Networks and Complexity

Exercise Sheet 16: The Critical Brain

This is an exercise sheet from the forthcoming book Networks and Complexity. Find more exercises and solutions at https://github.com/NC-Book/NCB

Ex 16.1: A fancy derivative [1]
Consider the equation

$$a = bc + \sqrt{b}$$

where

$$b = 3c + 1$$

Compute $\mathrm{d}a/\mathrm{d}c$ at $c = 1$.

Ex 16.2: Transcritical bifurcation [2]
We can write the normal form of the transcritical bifurcation as

$$\dot{x} = apx + bx^2 \tag{7}$$

Find the steady states of this dynamical system, analyze their stability and draw the bifurcation diagram.

Ex 16.3: Abstract sensitivity example [2]
The following model is quite complicated:

$$\dot{x} = -x^4 + 2px^3 + 2p^2x - px^2 + x - 2p$$

It would be a pain to compute the steady states by hand. However, suppose that based on your understanding of the system you suspect that for $p^* = 1$ there is a steady state at $x^* = 2$.

a) Verify that $x^* = 2$ is indeed a steady state at $p^* = 1$.
b) Use the equation for the sensitivity of steady states from the lecture to compute how the steady state is affected if we change p a little bit.

Ex 16.4: Pitchfork bifurcation [3]
To explore the pitchfork bifurcation, we consider again a system of the form

$$\dot{x} = f(x, p) \tag{26}$$

where a physical symmetry stipulates that

$$f(x, p) = -f(-x, p) \tag{27}$$

a) Use the symmetry condition to show that there is a steady state at $x^* = 0$.
b) Now use a similar idea to also show

$$\left.\frac{\partial^n}{\partial x^n} f(x, p)\right|_{x=0} = 0 \tag{31}$$

for all even n.

c) Consider the stability of the steady state at 0, by setting $x = \delta \ll 1$ and analyzing a suitable Taylor expansion of the dynamics.
d) Now assume that the steady state at $x = 0$ changes stability at p^* and consider the dynamics close to the bifurcation by setting $p = p^* + \rho$. Taylor expand the dynamics in both ρ and δ to find the a normal form for the bifurcation.
e) Write the normal form in terms of more convenient variables by defining δ as the new x, ρ as the new p and using a and b for normal form parameters, like we did above. Then compute the steady states and analyze their stability to draw the bifurcation diagram.

Ex 16.5: Fold normal form revisited [4]

Consider the normal from of the fold bifurcation, which we can write as

$$\dot{x} = ax^2 + bp$$

where a and b are normal form coefficients, p is the control parameter, and x is the dynamical variable. Find the steady states, determine their stability, and draw the bifurcation diagram. Are there actually different forms (e.g. subcritical, supercritical) of this bifurcation as well?