Networks and Complexity

Exercise Sheet 15: Social Distancing

This is an exercise sheet from the forthcoming book Networks and Complexity. Find more exercises and solutions at https://github.com/NC-Book/NCB

Ex 15.1: Minority game [2]

Consider a network of agents which can be in either of two states A, or B. No agent wants to be in the same state as its neighbors. Links in the network are checked at rate r. When a link is checked, the agents connected by the link compare their state. If they are in the same state, one of them chosen randomly switches to the respectively other state. For example if an AA-link is checked one of the agents would switch to state B. Write a mean field model, find its steady state and check the stability.

Ex 15.2: Voter model mean field [2]

Consider a network with N nodes and K links, where nodes represent agents and links represent social contacts. Each agent is in either of two states, say favoring one of two political parties. We call these states A and B. Further we say a link is an active link if it connects agents with different opinions, i.e. it is an AB-link. Active links get updated at rate r. In an update one of the two agents connected by the link is chosen at random and that agent's opinion is copied to the other agent. Use a mean field approximation to derive a system of differential equations for this model. (The result may be surprising)

Ex 15.3: Rock-paper-scissors [2]

Consider a network of agents playing the rock-paper-scissors game. Each agent has a state which can be either rock (R), paper (P), or scissors (S). At some rate we pick a random link and the game is played by the two agents that the link connects: If both players are in the same state the game is a draw and nothing happens. Otherwise R wins against S, S wins against P, and P wins against R. Whoever is the loser then changes it's state to the strategy that would have won the game. For example if R plays against P, R loses and switches to S. Write mean field equations for the proportion of players with a given strategy, $[R]$, $[P]$, $[S]$, and find the steady states.

Ex 15.4: A Meta-community [3]

Consider an archipelago containing many islands. Some islands are Barren (B), on some islands there is grass (G), and some islands there is not only grass, but even some sheep (S). We can think of the islands as a network with N nodes and K links. The links are actually routes of human travel, and from time seeds of grass stuck to human shoes and clothing allow the grass from one island to colonize a neighboring barren island. Thus grass can spread from one island where it is present (type G or S) to a neighboring barren island at rate c.

Sometimes humans also carry sheep from one island to the another. Thus sheep can spread from an island of type S to a neighboring island of type G at a low rate l.

However life on the islands is dangerous and at a rate m the grass on an island of type G or S goes extinct and if there is a population of sheep on the island they also go extinct.

a) Write differential equations that govern the variables $[B]$, $[G]$, S, denoting the proportion of barren, grassy and sheep islands as a function of link densities, e.g. [BG]

b) Use a meanfield approximation to replace the link densities in your equations with functions of the node densities.
c) Now identify a suitable conservation law to remove the variable $[B]$ from the system.
d) You may have noted that $[G]+[S]$ appears quite a lot. This makes sense since $[G]+[S]$ is the proportion of all islands with grass. Let's define a new variable $[T]=[G]+[S]$. Derive a differential equation for this variable, and use it to replace the differential equation for $[G]$.
e) We now want to find the steady states of the system. Ecological intuition tells us that there should be at least three of them, which we can call "no grass", "no sheep" and "grass and sheep". Compute these steady states.
f) Find the minimal connectivity z_g from which on grass can persist in the archipelago and the also the minimal connectivity z_s from which on sheep can persist.

Ex 15.5: Competing epidemics [3]

Consider two competing SIS diseases I and J, which spread at different rates $p_1 \neq p_2$ but from which people recover at the same rate r. Assume that being infected with one of the diseases gives the infected person immunity to the other disease (This is actually the case for many pairs of diseases, e.g. tuberculosis and leprosy). Use a mean field approximation to show that the system cannot be in a stationary state where both diseases persist.

Ex 15.6: Adaptive Voter Model [3]

We now continue the adaptive voter model. In this model the nodes represent agents, each of whom holds one of two possible opinions, A or B. Links represent social interaction and a link is considered active if it connects agents holding different opinions. At rate 1 these AB-links are updated. In the update, with probability p, one of the agents copies the others opinion. Otherwise, with probability $\bar{p}=1-p$, the link is rewired. In the former case (opinion update) one of the agents is chosen at random who will keep their opinion, which the respectively other agent now copies. In the latter event (rewiring) the link connecting the agents is cut. A new link is then made from one of the two agents (chosen randomly) to a new partner picked randomly from all agents holding the same opinion (e.g. If an agent of opinion A gets to create a new link, they always link to a random agent who holds also opinion A)

a) As in the non-adaptive voter model, it is easy to show that the number of agents holding a certain opinion, does not change deterministically ($\dot{[A]}=\dot{[B]}=0$). However there is interesting dynamics of the links. Dervie differential equations for $\dot{[AA]}$ and $\dot{[BB]}$ and close them. (Don't use the conservation law yet.)
b) Find the steady states of the system. (Hint: You will need to use a conservation law, but delay this as long as possible to preserve the symmetry)
c) Find an estimate for the value of p at which the network fragments.

