Convexitonic Maps and some of properties in 2 and 3 (and some 4) variables

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This notebook is for “Bianalytic Maps Between Free Spectrahedra.” M. Augat, J.W. Helton, I. Klep, S. McCullough

https://arxiv.org/abs/1604.04952

Warning: This nb. runs on the 4Oct17

devel version of NC and will not run on older versions. Newer versions should be fine.

Basic Commands

```
(* Basic NC calls used in the notebook *)
<< NC`;
<< NCAlgebra`;
<< NCGBX`;
<< NCPolyAssociationGraded`

(* SetOptions[inv,Distribute→ True] causes NCAlgebra to consider

  inv[x**y]=inv[y]**inv[x].

   By default this option is sometimes set to false. Setting the command to true allows
  for more simplification but assumes soome expressions are invertible. *)
SetOptions[inv, Distribute → True]

SNC[R];
mf = MatrixForm;

(* The following aliases are used since NCGBX may not recognize variables of
 the form R[i]. NCGBX does recognize variables of the form Subsript[R,i]. *)
R[i_] := Subscript[R, i];
Z[i_] := Subscript[Z, i];
x[i_] := Subscript[x, i];
y[i_] := Subscript[y, i];
```

```
You are using the version of NCAlgebra which is found in:

  C:\Users\Owner\NC\

You can now use "<< NCAlgebra`" to load NCAlgebra.

------------------------------------------------------------------
NCAlgebra - Version 5.0.4
Compatible with Mathematica Version 10 and above

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  Helton and de Oliveira 2017
  Helton 2002
  Helton and Miller June 1991
  All rights reserved.

The program was written by the authors and by:
  David Hurst, Daniel Lamm, Orlando Merino, Robert Obar,
  Henry Pfister, Mike Walker, John Wavrik, Lois Yu,
  J. Camino, J. Griffin, J. Ovall, T. Shaheen, John Shopple.
  The beginnings of the program come from eran@slac.
  Considerable recent help came from Igor Klep.

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  and US Department of Education.

For NCAlgebra updates see:

  www.github.com/NCAlgebra/NC
  www.math.ucsd.edu/~ncalg

------------------------------------------------------------------
```

All functions that need to be compiled for map computation including the defining relations for G=2 and

G=3 and included G=4 dimensional algebras.

Resulting Structure Constants for the algebras and their convexotonic maps

G=2

G=2, A1

```
RuAlg[2, 1]
XiAll[2, 1]
pAll[2, 1]
qAll[2, 1]
pofq[2, 1]
qofp[2, 1]
```

$\{R_1 ** R_1 \to R_2, R_2 ** R_2 \to 0, R_1 ** R_2 \to 0, R_2 ** R_1 \to 0\}$

$\left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$

$\{x_1, x_1 ** x_1 + x_2\}$

$\{y_1, -y_1 ** y_1 + y_2\}$

$\{y_1, y_2\}$

$\{x_1, x_2\}$

G=2, A2

```
RuAlg[2, 2]
XiAll[2, 2]
pAll[2, 2]
qAll[2, 2]
pofq[2, 2]
qofp[2, 2]
```

$\{R_1 ** R_1 \to R_1,\ R_2 ** R_2 \to 0,\ R_1 ** R_2 \to R_2,\ R_2 ** R_1 \to 0\}$

$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$

$\left\{x_1 ** \left(1 - x_1\right)^{-1},\ x_1 ** \left(1 - x_1\right)^{-1} ** x_2 + x_2\right\}$

$\left\{y_1 ** \left(1 + y_1\right)^{-1},\ -y_1 ** \left(1 + y_1\right)^{-1} ** y_2 + y_2\right\}$

$\{y_1,\ y_2\}$

$\{x_1,\ x_2\}$

G=2, A3

```
RuAlg[2, 3]
XiAll[2, 3]
pAll[2, 3]
qAll[2, 3]
pofq[2, 3]
qofp[2, 3]
```

$\{R_1 ** R_1 \to R_1,\ R_2 ** R_2 \to 0,\ R_1 ** R_2 \to 0,\ R_2 ** R_1 \to R_2\}$

$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$

$\left\{x_1 ** \left(1 - x_1\right)^{-1},\ x_2 ** \left(1 - x_1\right)^{-1}\right\}$

$\left\{y_1 ** \left(1 + y_1\right)^{-1},\ y_2 ** \left(1 + y_1\right)^{-1}\right\}$

$\{y_1,\ y_2\}$

$\{x_1,\ x_2\}$

G=2, A4

```
RuAlg[2, 4]
XiAll[2, 4]
pAll[2, 4]
qAll[2, 4]
pofq[2, 4]
qofp[2, 4]
```

$\{R_1 ** R_1 \to R_1,\ R_2 ** R_2 \to 0,\ R_1 ** R_2 \to R_2,\ R_2 ** R_1 \to R_2\}$

$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$

$\{x_1 ** (1 - x_1)^{-1},\ x_2 ** (1 - x_1)^{-1} + x_1 ** (1 - x_1)^{-1} ** x_2 ** (1 - x_1)^{-1}\}$

$\{y_1 ** (1 + y_1)^{-1},\ y_2 ** (1 + y_1)^{-1} - y_1 ** (1 + y_1)^{-1} ** y_2 ** (1 + y_1)^{-1}\}$

$\{y_1,\ y_2\}$

$\{x_1,\ x_2\}$

G=3

G=3, A1

```
RuAlg[3, 1]
XiAll[3, 1]
pAll[3, 1]
qAll[3, 1]
pofq[3, 1]
qofp[3, 1]
```

$\{R_1 ** R_1 \to 0,\ R_2 ** R_2 \to 0,\ R_3 ** R_3 \to 0,\ R_1 ** R_2 \to 0,$
$R_2 ** R_1 \to 0,\ R_1 ** R_3 \to R_2,\ R_3 ** R_1 \to R_2,\ R_2 ** R_3 \to 0,\ R_3 ** R_2 \to 0\}$

$\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\}$

$\{x_1,\ x_1 ** x_3 + x_3 ** x_1 + x_2,\ x_3\}$

$\{y_1,\ -y_1 ** y_3 - y_3 ** y_1 + y_2,\ y_3\}$

$\{y_1,\ y_2,\ y_3\}$

$\{x_1,\ x_2,\ x_3\}$

G=3, A2

```
RuAlg[3, 2]
XiAll[3, 2]
pAll[3, 2]
qAll[3, 2]
pofq[3, 2]
qofp[3, 2]
```

$\{R_1 ** R_1 \to 0,\ R_2 ** R_2 \to 0,\ R_3 ** R_3 \to 0,\ R_1 ** R_2 \to 0,$
$R_2 ** R_1 \to 0,\ R_1 ** R_3 \to R_2,\ R_3 ** R_1 \to \text{alpha}\, R_2,\ R_2 ** R_3 \to 0,\ R_3 ** R_2 \to 0\}$

$$\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \text{alpha} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\}$$

$\{x_1,\ x_1 ** x_3 + \text{alpha}\, x_3 ** x_1 + x_2,\ x_3\}$

$\{y_1,\ -y_1 ** y_3 - \text{alpha}\, y_3 ** y_1 + y_2,\ y_3\}$

$\{y_1,\ y_2,\ y_3\}$

$\{x_1,\ x_2,\ x_3\}$

G=3, A3

```
RuAlg[3, 3]
XiAll[3, 3]
pAll[3, 3]
qAll[3, 3]
pofq[3, 3]
qofp[3, 3]
```

$\{R_1 ** R_1 \to R_2,\ R_2 ** R_2 \to 0,\ R_3 ** R_3 \to 0,\ R_1 ** R_2 \to R_3,$
$R_2 ** R_1 \to R_3,\ R_1 ** R_3 \to 0,\ R_3 ** R_1 \to 0,\ R_2 ** R_3 \to 0,\ R_3 ** R_2 \to 0\}$

$$\left\{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\}$$

$\{x_1,\ x_1 ** x_1 + x_2,\ x_1 ** (x_1 ** x_1 + x_2) + x_2 ** x_1 + x_3\}$

$\{y_1,\ -y_1 ** y_1 + y_2,\ y_1 ** (y_1 ** y_1 - y_2) - y_2 ** y_1 + y_3\}$

$\{y_1,\ y_2,\ y_3\}$

$\{x_1,\ x_2,\ x_3\}$

G=3, A4

```
RuAlg[3, 4]
XiAll[3, 4]
pAll[3, 4]
qAll[3, 4]
pofq[3, 4]
qofp[3, 4]
```

$\{R_1 ** R_1 \to 0,\ R_2 ** R_2 \to 0,\ R_3 ** R_3 \to R_3,\ R_1 ** R_2 \to 0,$
$R_2 ** R_1 \to 0,\ R_1 ** R_3 \to R_2,\ R_3 ** R_1 \to 0,\ R_2 ** R_3 \to R_2,\ R_3 ** R_2 \to 0\}$

$$\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

$\left\{x_1,\ x_2 ** (1 - x_3)^{-1} + x_1 ** x_3 ** (1 - x_3)^{-1},\ x_3 ** (1 - x_3)^{-1}\right\}$

$\left\{y_1,\ y_2 ** (1 + y_3)^{-1} - y_1 ** y_3 ** (1 + y_3)^{-1},\ y_3 ** (1 + y_3)^{-1}\right\}$

$\{y_1,\ y_2,\ y_3\}$

$\{x_1,\ x_2,\ x_3\}$

G=3, A5

```
RuAlg[3, 5]
XiAll[3, 5]
pAll[3, 5]
qAll[3, 5]
pofq[3, 5]
qofp[3, 5]
```

$\{R_1 ** R_1 \to 0,\ R_2 ** R_2 \to 0,\ R_3 ** R_3 \to R_3,\ R_1 ** R_2 \to 0,$
$R_2 ** R_1 \to 0,\ R_1 ** R_3 \to 0,\ R_3 ** R_1 \to R_1,\ R_2 ** R_3 \to R_2,\ R_3 ** R_2 \to 0\}$

$$\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

$\left\{x_3 ** (1 - x_3)^{-1} ** x_1 + x_1,\ x_2 ** (1 - x_3)^{-1},\ x_3 ** (1 - x_3)^{-1}\right\}$

