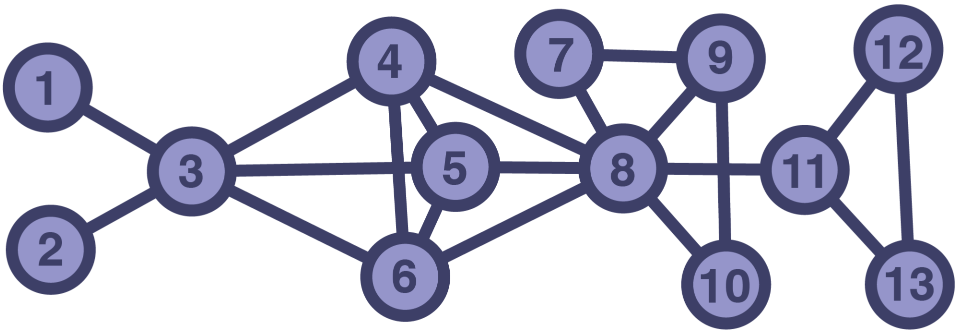
NETWORKS AND COMPLEXITY

Exercise Sheet 20: Motifs

This is an exercise sheet from the forthcoming book Networks and Complexity.
Find more exercises and solutions at https://github.com/NC-Book/NCB

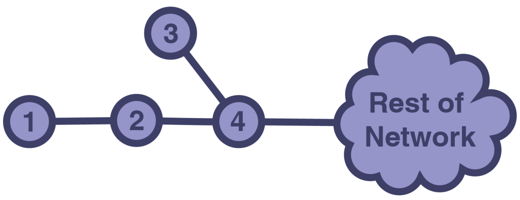
Ex 20.1: Symmetry [2]

Find at least 5 eigenvalues of the adjacency matrix of the following network:



Ex 20.2: Eigenvector in a Branch [2]

Suppose the adjacency matrix of the following network has an eigenvalue λ with the corresponding eigenvector $\boldsymbol{v}$:



a) Consider the eigenvector v_1 and v_2 element of $\boldsymbol{v}$ that correspond to node 1 and 2, respectively. Express v_2 as a function of v_1 and λ.
b) Also express v_2, v_3 and v_4 as functions of λ and v_1.
c) For which values of λ would v_1 and v_3 have the same sign.

Ex 20.3: Exploitative Competition [3]

We consider two species of predators, X_1, X_2, and a resource R, which are part of of much larger ecological food web. The dyanamics of the three variables are described by the differential equations

$$\begin{aligned}\dot{X}_1 &= g_1(R)X_1 - m_1X_1\\ \dot{X}_2 &= g_2(R)X_2 - m_2X_2\\ \dot{R} &= s(R) - g_1(R)X_1 - g_2(R)X_2 + \dots\end{aligned}$$

where g_1, g_2 and s are arbitrary functions, m_1 and m_2 are arbitrary mortality rates for the two predator species and '...' represents further terms that connect R to other variables of a large dynamical system.

a) Formally compute the 3×3-block of the Jacobian that describes the interaction between X_1, X_2, and R. Use the stationarity condition for X_1 and X_2 to simplify your matrix.
b) Show that in any large ODE system that contains the exploitative competition motif studied here, one eigenvalue of the Jacobian for the entire system will be zero.

Ex 20.4: Reactivity [4]

Reactivity was introduced by Neubert and Caswell in 1997 to measure the initial response of a dynamical system to perturbations. Specifically, reactivity is defined as the initial amplification of a perturbation by the dynamics. In this exercise we will re-derive reactivity, interpret it as a bespoke centrality for a given question and also discuss its role in motifs.

a) We are interested in the relative rate of growth a of the size of small perturbation $\boldsymbol{\delta}$ to a steady state, that is
$$a = \frac{1}{|\boldsymbol{\delta}|}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{\delta}|$$
Recall that $|\boldsymbol{\delta}| = \sqrt{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{\delta}}$, and $\dot{\boldsymbol{\delta}} = \mathbf{J}\boldsymbol{\delta}$. Then show
$$a = \frac{\boldsymbol{\delta}^{\mathrm{T}}\mathbf{H}\boldsymbol{\delta}}{\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{\delta}},$$
where $\mathbf{H} = (\mathbf{J} + \mathbf{J}^{\mathrm{T}})/2$ is the symmetric part of the Jacobian.
b) Consider the predator prey system
$$\begin{aligned} \dot{X} &= X - XY/(3+X) \\ \dot{Y} &= 2XY/(3+X) - Y. \end{aligned}$$
Compute $\mathbf{J}$ and $\mathbf{H}$ in the nontrivial steady state.
c) We define the reactivity of the system as the largest amplification a that can be observed in response to any small perturbation. Because $\mathbf{H}$ is a symmetric (hermitian) matrix, it's eigenvectors are orthogonal. Hence they can be normalized such that $\boldsymbol{v_n}^T\boldsymbol{v_m} = \delta_{nm}$, where δ is the Kronecker delta. Use this to show that the maximal value of a is observed when the perturbation is in the direction of the eigenvector of $\boldsymbol{v}$ with the largest eigenvalue.
d) A system is said to be reactive if it has positive reactivity, $a > 0$. Is our predator-prey system from above reactive. Furthermore, would a system with the following Jacobian be reactive:
$$\mathbf{J} = \begin{pmatrix} -8 & -8 \\ 2 & 0 \end{pmatrix}.$$
e) The reactivity of a system is at least as large as the reactivity found in any motif within the system. Explain why.