NC Grobner basis computation of NC polynomials annihilating the Arveson boundary of a free quadrilateral using NCGBX.

This notebook requires the NCAlgebra package available at http://math.ucsd.edu/~ncalg/. Relevant paper: https://arxiv.org/abs/2008.13250

In[∘]:=
```
<< NC`
<< NCAlgebra`
<< NCGBX`
RuleToPoly[rule_] := NCReplaceRepeated[rule, Rule → Subtract];
view[list_] := NCReplaceRepeated[list, NonCommutativeMultiply → Dot] // MatrixForm

(* Set the variables z1 and z2 noncommutative,
and set monomial order z1 << z2 << (c0+c1 z1+c2 z2)^-1.  *)
SNC[z1, z2]
SetMonomialOrder[z1, z2, inv[c0 + c1 * z1 + c2 * z2]]
Clear[c0, c1, c2]

InvPolys = {c0 + c1 * z1 + c2 * z2 - z1 ** inv[c0 + c1 * z1 + c2 * z2] ** z1,
   c0 + c1 * z1 + c2 * z2 - z2 ** inv[c0 + c1 * z1 + c2 * z2] ** z2};
view[
 InvPolys]
```

```
You are using the version of NCAlgebra which is found in:

  C:\Users\Eric\NC\

You can now use "<< NCAlgebra`" to load NCAlgebra.
```

Out[∘]= $\{\{z1\}, \{z2\}, \{(c0 + c1\, z1 + c2\, z2)^{-1}\}\}$

Out[∘]//MatrixForm=

$$\begin{pmatrix} c0 + c1\, z1 + c2\, z2 - z1.(c0 + c1\, z1 + c2\, z2)^{-1}.z1 \\ c0 + c1\, z1 + c2\, z2 - z2.(c0 + c1\, z1 + c2\, z2)^{-1}.z2 \end{pmatrix}$$

The generic case

In[∘]:=
```
(* We compute a GB from the rational functions defining
 the Arveson boundary of our free quadrilateral. The GB contains
 nine NC rational functions, four of which are NC polynomials. *)
```

In[∘]:=
```
InvGB = NCMakeGB[InvPolys, 4];
InvGB // view
```

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
* Symbolic coefficients detected
* Monomial order: z1 ≪ z2 ≪ (c0 + c1 z1 + c2 z2)^-1
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 7 polys in the basis, 11 obstructions
> MAJOR Iteration 2, 9 polys in the basis, 14 obstructions
* Found Groebner basis with 9 polynomials
* * * * * * * * * * * * * * * *
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \text{z1.z1.}(\text{c}\ldots \\ (\text{c0} + \text{c1 z}\ldots \\ \text{z2.z1.}(\text{c0}+\text{c1 z1}+\text{c2 z2})^{-1} \to -\frac{(-1+\text{c1}^2-\text{c2}^2)\,\text{z2}}{\text{c1}} - \frac{(-1+\text{c1}^2-\text{c2}^2)\,\text{z1.z2}}{\text{c0}} - \frac{\text{c0}\,(1-2\,\text{c1}^2+\text{c2}^2)\,\text{z1.}(\text{c0}+\text{c1 z1}+\text{c2 z2})^{-1}}{\text{c2}\,(-1-\text{c1}^2+\text{c2}^2)} - \frac{\text{c2}\,(\ldots}{} \\ \text{z1.z2.z1}\ldots \\ \text{z1.z2.z2} \to -\frac{\text{c0}^3\,\text{c1}}{\text{c2}^2\,(1+\text{c1}^2-\text{c2}^2)} - \frac{\text{c0}^2\,(-3\,\text{c1}^2+\text{c2}^2)\,\text{z1}}{\text{c2}^2\,(-1-\text{c1}^2+\text{c2}^2)}\ldots \\ \text{z2.z2.z1} \to -\frac{\text{c0}^3\,\text{c1}}{\text{c2}^2\,(1+\text{c1}^2-\text{c2}^2)} - \frac{\text{c0}^2\,(-3\,\text{c1}^2+\text{c2}^2)\,\text{z1}}{\text{c2}^2\,(-1-\text{c1}^2+\text{c2}^2)}\ldots \\ \text{z2.z2.z2}\ldots \end{pmatrix}$$

In[∘]:= **(* Note that the fourth equation in the GB above is trivial so long as c0+ c1*z1+c2*z2 is invertible *)**

In[∘]:=
```
InvGB[[4]]
NCExpand[RuleToPoly[InvGB[[4]]] ** (c0 + c1 z1 + c2 z2)]
```

