Time Stepping Algorithms in the MOM6 Ocean Model

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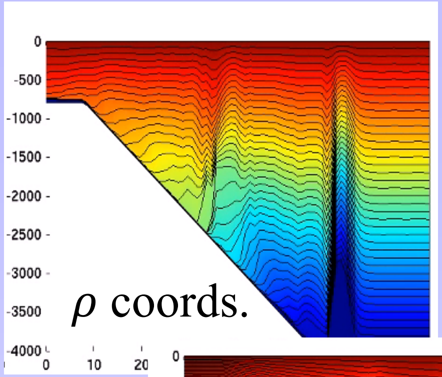
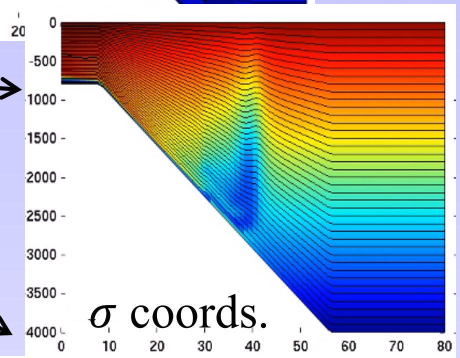
MOM6 is available via https://github.com/NOAA-GFDL/MOM6-examples

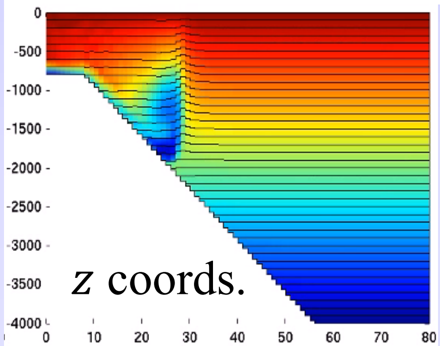
What is the MOM6 ocean model?

Community ocean model rooted in climate modeling
- Structured C-grid, Finite Volume Core
- Global climate modeling focus; Processes oriented origins
- Hydrostatic Primitive Equations
- Conservative, including wetting & drying (ice shelves)

Arbitrary Lagrangian Eulerian Method (ALE)
- General vertical coordinates
- No vertical CFL limit on timesteps/resolution
- Reduces numerical diapycnal mixing for some coordinates
- Efficiencies for biogeochemistry & passive tracers







Comprehensive set of physical process parameterizations
- Required for climate model projections into unobserved states
- Moving toward energetics-based formulations of params.

Many capabilities resulted from extensive collaborations
- 4 NSF/NOAA sponsored Climate Process Teams
- CVMix shared NCAR/GFDL/LANL vertical mixing code
- CLIVAR CORE/OMIP ocean-climate model comparisons
- Neutral mixing params. from Dr. A. Shao of U. Victoria, Canada

Embedding SIS2 sea-ice & GFDL icebergs in MOM6

Free Community Open Development with deliberate ocean model software design
- Design drew upon experience from MOM4/5, MITgcm, HIM, GOLD, Poseidon, …

Plan for this talk

- Description of MOM6 time stepping cycles
- Interactions of algorithms to allow varied length time-stepping cycles (in technical detail)
  - Tracer advection strategy in MOM6
  - Barotropic / baroclinic split explicit time-stepping
  - Challenges for split explicit time-stepping from nonlinear continuity solver & evolving layer thicknesses, and MOM6 solutions
- Some areas of potential improvement in the MOM6 time stepping and solver?

The Arbitrary Lagrangian Eulerian method

Solve equations in 2 phases:

- a **Lagrangian** dynamic update (shallow water eqns.)
- Vertical remapping to an arbitrary (**Eulerian**?) coordinate

Momentum eqn.:

$$\frac{\partial \vec{u}}{\partial t} + \dot{s}\frac{\partial \vec{u}}{\partial s} + (f + \nabla_s \times \vec{u})\hat{k} \times \vec{u} = -\frac{1}{\rho}\nabla_s p - \nabla_s\left(\phi + \tfrac{1}{2}\|\vec{u}\|^2\right) + \frac{1}{\rho}\nabla \cdot \tilde{\vec{\tau}}$$

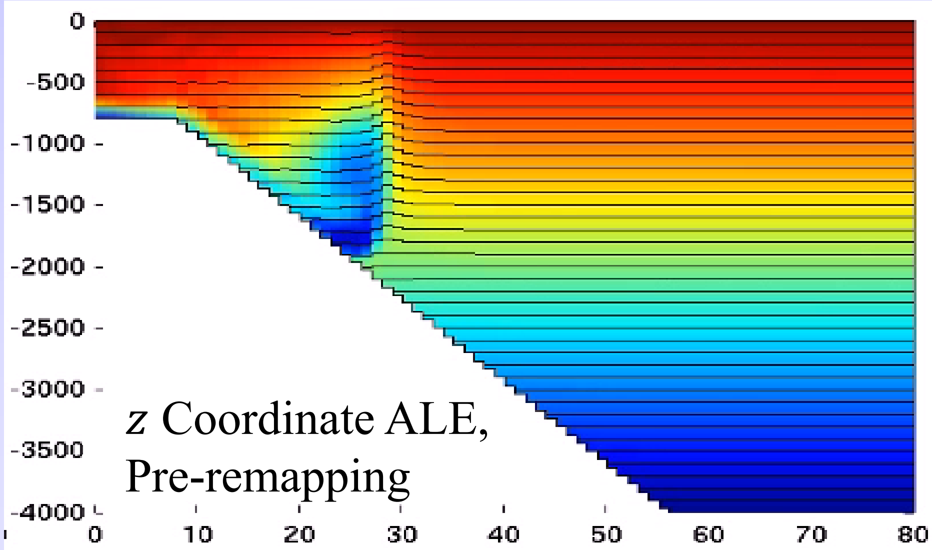
Continuity eqn.:

$$\left.\frac{\partial}{\partial t}\right|_s \left(\frac{\partial p}{\partial s}\right) + \nabla_s \cdot \left(\vec{u}\frac{\partial p}{\partial s}\right) + \frac{\partial}{\partial s}\left(\dot{s}\frac{\partial p}{\partial s}\right) = 0$$

Tracer eqn.:

$$\left.\frac{\partial}{\partial t}\right|_s \left(\frac{\partial p}{\partial s}\theta\right) + \nabla_s \cdot \left(\vec{u}\frac{\partial p}{\partial s}\theta\right) + \frac{\partial}{\partial s}\left(\dot{s}\frac{\partial p}{\partial s}\theta\right) = Q\frac{\partial p}{\partial s}$$

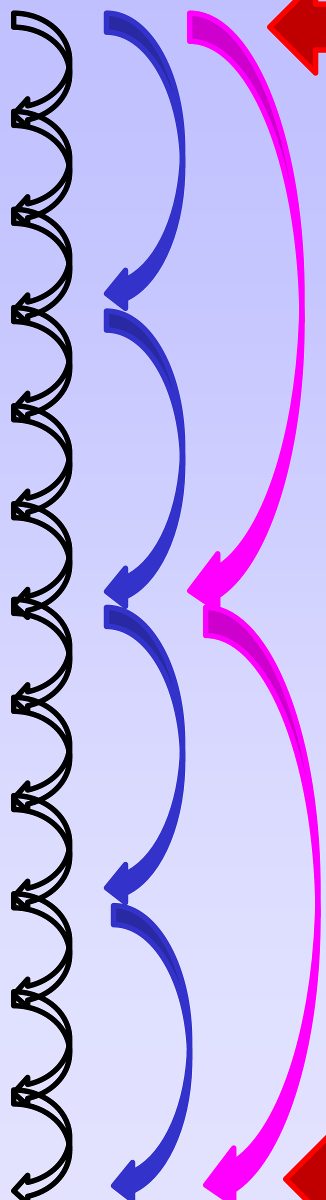
Dense-water Overflow Plume in Side-View



ALE advantages:

- Flexible vertical coordinates
- Remapping imposes no vertical CFL limit on timesteps
- Tracer advection not required to represent gravity waves

4 Time Stepping Cycles in MOM6



Barotropic time steps (2-d linear momentum, integrated continuity)

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \left((D+\eta) \bar{u}_{BT} h_k \right) = P - E \qquad \frac{\partial \bar{u}_{BT}}{\partial t} = -g\nabla\eta - f\hat{z} \times \bar{u}_{BT} + \bar{F}_{BT}$$

Lagrangian dynamics (3-d Stacked Shallow Water Eqns.)

$$\frac{\partial \vec{u}_k}{\partial t} + (f + \nabla_s \times \vec{u}_k)\hat{z} \times \vec{u}_k = -\frac{\nabla_s p_k}{\rho} - \nabla_s\left(\phi_k + \tfrac{1}{2}\|\vec{u}_k\|^2\right) + \frac{\nabla \cdot \tilde{\tilde{\tau}}_k}{\rho}$$

$$\frac{\partial h_k}{\partial t} + \nabla_s \cdot (\vec{u} h_k) = 0$$

Tracer Advection, **Thermodynamics** and Mixing (Column physics)

$$\frac{\partial h_k}{\partial t} = (P-E)_k$$

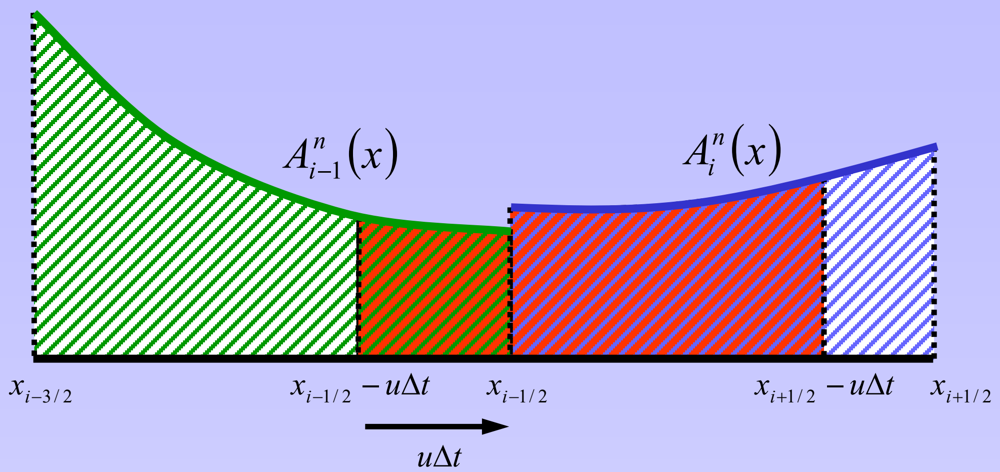
$$\frac{\partial}{\partial t}(h_k\theta_k) + \nabla_s \cdot (\vec{u} h_k \theta_k) = Q_k^\theta h_k + \frac{1}{h_k}\Delta\left(\kappa \frac{\partial \theta}{\partial z}\right) + \frac{1}{h_k}\nabla_s(h_k K \nabla_s \theta)$$

Remapping and coordinate restoration

$$h_k^{new} = \Delta_k z_{Coord} \qquad \sum h_k^{new} = \sum h_k^{old}$$

$$\vec{u}_k^{new} = \frac{1}{h_k}\int_{z_{k+\frac{1}{2}}}^{z_{k+\frac{1}{2}}+h_k} \vec{u}^{old}(z')dz' \qquad \theta_k^{new} = \frac{1}{h_k}\int_{z_{k+\frac{1}{2}}}^{z_{k+\frac{1}{2}}+h_k} \theta(z')dz'$$

1-D Finite Volume Advection of Scalars



Given a piecewise polynomial description of the tracer concentration, the new tracer cell concentration is the average of the fluid that will be in the cell after a timestep.

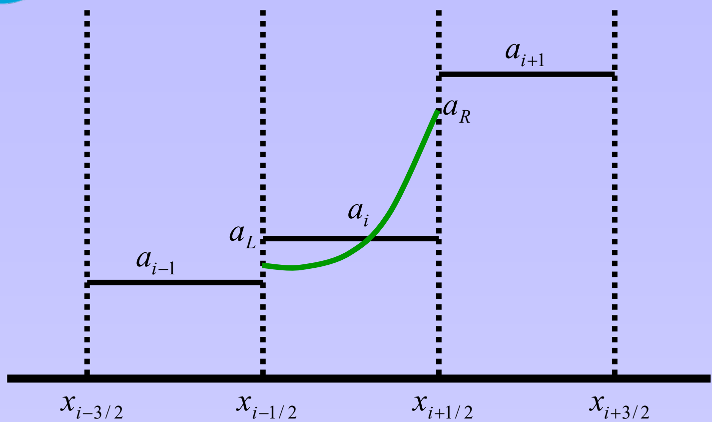
$$\int_{x_{i-1/2}}^{x_{i+1/2}} A_i^{n+1}(x)dx = \int_{x_{i-1/2}-u\Delta t}^{x_{i+1/2}-u\Delta t} A^n(x)dx = \int_{x_{i-1/2}}^{x_{i+1/2}} A_i^n(x)dx - \int_{x_{i+1/2}-u\Delta t}^{x_{i+1/2}} A_i^n(x)dx + \int_{x_{i-1/2}-u\Delta t}^{x_{i-1/2}} A_i^n(x)dx$$

Fluxes are found by analytically integrating the profile over the distance that is swept past the face within a timestep.

$$a_i^n = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} A_i^n(x)dx \qquad a_i^{n+1} = a_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2} - F_{i-1/2}\right) \qquad F_{i+1/2} = \frac{1}{\Delta t}\int_{x_{i+1/2}-u\Delta t}^{x_{i+1/2}} A_i^n(x)dx$$

With piecewise constant profiles, this approach gives 1st order upwind advection. Higher order polynomials (e.g., parabolas) can give higher order accuracy.

Piecewise Parabolic Method (PPM)



Colella &Woodward, 1984, *J. Comp. Phys.*, 1984, **54**, 174-201.
Carpenter et al., 1990, *Mon. Wea. Rev.*, **118**, 586-612.

Each cell is assumed to have a piecewise parabolic representation, which is uniquely prescribed by conservation and the two edge values.

$$x_i' \equiv \frac{x - x_{i-1/2}}{\Delta x_i} \qquad \Delta x_i \equiv x_{i+1/2} - x_{i-1/2} \qquad c \equiv u\Delta t / \Delta x_i$$

$$A_i(x') = a_L + (a_R - a_L)x_i' + a_6 x_i'(1 - x_i')$$

$$a_6 = 6a_i - 3(a_R + a_L)$$

$$a_i = \int_0^1 A_i(x_i')dx_i' = \int_0^1 a_L + (a_R - a_L)x_i' + a_6 x_i'(1 - x_i')dx_i'$$

$$= \left[a_L x_i' + \tfrac{1}{2}(a_R - a_L)x_i'^2 + a_6\left(\tfrac{1}{2}x_i'^2 - \tfrac{1}{3}x_i'^3\right)\right]_0^1 = \tfrac{1}{2}(a_R + a_L) + \tfrac{1}{6}a_6$$

$$F_{i+1/2} = \frac{1}{\Delta t}\int_{x_{i+1/2}-u\Delta t}^{x_{i+1/2}} A_i^n(x)dx = \frac{\Delta x}{\Delta t}\int_{1-c}^{1} A_i(x_i')dx_i' = \frac{\Delta x}{\Delta t}\left[a_L x_i' + \tfrac{1}{2}(a_R - a_L)x_i'^2 + a_6\left(\tfrac{1}{2}x_i'^2 - \tfrac{1}{3}x_i'^3\right)\right]_{1-c}^{1}$$

$$= \frac{\Delta x}{\Delta t}\left[a_L c + (a_R - a_L + a_6)\left(c - \tfrac{1}{2}c^2\right) - a_6\left(c - c^2 + \tfrac{1}{3}c^3\right)\right]$$

$$= u\left[a_R + \tfrac{1}{2}(a_L - a_R)c + a_6\left(\tfrac{1}{2}c - \tfrac{1}{3}c^2\right)\right]$$

By construction, the PPM approach is conservative.

The (unlimited) order of accuracy is determined by the estimates of the edge values.

Monotonicity is ensured by adjusting the edge values to flatten the profile.

Easter's Pseudocompressibility Multidimensional Tracer Advection Method

$$\frac{\partial h}{\partial t} + \vec{\nabla} \cdot (\vec{u} h) = 0 \equiv \frac{\partial h}{\partial t} + \vec{\nabla} \cdot (\vec{U}) \qquad \frac{\partial}{\partial t}(h\psi) + \vec{\nabla} \cdot (\vec{U}\psi) = 0 \qquad \frac{\partial \psi}{\partial t} + \vec{u} \cdot \vec{\nabla} \psi = 0$$

$$\frac{\partial h}{\partial t} = \frac{1}{\Delta x}\left(U_{i-\frac{1}{2},j} - U_{i+\frac{1}{2},j}\right) + \frac{1}{\Delta y}\left(V_{i,j-\frac{1}{2}} - V_{i,j+\frac{1}{2}}\right) \qquad F_{i+1/2,j}(\psi) = U_{i+1/2,j}\psi_{i+1/2,j}^{Monotonic}$$

$$\tilde{h}_{i,j}\tilde{\psi}_{i,j} = h_{i,j}^{n}\psi_{i,j} + \frac{\Delta t}{\Delta x}\left(F_{i-\frac{1}{2},j}(\psi^{n}) - F_{i+\frac{1}{2},j}(\psi^{n})\right)$$

$$\tilde{h}_{i,j} = h_{i,j}^{n} + \frac{\Delta t}{\Delta x}\left(U_{i-\frac{1}{2},j} - U_{i+\frac{1}{2},j}\right)$$

$$\tilde{\psi}_{i,j} = \frac{\tilde{h}_{i,j}\tilde{\psi}_{i,j}}{\tilde{h}_{i,j}}$$ Monotonic.

$$h_{i,j}^{n+1}\psi_{i,j}^{n+1} = \tilde{h}_{i,j}\tilde{\psi}_{i,j} + \frac{\Delta t}{\Delta y}\left(G_{i,j-\frac{1}{2}}(\tilde{\psi}) - G_{i,j+\frac{1}{2}}(\tilde{\psi})\right)$$

$$h_{i,j}^{n+1} = \tilde{h}_{i,j} + \frac{\Delta t}{\Delta y}\left(V_{i,j-\frac{1}{2}} - V_{i,j+\frac{1}{2}}\right)$$

$$\psi_{i,j}^{n+1} = \frac{h_{i,j}^{n+1}\psi_{i,j}^{n+1}}{h_{i,j}^{n+1}}$$ Still Monotonic.

Strang splitting (alternating x-y order) can reduce directional splitting error.

Easter's Method gives an absolutely monotonic scheme based on a set of 1-D monotonic evaluations of fluxes. MOM6 uses PPM.

