

# Noncommutative geometry of foliations

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## Foliations: motivations, definitions and examples

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### 1.1 Motivations

#### 1.1.1 From Frobenius theorem to foliations

A *foliation* on a manifold is, roughly speaking, a decomposition of it into immersed submanifolds (called the *leaves* of the foliation), such that the leaves fit together nicely. The theory of foliations is a tremendous component in modern differential geometry, contributed by several big names: Ehresmann, Reeb, Haefliger, Novikov, Thurston, Molino, Sullivan, Connes and many others.

The idea of foliations comes from a much older task: solving differential equations. This can be best by the celebrated Frobenius theorem<sup>1</sup>. Let  $M$  be a smooth manifold. Let  $E$  be a smooth rank- $k$  distribution over  $M$ , that is, a rank- $k$  subbundle of  $TM$ . An *integral manifold* of  $E$  is an immersed submanifold  $N \subseteq M$  such that  $T_p N = E_p$  for every  $p \in N$ . A distribution  $E$  is said to be *integrable* if every  $x \in M$  is contained in an integral manifold of  $E$ . A distribution  $E$  is said to be *involutive* if  $\Gamma(E)$ , the space of smooth sections of  $E$ , is a Lie subalgebra of  $\Gamma(TM)$  under the Lie bracket.

**Theorem 1.1** (Frobenius). *A distribution is integrable iff it is involutive.*

The Frobenius theorem is a vast generalisation of the classical existence theorems in differential equations, that is, the existence (and uniqueness) of solution (integral curve) of linear partial differential equations. The involutivity of a distribution can be viewed as its “local integrability”, which by Frobenius theorem implies the existence of an integral manifold at every point. In modern language, the Frobenius theorem says that an involutive distribution gives a foliation of the base manifold, whose leaves are those maximal connected integral submanifolds.

Geometry serves as a big source of foliations. Every submersion of manifolds define a foliation on the total space, whose leaves are the connected components of fibres (Example 1.7). In particular, a fibre bundle defines a foliation of the total space. Another interesting example is the symplectic foliation of a Poisson manifold, which is generated by its Hamiltonian vector fields.

Dynamical systems also supply many interesting instances of foliations. Let  $G$  be a Lie group that acts on a smooth manifold  $M$ . Suppose that the stabiliser groups have constant dimension. Then  $M$  is foliated by the connected components of  $G$ -orbits (Example 1.8). The dimension condition is necessary here as otherwise the leaves will not have the same dimension. That would be an instance of a singular foliation and an orbifold.

#### 1.1.2 Why noncommutative geometry?

Let  $(M, \mathcal{F})$  be a foliated manifold, that is, a manifold together with a foliation thereon. We wish to study the geometry of two “spaces”:

1. The geometry of the leaves (generic fibre)  $L$ .
2. The geometry of the space of leaves  $M/\mathcal{F}$ , obtained from the equivalence relation generated by leaves.

However, there is several technical difficulty for us to study them, even in many simplest examples.

- A leaf  $L$  of a foliation can be usually non-compact. For example, every leaf of the Kronecker foliation (Example 1.17) is diffeomorphic to  $\mathbb{R}$ . This non-compactness, in many cases, obstructs us to pass from local (leaves) to global (the space  $M$ ).

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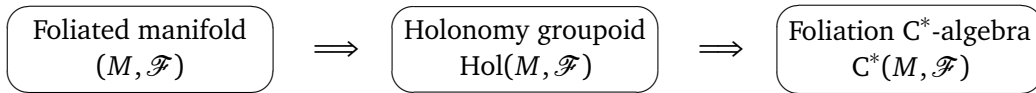
<sup>1</sup>Despite of the name, this theorem is indeed not proven by Frobenius, at whose age the concept of manifolds were not yet established.

- The quotient space  $M/\mathcal{F}$  is usually badly behaved as a topological space. In the case of the Kronecker foliation, every leaf is dense in  $\mathbb{T}^2$ . So the quotient space is necessarily non-Hausdorff.
- We would like to define a measure on the quotient space, so that we can (1) talk about integrals (2) make the word “generic” precise, in the sense of probability. But  $M/\mathcal{F}$  is also quite terrible as a measurable space due to ergodicity. That is, every measurable function of  $M/\mathcal{F}$  is necessarily constant almost everywhere. Hence  $L^p(M/\mathcal{F}) = \mathbb{C}$  for every  $p \in [1, \infty]$ .

These force us to seek other geometric objects to replace  $M/\mathcal{F}$ , and leads to Connes’  $C^*$ -algebras of foliations and noncommutative geometry. Roughly, these can be summarised as follows:

- The leaves of a foliation generate an equivalence relation of the manifold, and hence a *groupoid*  $\text{Hol}(M, \mathcal{F})$ , called the holonomy groupoid of the foliation. This is a noncommutative object which encodes the topology of a foliation.
- By some standard techniques introduced by Renault, one can define a  $C^*$ -algebra  $C^*(M, \mathcal{F})$  of this groupoid. This is known as Connes’  $C^*$ -algebra of foliation.
- A measure on  $M/\mathcal{F}$  should be by a measure on  $M$  that is holonomy invariant. This is called a transverse measure of the foliation. With such as input, one may get a trace on  $C^*(M, \mathcal{F})$  and hence an “index map”  $K_*(C^*(M, \mathcal{F})) \rightarrow \mathbb{R}$ . This reveals the index theory of a foliation.

Thus we have the following nice machine for foliations:



However, a new question emerges: what are the correct *morphisms*? Note that assigning the groupoid  $C^*$ -algebra to a groupoid is not functorial. For example, topological spaces and topological groups are both topological groupoids. Whereas groupoid  $C^*$ -algebras for spaces are contravariantly functorial, those for groupoids are covariantly functorial. So homomorphisms of groupoids are not satisfying, and we will replace them by a more general concepts of morphisms: the *groupoid correspondences*. These were first introduced in [4] and later referred to as “Hilsum–Skandalis morphisms”. A brief account of this is Bram’s talk at the groupoid seminar [7].

## 1.2 Definitions and first examples

### 1.2.1 By foliation atlas

**Definition 1.2** (First definition). Let  $M$  be a smooth manifold of dimension  $n$ . A (regular) *foliation atlas* of codimension  $q$  is an atlas of  $M$ , each chart of the form

$$(U_i, \varphi_i: U_i \rightarrow \mathbb{R}^{n-q} \times \mathbb{R}^q),$$

such that the transition functions

$$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1}: \varphi_j(U_i \cap U_j) \rightarrow \varphi_i(U_i \cap U_j)$$

have the form

$$\varphi_{ij}(x, y) = (g_{ij}(x, y), h_{ij}(y)), \quad x \in \mathbb{R}^{n-q}, y \in \mathbb{R}^q$$

for some smooth functions  $g_{ij}: \mathbb{R}^n \rightarrow \mathbb{R}^{n-q}$  and  $h_{ij}: \mathbb{R}^q \rightarrow \mathbb{R}^q$ .

