Networks and Complexity

Solution 10-10

This is an example solution from the forthcoming book Networks and Complexity. Find more exercises at https://github.com/NC-Book/NCB

Ex 10.10: Viral attacks in general [3]

Now, we derive some results on the viral attack. Let G be the degree generating function before the attack, and y the proportion of links that are not part of the giant conducting component.

a) Show that a viral attack starting within the giant conducting component is described by the attack generating function

$$R = G - G(xy)$$

Note that we have omitted the argument (x) behind R and the first G, in accordance with our notation.

Solution

From the chapter we know that the probability that a randomly picked node has degree k and is subsequently removed in the viral attack is

$$r_k = p_k(1 - y^k). \tag{1}$$

Furthermore we know the definition of the attack function

$$R = \sum r_k x^k. \tag{2}$$

Substituting the r_k we can write

$$\begin{aligned} R &= \sum p_k(1 - y^k)x^k & (3) \\ &= \left(\sum k p_k x^k\right) - \left(\sum p_k (xy)^k\right) & (4) \\ &= G(x) - G(xy) = G - G(xy) & (5) \end{aligned}$$

where we used $G(x) = sump_k x^k$ in the last step.

b) Now start from $R = G - G(xy)$ and show that the properties of the network after the viral attack are given by

$$N_\mathrm{a} = G(y)N,$$

$$z_\mathrm{a} = y^2 \frac{Q(y)}{G(y)} G'(y), \qquad q_\mathrm{a} = y^2 Q'(y),$$

$$G_\mathrm{a} = \frac{G(\tilde{A}y)}{G(y)}, \qquad Q_\mathrm{a} = \frac{Q(\tilde{A}y)}{Q(y)},$$

where $\tilde{A} = \tilde{c}x + \tilde{r}$ and $\tilde{c} = 1 - \tilde{r} = yG'(y)/z = yQ(y)$.

Solution

Number of nodes. To find the equation for N we need to work out what proportion of the network is removed in the attack:

$$r = R(1) = G(1) - G(y) = 1 - G(y), \tag{6}$$

where we used $G(1) = 1$. The surviving fraction of the nodes is

$$c = 1 - r = G(y). \tag{7}$$

Multiplying this surviving fraction with the total number of nodes gives us the desired result

$$N_\mathrm{a} = cN = G(y)N. \tag{8}$$

Mean degree. To find the result we use the respective result for degree targeted attacks

$$z_\mathrm{a} = \frac{z\tilde{c}^2}{c}. \tag{9}$$

In our calculation of the surviving number of nodes we have just seen $c = G(y)$ and from our analysis of the degree-targeted attack we know

$$\tilde{r} = \frac{R'(1)}{z}. \tag{10}$$

Substituting $R = G - G(xy)$ yields

$$\begin{aligned}
\tilde{r} &= \frac{G'(1) - yG'(y)}{z} && (11)\\
&= \frac{z - yG'(y)}{z} && (12)\\
&= 1 - \frac{yG'(y)}{z} && (13)\\
&= 1 - yQ(y) && (14)
\end{aligned}$$

where the factor y appears as the inner derivative from differentiating $G(xy)$ with respect to x. We can now find the $\tilde{c}$ as the complement of $\tilde{r}$,

$$\tilde{c} = 1 - \tilde{r} = yQ(y). \tag{15}$$

This gives us all the parts that we need to put the solution together, we start the equation for z_a and substitute c and $\tilde{c}$,

$$\begin{aligned}
z_\mathrm{a} &= \frac{z\tilde{c}^2}{c} && (16)\\
&= \frac{z(yQ(y))^2}{G(y)} && (17)\\
&= y^2 \frac{Q(y)}{G(y)} zQ(y) && (18)\\
&= y^2 \frac{Q(y)}{G(y)} G'(y) && (19)
\end{aligned}$$

where we have used $Q = G'/z$ in the last step to make the result a little bit nicer. This form is appealing because $z = G'(1)$, so having a G' appear on the right hand side is neat.

An alternative way is to first derive

$$G_{\rm a} = \frac{G(\tilde{A}y)}{G(y)} \tag{20}$$

and then compute $z_{\rm a}$ using the relationship between degree generating function and the mean degree

$$z_{\rm a} = G'_{\rm a}(1) \tag{21}$$

In this case we have take care with the derivative as we are differentiating with respect to x, which appears as the argument of $\tilde{A}$, (remember $\tilde{A} = \tilde{A}(x)$). We find

$$\begin{aligned}
z_{\rm a} &= \left.\frac{\partial}{\partial x}\frac{G(\tilde{A}y)}{G(y)}\right|_{x=1} && (22)\\
&= \left.\frac{G'(\tilde{A}y)\tilde{A}'y}{G(y)}\right|_{x=1} && (23)\\
&= \frac{G'(\tilde{A}(1)y)\tilde{A}'(1)y}{G(y)} && (24)\\
&= \frac{G'(y)\tilde{A}'(1)y}{G(y)} && (25)\\
&= \frac{G'(y)\tilde{c}y}{G(y)} && (26)\\
&= \frac{G'(y)y^2Q(y)}{G(y)} && (27)\\
&= y^2\frac{Q(y)}{G(y)}G'(y) && (28)
\end{aligned}$$

where we used $\tilde{A}(1) = 1$ and $\tilde{A}'(1) = \tilde{c}$ and $\tilde{c} = yQ(y)$, derived above.