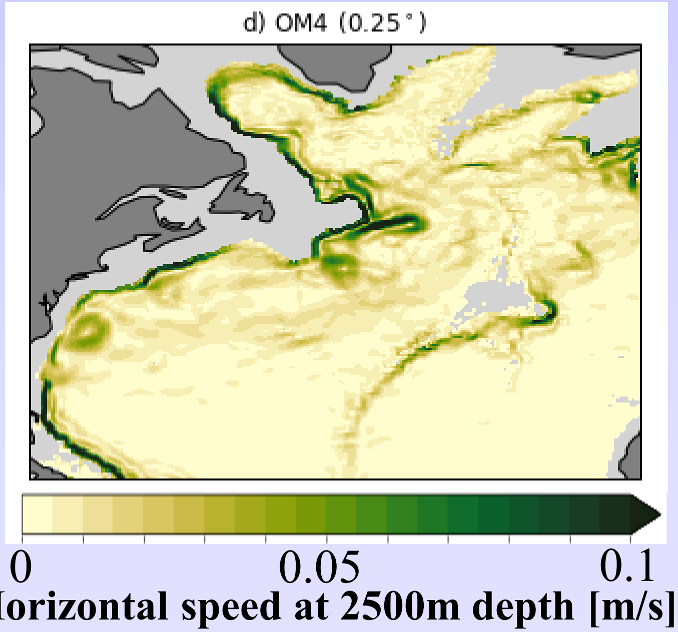
[see Easter (1993) *Mon. Wea. Rev.*; Durran, 2010, §5.9.4; Russell & Learner (1981) *J. Appl. Met.*]

Tracer Advection & Dynamics Cycles

- Flux-form pseudo-compressibility advection is based on accumulated mass (or volume) fluxes, not velocities.
- Additional pseudo-compressibility passes can be added to accommodate transports exceeding cell masses.

- Explicit layered dynamics time-steps are limited by Doppler-shifted internal gravity wave speeds or inertial oscillations.
- Flow speed in most of the ocean volume are much smaller than the peak internal wave speeds.

- Advective mass fluxes in MOM6 are often accumulated over multiple dynamic steps.
- Extra passes of tracer advection are used in MOM6 in the small fraction of cells where the mass fluxes exceed cell masses.



Split Time Stepping of the Barotropic Mode in Ocean Models

Making the hydrostatic approximation eliminates the sound waves. The external gravity waves are ~100 times faster than the Doppler-shifted internal wave.

$$\frac{D\vec{u}}{Dt} + f\hat{k}\times\vec{u} = -\frac{1}{\rho_o}\nabla p_{BC} - \frac{1}{\rho_o}\nabla g\eta$$

$$\frac{\partial\eta}{\partial t} + \nabla\cdot\left(\int_{-D}^{\eta}\vec{u}\,dz\right) \equiv \frac{\partial\eta}{\partial t} + \nabla\cdot\left((D+\eta)\vec{u}_{BT}\right) = 0$$

$$\vec{u}_{BT} \equiv \frac{1}{D+\eta}\int_{-D}^{\eta}\vec{u}\,dz$$

$$\frac{\partial p_{BC}}{\partial z} = -\rho g \qquad g_{Eff} = g + \frac{\partial}{\partial\eta}\left[\frac{1}{D+\eta}\int_{-D}^{\eta} p_{BC}\,dz\right]$$

Simplified equations:

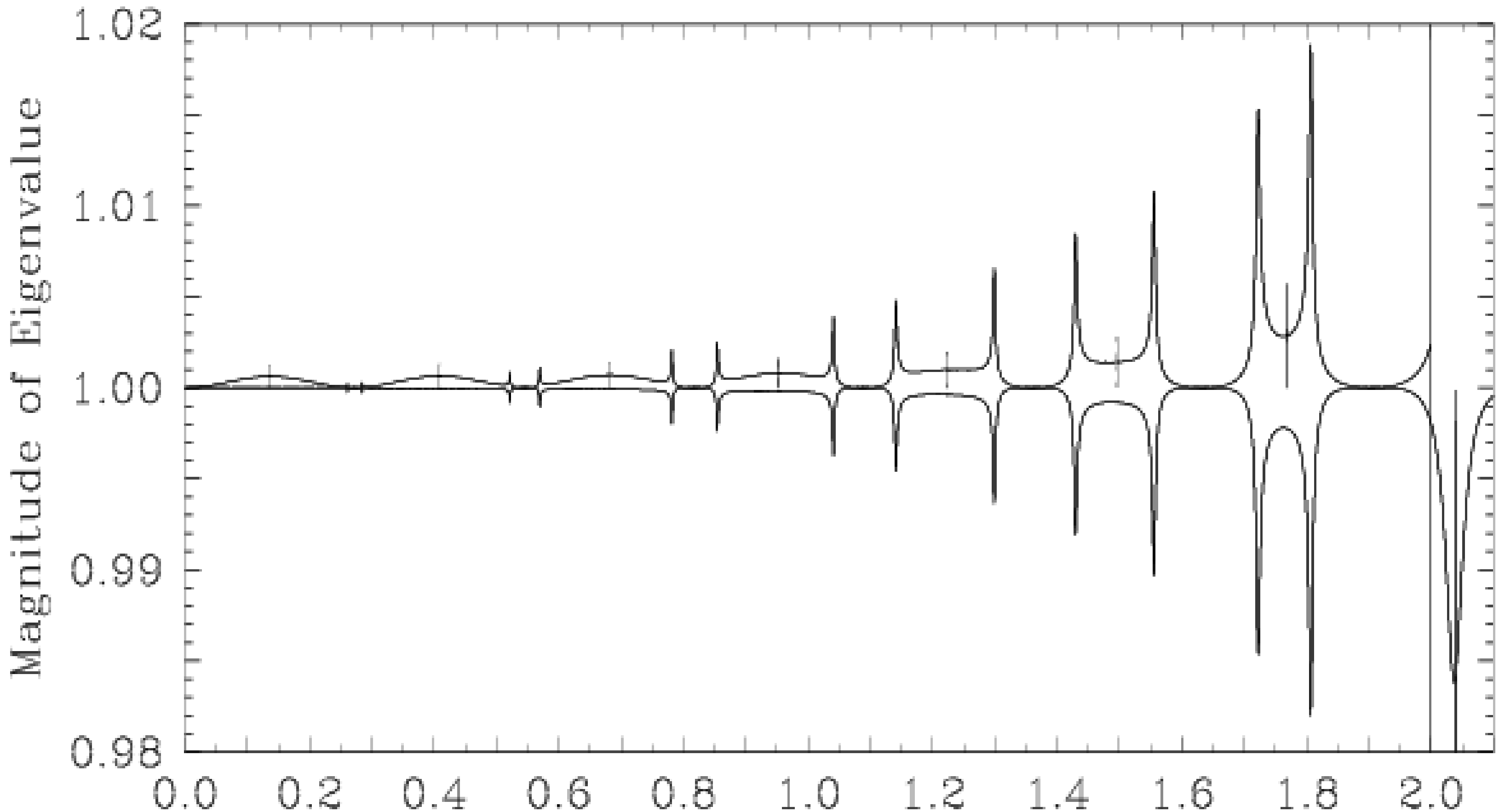
$$\frac{\partial\vec{u}_{BT}}{\partial t} + f\hat{k}\times\vec{u}_{BT} + \frac{1}{\rho_o}\nabla g_{Eff}\eta = \text{Residual}$$

$$\frac{\partial\eta}{\partial t} + \nabla\cdot\left((D+\eta)\vec{u}_{BT}\right) = 0$$

$$\Delta t\frac{\partial\vec{u}}{\partial t} = \Delta t\left(\frac{\partial\vec{u}}{\partial t} - \frac{\partial\vec{u}_{BT}}{\partial t}\right)^{n} + \Delta t\overline{\frac{\partial\vec{u}_{BT}}{\partial t}}^{\Delta t}$$

$$\Delta t\frac{\partial\theta}{\partial t} + \Delta t\left(\tilde{\vec{u}}\cdot\nabla\theta + \tilde{w}\frac{\partial\theta}{\partial z}\right) \qquad \tilde{\vec{u}} = \vec{u}_{BC} + \overline{\vec{u}_{BT}}^{\Delta t} \qquad \frac{\partial\tilde{w}}{\partial z} = -\nabla\cdot\tilde{\vec{u}}$$

Instability with Bleck & Smith (1990) Spit-explicit Time Stepping Scheme (Forward Baroclinic/Analytic Barotropic)

