Networks and Complexity

Exercise Sheet 12: Models of Growth

This is an exercise sheet from the forthcoming book Networks and Complexity. Find more exercises and solutions at https://github.com/NC-Book/NCB

Ex 12.1: Integration [1]

Find the general solutions for the following differential equations:

a.
$$\dot{x} = 4x$$

b.
$$\dot{x} = \frac{1}{2nx}$$

c.
$$\dot{x} = \frac{x}{a+t}$$

d.
$$\frac{\mathrm{d}c_k}{\mathrm{d}k} = -\frac{2c_k}{k}$$

Ex 12.2: Mass action laws [1]

Write the differential equations for x, y and z that correspond to the reaction diagram

$$\begin{aligned} \mathrm{X} + 2\mathrm{Y} &\xrightarrow{r} \mathrm{Z} \\ 2\mathrm{Z} &\xrightarrow{s} \emptyset \end{aligned}$$

Ex 12.3: Infinite excess degree [1]

Consider a network with $p_0 = 0$ and $p_k = ak^{-3}$ for $k > 0$. Show that this network has finite mean degree but infinite mean excess degree. You can use the $\sum_{k=1}^{\infty} k^{-2}$ is finite while $\sum_{k=1}^{\infty} k^{-1} = \infty$. [Hint: Don't try to determine a, it's complicated.]

Ex 12.4: Population growth [2]

Sexual reproduction requires that two individuals of a species meet. If we apply the law of mass action naively we may arrive at a model of the form $\dot{x} = x^2$. Solve the initial value problem for $x(0) = 1$ and explore what happens.

Ex 12.5: Birth and death [3]

A population of bacteria X grows by cell division at rate g

$$\mathrm{X} \xrightarrow{g} 2\ \mathrm{X}$$

In addition the bacteria die spontaneously at rate l

$$\mathrm{X} \xrightarrow{l} \emptyset$$

Derive an ODE for the population size x and solve it.

Ex 12.6: The leaky sink [3]

Water is flowing into a sink at a rate a liters per second. But the stopper that is supposed to plug the outflow at the bottom does not fit well such that a little bit of water can leaks out. The rate at which water is proportional to the water pressure at the bottom of the sink. Hence the outflow is b liters per second for every liter of water that is currently in the sink.

a) Write a differential equation for x, the volume of water that is in the sink.
b) Find the steady state.
c) Find the general solution to the differential equation. (Hint: If stuck consider the variable $\delta = x^* - x$)
d) Solve the initial value problem for a sink that is initially empty.

Ex 12.7: Condensation [3]

Imagine a bathroom mirror. When you take a shower drops of condensation begin to form on the cold surface. The drops grow by absorbing water vapor through their surface. Hence the rate at which volume is absorbed is proportional to the radius r of the droplet squared. Because the droplet is a three-dimensional object its volume scales as the r^3. Hence it is reasonable to model the growth of the droplet by

$$\dot{v} = ar^2 = bv^{2/3} \tag{77}$$

where a and b are parameters and v is the volume. Solve the initial value problem with $v(0) = 0$. Is this the only solution? (more about this in the solutions).

Ex 12.8: Iterative integration [3]

Consider the equation

$$\dot{x} = rx$$

where $x(0) = x_0 = 1$. We know that we can't solve this equation by direct integration. However, let's try nevertheless ...

a) Assume that x on the right hand side of the equation is a constant and directly integrate the equation. (This will prove the assumption wrong)
b) Now take your solution from (a) and use this as a new assumption, for x and integrate again.
c) Iterate this process to find successively better approximations to the solution. Can you spot the pattern that is developing?

Ex 12.9: Price's model [3]

When a scientific paper is published it cites m other papers. Papers that have already been cited more often have higher visibility and hence attract new citations at a higher rate. We assume that a paper with k citations has the visibility $v = a + k$, where a represents an intrinsic visibility the paper has even without citations.

a) Let p_k be the in-degree distribution of the papers, i.e. p_k is the proportion of papers that have been cited k times. Consider a single citation from a new paper. Assume that the chance that a paper is cited is proportional to its visibility. Show that the probability v_k that a single new citation cites a paper that has previously had k citations is

$$v_k = p_k \frac{a + k}{a + z}$$

b) Show that in the long run the in-degree distribution p_k approaches a power-law for $k \gg 1$, Also determine the power-law exponent.

Ex 12.10: Change of variables [3]

Consider a system of differential equations where

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x \end{aligned}$$

We want to transform these equations into polar coordinates, where

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \phi &= \arctan(y/x) \end{aligned}$$

Find (closed) differential equation for ϕ and r. Simplify where possible. Then solve the differential equations.

Ex 12.11: Box of bolts [4]

In a factory there is a big box of bolts, some of which are broken. At a rate r a worker takes a bolt that is not broken out of the box.

a) Write a differential equation for the number of bolts N that are in the box.
b) Express the proportion of broken bolts x as a function of N and the number of broken bolts B.
c) Differentiate the equation from part (b) with respect to time to find a differential equation for the number of broken bolts. (Note that B is constant in time). Then use substitution to express the right hand side of the differential equation in terms of x (There can be a B on the right hand side because B is a constant but we want to replace the dynamical variable N)
d) Solve the differential equation for x.
e) Bonus: Check your result by computing $x(t)$ in a different way.

Ex 12.12: Solving SIS and Logistic Growth [4]

Many systems in nature exhibit logistic growth, which is described by the logistic differential equation $\dot{x} = Ax(K - x)$.

a) Show that the equation for the SIS model can be written in the form of logistic growth and determine the values of A and K.
b) Show that the logistic differential equation can be solved by separation of variables, identify the two integrals that need to be solved, and solve the simpler one of them.
c) We are now left with a more difficult integral to solve. To do this, first show that the term under the integral can be written in the form

$$\frac{a}{x} + \frac{b}{K - x}$$

where a and b are constants. In other words: Find a and b such that such that the expression above becomes identical to the term under the integral. (Hint: After the first steps, isolate the x.)
d) Use the result from (c) to split the integral that we still need to solve into two integrals and solve them.
e) Finally put all the parts together and solve for x.
f) Substitute K and A into the solution to find a general solution for the SIS model.