Networks and Complexity

Exercise Sheet 10: Attacks on Networks

This is an exercise sheet from the forthcoming book Networks and Complexity.
Find more exercises and solutions at https://github.com/NC-Book/NCB

Ex 10.1: Polynomial Long Division [1]

In the chapter we needed to solve $v = (0.8v + 0.2)^3$. Use the known solution $v = 1$ to reduce this equation to a quadratic polynomial by polynomial long division. [This is not essential, but a neat trick. Just check the solution if you haven't seen it before.]

Ex 10.2: Excess degree rule [2]

Use a generating function calculation to show that removing a proportion r of the links from a network at random, also decreases the mean excess degree by a proportion r.

Ex 10.3: Some quick calculations [2]

The following questions can be answered with very short calculations:

a) A configuration model network has mean excess degree $q = 5$. We attack it by random node removal. What proportion of nodes doe we have to remove to break the giant component?

b) A configuration model network has mean degree 3 and mean excess degree 7. We attack it by removing nodes of degree 10. What proportion of nodes do we need to remove to break the giant component?

c) Show that removing nodes of degree 0 or 1 from a configuration model network does not change the mean excess degree.

Ex 10.4: Mail worm [2]

In a company employees have in average regular email contact to 6 other employees. A survey finds that 50% of employees are careless when opening email attachments, which may allow mail worms to spread. Assume that the network is an ER graph, if a careless employee receives the mail worm, the employee will open it, becoming infected, and the worm will be sent to all of the employee's contacts. Careful employees will not become infected and will not allow the worm to spread. What is the chance that there is a major outbreak if a random employee receives the worm? How large will the outbreak be?

Ex 10.5: Abstract Targeted Attack [2]

Consider a ER random graph of N nodes with mean degree z. An attack removes all nodes of degree 0, half the nodes of degree 1, a quarter of the nodes of degree 2 and so on. Compute the size of the network N_{a}, and the mean excess degree q_{a} after the attack.

Ex 10.6: Abstract viral attack [3]

Consider a configuration model network with degree distribution $p_k = 0.8\delta_{k,2} + 0.2\delta_{k,8}$. A viral attack can spread over 40% of the links in the network. Is there still a giant component after the nodes affected by the viral attack have been removed?

Ex 10.7: Too big to fail [3]

Consider a network where nodes are banks and links represent financial relations. If one bank defaults on obligations there is a risk that it's neighbors in the network default as well. Consider a network where 95% of the nodes have degree 5 while 5% have degree 25. However, let's assume that only 20% of links (distributed randomly) transmit defaults to neighboring institutions, whereas the rest of the links are weak enough that a default of the linked banks is not transmitted to its partner. What is the chance that the default of a random bank will lead to many bank failures? Also, how could we make this model more realistic?

Ex 10.8: Island Meta-community [3]

Consider a system of grassy small islands. Individually each island is too small to sustain animal populations for an extended amount of time, due to inbreeding. However, sometimes, during an especially low tide, animals can cross between certain islands. We can describe this system as a network where nodes are islands and links are possible crossings. Assume that this system is an ER network with mean degree $z = 6$.

a) A species of sheep live on these islands, but they can only maintain the necessary genetic diversity on those islands that are in genetic exchange with many other islands (i.e. those forming a giant component of sheep). Moreover, sheep are not good swimmers so they can cross 20% of the network links. Compute the proportion of islands on which sheep can persist.
b) There is also a species of rabbits on the islands. Again they can only survive by forming a giant component of connected populations. They are surprisingly good swimmers so they can use all the paths between islands. However, the rabbits can't colonize any islands where sheep are present.

Ex 10.9: Targeted attacks in general [3]

In this exercise we derive some results on degree-targeted attacks in general. Consider a configuration model network, described by a degree generating function G, that is subject to an attack described by a removal function R. Try to do the calculations below in the forward direction (deriving the result) instead of the backward direction (proving the result is true).

a) Show that the degree distribution of the surviving nodes after the first step of the attack is generated by $G_{\mathrm{h}} = \frac{G-R}{c}$.
b) Show that the mean degree after the attack is $z_{\mathrm{a}} = z\tilde{c}^2/c$.
c) Show that the excess degree distribution after the attack is generated by $Q_{\mathrm{a}} = (G'(\tilde{A}) - R'(\tilde{A}))/\tilde{c}z$.
d) Show that the mean excess degree after the attack is $q_{\mathrm{a}} = q - R''(1)/z$.

Ex 10.10: Viral attacks in general [3]

Now, we derive some results on the viral attack. Let G be the degree generating function before the attack, and y the proportion of links that are not part of the giant conducting component.

a) Show that a viral attack starting within the giant conducting component is described by the attack generating function

$$R = G - G(xy)$$

Note that we have omitted the argument (x) behind R and the first G, in accordance with our notation.

b) Now start from $R = G - G(xy)$ and show that the properties of the network after the viral attack are given by

$$N_{\mathrm{a}} = G(y)N,$$

$$z_{\mathrm{a}} = y^2 \frac{Q(y)}{G(y)} G'(y), \qquad q_{\mathrm{a}} = y^2 Q'(y),$$

$$G_{\mathrm{a}} = \frac{G(\tilde{A}y)}{G(y)}, \qquad Q_{\mathrm{a}} = \frac{Q(\tilde{A}y)}{Q(y)},$$

where $\tilde{A} = \tilde{c}x + \tilde{r}$ and $\tilde{c} = 1 - \tilde{r} = yG'(y)/z = yQ(y)$.

Ex 10.11: Messaging App [4]

I have been developing a new messaging app. Of course the last thing everybody needs is another messaging app. However, I think that people will adopt my app if at least one quarter of their friends use the app. I can't really afford advertisement so I will just hope that it spreads on its own.

a) Consider a configuration model network where the nodes are users and links mean that the users regularly exchange messages. Based on your understanding of random networks what degrees do you expect the initial adopters of the messaging app to have.

b) Assume that the network is described by a general degree distribution p_k. Remove all people from the network who are not likely early adopters. Then find the conditions under which there is still a giant component such that the app is able to spread.

Ex 10.12: The most robust network? [4]

Excess degree makes a network robust against breaking of the giant component. But consider that an ER network with $N = 10,000$ nodes and $K = 20,000$ links has only excess degree $q = 4$. Given this number of nodes and links, how would you build a connected network that has maximally high excess degree? Show that you can reach $q = 4999.00075$.