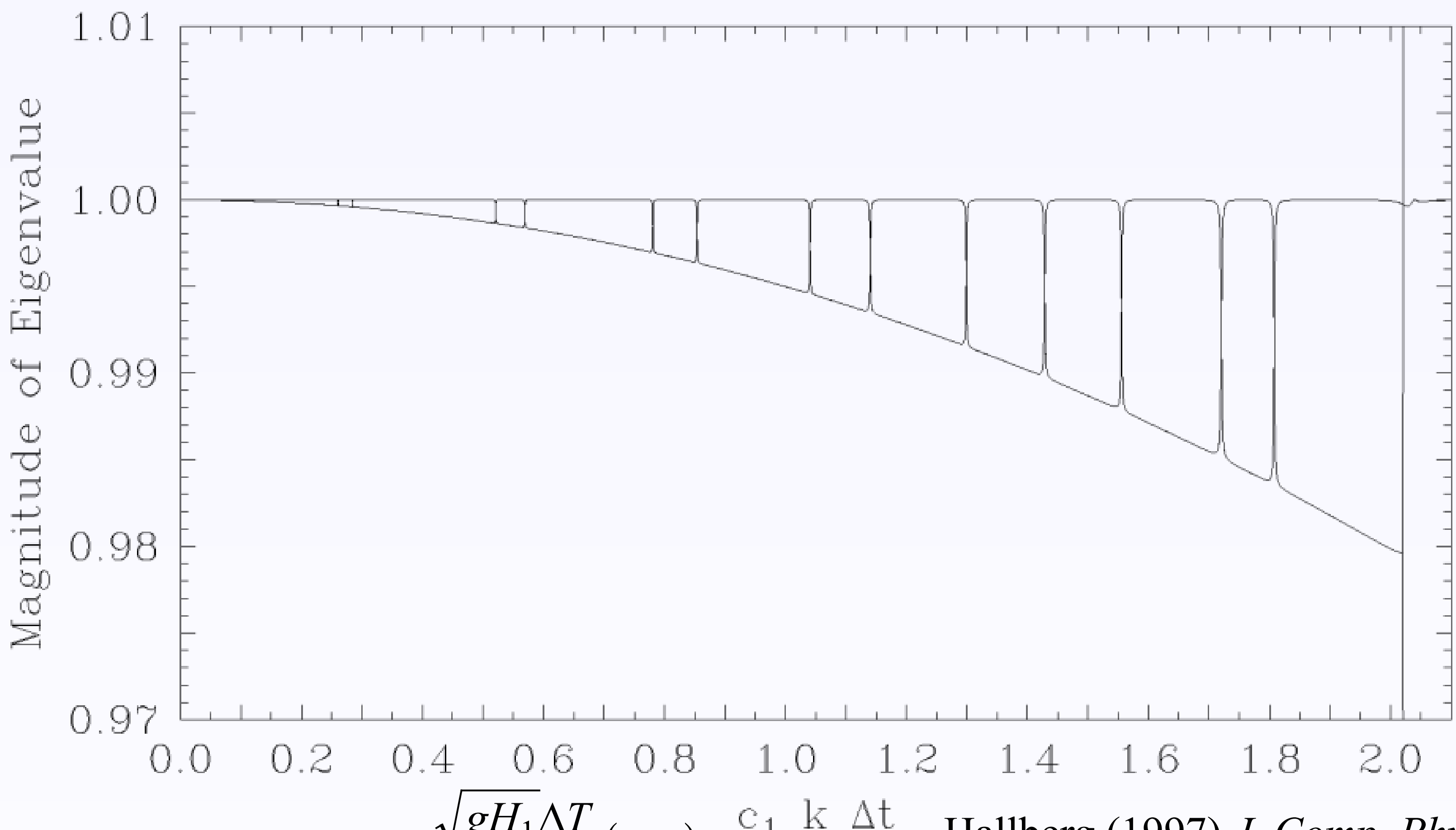


$$\frac{\sqrt{gH_1}\Delta T}{\Delta x}(k\Delta x)$$ $c_1\ k\ \Delta t$ Hallberg (1997) *J. Comp. Phys.*

Analysis of 2-layer shallow water model.

Magnitude of instability in mode splitting scales with $\dfrac{\sqrt{g' H_1}}{\sqrt{g(H_1+H_2)}}$

Stable Spit-explicit Time Stepping Scheme in MOM6 (Forward Baroclinic/Analytic Barotropic)



$$\frac{\sqrt{gH_1}\Delta T}{\Delta x}(k\Delta x)$$

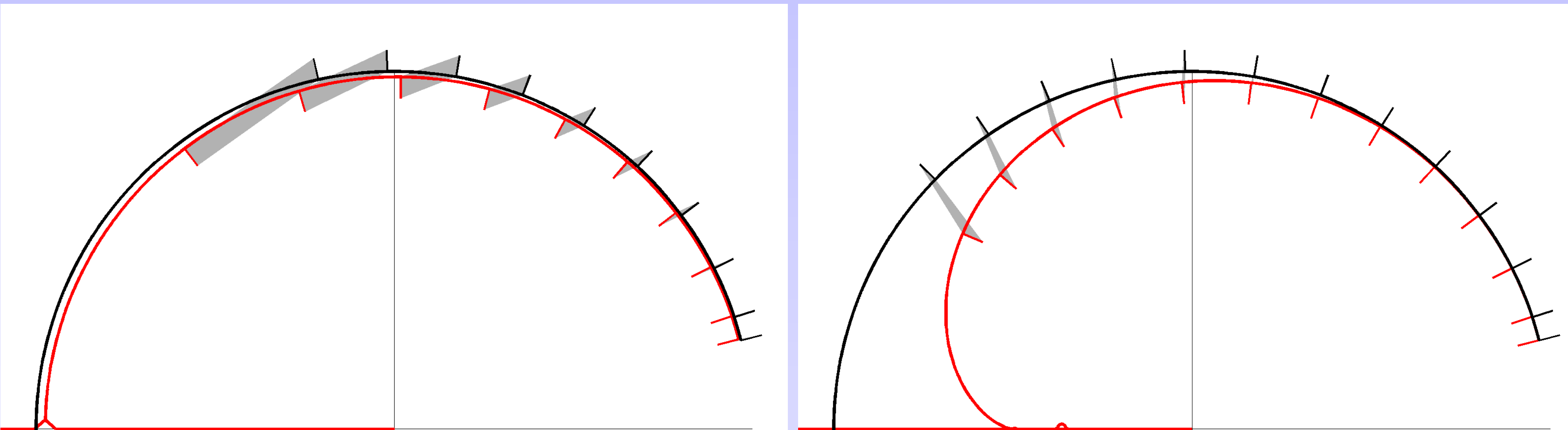
Hallberg (1997) *J. Comp. Phys.*

All fast motions included in barotropic solver
Frequency-dependent damping in stepping for baroclinic mode (1st order accurate)
Some damping of barotropic mode can be helpful as well

Damping and Order of Accuracy of Split Explicit Time Stepping Schemes

Hallberg (1997) *JCP*
1st order accurate (quasi-2nd)
Weak damping (HIM/GOLD/MOM6)

Shchepetkin & McWilliams (2005) *Oce. Mod.*
3rd order accurate
Strong damping at high freq. (ROMS)



Change in complex phase for baroclinic inertia-gravity waves with various length timesteps. The analytic solution (black) stays on the unit circle.

The MOM6 Runge-Kutta-~2 / Forward-Backward scheme calls the barotropic solver twice per time-step.

The ROMS scheme uses Adams-Bashforth techniques that are incompatible with the ALE remapping used in MOM6.

Two Estimates of the Free Surface Height

Vertically Discrete, Temporally Continuous Equations

- Layer Continuity Equations:

$$\frac{\partial h_k}{\partial t} = -\nabla \cdot \left(u h_k\right) = -\nabla \cdot \boldsymbol{F}\left(u_k, h_k\right)$$

- Relationship between free surface height and layer thicknesses:

$$\eta = \sum_{k=1}^{N} h_k - D$$

- Evolution equation for free surface height:

$$\frac{\partial \eta}{\partial t} = \sum_{k=1}^{N} \frac{\partial h_k}{\partial t} = -\nabla \cdot \sum_{k=1}^{N} F\left(u_k, h_k\right)$$

- If the algorithms for the fluxes in the continuity equations are *linear* in the velocity, the free surface height can be rewritten as:

$$\frac{\partial \eta}{\partial t} = -\nabla \cdot \sum_{k=1}^{N} F\left(u_k, h_k\right) = -\nabla \cdot \sum_{k=1}^{N} \left(u_k h_k\right)$$

$$= -\nabla \cdot \left[\sum_{k=1}^{N} h_k \frac{\sum_{k=1}^{N} \left(u_k h_k\right)}{\sum_{k=1}^{N} h_k} \right] \equiv -\nabla \cdot HU \qquad U \equiv \frac{\sum_{k=1}^{N} \left(u_k h_k\right)}{\sum_{k=1}^{N} h_k} \qquad H \equiv \sum_{k=1}^{N} h_k$$

But ALE models like MOM6 require positive-definite, nonlinear continuity solvers.

Two Estimates of the Free Surface Height

Time-split Equations

- External gravity waves are ~100 times faster than the fastest internal motions.
- All practical ocean models use a split-explicit or split-implicit time stepping scheme to avoid resolving the external gravity waves.

$$\frac{\eta^{n+1}-\eta^{n}}{\Delta t}+\nabla\cdot\left(\overline{UH}\right)=0 \qquad\qquad \overline{U}=\overline{UH}/\overline{H}$$

- The layer thicknesses are stepped forward with baroclinic velocities

$$\frac{h_k^{n+1}-h_k^{n}}{\Delta t}=-\nabla\cdot\boldsymbol{F}\left(\tilde{u}_k,h_k\right) \qquad\qquad \eta_h^{n+1}=\sum_{k=1}^{N}h_k^{n+1}-D$$

- Consistency between the two estimates η^{n+1} and η_h^{n+1} requires that

$$\nabla\cdot\sum_{k=1}^{N}F\left(\tilde{u}_k,h_k\right)=\nabla\cdot\left(\overline{UH}\right)$$

which can be satisfied for

$$\tilde{u}_k=u'_k+\overline{U} \qquad u'_k\equiv u_k-\sum_{k=1}^{N}\tilde{h}_k u_k\Big/\sum_{k=1}^{N}\tilde{h}_k \qquad \hat{h}_k\equiv\frac{F\left(\tilde{u}_k,h_k\right)}{\tilde{u}_k}$$

provided that $\hat{h}_k=\tilde{h}_k$ and $\overline{H}=\sum_{k=1}^{N}\tilde{h}_k$ (Perhaps $\hat{h}_k=\tilde{h}_k=\overline{h_k^n}^{x}\dfrac{\overline{H}}{H^n}$)

This is easily satisfied with z- or s-coordinate models, but not ALE models like MOM6.