$\left\{-y_3 ** (1 + y_3)^{-1} ** y_1 + y_1,\ y_2 ** (1 + y_3)^{-1},\ y_3 ** (1 + y_3)^{-1}\right\}$

$\{y_1,\ y_2,\ y_3\}$

$\{x_1,\ x_2,\ x_3\}$

G=3, A6

```
RuAlg[3, 6]
XiAll[3, 6]
pAll[3, 6]
qAll[3, 6]
pofq[3, 6]
qofp[3, 6]
```

$\{R_1 ** R_1 \to 0,\ R_2 ** R_2 \to 0,\ R_3 ** R_3 \to R_3,\ R_1 ** R_2 \to 0,$
$R_2 ** R_1 \to 0,\ R_1 ** R_3 \to 0,\ R_3 ** R_1 \to R_2,\ R_2 ** R_3 \to 0,\ R_3 ** R_2 \to R_2\}$

$$\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

$\{x_1,\ x_3 ** (1 - x_3)^{-1} ** (x_1 + x_2) + x_2,\ x_3 ** (1 - x_3)^{-1}\}$

$\{y_1,\ y_3 ** (1 + y_3)^{-1} ** (-y_1 - y_2) + y_2,\ y_3 ** (1 + y_3)^{-1}\}$

$\{y_1,\ y_2,\ y_3\}$

$\{x_1,\ x_2,\ x_3\}$

G=3, A7

```
RuAlg[3, 7]
XiAll[3, 7]
pAll[3, 7]
qAll[3, 7]
pofq[3, 7]
qofp[3, 7]
```

$\{R_1 ** R_1 \to 0,\ R_2 ** R_2 \to R_2,\ R_3 ** R_3 \to R_3,\ R_1 ** R_2 \to R_1,$
$R_2 ** R_1 \to 0,\ R_1 ** R_3 \to 0,\ R_3 ** R_1 \to R_1,\ R_2 ** R_3 \to 0,\ R_3 ** R_2 \to 0\}$

$$\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

$\{x_1 ** (1 - x_2)^{-1} + x_3 ** (1 - x_3)^{-1} ** x_1 ** (1 - x_2)^{-1},\ x_2 ** (1 - x_2)^{-1},\ x_3 ** (1 - x_3)^{-1}\}$

$\{y_1 ** (1 + y_2)^{-1} - y_3 ** (1 + y_3)^{-1} ** y_1 ** (1 + y_2)^{-1},\ y_2 ** (1 + y_2)^{-1},\ y_3 ** (1 + y_3)^{-1}\}$

$\{y_1,\ y_2,\ y_3\}$

$\{x_1,\ x_2,\ x_3\}$

G=3, A8

```
RuAlg[3, 8]
XiAll[3, 8]
pAll[3, 8]
qAll[3, 8]
pofq[3, 8]
qofp[3, 8]
```

$\{R_1 ** R_1 \to 0,\ R_2 ** R_2 \to 0,\ R_3 ** R_3 \to R_3,\ R_1 ** R_2 \to 0,$
$R_2 ** R_1 \to 0,\ R_1 ** R_3 \to R_1,\ R_3 ** R_1 \to R_1,\ R_2 ** R_3 \to R_2,\ R_3 ** R_2 \to 0\}$

$$\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

$\{x_1 ** (1 - x_3)^{-1} + x_3 ** (1 - x_3)^{-1} ** x_1 ** (1 - x_3)^{-1},\ x_2 ** (1 - x_3)^{-1},\ x_3 ** (1 - x_3)^{-1}\}$

$\{y_1 ** (1 + y_3)^{-1} - y_3 ** (1 + y_3)^{-1} ** y_1 ** (1 + y_3)^{-1},\ y_2 ** (1 + y_3)^{-1},\ y_3 ** (1 + y_3)^{-1}\}$

$\{y_1,\ y_2,\ y_3\}$

$\{x_1,\ x_2,\ x_3\}$

G=3, A9

```
RuAlg[3, 9]
XiAll[3, 9]
pAll[3, 9]
qAll[3, 9]
pofq[3, 9]
qofp[3, 9]
```

$\{R_1 ** R_1 \to 0,\ R_2 ** R_2 \to 0,\ R_3 ** R_3 \to R_3,\ R_1 ** R_2 \to 0,$
$R_2 ** R_1 \to 0,\ R_1 ** R_3 \to 0,\ R_3 ** R_1 \to R_1,\ R_2 ** R_3 \to R_2,\ R_3 ** R_2 \to R_2\}$

$$\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

$\{x_3 ** (1 - x_3)^{-1} ** x_1 + x_1,\ x_2 ** (1 - x_3)^{-1} + x_3 ** (1 - x_3)^{-1} ** x_2 ** (1 - x_3)^{-1},\ x_3 ** (1 - x_3)^{-1}\}$

$\{-y_3 ** (1 + y_3)^{-1} ** y_1 + y_1,\ y_2 ** (1 + y_3)^{-1} - y_3 ** (1 + y_3)^{-1} ** y_2 ** (1 + y_3)^{-1},\ y_3 ** (1 + y_3)^{-1}\}$

$\{y_1,\ y_2,\ y_3\}$

$\{x_1,\ x_2,\ x_3\}$

G=3, A10

```
RuAlg[3, 10]
XiAll[3, 10]
pAll[3, 10]
qAll[3, 10]
pofq[3, 10]
qofp[3, 10]
```

$\{R_1 ** R_1 \to 0,\ R_2 ** R_2 \to 0,\ R_3 ** R_3 \to R_3,\ R_1 ** R_2 \to 0,\ R_2 ** R_1 \to 0,\ R_1 ** R_3 \to R_1,\ R_3 ** R_1 \to R_1,\ R_2 ** R_3 \to R_2,\ R_3 ** R_2 \to R_2\}$

$$\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

$\{x_1 ** (1 - x_3)^{-1} + x_3 ** (1 - x_3)^{-1} ** x_1 ** (1 - x_3)^{-1},\ x_2 ** (1 - x_3)^{-1} + x_3 ** (1 - x_3)^{-1} ** x_2 ** (1 - x_3)^{-1},\ x_3 ** (1 - x_3)^{-1}\}$

$\{y_1 ** (1 + y_3)^{-1} - y_3 ** (1 + y_3)^{-1} ** y_1 ** (1 + y_3)^{-1},\ y_2 ** (1 + y_3)^{-1} - y_3 ** (1 + y_3)^{-1} ** y_2 ** (1 + y_3)^{-1},\ y_3 ** (1 + y_3)^{-1}\}$

$\{y_1, y_2, y_3\}$

$\{x_1, x_2, x_3\}$

G=3, A11

```
RuAlg[3, 11]
XiAll[3, 11]
pAll[3, 11]
qAll[3, 11]
pofq[3, 11]
qofp[3, 11]
```

$\{R_1 ** R_1 \to 0,\ R_2 ** R_2 \to 0,\ R_3 ** R_3 \to R_3,\ R_1 ** R_2 \to 0,\ R_2 ** R_1 \to 0,\ R_1 ** R_3 \to R_2,\ R_3 ** R_1 \to R_2,\ R_2 ** R_3 \to R_2,\ R_3 ** R_2 \to R_2\}$

$$\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

$\{x_1,\ x_2 ** (1 - x_3)^{-1} + x_1 ** x_3 ** (1 - x_3)^{-1} - x_3 ** (1 - x_3)^{-1} ** (-x_1 - x_2) ** (1 - x_3)^{-1},\ x_3 ** (1 - x_3)^{-1}\}$

$\{y_1,\ y_2 ** (1 + y_3)^{-1} - y_1 ** y_3 ** (1 + y_3)^{-1} - y_3 ** (1 + y_3)^{-1} ** (y_1 + y_2) ** (1 + y_3)^{-1},\ y_3 ** (1 + y_3)^{-1}\}$

$\{y_1, y_2, y_3\}$

$\{x_1, x_2, x_3\}$

G=3, A12

```
RuAlg[3, 12]
XiAll[3, 12]
pAll[3, 12]
qAll[3, 12]
pofq[3, 12]
qofp[3, 12]
```

$\{R_1 ** R_1 \to R_2,\ R_2 ** R_2 \to 0,\ R_3 ** R_3 \to R_3,\ R_1 ** R_2 \to 0,$
$R_2 ** R_1 \to 0,\ R_1 ** R_3 \to R_1,\ R_3 ** R_1 \to R_1,\ R_2 ** R_3 \to R_2,\ R_3 ** R_2 \to R_2\}$

$$\left\{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

$\{x_1 ** (1 - x_3)^{-1} + x_3 ** (1 - x_3)^{-1} ** x_1 ** (1 - x_3)^{-1},$
$-x_1 ** (1 - x_3)^{-1} ** (-(1 - x_3) ** (x_1)^{-1} ** x_2 - x_1) ** (1 - x_3)^{-1} -$
$x_3 ** (1 - x_3)^{-1} ** x_1 ** (1 - x_3)^{-1} ** (-(1 - x_3) ** (x_1)^{-1} ** x_2 - x_1) ** (1 - x_3)^{-1},\ x_3 ** (1 - x_3)^{-1}\}$

$\{y_1 ** (1 + y_3)^{-1} - y_3 ** (1 + y_3)^{-1} ** y_1 ** (1 + y_3)^{-1},$
$-y_1 ** (1 + y_3)^{-1} ** (-(1 + y_3) ** (y_1)^{-1} ** y_2 + y_1) ** (1 + y_3)^{-1} +$
$y_3 ** (1 + y_3)^{-1} ** y_1 ** (1 + y_3)^{-1} ** (-(1 + y_3) ** (y_1)^{-1} ** y_2 + y_1) ** (1 + y_3)^{-1},\ y_3 ** (1 + y_3)^{-1}\}$

