Networks and Complexity

Solution 7-10

This is an example solution from the forthcoming book Networks and Complexity. Find more exercises at https://github.com/NC-Book/NCB

Ex 7.10: Poisson approximation [4]

Using approximations from the chapter and

$$(1+x)^n \approx \mathrm{e}^{xn} \tag{1}$$

for small x show that the Binomial distribution can be approximated by the Poisson distribution.

Solution

We start with the Binomial distribution

$$p_k = p^k q^{N-1-k} B(N-1-k) \tag{2}$$

and use the approximation for B from the chapter

$$B(N,n) \approx \frac{N^n}{n!}. \tag{3}$$

So in the present case

$$B(N-1,k) \approx \frac{(N-1)^k}{k!}. \tag{4}$$

Substituting into p_k this yields

$$\begin{aligned} p_k &\approx p^k q^{N-1-k} \frac{(N-1)^k}{k!} && (5)\\ &= \frac{(p(N-1))^k}{k!} q^{N-1-k} && (6)\\ &= \frac{z^k}{k!} q^{N-1-k} && (7) \end{aligned}$$

The fraction that appears is starting to look like a Poisson distribution, but we are still lacking a factor e^{-z}. The question actually gives us a hint where to find it. To use

$$(1+x)^n \approx \mathrm{e}^{xn} \tag{8}$$

We rewrite the factor q^{N-1-k} that appears in the distribution:

$$\begin{aligned} q^{N-1-k} &= (1-p)^{N-1-k} && (9)\\ &\approx \mathrm{e}^{-p(N-1-k)} && (10)\\ &= \mathrm{e}^{-z}\mathrm{e}^{pk} && (11)\\ &\approx \mathrm{e}^{-z} && (12) \end{aligned}$$

In the last step we have used that for large N, $pk = zk/(1-N)$ goes to zero and hence $\mathrm{e}^{pk} \approx 1$. Substituting

$$q^{N-1-k} \approx \mathrm{e}^{-z} \tag{13}$$

into our equation for p_k yields the desired Poisson distribution

$$p_k = \frac{z^k \mathrm{e}^{-z}}{k!} \tag{14}$$

As approximations go, this derivation was relatively crude. A more careful analysis reveals that the Poisson approximation is actually much better than this derivation suggests: The inaccuracies introduced by the three approximation steps we used here actually largely cancel each other such that the result is very precise.