Networks and Complexity

Solution 14-12

This is an example solution from the forthcoming book Networks and Complexity. Find more exercises at https://github.com/NC-Book/NCB

Ex 14.12: The mechanical hydra [5]

This exercise proposes a hypothetical mechanical device. The first subquestions should be easy to solve. The final one is unsolved but probably solvable. We consider the ODE system

$$\begin{aligned}\dot{x} &= 2x - 4y \\ \dot{y} &= 3x - 5y.\end{aligned}$$

a) Determine the stability of the steady state at $x = y = 0$.

Solution

We compute the Jacobian matrix

$$\mathbf{J} = \begin{pmatrix} 2 & -4 \\ 3 & -5 \end{pmatrix}.$$

We compute the characteristic polynomial as

$$\left| \begin{pmatrix} 2-\lambda & -4 \\ 3 & -5-\lambda \end{pmatrix} \right| = \lambda^2 + 3\lambda + 2$$

Then, we solve

$$\begin{aligned} 0 &= \lambda^2 + 3\lambda + 2 \\ 0 &= (\lambda + 3/2)^2 + 2 - 9/4 \\ 1/4 &= (\lambda + 3/2)^2 \\ \pm 1/2 &= \lambda + 3/2 \\ \lambda &= -3/2 \pm 1/2 \end{aligned}$$

So, we have two eigenvalues

$$\lambda_1 = -1 \qquad \lambda_2 = -2.$$

Since both eigenvalues are negative the steady state is stable.

b) Now imagine that we changed this system by clamping some part in place. As a result y becomes fixed at $y = 0$ and cannot change anymore, which removes it as a variable. Write the differential equation that describes the dynamics of the remaining system after y has been fixed.

Solution

As the value of y is not prescribed by the algebraic equation $y = 0$ we don't need a differential equation for y anymore. In the differential equation for x we can substitute $y = 0$ to remove the remaining y, which leaves us with

$$\dot{x} = 2x$$

c) Determine the stability of the steady state in the resulting system. Is the result surprising?

Solution

The system still has a steady state at $x = 0$, which isn't very surprising. However if we compute the Jacobian of the new system we find

$$\mathbf{J} = 2,$$

hence

$$\lambda = 2 \tag{1}$$

which shows that the system is now unstable. This result is surprising: We started with a system were x and y sat stably at 0, but if we try to hold y there, then the stability is lost.

d) This system exhibits the so-called *hydra effect*. If left alone it settles into a stable steady state, but if we try to hold a certain part in place in this state, then the rest of the system becomes unstable. Your task is to design a mechanical system that exhibits this effect.

Solution

This is an unsolved question. But there doesn't seem to be a fundamental reason that would make such a system impossible. It would be really interesting to have a mechanical toy that is stable if it isn't held in place.