Networks and Complexity

Exercise Sheet 19: The Most Important Node

This is an exercise sheet from the forthcoming book Networks and Complexity.
Find more exercises and solutions at https://github.com/NC-Book/NCB

Ex 19.1: Spectral shift [1]

We claimed that adding the identity matrix $\mathbf{I}$ to a matrix $\mathbf{A}$ shifts the spectrum of $\mathbf{A}$ by 1. More generally we can say the following: Assume we have two matrices $\mathbf{K}$ and $\mathbf{L}$, such that

$$\mathbf{L} = \mathbf{K} + c\mathbf{I}. \tag{1}$$

Then for every eigenvalue λ of $\mathbf{K}$ there is an eigenvalue $\lambda + c$ of $\mathbf{L}$. The eigenvectors corresponding to these to eigenvalues are identical.

Can you actually prove this? (Hint: You want to show $\mathbf{K}\boldsymbol{v} = \kappa\boldsymbol{v}$, where $\kappa = \lambda + c$. You define $\boldsymbol{v}$ as an eigenvector of $\mathbf{L}$ with eigenvalue λ.)

Ex 19.2: When things go wrong [2]

We discovered that the iterative procedure for the leading eigenvalue fails in bipartite networks. Let's try it nevertheless.

a) Consider

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the initial vector

$$\boldsymbol{v}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Use the iteration

$$\boldsymbol{v}_{i+1} = \mathbf{A}\boldsymbol{v}_i$$

a few times and see what happens.

b) Define the shifted matrix $\mathbf{B} = \mathbf{A} + \mathbf{I}$ and try again.

c) Based on the result from (b) state the largest eigenvalue and the eigenvector of $\mathbf{A}$.

d) Bonus: This one is more tricky, but note that in this case the iteration lands us exactly on the eigenvalue in the first step. This actually tells us what the second eigenvalue of $\mathbf{A}$ is. Explain!

Ex 19.3: Epidemic Centrality [3]

Let's find the most important node for a spreading process.

a) Consider an SIS epidemic on a network described by an adjacency matrix $\mathbf{A}$. Explain why $x_i(t)$, the probability that node i is infected at time t can be modelled by

$$\dot{x}_i = -rx_i + p(1 - x_i)\sum_j A_{ij}x_j \tag{13}$$

b) So far the equation is non-linear. To use spectral methods we need to linearize it. Consider the limit in which only a small proportion of nodes is infected at any time and use it to find a linear approximation to the differential equations. (There are two ways to do this, if you see only one check the solution for the other.)

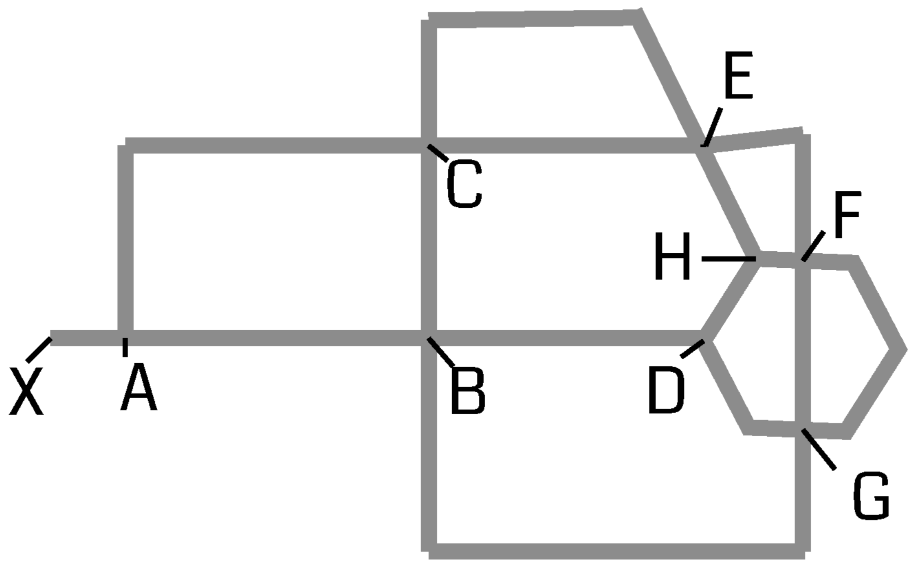
c) Show that the system can now be written in the matrix form $\dot{\boldsymbol{x}} = \mathbf{J}\boldsymbol{x}$. Find a way to express the matrix $\mathbf{J}$ that appears as a function of $\mathbf{A}$.

d) The elements of the leading eigenvector of $\mathbf{J}$ now indicate who is most at risk in the epidemic. Explore how the largest eigenvalue λ and the corresponding eigenvector $\boldsymbol{v}$ is linked to the adjacency matrix $\mathbf{A}$.

e) Bonus questions: In an epidemic the nodes who are most at risk also poses the greatest risk to others. This may be intuitive but how can we see it mathematically?

Ex 19.4: Traffic: Spectral Centralities [3]

Consider the following road network



Find the relative importance of the nodes using the spectral centrality metric (Hint: mind the parallel links). Is the result reasonable?

Ex 19.5: Traffic: Degree centrality [3]

Consider the network from the previous exercise again, but this time determine the degree centrality. Is the result reasonable?

Ex 19.6: Traffic: Closeness centrality [3]

Consider the traffic network again, but this time compute the closeness centrality. (You can consider the length of each link to be one.) Is the result reasonable?

Ex 19.7: Traffic: Betweenness centrality []

Consider the traffic a final time, and this time rank the nodes by betweenness centrality. (If there is multiple alternative shortest paths between nodes assign importance proportionally, e.g. half a point if there are two paths.) Is the result reasonable?

Ex 19.8: Traffic: Summary [3]

In the past exercises we have computed several different types of centralities. For each of these state a context where the respective type of centrality of the road network would be relevant.

Ex 19.9: Problem with multiple components []

If our network has multiple components it is hard to compare the importance of nodes in different components. When the iterative eigensolver from the lecture is applied then the result that we get for the spectral centrality can depend on the initial vector that we use to start the iteration. Construct an example to illustrate this.

Ex 19.10: Five star [4]

Consider a 5-star, a single central node of degree 5 that is connected to 5 nodes of degree 1. Start using the procedure from the lecture to start compute the leading eigenvector. After a few steps use the insight gained to compute the eigenvector exactly. (This might be very hard or very easy)