Solution 13-6

This is an example solution from the forthcoming book Networks and Complexity.
Find more exercises at https://github.com/NC-Book/NCB

Ex 13.6: Mothers and fathers [3]

In the chapter we modeled human ancestors in terms of people while for bees we differentiated between drones and queens. Do the calculation for humans again but this time differentiate between the number of men m_i, and women w_i in generation i. Confirm that you get the expected result. (Bonus: Is there an intuitive interpretation of the eigenvectors and the corresponding expansion coefficients c_1, c_2 that appear.)

Solution

Okay, male humans have a mother and a father, female humans also have a mother and a father. Hence we can write the dynamical system like this

$$\begin{pmatrix} m_{t+1} \\ f_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} m_t \\ f_t \end{pmatrix} \tag{1}$$

where m_t and f_t are respectively the number of male and female humans in generation t (counting backwards). To solve the system using the procedure from the chapter, we need to compute the eigenvectors and eigenvalues. So let's solve

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{1b} = \lambda \mathbf{1b} \tag{2}$$

where we already set the first element of the vector to 1. In components the equations read

$$1 + b = \lambda \tag{3}$$

$$1 + b = \lambda b \tag{4}$$

And hence $\lambda = \lambda b$. One solution to this equation is $b = 1$, another is $\lambda = 0$. In the first case $b = 1$ our system of equations tells us straight away $\lambda = 2$. So one eigenvalue is $\lambda_1 = 1$ with the corresponding eigenvector $\boldsymbol{v_1} = (1, 1)^{\mathrm{T}}$.

In second case, $\lambda = 0$ the system reads $1 + b = 0$ so $b = -1$, and hence we have an eigenvalue $\lambda = 0$ with the corresponding eigenvector $\boldsymbol{v_2} = (1, -1)^{\mathrm{T}}$.

We can now write the general solution

$$\begin{pmatrix} m_t \\ f_t \end{pmatrix} = c_1 2^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 0^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{5}$$

The c_1 and c_2 depend on your initial generation ($t = 0$) and are $c_1 = c_2 = 1/2$ if you start with one male individual, and $C_1 = 1/2$, $c_2 = -1/2$ if you start with one female individual.

It is interesting to note how the eigenvalue of 2 doubles the number of people in each generation that we go back. But perhaps even nicer is how the eigenvalue of 0 erases the information if we started with a man or a woman as soon as we take one step into the past. Unless we look at the starting generation, the second term will always be zero.