Two Estimates of the Free Surface Height

- Discrepancies between the two estimates arise from the use of a nonlinear continuity solver.
- MOM6 continuity solver uses PPM with a monotonic or positive definite limiter.

$$\tilde{u}_k = \overline{U} + u_k - \sum_{k=1}^{N} \tilde{h}_k u_k \Big/ \sum_{k=1}^{N} \tilde{h}_k \qquad \hat{h}_k \equiv \frac{F\left(\tilde{u}_k, h_k\right)}{\tilde{u}_k}$$

$$\eta_h^{n+1} - \eta^{n+1} = \left(\eta_h^n - \eta^n\right) + \Delta t \nabla \cdot \left(\overline{UH}\right) - \Delta t \nabla \cdot \sum_{k=1}^{N} \left(\hat{h}_k \tilde{u}_k\right)$$

$$\eta_h^{n+1} - \eta^{n+1} = \left(\eta_h^n - \eta^n\right) + \Delta t \nabla \cdot \left[\overline{U}\left(\overline{H} - \sum_{k=1}^{N} \hat{h}_k\right)\right] - \Delta t \nabla \cdot \sum_{k=1}^{N} \left[\left(\hat{h}_k - \left(\sum_{j=1}^{N} \hat{h}_j \Big/ \sum_{j=1}^{N} \tilde{h}_j\right) \tilde{h}_k\right) u_k\right]$$

- $\overline{H}$ is given by the barotropic algorithm, and $\tilde{h}_k$ can be specified to be anything convenient.

- $\hat{h}_k \equiv \frac{F(\tilde{u}_k, h_k)}{\tilde{u}_k}$ is only known once $\tilde{u}_k$ is!

Past Strategies for Reconciling the Two Estimates of the Free Surface Height

1. Dilate the layers to agree with the barotropic solution.

- First proposed by Bleck & Smith (1990), and used in MICOM and Hycom ever since.

$$h_k^{n+1} = S h_k^n - \Delta t S \nabla \cdot \boldsymbol{F}\left[\left(u'_k + \overline{U}\right), h_k\right] \qquad u'_k \equiv u_k - \sum_{k=1}^{N} \tilde{h}_k u_k \Big/ \sum_{k=1}^{N} \tilde{h}_k$$

$$S = \frac{D + \eta^{n+1}}{\sum_{k=1}^{N} \left\{ h_k^n - \Delta t \nabla \cdot \boldsymbol{F}\left[\left(u'_k + \overline{U}\right), h_k\right]\right\}} \quad \Rightarrow \quad \eta_h^{n+1} = \eta^{n+1}$$

- The two estimates of sea surface height agree exactly.
- Dilation adds minimal computational expense.
- Total mass (or volume) is conserved because it is conserved by the barotropic solution.
- Layer equations are no longer in flux-form.
- Total salt, heat, and other tracers are not conserved.
- Spurious diapycnal mass/volume exchange is introduced.
- Probably acceptable for short runs; probably unacceptable for long-term climate studies.

Past Strategies for Reconciling the Two Estimates of the Free Surface Height

2. Drive the solutions toward each other with a fake mass source in the barotropic equations.
   - Used in HIM solutions from ~1996-2006; never published.

$$\frac{\eta^{n+1} - \eta^{n}}{\Delta t} + \nabla \cdot \left(\overline{UH}\right) = \lambda \frac{\eta_h^n - \eta^n}{\Delta t}$$

$$u'_k \equiv u_k - \sum_{k=1}^{N} \tilde{h}_k u_k \Big/ \sum_{k=1}^{N} \tilde{h}_k$$

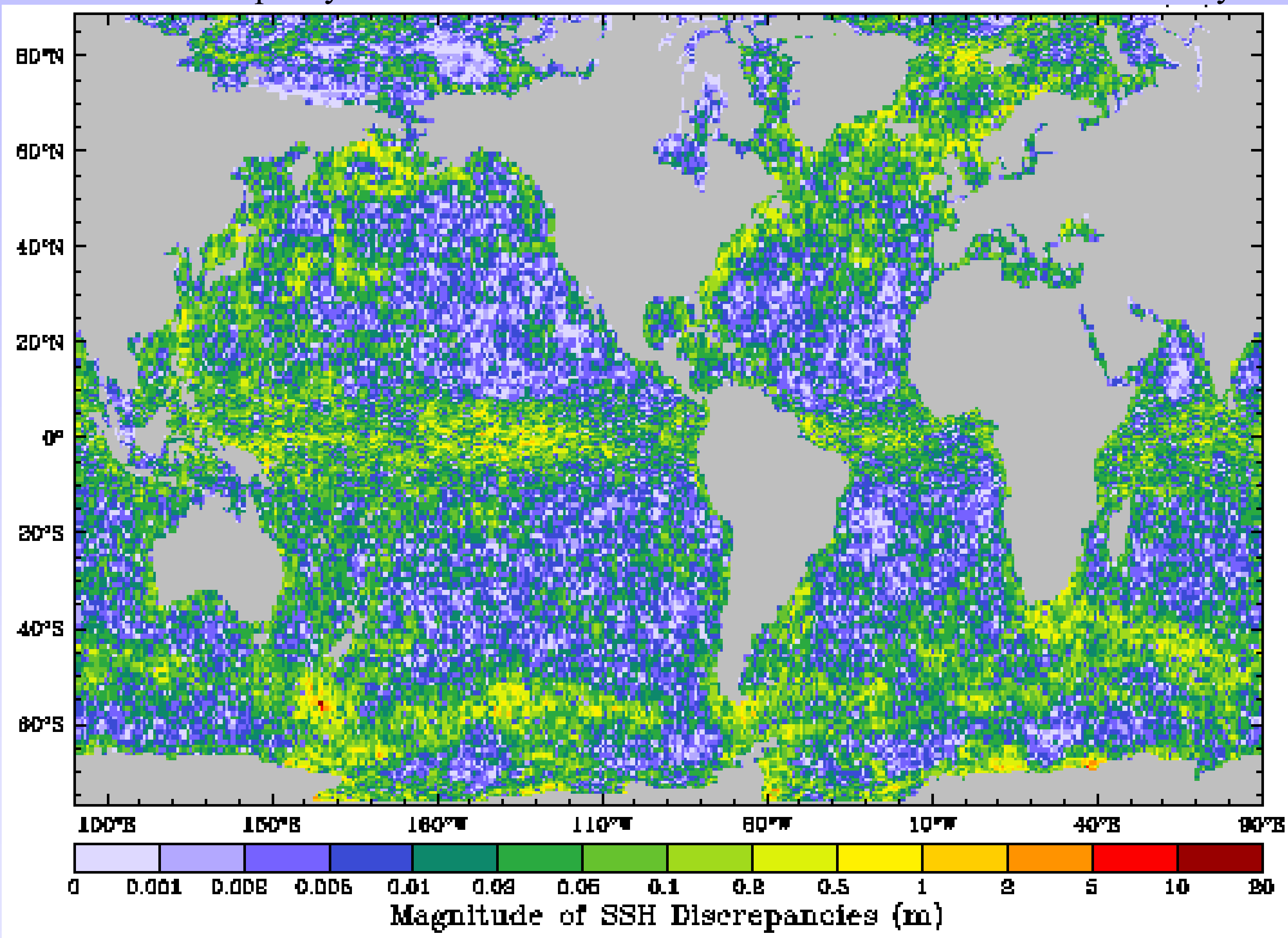
$$h_k^{n+1} = h_k^n - \Delta t \nabla \cdot \mathrm{F}\left[\left(u'_k + \overline{U}\right), h_k\right]$$

- Total mass is conserved layer-wise.
- Layer equations retain their flux-form.
- Total salt, heat, and other tracers are exactly conserved.
- Long-term mean free surface heights tend to agree – total layer thicknesses are driven toward the external mode height by the divergent barotropic velocities.
- The instantaneous free surface heights can be quite different.
- Energetic external gravity waves can be generated.
- Probably unacceptable for short runs; maybe OK for long-term climate studies.

Instantaneous SSH Discrepancies – Old HIM Scheme

Global 1°, 48-Layer CORE Run ; 3600 s Timestep ; λ=1

RMS Discrepancy: 0.132 m January 1



Past Strategies for Reconciling the Two Estimates of the Free Surface Height

3. Add an upwind corrective step to the layer equations.
   - Suggested by Paul Schopf ~2003; Higdon (2005)

$$\frac{\eta^{n+1}-\eta^{n}}{\Delta t}+\nabla\cdot\left(\overline{UH}\right)=0$$

$$h_k^{*}=h_k^{n}-\Delta t\nabla\cdot \mathrm{F}\left[\left(u'_k+\overline{U}\right),h_k\right]$$

$$h_k^{n+1}=h_k^{*}-\Delta t\nabla\cdot\left[U_C h_{k,Upwind}^{*}\right]$$

$$U_C=\frac{\overline{UH}-\sum_{k=1}^{N}\mathrm{F}\left[\left(u'_k+\overline{U}\right),h_k\right]}{\sum_{k=1}^{N}h_{k,Upwind}^{*}}$$

   - Total mass is conserved layer-wise.
   - Layer equations retain their flux-form.
   - Total salt, heat, and other tracers are exactly conserved.
   - Free surface heights exactly agree.
   - Requires additional communication/wider halos.
   - The corrections may not correspond to the layers whose flow was deficient.
   - There is a tendency to move the densest layers upslope, leading to large spurious "Neptune" circulations.

