Networks and Complexity

Exercise Sheet 22: Stripes on a Tiger

This is an exercise sheet from the forthcoming book Networks and Complexity.
Find more exercises and solutions at https://github.com/NC-Book/NCB

Ex 22.1: Konecker products [1]
Consider the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 4 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1}$$

Compute a) $\mathbf{A} \otimes \mathbf{B}$, b) $\mathbf{A} \otimes \mathbf{C}$ and c) $\mathbf{C} \otimes \mathbf{A}$

Ex 22.2: Abstract MSF [2]
A dynamical system has a steady state whose Jacobian matrix is given by

$$\mathbf{P} = \begin{pmatrix} 2 & 3 \\ -3 & -4 \end{pmatrix} \tag{5}$$

We place copies of this system in the nodes of a network and couple them diffusively such that the coupling matrix is

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix} \tag{6}$$

Compute and sketch the master stability function for this system. Describe what the result means for the network.

Ex 22.3: Product Formula [3]
Consider the Kronecker product

$$\mathbf{J} = \mathbf{A} \otimes \mathbf{B} \tag{14}$$

where $\mathbf{A}$ and $\mathbf{B}$ are matrices. Find a formula that relates the eigenvalues of $\mathbf{J}$ to the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$. (This is very similar, but simpler than the derivation in the lecture. Formulate an Ansatz for the eigenvector $\boldsymbol{v}$, then show that it works and find the eigenvalue on the way).

Ex 22.4: A different Predator-Prey model [3]
This exercise is good to practice mathematical technique. The challenge in the later subquestions is to follow the right path. Without being distracted by details of expressions. Try to keep your notation simple, it will make it easier.

A predator-prey system is described by

$$\begin{aligned} \dot{X} &= X - 2XY \\ \dot{Y} &= \frac{XY}{1+X} - Y^2 \end{aligned}$$

a) Compute the nontrivial steady state of the system.
b) Compute the Jacobian matrix $\mathbf{P}$ of the non-trivial steady state.

c) Now consider that the system is placed in the nodes of a network, where links represent diffusive coupling between nodes with a coupling matrix

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Compute $\mathbf{P} - \kappa\mathbf{C}$ and determine when the eigenvalues λ of this matrix are real.

d) For the case when eigenvalues are complex, compute the real part of the largest eigenvalue, and show that the system is always stable in this range of κ.

e) Now consider the case where the eigenvalues are real. Can the system become unstable for certain values of κ in this range?

Ex 22.5: Flight response? []

Let us consider again the predator-prey system above. Again we place the system on a network but in this case the coupling between patches is a bit different. After placing the system on a network the dynamics on a node i are

$$\dot{X}_i = X_i - 2X_iY_i + \sum_j A_{ij}(Y_i - Y_j) \tag{44}$$

$$\dot{Y}_i = \frac{X_iY_i}{1+X_i} - Y_i^2 \tag{45}$$

Show that the nontrivial steady state from the predator-prey exercise above is still a steady state, and compute the master stability function. Which condition must a network meet for this steady state to be stable?

Ex 22.6: Spectra of Hypercubes [4]

A hypercube is a generalization of cubes to arbitrary dimensions. The 0-dimensional hypercube is just a single node.

a) Compute the eigenvalue of the adjacency matrix of the zero-dimenional hypercube, $\mathbf{A_0}$.

b) To make a 1-dimenional hypercube we take the zero-dimensional hypercube and make a copy of it. Then we connect every node in the original cube with the same node in the copy. Compute the eigenvalues of the corresponding adjacency matrix $\mathbf{A_1}$.

c) To make a 2-dimenional hypercube we take the one-dimensional hypercube and make a copy of it. Then we connect every node in the original 1D-cube with the same node in the copy. Write the corresponding adjacency matrix $\mathbf{A_2}$ as a sum of two Kronecker products containing $\mathbf{A_1}$, the identity matrix $\mathbf{I}$, and the matrix

$$\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{61}$$

Explain the resulting formula in words and use it to compute the eigenvalues. (Hint: $\mathbf{M}$ and $\mathbf{A_1}$ look the same but they fill a different role. If you used them cleverly you can shift the indices in your formula and you will get an equation that relates $\mathbf{A_0}$ and $\mathbf{A_1}$ correctly.)

d) To make a 3-dimenional hypercube we take the two-dimensional hypercube and make a copy of it. Then we connect every node in the original 2D-cube with the same node in the copy. Compute its spectrum.

e) We can also find the Laplacian spectrum very easily. Note that the hypercubes are regular graphs, with a degree that is identical to the dimension of the cube. So for example the three-dimendional hypercube is a 3-regular graph. Quickly compute the spectrum of $\mathbf{L_3} = 3\mathbf{I} - \mathbf{A_3}$.

f) Now adapt our previous rule to compute the Laplacian spectrum of the 4-dimensional hypercube.

g) Now that we have spotted the pattern, consider the generating function

$$G_i(x) = \sum c_{i,n} x^n$$

where $c_{i,n}$ is the multiplicity of the eigenvalue $\lambda = 2n$ in the Laplacian spectrum of the i-dimensional hypercube. Write an iteration rule that relates G_i and G_{i+1} and hence find a closed form for G_i