A (regular) *foliation* of codimension  $q$  of  $M$  is a maximal foliation atlas  $\mathcal{F}$  of  $M$ . We write  $\text{codim } \mathcal{F}$  for the codimension of  $\mathcal{F}$ .

A *foliated manifold* is a pair  $(M, \mathcal{F})$  of a manifold  $M$  together with a foliation  $\mathcal{F}$  of it.

**Definition 1.3.** A *plaque* of a foliation chart  $(U_i, \varphi_i: U_i \rightarrow \mathbb{R}^{n-q} \times \mathbb{R}^q)$  is a connected component of the submanifold  $\varphi^{-1}(\mathbb{R}^{n-q} \times \{y\})$  for some  $y \in \mathbb{R}^q$ . Two points  $x, y \in M$  belong to the same *leaf* if there exist a sequence of foliation charts  $U_1, \dots, U_k$  and a sequence of points  $x = p_1, \dots, p_k = y$  with  $p_i \in U_i$  and such that  $p_{i-1}$  and  $p_i$  belong to the same plaque.

A *leaf* is a collection of points that belong to the same leaf.

The *space of leaves* is the quotient space  $M/\mathcal{F}$  obtained by the equivalence relation generated by the leaves.

One can easily show that leaves are immersed submanifolds of  $M$ :

**Proposition 1.4.** Let  $(M, \mathcal{F})$  be a codimension- $q$  foliated manifold. Then every leaf is an immersed submanifold of  $M$  of dimension  $n - q$ .

*Proof.* Choose  $x \in M$  and let  $L$  be the leaf of  $x$ . Every plaque is a smooth chart of dimension  $n - q$ . If  $x$  belongs to a plaque, then every point of the plaque belongs to  $L$  by definition. This holds for other plaques containing a point that belongs to the same leaf with  $x$ . Therefore  $L$  is covered by these plaques and they are smoothly compatible. The plaques are immersed submanifolds of  $M$ , and this holds for  $L$  as well.  $\square$

**Definition 1.5.** Let  $(M, \mathcal{F})$  and  $(N, \mathcal{G})$  be foliated manifolds. A morphism between them is a smooth map  $f: M \rightarrow N$  which preserves the foliation. That means, every leaf of  $\mathcal{F}$  is mapped into a leaf of  $\mathcal{G}$ . We also say such a map is foliated.

*Example 1.6* (Trivial foliation). The space  $\mathbb{R}^n$  admits a trivial foliation: the foliated atlas consists of a single chart  $\text{id}: \mathbb{R}^n \rightarrow \mathbb{R}^{n-q} \times \mathbb{R}^q$ . Similarly, any linear isomorphism  $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$  determines a foliation. The leaves are given by  $A^{-1}(\mathbb{R}^{n-q} \times \{y\})$ .

*Example 1.7* (Submersions). Let  $\pi: E \rightarrow M$  be a submersion. This defines a foliation of  $E$  whose leaves are connected components of the fibres of  $\pi$ . The codimension of the foliation equals  $\dim M$ . If every fibre is connected, then  $E/\mathcal{F} = M$ . If some fibres are not connected, then  $E/\mathcal{F}$  is a quotient of  $M$  which is necessarily non-Hausdorff.

In particular, a rank- $k$  vector bundle  $\pi: E \rightarrow M$  gives a foliation of  $E$  whose leaves are given by  $\pi^{-1}(x) \simeq \mathbb{R}^k$ .

*Example 1.8* (Lie group actions). Another important source of interesting foliations is smooth dynamical systems. Let  $G$  be a Lie group which acts on a smooth manifold  $M$ . We wish to foliate  $M$  into the connected components of the orbits under the  $G$ -action. But this cannot be always true. For example, consider the canonical action of  $\text{GL}(2, \mathbb{R})$ , or  $\text{SL}(2, \mathbb{R})$ , or  $\mathbb{C}^*$  (viewed as a submanifold of  $\mathbb{R}^2$  equipped with its usual product) on  $\mathbb{R}^2$ . The orbits are  $\{0\}$  and  $\mathbb{R}^2 \setminus \{0\}$ . However, if it were a foliation, then these orbits should have the same dimension.

In order to exclude such singular cases, consider the isotropy subgroups at  $x$ :

$$G_x := \{g \in G \mid gx = x\}$$

This is a closed subgroup, hence a Lie subgroup. The orbit of  $x$  is identified with the homogeneous space  $G/G_x$ , and immersed into  $M$ . We say that the action of  $G$  on  $M$  is *foliated*, if  $G_x$  has constant dimension. Then all orbits have the same dimension, and the orbits form a foliation of  $M$ .

For example, let  $G = \mathbb{R}$ . Then an  $\mathbb{R}$ -action on  $M$

$$\mu: \mathbb{R} \times M \rightarrow M$$

is also called a *flow*. The vector field associated to the flow is

$$X(x) := \left. \frac{\partial \mu(t, x)}{\partial t} \right|_{t=0}.$$

An  $\mathbb{R}$ -action is foliated, iff its associated vector field  $X$  is nowhere vanishing. Then the leaves of the foliation are given by the integral curves of  $X$ .

### 1.2.2 By Haefliger structure

A foliation can be equivalently described in several equivalent ways. The following definition is based on a “Haefliger structure”.

**Definition 1.9** (Second definition). A codimension- $q$  foliation of a manifold  $M$  is given by an open cover  $\{U_i\}$  of  $M$  together with submersions

$$s_i: U_i \rightarrow \mathbb{R}^q,$$

such that there are (necessarily unique) diffeomorphisms

$$\gamma_{ij}: s_j(U_i \cap U_j) \rightarrow s_i(U_i \cap U_j)$$

satisfying:

$$\gamma_{ij} \circ s_j = s_i, \quad \gamma_{ij} \circ \gamma_{jk} = \gamma_{ik}.$$

These maps  $\gamma_{ij}$  are called the *Haefliger cocycles* of the foliation. We briefly call the data  $\{U_i, s_i, \gamma_{ij}\}_{ij}$  a *Haefliger structure*.