MOM6 Strategy for Reconciling the Two Estimates of the Free Surface Height

4. Determine the barotropic adjustment so that the fluxes agree.

- First used in GFDL's GOLD (predecessor to MOM6) ~2006

$$\frac{\eta^{n+1} - \eta^{n}}{\Delta t} + \nabla \cdot \left(\overline{UH}\right) = 0$$

Determine the ΔU such that $\sum_{k=1}^{N} F\left(\tilde{u}_k, h_k\right) = \left(\overline{UH}\right)$ where $\tilde{u}_k = u_k + \Delta U$

$$h_k^{n+1} = h_k^{n} - \Delta t \nabla \cdot \mathrm{F}\left(\tilde{u}_k, h_k\right) \quad \Rightarrow \quad \eta_h^{n+1} = \eta^{n+1}$$

- Total mass is conserved layer-wise.
- Layer equations retain their flux-form.
- Total salt, heat, and other tracers are exactly conserved.
- Free surface heights exactly agree.
- Requires (very few) completely local iterations.
- The velocity corrections are barotropic, and more likely to correspond to the layers whose flow was deficient.

Hallberg, R., and A. Adcroft, 2009: Reconciling estimates of the free surface height in Lagrangian vertical coordinate ocean models with mode-split time stepping. *Ocean Modelling*, **29**, doi:10.1016/j.ocemod.2009.02.008.

Iterations to find ΔU

Determining the ΔU such that $\sum_{k=1}^{N} F(\tilde{u}_k, h_k) = (\overline{UH})$ where $\tilde{u}_k = u_k + \Delta U$

- The solution is unique provided that $\frac{\partial}{\partial \tilde{u}_k} F(\tilde{u}_k, h_k) > 0$
- This is a reasonable requirement for any discretization of the continuity equation. In the continuous limit, $F(u,h) = uh$, so one interpretation is

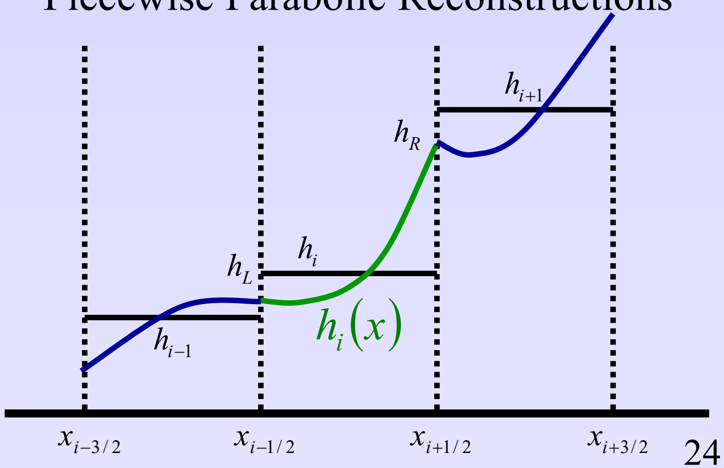
$$\frac{\partial}{\partial \tilde{u}_k} F(\tilde{u}_k, h_k) = h_{k,\mathrm{Marginal}}$$

With a the PPM continuity scheme:

$$F_{i+1/2} = \frac{1}{\Delta t}\int_{x_{i+1/2}-u\Delta t}^{x_{i+1/2}} h_i^n(x)\,dx \quad \Rightarrow \quad \frac{\partial F_{i+1/2}}{\partial u_{i+1/2}} = h_i^n\left(x_{i+1/2} - u_{i+1/2}\Delta t\right) \equiv h_{k,\mathrm{Marginal}}$$

$h_i(x) > 0$ is already required for positive definiteness.

Piecewise Parabolic Reconstructions



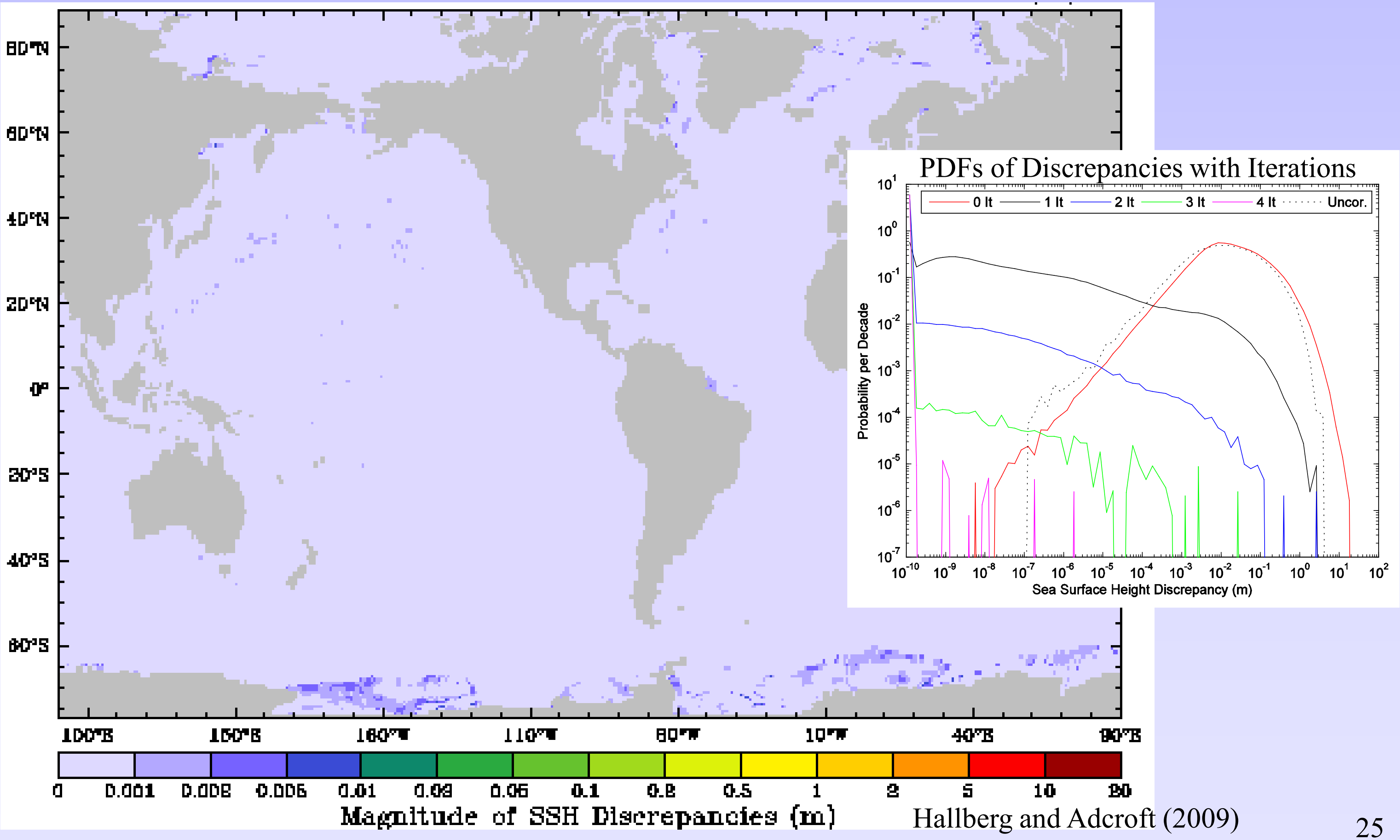
- Newton's method iterations quickly give ΔU

$$\Delta U^{m+1} = \Delta U^m + \frac{(\overline{UH}) - \sum_{k=1}^{N} F(u_k + \Delta U^m, h_k)}{\sum_{k=1}^{N} h_{k,\mathrm{Marginal}}}$$

Instantaneous SSH Discrepancies MOM6 Iterative PPM Scheme

Global 1°, 48-Layer CORE Run ; 3600 s Timestep ; 3 Iterations ; 10^{-6} m Tolerence

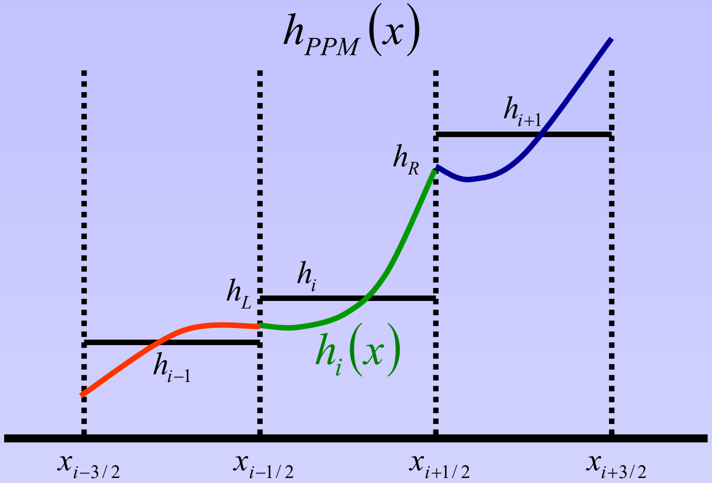
RMS Discrepancy: 0.00033 m after 3 Iterations January 1

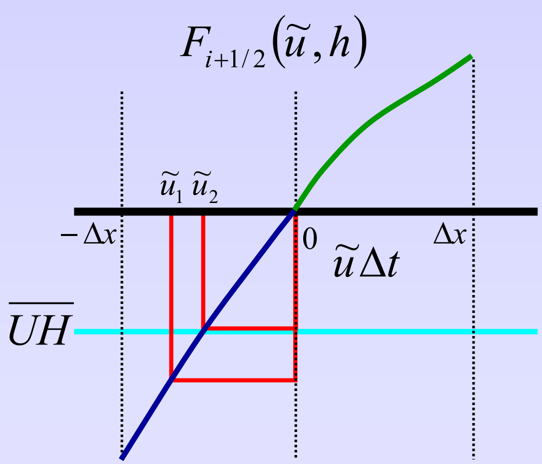


Hallberg and Adcroft (2009)

How practical is this iterative approach?

Piecewise Parabolic Reconstructions





$$F_{i+1/2} = \frac{1}{\Delta t}\int_{x_{i+1/2}-u\Delta t}^{x_{i+1/2}} h_{PPM}(x)\,dx$$

A "Piecewise Parabolic Method" continuity solver uses two steps:

1. Setting up the positive-definite subgridscale profiles, $h_{PPM}(x)$.
2. Integrating the profiles to determine F.

Step 1. is typically ~3x as expensive as Step 2.

$F(u,h)$ is piecewise cubic in u, but often nearly linear.

Newton's method iterations converge quickly.

$$\Delta U^{m+1} = \Delta U^{m} + \frac{\left(\overline{UH}\right) - \sum_{k=1}^{N} F\left(u_k + \Delta U^{m}, h_k\right)}{\sum_{k=1}^{N} h_{k,\mathrm{Marginal}}}$$

In a global model SSHs converge everywhere to a tolerance of 10^{-6} m within 5 iterations.

These 5 iterations add ~1.6 times more CPU time to the PPM continuity solver.

The continuity solver is just 12% of the total model time.

One Added Wrinkle from Considering Viscosity

Barotropic accelerations do not lead to barotropic flows after a timestep when vertical viscosity is taken into account!

$$u_k^{n+1} = u_k^n + \Delta t A_k + \Delta t \frac{\tau_{k-1/2} - \tau_{k+1/2}}{h_k}$$

$$\gamma_k \equiv \frac{1}{\Delta t} \frac{\delta u_k^{n+1}}{\delta \overline{A}}$$

$$\tau_{1/2} = \tau_{Wind}$$

$$\tau_{k+1/2} = \nu_{k+1/2} \frac{u_k^{n+1} - u_{k+1}^{n+1}}{h_{k+1/2}}$$

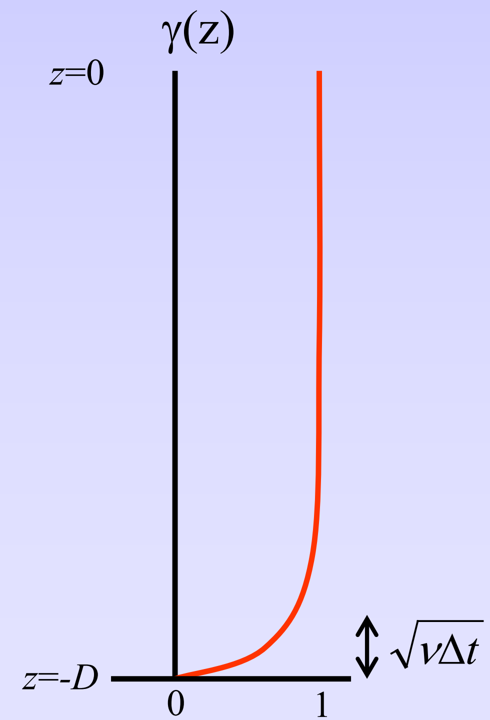
$$\tau_{N+1/2} = \nu_{N+1/2} \frac{2u_N^{n+1}}{h_{N+1/2}}$$

A tridiagonal equation for γ_k results, going from 0 for thin layers near the bottom to 1 far above the bottom.

$$\gamma_1 = 1 + \frac{1}{h_1}\left[-\frac{\nu_{3/2}\Delta t}{h_{3/2}} (\gamma_1 - \gamma_2) \right]$$

$$\gamma_k = 1 + \frac{1}{h_k}\left[\frac{\nu_{k-1/2}\Delta t}{h_{k-1/2}} (\gamma_{k-1} - \gamma_k) - \frac{\nu_{k+1/2}\Delta t}{h_{k+1/2}} (\gamma_k - \gamma_{k+1}) \right]$$

$$\gamma_N = 1 + \frac{1}{h_N}\left[\frac{\nu_{N-1/2}\Delta t}{h_{N-1/2}} (\gamma_{N-1} - \gamma_N) - \frac{2\nu_{N+1/2}\Delta t}{h_{N+1/2}} \gamma_N \right]$$



In the continuous limit: $\gamma(z) = 1 + \Delta t \frac{d}{dz}\left(\nu \frac{d\gamma}{dz} \right)$

with boundary conditions: $\frac{d\gamma}{dz}(0) = 0 \qquad \gamma(-D) = 0$

Deep water, constant ν: $\gamma(z) = 1 - \exp\left(-\sqrt{\nu \Delta t} (z + D) \right)$

MOM6 Strategy for Reconciling the Two Estimates of the Free Surface Height

4b. Determine the barotropic *accelerations* so that the fluxes agree.

- Used in MOM6; in use in GFDL's GOLD since ~2008

$$\frac{\eta^{n+1}-\eta^{n}}{\Delta t}+\nabla\cdot\left(\overline{UH}\right)=0$$

Determine the $\Delta\overline{A}$ such that $\sum_{k=1}^{N} F\left(\tilde{u}_k, h_k\right)=\left(\overline{UH}\right)$ where $\tilde{u}_k = u_k + \gamma_k \Delta\overline{A}\Delta t$

$$h_k^{n+1} = h_k^n - \Delta t\nabla\cdot \mathrm{F}\left(\tilde{u}_k, h_k\right) \qquad \Rightarrow \qquad \eta_h^{n+1} = \eta^{n+1}$$

- Total mass is conserved layer-wise.
- Layer equations retain their flux-form.
- Total salt, heat, and other tracers are exactly conserved.
- Free surface heights exactly agree.
- Requires (very few) completely local iterations.
- The corrective *accelerations* are barotropic, and more likely to correspond to the layers whose flow was deficient.

Additional details concerning MOM6 baroclinic-barotropic split time stepping

- Transports used as input and output to the barotropic solver. The continuity solver is inverted to determine velocities.

$$\frac{\partial \eta}{\partial t} = \nabla \cdot \overline{U} + M \qquad \overline{U}\left(\overline{u}\right) = \frac{1}{\Delta T}\int_0^{\overline{u}\Delta T} H(x)\,dx \qquad \overline{u}^n = \overline{U}^{-1}\left(\sum_k U_k^n\right)$$

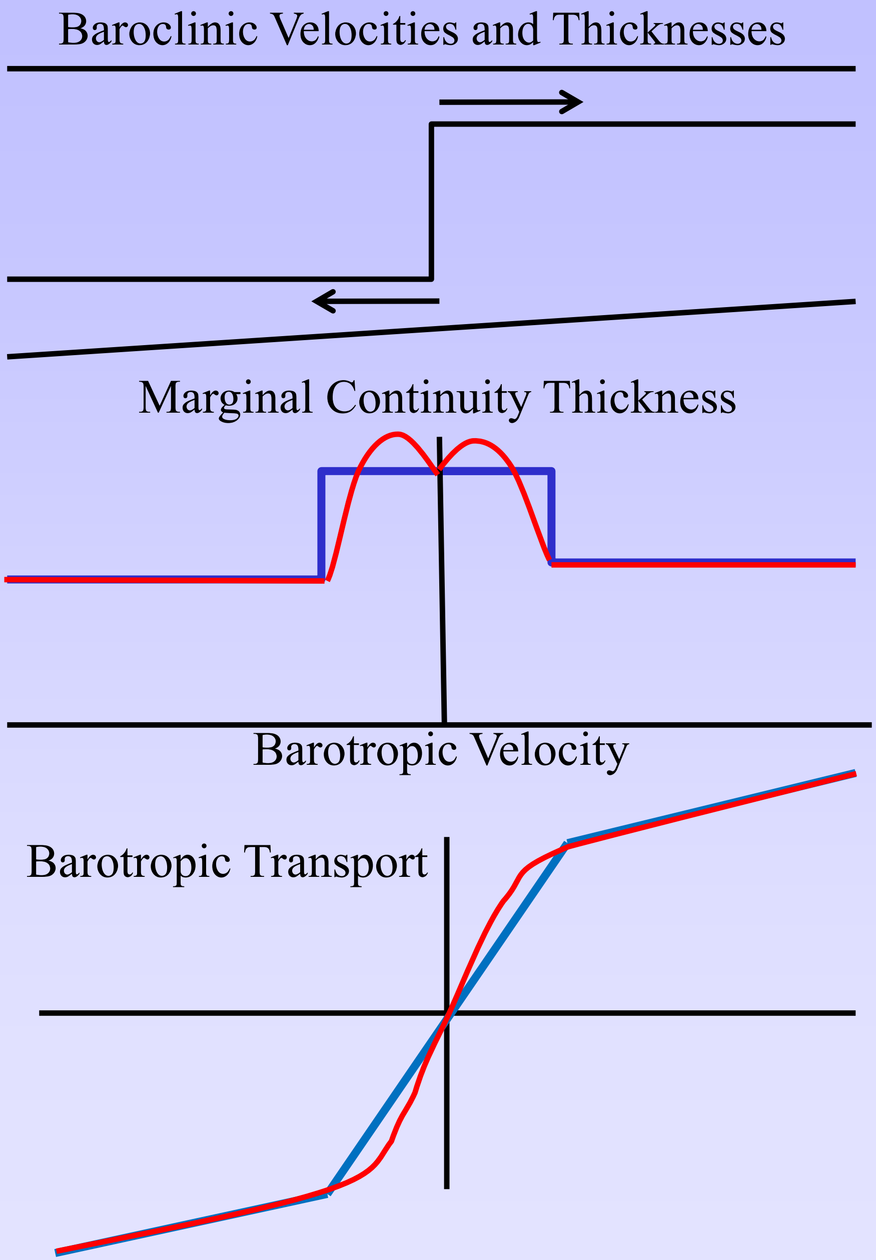
$$u_k^{n+1} = \tilde{u}_k^{n+1} + \Delta\overline{u} \qquad \text{Find } \Delta\overline{u} \text{ such that } \sum_k U_k\left(\tilde{u}_k^{n+1} + \Delta\overline{u}\right) = \overline{U}^{n+1}$$

