Networks and Complexity

Exercise Sheet 6: Friendship Paradox

This is an exercise sheet from the forthcoming book Networks and Complexity.
Find more exercises and solutions at https://github.com/NC-Book/NCB

Ex 6.1: From case-wise to Kronecker and back [1]

Write the following case-wise distributions using the Kronecker delta notation:

$$\text{a)} \quad p_k = \begin{cases} 3/4 & \text{if } k = 4 \\ 1/4 & \text{if } k = 8 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{b)} \quad p_k = \begin{cases} 1/3 & \text{if } k = 3 \\ 1/3 & \text{if } k = 4 \\ 1/3 & \text{if } k = 14 \\ 0 & \text{otherwise} \end{cases}$$

Then, write the following distributions using the case-wise notation:

$$\text{c)} \quad p_k = \frac{1}{6}\delta_{k,1} + \frac{1}{3}\delta_{k,2} + \frac{1}{2}\delta_{k,3} \qquad \text{d)} \quad p_k = \frac{1}{5}\delta_{k,5} + \frac{4}{5}\delta_{k,10}$$

Ex 6.2: Kroneckerisms [1]

Compute the following sums:

$$\text{a)} \sum k\delta_{k,3} \qquad \text{b)} \sum k\delta_{k+1,2} \qquad \text{c)} \sum 3\delta_{k,5} \qquad \text{d)} \sum 4k(\delta_{k,2} - \delta_{k,1})$$

Ex 6.3: Norm [2]

We now consider the two distributions from Ex. 6.1 (a) and (b) as degree distributions of networks. Show that these distributions are correctly normalized, i.e. $\sum p_k = 1$.

Ex 6.4: Mean degree [2]

For the networks described by the two degree distributions from Ex. 6.1 (a) and (b) compute the mean degree, $z = \sum kp_k$.

Ex 6.5: Mean excess degree [2]

We continue to explore the two networks from Ex. 6.1 (a) and (b). Now, compute the excess degree distribution $q_k = (k+1)p_{k+1}/z$ and the mean excess degree $q = \sum kq_k$. Note that we want to write the excess degree distribution such that k appears only in the index of Kronecker deltas.

Ex 6.6: Another abstract example [2]

Now consider the networks described by the two degree distributions from Ex. 6.1 (c) and (d). Compute the excess degree distributions q_k for these networks and then show that it is a correctly normalized probability distribution.

Ex 6.7: Manhattan [3]

In a city every road junction is a 4-way intersection. Write the degree distribution p_k for the network where intersections are represented by nodes and the road segments connecting them are links. Then show mathematically that $z = 4$ and $q = 3$.

Ex 6.8: Rumors [3]

In a company 75% of employees have three friends who are also working for the company. 20% have 6 friends in the company, and 5% have 71 friends in the company. All employees attend the company Christmas party. After some drinks Ada tells her friend Bob, an embarrassing secret about her private life. Over the course of the next week Bob shares Ada's secret with all his friends in the company. Find the expectation number for the number of people who know of Ada's secret? Would the situation be different if Ada had shared her secret with a random person at the party, say Chris, who had then told all his friends about it?

Ex 6.9: Pandemic [3]

During the zombie apocalypse 83% of the population self-isolate. They reduce their social contacts to a minimum and use protective measures. As a result they don't have any contacts in which the disease could be transmitted. Furthermore, 9% of the population carry on with their normal lives, which means they maintain contact with 20 people. The remaining 8% celebrate the end of days in wild parties, let's assume they have 90 contacts. The zombie disease is highly virulent and is guaranteed to spread to all people in contact with an infected person. Suppose person X catches the disease from a friend during the early stage of the pandemic, when almost nobody is infected.

a) What is the expected number of people that they will infect?
b) What is the expected number of people that are infected by the people that X infects (assume a locally treelike network $c = 0$)
c) Why are these estimates only valid for the early stage of the pandemic?

Ex 6.10: Diameter of heterogeneous networks [4]

Revisit our estimate of the network diameter (average node distance) from the previous chapter. In the derivation we assumed that our initial node has degree z. Further we assumed that if follow the links form this node we reach nodes that have in average z links in addition to the one we are arriving on. This means that we assumed $q = z$. Let us now relax this assumption. Follow the reasoning of Chap. 5 to derive a formula for the diameter of a locally treelike network with a given number of nodes N, mean degree z, and mean excess degree q, but clustering coefficient $c = 0$. Then, use your formula to estimate the diameter of a network with 10000 nodes, of which half have degree 2 and half have degree 8.

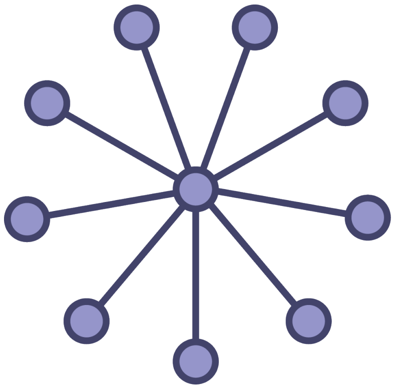
Ex 6.11: Normalization proof [4]

Suppose we start with a correctly normalized degree distribution p_k, such that $\sum p_k = 1$. If we calculate the corresponding excess degree distribution q_k we should get a distribution that is a proper probability distribution. Meaning it is also correctly normalized. But can you actually show that $\sum q_k = 1$, for every p_k that meets the normalization condition?

Ex 6.12: Degree Correlations [4]

In a sufficiently random network the expected degree of a random neighbor of a random node is $k_{\mathrm{nn}} = q + 1$. However, this isn't true if the nodes are connected in a specific way that causes strong assortativity/disassortativity.

a) For illustration, compute the average degree of a random neighbor of a random node, k_{nn}, for the star network from the following network:



Compare k_{nn} to q.

b) Suppose the degree distribution of a network is p_k and the probability that a neighbor of a node with degree k has degree j is $x_{k,j}$. Find a general formula for k_{nn}.

c) Write p_k and $x_{k,j}$ for network B in terms of Kronecker deltas and show that your formula for k_{nn} yields the expected result.