Networks and Complexity

Solution 13-5

This is an example solution from the forthcoming book Networks and Complexity.
Find more exercises at https://github.com/NC-Book/NCB

Ex 13.5: Abstract linear differential equation system [2]

Consider the dynamical system

$$\begin{aligned} \dot{x} &= 2x - 9y \\ \dot{y} &= -x + 2y \end{aligned}$$

where the initial state is $x = y = 1$.

a) Write the system in matrix form.
b) Write the initial state as a linear combination of the eigenvectors of the matrix that appears in (a).
c) Solve the system (i.e. compute $x(t)$ and $y(t)$).

Solution

We write the system in the form

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 2 & -9 \\ -1 & 2 \end{pmatrix} \boldsymbol{x}$$

where $\boldsymbol{x} = (x, y)^T$. We write the characteristic polynomial

$$\begin{aligned} 0 &= (2-\lambda)^2 - 9 \\ \lambda &= 2 \pm 3 \end{aligned}$$

We find the first eigenvector from

$$\begin{pmatrix} 2 & -9 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ a \end{pmatrix} = 5 \begin{pmatrix} 1 \\ a \end{pmatrix}$$

the first line reads

$$2 - 9a = 5 \tag{1}$$

which means $a = -1/3$ so the eigenvector for the $\lambda_1 = 5$ is

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -1/3 \end{pmatrix}$$

Because an eigenvector stays an eigenvector if we rescale it we can multiply it by 3 which gives a nicer version of the same eigenvector

$$\begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

but for the purpose of this exercise we will stick with the $\boldsymbol{v}_1$ that we found above.

For the second eigenvector we write

$$\begin{pmatrix} 2 & -9 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ a \end{pmatrix} = -1 \begin{pmatrix} 1 \\ a \end{pmatrix}$$

the first line reads

$$2 - 9a = -1$$

and hence $a = 1/3$. So the eigenvector for $\lambda_2 = -1$ is

$$\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 1/3 \end{pmatrix}$$

We now need to write the initial condititon as a combination of eigenvectors. We are looking for c_1 and c_2 such that

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1/3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1/3 \end{pmatrix}$$

Form the first row we can see that

$$c_1 + c_2 = 1$$

. and hence

$$c_2 = 1 - c_1$$

From the second row we get

$$\frac{1}{3}c_2 - \frac{1}{3}c_1 = 1$$

We multiply the equation by 3 which gives

$$c_2 - c_1 = 3$$

Substituting $c_2 = 1 - c_1$ from above yields

$$1 - 2c_1 = 3$$

Solving this we find

$$c_1 = -1$$

And using either of the relations above

$$c_2 = 2$$

We know from the lecture that the solution has the form

$$\boldsymbol{x}(t) = c_1 \mathrm{e}^{\lambda_1 t} \boldsymbol{v}_1 + c_2 \mathrm{e}^{\lambda_2 t} \boldsymbol{v}_2$$

so in this case

$$\boldsymbol{x}(t) = -\mathrm{e}^{5t} \begin{pmatrix} 1 \\ -1/3 \end{pmatrix} + 2\mathrm{e}^{-t} \begin{pmatrix} 1 \\ 1/3 \end{pmatrix}$$