Excess degree. As for the mean degree there are two ways to reach the desired result. First, we can use the equation for the excess degree after a degree targeted attack $q_{\rm a} = q - R''(1)/z$. We start by computing the second derivative of R,

$$\begin{aligned}
R'' &= \frac{\partial^2}{\partial x^2}(G - G(xy)) && (29)\\
&= G'' - y^2G''(xy) && (30)
\end{aligned}$$

Substituting into the equation for $q_{\rm a}$ yields

$$\begin{aligned}
q_{\rm a} &= q - \frac{R''(1)}{z} && (31)\\
&= q - \frac{G''(1) - y^2G''(y)}{z} && (32)\\
&= q - \frac{G''(1)}{z} + \frac{y^2G''(y)}{z} && (33)\\
&= q - q + \frac{y^2G''(y)}{z} && (34)\\
&= \frac{y^2G''(y)}{z} && (35)\\
&= y^2Q'(y) && (36)
\end{aligned}$$

where we used $Q' = G''/z$ in the last step, which follows from $Q = G'/z$.

Alternatively we can first derive the equation for the excess degree generating function $Q_{\mathrm{a}} = Q(\tilde{A}y)/Q(y)$ and then compute the mean excess degree in the usual way,

$$\begin{aligned}
q_{\mathrm{a}} &= Q'_{\mathrm{a}}(1) && (37)\\
&= \frac{\partial}{\partial x}\frac{Q(\tilde{A}y)}{Q(y)}\bigg|_{x=1} && (38)\\
&= \frac{Q'(\tilde{A}y)\tilde{A}'y}{Q(y)}\bigg|_{x=1} && (39)\\
&= \frac{Q'(\tilde{A}(1)y)\tilde{A}'(1)y}{Q(y)} && (40)\\
&= \frac{Q'(y)\tilde{c}y}{Q(y)} && (41)\\
&= \frac{Q'(y)Q(y)y^2}{Q(y)} && (42)\\
&= y^2Q'(y) && (43)
\end{aligned}$$

which is again the desired result.

Degree generating function. Here we use the result from degree-targeted attacks, which tells us

$$\begin{aligned}
G_{\mathrm{a}} &= \frac{G(\tilde{A}) - R(\tilde{A})}{c} && (44)\\
&= \frac{G(\tilde{A}) - (G(\tilde{A}) - G(\tilde{A}y))}{c} && (45)\\
&= \frac{G(\tilde{A}y))}{c} && (46)\\
&= \frac{G(\tilde{A}y))}{G(y)}. && (47)
\end{aligned}$$

Excess degree generating function. Here we have again the choice between two different ways. Using the result from degree-targeted attacks we can write

$$\begin{aligned}
Q_{\mathrm{a}} &= \frac{G'(\tilde{A}) - R'(\tilde{A})}{z\tilde{c}} && (48)\\
&= \frac{G'(\tilde{A}) - (G'(\tilde{A}) - G'(\tilde{A}y))}{z\tilde{c}} && (49)\\
&= \frac{yG'(\tilde{A}y)}{z\tilde{c}} && (50)\\
&= \frac{yG'(\tilde{A}y)}{z}\frac{z}{yG'(y)} && (51)\\
&= \frac{G'(\tilde{A}y)}{G'(y)}. && (52)
\end{aligned}$$

To bring this into the desired form we need to expand the fraction by $1/z$ in order to use

$Q = G'/z$, which gives us

$$\begin{aligned} Q_{\rm a} &= \frac{G'(\tilde{A}y)}{G'(y)} && (53) \\ &= \frac{G'(\tilde{A}y)/z}{G'(y)/z} && (54) \\ &= \frac{Q(\tilde{A}y)}{Q(y)} && (55) \end{aligned}$$

which is what we wanted.

The alternative way to construct the excess degree generating function after the attack is to compute it from the degree generating function using $Q = G'/z$ applied to the functions after the attack, so

$$Q_{\rm a} = \frac{G'_{\rm a}}{z_{\rm a}} = \frac{G'_{\rm a}}{G'_{\rm a}(1)} \qquad (56)$$

To make use of this relationship, lets start by computing

$$\begin{aligned} G'_{\rm a} &= \frac{\partial}{\partial x}\frac{G(\tilde{A}y)}{G(y)} && (57) \\ &= \frac{G'(\tilde{A}y)\tilde{A}y}{G(y)} && (58) \\ &= \frac{G'(\tilde{A}y)\tilde{A}'y}{G(y)} && (59) \\ &= \frac{G'(\tilde{A}y)\tilde{c}y}{G(y)} && (60) \end{aligned}$$

It is now tempting to substitute the equation for $\tilde{c}$ that we derived above, but in this case it is unnecessary because this factor is about to get cancelled anyway as we substitute into the equation for $Q_{\rm a}$. Cancel culture, here we come,

$$\begin{aligned} Q_{\rm a} &= \frac{G'_{\rm a}}{G'_{\rm a}(1)} && (61) \\ &= \frac{G'(\tilde{A}y)\tilde{c}y}{G(y)}\frac{G(y)}{G'(\tilde{A}(1)y)\tilde{c}y} && (62) \\ &= \frac{G'(\tilde{A}y)}{G'(\tilde{A}(1)y)} && (63) \\ &= \frac{G'(\tilde{A}y)}{G'(y)} && (64) \\ &= \frac{G'(\tilde{A}y)/z}{G'(y)/z} && (65) \\ &= \frac{Q(\tilde{A}y)}{Q(y)} && (66) \end{aligned}$$