*Equivalence with first definition.* Suppose we are given a Haefliger structure  $\{U_i, s_i, \gamma_{ij}\}_{ij}$ . Let  $\{(V_k, \varphi_k)\}_k$  be an atlas of  $M$ . Up to refinement, we may assume that each  $V_k$  is contained in a single  $U_{i_k}$ . Then  $s_{i_k}|_{V_k}$  is also a submersion, hence there is a diffeomorphism  $\kappa_k: s_{i_k}(V_k) \rightarrow \mathbb{R}^q$  such that  $\kappa_k \circ s_{i_k}$  retracts to the projection  $\mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q \rightarrow \mathbb{R}^q$  on local charts. That is,  $\kappa_k \circ s_{i_k} \circ \varphi_k^{-1} = \text{pr}_{\mathbb{R}^q}$ . We claim that  $\{V_k, \varphi_k\}$  is a foliation chart in the sense of Definition 1.2: we have

$$\begin{aligned} \text{pr}_{\mathbb{R}^q} \circ \varphi_{kl}(x, y) &= \text{pr}_{\mathbb{R}^q} \circ \varphi_k \circ \varphi_l^{-1}(x, y) \\ &= \kappa_k \circ s_{i_k} \circ \varphi_l^{-1}(x, y) \\ &= \kappa_k \circ \gamma_{i_k i_l} \circ s_{i_l} \circ \varphi_l^{-1}(x, y) \\ &= \kappa_k \circ \gamma_{i_k i_l} \circ \kappa_l^{-1}(y). \end{aligned}$$

Conversely, given a foliation atlas  $\{U_i, \varphi_i: U_i \rightarrow \mathbb{R}^{n-q} \times \mathbb{R}^q\}$  such that  $\varphi_{ij}(x, y) = (g_{ij}(x, y), h_{ij}(y))$ . Set  $s_i := \text{pr}_{\mathbb{R}^q} \circ \varphi_i$  and  $\gamma_{ij} := h_{ij}$ . These render the desired Haefliger cocycles.  $\square$

*Example 1.10.* Let  $\pi: E \rightarrow M$  be a rank- $(n-q)$  vector bundle and  $\dim M = q$ . This gives a codimension- $q$  foliation on  $M$  whose leaves are  $\pi^{-1}(x)$  for  $x \in M$ . Let  $\{U_i, \varphi_i: U_i \rightarrow \mathbb{R}^q\}$  be an atlas of  $M$ , and  $\{\Phi_i: \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^{n-q}\}$  be the associated local trivialisations of  $E$ . Then the foliation chart on  $\pi^{-1}(U_i)$  is just the composition of the local trivialisations  $\Phi_i$ , with the chart of  $U_i$ . The Haefliger cocycles are just the transition functions  $h_{ij}: U_i \cap U_j \rightarrow \text{GL}(q, \mathbb{R})$  of the base manifold  $M$ .

### 1.2.3 By involutive distribution

The alternative definition below justifies the connection between foliations and the Frobenius theorem:

**Definition 1.11** (Third definition). A codimension- $q$  foliation  $\mathcal{F}$  of a manifold  $M$  is given by an involutive distribution.

The involutive distribution is called the *tangent bundle* of the foliation  $\mathcal{F}$  and denoted by  $T\mathcal{F}$ .

The equivalence with the first definition is heavily based on the Frobenius theorem. We will omit the proof and only sketch how to derive these definitions from each other. The proof can be found in [6, Section 1.2]; see also [5, Theorem 19.21].

Let  $E \subseteq TM$  be an involutive distribution. By Frobenius theorem, it is integrable: every point  $x \in M$  belongs to an integral manifold. These manifolds form a foliation of  $M$ .

Conversely, if  $(M, \mathcal{F})$  is a foliated manifold. For every  $x \in M$ , let  $L$  be the leaf of  $x$ . Then the collection  $\{T_x L\}_{x \in M}$  is a vector subbundle of  $TM$  which is involutive. We denote this bundle by  $T_x \mathcal{F}$  and also write  $T_x \mathcal{F}$  for  $T_x L$ .

*Example 1.12.* A vector bundle  $E \rightarrow M$  gives a foliation  $\mathcal{F}$  on the total space  $E$ . The tangent bundle of the foliation  $T\mathcal{F}$  is just the vertical bundle of  $E \rightarrow M$ :  $VE := \ker T\pi$ . Note that every leaf is vertical in the sense that  $T\mathcal{F} \subseteq \ker T\pi$ .

*Example 1.13.* Let  $G$  be a Lie group which has a foliated action on  $M$ . Then  $T\mathcal{F}$  is spanned by the fundamental vector fields of this group action. Namely, the image of the bundle homomorphism

$$\mathfrak{g} \times M \rightarrow TM, \quad (\xi, x) \mapsto \left. \frac{d}{dt} \right|_{t=0} x \exp(t\xi).$$

*Remark 1.14.* We have seen so far that a Lie group action generates a foliation necessarily when the dimensions of the stabliser groups are constant. If this is not the case, then we speak of a *singular foliation*. The following definition (due to Peter Stefan, Iakovos Androulidakis and Georges Skandalis) is modelled on the third definition that we have introduced above.

**Definition 1.15.** A (singular) foliation is a locally finitely generated, involutive  $C^\infty(M)$ -submodule of  $\Gamma_c(M, TM)$ .

Singular foliations are much more complicated to deal with as opposed to those regular ones. For example, the holonomy groupoid of a singular foliation is hard to define. It seems that the most general definition so far is the one given in [1]. We might talk about this in a later talk.

### 1.3 Constructions and more examples

#### 1.3.1 Product foliations

Let  $(M, \mathcal{F})$  and  $(N, \mathcal{G})$  be foliated manifolds, given by a Haefliger structure  $(U_i, s_i, \gamma_{ij})_{ij}$  on  $M$  and  $(U'_k, s'_k, \gamma'_{kl})$  on  $N$ . There is a *product foliation*  $\mathcal{F} \times \mathcal{G}$  on  $M \times N$ , given by the Haefliger structure

$$(U_i \times U'_k, s_i \times s'_k, \gamma_{ij} \times \gamma'_{kl})_{ijkl}.$$

The product foliation  $\mathcal{F} \times \mathcal{G}$  has codimension  $\text{codim } \mathcal{F} + \text{codim } \mathcal{G}$ . The tangent bundle  $T(\mathcal{F} \times \mathcal{G}) \simeq T\mathcal{F} \oplus T\mathcal{G} \subseteq TM \oplus TN \simeq T(M \times N)$ .

#### 1.3.2 Pullback foliations along transverse maps

**Definition 1.16.** Let  $(M, \mathcal{F})$  be a foliated manifold. A smooth map  $f : N \rightarrow M$  is *transverse* to  $\mathcal{F}$ , if  $f$  is transverse to all the leaves of  $\mathcal{F}$ . That is, for every  $x \in N$ ,

$$T_{f(x)}M = T_{f(x)}\mathcal{F} + T_x f(T_x N).$$

Let  $\mathcal{F}$  be given by the Haefliger structure  $(U_i, s_i, \gamma_{ij})_{ij}$ . We claim that the data

$$(V_i := f^{-1}(U_i), \quad s'_i := s_i \circ f|_{V_i}, \quad \gamma_{ij})_{ij}$$

defines a Haefliger structure. Its determined foliation is called the *pullback foliation* on  $N$  along  $f$ , denoted by  $f^*\mathcal{F}$ .

