Networks and Complexity

Solution 19-1

This is an example solution from the forthcoming book Networks and Complexity.
Find more exercises at https://github.com/NC-Book/NCB

Ex 19.1: Spectral shift [1]

We claimed that adding the identity matrix $\mathbf{I}$ to a matrix $\mathbf{A}$ shifts the spectrum of $\mathbf{A}$ by 1. More generally we can say the following: Assume we have two matrices $\mathbf{K}$ and $\mathbf{L}$, such that

$$\mathbf{L} = \mathbf{K} + c\mathbf{I}. \tag{1}$$

Then for every eigenvalue λ of $\mathbf{K}$ there is an eigenvalue $\lambda + c$ of $\mathbf{L}$. The eigenvectors corresponding to these to eigenvalues are identical.

Can you actually prove this? (Hint: You want to show $\mathbf{K}\boldsymbol{v} = \kappa\boldsymbol{v}$, where $\kappa = \lambda + c$. You define $\boldsymbol{v}$ as an eigenvector of $\mathbf{L}$ with eigenvalue λ.)

Solution

Let $\boldsymbol{v}$ be an eigenvector of $\mathbf{L}$ with eigenvalue λ, ie. $\mathbf{L}\boldsymbol{v} = \lambda\boldsymbol{v}$ We now write

$$\begin{aligned} \mathbf{K}\boldsymbol{v} &= (\mathbf{L} + c\mathbf{I})\boldsymbol{v} && (2)\\ &= \mathbf{L}\boldsymbol{v} + c\mathbf{I}\boldsymbol{v} && (3)\\ &= \lambda\boldsymbol{v} + c\boldsymbol{v} && (4)\\ &= (\lambda + c)\boldsymbol{v} && (5)\\ &= \kappa\boldsymbol{v} && (6) \end{aligned}$$

We have shown

$$\mathbf{B}\boldsymbol{v} = (\lambda + \alpha)\boldsymbol{v} \tag{7}$$

which means that $\boldsymbol{v}$ is an eigenvector of $\mathbf{K}$ with eigenvalue $\lambda + c$. So every eigenvector of $\mathbf{L}$ is also an eigenvector of $\mathbf{K}$ with the eigenvalue shifted by c.