drawer in which I keep charging cables for various devices. In my experience half the cables are tangled up with one other cable, a quarter of the cables is tangled up with two other cables, whereas the final quarter are not tangled up with any other cable. When one pulls on one cable, one usually ends up pulling a big tangle of cables out of the drawer. Translate this situation into a network problem. Use generating function methods to compute C, the generating function of component sizes and use it to find the expected number of cables in the tangle that one extracts from the drawer by pulling on one cable.

Ex 9.7: Expansion formula [3]

Apply the same reasoning that we used to find the Taylor coefficients c_0 and c_1 for to derive

$$c_k = \frac{1}{k!} f^{(k)}(0)$$

where $f^{(k)}$ is the kth derivative of f. If this looks too complicated. Try to convince yourself the equation is correct and then check out the solution.

Ex 9.8: Series expansions [3]

Find the series expansions of the following functions

$$\text{a) } \mathrm{e}^x \qquad \text{b) } 1/(1+x) \qquad \text{c) } 1/(1-x)$$

Ex 9.9: Solving series [3]

Use generating functions to find the solutions of $\sum c_k$, where $c_0 = 1$ and

$$\text{a) } c_{k+1} = \frac{c_k}{2} \qquad \text{b) } c_{k+1} = -c_k/3 + 1$$

Hint: You might want to use the result of Ex. 9.6

Ex 9.10: Excess distribution formula [3]

Starting from $q_k = (k+1)p_{k+1}/z$, show that $Q = G'/G'(1)$, where is the generating function for the excess degree distribution and $G = \sum p_k x^k$ is the generating function for the degree distribution.

Ex 9.11: Average component size [4]

In this chapter we derived a method to compute c_k, the probability that a randomly picked node is in a component of k nodes. Now, let s_k be the probability that a randomly picked component has k nodes. Formulate an equation that relates c_k and s_k. Then translate your equation into an equation that relates the C and S, the generating functions of c_k and s_k. Use your equations to compute the average component size (the expectation value of s_k, for the networks from Ex. 9.5 and Ex. 9.6) [This one has some pitfalls, proceed with caution.]

Ex 9.12: Attacks on networks [4]

In the next chapter we study attacks on networks. In this exercise we can explore some relevant calculations already. Let us consider a network where every node has initially degree 4. Then an attack removes half the links at random.

a) Construct the degree distribution after the attack, and use it to compute the mean excess degree after the attack without using generating functions

b) Now we do the same calculation with generating functions: Write the generating function G of the degree distribution before the attack. Also write the generating function A for a coin flip that gives us a result of 0 or 1 with equal probability. Use G and A to write the generating function G_{a} for the degree distribution after the attack. Finally, use this to construct Q_{a} the generating function for the excess degree distribution after the attack and use it to compute the mean excess degree after the attack. (the calculation will be short)