Networks and Complexity

Exercise Sheet 13: A Secret of Bees

This is an exercise sheet from the forthcoming book Networks and Complexity. Find more exercises and solutions at https://github.com/NC-Book/NCB

Ex 13.1: Quadratic polynomials [1]

Solve the following quadratic polynomials:

a) $1 + \lambda = \lambda^2$,
b) $(6 - \lambda)(2 - \lambda) - 60 = 0$,
c) $(2 - \lambda)^2 - 9 = 0$.

Ex 13.2: Eigenvalues and eigenvectors [1]

Compute the eigenvalues and eigenvectors of the matrices

$$\mathbf{A} = \begin{pmatrix} 6 & 5 \\ 12 & 2 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 8 \\ 0 & 3 \end{pmatrix}$$

Then also write the vector $(8, 9)^{\mathrm{T}}$ as a linear combination of the eigenvectors of matrix $\mathbf{A}$.

Ex 13.3: Fibonacci numbers [1]

Let's do two quick tests of our Fibonacci results:

a) Compute the first Fibonacci numbers up to F_4 using

$$F_i = \frac{\phi^i - \bar{\phi}^i}{\sqrt{5}}.$$

Convince yourself that this formula produces integer results.

b) Use $D_{i+1} = Q_i$, $Q_{i+1} = Q_i + B_i$, and $F_i = Q_i + B_i$ to show

$$F_{i+1} = F_i + F_{i-1}.$$

Ex 13.4: Abstract two-dimensional map [2]

Consider the following system:

$$\begin{aligned} X_{i+1} &= -X_i + Y_i \\ Y_{i+1} &= X_i - Y_i \end{aligned}$$

a) Write the system in matrix form.
b) Compute the eigenvalues of the matrix that appears in (a).
c) Find the corresponding eigenvectors and write the initial state $X_0 = 2$, $Y_0 = 0$ as a linear combination of eigenvectors.
d) Hence, solve the initial value problem (i.e. find X_i, Y_i for all $i > 0$).
e) Verify your solution by computing the first X_1, Y_1, X_2, Y_2 by hand.
f) Bonus: Using matrix methods, show that for any initial condition $X_i = -Y_i$ must hold for all $i \geq 1$.

Ex 13.5: Abstract linear differential equation system [2]
Consider the dynamical system

$$\begin{aligned} \dot{x} &= 2x - 9y \\ \dot{y} &= -x + 2y \end{aligned}$$

where the initial state is $x = y = 1$.

a) Write the system in matrix form.
b) Write the initial state as a linear combination of the eigenvectors of the matrix that appears in (a).
c) Solve the system (i.e. compute $x(t)$ and $y(t)$).

Ex 13.6: Mothers and fathers [3]
In the chapter we modeled human ancestors in terms of people while for bees we differentiated between drones and queens. Do the calculation for humans again but this time differentiate between the number of men m_i, and women w_i in generation i. Confirm that you get the expected result. (Bonus: Is there an intuitive interpretation of the eigenvectors and the corresponding expansion coefficients c_1, c_2 that appear.)

Ex 13.7: Going in circles, again [3]
In Ex. 12.10 we studied the following system using a change of variables to disentangle the differential equations:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x \end{aligned}$$

The previous solution was elegant but required a lot of insight. Now, solve the system also using the eigendecomposition method from this chapter. Also find the particular solution starting at $x = 1$, $y = 0$. (There may be a surprise on the way but eventually you should reach the same result as before. If it is too difficult now read the next chapter first, and then perhaps check the solution.)

Ex 13.8: Fixed points and the logistic map [3]
A famous map, is the so-called logistic map

$$x_{i+1} = px_i(1 - x_i)$$

where p is a parameter. This is a nonlinear map, so we can't just solve it like the bee system. However, we can compute *fixed points*, i.e. points which remain stationary under the action of the map, analogous to the steady states in differential equations.

a) For a general map $\boldsymbol{x_{i+1}} = f(\boldsymbol{x_i})$ formulate the condition that a value $\boldsymbol{x}^*$ has to meet to be be considered a stationary solution (Hint: It's not $f(\boldsymbol{x}^*) = \mathbf{0}$.)
b) Compute the fixed points of the logistic map. Verify your result by showing that if you substitute the fixed points into the map you get the same result back.

Ex 13.9: The Zombie Apocalypse ... will be weird [3]
During the zombie apocalypse each zombie apocalypse each zombie is killed within a day, but not before infecting 3 people with the zombie virus. Within a day the infected people turn to zombies.

a) Model the system as a discrete-time map where the Z_t and I_t are the numbers of zombies and infected on day t.
b) Use the method from the chapter to compute Z_t, I_t for all $t > 0$.
c) Using your solution, consider the case where the outbreak starts with one infected on day $t = 0$ and compute the number of zombies on days 1,2,20, and 21.

Ex 13.10: General solution for linear maps [4]

In this chapter we derived a general solution for linear differential equations, but for linear maps we have only solved examples. Find a general solution for linear discrete-time maps, i.e. systems of the form $\boldsymbol{x}_{i+1} = \mathbf{U}\boldsymbol{x}_i$.

Ex 13.11: Touch down [4]

After a successful scientific mission a spacecraft lands on earth. The moment its legs touch the ground the pilot shuts off the thrusters and the full weight of the spacecraft settles on the legs. The legs aren't rigid though, but are designed to absorb the shock of the landing by compressing gas filled cylinders. These cylinders act as springs such that the net force by that they generate is $F = -cx - bv$, where b and c are constants, and x is the length of the leg relative to the rest position into which it eventually settles, and $v = \dot{x}$ is the velocity at which the leg is currently contracting or extending. The spacecraft obeys Newton's law $F = ma$ where a is the acceleration and hence $a = \ddot{x}$.

a) Write the equation of motion for a system in the form of two first-order differential equations for the change of x and v.
b) Define $\boldsymbol{x} = (x, v)^{\mathrm{T}}$ to write the two-dimensional dynamical systems from (a) in matrix form.
c) Compute the eigenvectors and eigenvalues of the matrix and then state the general solution for this system. (Introduce new parameters such as $\tilde{b} = b/2m$ and $\tilde{c} = c/m$ if this saves you work).
d) Find the particular solutions for the initial conditions $v(0) = 0$, $x(0) = 1$, and the parameters $m = 1/2$, $b = 1$ and $c = 5$. What does this solution look like? (If this is too tedious, consider the initial condition $v(0) = 1$, $x(0) = 0$ instead.)
e) In the particular solution the spacecraft bounces up and down–we don't want that. Consider again the general solution and determine how the damping b has to be chosen such that the spacecraft settles to the resting position as quickly as possible without bouncing.
f) Check the solution for some more information on this case and its applications.

Ex 13.12: Generating Bees [4]

We can also write the construction rule of the Fibonacci sequence as $F_{n+2} = F_{n+1} + F_n$ starting from $F_0 = 0$, $F_1 = 1$. Define a generating function F for the sequence $F_0, F_1 \ldots$, then use the construction rule to find an equation for the generating function. Use this to solve for the generating function and finally find a closed form for the sequence.

(Hints: This one requires more mathematical technique than most exercises. Mind the edge terms when shifting indices, don't forget to use the initial condition. Once you can express the generating function as a fraction try factorizing the denominator, use partial fractions, also recall $\sum x^n = 1/(1-x)$ and note $\phi = -1/\bar{\phi}$.)