Networks and Complexity

Exercise Sheet 14: Complexity-Stability Debate

This is an exercise sheet from the forthcoming book Networks and Complexity.
Find more exercises and solutions at https://github.com/NC-Book/NCB

Ex 14.1: Trace and determinant [1]

Compute the trace and determinant of the following matrices:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

Ex 14.2: Stability analysis in one dimension [2]

Find the steady states of the following one-dimensional systems:

a)
$$\dot{x} = 5 - 3x$$

b)
$$\dot{x} = (3 - x)(x + 5)(x - 9)(x - 1)$$

c)
$$\dot{x} = \sin(x)$$

Ex 14.3: Stability of the SIS system [2]

In the text of this chapter we studied the stability of our one-dimensional version of the SIS model

$$\dot{I} = p(N - I)I - rI$$

specifically we studied the stability of the trivial steady state $I^* = 0$ and found that this state is stable if $pN < r$ and unstable otherwise. Now consider the non-trivial steady state and determine when it is stability.

Ex 14.4: Two abstract two-dimensional system [2]

Let's start with the example system from the chapter

$$\begin{aligned} \dot{x} &= y^5 - x^2 \\ \dot{y} &= 1 - x \end{aligned}$$

a) Compute the steady state
b) Compute the Jacobian in the steady state
c) Compute the eigenvalues and eigenvectors of the Jacobian
d) Compute the real part of the eigenvalues and decide whether the steady state is stable or not.

e) Use the same approach to find the steady state of the following system, and determine its stability:

$$\begin{aligned}\dot{x} &= x - y + 2\\ \dot{y} &= x - 3y + 8\end{aligned}$$

Ex 14.5: Walk through a more complex two-dimensional analysis [3]

Let's go step-by-step through the analysis of the following system:

$$\begin{aligned}\dot{x} &= a + bxy\\ \dot{y} &= \frac{1}{2} - x\end{aligned}$$

a) Compute the steady state of the system. Then find the Jacobian in the steady state, and compute the eigenvalues:
b) You should now have an expression for the eigenvalues that contains a square root. To make sense of it, lets determine for which values of a and b the eigenvalues are complex. (i.e. when is the term under the root negative such that it the root becomes imaginary.)
c) Now, assuming the eigenvalues are complex, consider the real part of the two eigenvalues and hence find a condition for stability of the steady state.
d) Assuming eigenvalues are real, consider the expression for the leading (i.e. largest) eigenvalue λ_1 and hence determine the stability in this case.
e) Summarize your results. Draw a diagram where a and b are the axis. In this diagram draw lines that correspond to the stability conditions and the points where the eigenvalues change from real to complex. Then color in the area where the steady state is stable.

Ex 14.6: Allee effect [3]

The change in the population size x of an ecological population is described by the differential equation

$$\dot{x} = \frac{x^2}{1+x^2} - \frac{m}{2}x,$$

where m is a mortality rate.

a) Compute the three branches of steady states of this system.
b) Determine the stability of the steady states. (This requires a bit more mathematical technique than most exercises in this book. Give it a try, if you get tangled up in too many square roots check the solution and see what you could have done differently.)
c) Draw a bifurcation diagram: Plot the steady states as a function of m between $m = 0.5$ and $m = 1.5$ (m should be the x-axis and x the y-axis). Indicate the stability of the steady states in the diagram. Also color the area between the branches of steady states according to wether the population size x is increasing or decreasing in the respective areas.
d) Now assume we start the system at $m = 0.5$ in a stable steady state with $x^* > 0$. What happens if we slowly increase m? Once the population goes extinct can we bring it back by decreasing m? (This is the so-called Allee effect, an example of a tipping point in an ecological system.)

Ex 14.7: Paradox of Enrichment [3]

A fundamental building block of theoretical ecology is the study of predator-prey interaction.

A simple model of this interaction is provided by the Rosenzweig-Mac-Arthur system [?]:

$$\begin{aligned}\dot{x} &= 5x\left(1-\frac{x}{K}\right)-\frac{25xy}{2+3x}\\ \dot{y} &= \frac{5xy}{2+3x}-y\end{aligned}$$

where x is the population size of the prey population, y is a population size of the predator population and K is the co-called carrying capacity. It is the maximum population size that which can be interpreted as an indicator for the amount of nutrients of available to the prey. All three of these quantitites are measured in some arbitrary units (e.g. tonnes of biomass, not number of individuals).

a) Show that there is one steady state, where neither predator nor-prey survives in the system, and another one where only the prey survives. Then find the steady state in which predator and prey coexist in the system.

b) Show that for $0 < K < 1$, the state where the prey is present in the system but the predator is extinct, is the only relevant stable steady state.

c) If we add nutrients to increase K above 1 then the predator can enter the system. Show that the steady state in which both species coexist, becomes unstable if we raise K beyond a critical point K^*. (This is the paradox of enrichment: Too many nutrients destabilize the system.)

