Networks and Complexity

# Solution 15-9

*This is an example solution from the forthcoming book Networks and Complexity. Find more exercises at https://github.com/NC-Book/NCB*

**Ex 15.9: Heterogeneous Expansion [4]**

Consider a network with $N$ nodes and mean degree $z$. Existing links in this network are broken at a rate 1 per link. At rate $r$ per node a node establishes a link to a randomly chosen partner.

a) The dynamics of the degree distribution are captured by the differential equation

$$\dot{p_k} = -\underbrace{2rp_k}_{1} - \underbrace{kp_k}_{2} + \underbrace{2rp_{k-1}}_{3} + \underbrace{(k+1)p_{k+1}}_{4} \tag{1}$$

Explain the meaning of the terms 1-4.

*Solution*

The terms capture the following processes:

1. We lose nodes of degree $k$ because they gain an additional link
2. We lose nodes of degree $k$ because they lose one of their links
3. We gain nodes of degree $k$ because nodes of degree $k-1$ gain a link
4. We gain nodes of degree $k$ because nodes of degree $k+1$ lose a link

b) Define $G(x)$ as the generating function of the degree distribution. And derive a differential equation that captures the dynamics of $G$.

*Solution*

We define

$$G(x) = \sum p_k x^k \tag{2}$$

and write

$$\begin{aligned} \dot{G} &= \sum \dot{p}_k x^k & (3)\\ &= \sum(-2rp_k - kp_k + 2rp_{k-1} + (k+1)p_{k+1})x^k & (4)\\ &= -2rG - xG' + 2rxG + G' & (5)\\ &= 2r(x-1)G + (1-x)G' & (6) \end{aligned}$$

c) Consider the steady state of your equation and integrate to obtain $G(x)$. (Use the normalization conditions to fix the constant of integration.)

*Solution*

Setting the rate of change to zero yields

$$0 = 2r(x-1)G + (1-x)G' \tag{7}$$

Which we can also write as

$$G' = 2rG \tag{8}$$

and hence

$$G(x) = G(0)\mathrm{e}^{2rx} \tag{9}$$

We don't know $G(0)$ straight away, but we know $G(1) = 1$ and hence

$$1 = G(0)\mathrm{e}^{2r} \tag{10}$$

Solving for $G(0)$ yields

$$G(0) = \mathrm{e}^{-2r} \tag{11}$$

which means

$$G(x) = \mathrm{e}^{2r(x-1)} \tag{12}$$

d) We have now found the generating function that the network approaches in the long run. What does it tell you about the network.

*Solution*
This is the generating function of en ER graph with $z = 2r$.