Networks and Complexity

Solution 12-4

This is an example solution from the forthcoming book Networks and Complexity. Find more exercises at https://github.com/NC-Book/NCB

Ex 12.4: Population growth [2]

Sexual reproduction requires that two individuals of a species meet. If we apply the law of mass action naively we may arrive at a model of the form $\dot{x} = x^2$. Solve the initial value problem for $x(0) = 1$ and explore what happens.

Solution

Starting from

$$\frac{\mathrm{d}}{\mathrm{d}t} x = x^2 \tag{1}$$

separation of variables leads to

$$\frac{1}{x^2} \mathrm{d}x = \mathrm{d}t. \tag{2}$$

Integrating we find

$$\int \frac{1}{x^2} \mathrm{d}x = \int \mathrm{d}t \tag{3}$$

$$-\frac{1}{x} = t + C \tag{4}$$

where C is the constant of integration. Solving for x yields

$$x(t) = -\frac{1}{t + C}. \tag{5}$$

This is already the general solution to the problem. To find the particular solution that solves the initial value problem we use

$$x(0) = 1 = -\frac{1}{C}, \tag{6}$$

which we can solve for

$$C = -1. \tag{7}$$

Hence the solution to the initial value problem is

$$x(t) = -\frac{1}{t-1} = \frac{1}{t-1}. \tag{8}$$

From this solution we can see that something interesting is happening here. At $t = 1$ the denominator of the fraction is zero and the population size x diverges toward $+\infty$. This is indeed a correct result, quadratic growth leads to an explosion that reaches ∞ in finite time.

We can see from this result that geometric growth ($\dot{x} \sim x^\gamma$ with $\gamma > 1$) is much much more violent than exponential growth, which also reaches ∞, but only after infinite time.

For biological populations quadratic growth is a bad model for reasons that we discover in Chap. 17.