Out[∘]= $\text{z2} ** (\text{c0}+\text{c1 z1}+\text{c2 z2})^{-1} \to \frac{1}{\text{c2}} - \frac{\text{c0}\,(\text{c0}+\text{c1 z1}+\text{c2 z2})^{-1}}{\text{c2}} - \frac{\text{c1 z1} ** (\text{c0}+\text{c1 z1}+\text{c2 z2})^{-1}}{\text{c2}}$

Out[∘]= 0

In[∘]:= **(* We will show that the polynomials in the GB above together reduce the other equations in the GB, i.e. if the polynomials evaluate to zero, then all the rational functions evaluate to zero. To accomplish this we form a Grobner basis from these polynomials together with the rational function InvGB[[4]] and use it to reduce InvGB *)**

In[∘]:=
```
GBpolys = Union[InvGB[[6 ;; 9]], {InvGB[[4]]}];
PolysGB = NCMakeGB[GBpolys, 4];
```

```
* * * * * * * * * * * * * * * * 
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * * 
* Symbolic coefficients detected
* Monomial order: z1 ≪ z2 ≪ (c0 + c1 z1 + c2 z2)^-1
* Reduce and normalize initial set
> Initial set reduced to '6' out of '7' polynomials
* Computing initial set of obstructions
> MAJOR Iteration 1, 10 polys in the basis, 25 obstructions
> MAJOR Iteration 2, 9 polys in the basis, 6 obstructions
* Found Groebner basis with 9 polynomials
* * * * * * * * * * * * * * * * 
```

In[∘]:=

```
(* All elements of InvGB are reduced to 0 by a first pass of the GB formed from
 our polynomials except for one. The remaining requation can be reduced to 0
 by left multiplying by z1 and performing several substituions of by PolysGB *)
```

In[∘]:=

```
InvPolys = Map[RuleToPoly, InvGB];
InvGBSimplified = NCReplaceRepeated[InvPolys, PolysGB] // Simplify
```

