Groupoid C*-algebras

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Contents

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1 Groupoids: motivations, definitions and examples (Yuezhao, Sep 13)

1.1 Motivations

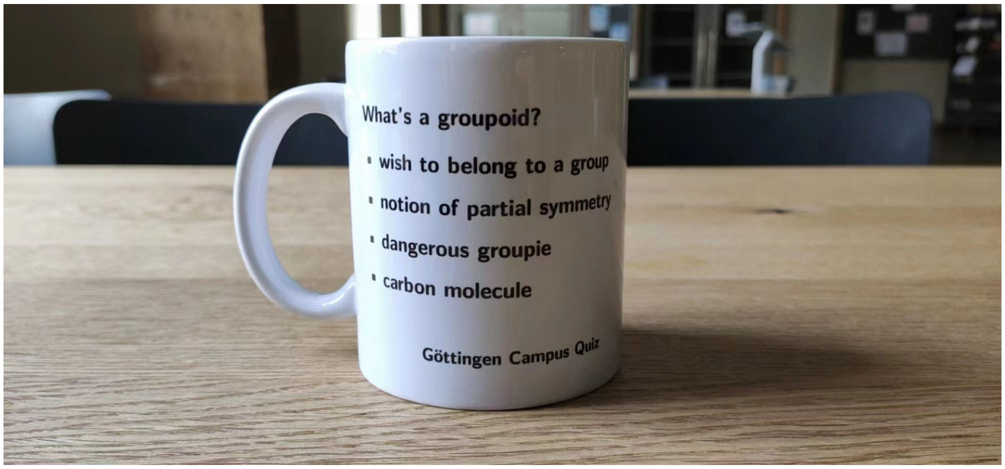

Figure 1: What is a groupoid?

1.1.1 What is a groupoid?

There are at least two answers. One way is to think of a groupoid as '*a generalised group, but the multiplication is only partially defined*. Another is to view a groupoid as *a group with more than one units*. For the second viewpoint, recall that a group can be understood as a category with only one object, and all of its arrows are invertible. Then the group elements correspond to the arrows, and the product of group elements correspond to the composition of arrows. A groupoid, on the other hand, has a set of objects and arrows, each object corresponding to a unit by identifying this object with the identity arrow associated to it.

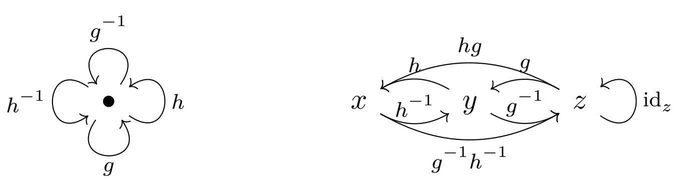

Figure 2: A group is a category with one object and all arrows are invertible. A groupoid is a small category whose all arrows are invertible.

Our main reference [Wil19] adopts the first viewpoint, but I feel that the second is more concise, and even more useful in many cases, too. The source and range maps s and r have explicit geometric meanings in the second picture. This is not only for convenience, but also crucial while working with Lie groupoids: in the Lie groupoid the source and range maps are required to be surjective submersions, and the unit space is required to be a smooth manifold. I also do not quite agree with [Wil19, Remark 1.7] where the author calls the categorical viewpoint an abstract nonsence: this seems to be a misuse of the word. An abstract nonsense is a formal proof based on techniques from category theory, usually not specific to a fixed context. So I would say an abstract nonsense is a *method*, and a definition *cannot* be an abstract nonsense. — Commented by Y. Li

1.1.2 Why do we study groupoids?

Before going into the details, it is worthwhile to explain why we care about groupoids. A short answer is that groupoids interact strongly with other fields like dynamical systems and differential geometry, and they themselves are also vital as geometric models of noncommutative spaces.

Dynamical systems and groupoids . Starting from a dynamical system, you can usually construct a groupoid. Depending on the type of the dynamical system (measurable, topological, smooth, ...) you have different structures of the groupoid (Borel, topological, Lie, ...). This groupoid tells you the information of the original dynamical system. For example, let X be a set and G be a group. It is granted that there is a one-to-one correspondence between

$$\text{transitive } G\text{-actions on } X \qquad \text{and} \qquad \text{conjugate classes of subgroups of } G.$$

Now if we work with a Borel group G and a Borel space X. Can we say something about the ergodic actions of G on X? This shall be more difficult than transitive actions, becase ergodic actions involve the data of both the group and the space. Then one needs to construct a groupoid $G \ltimes X$, the *action* groupoid. This is a Borel groupoid, whose Borel structure comes from those of G and X. The answer is that, an ergodic action of G on X corresponds precisely to an *ergodic groupoid*.

