Networks and Complexity

Solution 23-3

This is an example solution from the forthcoming book Networks and Complexity.
Find more exercises at https://github.com/NC-Book/NCB

Ex 23.3: Paper supply [2]

In an office the copier paper is stored in a cabinet. At an average rate of once a week an employee takes paper out of the cabinet and loads it into the copier. There is a 50% chance that the employee tries to load only one packet of paper, and a 50% chance that the employee tries to load two packets of paper. If the cabinet is empty, the office manager immediately orders 4 new packets of paper which arrive in average within 2 weeks.

a) Consider this system as a network of states, numbered 0 to 4. The system is in state i if there are i packets of paper in the cabinet. Write a system of differential equations for the variables $x_0, \ldots, x_4$, where x_i is the probability that the system is in state i.

Solution

The system of equations is

$$\begin{aligned} \dot{x}_0 &= -0.5x_0 + x_1 + 0.5x_2 && (1) \\ \dot{x}_1 &= -x_1 + 0.5x_2 + 0.5x_3 && (2) \\ \dot{x}_2 &= -x_2 + 0.5x_3 + 0.5x_4 && (3) \\ \dot{x}_3 &= -x_3 + 0.5x_4 && (4) \\ \dot{x}_4 &= -x_4 + 0.5x_0 && (5) \end{aligned}$$

b) Write the differential equation system in the form

$$\dot{\boldsymbol{x}} = -\mathbf{L}\boldsymbol{x} \qquad (6)$$

where $\boldsymbol{x} = (x_0, \ldots, x_4)^{\mathrm{T}}$. Compute the entries of the matrix $\mathbf{L}$.

Solution

We find

$$\mathbf{L} = \begin{pmatrix} 0.5 & -1 & -0.5 & 0 & 0 \\ 0 & 1 & -0.5 & -0.5 & 0 \\ 0 & 0 & 1 & -0.5 & -0.5 \\ 0 & 0 & 0 & 1 & -0.5 \\ -0.5 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (7)$$

c) Use Kirchhoff's theorem to compute the probability that there isn't any paper in the cabinet. (i.e. compute x_0 in the steady state).

Solution

The easiest solution is to use Kirchhoff together with the matrix-tree theorem. This yields

$$S_1 = \begin{vmatrix} 1 & -0.5 & -0.5 & 0 \\ 0 & 1 & -0.5 & -0.5 \\ 0 & 0 & 1 & -0.5 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \tag{8}$$

where we have used that the determinant is the product of the eigenvalues, and in this case we can see immediately that all the eigenvalues are 1 due to the upper triangular form of the matrix.

$$S_2 = \begin{vmatrix} 0.5 & -0.5 & 0 & 0 \\ 0 & 1 & -0.5 & -0.5 \\ 0 & 0 & 1 & -0.5 \\ -0.5 & 0 & 0 & 1 \end{vmatrix} \tag{9}$$

$$= 0.5 \begin{vmatrix} 1 & -0.5 & -0.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 1 \end{vmatrix} + 0.5 \begin{vmatrix} 0.5 & -0.5 & 0 \\ 0 & 1 & -0.5 \\ 0 & 0 & 1 \end{vmatrix} = 0.75 \tag{10}$$

$$S_3 = \begin{vmatrix} 0.5 & -1 & 0 & 0 \\ 0 & 1 & -0.5 & 0 \\ 0 & 0 & 1 & -0.5 \\ -0.5 & 0 & 0 & 1 \end{vmatrix} = 0.75 \tag{11}$$

$$S_4 = \begin{vmatrix} 0.5 & -1 & -0.5 & 0 \\ 0 & 1 & -0.5 & 0 \\ 0 & 0 & 1 & -0.5 \\ -0.5 & 0 & 0 & 1 \end{vmatrix} = 0.75 \tag{12}$$

$$S_5 = \begin{vmatrix} 0.5 & -1 & -0.5 & 0 \\ 0 & 1 & -0.5 & -0.5 \\ 0 & 0 & 1 & -0.5 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0.5 \tag{13}$$

In summary we have

$$S_1 = \frac{4}{4} \quad S_2 = S_3 = S_4 = \frac{3}{4} \quad S_5 = \frac{2}{4} \tag{14}$$

hence

$$\sum S_i = \frac{15}{4} \tag{15}$$

so we can find the steady state

$$X_1^* = \frac{4}{15} \quad X_2^* = X_3^* = X_4^* = \frac{1}{5} \quad X_5^* = \frac{2}{15}. \tag{16}$$