Out[∘]=

$$
\begin{aligned}
&\Big\{0,\ 0,\ 0,\ 0,\ \frac{1}{c0\ c1\ c2\ \left(1+c1^{2}-c2^{2}\right)} \\
&\Big(c0^{2}+c0^{2}\,c1^{4}+2\,c0^{2}\,c2^{2}-3\,c0^{2}\,c1^{2}\,c2^{2}-c0^{2}\,c1^{4}\,c2^{2}+c0^{2}\,c2^{4}+c0^{2}\,c1^{2}\,c2^{4}- \\
&\quad c0\,c1\,z1+c0\,c1^{3}\,z1+c0\,c1\,c2^{2}\,z1-2\,c0\,c1^{3}\,c2^{2}\,z1-2\,c0\,c1^{5}\,c2^{2}\,z1+c0\,c1\,c2^{4}\,z1+ \\
&\quad 3\,c0\,c1^{3}\,c2^{4}\,z1-c0\,c1\,c2^{6}\,z1-c0\,c2\,z2+2\,c0\,c1^{2}\,c2\,z2+c0\,c1^{4}\,c2\,z2-3\,c0\,c1^{2}\,c2^{3}\,z2- \\
&\quad c0\,c1^{4}\,c2^{3}\,z2+c0\,c2^{5}\,z2+c0\,c1^{2}\,c2^{5}\,z2+c0^{3}\left(-1+c1^{2}-c2^{2}\right)\left(c0+c1\,z1+c2\,z2\right)^{-1}- \\
&\quad c1^{2}\left(1+c2^{2}+c2^{4}+c2^{6}+c1^{4}\left(1+c2^{2}\right)-2\,c1^{2}\left(1+c2^{2}+c2^{4}\right)\right)z1\ast\ast z1-c1\,c2^{3}\,z1\ast\ast z2- \\
&\quad 2\,c1^{3}\,c2^{3}\,z1\ast\ast z2-c1^{5}\,c2^{3}\,z1\ast\ast z2+2\,c1\,c2^{5}\,z1\ast\ast z2+2\,c1^{3}\,c2^{5}\,z1\ast\ast z2-c1\,c2^{7}\,z1\ast\ast z2+ \\
&\quad 2\,c0^{2}\,c1^{3}\,z1\ast\ast\left(c0+c1\,z1+c2\,z2\right)^{-1}-2\,c0^{2}\,c1\,c2^{2}\,z1\ast\ast\left(c0+c1\,z1+c2\,z2\right)^{-1}- \\
&\quad c0\,c1\,c2^{2}\,z1\ast\ast\left(-1+c0\left(c0+c1\,z1+c2\,z2\right)^{-1}+c1\,z1\ast\ast\left(c0+c1\,z1+c2\,z2\right)^{-1}\right)- \\
&\quad c0\,c1^{3}\,c2^{2}\,z1\ast\ast\left(-1+c0\left(c0+c1\,z1+c2\,z2\right)^{-1}+c1\,z1\ast\ast\left(c0+c1\,z1+c2\,z2\right)^{-1}\right)+ \\
&\quad c0\,c1\,c2^{4}\,z1\ast\ast\left(-1+c0\left(c0+c1\,z1+c2\,z2\right)^{-1}+c1\,z1\ast\ast\left(c0+c1\,z1+c2\,z2\right)^{-1}\right)+ \\
&\quad c1\,z1\ast\ast\Big(c0\,c1^{2}-c0\,c2^{2}-c1\,z1+c1^{3}\,z1-c1\,c2^{2}\,z1+c2\,z2+c1^{2}\,c2\,z2-c2^{3}\,z2+ \\
&\qquad c0\,c1\,z1\ast\ast\left(c0+c1\,z1+c2\,z2\right)^{-1}\Big)+c1^{3}\,z1\ast\ast\Big(c0\,c1^{2}-c0\,c2^{2}-c1\,z1+c1^{3}\,z1- \\
&\qquad c1\,c2^{2}\,z1+c2\,z2+c1^{2}\,c2\,z2-c2^{3}\,z2+c0\,c1\,z1\ast\ast\left(c0+c1\,z1+c2\,z2\right)^{-1}\Big)- \\
&\quad c1\,c2^{2}\,z1\ast\ast\Big(c0\,c1^{2}-c0\,c2^{2}-c1\,z1+c1^{3}\,z1-c1\,c2^{2}\,z1+c2\,z2+c1^{2}\,c2\,z2-c2^{3}\,z2+ \\
&\qquad c0\,c1\,z1\ast\ast\left(c0+c1\,z1+c2\,z2\right)^{-1}\Big)+c1\,c2\,z2\ast\ast z1-c1^{5}\,c2\,z2\ast\ast z1+2\,c1^{3}\,c2^{3}\,z2\ast\ast z1- \\
&\quad c1\,c2^{5}\,z2\ast\ast z1+c1\Big(c0-c0\,c1^{2}+c0\,c2^{2}+c1\,z1-c1^{3}\,z1+c1\,c2^{2}\,z1-c2\,z2-c1^{2}\,c2\,z2+ \\
&\qquad c2^{3}\,z2-c0^{2}\left(c0+c1\,z1+c2\,z2\right)^{-1}-c0\,c1\,z1\ast\ast\left(c0+c1\,z1+c2\,z2\right)^{-1}\Big)\ast\ast z1- \\
&\quad c1^{3}\Big(c0-c0\,c1^{2}+c0\,c2^{2}+c1\,z1-c1^{3}\,z1+c1\,c2^{2}\,z1-c2\,z2-c1^{2}\,c2\,z2+c2^{3}\,z2- \\
&\qquad c0^{2}\left(c0+c1\,z1+c2\,z2\right)^{-1}-c0\,c1\,z1\ast\ast\left(c0+c1\,z1+c2\,z2\right)^{-1}\Big)\ast\ast z1+ \\
&\quad c1\,c2^{2}\Big(c0-c0\,c1^{2}+c0\,c2^{2}+c1\,z1-c1^{3}\,z1+c1\,c2^{2}\,z1-c2\,z2-c1^{2}\,c2\,z2+c2^{3}\,z2- \\
&\qquad c0^{2}\left(c0+c1\,z1+c2\,z2\right)^{-1}-c0\,c1\,z1\ast\ast\left(c0+c1\,z1+c2\,z2\right)^{-1}\Big)\ast\ast z1\Big),\ 0,\ 0,\ 0,\ 0\Big\}
\end{aligned}
$$

```
In[ ]:= NCExpand[NCReplaceRepeated[NCExpand[NCReplaceRepeated[
          NCExpand[NCReplaceRepeated[NCExpand[z1 ** InvGBSimplified[[5]]], PolysGB]],
          PolysGB]], PolysGB]] // Simplify
Out[ ]= 0
```

Special cases

We now consider several special cases of c0, c1, and c2 in the equations defining InvGB are not defined. Namely we consider the cases c1=0 and c2=0 and c1=c2=0 and $1 + c1^2 - c2^2$=0. As described in the main article, the c0=0 cannot occur, so we do not consider it here.