Ex 15.7: aSIS model [4]

We now consider the adaptive SIS model. This model is identical to the network SIS model that we studied before, but additionally susceptible nodes try to distance themselves from infected neighbors. FOr every SI link,t the S node cuts the links at rate w. Whenever this happens the S-node establishes a new link to a randomly chosen S node. Hence, the number of links in the model remains constant.

a) Write differential equations for the density of infected nodes $[I]$ and a conservation law that governs the density of susceptibles $[S]$. Express the right hand side of the differential equation in terms of node and link densities, don't close it yet.

b) Derive differential equations for the $[SS]$ and $[II]$ link densities and find a conservation law that governs the $[SI]$ link density.
c) Use a pair approximation to close your differential equation system.
d) We now have a dynamical system with 3 dynamical variables $[I], [SS], [II]$ and two auxilliary variables governed by conservation laws. Compute the Jacobian matrix that governs of this system. (Note that you either have to substitute the conservation laws or consider the change of the auxilliary variables in your derivatives.)
e) Find the steady state where the epidemic is extinct. Evaluate the Jacobi matrix in that steady state.
f) You should be able to read off one eigenvalue of the Jacobian straight away. For reasonable choices of parameters this eigenvalue is always negative. The other two eigenvalues are given by a two by two matrix. Identify this reduced matrix.
g) We could now compute the eigenvalues of the remaining 2x2 matrix. But we can also make our life a bit easier. The symmetry of the system (extinct steady state is fixed at zero) tells us that a bifurcation occurring on the extinct state will likely be a transcritical. Hence we expect the epidemic threshold to be a transcritical bifurcation. In this bifurcation one eigenvalue becomes zero and therefore also the determinant becomes zero. Use this to find the epidemic threshold and express it in terms of the critical value of p from which on the disease can invade a healthy network.

Ex 15.8: SIS percolation []

In this exercise we find the epidemic threshold of the SIS model using a different approach:

a) Consider a single infected node in a large network where every other node is susceptible. How long do you expect the infected node to stay infected until it returns to the susceptible state?
b) While the initial node is infected it can infect other nodes. Estimate the initial rate at which it causes such infections.
c) Now estimate the basic reproductive number R_0, i.e. the number of infections that are directly caused by our initial infected node by multiplying the time that the initial node is infected with the rate at which it causes infections.
d) The disease can invade the population when $R_0 > 1$, hence $R_0 = 1$ is the epidemic threshold. Use this to find the critical value of p where the epidemic threshold occurs.

Ex 15.9: Heterogeneous Expansion [4]

Consider a network with N nodes and mean degree z. Existing links in this network are broken at a rate 1 per link. At rate r per node a node establishes a link to a randomly chosen partner.

a) The dynamics of the degree distribution are captured by the differential equation

$$\dot{p_k} = -\underbrace{2rp_k}_{1} - \underbrace{kp_k}_{2} + \underbrace{2rp_{k-1}}_{3} + \underbrace{(k+1)p_{k+1}}_{4} \tag{152}$$

Explain the meaning of the terms 1-4.

b) Define $G(x)$ as the generating function of the degree distribution. And derive a differential equation that captures the dynamics of G.
c) Consider the steady state of your equation and integrate to obtain $G(x)$. (Use the normalization conditions to fix the constant of integration.)
d) We have now found the generating function that the network approaches in the long run. What does it tell you about the network.

Ex 15.10: Triplets! [4]

Derive a differential equation for $[ABA]$ in the adaptive voter model. (you do not need to close the expansion)

Ex 15.11: SIS ISI ODE TLA [4]

For the aSIS model from above. Derive a differential equation for $[ISI]$ the density of ISI-chains. (Hint, you might need symbols of the form $[ABCD]$, density of four node chains with states A,B,C,and D, and $[{}^{B}A^{C}_{D}]$, density of nodes in state A with neighbors in state B, C and D.)

Ex 15.12: Higher closures [5]

It is easy to find different plausible closure approximations for motifs such as $[{}^{B}A^{C}_{D}]$, but which one is the right one?