Dynamics, topological groupoids and groupoid C^*-algebras . If we start with a topological dynamical system, say, a locally-compact topological group G acting on a locally-compact space X. Then $G \ltimes X$ is a topological groupoid, and we can construct C^*-algebras $\mathrm{C}^*(G \ltimes X)$ and $\mathrm{C}^*_{\mathrm{r}}(G \ltimes X)$, the full and reduced groupoid C^*-algebra of the groupoid $G \ltimes X$. These C^*-algebras are isomorphic to the full and reduced crossed product C^*-algebras, and encode a lot of data of the dynamics: for example, $\mathrm{C}^*_{\mathrm{r}}(G \ltimes X)$ is simple iff the action of G on X is topological free and minimal.

Groupoids and noncommutative spaces . Another important reason to study groupoids is that they are viewed as geometric models for *noncommutative spaces*. Recall that a locally-compact Hausdorff space X corresponds to a commutative C^*-algebra $\mathrm{C}_0(X)$: this is the well-known Gelfand duality. Then noncommutative C^*-algebras play the role of "non-commutative spaces" in the algebraic setting. But sometimes it is desirable to find geometric models of noncommutative spaces. Of course, they cannot be the usual topological spaces because $\mathrm{C}_0(X)$ is always commutative. One attempt is to seek a topological groupoid $\mathcal{G}$, such that its groupoid C^*-algebra is isomorphic to the noncommutative C^*-algebra that we start with. Then $\mathcal{G}$ is a good geometric model for our noncommutative space. If we view a topological space X as a groupoid (see Example 1.15), then its groupoid C^*-algebra is precisely $\mathrm{C}_0(X)$, complying nicely with the classical Gelfand theory.

Foliation, groupoids and index theory . Index theory studies the connection between indices of (pseudo)differential operators and the topology or geometry of the spaces they live in. One of the most celabrated index theorem is the Atiyah–Singer index theorem. The family index theorem is a variant of the Atiyah–Singer index theorem. The set-up is a fibration $E \to X$ over a compact base, and a family of elliptic operators $\{D_x\}_{x\in X}$ parametrised by X, and such that each D_x acts on the vertical fibre E_x. The family index theorem computes the index of the family $\mathrm{Index}(D_x)$, which is an element in $\mathrm{K}^0(X)$, the K-theory of X.

Fibrations are special cases of *foliations*, and one might wish to generalise the theory to arbitrary foliations. However, foliations can be quite well-behaved in general. For example, consider the *Kronecker foliation* of $\mathbb{T}^2 \cong \mathbb{R}^2/2\pi\mathbb{Z}$ defined by the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \theta$. If θ is rational. Then every orbit (leaf) is closed and homeomorphic to a circle. However, if θ is irrational, then every leaf is dense in $\mathbb{T}^2$, and the leaf space with the quotient topology is homeomorphic to a single point. This makes the family index theorem useless.

The problem arises because the leaf space is badly-behaved. The solution is to replace this space by a "noncommutative space" — the *foliation groupoid*. Thus the K-theory of the C*-algebra of the foliation groupoid becomes a nice receptacle of the family index. This is the now well-known longitudinal index theorem of Connes and Skandalis [CS84].

1.2 Groupoids

Definition 1.1 (First definition of groupoids)**.** A groupoid is a set $\mathcal{G}$ together with:

- a set $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$ of "composable arrows";
- a "multiplication map" $\mathcal{G}^{(2)} \to \mathcal{G}$, $(a, b) \mapsto ab$;
- an "inverse" map $\mathcal{G} \to \mathcal{G}$, $a \mapsto a^{-1}$,

such that:

1. (Associativity) If $(a, b) \in \mathcal{G}^{(2)}$ and $(b, c) \in \mathcal{G}^{(2)}$. Then (ab, c), $(a, bc) \in \mathcal{G}^{(2)}$ and $(ab)c = a(bc)$.
2. (Involutivity) $(a^{-1})^{-1} = a$.
3. (Unit) For any $a \in \mathcal{G}$, $(a^{-1}, a) \in \mathcal{G}^{(2)}$; if $(a, b) \in \mathcal{G}^{(2)}$, then $abb^{-1} = a$ and $a^{-1}ab = b$.