---

c1 = 0

```
In[ ]:= InvPolys = {c0 + c1 * z1 + c2 * z2 - z1 ** inv[c0 + c1 * z1 + c2 * z2] ** z1,
          c0 + c1 * z1 + c2 * z2 - z2 ** inv[c0 + c1 * z1 + c2 * z2] ** z2} //. {c1 → 0};
        view[InvPolys]
        SetMonomialOrder[z1, z2, inv[c0 + c2 * z2]]
        InvGB = NCMakeGB[InvPolys, 4];
        InvGB // view
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} c0 + c2\,z2 - z1.\left(c0 + c2\,z2\right)^{-1}.z1 \\ c0 + c2\,z2 - z2.\left(c0 + c2\,z2\right)^{-1}.z2 \end{pmatrix}$$

Out[ ]= $\left\{\{z1\},\ \{z2\},\ \left\{\left(c0 + c2\,z2\right)^{-1}\right\}\right\}$

```
* * * * * * * * * * * * * * * *
* * *    NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
* Symbolic coefficients detected
```

* Monomial order: $z1 \ll z2 \ll (c0 + c2\,z2)^{-1}$

```
* Reduce and normalize initial set
> Initial set reduced to '3' out of '4' polynomials
* Computing initial set of obstructions
* Found Groebner basis with 3 polynomials
* * * * * * * * * * * * * * * *
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} z1.z2.z1 \to \frac{c0^3}{c2\left(-1+c2^2\right)} + \frac{c0^2\,z2}{-1+c2^2} - \frac{c0\left(1+c2^2\right)z1.z1}{c2\left(-1+c2^2\right)} \\ \left(c0 + c2\,z2\right)^{-1} \to -\frac{-1-c2^2}{c0} + \frac{c2\left(-1+c2^2\right)z2}{c0^2} \\ z2.z2 \to -\frac{c0^2}{-1+c2^2} - \frac{2\,c0\,c2\,z2}{-1+c2^2} \end{pmatrix}$$

```
In[ ]:= (* In this case we obtain 3 equation two of which are
         polynomials. Using the invertibility of (c0+c2*z2) we see
         that the rational equation is reduced by the polynomials. *)
```

In[∘]:= **GBpolys = {InvGB[[1]], InvGB[[3]]}**

Out[∘]= $\left\{z1 ** z2 ** z1 \to \frac{c0^3}{c2\,(-1+c2^2)} + \frac{c0^2\,z2}{-1+c2^2} - \frac{c0\,(1+c2^2)\,z1 ** z1}{c2\,(-1+c2^2)},\; z2 ** z2 \to -\frac{c0^2}{-1+c2^2} - \frac{2\,c0\,c2\,z2}{-1+c2^2}\right\}$

In[∘]:= **InvGB[[2]]**
NCReplaceRepeated[NCExpand[RuleToPoly[InvGB[[2]]] ** (c0 + c2 z2)], GBpolys] //
FullSimplify

Out[∘]= $(c0 + c2\,z2)^{-1} \to -\frac{-1-c2^2}{c0} + \frac{c2\,(-1+c2^2)\,z2}{c0^2}$

