Networks and Complexity

Solution 9-3

This is an example solution from the forthcoming book Networks and Complexity.
Find more exercises at https://github.com/NC-Book/NCB

Ex 9.3: Degree generating functions [2]

Derive the generating functions G for the degree distribution for the following networks:

a) A regular graph where every node has degree 3.

<u>*Solution*</u>
For this network the degree distribution is

$$p_k = \delta_{k,3}. \tag{1}$$

We can directly write the degree generating function as

$$G = \sum p_k x^k = \sum \delta k, 3 = x^3 \tag{2}$$

b) A network where half the nodes have degree 10 and the other half has degree 20.

<u>*Solution*</u>
In this case we have

$$p_k = \frac{1}{2}\delta_{k,10} + \frac{1}{2}\delta_{k,20} \tag{3}$$

and hence

$$G = \sum p_k x^k = \frac{1}{2}\sum(\delta_{k,10} + \delta_{k,20})x^k = \frac{x^{10} + x^{20}}{2} \tag{4}$$

c) An Erdős-Rényi random graph with mean degree z (find a nice form for the result).

<u>*Solution*</u>
In this case the degree distribution is

$$p_k = \frac{z^k \mathrm{e}^{-z}}{k!}. \tag{5}$$

We can write the generating function as

$$G = \sum p_k x^k = \sum \frac{z^k \mathrm{e}^{-z}}{k!} x^k \tag{6}$$

Since we have now a sum over something related to the Poisson distribution our intuition should be to use our knowledge of the exponential series to simplify it. We write

$$G = \mathrm{e}^{-z} \sum \frac{(zx)^k}{k!} = \mathrm{e}^{-z}\mathrm{e}^{zx}, \tag{7}$$

which we can write also write as

$$G = \mathrm{e}^{z(x-1)}. \tag{8}$$

d) A network with the degree distribution

$$p_k = \frac{\mathrm{e}^{-5}5^k}{2k!} + \frac{1}{2}\delta_{5,k}$$

<u>*Solution*</u>
We define

$$\begin{aligned}
G(x) &= \sum p_k x^k && (9)\\
&= \sum \left(\frac{\mathrm{e}^{-5}5^k}{2k!} + \frac{1}{2}\delta_{5,k} \right) x^k && (10)\\
&= \left(\frac{\mathrm{e}^{-5}}{2} \sum \frac{(5x)^k}{k!} \right) + \frac{x^5}{2} && (11)\\
&= \frac{\mathrm{e}^{-5}}{2}\mathrm{e}^{5x} + \frac{x^5}{2} && (12)\\
&= \frac{\mathrm{e}^{5(x-1)} + x^5}{2} && (13)
\end{aligned}$$