The unit axiom asserts that, unlike a group, a groupoid can have many (one-sided) units; the units of a groupoid $\mathcal{G}$ forms a subset $\mathcal{G}^{(0)} \subseteq \mathcal{G}$, which comes together with a pair of maps $s, r \colon \mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$.

Definition 1.2. Let $\mathcal{G}$ be a groupoid.

- The *unit space* $\mathcal{G}^{(0)}$ of $\mathcal{G}$ is
$$\mathcal{G}^{(0)} := \{a \in \mathcal{G} \mid a = a^{-1} = a^2\}.$$
- The *source* map of $\mathcal{G}$ is $s \colon \mathcal{G} \to \mathcal{G}^{(0)}$, $a \mapsto a^{-1}a$.
The *range* map of $\mathcal{G}$ is $r \colon \mathcal{G} \to \mathcal{G}^{(0)}$, $a \mapsto aa^{-1}$.

We have the following:

Lemma 1.3. *Let $\mathcal{G}$ be a groupoid.*

1. $\mathcal{G}^{(0)} = \{aa^{-1} \mid a \in \mathcal{G}\}$.
2. $\mathcal{G}^{(2)} = \{(a, b) \in \mathcal{G} \times \mathcal{G} \mid s(a) = r(b)\}$.
3. *If $a, b \in \mathcal{G}$ and $(a, b) \in \mathcal{G}^{(2)}$. Then:*
$$s(a) = r(a^{-1}), \qquad s(ab) = s(b), \qquad r(ab) = r(a),$$
$$(b^{-1}, a^{-1}) \in \mathcal{G}^{(2)}, \qquad b^{-1}a^{-1} = (ab)^{-1}.$$

Proof.

- Clearly $aa^{-1} \in \mathcal{G}^{(0)}$ for any $a \in \mathcal{G}$. If $a \in \mathcal{G}^{(0)}$, then $a = a^2 = aa^{-1}$.
- If $s(a) = r(b)$, then $a^{-1}a = bb^{-1}$. Since $(a, a^{-1}), (a^{-1}, a), (b^{-1}, b), (b, b^{-1}) \in \mathcal{G}^{(2)}$. We have $(a, a^{-1}a) = (a, bb^{-1}) \in \mathcal{G}^{(2)}$ and $(bb^{-1}, b) \in \mathcal{G}^{(2)}$. Then $(a, bb^{-1}b) = (a, b) \in \mathcal{G}^2$. Conversely, if $(a, b) \in \mathcal{G}^{(2)}$. Since $(a^{-1}, a), (b, b^{-1}) \in \mathcal{G}^{(2)}$, the product $a^{-1}abb^{-1}$ makes sense, which equals both bb^{-1} and $a^{-1}a$.

- The three equations in the first line can be quickly checked. If $(a,b) \in \mathcal{G}^{(2)}$. Then $r(a^{-1}) = s(a) = r(b) = s(b^{-1})$ and hence $(b^{-1}, a^{-1}) \in \mathcal{G}^{(2)}$. The product $b^{-1}a^{-1}ab(ab)^{-1}$ makes sense and equals both $b^{-1}a^{-1}$ and $(ab)^{-1}$. $\square$

Remark 1.4. The previous lemma states that, a groupoid can equivalently be described by the data $(\mathcal{G}, \mathcal{G}^{(0)}, s, r, ^{-1})$. This leads to an alternative definitions of groupoids.

Definition 1.5 (Second definition of groupoids)**.** A groupoid is a set $\mathcal{G}$ together with:

- a distinguished subset $\mathcal{G}^{(0)} \subseteq \mathcal{G}$;
- a pair of maps $r, s\colon \mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$;
- a map $\mathcal{G}^2 \to \mathcal{G}$, $(a,b) \mapsto ab$, where

$$\mathcal{G}^{(2)} := \{(a,b) \in \mathcal{G} \times \mathcal{G} \mid s(a) = r(b)\};$$

- a map $\mathcal{G} \to \mathcal{G}$, $a \mapsto a^{-1}$,

such that

1. $r(x) = x = s(x)$ for all $x \in \mathcal{G}^{(0)}$.
2. $r(a)a = a = as(a)$ for all $a \in \mathcal{G}$.
3. $r(a^{-1}) = s(a)$ for all $a \in \mathcal{G}$.
4. $s(a) = a^{-1}a$ and $r(a) = aa^{-1}$ for all $a \in \mathcal{G}$.
5. $r(ab) = r(a)$ and $s(ab) = s(b)$ for all $(a,b) \in \mathcal{G}^{(2)}$.
6. $(ab)c = a(bc)$ whenever $s(a) = r(b)$ and $s(b) = r(c)$.