Out[∘]= 0

---

c2 = 0

In[∘]:=
```
InvPolys = {c0 + c1 * z1 + c2 * z2 - z1 ** inv[c0 + c1 * z1 + c2 * z2] ** z1,
    c0 + c1 * z1 + c2 * z2 - z2 ** inv[c0 + c1 * z1 + c2 * z2] ** z2} //. {c2 → 0};
view[InvPolys]
SetMonomialOrder[z1, z2, inv[c0 + c1 * z1]]
InvGB = NCMakeGB[InvPolys, 4];
InvGB // view
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} c0 + c1\,z1 - z1.(c0 + c1\,z1)^{-1}.z1 \\ c0 + c1\,z1 - z2.(c0 + c1\,z1)^{-1}.z2 \end{pmatrix}$$

Out[∘]= $\{\{z1\}, \{z2\}, \{(c0 + c1\,z1)^{-1}\}\}$

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
* Symbolic coefficients detected
* Monomial order: z1 ≪ z2 ≪ (c0 + c1 z1)^-1
* Reduce and normalize initial set
> Initial set reduced to '3' out of '4' polynomials
* Computing initial set of obstructions
* Found Groebner basis with 3 polynomials
* * * * * * * * * * * * * * * *
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} (c0 + c1\,z1)^{-1} \to -\frac{-1-c1^2}{c0} + \frac{c1\,(-1+c1^2)\,z1}{c0^2} \\ z2.z1.z2 \to \frac{c0^3}{c1\,(-1+c1^2)} + \frac{c0^2\,z1}{-1+c1^2} - \frac{c0\,(1+c1^2)\,z2.z2}{c1\,(-1+c1^2)} \\ z1.z1 \to -\frac{c0^2}{-1+c1^2} - \frac{2\,c0\,c1\,z1}{-1+c1^2} \end{pmatrix}$$

In[∘]:= **(* In this case we obtain 3 equation two of which are polynomials. Using the invertibility of (c0+c1*z1) we see that the rational equation is reduced by the polynomials. *)**

In[ ]:=
```
GBpolys = {InvGB[[2]], InvGB[[3]]}
```

Out[ ]=
$$\left\{z2 ** z1 ** z2 \to \frac{c0^3}{c1\left(-1+c1^2\right)} + \frac{c0^2\, z1}{-1+c1^2} - \frac{c0\left(1+c1^2\right) z2 ** z2}{c1\left(-1+c1^2\right)},\ z1 ** z1 \to -\frac{c0^2}{-1+c1^2} - \frac{2\, c0\, c1\, z1}{-1+c1^2}\right\}$$

In[ ]:=
```
InvGB[[1]]
NCReplaceRepeated[NCExpand[RuleToPoly[InvGB[[1]]] ** (c0 + c1 z1)], GBpolys] //
 FullSimplify
```

Out[ ]=
$$(c0 + c1\, z1)^{-1} \to -\frac{-1-c1^2}{c0} + \frac{c1\left(-1+c1^2\right) z1}{c0^2}$$

Out[ ]= 0

In[ ]:=

---

c1 = c2 = 0

In[ ]:=
```
InvPolys = {c0 + c1 * z1 + c2 * z2 - z1 ** inv[c0 + c1 * z1 + c2 * z2] ** z1,
    c0 + c1 * z1 + c2 * z2 - z2 ** inv[c0 + c1 * z1 + c2 * z2] ** z2} //. {c1 → 0, c2 → 0};
view[InvPolys]
SetMonomialOrder[z1, z2]
InvGB = NCMakeGB[InvPolys, 4];
InvGB // view
```

Out[ ]//MatrixForm=
$$\begin{pmatrix} c0 - \frac{z1.z1}{c0} \\ c0 - \frac{z2.z2}{c0} \end{pmatrix}$$

Out[ ]= {{z1}, {z2}}

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner   * * *
* * * * * * * * * * * * * * * *
* Symbolic coefficients detected
* Monomial order: z1 ≪ z2
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
* Found Groebner basis with 2 polynomials
* * * * * * * * * * * * * * * *
```

Out[ ]//MatrixForm=
$$\begin{pmatrix} z1.z1 \to c0^2 \\ z2.z2 \to c0^2 \end{pmatrix}$$

In[ ]:=
```
(* Here we obtain 2 polynomials and are done *)
```

$1 + c1^2 - c2^2 = 0$

Here we consider two subcases.

$c2 = -\sqrt{1 + c1^2}$

In[∘]:=
```
InvPolys = {c0 + c1 * z1 + c2 * z2 - z1 ** inv[c0 + c1 * z1 + c2 * z2] ** z1,
    c0 + c1 * z1 + c2 * z2 - z2 ** inv[c0 + c1 * z1 + c2 * z2] ** z2} //. {c2 → - √(1 + c1^2)};
view[InvPolys]
SetMonomialOrder[z1, z2, inv[c0 + c1 * z1 - √(1 + c1^2) * z2]]
InvGB = NCMakeGB[InvPolys, 4];
InvGB // view
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} c0 + c1\,z1 - \sqrt{1 + c1^2}\,z2 - z1.\left(c0 + c1\,z1 - \sqrt{1 + c1^2}\,z2\right)^{-1}.z1 \\ c0 + c1\,z1 - \sqrt{1 + c1^2}\,z2 - z2.\left(c0 + c1\,z1 - \sqrt{1 + c1^2}\,z2\right)^{-1}.z2 \end{pmatrix}$$

Out[∘]= $\left\{\{z1\}, \{z2\}, \left\{\left(c0 + c1\,z1 - \sqrt{1 + c1^2}\,z2\right)^{-1}\right\}\right\}$

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner   * * *
* * * * * * * * * * * * * * * *
* Symbolic coefficients detected
```