$\{y_1,\ y_2,\ y_3\}$

$\{x_1,\ x_2,\ x_3\}$

G=4, ten examples (There are 46 of these according to referenced paper)

```
alpha ≠ 1;
RuAlg[4, 1] = {R[1] ** R[1] → 0 , R[2] ** R[2] → 0, R[3] ** R[3] → 0, R[4] ** R[4] → 0,
   R[1] ** R[2] → R[3], R[2] ** R[1] → R[4], R[1] ** R[3] → 0, R[3] ** R[1] → 0,
   R[2] ** R[3] → 0, R[3] ** R[2] → 0, R[1] ** R[4] → 0, R[4] ** R[1] → 0,
   R[2] ** R[4] → 0, R[4] ** R[2] → 0, R[3] ** R[4] → 0, R[4] ** R[3] → 0};
RuAlg[4, 2] = {R[1] ** R[1] → 0 , R[2] ** R[2] → 0, R[3] ** R[3] → 0, R[4] ** R[4] → 0,
   R[1] ** R[2] → R[4], R[2] ** R[1] → 0, R[1] ** R[3] → 0, R[3] ** R[1] → R[4],
   R[2] ** R[3] → 0, R[3] ** R[2] → 0, R[1] ** R[4] → 0, R[4] ** R[1] → 0,
   R[2] ** R[4] → 0, R[4] ** R[2] → 0, R[3] ** R[4] → 0, R[4] ** R[3] → 0};
RuAlg[4, 3] = {R[1] ** R[1] → 0 , R[2] ** R[2] → -R[3], R[3] ** R[3] → 0,
   R[4] ** R[4] → 0, R[1] ** R[2] → R[3], R[2] ** R[1] → R[4], R[1] ** R[3] → 0,
   R[3] ** R[1] → 0, R[2] ** R[3] → 0, R[3] ** R[2] → 0, R[1] ** R[4] → 0, R[4] ** R[1] → 0,
   R[2] ** R[4] → 0, R[4] ** R[2] → 0, R[3] ** R[4] → 0, R[4] ** R[3] → 0};
RuAlg[4, 4] = {R[1] ** R[1] → 0 , R[2] ** R[2] → R[4], R[3] ** R[3] → 0, R[4] ** R[4] → 0,
   R[1] ** R[2] → R[3], R[2] ** R[1] → -R[3], R[1] ** R[3] → 0, R[3] ** R[1] → 0,
   R[2] ** R[3] → 0, R[3] ** R[2] → 0, R[1] ** R[4] → 0, R[4] ** R[1] → 0,
   R[2] ** R[4] → 0, R[4] ** R[2] → 0, R[3] ** R[4] → 0, R[4] ** R[3] → 0};
RuAlg[4, 5] = {R[1] ** R[1] → 0 , R[2] ** R[2] → 0, R[3] ** R[3] → R[4], R[4] ** R[4] → 0,
   R[1] ** R[2] → R[4], R[2] ** R[1] → -R[4], R[1] ** R[3] → 0, R[3] ** R[1] → 0,
   R[2] ** R[3] → 0, R[3] ** R[2] → 0, R[1] ** R[4] → 0, R[4] ** R[1] → 0,
   R[2] ** R[4] → 0, R[4] ** R[2] → 0, R[3] ** R[4] → 0, R[4] ** R[3] → 0};
RuAlg[4, 6] = {R[1] ** R[1] → 0 , R[2] ** R[2] → R[3], R[3] ** R[3] → 0, R[4] ** R[4] → 0,
   R[1] ** R[2] → R[4], R[2] ** R[1] → (1 + alpha) / (1 - alpha) * R[4], R[1] ** R[3] → 0,
   R[3] ** R[1] → 0, R[2] ** R[3] → 0, R[3] ** R[2] → 0, R[1] ** R[4] → 0, R[4] ** R[1] → 0,
   R[2] ** R[4] → 0, R[4] ** R[2] → 0, R[3] ** R[4] → 0, R[4] ** R[3] → 0};

(* Algebra with all relations going to
 zero (for easier input of new algebras if desired *)

RuAlg[4, 0] = {R[1] ** R[1] → 0 , R[2] ** R[2] → 0, R[3] ** R[3] → 0, R[4] ** R[4] → 0,
   R[1] ** R[2] → 0, R[2] ** R[1] → 0, R[1] ** R[3] → 0, R[3] ** R[1] → 0,
   R[2] ** R[3] → 0, R[3] ** R[2] → 0, R[1] ** R[4] → 0, R[4] ** R[1] → 0,
   R[2] ** R[4] → 0, R[4] ** R[2] → 0, R[3] ** R[4] → 0, R[4] ** R[3] → 0};
```

G=4, A1

```
RuAlg[4, 1]
XiAll[4, 1]
pAll[4, 1]
qAll[4, 1]
pofq[4, 1]
qofp[4, 1]
```

$\{R_1 ** R_1 \to 0,\ R_2 ** R_2 \to 0,\ R_3 ** R_3 \to 0,\ R_4 ** R_4 \to 0,\ R_1 ** R_2 \to R_3,\ R_2 ** R_1 \to R_4,\ R_1 ** R_3 \to 0,\ R_3 ** R_1 \to 0,\ R_2 ** R_3 \to 0,\ R_3 ** R_2 \to 0,\ R_1 ** R_4 \to 0,\ R_4 ** R_1 \to 0,\ R_2 ** R_4 \to 0,\ R_4 ** R_2 \to 0,\ R_3 ** R_4 \to 0,\ R_4 ** R_3 \to 0\}$

$$\left\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right\}$$

$\{x_1,\ x_2,\ x_1 ** x_2 + x_3,\ x_2 ** x_1 + x_4\}$

$\{y_1,\ y_2,\ -y_1 ** y_2 + y_3,\ -y_2 ** y_1 + y_4\}$

$\{y_1,\ y_2,\ y_3,\ y_4\}$

$\{x_1,\ x_2,\ x_3,\ x_4\}$

G=4, A2

```
RuAlg[4, 2]
XiAll[4, 2]
pAll[4, 2]
qAll[4, 2]
pofq[4, 2]
qofp[4, 2]
```

$\{R_1 ** R_1 \to 0,\ R_2 ** R_2 \to 0,\ R_3 ** R_3 \to 0,\ R_4 ** R_4 \to 0,\ R_1 ** R_2 \to R_4,\ R_2 ** R_1 \to 0,\ R_1 ** R_3 \to 0,\ R_3 ** R_1 \to R_4,\ R_2 ** R_3 \to 0,\ R_3 ** R_2 \to 0,\ R_1 ** R_4 \to 0,\ R_4 ** R_1 \to 0,\ R_2 ** R_4 \to 0,\ R_4 ** R_2 \to 0,\ R_3 ** R_4 \to 0,\ R_4 ** R_3 \to 0\}$

$$\left\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right\}$$

$\{x_1,\ x_2,\ x_3,\ x_1 ** x_2 + x_3 ** x_1 + x_4\}$

$\{y_1,\ y_2,\ y_3,\ -y_1 ** y_2 - y_3 ** y_1 + y_4\}$

$\{y_1,\ y_2,\ y_3,\ y_4\}$

$\{x_1,\ x_2,\ x_3,\ x_4\}$

G=4, A3

```
RuAlg[4, 3]
XiAll[4, 3]
pAll[4, 3]
qAll[4, 3]
pofq[4, 3]
qofp[4, 3]
```

$\{R_1 ** R_1 \to 0,\ R_2 ** R_2 \to -R_3,\ R_3 ** R_3 \to 0,\ R_4 ** R_4 \to 0,\ R_1 ** R_2 \to R_3,$
$R_2 ** R_1 \to R_4,\ R_1 ** R_3 \to 0,\ R_3 ** R_1 \to 0,\ R_2 ** R_3 \to 0,\ R_3 ** R_2 \to 0,$
$R_1 ** R_4 \to 0,\ R_4 ** R_1 \to 0,\ R_2 ** R_4 \to 0,\ R_4 ** R_2 \to 0,\ R_3 ** R_4 \to 0,\ R_4 ** R_3 \to 0\}$

$$\left\{\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&0&-1&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\right\}$$

$\{x_1,\ x_2,\ x_1 ** x_2 - x_2 ** x_2 + x_3,\ x_2 ** x_1 + x_4\}$

$\{y_1,\ y_2,\ -y_1 ** y_2 + y_2 ** y_2 + y_3,\ -y_2 ** y_1 + y_4\}$

$\{y_1,\ y_2,\ y_3,\ y_4\}$

$\{x_1,\ x_2,\ x_3,\ x_4\}$

G=4, A4

```
RuAlg[4, 4]
XiAll[4, 4]
pAll[4, 4]
qAll[4, 4]
pofq[4, 4]
qofp[4, 4]
```

$\{R_1 ** R_1 \to 0,\ R_2 ** R_2 \to R_4,\ R_3 ** R_3 \to 0,\ R_4 ** R_4 \to 0,\ R_1 ** R_2 \to R_3,$
$R_2 ** R_1 \to -R_3,\ R_1 ** R_3 \to 0,\ R_3 ** R_1 \to 0,\ R_2 ** R_3 \to 0,\ R_3 ** R_2 \to 0,$
$R_1 ** R_4 \to 0,\ R_4 ** R_1 \to 0,\ R_2 ** R_4 \to 0,\ R_4 ** R_2 \to 0,\ R_3 ** R_4 \to 0,\ R_4 ** R_3 \to 0\}$

$$\left\{\begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\right\}$$

$\{x_1,\ x_2,\ x_1 ** x_2 - x_2 ** x_1 + x_3,\ x_2 ** x_2 + x_4\}$

$\{y_1,\ y_2,\ -y_1 ** y_2 + y_2 ** y_1 + y_3,\ -y_2 ** y_2 + y_4\}$

$\{y_1,\ y_2,\ y_3,\ y_4\}$

$\{x_1,\ x_2,\ x_3,\ x_4\}$

G=4, A5

```
RuAlg[4, 5]
XiAll[4, 5]
pAll[4, 5]
qAll[4, 5]
pofq[4, 5]
qofp[4, 5]
```

