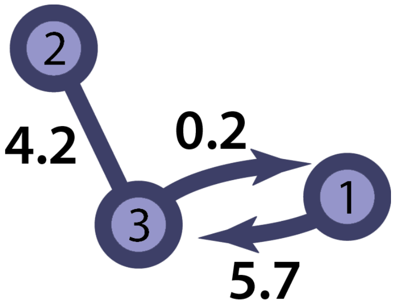
Networks and Complexity

Solution 23-2

This is an example solution from the forthcoming book Networks and Complexity. Find more exercises at https://github.com/NC-Book/NCB

Ex 23.2: Weighted Graph [2]

Consider particles diffusing on the following network:



a) Use Kirchhoffs's theorem to find the proportion of particles in each of the nodes in the steady state.

<u>*Solution*</u>
The system is described by the Laplacian matrix

$$\mathbf{L} = \begin{pmatrix} 5.7 & 0 & -0.2 \\ 0 & 4.2 & -4.2 \\ -5.7 & -4.2 & 4.4 \end{pmatrix} \tag{1}$$

We can now compute

$$S_1 = \begin{vmatrix} 4.2 & -4.2 \\ -4.2 & 4.4 \end{vmatrix} = 0.84 \tag{2}$$

$$S_2 = \begin{vmatrix} 5.7 & -0.2 \\ -5.7 & 4.4 \end{vmatrix} = 23.94 \tag{3}$$

$$S_3 = \begin{vmatrix} 5.7 & 0 \\ 0 & 4.2 \end{vmatrix} = 23.94 \tag{4}$$

We compute

$$S = 0.84 + 23.94 + 23.94 = 48.72 \tag{5}$$

and hence

$$p_1 = 0.84/48.72 \approx 0.02 \tag{6}$$
$$p_2 = 23.94/48.72 \approx 0.49 \tag{7}$$
$$p_3 = 23.94/48.72 \approx 0.49 \tag{8}$$

b) Is the system in equilibrium in the steady state?

Solution

Yes, if the network of states is a tree it always reaches equilibrium. The network from this exercise is still a tree in this sense despite the directed links.

We can verify the equilibrium explicitly by checking the detailed balance on the links. There is no flow between node 1 and 2 and the balance on the link 2-3 is immediately obvious. So we need to test the balance on the link 1-3. The flow from 1 to 3 is $0.02 \cdot 5.7 \approx 0.1$ and the flow from 3 to 1 is $0.49 \cdot 0.2 \approx 0.1$.