Ex 14.8: Stability condition for maps [4]

In the chapter we derived a general condition for the stability of steady states in ODE systems. Now use the same procedure to find a condition for the stability of fixed points in maps.

a) Start by finding a condition for fixed points in maps. Don't try to guess the result, instead proceed analogously to our approach from the chapter: Define the general one-dimensional map $x_{i+1} = f(x_i)$, consider a fixed point x^*, define a deviation δ_i from the fixed point, find an iteration rule for δ_i, Taylor expand around x^*, eliminate all terms that can be eliminated and solve the resulting equation to find the stability condition. (If you get stuck, check the solution)

b) Now, find the stability condition for multi-dimensional maps $\boldsymbol{x_{i+1}} = f(\boldsymbol{x_i})$, where $\boldsymbol{x_i}$ is a vector of dimension N. (Proceed as in (a), when you arrive at a multi-dimensional linear map, solve it using eigendecomposition.)

c) In Ex. 13.8 we found that the logistic map $x_{i+1} = px_i(1-x_i)$ has fixed points at $x_1^* = 0$ and $x_2^* = (p-1)/p$. Determine the stability of these fixed points.

d) The equation $5x = x^2$ is easy to solve, but suppose you wanted to solve it by the fixed-point iteration $x_{i+1} = {x_i}^2/5$. If you give it a try, you'll find one of the solutions but not the other. Explain why this happens. (Bonus: Come up with a way to nevertheless find the second fixed point by fixed-point iteration.)

Ex 14.9: A puzzle [4]

Determine which of the following Jacobians corresponds to a stable steady state (you won't need a computer):

$$\mathbf{J_1} = \begin{pmatrix} 7 & 0 & 3 & 0 & 9\\ 0 & -3 & 0 & -4 & 4\\ 1 & -2 & -1 & 4 & 4\\ 0 & -3 & 2 & -1 & 2\\ -1 & -2 & 1 & 1 & -1 \end{pmatrix}, \quad \mathbf{J_2} = \begin{pmatrix} -4 & 0 & 0 & 1 & 0\\ 5 & 1 & 0 & 2 & 0\\ 1 & -2 & -2 & 4 & 4\\ 6 & 0 & 0 & -1 & 0\\ -1 & -2 & 0 & 1 & -1 \end{pmatrix}$$

$$\mathbf{J_3} = \begin{pmatrix} -9 & -1 & 2 & 0 & 1 \\ 0 & -5 & 0 & 1 & 0 \\ 1 & 1 & -4 & 0 & 0 \\ 4 & 1 & 0 & -3 & 0 \\ 3 & 1 & 1 & 1 & -2 \end{pmatrix}, \quad \mathbf{J_4} = \begin{pmatrix} -2 & 1 & 1 & -4 & 3 \\ 0 & -9 & 1 & 1 & 0 \\ 0 & 0 & -9 & 1 & 0 \\ 0 & 0 & 0 & -9 & 0 \\ 3 & -2 & 2 & 1 & -2 \end{pmatrix}$$

Ex 14.10: Glycolysis [4]

Glycolysis is a process by which your cells turn sugar into energy. The essence of glycolysis can be captured in the following conceptual model:

$$\begin{aligned} \text{A} &\xrightarrow{r} B \\ \text{B} &\xrightarrow{s} 2A \\ \text{A} &\xrightarrow{v} \oslash \end{aligned}$$

Here, A describes the amount of available energy, which the cell stores in form of a molecule, adenosine-triphosphate, or, for short, ATP. The first of these reactions captures that energy is needed to start the process. In the second reaction more energy is produced as a result of the process, while the final reaction represents energy being used in the body. By measuring all rates in multiples of the energy consumption rate v we can fix $v = 1$.

a) Define a as the amount of A, and b as the amount of B in the system. Then, write the ODE system that captures the dynamics of these variables, and show that there's in general no steady state in which $a > 0$.

b) The absence of a steady state where we have a constant level of stored energy, is a problem. Your cells solve this problem by making the enzyme that catalyzes the first reaction dependent on the concentration of its substrate a. So let's assume that $r = a^p$. Show that this leads to a steady state with $a > 0$.

c) Show that the steady state is unstable for $p > 0$. (This leads up to a nice biological insight, check the solution).

Ex 14.11: A simple controller [4]

Consider the following system:

$$\begin{aligned} \dot{x} = \quad & f(x,y) & = x - 2y \\ \dot{y} = \quad & g(x,y) & = 2x + y \end{aligned}$$

a) Show that the system has a single unstable steady state.

b) We now want to stabilize the system for this purpose we will modify the system by adding a feedback onto the equation for y. In the simplest case this feedback acts instantaneously and directly such that the equations become

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) + h(y), \end{aligned}$$

where g and y are as above but h describes the newly added feedback. Find a function $h(y)$ such that $h(0) = 0$ and the steady state at $(0,0)$ is stable.

Ex 14.12: The mechanical hydra [5]

This exercise proposes a hypothetical mechanical device. The first subquestions should be easy to solve. The final one is unsolved but probably solvable. We consider the ODE system

$$\begin{aligned} \dot{x} &= 2x - 4y \\ \dot{y} &= 3x - 5y. \end{aligned}$$

a) Determine the stability of the steady state at $x = y = 0$.
b) Now imagine that we changed this system by clamping some part in place. As a result y becomes fixed at $y = 0$ and cannot change anymore, which removes it as a variable. Write the differential equation that describes the dynamics of the remaining system after y has been fixed.
c) Determine the stability of the steady state in the resulting system. Is the result surprising?
d) This system exhibits the so-called *hydra effect*. If left alone it settles into a stable steady state, but if we try to hold a certain part in place in this state, then the rest of the system becomes unstable. Your task is to design a mechanical system that exhibits this effect.