Networks and Complexity

Solution 8-3

This is an example solution from the forthcoming book Networks and Complexity. Find more exercises at https://github.com/NC-Book/NCB

Ex 8.3: Abstract giant component [2]

Compute the size of the giant component in a network nodes in the following networks:

a) An ER-graph with $z = 2.5$ with 100.000 nodes.

<u>*Solution*</u>

We know that the giant component size in an ER network obeys

$$s = 1 - \mathrm{e}^{-sz} \tag{1}$$

For $z = 2.5$ we solve this by iteration, which yields $s \approx 0.89264$ as the fraction of nodes in the giant component. To get the number we multiply by N which yields $89,264$ nodes.

[There is an 11% chance that a randomly picked node is not in the giant component. All other components are small in comparison, containing only a few nodes. Hence the expected number of nodes that we find in the component of a randomly picked nodes is approximately 79.000 but this wasn't the question.]

b) A network consisting of 50.000 nodes of degree 1 and 50.000 nodes of degree 3.

<u>*Solution*</u>

The degree distribution is

$$p_k = \frac{1}{2}(\delta_{k,1} + \delta_{k,3}) \tag{2}$$

We compute the mean degree

$$z = \sum k p_k = \frac{1}{2}(1 + 3) = 2 \tag{3}$$

This leads to the excess degree distribution

$$q_k = \frac{(k+1)p_{k+1}}{z} = \frac{1}{4}(\delta_{k,0} + 3\delta_{k,2}) \tag{4}$$

Let us also compute the mean excess degree, we don't actually need it here but it will be useful in a later exercise

$$q = \sum k q_k = \sum k \frac{1}{4}(\delta_{k,0} + 3\delta_{k,2}) = \frac{0+6}{4} = 1.5 \tag{5}$$

To compute the probability that a link does not lead to the giant component, we use the self-consistency equation

$$\begin{aligned} v &= \sum q_k v^k && (6)\\ &= \sum \frac{1}{4}(\delta_{k,0} + 3\delta_{k,2}) v^k && (7)\\ &= \frac{1+3v^2}{4} && (8) \end{aligned}$$

We can write this as quadratic polynomial

$$0 = 3v^2 - 4v + 1. \tag{9}$$

We could now solve this in the usual way (e.g. completing the square). A more insightful solution is to realize that we know one solution already as $v = 1$ is always a solution due to the nature of the self-consistency approach. We are not interested in this solution itself but we can divide the corresponding factor $(v - 1)$ out of the polynomial, using polynomial long division:

$$(3v^2 - 4v + 1)/(v - 1) = 3v - 1 \tag{10}$$

and hence the solution that we are actually interested in is

$$v = \frac{1}{3} \tag{11}$$

[There will actually a longer exercise on the polynomial long dividion in Chap. 10]

Now that we know v we use the equation for the giant component size

$$\begin{aligned}
s &= 1 - \sum p_k v^k && (12)\\
&= 1 - \sum \frac{1}{2}(\delta_{k,1} + \delta_{k,3})v^k && (13)\\
&= 1 - \frac{v + v^3}{2} && (14)\\
&= 1 - \frac{1/3 + 1/27}{2} && (15)\\
&= \frac{54}{54} - \frac{9}{54} - \frac{1}{54} && (16)\\
&= \frac{22}{27} = 0.\overline{814} && (17)
\end{aligned}$$

In other words we expect 81.481 nodes to be in the giant component.