Networks and Complexity

Solution 13-10

This is an example solution from the forthcoming book Networks and Complexity. Find more exercises at https://github.com/NC-Book/NCB

Ex 13.10: General solution for linear maps [4]

In this chapter we derived a general solution for linear differential equations, but for linear maps we have only solved examples. Find a general solution for linear discrete-time maps, i.e. systems of the form $\boldsymbol{x}_{i+1} = \mathbf{U}\boldsymbol{x}_i$.

<u>*Solution*</u>

Any linear map can be written in the form

$$\boldsymbol{x}_{t+1} = \mathbf{J}\boldsymbol{x}_t. \tag{1}$$

Since $\mathbf{J}$ advances the time by one step we can also write

$$\boldsymbol{x}_t = \mathbf{J}^t \boldsymbol{x}_0 \tag{2}$$

We now decompose $\boldsymbol{x}_0$ into eigenvectors of $\mathbf{J}$, that is we determine c_n such that

$$\boldsymbol{x}_0 = \sum c_n \boldsymbol{v_n} \tag{3}$$

where

$$\mathbf{J}\boldsymbol{v_n} = \lambda_n \boldsymbol{v_n}. \tag{4}$$

The time dependent solution is now

$$\begin{aligned} \boldsymbol{x}_t &= \mathbf{J}^t \boldsymbol{x}_0 && (5)\\ &= \mathbf{J}^t \sum c_n \boldsymbol{v_n} && (6)\\ &= \sum c_n \mathbf{J}^t \boldsymbol{v_n} && (7)\\ &= \sum c_n {\lambda_n}^t \boldsymbol{v_n} && (8)\\ & && (9) \end{aligned}$$

So the general solution for linear maps is

$$\boldsymbol{x}_t = \sum c_n {\lambda_n}^t \boldsymbol{v_n}. \tag{10}$$