$\{R_1 ** R_1 \to 0,\ R_2 ** R_2 \to 0,\ R_3 ** R_3 \to R_4,\ R_4 ** R_4 \to 0,\ R_1 ** R_2 \to R_4,\ R_2 ** R_1 \to -R_4,\ R_1 ** R_3 \to 0,\ R_3 ** R_1 \to 0,\ R_2 ** R_3 \to 0,\ R_3 ** R_2 \to 0,\ R_1 ** R_4 \to 0,\ R_4 ** R_1 \to 0,\ R_2 ** R_4 \to 0,\ R_4 ** R_2 \to 0,\ R_3 ** R_4 \to 0,\ R_4 ** R_3 \to 0\}$

$$\left\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right\}$$

$\{x_1,\ x_2,\ x_3,\ x_1 ** x_2 - x_2 ** x_1 + x_3 ** x_3 + x_4\}$

$\{y_1,\ y_2,\ y_3,\ -y_1 ** y_2 + y_2 ** y_1 - y_3 ** y_3 + y_4\}$

$\{y_1,\ y_2,\ y_3,\ y_4\}$

$\{x_1,\ x_2,\ x_3,\ x_4\}$

G=4, A6

```
RuAlg[4, 6]
XiAll[4, 6]
pAll[4, 6]
qAll[4, 6]
pofq[4, 6]
qofp[4, 6]
```

$$\left\{R_1 ** R_1 \to 0,\ R_2 ** R_2 \to R_3,\ R_3 ** R_3 \to 0,\ R_4 ** R_4 \to 0,\ R_1 ** R_2 \to R_4,\ R_2 ** R_1 \to \frac{(1+\text{alpha})\ R_4}{1-\text{alpha}},\ R_1 ** R_3 \to 0,\ R_3 ** R_1 \to 0,\ R_2 ** R_3 \to 0,\ R_3 ** R_2 \to 0,\ R_1 ** R_4 \to 0,\ R_4 ** R_1 \to 0,\ R_2 ** R_4 \to 0,\ R_4 ** R_2 \to 0,\ R_3 ** R_4 \to 0,\ R_4 ** R_3 \to 0\right\}$$

$$\left\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1-\text{alpha}}{-1+\text{alpha}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right\}$$

$$\left\{x_1,\ x_2,\ x_2 ** x_2 + x_3,\ x_1 ** x_2 + \frac{(-1-\text{alpha})\ x_2 ** x_1}{-1+\text{alpha}} + x_4\right\}$$

$$\left\{y_1,\ y_2,\ -y_2 ** y_2 + y_3,\ -y_1 ** y_2 - \frac{(-1-\text{alpha})\ y_2 ** y_1}{-1+\text{alpha}} + y_4\right\}$$

$$\{y_1,\ y_2,\ y_3,\ y_4\}$$

$$\{x_1,\ x_2,\ x_3,\ x_4\}$$

G=4, A7

```
RuAlg[4, 7]
XiAll[4, 7]
pAll[4, 7]
qAll[4, 7]
pofq[4, 7]
qofp[4, 7]
```

$\{R_1 ** R_1 \to R_1,\ R_2 ** R_2 \to 0,\ R_3 ** R_3 \to 0,\ R_4 ** R_4 \to 0,\ R_1 ** R_2 \to 0,$
$R_2 ** R_1 \to R_2,\ R_1 ** R_3 \to 0,\ R_3 ** R_1 \to 0,\ R_2 ** R_3 \to 0,\ R_3 ** R_2 \to 0,$
$R_1 ** R_4 \to R_4,\ R_4 ** R_1 \to 0,\ R_2 ** R_4 \to R_3,\ R_4 ** R_2 \to 0,\ R_3 ** R_4 \to 0,\ R_4 ** R_3 \to 0\}$

$$\left\{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\right\}$$

$\left\{x_1 ** (1-x_1)^{-1},\ x_2 ** (1-x_1)^{-1},\ x_2 ** (1-x_1)^{-1} ** x_4 + x_3,\ x_1 ** (1-x_1)^{-1} ** x_4 + x_4\right\}$

$\left\{y_1 ** (1+y_1)^{-1},\ y_2 ** (1+y_1)^{-1},\ -y_2 ** (1+y_1)^{-1} ** y_4 + y_3,\ -y_1 ** (1+y_1)^{-1} ** y_4 + y_4\right\}$

$\{y_1,\ y_2,\ y_3,\ y_4\}$

$\{x_1,\ x_2,\ x_3,\ x_4\}$

G=4, A7

```
RuAlg[4, 7]
XiAll[4, 7]
pAll[4, 7]
qAll[4, 7]
pofq[4, 7]
qofp[4, 7]
```

$\{R_1 ** R_1 \to R_1,\ R_2 ** R_2 \to 0,\ R_3 ** R_3 \to 0,\ R_4 ** R_4 \to 0,\ R_1 ** R_2 \to 0,$
$R_2 ** R_1 \to R_2,\ R_1 ** R_3 \to 0,\ R_3 ** R_1 \to 0,\ R_2 ** R_3 \to 0,\ R_3 ** R_2 \to 0,$
$R_1 ** R_4 \to R_4,\ R_4 ** R_1 \to 0,\ R_2 ** R_4 \to R_3,\ R_4 ** R_2 \to 0,\ R_3 ** R_4 \to 0,\ R_4 ** R_3 \to 0\}$

$$\left\{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\right\}$$

$\left\{x_1 ** (1-x_1)^{-1},\ x_2 ** (1-x_1)^{-1},\ x_2 ** (1-x_1)^{-1} ** x_4 + x_3,\ x_1 ** (1-x_1)^{-1} ** x_4 + x_4\right\}$

$\left\{y_1 ** (1+y_1)^{-1},\ y_2 ** (1+y_1)^{-1},\ -y_2 ** (1+y_1)^{-1} ** y_4 + y_3,\ -y_1 ** (1+y_1)^{-1} ** y_4 + y_4\right\}$

$\{y_1,\ y_2,\ y_3,\ y_4\}$

$\{x_1,\ x_2,\ x_3,\ x_4\}$

G=4, A8

```
RuAlg[4, 8]
XiAll[4, 8]
pAll[4, 8]
qAll[4, 8]
pofq[4, 8]
qofp[4, 8]
```

$\{R_1 ** R_1 \to R_1,\ R_2 ** R_2 \to 0,\ R_3 ** R_3 \to 0,\ R_4 ** R_4 \to 0,\ R_1 ** R_2 \to 0,$
$R_2 ** R_1 \to R_2,\ R_1 ** R_3 \to 0,\ R_3 ** R_1 \to R_3,\ R_2 ** R_3 \to 0,\ R_3 ** R_2 \to 0,$
$R_1 ** R_4 \to 0,\ R_4 ** R_1 \to 0,\ R_2 ** R_4 \to 0,\ R_4 ** R_2 \to 0,\ R_3 ** R_4 \to 0,\ R_4 ** R_3 \to R_2\}$

$$\left\{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\right\}$$

$\left\{x_1 ** \left(1 - x_1\right)^{-1},\ x_2 ** \left(1 - x_1\right)^{-1} + x_4 ** x_3 ** \left(1 - x_1\right)^{-1},\ x_3 ** \left(1 - x_1\right)^{-1},\ x_4\right\}$

$\left\{y_1 ** \left(1 + y_1\right)^{-1},\ y_2 ** \left(1 + y_1\right)^{-1} - y_4 ** y_3 ** \left(1 + y_1\right)^{-1},\ y_3 ** \left(1 + y_1\right)^{-1},\ y_4\right\}$

$\{y_1,\ y_2,\ y_3,\ y_4\}$

$\{x_1,\ x_2,\ x_3,\ x_4\}$

G=4, A29

```
RuAlg[4, 29]
XiAll[4, 29]
pAll[4, 29]
qAll[4, 29]
pofq[4, 29]
qofp[4, 29]
```

$\{R_1 ** R_1 \to R_1,\ R_2 ** R_2 \to R_2,\ R_3 ** R_3 \to 0,\ R_4 ** R_4 \to 0,\ R_1 ** R_2 \to 0,$
$R_2 ** R_1 \to 0,\ R_1 ** R_3 \to R_3,\ R_3 ** R_1 \to R_3,\ R_2 ** R_3 \to 0,\ R_3 ** R_2 \to 0,$
$R_1 ** R_4 \to 0,\ R_4 ** R_1 \to R_4,\ R_2 ** R_4 \to R_4,\ R_4 ** R_2 \to 0,\ R_3 ** R_4 \to 0,\ R_4 ** R_3 \to 0\}$

$$\left\{\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}\right\}$$

$\left\{x_1 ** \left(1 - x_1\right)^{-1},\ x_2 ** \left(1 - x_2\right)^{-1},\ x_3 ** \left(1 - x_1\right)^{-1} + x_1 ** \left(1 - x_1\right)^{-1} ** x_3 ** \left(1 - x_1\right)^{-1},\right.$
$\left.x_4 ** \left(1 - x_1\right)^{-1} + x_2 ** \left(1 - x_2\right)^{-1} ** x_4 ** \left(1 - x_1\right)^{-1}\right\}$

$\left\{y_1 ** \left(1 + y_1\right)^{-1},\ y_2 ** \left(1 + y_2\right)^{-1},\ y_3 ** \left(1 + y_1\right)^{-1} - y_1 ** \left(1 + y_1\right)^{-1} ** y_3 ** \left(1 + y_1\right)^{-1},\right.$
$\left.y_4 ** \left(1 + y_1\right)^{-1} - y_2 ** \left(1 + y_2\right)^{-1} ** y_4 ** \left(1 + y_1\right)^{-1}\right\}$

$\{y_1,\ y_2,\ y_3,\ y_4\}$

$\{x_1,\ x_2,\ x_3,\ x_4\}$

G=4, A40

```
RuAlg[4, 29]
XiAll[4, 29]
pAll[4, 29]
qAll[4, 29]
pofq[4, 29]
qofp[4, 29]
```

$\{R_1 ** R_1 \to R_1,\ R_2 ** R_2 \to R_2,\ R_3 ** R_3 \to 0,\ R_4 ** R_4 \to 0,\ R_1 ** R_2 \to 0,$
$R_2 ** R_1 \to 0,\ R_1 ** R_3 \to R_3,\ R_3 ** R_1 \to R_3,\ R_2 ** R_3 \to 0,\ R_3 ** R_2 \to 0,$
$R_1 ** R_4 \to 0,\ R_4 ** R_1 \to R_4,\ R_2 ** R_4 \to R_4,\ R_4 ** R_2 \to 0,\ R_3 ** R_4 \to 0,\ R_4 ** R_3 \to 0\}$

$$\left\{\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}\right\}$$

$\{x_1 ** (1-x_1)^{-1},\ x_2 ** (1-x_2)^{-1},\ x_3 ** (1-x_1)^{-1} + x_1 ** (1-x_1)^{-1} ** x_3 ** (1-x_1)^{-1},$
$x_4 ** (1-x_1)^{-1} + x_2 ** (1-x_2)^{-1} ** x_4 ** (1-x_1)^{-1}\}$

$\{y_1 ** (1+y_1)^{-1},\ y_2 ** (1+y_2)^{-1},\ y_3 ** (1+y_1)^{-1} - y_1 ** (1+y_1)^{-1} ** y_3 ** (1+y_1)^{-1},$
$y_4 ** (1+y_1)^{-1} - y_2 ** (1+y_2)^{-1} ** y_4 ** (1+y_1)^{-1}\}$

$\{y_1, y_2, y_3, y_4\}$

$\{x_1, x_2, x_3, x_4\}$

Null Space and Corange Calculation

Which finite dimensional algebras have a faithful representation wwith a common nullspace?