In this definition, the roles of the range and source maps are highlighted: this is actually more important if we want to study a topological groupoid or a Lie groupoid. I feel that it is sometimes more convenient to denote a groupoid by a diagram $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} \mathcal{G}^{(0)}$.

Example 1.6.

1. A group G is a groupoid $G \rightrightarrows \mathrm{pt}$.
2. A set X is a groupoid $X \overset{\mathrm{id}}{\underset{\mathrm{id}}{\rightrightarrows}} X$ together with the trivial multiplication and inverse maps.
3. Group bundle. A group bundle consists of two sets E, X and a surjective map $\pi\colon E \twoheadrightarrow X$ such that $\pi^{-1}(x)$ is a group for every $x \in X$. A group bundle can be viewed as a groupoid $E \overset{\pi}{\underset{\pi}{\rightrightarrows}} X$. In particular: a vector bundle is a groupoid.
4. *Action groupoid.* Let X be a (left) G-set. That is, G acts on X on the left. The action groupoid $G \ltimes X \rightrightarrows X$ is defined as follows. We set $G \ltimes X := G \times X$ as a set and $(G \ltimes X)^{(0)} := X$. The source, range, multiplication and inverse maps are
$$s(g,x) := x, \qquad r(g,x) := g \cdot x, \qquad (h, gx)(g,x) := (hg, x), \qquad (g,x)^{-1} := (g^{-1}, gx).$$
5. *Pair groupoid.* Let X be a set. The pair groupoid is given by $X \times X \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} X$. The multiplication map is given by $(x,y)(y,z) := (x,z)$ and the inverse map is $(x,y)^{-1} := (y,x)$.

6. *Equivalence relations.* Let X be a set and $\mathcal{R} \subseteq X \times X$ be an equivalence relation on X. Then $\mathcal{R} \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X$ is a groupoid, with multiplication $(x, y)(y, z) := (x, z)$ and inverse $(x, y)^{-1} := (y, x)$.
   - If we set $\mathcal{R} := X \times X$, then we recover the pair groupoid as a special case. If we set $\mathcal{R} := \emptyset$, then we recover the groupoid $X \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightrightarrows}} X$.
   - Let $\mathcal{G}$ be any groupoid. We can define an equivalence relation on $\mathcal{G}^{(0)}$ by claiming two units are equivalent iff they are connected by a groupoid element. Equivalently, this is the subset $\mathcal{R}(\mathcal{G}) := \{(r(a), s(a)) \mid a \in \mathcal{G}\} \subseteq \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$. Thus we obtain a groupoid $\mathcal{R}(\mathcal{G}) \rightrightarrows \mathcal{G}^{(0)}$. We say a groupoid $\mathcal{G}$ is *principal* if $\mathcal{G}$ is isomorphic to $\mathcal{R}(\mathcal{G})$ as a groupoid. Equivalently, this means there exists at least one arrow between two units in $\mathcal{G}$.

7. *Fundamental groupoid.* Let X be a topological space and $x \in X$. An important invariant in algebraic topology is the *fundamental group* of X (with basepoint x), defined as the group of (basepoint-fixed homotopy) equivalence classes of loops in X with basepoint x:
$$\pi_1(X, x) := \frac{\{\text{Loop } \gamma \text{ in } X \mid \gamma(0) = \gamma(1) = x\}}{\gamma \sim \gamma' \text{ iff } \gamma \text{ and } \gamma' \text{ are homotopic with basepoint fixed}}.$$
   This definition is not completely satisfying due to the following issues. If X is not path-connected, given $x, y \in X$, $\pi_1(X, x)$ and $\pi_1(X, y)$ may not be isomorphic if x and y do not lie in the same path component. If X is path-connected, then X has a unique path component and a different basepoint gives rise to an isomorphic fundamental group. However, the isomorphism between these two groups depend on the choice of the basepoints and on the specified path connecting them, hence not canonical.

