Solution 23-1

This is an example solution from the forthcoming book Networks and Complexity.
Find more exercises at https://github.com/NC-Book/NCB

Ex 23.1: Circular reasoning [2]

While I worked in Davis I was struggling to buy enough drinking water. Basically I distinguish three states: a) "I have enough water", b) "Water in short supply", c) "Not a single drop". When I am in state a, I usually transition to state b at rate 1. If I am in state b, then I try to buy water. This happens happens also at rate 1 and takes me back to state a. More commonly, at rate 2, I transition to state c. In state c I have a good incentive to go shopping. This happens at rate 3 and takes me back to state a. However, sometimes when I am in state c, I just grab one bottle on the way back from work. This takes me back to state b at rate 1. Use Kirchhoff's theorem to determine the proportion of the time I spend in state c.

Solution

The setting from the question is described by the Adjacency matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \tag{1}$$

this leads to the Laplacian

$$\mathbf{L} = \begin{pmatrix} 1 & -1 & -3 \\ -1 & 3 & -1 \\ 0 & -2 & 4 \end{pmatrix} \tag{2}$$

We now compute the determinants of the minors. For node 1 this is

$$m_1 = \begin{vmatrix} 3 & -1 \\ -2 & 4 \end{vmatrix} = 12 - 2 = 10 \tag{3}$$

For node 2

$$m_2 = \begin{vmatrix} 1 & -3 \\ 0 & 4 \end{vmatrix} = 4 - 0 = 4 \tag{4}$$

And, for node 3

$$m_3 = \begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} = 3 - 1 = 2 \tag{5}$$

So the total proportion of time I spend without water is

$$T_3 = \frac{m_3}{m_1 + m_2 + m_3} = \frac{2}{2 + 4 + 10} = \frac{1}{8} \tag{6}$$

On the other hand I spend half the time with plenty of water.