Solution 12-3

This is an example solution from the forthcoming book Networks and Complexity. Find more exercises at https://github.com/NC-Book/NCB

Ex 12.3: Infinite excess degree [1]

Consider a network with $p_0 = 0$ and $p_k = ak^{-3}$ for $k > 0$. Show that this network has finite mean degree but infinite mean excess degree. You can use the $\sum_{k=1}^{\infty} k^{-2}$ is finite while $\sum_{k=1}^{\infty} k^{-1} = \infty$. [Hint: Don't try to determine a, it's complicated.]

Solution

The mean degree of our network is

$$z = \sum kp_k \quad (1)$$

$$= \sum_{k=1}^{\infty} kak^{-3} \quad (2)$$

$$= a\sum_{k=1}^{\infty} ak^{-2} \quad (3)$$

which is finite. We now compute the excess degree distribution

$$q_k = \frac{(k+1)p_{k+1}}{z} \quad (4)$$

$$= \frac{a}{z}(k+1)(k+1)^{-3} = \frac{a}{z}(k+1)^{-2}. \quad (5)$$

Note that we can use this equation for all k including 0. In the next step we want to show that the mean excess degree is infinite, so our strategy is to create an expression of the form

$\sum_{k=1}^{\infty} k^{-1}$. We compute

$$\begin{aligned}
q &= \sum k q_k && (6)\\
&= \sum \frac{a}{z} k(k+1)^{-2} && (7)\\
&= \sum \frac{a}{z}(k+1)(k+1)^{-2} - \frac{a}{z}(1)(k+1)^{-2} && (8)\\
&= \left(\sum \frac{a}{z}(k+1)(k+1)^{-2}\right) - \left(\sum \frac{a}{z}\frac{a}{z}(k+1)^{-2}\right) && (9)\\
&= \frac{a}{z}\left(\sum (k+1)^{-1}\right) - \frac{1}{z}\left(\sum a(k+1)^{-2}\right) && (10)\\
&= \frac{a}{z}\left(\sum_{k=1}^{\infty} k^{-1}\right) - \frac{1}{z}\left(\sum_{k=1}^{\infty} ak^{-2}\right) && (11)\\
&= \frac{a}{z}\left(\sum_{k=1}^{\infty} k^{-1}\right) - \frac{z}{z} && (12)\\
&= \frac{a}{z}\left(\sum_{k=1}^{\infty} k^{-1}\right) - 1 && (13)\\
&= \infty - 1 = \infty && (14)\\
& && (15)
\end{aligned}$$