Networks and Complexity

Exercise Sheet 23: No current without heat

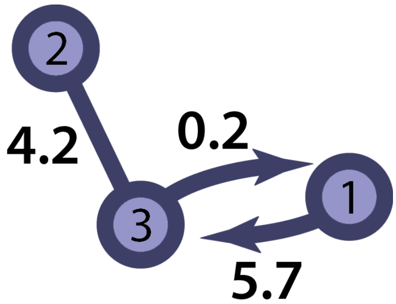
This is an exercise sheet from the forthcoming book Networks and Complexity. Find more exercises and solutions at https://github.com/NC-Book/NCB

Ex 23.1: Circular reasoning [2]

While I worked in Davis I was struggling to buy enough drinking water. Basically I distinguish three states: a) "I have enough water", b) "Water in short supply", c) "Not a single drop". When I am in state a, I usually transition to state b at rate 1. If I am in state b, then I try to buy water. This happens happens also at rate 1 and takes me back to state a. More commonly, at rate 2, I transition to state c. In state c I have a good incentive to go shopping. This happens at rate 3 and takes me back to state a. However, sometimes when I am in state c, I just grab one bottle on the way back from work. This takes me back to state b at rate 1. Use Kirchhoff's theorem to determine the proportion of the time I spend in state c.

Ex 23.2: Weighted Graph [2]

Consider particles diffusing on the following network:



a) Use Kirchhoffs's theorem to find the proportion of particles in each of the nodes in the steady state.

b) Is the system in equilibrium in the steady state?

Ex 23.3: Paper supply [2]

In an office the copier paper is stored in a cabinet. At an average rate of once a week an employee takes paper out of the cabinet and loads it into the copier. There is a 50% chance that the employee tries to load only one packet of paper, and a 50% chance that the employee tries to load two packets of paper. If the cabinet is empty, the office manager immediately orders 4 new packets of paper which arrive in average within 2 weeks.

a) Consider this system as a network of states, numbered 0 to 4. The system is in state i if there are i packets of paper in the cabinet. Write a system of differential equations for the variables $x_0, \ldots, x_4$, where x_i is the probability that the system is in state i.

b) Write the differential equation system in the form

$$\dot{\boldsymbol{x}} = -\mathbf{L}\boldsymbol{x} \tag{20}$$

where $\boldsymbol{x} = (x_0, \ldots, x_4)^{\mathrm{T}}$. Compute the entries of the matrix $\mathbf{L}$.

c) Use Kirchhoff's theorem to compute the probability that there isn't any paper in the cabinet. (i.e. compute x_0 in the steady state).

Ex 23.4: Kirchhoff proof [5]

Find a 'forward' proof for Kirchhoff's theorem, the matrix-tree theorem for weighted directed matrices. (By forward proof we mean a line of reasoning that leads to the result, rather than postulating the result and then proving that it is true.)