*Proof.* We must show that the maps  $s'_i : V_i \rightarrow \mathbb{R}^q$  are submersion, that is,  $T_x s'_i = T_{f(x)} s_i \circ T_x f$  is surjective for every  $x \in V_i$ . Since  $f$  is transverse to  $\mathcal{F}$ , we have that the map

$$\tilde{f} : T_x V_i \xrightarrow{T_x f} T_{f(x)} U_i \rightarrow T_{f(x)} U_i / T_{f(x)} \mathcal{F}$$

is surjective. The submersion  $s_i$  is trivial along the leaves, that is,  $T_{f(x)} s_i = 0$  on  $T_{f(x)} \mathcal{F} \subseteq T_{f(x)} U_i$ . So  $T_x s'_i$  factors through the quotient map  $T_{f(x)} U_i \rightarrow T_{f(x)} U_i / T_{f(x)} \mathcal{F}$ . Thus the diagram

$$\begin{array}{ccccc} T_x V_i & \xrightarrow{T_x f} & T_{f(x)} U_i & \xrightarrow{T_{f(x)} s_i} & \mathbb{R}^q \\ & \searrow \tilde{f} & \downarrow & \nearrow \tilde{s}_i & \\ & & T_{f(x)} U_i / T_{f(x)} \mathcal{F} & & \end{array}$$

commutes. So  $T_{f(x)}s_i \circ T_x f = \tilde{s}_i \circ \tilde{f}$  is the composition of two surjective maps, hence surjective.  $\square$

We have  $\text{codim } f^* \mathcal{F} = \text{codim } \mathcal{F}$  and  $T f^* \mathcal{F} = (T f)^{-1}(T \mathcal{F})$ .

### 1.3.3 Quotient foliations of covering space actions

Let  $(M, \mathcal{F})$  be a foliated manifold such that  $M$  carries a free, properly discontinuous action of a discrete Lie group  $G$ . Then the quotient manifold  $M/G$  is Hausdorff. Assume that  $\mathcal{F}$  is *invariant* under the  $G$ -action, that is, every  $g \in G$  is an automorphism of the foliation  $(M, \mathcal{F})$ . Then there is an induced foliation  $\mathcal{F}/G$  on  $M/G$  as follows.

Since  $G$  acts freely and properly discontinuously, the quotient map  $\pi: M \rightarrow M/G$  defines a covering space. Let  $\{(U_i, \varphi_i: U_i \rightarrow \mathbb{R}^{n-q} \times \mathbb{R}^q)\}_i$  be a foliation atlas of  $\mathcal{F}$ . Then  $\pi$  is a local diffeomorphism. We may assume that  $\pi|_{U_i}$  is a diffeomorphism by replacing  $U_i$  by its refinement. Then it has an inverse section  $s_i$  and

$$\{(\varphi_i \circ s_i: \pi(U_i) \rightarrow \mathbb{R}^{n-q} \times \mathbb{R}^q)\}_i$$

renders a foliation  $\mathcal{F}/G$  for  $M/G$ . The leaves of  $\mathcal{F}/G$  are quotient manifolds  $L/G_L$ , where  $L$  is a leaf of  $\mathcal{F}$  and  $G_L$  is the isotropy subgroup of the leaf  $L$ . The latter is well-defined because every  $g \in G$  maps a leaf into a leaf.

We have  $\text{codim}(\mathcal{F}/G) = \text{codim}(\mathcal{F})$  and  $T(\mathcal{F}/G) = T\pi(T\mathcal{F})$ .

*Example 1.17* (Kronecker foliation). Let  $\vartheta \in \mathbb{R} \setminus \mathbb{Q}$  and define the submersion

$$s: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad s(x, y) = x - \vartheta y.$$

Then it defines a foliation on  $\mathbb{R}^2$  following Example 1.6. The group  $\mathbb{Z}^2$  acts on  $\mathbb{R}^2$  freely and properly discontinuously, and the aforementioned foliation on  $\mathbb{R}^2$  is  $\mathbb{Z}^2$ -invariant. So there is an induced foliation  $\mathcal{F}$  on the quotient manifold  $\mathbb{T}^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$ . This is known as the *Kronecker foliation* on  $\mathbb{T}^2$ . Equivalently, one can consider the integral curves generated by the differential equation  $\frac{dx}{dy} = \vartheta$ .

*Example 1.18* (Foliation of the Möbius band). Recall that the Möbius bundle is the quotient space of  $\mathbb{R}^2$ :

$$\text{Möb} := \mathbb{R}^2/\sim, \quad (x, y) \sim (x + 1, -y).$$

The projection  $\mathbb{R}^2 \rightarrow \text{Möb}$  is a two-fold covering space. The trivial codimension-1 foliation of  $\mathbb{R}^2$  is invariant under this group action, hence there is a quotient foliation of Möb. The interesting property of this foliation is that every leaf (which is diffeomorphic to  $\mathbb{T}$ ) will wrap around the whole band twice except for the middle one generated by the plaques with  $y = 0$ .

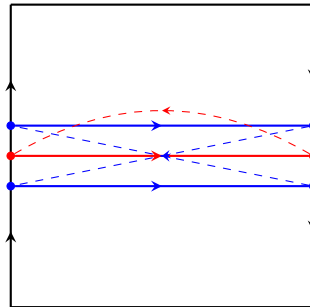


Figure 1.1: A foliation of the Möbius band

### 1.3.4 Suspensions

Suspensions of diffeomorphism groups are special cases of the quotient foliation. They constitute a large class of interesting foliations.



Let  $F$  be a smooth manifold and  $f : F \rightarrow F$  be an automorphism. Consider the trivial 1-dimensional foliation on  $M = \mathbb{R} \times F$ , i.e. the leaves are  $\mathbb{R} \times \{x\}$  for  $x \in F$ . There is a  $\mathbb{Z}$ -action on  $M$  generated by the diffeomorphism

$$(t, x) \mapsto (t + 1, f(x)).$$

The action is free and proper, and leaves the foliation on  $M$  invariant. So there is a quotient foliation on the quotient manifold  $\mathbb{R} \times_{\mathbb{Z}} F$ . This is called the *suspension* of the diffeomorphism  $f$ , or the suspension of the group  $\mathbb{Z}$  generated by  $f$ .

### 1.3.5 Flat bundles

The suspensions are a special case of flat bundles. Let  $M$  be a manifold and  $\tilde{M}$  be the universal cover of  $M$ . Then  $\tilde{M} \rightarrow M$  is a (right) principal  $\pi_1(M)$ -bundle. Assume that  $\pi_1(M)$  acts on another manifold  $F$ . Then there is an associated fibre bundle

$$\tilde{M} \times_{\pi_1(M)} F \rightarrow M.$$

The space  $\tilde{M} \times_{\pi_1(M)} F$  is the quotient of  $\tilde{M} \times F$  under the equivalence relation  $(x, y) \sim (xg^{-1}, gy)$ .

The submersion  $\tilde{M} \times F \rightarrow F$  generates a foliation of  $\tilde{M} \times F$  which is invariant under the action of  $\pi_1(M)$ . So there is a quotient foliation on  $\tilde{M} \times_{\pi_1(M)} F$ . This is called a *flat bundle*. In contrast to the foliation coming from a submersion, in which every leaf is vertical in the sense that  $T\mathcal{F} \subseteq \ker T\pi$ , in a flat bundle all leaves are horizontal, that is,  $T_x\mathcal{F} \simeq T_{\pi(x)}M$ .

The suspension is the special case where  $M = \mathbb{T}$  and the action of  $\pi_1(M) = \mathbb{Z}$  on  $F$  is generated by the diffeomorphism  $f$ .

### 1.3.6 Reeb foliation

Foliations can be defined on manifolds with boundary as well. But this requires that the foliation behaves nicely near the boundary, that is, the foliation is either *transverse* to the boundary or *tangent* to the boundary.

**Definition 1.19** ([3, Definition 1.1.11]). Let  $(M, \mathcal{F})$  be a foliated manifold, and  $N \subseteq M$  be a submanifold. We say that:

- $\mathcal{F}$  is *transverse* to  $N$ , denoted by  $\mathcal{F} \pitchfork N$ , if every leaf of  $\mathcal{F}$  is transverse to  $N$ . That is,

$$T_x M = T_x \mathcal{F} + T_x N$$

for every  $x \in N$ .

- $\mathcal{F}$  is *tangent* to  $N$ , if  $T_x \mathcal{F} \subseteq T_x N$  for every  $x \in N$ . Equivalently this means every leaf  $L$  of  $\mathcal{F}$  is either disjoint from  $N$ , or contained in  $N$ .

The *Reeb foliation* of the solid torus  $X = \mathbb{D}^2 \times \mathbb{T}$  is a foliation on a manifold with boundary, of the second sort. Let

$$\mathbb{D}^2 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

be the unit disk. Foliate the solid cylinder  $\mathbb{D}^2 \times \mathbb{R}$  by the submersion

$$f : \text{Int}(\mathbb{D}^2) \times \mathbb{R} \rightarrow \mathbb{R}, \quad f(x, y, t) = \exp\left(\frac{1}{1 - x^2 - y^2}\right) - t$$

and require that the boundary of the solid cylinder is another leaf. This gives a foliation of the solid cylinder.

The translation action of  $\mathbb{Z}$  on the second entry of  $\mathbb{D}^2 \times \mathbb{R}$  preserves this foliation. So there is a quotient foliation on solid torus  $X = \mathbb{D}^2 \times \mathbb{R}/\mathbb{Z} \simeq \mathbb{D}^2 \times \mathbb{T}$ . This is the *Reeb foliation* on the solid torus. Note that every leaf is diffeomorphic to  $\mathbb{R}^2$  except for the boundary leaf  $\mathbb{T}^2$ .

Now we describe the Reeb foliation on  $\mathbb{S}^3$ . A statement from topology says that  $\mathbb{S}^3$  can be obtained by gluing together two copies of the solid torus along the boundary  $\mathbb{T}^2$ :

$$\mathbb{S}^3 = X \cup_{\partial X} X, \quad X = \mathbb{D}^2 \times \mathbb{T}, \quad \partial X = \mathbb{T}^2.$$

The Reeb foliation can be glued together to form a foliation on  $\mathbb{S}^3$  as well. This does happen in general: one needs certain “tameness” conditions in order to glue two foliations together along the boundary; we shall, however, not talk about this in detail. Then we obtain the *Reeb foliation* on  $\mathbb{S}^3$ . It has all leaves diffeomorphic to  $\mathbb{R}^2$  except for the “boundary” leaf of solid cylinder, which is compact.

The importance of the Reeb foliation is illustrated by the following celebrated theorem:

**Theorem 1.20** (Novikov’s compact leaf theorem). *Every codimension-1 foliation of  $\mathbb{S}^3$  has a compact leaf, bounding a solid torus with the Reeb foliation.*

So the Reeb foliation plays an important role in the foliation theory of  $\mathbb{S}^3$ .

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## Holonomy and stability

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### 2.1 Holonomy

#### 2.1.1 Motivation: Poincaré return map

The idea of holonomy goes back to the Poincaré return map. Let  $(\phi, M, \mathbb{R})$  be a global dynamical system, where

$$\phi : \mathbb{R} \times M \rightarrow M$$

is the flow. Suppose that  $\gamma$  is a periodic orbit, that is, there exists a time  $T$  such that  $\phi(T, x) = x$  for every  $x \in \gamma$ . The orbit is thus a circle. One is then motivated to study the property of this closed orbit. The idea of Poincaré is to look at how the points near this closed orbit behave with time. This is done through the Poincaré section and the Poincaré return map.

Let  $p \in \gamma$  and let  $S$  be a local smooth section through  $p$  which is transverse to the flows of  $\phi$ . Such an  $S$  is called a *Poincaré section*. A function  $P : U \rightarrow S$  from an open and connected neighbourhood  $U \subseteq S$  of  $p$  to  $S$  is called a *Poincaré map*, if the followings hold:

- $P(p) = p$ .
- $P(U)$  is diffeomorphism to  $U$ .
- For every  $x \in U$ ,  $P(x) = \phi(t_0, x)$ , where  $t_0 = \min\{t \in \mathbb{R} \mid \phi(t, x) \in S\}$ . Namely,  $t_0$  is the first time that  $x$  returns to  $S$ .

The idea of holonomy is reminiscent of that of the Poincaré return maps, which describes the behaviour of all the nearby leaves of a given leaf.

#### 2.1.2 The holonomy map over a path

Let  $(M, \mathcal{F})$  be a foliated manifold. Let  $\alpha$  be a smooth path in a leaf  $L$  of  $\mathcal{F}$  from  $x$  to  $y$ . That is,

$$\alpha : [0, 1] \rightarrow L, \quad \alpha(0) = x, \quad \alpha(1) = y.$$

We wish to define the holonomy map as a local map “along” the leaves, “over” the path  $\alpha$ , which takes  $x$  to  $y$ . More precisely, it is defined as an equivalence class of smooth maps called a germ:

**Definition 2.1.** Let  $M$  and  $N$  be smooth manifolds. Let  $x \in M$  and  $y \in N$ . A *germ of smooth maps* from  $x$  to  $y$  is an equivalence class of smooth maps

$$f : U \rightarrow V, \quad \text{satisfying } f(x) = y$$

where  $U$  is an open neighbourhood of  $x$  and  $V$  is an open neighbourhood of  $y$ . Two smooth maps  $f : U \rightarrow V$  and  $f' : U' \rightarrow V'$  are equivalent as germs if there exists an open neighbourhood  $W \subseteq U \cap U'$  of  $x$  such that  $f|_W = f'|_W$ . We write  $\text{germ}(f)$  for the germ of  $f$ .

Similarly, one can define a *germ of local diffeomorphisms* from  $x$  to  $y$ , by requiring in addition that  $f$  is a local diffeomorphism, that is, a diffeomorphism from an open neighbourhood of  $x$  to its image. Clearly, if  $f$  is a local diffeomorphism, then the germ (of diffeomorphisms) of  $f$  is the same thing as the germ of smooth maps of  $f$ .

Let  $x \in M$ . Denote the set of germs of local diffeomorphisms  $f : (M, x) \rightarrow (M, x)$  by  $\text{Diff}_x(M)$ . Clearly this is a group under the composition of smooth maps.

The holonomy of a leaf  $L$  along a path  $\alpha : [0, 1] \rightarrow L$  will be defined to be a germ of smooth maps from  $x = \alpha(0)$  to  $y = \alpha(1)$ . In order to do that, a choice of transverse section across  $x$ , resp.  $y$ , is needed.

**Definition 2.2.** Let  $(M, \mathcal{F})$  be a foliated manifold of codimension  $q$ . A *transverse section*, or a *transversal*, is a  $q$ -dimensional submanifold of  $M$  which is transverse to  $\mathcal{F}$  (Definition 1.19).

**Definition and Lemma 2.3.** Let  $L$  be a leaf of  $(M, \mathcal{F})$  and  $\alpha : [0, 1] \rightarrow L$  be a smooth path in  $L$ . Let  $x = \alpha(0)$  and  $y = \alpha(1)$ . Let  $T$  and  $S$  be transverse sections of  $(M, \mathcal{F})$  such that  $x \in T$  and  $y \in S$ .

The *holonomy map*  $\text{hol}^{S,T}(\alpha)$  of the leaf  $L$  along the path  $\alpha$ , with respect to the transversals  $T$  and  $S$ , is defined as follows.

1. Assume first that  $\alpha([0, 1])$  is contained in a single foliation chart. Then  $x$  and  $y$  belong to the same plaque. *There exists a smooth map  $f : A \rightarrow S$ , where  $A \subseteq U$  is an open subset containing  $x$ , such that:*

- $f(x) = y$ ;
- For any  $x' \in A$ ,  $f(x')$  and  $x'$  belongs to the same plaque.
- $f$  is a local diffeomorphism at  $x$ .

*In particular, the germ of  $f$  is uniquely determined.* Then we define the holonomy map as the germ of  $f$ :

$$\text{hol}^{S,T}(\alpha) := \text{germ}(f).$$

2. In the general case, *there exists a finite sequence of foliation charts  $\{U_0, \dots, U_k\}$  such that  $\alpha([\frac{i}{k}, \frac{i+1}{k}]) \subseteq U_i$ . Set  $x_i = \alpha(\frac{i}{k})$  and  $\alpha_i$  to be the restriction of  $\alpha$  to  $[\frac{i}{k}, \frac{i+1}{k}]$ . Choose transversals  $T = T_0, T_1, \dots, T_n = S$  of  $\mathcal{F}$  at  $x = x_0, x_1, \dots, x_n = y$ . Define*

$$\text{hol}^{S,T}(\alpha) := \text{hol}^{T_k, T_{k-1}}(\alpha_{k-1}) \circ \text{hol}^{T_{k-1}, T_{k-2}}(\alpha_{k-2}) \circ \dots \circ \text{hol}^{T_1, T_0}(\alpha_0).$$

*In particular,  $\text{hol}^{S,T}(\alpha)$  does not depend on the intermediate transversals  $T_1, \dots, T_{k-1}$ .*

Detailed proofs of the statements above can be found in [2, Chapter IV], which we omit here. To understand this construction, let us work in a foliation chart  $(U, \varphi : U \rightarrow \mathbb{R}^{n-q} \times \mathbb{R}^q)$  centered at  $x$ . Then  $x$  belongs to the plaque  $\varphi^{-1}(\mathbb{R}^{n-q} \times \{0\})$ . A transverse section  $S$  at  $x$  has dimension  $q$  and intersects  $T_x \mathcal{F}$  transversely, which means that

$$T_x M = T_x S \oplus T_x \mathcal{F}.$$

Therefore, locally  $S$  intersects the leaf of  $x$ , and *every leaf* in a neighbourhood of  $x$ , at a single point. We may assume that  $U$  is small enough so that  $S$  intersects every leaf in  $U$  at a unique point. Then  $S$  is given by  $\varphi^{-1}(\{0\} \times \mathbb{R}^q)$  up to a coordinate change.

Now assume that  $\alpha: [0, 1] \rightarrow L$  is contained in a single chart  $U$ . Assume without loss of generality that the transverse sections  $T$  at  $x = \alpha(0)$  and  $S$  at  $y = \alpha(1)$  intersect the every leaf in  $U$  at a unique point. Then the holonomy map  $\text{hol}^{S,T}(\alpha)$  is defined as follows: for any  $x' \in T$ , which belongs to a leaf  $L'$ . Then  $\text{hol}^{S,T}(\alpha)(x')$  is the unique point that belongs to the intersection  $S \cap L'$ .

These process can be done for paths which do not lie in a unique foliation chart as well, whereas the *holonomy groups* become significant as they might be non-trivial.

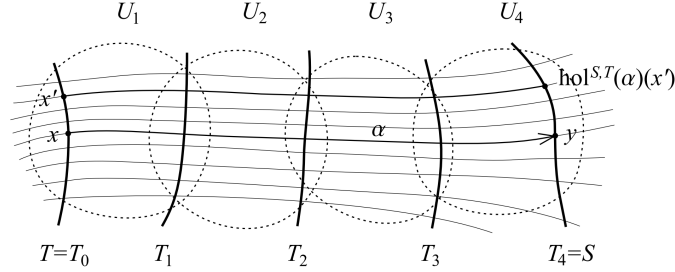


Figure 2.1: Holonomy maps. This figure is taken from [6, Fig 2.1].

**Proposition 2.4.** *The holonomy maps have the following properties:*

1. If  $\alpha$  is a path in  $L$  from  $x$  to  $y$ , and  $\beta$  is a path in  $L$  from  $y$  to  $z$ . Let  $T, S, R$  be transverse sections of  $\mathcal{F}$  at  $x, y, z$ . Then

$$\text{hol}^{R,T}(\beta\alpha) = \text{hol}^{R,S}(\beta) \circ \text{hol}^{S,T}(\alpha).$$

2. Homotopy invariance. If  $\alpha$  and  $\alpha'$  are paths which are homotopic in a leaf with basepoints fixed. Then  $\text{hol}^{S,T}(\alpha) = \text{hol}^{S,T}(\alpha')$ .

3. Let  $\alpha$  be a path in  $L$  from  $x$  to  $y$ . Let  $T, T'$  be transverse sections at  $x$  and  $S, S'$  be transverse sections at  $y$ . Then

$$\text{hol}^{S',T'}(\alpha) = \text{hol}^{S',S}(\bar{y}) \circ \text{hol}^{S,T}(\alpha) \circ \text{hol}^{T,T'}(\bar{x}),$$

where  $\bar{x}$  (resp.  $\bar{y}$ ) stands for the identity path at  $x$  (resp. at  $y$ ).