Here we implement a test which insures an albegra does not. It is based on the equation

$\sum_{i=1}^{g} R_i ** Z_i = 1.$

in a basis R_i for the algebra and uknowns Z_i.Similarly we test for a common corange using the equation

$\sum_{i=1}^{g} Z_i ** R_i = 1.$

If a faithful representation for A exists with a null space, then clearly no solution Z_i exists. The converse also holds.

We search for a solution by forming a Grobner basis for the ideal generated by the defining relations for the algebra and the equation $\sum_{i=1}^{g} R_i ** Z_i = 1.$ The ExistsNullQ and ExistsTransNullQ commands look for contradictions of the form 1=0 or R_i=0 for some i. There may be other contradictions which are not checked for.

Null prep creates the list of equations needed to run the gb computation. The output of ExistsNullQ and ExistsTransNullQ is the generated gb.

```
SNC[R, Z, X, Y, Z]
SetMonomialOrder[{Z[1], tp[Z[1]]}, {Z[2], tp[Z[2]]}, {Z[3], tp[Z[3]]}, {Z[4], tp[Z[4]]},
  {R[1], tp[R[1]]}, {R[2], tp[R[2]]}, {R[3], tp[R[3]]}, {R[4], tp[R[4]]}];

NullPrep[k_, m_] := Union[NCRuleToPoly[RuAlg[k, m]], {Sum[Z[i] ** R[i], {i, k}] - 1}]
TransNullPrep[k_, m_] := Union[NCRuleToPoly[RuAlg[k, m]], {Sum[R[i] ** Z[i], {i, k}] - 1}]
symDiff[x_, y_] := Complement[Union[x, y], Intersection[x, y]]
ExistsNullQ[k_, m_, n_] := Block[{NullGB, PossibleContradictions, ContradictionList},
  NullGB = NCMakeGB[NullPrep[k, m], n];
  PossibleContradictions = Union[Table[R[i] → 0, {i, k}], {1 → 0}];
  ContradictionList = Intersection[PossibleContradictions, NullGB];
  If[Length[ContradictionList] == 0,
   Print["There no simple contradictions were found. More detailed analysis needed. "],
   Print[
    "Faithful representations of the algebra have a common Null Space. The following
      is a list of contradictions in the GB"];
   Print[ContradictionList]];
  Return[NullGB]
 ]

ExistsTransNullQ[k_, m_, n_] := Block[{NullGB, PossibleContradictions, ContradictionList},
  NullGB = NCMakeGB[TransNullPrep[k, m], n];
  PossibleContradictions = Union[Table[R[i] → 0, {i, k}], {1 → 0}];
  ContradictionList = Intersection[PossibleContradictions, NullGB];
  If[Length[ContradictionList] == 0,
   Print["There no simple contradictions were found. More detailed analysis needed. "],
   Print["Faithful representations of the algebra have a common
      corange. The following is a list of contradictions in the GB"];
   Print[ContradictionList]];
  Return[NullGB]
 ]
```

G=2 Null Space And Corange Calculations

G=2, A1

Null Space

```
ExistsNullQ[2, 1, 1]
```

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set reduced to '3' out of '5' polynomials
* Computing initial set of obstructions
* Found Groebner basis with 1 polynomials
* * * * * * * * * * * * * * * *
```

Faithful representations of the algebra have a
common Null Space. The following is a list of contradictions in the GB

$\{1 \to 0\}$

$\{1 \to 0\}$

Corange

ExistsTransNullQ[2, 1, 2]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set reduced to '3' out of '5' polynomials
* Computing initial set of obstructions
* Found Groebner basis with 1 polynomials
* * * * * * * * * * * * * * * *
```

Faithful representations of the algebra have a
common corange. The following is a list of contradictions in the GB

$\{1 \to 0\}$

$\{1 \to 0\}$

G=2, A2

Null Space

ExistsNullQ[2, 2, 2]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

$* \text{ Monomial order: } Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 7 polys in the basis, 4 obstructions
* Found Groebner basis with 7 polynomials
* * * * * * * * * * * * * * * *
There no simple contradictions were found. More detailed analysis needed.
```

$\{R_2 ** R_1 \to 0,\ R_2 ** R_2 \to 0,\ Z_2 ** R_2 \to 1 - Z_1 ** R_1,$
$R_1 ** R_1 \to R_1,\ R_1 ** R_2 \to R_2,\ Z_1 ** R_1 \to R_1,\ Z_1 ** R_2 \to R_2\}$

Corange

ExistsTransNullQ[2, 2, 2]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

$* \text{ Monomial order: } Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 3 polys in the basis, 1 obstructions
* Found Groebner basis with 3 polynomials
* * * * * * * * * * * * * * * *
Faithful representations of the algebra have a
  common corange. The following is a list of contradictions in the GB
```

$\{R_2 \to 0\}$

$\{R_2 \to 0,\ R_1 \to 1,\ Z_1 \to 1\}$

G=2, A3

Null Space

ExistsNullQ[2, 3, 2]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

$* $ Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 3 polys in the basis, 1 obstructions
* Found Groebner basis with 3 polynomials
* * * * * * * * * * * * * * * *
Faithful representations of the algebra have a
  common Null Space. The following is a list of contradictions in the GB
```

$\{R_2 \to 0\}$

$\{R_2 \to 0, R_1 \to 1, Z_1 \to 1\}$

Corange

ExistsTransNullQ[2, 3, 2]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

$* $ Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 7 polys in the basis, 4 obstructions
* Found Groebner basis with 7 polynomials
* * * * * * * * * * * * * * * *
There no simple contradictions were found. More detailed analysis needed.
```

$\{R_1 ** R_2 \to 0, R_2 ** R_2 \to 0, R_2 ** Z_2 \to 1 - R_1 ** Z_1,$
$R_1 ** R_1 \to R_1, R_2 ** R_1 \to R_2, R_1 ** Z_1 \to R_1, R_2 ** Z_1 \to R_2\}$

G=2, A4

Null Space

ExistsNullQ[2, 4, 2]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 4 polys in the basis, 1 obstructions
* Found Groebner basis with 4 polynomials
* * * * * * * * * * * * * * * *
There no simple contradictions were found. More detailed analysis needed.
```

$\{R_2 ** R_2 \to 0,\ Z_2 ** R_2 \to 1 - Z_1 ** R_1,\ Z_1 ** R_2 \to R_2,\ R_1 \to 1\}$

Corange

ExistsTransNullQ[2, 4, 2]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 4 polys in the basis, 1 obstructions
* Found Groebner basis with 4 polynomials
* * * * * * * * * * * * * * * *
There no simple contradictions were found. More detailed analysis needed.
```

$\{R_2 ** R_2 \to 0,\ R_2 ** Z_2 \to 1 - R_1 ** Z_1,\ R_2 ** Z_1 \to R_2,\ R_1 \to 1\}$

G=3 Null Space And Corange Calculations

G=3, A1

Null Space

ExistsNullQ[3, 1, 2]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

$\star$ Monomial order: $Z_1 < Z_1{}^T \ll Z_2 < Z_2{}^T \ll Z_3 < Z_3{}^T \ll Z_4 < Z_4{}^T \ll R_1 < R_1{}^T \ll R_2 < R_2{}^T \ll R_3 < R_3{}^T \ll R_4 < R_4{}^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 4 polys in the basis, 1 obstructions
* Found Groebner basis with 1 polynomials
* * * * * * * * * * * * * * * *
Faithful representations of the algebra have a
  common Null Space. The following is a list of contradictions in the GB
```

$\{1 \to 0\}$

$\{1 \to 0\}$

Corange

ExistsTransNullQ[3, 1, 2]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

$\star$ Monomial order: $Z_1 < Z_1{}^T \ll Z_2 < Z_2{}^T \ll Z_3 < Z_3{}^T \ll Z_4 < Z_4{}^T \ll R_1 < R_1{}^T \ll R_2 < R_2{}^T \ll R_3 < R_3{}^T \ll R_4 < R_4{}^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 4 polys in the basis, 1 obstructions
* Found Groebner basis with 1 polynomials
* * * * * * * * * * * * * * * *
Faithful representations of the algebra have a
  common corange. The following is a list of contradictions in the GB
```

$\{1 \to 0\}$

$\{1 \to 0\}$

G=3, A2

Null Space

ExistsNullQ[3, 2, 2]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Symbolic coefficients detected

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

* Reduce and normalize initial set

> Initial set could not be reduced

* Computing initial set of obstructions

> MAJOR Iteration 1, 4 polys in the basis, 1 obstructions

* Found Groebner basis with 1 polynomials

```
* * * * * * * * * * * * * * * *
```

Faithful representations of the algebra have a
  common Null Space. The following is a list of contradictions in the GB

{1 → 0}

{1 → 0}

Corange

ExistsTransNullQ[**3**, **2**, **2**]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Symbolic coefficients detected

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

* Reduce and normalize initial set

> Initial set could not be reduced

* Computing initial set of obstructions

> MAJOR Iteration 1, 4 polys in the basis, 1 obstructions

* Found Groebner basis with 1 polynomials

```
* * * * * * * * * * * * * * * *
```

Faithful representations of the algebra have a
  common corange. The following is a list of contradictions in the GB

{1 → 0}

{1 → 0}

G=3, A3

Null Space

ExistsNullQ[3, 3, 2]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

$*$ Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set reduced to '4' out of '10' polynomials
* Computing initial set of obstructions
* Found Groebner basis with 1 polynomials
* * * * * * * * * * * * * * * *
Faithful representations of the algebra have a
  common Null Space. The following is a list of contradictions in the GB
```

$\{1 \to 0\}$

$\{1 \to 0\}$

Corange

ExistsTransNullQ[3, 3, 2]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

$*$ Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set reduced to '4' out of '10' polynomials
* Computing initial set of obstructions
* Found Groebner basis with 1 polynomials
* * * * * * * * * * * * * * * *
Faithful representations of the algebra have a
  common corange. The following is a list of contradictions in the GB
```

$\{1 \to 0\}$

$\{1 \to 0\}$

G=3, A4

Null Space

ExistsNullQ[**3**, **4**, **2**]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
* Found Groebner basis with 4 polynomials
* * * * * * * * * * * * * * * *
```

Faithful representations of the algebra have a
common Null Space. The following is a list of contradictions in the GB

$\{R_1 \to 0, R_2 \to 0\}$

$\{R_1 \to 0, R_2 \to 0, R_3 \to 1 - Z_1 ** R_1 - Z_2 ** R_2, Z_3 \to 1\}$

Corange

ExistsTransNullQ[**3**, **4**, **3**]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 7 polys in the basis, 6 obstructions
> MAJOR Iteration 2, 6 polys in the basis, 6 obstructions
* Found Groebner basis with 6 polynomials
* * * * * * * * * * * * * * * *
```

There no simple contradictions were found. More detailed analysis needed.

$\{R_1 ** R_1 \to 0, R_2 \to -R_1 ** R_1 ** Z_1 + R_1, R_1 ** Z_3 \to R_1, R_3 \to 1 - R_1 ** Z_1 - R_2 ** Z_2,$
$R_1 ** Z_2 ** R_1 \to -R_1 ** Z_1 ** R_1 + R_1, R_1 ** Z_2 ** Z_3 \to -1 + R_1 ** Z_1 + R_1 ** Z_2 - R_1 ** Z_1 ** Z_3 + Z_3\}$

G=3, A5

Null Space

ExistsNullQ[3, 5, 2]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 6 polys in the basis, 7 obstructions
* Found Groebner basis with 6 polynomials
* * * * * * * * * * * * * * * *
Faithful representations of the algebra have a
  common Null Space. The following is a list of contradictions in the GB
```

$\{R_2 \to 0\}$

$\{R_1 ** R_1 \to 0, R_2 \to 0, Z_3 ** R_1 \to R_1, R_3 \to 1 - Z_1 ** R_1 - Z_2 ** R_2,$
$R_1 ** Z_1 ** R_1 \to R_1, Z_3 ** Z_1 ** R_1 \to -1 + Z_1 ** R_1 + Z_3\}$

Corange

ExistsTransNullQ[3, 5, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 6 polys in the basis, 7 obstructions
* Found Groebner basis with 6 polynomials
* * * * * * * * * * * * * * * *
```

Faithful representations of the algebra have a
common corange. The following is a list of contradictions in the GB

$\{R_1 \to 0\}$

$\{R_2 ** R_2 \to 0, R_1 \to 0, R_2 ** Z_3 \to R_2, R_3 \to 1 - R_1 ** Z_1 - R_2 ** Z_2,$
$R_2 ** Z_2 ** R_2 \to -R_1 ** Z_1 ** R_2 + R_2, R_2 ** Z_2 ** Z_3 \to -1 + R_1 ** Z_1 + R_2 ** Z_2 - R_1 ** Z_1 ** Z_3 + Z_3\}$

G=3, A6

Null Space

ExistsNullQ[3, 6, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 7 polys in the basis, 6 obstructions
> MAJOR Iteration 2, 6 polys in the basis, 6 obstructions
* Found Groebner basis with 6 polynomials
* * * * * * * * * * * * * * * *
```

There no simple contradictions were found. More detailed analysis needed.

$\{R_1 ** R_1 \to 0, R_2 \to -Z_1 ** R_1 ** R_1 + R_1, Z_3 ** R_1 \to R_1, R_3 \to 1 - Z_1 ** R_1 - Z_2 ** R_2,$
$R_1 ** Z_2 ** R_1 \to -R_1 ** Z_1 ** R_1 + R_1, Z_3 ** Z_2 ** R_1 \to -1 + Z_1 ** R_1 + Z_2 ** R_1 - Z_3 ** Z_1 ** R_1 + Z_3\}$

Corange

ExistsTransNullQ[3, 6, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

$* \text{ Monomial order: } Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
* Found Groebner basis with 4 polynomials
* * * * * * * * * * * * * * * *
```

Faithful representations of the algebra have a
common corange. The following is a list of contradictions in the GB

$\{R_1 \to 0, R_2 \to 0\}$

$\{R_1 \to 0, R_2 \to 0, R_3 \to 1 - R_1 ** Z_1 - R_2 ** Z_2, Z_3 \to 1\}$

G=3, A7

Null Space

ExistsNullQ[3, 7, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

$* \text{ Monomial order: } Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 8 polys in the basis, 6 obstructions
* Found Groebner basis with 8 polynomials
* * * * * * * * * * * * * * * *
```

There no simple contradictions were found. More detailed analysis needed.

$\{R_1 ** R_1 \to 0, R_2 ** R_1 \to 0, R_1 ** R_2 \to R_1, R_2 ** R_2 \to R_2, Z_2 ** R_2 \to -Z_1 ** R_1 + R_2, Z_3 ** R_1 \to R_1, R_3 \to 1 - Z_1 ** R_1 - Z_2 ** R_2, Z_3 ** R_2 \to -1 + Z_1 ** R_1 + Z_2 ** R_2 + Z_3\}$

Corange

ExistsTransNullQ[3, 7, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 8 polys in the basis, 6 obstructions
* Found Groebner basis with 8 polynomials
* * * * * * * * * * * * * * * *
There no simple contradictions were found. More detailed analysis needed.
```

$\{R_1 ** R_1 \to 0, R_2 ** R_1 \to 0, R_1 ** R_2 \to R_1, R_2 ** R_2 \to R_2, R_1 ** Z_2 \to R_1, R_2 ** Z_2 \to R_2, R_3 \to 1 - R_2 ** Z_2, R_2 ** Z_3 \to -1 + R_1 ** Z_1 + R_2 ** Z_2 + Z_3\}$

G=3, A8

Null Space

ExistsNullQ[3, 8, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 5 polys in the basis, 2 obstructions
* Found Groebner basis with 5 polynomials
* * * * * * * * * * * * * * * *
Faithful representations of the algebra have a
  common Null Space. The following is a list of contradictions in the GB
```

$\{R_2 \to 0\}$

$\{R_1 ** R_1 \to 0, R_2 \to 0, Z_3 ** R_1 \to R_1, R_3 \to 1 - Z_2 ** R_2, Z_1 ** R_1 \to 1 - Z_3\}$

Corange

ExistsTransNullQ[3, 8, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

$* $ Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 10 polys in the basis, 19 obstructions
* Found Groebner basis with 10 polynomials
* * * * * * * * * * * * * * * *
There no simple contradictions were found. More detailed analysis needed.
```

$\{R_1 ** R_1 \to 0, R_1 ** R_2 \to 0, R_2 ** R_1 \to 0, R_2 ** R_2 \to 0, R_1 ** Z_3 \to R_1, R_2 ** Z_3 \to R_2, R_3 \to 1 - R_2 ** Z_2, R_2 ** Z_2 ** R_2 \to R_2, R_2 ** Z_2 ** Z_3 \to -1 + R_1 ** Z_1 + R_2 ** Z_2 + Z_3, R_2 ** Z_2 ** R_1 \to 0\}$

G=3, A9

Null Space

ExistsNullQ[3, 9, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

$* $ Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 10 polys in the basis, 19 obstructions
* Found Groebner basis with 10 polynomials
* * * * * * * * * * * * * * * *
There no simple contradictions were found. More detailed analysis needed.
```

$\{R_1 ** R_1 \to 0, R_1 ** R_2 \to 0, R_2 ** R_1 \to 0, R_2 ** R_2 \to 0, Z_3 ** R_1 \to R_1, Z_3 ** R_2 \to R_2, R_3 \to 1 - Z_1 ** R_1, R_1 ** Z_1 ** R_1 \to R_1, Z_2 ** R_2 \to 1 - Z_1 ** R_1 + Z_3 ** Z_1 ** R_1 - Z_3, R_2 ** Z_1 ** R_1 \to 0\}$

Corange

ExistsTransNullQ[3, 9, 3]

* * * * * * * * * * * * * * * *

* * * NCPolyGroebner * * *

* * * * * * * * * * * * * * * *

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

* Reduce and normalize initial set

> Initial set could not be reduced

* Computing initial set of obstructions

> MAJOR Iteration 1, 5 polys in the basis, 2 obstructions

* Found Groebner basis with 5 polynomials

* * * * * * * * * * * * * * * *

Faithful representations of the algebra have a
common corange. The following is a list of contradictions in the GB

$\{R_1 \to 0\}$

$\{R_2 ** R_2 \to 0,\ R_1 \to 0,\ R_2 ** Z_3 \to R_2,\ R_3 \to 1 - R_1 ** Z_1,\ R_2 ** Z_2 \to 1 - R_1 ** Z_1 + R_1 ** Z_1 ** Z_3 - Z_3\}$

G=3, A10

Null Space

ExistsNullQ[3, 10, 3]

* * * * * * * * * * * * * * * *

* * * NCPolyGroebner * * *

* * * * * * * * * * * * * * * *

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

* Reduce and normalize initial set

> Initial set could not be reduced

* Computing initial set of obstructions

> MAJOR Iteration 1, 8 polys in the basis, 6 obstructions

* Found Groebner basis with 8 polynomials

* * * * * * * * * * * * * * * *

There no simple contradictions were found. More detailed analysis needed.

$\{R_1 ** R_1 \to 0,\ R_1 ** R_2 \to 0,\ R_2 ** R_1 \to 0,\ R_2 ** R_2 \to 0,$
$Z_3 ** R_1 \to R_1,\ Z_3 ** R_2 \to R_2,\ R_3 \to 1,\ Z_2 ** R_2 \to 1 - Z_1 ** R_1 - Z_3\}$

Corange

ExistsTransNullQ[3, 10, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 8 polys in the basis, 6 obstructions
* Found Groebner basis with 8 polynomials
* * * * * * * * * * * * * * * *
There no simple contradictions were found. More detailed analysis needed.
```