$\star$ Monomial order: $z1 \ll z2 \ll \left(c0 + c1\,z1 - \sqrt{1 + c1^2}\,z2\right)^{-1}$

```
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 6 polys in the basis, 6 obstructions
* Found Groebner basis with 6 polynomials
* * * * * * * * * * * * * * * *
```

Out[∘]//MatrixForm=

$$\begin{pmatrix}
z1.z1.\left(c0 + c1\,z1 - \sqrt{1 + c1^2}\,z2\right)^{-1} \to -c0 - \frac{\left(-2+c1^2\right) z1}{c1} + \sqrt{1 + c1^2}\,z2 + \frac{2\,z1.z1}{c0} - \frac{c0\,z1.\left(c0+c1\,z1-\sqrt{1+c1^2}\,z2\right)^{-}}{c1} \\
\left(c0 + c1\,z1 - \sqrt{1 + c1^2}\,z2\right)^{-1}.z1 \to \frac{2}{c1} + \frac{2\,z1}{c0} - z1.\left(c0 + c1\,z1 - \sqrt{1 + c1^2}\,z2\right)^{-1} - \frac{c0\left(c0+c1\,z1-\sqrt{1+c1^2}\,z2\right)^{-1}}{c1} \\
\left(c0 + c1\,z1 - \sqrt{1 + c1^2}\,z2\right)^{-1}.z2 \to -\frac{1}{\sqrt{1+c1^2}} + \frac{c1\left(c0+c1\,z1-\sqrt{1+c1^2}\,z2\right)^{-1}.z1}{\sqrt{1+c1^2}} + \frac{c0\left(c0+c1\,z1-\sqrt{1+c1^2}\,z2\right)^{-1}}{\sqrt{1+c1^2}} \\
z2.\left(c0 + c1\,z1 - \sqrt{1 + c1^2}\,z2\right)^{-1} \to -\frac{1}{\sqrt{1+c1^2}} + \frac{c1\,z1.\left(c0+c1\,z1-\sqrt{1+c1^2}\,z2\right)^{-1}}{\sqrt{1+c1^2}} + \frac{c0\left(c0+c1\,z1-\sqrt{1+c1^2}\,z2\right)^{-1}}{\sqrt{1+c1^2}} \\
z2.z1 \to \frac{c0^2}{c1\,\sqrt{1+c1^2}} + \frac{3\,c0\,z1}{\sqrt{1+c1^2}} - \frac{c0\,z2}{c1} + \frac{2\left(-1+c1^2\right) z1.z1}{c1\,\sqrt{1+c1^2}} - z1.z2 - \frac{2\,z1.z1.z1}{c0\,\sqrt{1+c1^2}} \\
z2.z2 \to -\frac{c0^2}{1+c1^2} - \frac{\left(-c0+2\,c0\,c1^2\right) z1}{c1\left(1+c1^2\right)} + \frac{2\,c0\,z2}{\sqrt{1+c1^2}} - \frac{\left(-3+c1^2\right) z1.z1}{1+c1^2} - \frac{\left(2-c1^2\right) z1.z2}{c1\,\sqrt{1+c1^2}} + \frac{c1\,z2.z1}{\sqrt{1+c1^2}} + \frac{2\,c1\,z1.z1.z1}{c0\left(1+c1^2\right)} - \frac{2\,z1.z1.z2}{c0\,\sqrt{1+c1^2}}
\end{pmatrix}$$

In[∘]:= **(* We have two polynomials. In addition the 4th rational function is trivial using the invertibility of $\left(c0+c1\,z1-\sqrt{1+c1^2}\,z2\right)$ *).**

In[∘]:= NCExpand[RuleToPoly[InvGB[[4]]] ** $\left(c0 + c1\, z1 - \sqrt{1 + c1^2}\; z2\right)$]

Out[∘]= 0

In[∘]:= (* As in the generic case we form a Grobner basis using the polynomials and trivial rational equation above and show it reduces the remaining rational equations *)

In[∘]:=
```
GBpolys = InvGB[[4 ;; 6]];
PolysGB = NCMakeGB[GBpolys, 4];
InvPolys = Map[RuleToPoly, InvGB];
InvGBSimplified = NCReplaceRepeated[InvPolys, PolysGB] // Simplify
```

* * * * * * * * * * * * * * * *

* * *   NCPolyGroebner   * * *

* * * * * * * * * * * * * * * *

* Symbolic coefficients detected

* Monomial order: $z1 \ll z2 \ll \left(c0 + c1\, z1 - \sqrt{1 + c1^2}\; z2\right)^{-1}$

