Networks and Complexity

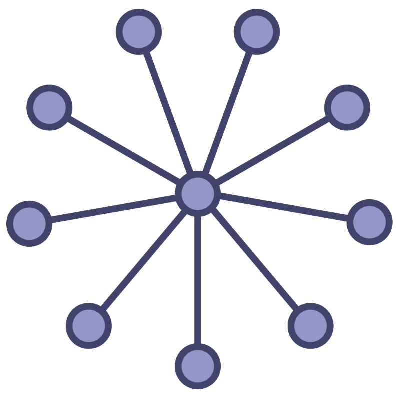
Solution 6-12

This is an example solution from the forthcoming book Networks and Complexity.
Find more exercises at https://github.com/NC-Book/NCB

Ex 6.12: Degree Correlations [4]

In a sufficiently random network the expected degree of a random neighbor of a random node is $k_{\text{nn}} = q + 1$. However, this isn't true if the nodes are connected in a specific way that causes strong assortativity/disassortativity.

a) For illustration, compute the average degree of a random neighbor of a random node, k_{nn}, for the star network from the following network:



Compare k_{nn} to q.

Solution

If we pick a random node in the network then we get the hub node with probability 1/10. If we then pick a random neighbor we are guaranteed to get a neighbor with degree 1. With probability 9/10 we pick one of the peripheral nodes, if we then pick a random neighbor this neighbor will always be the hub, so we will end up with a node of degree 9. So the expected degree of a random neighbor of a random node is

$$k_{\text{nn}} = \frac{1}{10}1 + \frac{9}{10}9 = 8.2 \tag{1}$$

Due to the strong disassortativity in this network this significantly greater than $q = 4$ which we know from the lecture.

b) Suppose the degree distribution of a network is p_k and the probability that a neighbor of a node with degree k has degree j is $x_{k,j}$. Find a general formula for k_{nn}.

Solution

To arrive at the general formula, let's first compute the expected degree of the neighbors of a node of degree k. We can write this as

$$\sum_j j x_{k,j}. \tag{2}$$

Now if we randomly pick a node we end up picking a node with degree k with probability p_k. Summing over all possibilities we find

$$k_{\mathrm{nn}} = \sum_k p_k \sum_j j x_{k,j}. \tag{3}$$

c) Write p_k and $x_{k,j}$ for network B in terms of Kronecker deltas and show that your formula for k_{nn} yields the expected result.

Solution

We already know that the degree distribution of the network can be written as

$$p_k = \frac{1}{10}\delta_{k,9} + \frac{9}{10}\delta_{k,1} \tag{4}$$

To also find an expression for $x_{k,j}$, we can see from the network that the neighbors of all nodes of degree one have degree 9 with probability 1

$$x_{1,9} = 1. \tag{5}$$

Likewise, a random neighbor of a node of degree 9 has degree 1 with probability 1, i.e.

$$x_{9,1} = 1. \tag{6}$$

All other $x_{k,j}$ are zero. (Well, actually properties such as $x_{4,5}$ are undefined as there are no nodes of degree 4 in the network, but we can treat them as zero). Hence we can write

$$x_{k,j} = \delta_{k,1}\delta_{j,9} + \delta_{k,9}\delta_{j,1} \tag{7}$$

Have you figured this out yourself? If yes, well done, if no take a moment to substitute some pairs of numbers (k, j) into this formula and to convince yourself that this is correct.

Let's substitute $x_{k,j}$ and into our formula

$$\begin{aligned}
k_{\mathrm{nn}} &= \sum_k p_k \sum_j j x_{k,j} && (8)\\
&= \sum_k p_k \sum_j j\left(\delta_{k,1}\delta_{j,9} + \delta_{k,9}\delta_{j,1}\right) && (9)\\
&= \sum_k p_k \sum_j j\delta_{k,1}\delta_{j,9} + j\delta_{k,9}\delta_{j,1}. && (10)
\end{aligned}$$

We use the substitution trick-vanishing act combo, first for j and then for k, which yields

$$\begin{aligned}
k_{\mathrm{nn}} &= \sum_k p_k\left(9\delta_{k,1} + \delta_{k,9}\right) && (11)\\
&= \sum_k p_k\left(9\delta_{k,1} + \delta_{k,9}\right) && (12)\\
&= \left(\sum_k 9p_k\delta_{k_1}\right) + \left(\sum_k p_k\delta_{k,9}\right) && (13)\\
&= 9p_1 + p_9. && (14)
\end{aligned}$$

Of course we could have substituted p_k in already earlier, but waiting with this saved us some writing. So, the last thing to do now is to plug in p_k to find

$$k_{\mathrm{nn}} = 9\frac{9}{10} + \frac{1}{10} = 8.2, \tag{15}$$

which is the expected result.