### 2.1.3 The honolomy group

**Definition 2.5.** Let  $(M, \mathcal{F})$  be a foliated manifold and  $L$  a leaf of  $\mathcal{F}$ . The *holonomy group* of  $L$  at  $x$  with respect to the transverse section  $T$ , denoted by  $\text{Hol}(L, T, x)$ , is defined as the image of the map

$$\text{hol}^{T,T}: \pi_1(L, x) \rightarrow \text{Diff}_x(T) \simeq \text{Diff}_0(\mathbb{R}^q).$$

It is clear (indeed, from 3 of the previous proposition) that a different choice of  $T$ , or a different choice of  $x \in L$ , results in a conjugacy of the holonomy map by a germ of diffeomorphism. Therefore we may define  $\text{Hol}(L)$  to be the conjugacy class of the holonomy maps in all of those  $\text{Hol}(L, T, x)$  with  $x \in L$  and  $T$  a transverse section at  $x$ .

The following result guarantees that it is sufficient to consider only  $\text{Im}(\text{hol}^{T,T})$ :

**Theorem 2.6** (Transverse uniformity of leaves, [2, §III.2, Theorem 3]). *Let  $L$  be a leaf of  $\mathcal{F}$  and  $x, y \in L$ . Then there exists transversals  $T$  at  $x$  and  $S$  at  $y$ , together with a local diffeomorphism  $f$  from an open subset  $U \subseteq T$  to  $S$ , such that for any leaf  $L'$ :*

$$f(L' \cap T) = L' \cap S.$$

Therefore, we only need to consider the first return map.

**Example 2.7.** If  $L$  is simply connected. Then  $\pi_1(L, x) = 0$  and hence  $\text{Hol}(L) = 0$ .

**Example 2.8** (Foliation of the Möbius band). A foliation of the Möbius band, as in Example 1.18, has two distinct classes of leaves: the middle leaf “wraps around” the base only once, whereas every other leaf wraps around the base twice. This can be made precise with holonomy.

Identify Möbius bundle with the quotient space of  $\mathbb{R}^2$ :

$$\text{Möb} := \mathbb{R}^2 / \sim, \quad (x, y) \sim (x + 1, -y).$$

Every leaf is diffeomorphic to  $\mathbb{S}^1$ , whose fundamental group  $\pi_1(\mathbb{S}^1) = \mathbb{Z}$  is generated by a loop of it with winding number 1. The holonomy group of a leaf is thus a quotient of  $\mathbb{Z}$ . For every other leaf  $L$  except for the middle one and choose  $x \in L$ . There exists a transverse section at  $x$  which does not cross the middle leaf  $L_{\text{mid}}$ . Those leaves which intersect the transverse section are leaves of a trivial foliation of  $\mathbb{S}^1 \times I$  for some open interval  $I \subseteq \mathbb{R}$  describing this transverse section. Thus  $\text{Hol}(L) = 0$ .

The middle leaf  $L_{\text{mid}}$  is the image of the line  $y = 0$  under the quotient map  $\mathbb{R}^2 \rightarrow \text{Möb}$ . Choose the basepoint to be  $(-1, 0) \in L_{\text{mid}}$  and the transverse section be the line  $x = -1$ . Then  $\pi_1(L_{\text{mid}})$  is generated by the straightline path  $\alpha$  connecting  $x$  and  $x' = (1, 0)$ , which becomes a loop of winding number 1 in Möb. Now for any other point  $(-1, -\varepsilon)$  in an open neighbourhood of  $(-1, 0)$ , the holonomy map over  $\alpha$  sends  $(-1, -\varepsilon)$  to  $(1, -\varepsilon) \sim (-1, \varepsilon)$ . This means that the generator of  $\pi_1(L_{\text{mid}})$  is sent to a generator of degree 2. This implies that  $\text{Hol}(L_{\text{mid}}) = \mathbb{Z}/2$ .

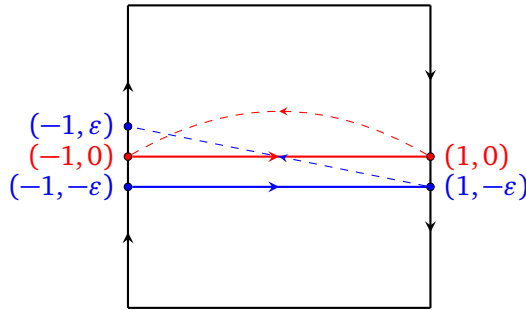


Figure 2.2: Holonomy of the middle leaf of the Möbius band

**Example 2.9** (Reeb foliation). Consider the Reeb foliation of the solid torus as in Example 1.3.6. All non-compact leaves (which are diffeomorphic to  $\mathbb{R}^2$ ) are contractible and have trivial holonomy group. The interesting case is the unique compact leaf  $L = \mathbb{T}^2$ . Identify a transverse section at  $x \in L$  with  $[0, +\infty)$  where  $x$  is identified with 0. Then  $\text{Hol}(L)$  is generated by  $f : [0, +\infty) \rightarrow [0, +\infty)$  with  $f(0) = 0$  and  $f(t) < t$  for all  $t > 0$ .

Likewise, the Reeb foliation of the unique compact leaf  $L$  of  $\mathbb{S}^3$  has holonomy group  $\mathbb{Z}^2$  generated by

$$\begin{aligned} f : \mathbb{R} \rightarrow \mathbb{R}, \quad & \begin{cases} f(t) = t, & t \leq 0; \\ f(t) < t, & t > 0. \end{cases} \\ h : \mathbb{R} \rightarrow \mathbb{R}, \quad & \begin{cases} h(t) = t, & t \geq 0; \\ h(t) > t, & t < 0. \end{cases} \end{aligned}$$

Here the transverse section at  $x \in L$  is identified with  $\mathbb{R}$ .

## 2.2 Stability

**Definition 2.10** ([8, Definition 4.2]). Let  $(M, \mathcal{F})$  be a foliated manifold. A subset  $B \subseteq M$  is called *stable*, if for every open neighbourhood  $W$  of  $B$  in  $M$ , there exists an open neighbourhood  $W' \subseteq W$  of  $B$  in  $M$ , such that every leaf intersecting  $W'$  is contained in  $W$ .

One of the classical problems in foliation theory is to find necessary and sufficient conditions for a leaf  $L \subseteq M$  to be stable. These are given by stability theorems. First of all, there is a close relation between stability and finiteness of the holonomy group.

*Example 2.11.* Consider the foliation of  $\mathbb{R}^2 \setminus \{(0,0)\}$  by cocentred circles  $\{(x,y) \mid x^2 + y^2 = r\}$ . Then each leaf is stable and compact, and each leaf has trivial holonomy group.

*Example 2.12.* The Reeb foliation of either the solid torus or of  $\mathbb{S}^3$  has a unique compact leaf. It is unstable, and has infinite holonomy group.

Indeed, we have the following sufficient condition.

**Theorem 2.13** ([8, Theorem 4.2]). *A compact leaf with finite holonomy group is stable.*

*Idea of proof.* Let  $L$  be a compact leaf with finite holonomy group  $\text{Hol}(L)$ . Choose a basepoint  $x \in L$  and a transverse section  $\Sigma$  at  $x$ . The following proof is an excerpt from the reference. I might complete it at some point.