* Reduce and normalize initial set

> Initial set reduced to '4' out of '5' polynomials

* Computing initial set of obstructions

> MAJOR Iteration 1, 7 polys in the basis, 12 obstructions

> MAJOR Iteration 2, 6 polys in the basis, 5 obstructions

* Found Groebner basis with 6 polynomials

* * * * * * * * * * * * * * * *

Out[∘]= {0, 0, 0, 0, 0, 0}

$c2 = \sqrt{1 + c1^2}$

In[∘]:= InvPolys = {c0 + c1 * z1 + c2 * z2 - z1 ** inv[c0 + c1 * z1 + c2 * z2] ** z1, c0 + c1 * z1 + c2 * z2 - z2 ** inv[c0 + c1 * z1 + c2 * z2] ** z2} //. $\left\{c2 \to \sqrt{1 + c1^2}\right\}$;
view[InvPolys]
SetMonomialOrder[z1, z2, inv[c0 + c1 * z1 + $\sqrt{1 + c1^2}$ * z2]]
InvGB = NCMakeGB[InvPolys, 4];
InvGB // view

Out[∘]//MatrixForm=

$$\begin{pmatrix} c0 + c1\, z1 + \sqrt{1 + c1^2}\; z2 - z1.\left(c0 + c1\, z1 + \sqrt{1 + c1^2}\; z2\right)^{-1}.z1 \\ c0 + c1\, z1 + \sqrt{1 + c1^2}\; z2 - z2.\left(c0 + c1\, z1 + \sqrt{1 + c1^2}\; z2\right)^{-1}.z2 \end{pmatrix}$$

Out[∘]= $\left\{\{z1\}, \{z2\}, \left\{\left(c0 + c1\, z1 + \sqrt{1 + c1^2}\; z2\right)^{-1}\right\}\right\}$