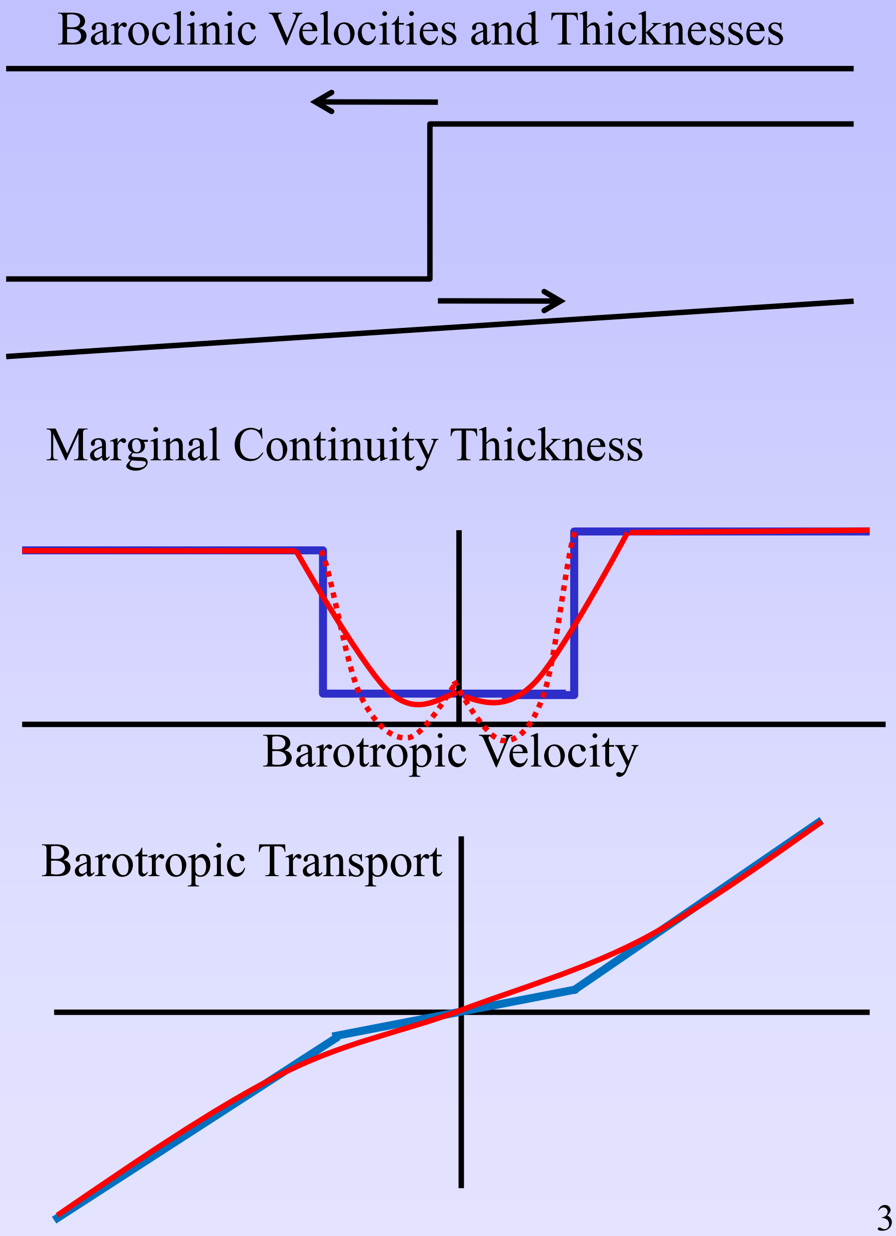
- Barotropic accelerations are treated as anomalies from the baroclinic state.

$$\frac{\partial \overline{u}}{\partial t} = -f\hat{k}\times\left(\overline{u} - \overline{u}_{Cor}\right) - \nabla \overline{g}\left(\eta - \eta_{PF}\right) - \frac{c_D\left(\left\|u_{Bot}\right\| + \left\|u_{Shelf}\right\|\right)}{\sum_k h_k}\left(\overline{u} - \overline{u}_{Drag}\right) + \frac{\sum_k h_k \frac{\partial u_k}{\partial t}}{\sum_k h_k}$$

- Bottom-drag (& under ice surface-drag) is treated implicitly.
- The barotropic continuity solver uses flow-dependent thicknesses fits which approximate the sum of the layer thickness transports, to accommodate wetting & drying.

Parametrically Representing Dependence of Summed Layer Transport on Barotropic Velocities





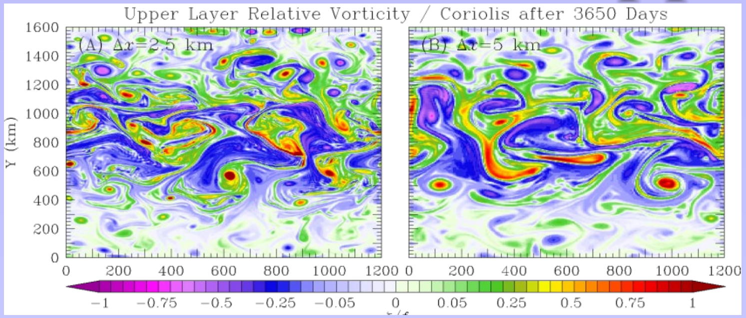
Takeaways Regarding MOM6 Split Time Stepping

- Efficient approach for handling fast modes via simplified 2-d equations, while 3-d baroclinic timestep is determined by baroclinic dynamics

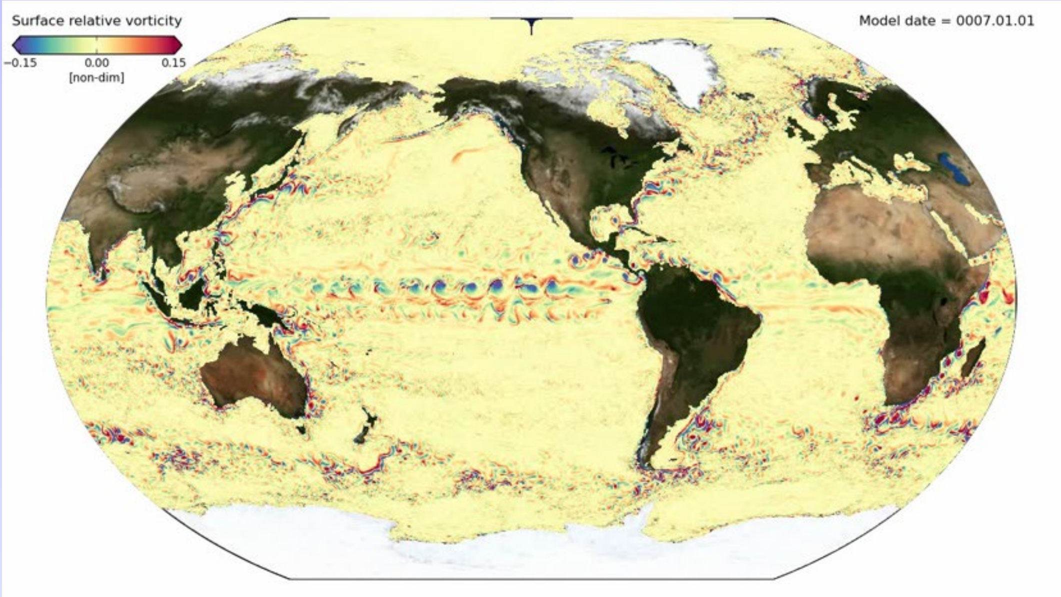
- Barotropic solver determines free surface height and time-averaged depth-integrated transports
- By using anomalies, the MOM6 split solver gives identical answers to an equivalent unsplit scheme for short time steps
- Demonstrated to work with wetting & drying (e.g., under ice-shelves)
- Linear barotropic solver allows MOM6 to automatically set stable barotropic timestep (e.g. to 98% of maximum)

Examples of Important MOM6 Applications

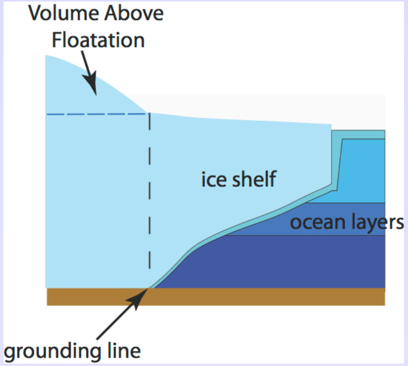
- Idealized process studies
- GFDL's ESM4/CM4 coupled climate model
- Ice-sheet ocean interactions
- NCAR CESM3
- Australian COSIMA Consortium
- NOAA/NCEP GFS-v3 coupled seasonal forecasts
- U.S. Navy HYCOM successor in near-term forecasts (~2025?)



Vorticity in study of resolution dependent eddy params. Hallberg (2013) *Ocean Modelling*



Surface Vorticity in CM4 prototype Adcroft et al. (2019) *JAMES*



Schematic illustration of dynamically coupled ice-sheet & ocean models. Goldberg et al. (2012) *J. Geophys. Res.*

Potential Areas of Improvement in MOM6 Solvers

- More efficient message passing to accommodate iterative tracer advection
- Single step baroclinic / barotropic solver (~2x speedup in dynamics stepping?)
- Higher order multi-step baroclinic / barotropic solver (better for forecasting but slower?)
- Incorporate fast ice (rheology & transport) solver into MOM6 barotropic solver to avoid coupled instabilities
- Thicknesses at velocity points used in viscosity terms taken from a linearization of continuity solver