   It is desirable to obtain a mathematical object similar to the fundamental group but does not depend on the choice of a basepoint. A natural idea is to choose (equivalence classes of) paths instead of loops. Unlike loops which are based at a certain point, paths are not concatenatable, unless the starting point of one coincides with the endpoint of another. This is precisely the axiom of a groupoid. So we may define the fundamental groupoid of X as:
$$\Pi_1(X) := \frac{\{\text{Path } \gamma \text{ in } X\}}{\gamma \sim \gamma' \text{ iff } \gamma \text{ and } \gamma' \text{ are homotopic with basepoints fixed}}.$$
   The fundamental groupoid is an important object in algebraic topology.

8. *Tangent groupoid.* Tangent groupoids were introduced by Alain Connes as an approach to study index theory. We briefly mention his construction. The interplay between tangent groupoids and index theory will be discussed in a future talk.

   Let M be a smooth manifold. The tangent groupoid of M is the groupoid
$$\mathbb{T}M := TM \times \{0\} \coprod M \times M \times (0, 1] \underset{s}{\overset{r}{\rightrightarrows}} M \times [0, 1],$$
   where
$$\begin{aligned} r(x, v, 0) &= (x, 0), \qquad & s(x, v, 0) &= (x, 0); \\ r(x, y, \epsilon) &= (x, \epsilon), \qquad & s(x, y, \epsilon) &= (y, \epsilon), \qquad \epsilon \in (0, 1]. \end{aligned}$$

Definition 1.7 (Subgroupoids)**.** A subgroupoid of a groupoid is a subset $\mathcal{H} \subseteq \mathcal{G}$ such that, the multiplication and inverse maps of $\mathcal{G}$ restricted to $\mathcal{H}$ turns it into a groupoid.

Definition 1.8 (Groupoid homomorphism)**.** A (strict) groupoid homomorphism $f\colon \mathcal{G} \to \mathcal{H}$ is a map such that $f \times f(\mathcal{G}^{(2)}) \subseteq \mathcal{H}^{(2)}$ and $f(ab) = f(a)f(b)$ for all $(a,b) \in \mathcal{G}^{(2)}$. It is an *isomorphism* if there exists another groupoid homomorphism $g\colon \mathcal{H} \to \mathcal{G}$ such that $f \circ g = \mathrm{id}_{\mathcal{H}}$ and $g \circ f = \mathrm{id}_{\mathcal{G}}$.

Remark 1.9. In later talks we shall define another larger class of morphisms between groupoids called *groupoid correspondences*. For clarity we will frequently refer to groupoid homomorphisms as *strict homomorphisms*.

Definition 1.10.

- Let $x, y \in \mathcal{G}^0$. We define the *source fibre at y* to be $\mathcal{G}_y := s^{-1}(y)$, the *range fibre at x* to be $\mathcal{G}^x := r^{-1}(x)$, and $\mathcal{G}_y^x := \mathcal{G}^x \cap \mathcal{G}_y$.
- Let $A, B \subseteq \mathcal{G}^{(0)}$. We define $\mathcal{G}_B := s^{-1}(\mathcal{G})$, $\mathcal{G}^A := r^{-1}(A)$ and $\mathcal{G}_B^A := \mathcal{G}^A \cap \mathcal{G}_B$.

Definition 1.11.

- Let $A \subseteq \mathcal{G}^{(0)}$. Then $\mathcal{G}_A^A \subseteq \mathcal{G}$ is a subgroupoid, called the restriction of $\mathcal{G}$ to A.
- Let $x \in \mathcal{G}^0$. Then $\mathcal{G}_x^x$ is a group, called the *isotropy group* at x.
- The *isotropy groupoid* is the subgroupoid of $\mathcal{G}$:

$$\mathrm{Iso}(\mathcal{G}) := \bigcup_{x \in \mathcal{G}^{(0)}} \mathcal{G}_x^x \rightrightarrows \mathcal{G}^{(0)}.$$

1.3 Topological groupoids

Now we turn to groupoids with extra structures. Charles Ehresmann was the first person to endow groupoids with extra structures while applying them to the study of foliation. Examples include topological groupoids, Borel groupoids (groupoids with measurable structures) and Lie groupoids.

Topological groupoids are the central objects that we will care about in the seminar talks. Let $\mathcal{G}$ be a groupoid that is also a topological space. Then $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$ inherits the product topology of $\mathcal{G} \times \mathcal{G}$.

Definition 1.12. A groupoid $\mathcal{G}$ is a *topological groupoid* if $\mathcal{G}$ is a topological space, and the multiplication map and the inverse map are continuous.

Remark 1.13. Just as in the case of groups, we usually require that a topological groupoid $\mathcal{G}$ is such that:

1. $\mathcal{G}$ is locally-compact.
2. $\mathcal{G}^{(0)}$ is Hausdorff (in the subspace topology).

However, the groupoid $\mathcal{G}$ itself does not have to be Hausdorff. When $\mathcal{G}$ is Hausdorff, its unit space will be closed, see the following lemma.

In fact, non-Hausdorff groupoids arise naturally from dynamical systems and differenrial geometry (e.g. singular foliations). They give rise to interesting C*-algebras.

Lemma 1.14. *Let $\mathcal{G}$ be a topological groupoid. Then $\mathcal{G}$ is Hausdorff iff $\mathcal{G}^{(0)}$ is closed.*

Proof. If $\mathcal{G}$ is Hausdorff. Then every convergent net in $\mathcal{G}$ converges to a unique point. Let $\{a_i\}_{i \in I}$ be a net in $\mathcal{G}^{(0)}$ which converges to $a \in \mathcal{G}$. We claim that the limit a must lie in $\mathcal{G}^{(0)}$ as well. Since $\mathcal{G}$ is a topological groupoid, the source and range maps $s, r\colon \mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ are continuous. Hence $s(a_i) \to s(a)$ and $r(a_i) \to r(a)$. But $a_i \in \mathcal{G}^{(0)}$ for all i, that is, $s(a_i) = a_i = r(a_i)$. Then we have $a_i \to s(a)$, $a_i \to r(a)$ and $a_i \to a$. The Hausdorffness of $\mathcal{G}$ forces $s(a) = a = r(a)$, that is, $a \in \mathcal{G}^{(0)}$.

Now assume that $\mathcal{G}^{(0)}$ is closed. Let $\{a_i\}_{i \in I}$ be any convergent net in $\mathcal{G}$ which converges to both a and b. We must prove that $a = b$. Since the multiplication and the inverse maps are continuous, we have $a_i^{-1} a_i \to a^{-1} b$. But $a_i^{-1} a_i \in \mathcal{G}^{(0)}$ for all i and $\mathcal{G}^{(0)}$ is closed. Therefore, $a^{-1} b \in \mathcal{G}^{(0)}$, which implies that $a = b$. $\square$

The following examples of topological groupoids are just modifications of Example 1.6.

Example 1.15.

1. A topological group G is a topological groupoid $G \rightrightarrows \mathrm{pt}$.
2. A topological space X is a topological groupoid $X \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightrightarrows}} X$ together with the trivial multiplication and inverse maps.
3. A topological group bundle consists of two topological spaces E and X, and a quotient map $\pi\colon E \twoheadrightarrow X$, such that for any $x \in X$, $\pi^{-1}(x)$ is a topological group. This is a topological groupoid.
4. Let G be a topological group which acts continuously on a space X. Then the action groupoid $G \ltimes X$ is a topological groupoid.
5. Let X be a topological space, and $\mathcal{R} \subseteq X \times X$ be an equivalence relation on X, equipped with the subspace topology. Then $\mathcal{R} \rightrightarrows X$ is a topological groupoid. In particular, the pair groupoid of a topological space is a topological groupoid.
6. The fundamental groupoid $\Pi_1(X)$ of a topological space X is a bit tricky. There are various ways to topologise it, but the "correct" topology is defined only when X satisfy some nice conditions (path-connected, locally path-connected and semi-locally simply connected). Readers who are familiar with algebraic topology shall notice that these conditions are precisely what one needs to obtain a nice classification theory of covering spaces. In such situation, the fundamental groupoid is realised as a quotient of the pair groupoid and has the quotient topology.
7. The tangent groupoid is a topological groupoid. The topology is defined as follows: we require that $M \times M \times (0,1]$ is open, and require that a sequence $\{(x_n, y_n, \epsilon_n)\}_n$ in $M \times M \times (0,1]$ converges to $(x, v, 0)$ iff
$$x_n \to x, \qquad y_n \to x, \qquad \frac{x_n - y_n}{\epsilon_n} \to v.$$

Finally, we define a subclass of topological groupoids called étale groupoids. They are analogs of discrete groupoids, and easier to study than general topological groupoids. It is a bit surprising that étale groupoids are already interesting enough, in the sense that many interesting C*-algebras arise as the C*-algebra of an étale groupoid.

Definition 1.16. A topologial groupoid $\mathcal{G}$ is called an *étale groupoid*, if the source and range maps $s, r\colon \mathcal{G} \to \mathcal{G}$ are étale. That is, s and r are local homeomorphisms.

Remark 1.17. Be careful that the maps s and r are étale as maps $\mathcal{G} \to \mathcal{G}$, but not as maps $\mathcal{G} \to \mathcal{G}^{(0)}$. This is a stronger argument: it asserts that the inclusion map $\mathcal{G}^{(0)} \hookrightarrow \mathcal{G}$ is a topological embedding.

Lemma 1.18. *If $\mathcal{G}$ is an étale groupoid. Then $\mathcal{G}^{(0)} \subseteq \mathcal{G}$ is open.*

Proof. $\mathcal{G}$ is étale implies that, for every $a \in \mathcal{G}$, there is an open neighbourhood $U_a \subseteq \mathcal{G}$ of a such that $s|_{U_a}\colon U_a \to s(U_a)$ is a homeomorphism. Then $\mathcal{G}^{(0)} = \bigcup_{a \in \mathcal{G}} s(U_a)$ is open. □

The following lemma states that étale groupoids are "fibrewise discrete".

Lemma 1.19. *If $\mathcal{G}$ is étale. Then for any $x \in \mathcal{G}^{(0)}$, $\mathcal{G}_x$ and $\mathcal{G}^x$ are discrete.*

Proof. $\mathcal{G}$ is étale implies that, for every $a \in \mathcal{G}$, there is an open neighbourhood $U_a \subseteq \mathcal{G}$ of a such that $s|_{U_a}\colon U_a \to s(U_a)$ is a homeomorphism. In particular, s is a bijection.

We claim that $\{a\} = \mathcal{G}_x \cap U_a$. Clearly $\{a\} \subseteq \mathcal{G}_x \cap U_a$. Suppose $b \in \mathcal{G}_x \cap U_a$, then $s(b) = x = s(a)$. Since s is bijective on U_a, we must have $b = a$. Therefore, $\{a\} \subseteq \mathcal{G}_x \cap U_a$ is open in $\mathcal{G}_x$ in the relative topology. So $\mathcal{G}_x$ is discrete. The proof for $\mathcal{G}^x$ is essentially the same. □

2 Haar systems and groupoid C*-algebras (Yufan, Sep 20)

Groupoid C*-algebras were introduced by Renault [Ren80], which consist of a large class of interesting C*-algebras.

In this lecture, all topological groupoids are assumed to be *locally-compact* and *Hausdorff*.

2.1 Haar systems

A Haar system is the analog of a Haar measure of a topological group. Recall that

Definition 2.1. A *Radon* measure is a measure μ on a locally-compact space X, which is

1. Borel: all open subsets are measurable.
2. regular: μ is inner regular and outer regular. That is,
$$\mu(E) = \sup\{\mu(K) \mid K \subseteq E \text{ is compact}\} \qquad \text{and} \qquad \mu(E) = \inf\{\mu(U) \mid U \supseteq E \text{ is open}\}.$$
3. locally finite: for any $x \in X$, there exists a neighbourhood N of x, such that $\mu(N) < \infty$.

Measures on a locally-compact space X are related to linear functionals of $\mathrm{C_c}(X)$ by the following theorem:

Theorem 2.2 (Riesz–Markov–Kakutani representation theorem)**.** *Let X be a locally-compact Hausdorff space, and $\psi\colon \mathrm{C_c}(X) \to \mathbb{C}$ (or $\psi\colon \mathrm{C_c}(X) \to \mathbb{R}$) be a linear functional. There exists a complex (or real) Radon measure μ on X, such that*
$$\psi(f) = \int_X f \,\mathrm{d}\mu.$$

This theorem allows us a construct a Radon measure from a linear functional.