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
* Symbolic coefficients detected
* Monomial order: z1 ≪ z2 ≪ (c0 + c1 z1 + √(1 + c1²) z2)^-1
* Reduce and normalize initial set
> Initial set could not be reduced
* Computing initial set of obstructions
> MAJOR Iteration 1, 6 polys in the basis, 6 obstructions
* Found Groebner basis with 6 polynomials
* * * * * * * * * * * * * * * *
```

Out[∘]//MatrixForm=

$$
\begin{pmatrix}
\mathrm{z1.z1.}\left(\mathrm{c0}+\mathrm{c1\,z1}+\sqrt{1+\mathrm{c1}^2}\,\mathrm{z2}\right)^{-1} \to -\mathrm{c0}-\frac{\left(-2+\mathrm{c1}^2\right)\mathrm{z1}}{\mathrm{c1}}-\sqrt{1+\mathrm{c1}^2}\,\mathrm{z2}+\frac{2\,\mathrm{z1.z1}}{\mathrm{c0}}-\frac{\mathrm{c0\,z1.}\left(\mathrm{c0}+\mathrm{c1\,z1}+\sqrt{1+\mathrm{c1}^2}\,\mathrm{z2}\right)^{-}}{\mathrm{c1}} \\
\left(\mathrm{c0}+\mathrm{c1\,z1}+\sqrt{1+\mathrm{c1}^2}\,\mathrm{z2}\right)^{-1}\mathrm{.z1} \to \frac{2}{\mathrm{c1}}+\frac{2\,\mathrm{z1}}{\mathrm{c0}}-\mathrm{z1.}\left(\mathrm{c0}+\mathrm{c1\,z1}+\sqrt{1+\mathrm{c1}^2}\,\mathrm{z2}\right)^{-1}-\frac{\mathrm{c0}\left(\mathrm{c0}+\mathrm{c1\,z1}+\sqrt{1+\mathrm{c1}^2}\,\mathrm{z2}\right)^{-1}}{\mathrm{c1}} \\
\left(\mathrm{c0}+\mathrm{c1\,z1}+\sqrt{1+\mathrm{c1}^2}\,\mathrm{z2}\right)^{-1}\mathrm{.z2} \to \frac{1}{\sqrt{1+\mathrm{c1}^2}}-\frac{\mathrm{c1}\left(\mathrm{c0}+\mathrm{c1\,z1}+\sqrt{1+\mathrm{c1}^2}\,\mathrm{z2}\right)^{-1}\mathrm{.z1}}{\sqrt{1+\mathrm{c1}^2}}-\frac{\mathrm{c0}\left(\mathrm{c0}+\mathrm{c1\,z1}+\sqrt{1+\mathrm{c1}^2}\,\mathrm{z2}\right)^{-1}}{\sqrt{1+\mathrm{c1}^2}} \\
\mathrm{z2.}\left(\mathrm{c0}+\mathrm{c1\,z1}+\sqrt{1+\mathrm{c1}^2}\,\mathrm{z2}\right)^{-1} \to \frac{1}{\sqrt{1+\mathrm{c1}^2}}-\frac{\mathrm{c1\,z1.}\left(\mathrm{c0}+\mathrm{c1\,z1}+\sqrt{1+\mathrm{c1}^2}\,\mathrm{z2}\right)^{-1}}{\sqrt{1+\mathrm{c1}^2}}-\frac{\mathrm{c0}\left(\mathrm{c0}+\mathrm{c1\,z1}+\sqrt{1+\mathrm{c1}^2}\,\mathrm{z2}\right)^{-1}}{\sqrt{1+\mathrm{c1}^2}} \\
\mathrm{z2.z1} \to -\frac{\mathrm{c0}^2}{\mathrm{c1}\sqrt{1+\mathrm{c1}^2}}-\frac{3\,\mathrm{c0\,z1}}{\sqrt{1+\mathrm{c1}^2}}-\frac{\mathrm{c0\,z2}}{\mathrm{c1}}-\frac{2\left(-1+\mathrm{c1}^2\right)\mathrm{z1.z1}}{\mathrm{c1}\sqrt{1+\mathrm{c1}^2}}-\mathrm{z1.z2}+\frac{2\,\mathrm{z1.z1.z1}}{\mathrm{c0}\sqrt{1+\mathrm{c1}^2}} \\
\mathrm{z2.z2} \to -\frac{\mathrm{c0}^2}{1+\mathrm{c1}^2}-\frac{\left(-\mathrm{c0}+2\,\mathrm{c0\,c1}^2\right)\mathrm{z1}}{\mathrm{c1}\left(1+\mathrm{c1}^2\right)}-\frac{2\,\mathrm{c0\,z2}}{\sqrt{1+\mathrm{c1}^2}}-\frac{\left(-3+\mathrm{c1}^2\right)\mathrm{z1.z1}}{1+\mathrm{c1}^2}-\frac{\left(-2+\mathrm{c1}^2\right)\mathrm{z1.z2}}{\mathrm{c1}\sqrt{1+\mathrm{c1}^2}}-\frac{\mathrm{c1\,z2.z1}}{\sqrt{1+\mathrm{c1}^2}}+\frac{2\,\mathrm{c1\,z1.z1.z1}}{\mathrm{c0}\left(1+\mathrm{c1}^2\right)}+\frac{2\,\mathrm{z1.z1.z2}}{\mathrm{c0}\sqrt{1+\mathrm{c1}^2}}
\end{pmatrix}
$$

In[∘]:= **(* We have two polynomials. In addition the 4th rational function is trivial using the invertibility of $\left(\mathrm{c0}+\mathrm{c1\ z1}+\sqrt{1+\mathrm{c1}^2}\ \mathrm{z2}\right)$ *)**

In[∘]:=
```
NCExpand[RuleToPoly[InvGB[[4]]] ** (c0 + c1 z1 + √(1 + c1²) z2)]
```

Out[∘]= 0

In[∘]:= **(* As in the generic case we form a Grobner basis using the polynomials and trivial rational equation above and show it reduces the remaining rational equations *)**

In[∘]:=
```
GBpolys = InvGB[[4 ;; 6]];
PolysGB = NCMakeGB[GBpolys, 4];
InvPolys = Map[RuleToPoly, InvGB];
InvGBSimplified = NCReplaceRepeated[InvPolys, PolysGB] // Simplify
```

```
* * * * * * * * * * * * * * * *
* * *   NCPolyGroebner    * * *
* * * * * * * * * * * * * * * *
* Symbolic coefficients detected
```

* Monomial order: $z1 \ll z2 \ll \left(c0 + c1\, z1 + \sqrt{1 + c1^2}\, z2\right)^{-1}$

```
* Reduce and normalize initial set
> Initial set reduced to '4' out of '5' polynomials
* Computing initial set of obstructions
> MAJOR Iteration 1, 7 polys in the basis, 12 obstructions
> MAJOR Iteration 2, 6 polys in the basis, 5 obstructions
* Found Groebner basis with 6 polynomials
* * * * * * * * * * * * * * * *
```

Out[ ]= {0, 0, 0, 0, 0, 0}