NETWORKS AND COMPLEXITY

Solution 9-2

This is an example solution from the forthcoming book Networks and Complexity.
Find more exercises at https://github.com/NC-Book/NCB

Ex 9.2: Coordinate shift [2]

Consider the function $f(x) = \sqrt{x}$. We are going to approximate the function around the point $x^* = 1$.

a) Introduce a new variable y, such that the point of interest ($x^* = 1$) is at $y^* = 0$.

Solution

We define

$$y = x - 1. \tag{1}$$

Using this relation we can confirm that $x^* = 1$ corresponds to $y^* = 0$.

b) Write f in terms of y, call the resulting function $h(y)$.

Solution

We can solve the defining equation of y for x, which yields

$$x = y + 1. \tag{2}$$

Substituting this equation into f yields

$$f(x) = \sqrt{x} = \sqrt{y+1} = h(y) \tag{3}$$

c) Approximate $h(y)$ by a function of the form $g(y) = c_0 + c_1 y$ around $y^* = 0$.

Solution

To find c_0 we evaluate h at $y^* = 0$. Let's write this using the bar notation

$$c_0 = h(0) = \sqrt{y+1}\Big|_{y=0} = \sqrt{1} = 1. \tag{4}$$

To find c_1 we differentiate and thereafter substitute the $y = 0$

$$c_1 = h'(0) = \frac{\partial}{\partial y}\sqrt{y+1}\Big|_{y=0} = \frac{1}{2\sqrt{1}} = \frac{1}{2} \tag{5}$$

hence

$$h(y) \approx g(y) = 1 + \frac{1}{2}y. \tag{6}$$

d) Write the approximation in terms of the original variable x.

Solution

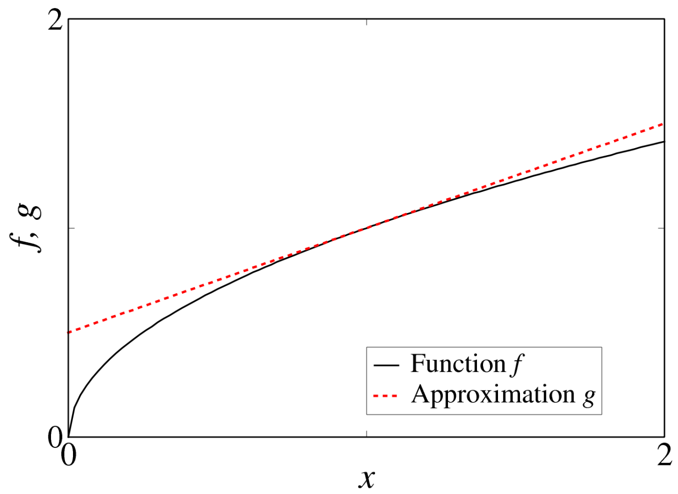
We translate back to x by using the definition of y in the form $y = x - 1$. Substituting into $g(y)$ yields

$$f(x) \approx 1 + \frac{1}{2}(x-1) \tag{7}$$

which we can simplify to

$$f(x) \approx \frac{1}{2} + \frac{1}{2}x \tag{8}$$

Here is a plot of the approximation for you:



e) Apply the same steps to approximate f up to quadratic order around $x^* = 2$.

Solution

In this case we need to define

$$y = x - 2, \tag{9}$$

Substituting $x = y + 2$ into f gives us

$$f(x) = \sqrt{x} = \sqrt{y+2} = h(y). \tag{10}$$

We are looking for an approximation of the form

$$g(y) = c_0 + c_1 y + c_2 y^2 \tag{11}$$

We find c_0 in the usual way

$$c_0 = h(0) = \sqrt{2}. \tag{12}$$

Furthermore we know,

$$c_1 = h'(0) = \frac{1}{2\sqrt{2}} \tag{13}$$

For the quadratic term we have to be careful because an extra factor of 2 appears in the Taylor formula

$$\begin{aligned}
c_2 &= \frac{1}{2!}\frac{\partial}{\partial y}h(y)\bigg|_{y=0} && (14)\\
&= \frac{1}{2}h''(0) && (15)\\
&= \frac{1}{2}\frac{\partial}{\partial y}\frac{1}{2\sqrt{y+2}}\bigg|_{y=0} && (16)\\
&= \frac{1}{2}\left(-\frac{1}{4}(y+2)^{-\frac{3}{2}}\right)_{y=0} && (17)\\
&= -\frac{1}{8\sqrt{8}} && (18)
\end{aligned}$$

So our solution in terms of y is

$$h(y) \approx \sqrt{2} + \frac{y}{2\sqrt{2}} - \frac{y^2}{16\sqrt{2}}. \qquad (19)$$

To find the solution in terms of x we substitute $y = x - 2$ which yields

$$\begin{aligned}
f(x) &\approx \sqrt{2} + \frac{x-2}{2\sqrt{2}} - \frac{(x-2)^2}{16\sqrt{2}} && (20)\\
&= \frac{1}{\sqrt{2}}\left(2 + \frac{x-2}{2} - \frac{(x-2)^2}{16}\right) && (21)\\
&= \frac{1}{\sqrt{2}}\left(2 + \frac{x}{2} - 1 - \frac{x^2}{16} + \frac{x}{4} - \frac{(1}{4}\right) && (22)\\
&= \frac{1}{\sqrt{2}}\left(\frac{3}{4} + \frac{3}{4}x - \frac{x^2}{16}\right) && (23)
\end{aligned}$$

This doesn't look like multiplying it out will make it simpler, so we leave it at that. Here is a plot for this result: