Networks and Complexity

Solution 13-4

This is an example solution from the forthcoming book Networks and Complexity. Find more exercises at https://github.com/NC-Book/NCB

Ex 13.4: Abstract two-dimensional map [2]

Consider the following system:

$$\begin{aligned} X_{i+1} &= -X_i + Y_i \\ Y_{i+1} &= X_i - Y_i \end{aligned}$$

a) Write the system in matrix form.

<u>*Solution*</u>
We write

$$\boldsymbol{x_{i+1}} = \mathbf{J}\boldsymbol{x_i}$$

where

$$\mathbf{J} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

and $\boldsymbol{x_i} = (X_i, Y_i)^{\mathrm{T}}$.

b) Compute the eigenvalues of the matrix that appears in (a).

<u>*Solution*</u>
We write the characteristic polynomial

$$\begin{aligned} 0 &= (-\lambda - 1)(-\lambda - 1) - 1 \\ 0 &= (\lambda + 1)(\lambda + 1) - 1 \\ 0 &= \lambda^2 + 2\lambda \end{aligned}$$

At this point we can see that $\lambda_1 = 0$ and $\lambda_2 = -2$.

c) Find the corresponding eigenvectors and write the initial state $X_0 = 2$, $Y_0 = 0$ as a linear combination of eigenvectors.

<u>*Solution*</u>
For the first eigenvector we consider the first line of the matrix which leads to the condition

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0 \begin{pmatrix} X \\ Y \end{pmatrix}$$

considering either line leads to

$$X = Y$$

and hence eigenvectors the eigenvector

$$\boldsymbol{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The second eigenvector must obey the condition

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = -2 \begin{pmatrix} X \\ Y \end{pmatrix}$$

Considering the first line leads to

$$-X + Y = -2X$$

or in other words $Y = -X$ and hence to

$$\boldsymbol{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

or multiples thereof. With these eigenvectors it is straight forward to expand the initial state as

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(If you don't see this immediately, try the ansatz $\boldsymbol{x_0} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2}$ then consider the lines separately. They will say $c_2 = c_1$ and $c_1 + c_2 = 2$ respectively which leads to $c_1 = c_2 = 1$.)

d) Hence, solve the initial value problem (i.e. find X_i, Y_i for all $i > 0$).

Solution
For $i > 0$ we can write

$$\begin{aligned} \boldsymbol{x_i} &= \mathrm{J}^i \boldsymbol{x_0} \\ &= \mathrm{J}^i (\boldsymbol{v_1} + \boldsymbol{v_2}) \\ &= \mathrm{J}^i \boldsymbol{v_1} + \mathrm{J}^i \boldsymbol{v_2} \\ &= {\lambda_1}^i \boldsymbol{v_1} + {\lambda_2}^i \boldsymbol{v_2} \\ &= (-2)^i \boldsymbol{v_2} \end{aligned}$$

Hence, the solution is

$$X_i = (-2)^i \qquad Y_i = -(-2)^i$$

e) Verify your solution by computing the first X_1, Y_1, X_2, Y_2 by hand.

Solution
Using the formula from the question

$$X_1 = -X_0 + Y_0 = -2 = (-2)^1$$

$$Y_1 = X_0 - Y_0 = 2 = -(-2)^1$$

$$X_2 = -X_1 + Y_1 = 2 + 2 = 4 = (-2)^2$$

$$Y_2 = X_1 - Y_1 = -2 - 2 = -4 = -(-2)^2$$

So, this works.

f) Bonus: Using matrix methods, show that for any initial condition $X_i = -Y_i$ must hold for all $i \geq 1$.

<u>*Solution*</u>

For a given initial state $\boldsymbol{x_0}$ we can find c_1 and c_2 such that

$$\boldsymbol{x_0} = c_1 \boldsymbol{v_1} + c_2 \boldsymbol{v_2} \tag{1}$$

Starting from this state the general solution is

$$\boldsymbol{x_i} = \mathbf{J}^i \boldsymbol{x_0} = c_1 {\lambda_1}^i \boldsymbol{v_1} + c_2 {\lambda_2}^i \boldsymbol{v_2} \tag{2}$$

Because $\lambda_1 = 0$ this simplifies to

$$\boldsymbol{x_i} = c_2 {\lambda_2}^i \boldsymbol{v_2} \tag{3}$$

which means

$$X_i = c_2(-2)^i \tag{4}$$
$$Y_i = -c_2(-2)^i \tag{5}$$

And hence $X_i = -Y_i$.