Networks and Complexity

Solution 16-3

This is an example solution from the forthcoming book Networks and Complexity.
Find more exercises at https://github.com/NC-Book/NCB

Ex 16.3: Abstract sensitivity example [2]
The following model is quite complicated:

$$\dot{x} = -x^4 + 2px^3 + 2p^2x - px^2 + x - 2p$$

It would be a pain to compute the steady states by hand. However, suppose that based on your understanding of the system you suspect that for $p^* = 1$ there is a steady state at $x^* = 2$.

a) Verify that $x^* = 2$ is indeed a steady state at $p^* = 1$.

Solution
Substituting $p = 1$ yields

$$\dot{x} = -x^4 + 2x^3 + 2x - x^2 + x - 2 \tag{1}$$

and then substituting $x = 2$ we find

$$\dot{x} = -2^4 + 2 \cdot 2^3 + 2 \cdot 2 - 2^2 + 2 - 2 = 0$$

so this is indeed a steady state.

b) Use the equation for the sensitivity of steady states from the lecture to compute how the steady state is affected if we change p a little bit.

Solution
We need

$$\begin{aligned}
f_{\mathrm{x}}(x^*, p^*) &= \frac{\partial}{\partial x}(-x^4 + 2px^3 + 2p^2x - px^2 + x - 2p)\Big|_* && (2)\\
&= (-4x^3 + 6px^2 + 2p^2 - 2px + 1)\big|_* && (3)\\
&= -4 \cdot 2^3 + 6 \cdot 1 \cdot 2^2 + 2 \cdot 1^2 - 2 \cdot 1 \cdot 2 + 1 && (4)\\
&= -32 + 24 + 2 - 4 + 1 && (5)\\
&= -9 && (6)
\end{aligned}$$

and also

$$\begin{aligned}
f_{\mathrm{p}}(x^*, p^*) &= \frac{\partial}{\partial p}(-x^4 + 2px^3 + 2p^2x - px^2 + x - 2p)\Big|_* && (7)\\
&= (2x^3 + 4px - x^2 - 2)\big|_* && (8)\\
&= 2 \cdot 2^3 + 4 \cdot 1 \cdot 2 - 2^2 - 2 && (9)\\
&= 16 + 8 - 4 - 2 && (10)\\
&= 18 && (11)
\end{aligned}$$

We can now compute

$$\left.\frac{\mathrm{d}x^*}{\mathrm{d}p}\right|_* = -\frac{f_\mathrm{p}}{f_\mathrm{x}} = -\frac{18}{-9} = 2 \tag{12}$$

In words: If we start from the steady state at $p^* = 1$, $x^* = 2$ and then change p by a small amount ρ we expect the steady state to shift by 2ρ.

[Bonus: Notice anything funny about the calculations in this exercise. There is a little bit more to discover here.]