

NETWORKS AND COMPLEXITY

Solution 15-11

*This is an example solution from the forthcoming book Networks and Complexity.
Find more exercises at <https://github.com/NC-Book/NCB>*

Ex 15.11: SIS ISI ODE TLA [4]

For the aSIS model from above. Derive a differential equation for $[ISI]$ the density of ISI-chains. (Hint, you might need symbols of the form $[ABCD]$, density of four node chains with states A,B,C,and D, and $[{}^B A_D^C]$, density of nodes in state A with neighbors in state B, C and D.)

Solution

To make sure we don't forget any terms lets go through the processes that occur systematically. The rewiring process occurs at a total rate $w[SI]$ the probability that a rewiring even will destroy an ISI-chain is $2[ISI]/[SI]$. Rewiring cannot create ISI-chains, so the total impact of rewiring is

$$a_{\text{rew.}} = w[SI] \left(\frac{2[ISI]}{[SI]} \right) = 2w[ISI] \quad (1)$$

Let's do recovery next. Recovery events occur at a total rate of $r[I]$. A recovery event can create an ISI-chain if the recovering node is the central I in an III chain. The probability that the recovering node is such a central I is $[III]/[I]$. (Note that there is no factor of 2 or 3: the III-chain contains three I but we need the central I to recover and not one of the peripheral ones to make an ISI, so only one chance not 3).

Recovery can also destroy an ISI chain if the recovering node was one of the I-nodes in an ISI chain, and the probability of that being the case is $2[ISI]/[I]$. In summary the impact of recovery is

$$a_{\text{rec.}} = r[I] \left(\frac{[III]}{[I]} - \frac{2[ISI]}{[I]} \right) = r[III] - 2r[ISI] \quad (2)$$

The most complex process is infection. Infection events happen at the total rate $p[SI]$. An infection can destroy ISI-chains if the SI-link on which it occurs is part of an ISI-chain. The probability of this happening is $2[ISI]/[SI]$ (factor of 2 because the SIS contains two $[SI]$ -links), it destroys an additional SSI chains if the SI-link was actually part of a star with an S in the center and 3 peripheral nodes (probability $3[{}^I S_I^I]/[SI]$)

Infection can also create an ISI-chain if it happens to turn an SSI-chain into an ISI-chain because the first S-node got infected from somewhere else. This happens whenever the infection event occurs in an ISSI-chain and the probability of this happening is $2[ISSI]/[SI]$. In summary the impact of infection is

$$a_{\text{inf}} = p[SI] \left(2 \frac{[ISSI]}{[SI]} - 2 \frac{[ISI]}{[SI]} - 3 \frac{[{}^I S_I^I]}{[SI]} \right) = 2p[ISSI] - 2p[ISI] - 3p[{}^I S_I^I] \quad (3)$$

Taking all terms into account, the differential equation for the ISI-chains is

$$[I\dot{S}I] = a_{\text{rew}} + a_{\text{rec}} + a_{\text{inf}} \quad (4)$$

$$= -2w[ISI] + r[III] - 2r[ISI] + 2p[ISSI] - 2p[ISI] - 3p[{}^I S_I^I] \quad (5)$$

$$= r[III] + 2p[ISSI] - 2(w + r + p)[ISI] - 3p[{}^I S_I^I] \quad (6)$$

(Getting the prefactors right is admittedly difficult, if you use this in the real world, always do a quick numerical check then continue analytically)