$\{R_1 ** R_1 \to 0,\ R_1 ** R_2 \to 0,\ R_2 ** R_1 \to 0,\ R_2 ** R_2 \to 0,$
$R_1 ** Z_3 \to R_1,\ R_2 ** Z_3 \to R_2,\ R_3 \to 1,\ R_2 ** Z_2 \to 1 - R_1 ** Z_1 - Z_3\}$

G=3, A11

Null Space

ExistsNullQ[3, 11, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 7 polys in the basis, 6 obstructions
> MAJOR Iteration 2, 5 polys in the basis, 1 obstructions
* Found Groebner basis with 5 polynomials
* * * * * * * * * * * * * * * *
There no simple contradictions were found. More detailed analysis needed.
```

$\{R_1 ** R_1 \to 0,\ R_2 \to -Z_1 ** R_1 ** R_1 + R_1,\ Z_3 ** R_1 \to R_1,$
$R_3 \to 1 - Z_1 ** R_1 + Z_1 ** R_2,\ Z_2 ** R_1 \to 1 - Z_1 ** R_1 - Z_3\}$

Corange

ExistsTransNullQ[3, 11, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 7 polys in the basis, 6 obstructions
> MAJOR Iteration 2, 5 polys in the basis, 1 obstructions
* Found Groebner basis with 5 polynomials
* * * * * * * * * * * * * * * *
```

There no simple contradictions were found. More detailed analysis needed.

$\{R_1 ** R_1 \to 0,\ R_2 \to -R_1 ** R_1 ** Z_1 + R_1,\ R_1 ** Z_3 \to R_1,$
$R_3 \to 1 - R_1 ** Z_1 + R_2 ** Z_1,\ R_1 ** Z_2 \to 1 - R_1 ** Z_1 - Z_3\}$

G=3, A12

Null Space

ExistsNullQ[3, 12, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set reduced to '6' out of '10' polynomials
* Computing initial set of obstructions
> MAJOR Iteration 1, 5 polys in the basis, 4 obstructions
> MAJOR Iteration 2, 6 polys in the basis, 2 obstructions
* Found Groebner basis with 6 polynomials
* * * * * * * * * * * * * * * *
```

There no simple contradictions were found. More detailed analysis needed.

$\{R_1 ** R_1 ** R_1 \to 0,\ R_2 \to R_1 ** R_1,\ Z_1 ** R_1 ** R_1 \to -Z_3 ** R_1 + R_1,$
$R_3 \to 1 - Z_2 ** R_2 + Z_2 ** R_1 ** R_1,\ Z_2 ** R_1 ** R_1 \to 1 - Z_1 ** R_1 - Z_3,\ Z_3 ** R_1 ** R_1 \to R_1 ** R_1\}$

Corange

ExistsTransNullQ[3, 12, 3]

```
* * * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * * *
```

$* $ Monomial order: $Z_1 < Z_1{}^T \ll Z_2 < Z_2{}^T \ll Z_3 < Z_3{}^T \ll Z_4 < Z_4{}^T \ll R_1 < R_1{}^T \ll R_2 < R_2{}^T \ll R_3 < R_3{}^T \ll R_4 < R_4{}^T$

```
* Reduce and normalize initial set
> Initial set reduced to '6' out of '10' polynomials
* Computing initial set of obstructions
> MAJOR Iteration 1, 5 polys in the basis, 4 obstructions
> MAJOR Iteration 2, 6 polys in the basis, 2 obstructions
* Found Groebner basis with 6 polynomials
* * * * * * * * * * * * * * * * *
There no simple contradictions were found. More detailed analysis needed.
```

$\{R_1 ** R_1 ** R_1 \to 0,\ R_2 \to R_1 ** R_1,\ R_1 ** R_1 ** Z_1 \to -R_1 ** Z_3 + R_1,$
$R_3 \to 1 - R_2 ** Z_2 + R_1 ** R_1 ** Z_2,\ R_1 ** R_1 ** Z_2 \to 1 - R_1 ** Z_1 - Z_3,\ R_1 ** R_1 ** Z_3 \to R_1 ** R_1\}$

G=4 Null Space And Corange Calculations

G=4, A1

Null Space

ExistsNullQ[4, 1, 3]

```
* * * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * * *
```

$* $ Monomial order: $Z_1 < Z_1{}^T \ll Z_2 < Z_2{}^T \ll Z_3 < Z_3{}^T \ll Z_4 < Z_4{}^T \ll R_1 < R_1{}^T \ll R_2 < R_2{}^T \ll R_3 < R_3{}^T \ll R_4 < R_4{}^T$

```
* Reduce and normalize initial set
> Initial set reduced to '7' out of '17' polynomials
* Computing initial set of obstructions
> MAJOR Iteration 1, 5 polys in the basis, 1 obstructions
* Found Groebner basis with 1 polynomials
* * * * * * * * * * * * * * * * *
Faithful representations of the algebra have a
  common Null Space. The following is a list of contradictions in the GB
```

$\{1 \to 0\}$

$\{1 \to 0\}$

Corange

ExistsTransNullQ[4, 1, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set reduced to '7' out of '17' polynomials
* Computing initial set of obstructions
* Found Groebner basis with 1 polynomials
* * * * * * * * * * * * * * * *
Faithful representations of the algebra have a
  common corange. The following is a list of contradictions in the GB
```

$\{1 \to 0\}$

$\{1 \to 0\}$

G=4, A2

Null Space

ExistsNullQ[4, 2, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set reduced to '10' out of '17' polynomials
* Computing initial set of obstructions
> MAJOR Iteration 1, 5 polys in the basis, 1 obstructions
* Found Groebner basis with 1 polynomials
* * * * * * * * * * * * * * * *
Faithful representations of the algebra have a
  common Null Space. The following is a list of contradictions in the GB
```

$\{1 \to 0\}$

$\{1 \to 0\}$

Corange

ExistsTransNullQ[4, 2, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1{}^T \ll Z_2 < Z_2{}^T \ll Z_3 < Z_3{}^T \ll Z_4 < Z_4{}^T \ll R_1 < R_1{}^T \ll R_2 < R_2{}^T \ll R_3 < R_3{}^T \ll R_4 < R_4{}^T$

```
* Reduce and normalize initial set
> Initial set reduced to '10' out of '17' polynomials
* Computing initial set of obstructions
* Found Groebner basis with 1 polynomials
* * * * * * * * * * * * * * * *
```

Faithful representations of the algebra have a
common corange. The following is a list of contradictions in the GB

$\{1 \to 0\}$

$\{1 \to 0\}$

G=4, A3

Null Space

ExistsNullQ[4, 3, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1{}^T \ll Z_2 < Z_2{}^T \ll Z_3 < Z_3{}^T \ll Z_4 < Z_4{}^T \ll R_1 < R_1{}^T \ll R_2 < R_2{}^T \ll R_3 < R_3{}^T \ll R_4 < R_4{}^T$

```
* Reduce and normalize initial set
> Initial set reduced to '7' out of '17' polynomials
* Computing initial set of obstructions
> MAJOR Iteration 1, 5 polys in the basis, 2 obstructions
* Found Groebner basis with 1 polynomials
* * * * * * * * * * * * * * * *
```

Faithful representations of the algebra have a
common Null Space. The following is a list of contradictions in the GB

$\{1 \to 0\}$

$\{1 \to 0\}$

Corange

ExistsTransNullQ[4, 3, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner   * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set reduced to '7' out of '17' polynomials
* Computing initial set of obstructions
> MAJOR Iteration 1, 7 polys in the basis, 6 obstructions
* Found Groebner basis with 1 polynomials
* * * * * * * * * * * * * * * *
```

Faithful representations of the algebra have a
  common corange. The following is a list of contradictions in the GB

$\{1 \to 0\}$

$\{1 \to 0\}$

G=4, A4

Null Space

ExistsNullQ[4, 4, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner   * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set reduced to '7' out of '17' polynomials
* Computing initial set of obstructions
> MAJOR Iteration 1, 7 polys in the basis, 6 obstructions
* Found Groebner basis with 1 polynomials
* * * * * * * * * * * * * * * *
```

Faithful representations of the algebra have a
  common Null Space. The following is a list of contradictions in the GB

$\{1 \to 0\}$

$\{1 \to 0\}$

Corange

ExistsTransNullQ[4, 4, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set reduced to '7' out of '17' polynomials
* Computing initial set of obstructions
* Found Groebner basis with 1 polynomials
* * * * * * * * * * * * * * * *
Faithful representations of the algebra have a
  common corange. The following is a list of contradictions in the GB
```

$\{1 \to 0\}$

$\{1 \to 0\}$

G=4, A5

Null Space

ExistsNullQ[4, 5, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set reduced to '10' out of '17' polynomials
* Computing initial set of obstructions
* Found Groebner basis with 1 polynomials
* * * * * * * * * * * * * * * *
Faithful representations of the algebra have a
  common Null Space. The following is a list of contradictions in the GB
```

$\{1 \to 0\}$

$\{1 \to 0\}$

Corange

ExistsTransNullQ[4, 5, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set reduced to '10' out of '17' polynomials
* Computing initial set of obstructions
* Found Groebner basis with 1 polynomials
* * * * * * * * * * * * * * * *
```

Faithful representations of the algebra have a
common corange. The following is a list of contradictions in the GB

