Networks and Complexity

Solution 10-9

This is an example solution from the forthcoming book Networks and Complexity. Find more exercises at https://github.com/NC-Book/NCB

Ex 10.9: Targeted attacks in general [3]

In this exercise we derive some results on degree-targeted attacks in general. Consider a configuration model network, described by a degree generating function G, that is subject to an attack described by a removal function R. Try to do the calculations below in the forward direction (deriving the result) instead of the backward direction (proving the result is true).

a) Show that the degree distribution of the surviving nodes after the first step of the attack is generated by $G_{\mathrm{h}} = \frac{G-R}{c}$.

<u>*Solution*</u>

We can start by recalling that the number of nodes with degree k is

$$n_k = Np_k. \tag{1}$$

Moreover the number of nodes of degree k that are removed in the attack are Nr_k. So the number of nodes of degree k that survive the first step of the attack are

$$n_{k,\mathrm{h}} = N(p_k - r_k) \tag{2}$$

We define

$$c = 1 - r = 1 - \sum r_k \tag{3}$$

as the proportion of nodes that survive the attack. Hence the total number of nodes in the network after the attack is

$$N_{\mathrm{h}} = cN \tag{4}$$

We can now compute the proportion of nodes of degree k after the first step of the attack as

$$p_{k,\mathrm{h}} = \frac{n_{k,\mathrm{h}}}{N_{\mathrm{h}}} = \frac{N(p_k - r_k)}{cN} = \frac{p_k - r_k}{c} \tag{5}$$

Now we multiply both sides of the equation with x^k and sum over all k,

$$\sum p_{k,\mathrm{h}} x^k = \sum \frac{p_k - r_k}{c} x^k \tag{6}$$

Finally using the definitions of the generating functions $G_{\mathrm{h}} = \sum p_{k,\mathrm{h}} x^k$, $G = \sum p_k x^k$, and $R = \sum r_k xk$, we arrive at the desired result

$$G_{\mathrm{h}} = \frac{G - R}{c} \tag{7}$$

b) Show that the mean degree after the attack is $z_{\mathrm{a}} = z\tilde{c}^2/c$.

<u>*Solution*</u>

First we recall that the degree generating function after the attack is

$$G_{\mathrm{a}} = \frac{G(\tilde{A}) - R(\tilde{A})}{c} \tag{8}$$

where

$$\tilde{A} = \tilde{r} + \tilde{c}x, \tag{9}$$

and $\tilde{r}$, $\tilde{c}$ are the removed and surviving proportions of endpoints respectively. They are defined by

$$\tilde{r} = \frac{R'(1)}{z}, \qquad \tilde{c} = 1 - \tilde{r} \tag{10}$$

Let's compute z_{a} as

$$z_{\mathrm{a}} = G'_{\mathrm{a}}(1) = \frac{G'(\tilde{A}(1)) - R'(\tilde{A}(1))}{c}\tilde{A}'(1) = \frac{G'(1) - R'(1)}{c}\tilde{c} = \frac{z - R'(1)}{c}\tilde{c} \tag{11}$$

At this point the $R'(1)$ is bothering us a little bit, so how can we get rid of it? We have last seen it in the definition of $\tilde{r}$ and using this definition we can replace

$$R'(1) = z\tilde{r} \tag{12}$$

and hence

$$z_{\mathrm{a}} = \frac{z - z\tilde{r}}{c}\tilde{c} = \frac{z(1-\tilde{r})\tilde{c}}{c} = z\frac{\tilde{c}^2}{c}, \tag{13}$$

which is the desired result.

c) Show that the excess degree distribution after the attack is generated by $Q_{\mathrm{a}} = (G'(\tilde{A}) - R'(\tilde{A}))/\tilde{c}z$.

<u>*Solution*</u>
We can now compute the excess degree generating function as

$$Q_{\mathrm{a}} = \frac{G'_{\mathrm{a}}}{z_{\mathrm{a}}} = \frac{G'(\tilde{A}) - R'(\tilde{A})}{cz_{\mathrm{a}}}\tilde{A}' = (G'(\tilde{A}) - R'(\tilde{A}))\frac{\tilde{c}}{z\tilde{c}^2} = \frac{G'(\tilde{A}) - R'(\tilde{A})}{z\tilde{c}} \tag{14}$$

d) Show that the mean excess degree after the attack is $q_{\mathrm{a}} = q - R''(1)/z$.

<u>*Solution*</u>
Since we already know the generating function Q_{a} we can compute

$$q_{\mathrm{a}} = Q'_{\mathrm{a}}(1) = \frac{G''(\tilde{A}(1)) - R''(\tilde{A}(1))}{z\tilde{c}}\tilde{A}'(1) = \frac{G''(1) - R''(1)}{z\tilde{c}}\tilde{c} = \frac{G''(1) - R''(1)}{z} \tag{15}$$

now we only need to recognize

$$\frac{G''(1)}{z} = q, \tag{16}$$

which leads us to the desired result

$$q_{\mathrm{a}} = q - \frac{R''(1)}{z}. \tag{17}$$