- By compactness, one can find a finite set of foliation charts  $\{U_1, \dots, U_k\}$  covering the leaf.
- Since  $\text{Hol}(L, x)$  is finite and  $\{U_1, \dots, U_k\}$  is a finite cover, we are able to shrink the domain, so that all holonomy maps  $h_j$  are defined on a common open subset of the transverse section  $\Sigma$ .
- Take the union of all plaques of the form

$$C_y^* = \{\pi_j^{-1}(h_j(f_{\beta_i}(y))) \mid 1 \leq i \leq k, 1 \leq j \leq r\}$$

Then one can show that  $C_y^*$  is the leaf containing  $y$ .

- Finally, set

$$W' = \bigcup_{y \in D'} F_y \subseteq W.$$

This is the set required in the definition of stability. □

### 2.2.1 Riemannian foliation

Let  $M$  be an  $n$ -dimensional manifold. Denote by  $\mathfrak{X}(M)$  the space of smooth vector fields over  $M$ . A symmetric,  $C^\infty(M)$ -bilinear form

$$g : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow C^\infty(M)$$

is said to be *positive*, if

$$g(X, X) \geq 0, \quad \text{for all } X \in \mathfrak{X}(M).$$

Let  $x \in M$ . Set

$$\ker g_x := \{\xi \in T_x M \mid g_x(\xi, T_x M) = 0\}.$$

The *Lie derivative*  $\mathcal{L}_X g$  of  $g$  in the direction of  $X \in \mathfrak{X}(M)$  is the symmetric  $C^\infty$ -bilinear form given by

$$\mathcal{L}_X g(Y, Z) := X g(Y, Z) - g([X, Y], Z) - g(Y, [X, Z]).$$

**Definition 2.14.** Let  $(M, \mathcal{F})$  be a foliated manifold. A *transverse metric* on  $M$  is a positive, symmetric  $C^\infty(M)$ -bilinear form  $g$  such that:

1.  $\ker g_x = T_x \mathcal{F}$  for all  $x \in M$ .
2.  $\mathcal{L}_X g = 0$  for any vector field  $X$  on  $M$  which is tangent to  $\mathcal{F}$ .

A *Riemannian foliation*  $(\mathcal{F}, g)$  of a manifold  $M$  is a foliation  $\mathcal{F}$  of  $M$  together with a transverse metric  $g$ .

*Remark 2.15.* Note that

1. 1 in the previous definition implies that  $g$  is the pullback of a Riemannian structure on the normal bundle  $N\mathcal{F}$ , along the projection  $TM \rightarrow N\mathcal{F}$ .

2. 2 in the previous definition is a local condition: consider a foliation chart  $(U, \varphi : U \rightarrow \mathbb{R}^{n-q} \times \mathbb{R}^q)$  and set  $\varphi(x) = (x_1, \dots, x_{n-q}, y_1, \dots, y_q)$ . From 1 we know that  $g$  is determined by  $g_{ij} := g(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j})$ . One can show, after some computation, that 2 holds iff  $\frac{\partial g_{ij}}{\partial x_k} = 0$  for all  $k, i, j$ . Equivalently,  $g|_U$  is the pullback of a Riemannian metric on  $\mathbb{R}^2$  along the projection  $\text{pr}_{\mathbb{R}^q} \circ \varphi : U \rightarrow \mathbb{R}^q$ .

In fact, we have:

**Theorem 2.16.** *Let  $(M, \mathcal{F})$  be a foliated manifold. Then a Riemannian structure on  $N\mathcal{F}$  determines a transverse metric iff it is holonomy invariant.*

**Theorem 2.17** ([6, Theorem 2.6]). *Let  $(\mathcal{F}, g)$  be a Riemannian foliation of  $M$  and assume that all leaves are compact. Then each leaf has finite holonomy group.*

*Idea of proof.* Let  $L$  be a leaf of  $\mathcal{F}$ ,  $x \in L$ , and  $S$  a transverse section at  $x$ . The induced Riemannian structure of  $S$  yields an exponential map

$$\exp_x : B(0, \varepsilon) \rightarrow S, \quad \text{such that } \exp_x(0) = x,$$

where  $B(0, \varepsilon)$  is the  $\varepsilon$ -ball centred at 0 of  $T_x M$ . Denote the image of  $\exp_x$  by  $U$ .

Since  $L$  is compact,  $\pi_1(L)$  is finitely generated, and hence  $\text{Hol}(L)$  is finitely generated, say, by holonomy maps  $\{h_1, \dots, h_n\}$  where  $h_i : V_i \rightarrow U$  is a diffeomorphism onto the image. We may replace all  $V_i$ 's by a common domain  $V := \cap_{i=1}^n V_i$ . By Theorem 2.16, every  $h_i$  is an isometry. Thus we may assume that  $V = \exp_x(B(0, \delta))$  for some  $\delta \leq \varepsilon$  and that  $h_i(V) = V$  for all  $i$ . Then we have represented  $\text{Hol}(L)$  as a group of isometry on  $V$ .

Since  $L$  is compact,  $L \cap S$  is a discrete set. But  $U$ , and hence  $V$ , is relatively compact in  $S$ . Thus  $L \cap V \subseteq L \cap S$  is finite. Therefore the orbit of any element in  $\text{Hol}(L)$ , viewed as an action of a group of isometry, is finite. The differential of this representation at  $x$  is a group of orthogonal linear transformations on  $T_x V$ , such that every orbit of the action is finite. This forces the group  $\text{Hol}(L)$  to be finite.  $\square$

**Corollary 2.18.** *Let  $(\mathcal{F}, g)$  be a Riemannian foliation of  $M$ . If all leaves are compact, then all leaves are stable.*

## 2.2.2 Reeb stability

Finally, we have the following Reeb stability theorems as consequences of Theorem 2.13.

**Theorem 2.19** (Local Reeb stability). *Let  $(M, \mathcal{F})$  be a foliated manifold. Let  $L$  be a compact leaf with finite holonomy group. Then for each neighbourhood  $W$  of  $L$ , there is an  $\mathcal{F}$ -invariant tubular neighbourhood  $W' \subseteq W$  with*

$$\pi : W' \rightarrow L$$

satisfying:

1. Every leaf  $L' \subseteq W'$  is compact with finite holonomy group, hence stable.
2. If  $L' \subseteq W'$  is a leaf. Then the restriction  $\pi|_{L'} : L' \rightarrow L$  is a finite covering map.

**Theorem 2.20** (Global Reeb stability). *Let  $\mathcal{F}$  be a codimension-1 foliation of a compact connected manifold  $M$ . If  $\mathcal{F}$  admits a compact leaf  $L$  with finite fundamental group, then all leaves of  $\mathcal{F}$  are compact with finite fundamental group. So all leaves have finite holonomy group and hence stable.*

*If  $\mathcal{F}$  is transversely orientable, that is,  $N\mathcal{F} = TM/T\mathcal{F}$  is orientable. Then every leaf of  $\mathcal{F}$  is diffeomorphic to  $L$ , and  $M$  is the total space of a fibration  $f : M \rightarrow \mathbb{S}^1$  with fibre  $L$ .*



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