$\{1 \to 0\}$

$\{1 \to 0\}$

G=4, A6

Null Space

ExistsNullQ[4, 6, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
* Symbolic coefficients detected
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set reduced to '7' out of '17' polynomials
* Computing initial set of obstructions
> MAJOR Iteration 1, 7 polys in the basis, 6 obstructions
* Found Groebner basis with 1 polynomials
* * * * * * * * * * * * * * * *
```

Faithful representations of the algebra have a
common Null Space. The following is a list of contradictions in the GB

$\{1 \to 0\}$

$\{1 \to 0\}$

Corange

ExistsTransNullQ[4, 6, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
* Symbolic coefficients detected
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set reduced to '7' out of '17' polynomials
* Computing initial set of obstructions
* Found Groebner basis with 1 polynomials
* * * * * * * * * * * * * * * *
Faithful representations of the algebra have a
  common corange. The following is a list of contradictions in the GB
```

$\{1 \to 0\}$

$\{1 \to 0\}$

RuAlg[4, 6]

$$\left\{R_1 ** R_1 \to 0,\ R_2 ** R_2 \to R_3,\ R_3 ** R_3 \to 0,\ R_4 ** R_4 \to 0,\ R_1 ** R_2 \to R_4,\ R_2 ** R_1 \to \frac{(1 + \text{alpha})\ R_4}{1 - \text{alpha}},\ R_1 ** R_3 \to 0,\ R_3 ** R_1 \to 0,\ R_2 ** R_3 \to 0,\ R_3 ** R_2 \to 0,\ R_1 ** R_4 \to 0,\ R_4 ** R_1 \to 0,\ R_2 ** R_4 \to 0,\ R_4 ** R_2 \to 0,\ R_3 ** R_4 \to 0,\ R_4 ** R_3 \to 0\right\}$$

RuAlg[3, 6]

$$\{R_1 ** R_1 \to 0,\ R_2 ** R_2 \to 0,\ R_3 ** R_3 \to R_3,\ R_1 ** R_2 \to 0,\ R_2 ** R_1 \to 0,\ R_1 ** R_3 \to 0,\ R_3 ** R_1 \to R_2,\ R_2 ** R_3 \to 0,\ R_3 ** R_2 \to R_2\}$$

G=4, A7

Null Space

ExistsNullQ[4, 7, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

$* \text{ Monomial order: } Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 9 polys in the basis, 8 obstructions
* Found Groebner basis with 9 polynomials
* * * * * * * * * * * * * * * *
```

Faithful representations of the algebra have a
common Null Space. The following is a list of contradictions in the GB

$\{R_2 \to 0,\ R_3 \to 0\}$

$\{R_4 ** R_1 \to 0,\ R_4 ** R_4 \to 0,\ Z_4 ** R_4 \to 1 - Z_1 ** R_1 - Z_2 ** R_2 - Z_3 ** R_3,$
$R_1 ** R_1 \to R_1,\ R_1 ** R_4 \to R_4,\ R_2 \to 0,\ R_3 \to 0,\ Z_1 ** R_1 \to R_1,\ Z_1 ** R_4 \to R_4\}$

Corange

ExistsTransNullQ[4, 7, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

$* \text{ Monomial order: } Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 7 polys in the basis, 2 obstructions
> MAJOR Iteration 2, 9 polys in the basis, 4 obstructions
* Found Groebner basis with 9 polynomials
* * * * * * * * * * * * * * * *
```

Faithful representations of the algebra have a
common corange. The following is a list of contradictions in the GB

$\{R_3 \to 0,\ R_4 \to 0\}$

$\{R_1 ** R_2 \to 0,\ R_2 ** R_2 \to 0,\ R_1 ** R_1 \to R_1,\ R_2 ** R_1 \to R_2,$
$R_3 \to 0,\ R_4 \to 0,\ R_2 ** Z_2 \to 1 - R_1 ** Z_1,\ R_1 ** Z_1 \to R_1,\ R_2 ** Z_1 \to R_2\}$

G=4, A8

Null Space

ExistsNullQ[4, 8, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
* Found Groebner basis with 5 polynomials
* * * * * * * * * * * * * * * *
Faithful representations of the algebra have a
  common Null Space. The following is a list of contradictions in the GB
```

$\{R_2 \to 0, R_3 \to 0, R_4 \to 0\}$

$\{R_2 \to 0, R_4 \to 0, R_1 \to 1, R_3 \to 0, Z_1 \to 1\}$

Corange

ExistsTransNullQ[4, 8, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 17 polys in the basis, 37 obstructions
* Found Groebner basis with 17 polynomials
* * * * * * * * * * * * * * * *
There no simple contradictions were found. More detailed analysis needed.
```

$\{R_1 ** R_2 \to 0, R_1 ** R_3 \to 0, R_2 ** R_2 \to 0, R_2 ** R_3 \to 0, R_3 ** R_2 \to 0,$
$R_3 ** R_3 \to 0, R_1 ** R_1 \to R_1, R_2 ** R_1 \to R_2, R_3 ** R_1 \to R_3, R_1 ** Z_1 \to R_1,$
$R_2 ** Z_1 \to R_2, R_3 ** Z_1 \to R_3, R_4 \to R_2 ** Z_3, R_2 ** Z_3 ** R_1 \to 0, R_2 ** Z_3 ** R_2 \to 0,$
$R_3 ** Z_3 \to 1 - R_1 ** Z_1 - R_2 ** Z_2 - R_2 ** Z_3 ** Z_4, R_2 ** Z_3 ** R_3 \to R_2\}$

G=4, A29

Null Space

ExistsNullQ[4, 29, 3]

* * * * * * * * * * * * * * * *

* * *   NCPolyGroebner   * * *

* * * * * * * * * * * * * * * *

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

* Reduce and normalize initial set

> Initial set could not be reduced

* Computing initial set of obstructions

> MAJOR Iteration 1, 14 polys in the basis, 11 obstructions

* Found Groebner basis with 14 polynomials

* * * * * * * * * * * * * * * *

There no simple contradictions were found. More detailed analysis needed.

$\{R_1 ** R_4 \to 0,\ R_3 ** R_3 \to 0,\ R_3 ** R_4 \to 0,\ R_4 ** R_3 \to 0,\ R_4 ** R_4 \to 0,$
$Z_4 ** R_4 \to 1 - Z_1 ** R_1 - Z_2 ** R_2 - Z_3 ** R_3,\ R_1 ** R_1 \to R_1,\ R_1 ** R_3 \to R_3,\ R_3 ** R_1 \to R_3,\ R_4 ** R_1 \to R_4,$
$Z_1 ** R_3 \to R_3,\ Z_2 ** R_4 \to R_4,\ R_2 \to 1 - Z_1 ** R_1 + Z_1 ** R_1 ** R_1 - R_1,\ Z_2 ** R_1 \to -1 + R_1 + Z_2\}$

Corange

ExistsTransNullQ[4, 29, 3]

* * * * * * * * * * * * * * * *

* * *   NCPolyGroebner   * * *

* * * * * * * * * * * * * * * *

* Monomial order: $Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

* Reduce and normalize initial set

> Initial set could not be reduced

* Computing initial set of obstructions

> MAJOR Iteration 1, 14 polys in the basis, 9 obstructions

* Found Groebner basis with 14 polynomials

* * * * * * * * * * * * * * * *

There no simple contradictions were found. More detailed analysis needed.

$\{R_1 ** R_4 \to 0,\ R_3 ** R_3 \to 0,\ R_3 ** R_4 \to 0,\ R_4 ** R_3 \to 0,\ R_4 ** R_4 \to 0,$
$R_4 ** Z_4 \to 1 - R_1 ** Z_1 - R_2 ** Z_2 - R_3 ** Z_3,\ R_1 ** R_1 \to R_1,\ R_1 ** R_3 \to R_3,$
$R_3 ** R_1 \to R_3,\ R_4 ** R_1 \to R_4,\ R_3 ** Z_3 \to -R_1 ** R_1 ** Z_1 - R_1 ** R_2 ** Z_2 + R_1,$
$R_3 ** Z_1 \to R_3,\ R_4 ** Z_1 \to R_4,\ R_2 \to 1 - R_1 ** Z_1 + R_1 ** R_1 ** Z_1 - R_1\}$

G=4, A40

Null Space

ExistsNullQ[4, 40, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

$* \text{ Monomial order: } Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 14 polys in the basis, 9 obstructions
* Found Groebner basis with 14 polynomials
* * * * * * * * * * * * * * * *
There no simple contradictions were found. More detailed analysis needed.
```

$\{R_2 ** R_2 \to 0,\ R_2 ** R_3 \to 0,\ R_2 ** R_4 \to 0,\ R_3 ** R_2 \to 0,$
$R_3 ** R_3 \to 0,\ R_3 ** R_4 \to 0,\ R_4 ** R_2 \to 0,\ R_4 ** R_3 \to 0,\ R_4 ** R_4 \to 0,$
$Z_4 ** R_4 \to 1 - Z_1 ** R_1 - Z_2 ** R_2 - Z_3 ** R_3,\ Z_1 ** R_2 \to R_2,\ Z_1 ** R_3 \to R_3,\ Z_1 ** R_4 \to R_4,\ R_1 \to 1\}$

Corange

ExistsTransNullQ[4, 40, 3]

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
```

$* \text{ Monomial order: } Z_1 < Z_1^T \ll Z_2 < Z_2^T \ll Z_3 < Z_3^T \ll Z_4 < Z_4^T \ll R_1 < R_1^T \ll R_2 < R_2^T \ll R_3 < R_3^T \ll R_4 < R_4^T$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 14 polys in the basis, 9 obstructions
* Found Groebner basis with 14 polynomials
* * * * * * * * * * * * * * * *
There no simple contradictions were found. More detailed analysis needed.
```

$\{R_2 ** R_2 \to 0,\ R_2 ** R_3 \to 0,\ R_2 ** R_4 \to 0,\ R_3 ** R_2 \to 0,$
$R_3 ** R_3 \to 0,\ R_3 ** R_4 \to 0,\ R_4 ** R_2 \to 0,\ R_4 ** R_3 \to 0,\ R_4 ** R_4 \to 0,$
$R_4 ** Z_4 \to 1 - R_1 ** Z_1 - R_2 ** Z_2 - R_3 ** Z_3,\ R_2 ** Z_1 \to R_2,\ R_3 ** Z_1 \to R_3,\ R_4 ** Z_1 \to R_4,\ R_1 \to 1\}$