Networks and Complexity

Solution 16-5

This is an example solution from the forthcoming book Networks and Complexity.
Find more exercises at https://github.com/NC-Book/NCB

Ex 16.5: Fold normal form revisited [4]

Consider the normal from of the fold bifurcation, which we can write as

$$\dot{x} = ax^2 + bp$$

where a and b are normal form coefficients, p is the control parameter, and x is the dynamical variable. Find the steady states, determine their stability, and draw the bifurcation diagram. Are there actually different forms (e.g. subcritical, supercritical) of this bifurcation as well?

Solution

We compute the steady states by solving

$$0 = ax^2 + bp \tag{1}$$

which yields

$$x^* = \pm\sqrt{\frac{b}{a}p} \tag{2}$$

This means we have either two steady states ($bp/a > 0$) or none ($bp/a < 0$). If $b/a > 0$ we find the steady states at positive values of p, otherwise they only exist at negative values of p.

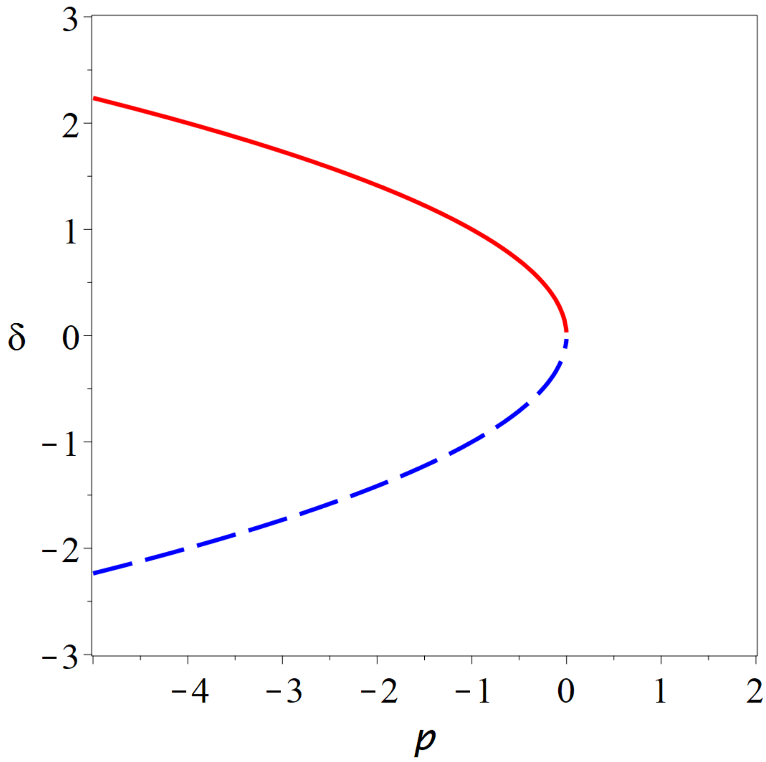
In this sense there are two forms of the fold bifurcation, but they are indistinguishable in practice.

To determine the stability of solutions we compute

$$f_{\mathrm{x}} = \frac{\partial}{\partial x} ax^2 + bp = 2ax \tag{3}$$

If $a > 0$ then the steady state at $+\sqrt{bp/a}$ is unstable and the one at $+\sqrt{bp/a}$ is stable. Otherwise its the other way around.

We can now draw the bifurcation diagram



This diagram corresponds to the case $a < 0$ (positive branch of steady states is stable) and $b > 0$ (steady states exist for negative p).