Networks and Complexity

Solution 12-8

This is an example solution from the forthcoming book Networks and Complexity.
Find more exercises at https://github.com/NC-Book/NCB

Ex 12.8: Iterative integration [3]

Consider the equation

$$\dot{x} = rx$$

where $x(0) = x_0 = 1$. We know that we can't solve this equation by direct integration. However, let's try nevertheless ...

a) Assume that x on the right hand side of the equation is a constant and directly integrate the equation. (This will prove the assumption wrong)

Solution

So we literally assume $x = 1$. substituting into the equation yields

$$\dot{x} = r \tag{1}$$

Integrating both sides we find

$$x(t) = 1 + rt. \tag{2}$$

b) Now take your solution from (a) and use this as a new assumption, for x and integrate again.

Solution

So we now assume

$$x(t) = 1 + rt \tag{3}$$

and hence integrate

$$\dot{x} = r(1 + rt) = r + r^2 t, \tag{4}$$

which yields

$$x(t) = 1 + rt + \frac{r^2 t^2}{2}. \tag{5}$$

c) Iterate this process to find successively better approximations to the solution. Can you spot the pattern that is developing?

Solution

Integrating a few more times we can see that we are approaching the solution

$$\begin{aligned} x(t) &= 1 + rt + \frac{r^2 t^2}{2} + \frac{r^3 t^3}{6} + \ldots & (6)\\ &= \sum_n \frac{(rt)^n}{n!} & (7)\\ &= \mathrm{e}^{rt} & (8) \end{aligned}$$

which is of course as it should be.