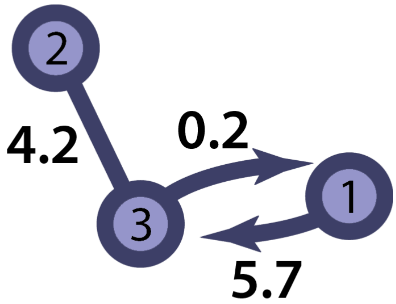
Networks and Complexity

Solution 21-3

This is an example solution from the forthcoming book Networks and Complexity.
Find more exercises at https://github.com/NC-Book/NCB

Ex 21.3: Weighted Laplacian [3]

Consider a number of walkers diffusion on the following network:



where the arrows are directed links. The numbers indicate the rates at which walkers at the respective source use the link.

a) Write the weighted adjacency and Laplacian matrices for this system. (Recall that A_{ij} is the weight of the link to i from j)

<u>*Solution*</u>

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0.2 \\ 0 & 0 & 4.2 \\ 5.7 & 4.2 & 0 \end{pmatrix} \tag{1}$$

$$\mathbf{L} = \begin{pmatrix} 5.7 & 0 & -0.2 \\ 0 & 4.2 & -4.2 \\ -5.7 & -4.2 & 4.4 \end{pmatrix} \tag{2}$$

b) Write a system of differential equations that governs x_i, the proportion of walkers in node i. Compute the Jacobian matrix $\mathbf{J}$ and verify $\mathbf{J} = -\mathbf{L}$.

<u>*Solution*</u>

We write the equation system

$$\dot{x}_1 = -5.7x_1 + 0.2x_3 \tag{3}$$

$$\dot{x}_2 = -4.2x_2 + 4.2x_3 \tag{4}$$

$$\dot{x}_3 = -4.4x_3 + 4.2x_2 + 5.7x_1 \tag{5}$$

The Jacobian is

$$\mathbf{J} = \begin{pmatrix} -5.7 & 0 & 0.2 \\ 0 & -4.2 & 4.2 \\ 5.7 & 4.2 & -4.4 \end{pmatrix} = -\mathbf{L} \tag{6}$$

c) Find the steady state distribution of walkers with a method of your choice.

Solution

An easy way to do this is to solve for the steady state of the dynamical system. From the second equation we can see that

$$x_2 = x_3 \tag{7}$$

and from the first equation we see that

$$5.7x_1 = 0.2x_3 \tag{8}$$

which means

$$x_3 = 28.5x_1 \tag{9}$$

So a possible solution would be $x_1 = 1$, $x_2 = x_3 = 28.5$, however the question asks for the proportion of walkers so we normalize these numbers such that they add up to 1. The result is.

$$x_1 = 0.01724 \tag{10}$$

$$x_2 = 0.49137 \tag{11}$$

$$x_3 = 0.49137 \tag{12}$$