Networks and Complexity

Solution 9-9

This is an example solution from the forthcoming book Networks and Complexity.
Find more exercises at https://github.com/NC-Book/NCB

Ex 9.9: Solving series [3]
Use generating functions to find the solutions of $\sum c_k$, where $c_0 = 1$ and

$$\text{a) } c_{k+1} = \frac{c_k}{2} \qquad \text{b) } c_{k+1} = -c_k/3 + 1$$

Hint: You might want to use the result of Ex. 9.6

Solution
For part (a) we define the generating function

$$G = \sum c_k x^k \tag{1}$$

We can write the iteration rule in the form

$$c_k = 2c_{k+1}. \tag{2}$$

Substitution into G yields

$$G = \sum 2c_{k+1} x^k. \tag{3}$$

We shift the index of the sum, which yields

$$G = -\frac{2c_0}{x} + \sum 2c_k x^{k-1}. \tag{4}$$

To make the right-hand side more similar to the definition of G we write it as

$$G = \frac{2}{x}\left(-c_0 + \sum c_k x^k\right). \tag{5}$$

We can now use the definition of G and $c_0 = 1$ to find the self-consistency condition

$$G = \frac{2}{x}(G-1), \tag{6}$$

which we can solve for

$$G = \frac{2}{2-x}. \tag{7}$$

We can now evaluate

$$\sum c_n = G(1) = \frac{2}{2-1} = 2, \tag{8}$$

which is the desired result.

For part (b) we start again by defining

$$G = \sum c_k x^k \tag{9}$$

and solving the iteration rule for c_k,

$$c_k = 3(1 - c_{k+1}). \tag{10}$$

Substitution into G yields

$$G = \sum 3(1 - c_{k+1})x^k. \tag{11}$$

It is useful to split this sum up and pull the factor of 3 out.

$$G = 3\left(\sum x^k\right) - 3\left(\sum c_{k+1}x^k\right) \tag{12}$$

We know from Ex. 9.8 that the term in the first bracket is the series expansion of $1/(1-x)$. To deal with the second term we start again with the index shift

$$G = \frac{3}{1-x} - 3\left(\sum c_k x^{k-1}\right) + \frac{3}{x}c_0. \tag{13}$$

After pulling $1/x$ from the sum we have

$$G = \frac{3}{x}\left(\frac{x}{1-x} + c_0 - \sum c_k x^k\right). \tag{14}$$

We can now use $c_0 = 1$ and identify G, which yields

$$G = \frac{3}{x}\left(\frac{x}{1-x} + 1 - G\right). \tag{15}$$

The 1 actually combines beautifully with the fraction that precedes it, so that

$$G = \frac{3}{x}\left(\frac{1}{1-x} - G\right). \tag{16}$$

Isolating G, we can write

$$(x-3)G = \frac{3}{1-x} \tag{17}$$

and hence the result

$$G = \frac{3}{(x-3)(1-x)}. \tag{18}$$

So, $G(1) = \infty$, this is a divergent series.