Definition 2.3 (Haar system)**.** A *Haar system* of a locally-compact Hausdorff groupoid $\mathcal{G}$ is a family of Radon measures $\{\mu^u\}_{u\in\mathcal{G}^{(0)}}$ indexed by $\mathcal{G}^{(0)}$, such that

(HS1) $\operatorname{supp}(\lambda^u) = \mathcal{G}^{(u)}$;

(HS2) For any $f \in \mathrm{C_c}(\mathcal{G})$, the function
$$\lambda_f\colon \mathcal{G}^{(0)} \to \mathbb{C}, \qquad u \mapsto \int_{\mathcal{G}} f(\gamma)\,\mathrm{d}\lambda^u(\gamma)$$
is continuous (and hence compactly-supported).

(HS3) For all $\eta \in \mathcal{G}$, the following "left-invariance" holds:
$$\int_{\mathcal{G}} f(\gamma)\,\mathrm{d}\lambda^{r(\eta)}(\gamma) = \int_{\mathcal{G}} f(\eta\gamma)\,\mathrm{d}\lambda^{s(\eta)}\gamma.$$

Example 2.4. Let X be a locally-compact space with Radon measure μ, G be a locally-compact group with a Haar measure μ. Define the groupoid $\mathcal{G} := X \times G \times X \overset{r}{\underset{s}{\rightrightarrows}} X$ with
$$r(x,g,y) := x, \qquad s(x,g,y) := y, \qquad (x,g,y)(y,h,z) := (x,gh,z).$$
Then $\{\lambda^x := \delta_x \times \mu \times \nu\}_{x\in X}$ is a Haar system of $\mathcal{G}$.

Example 2.5. Let $\mathcal{G} := G \ltimes X$ be the action groupoid. Define the linear functional

$$\lambda^x \colon \mathrm{C_c}(X) \to \mathbb{C}, \qquad \lambda^x(f) := \int_{\mathcal{G}} f(g, g^{-1}x) \, \mathrm{d}\mu(g).$$

This precribes a Haar system $\{\lambda^x\}_{x \in X}$ for $\mathcal{G}$ by Riesz–Markov–Kakutani representation theorem. It suffices to check conditions (HS1)–(HS3). For the first one, we need the following

Lemma 2.6. *Let* $\lambda \colon \mathrm{C_c}(X) \to \mathbb{C}$ *be a positive linear functional. Then*

$$x \in \operatorname{supp} \lambda \qquad \textit{iff} \qquad \lambda(f) > 0 \textit{ for all } f \in \mathrm{C_c}(X) \textit{ with } f(x) > 0.$$

This implies that $\operatorname{supp} \lambda^x = \mathcal{G}^x$.

Remark 2.7. Given a Haar system $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$ and a Radon measure μ on $\mathcal{G}^{(0)}$, we can define a linear functional on $\mathrm{C_c}(\mathcal{G})$:

$$\nu(f) := \int_{\mathcal{G}^{(0)}} \int_{\mathcal{G}} f(\gamma) \, \mathrm{d}\lambda^u(\gamma) \, \mathrm{d}\mu(u).$$

Hence defines a measure on $\mathcal{G}$ by Riesz–Markov–Kakutani representation theorem.

Remark 2.8. $\mathrm{C_c}(\mathcal{G})$ may be viewed as a right $\mathrm{C_0}(\mathcal{G}^{(0)})$-module via

$$f \cdot \psi(\gamma) := f(\gamma)\psi(s(\gamma)), \qquad f \in \mathrm{C_c}(\mathcal{G}),\ \psi \in \mathrm{C_c}(\mathcal{G}^{(0)}).$$

Then the Haar system $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$ allows for a $\mathrm{C_0}(\mathcal{G}^{(0)})$-valued inner product

$$\langle f, g \rangle(u) := \int_{\mathcal{G}} \overline{f(\gamma)} g(\gamma) \, \mathrm{d}\lambda^u(\gamma).$$

The condition (HS2) guarantees that the inner product actually lands in $\mathrm{C_0}(\mathcal{G}^{(0)})$. Endowed with these operations, $\mathrm{C_c}(\mathcal{G})$ becomes a pre Hilbert $\mathrm{C_0}(X)$-module, giving an alternative definition of groupoid C*-algebras (Remark 2.20)

Definition 2.9. A locally-compact Hausdorff groupoid $\mathcal{G}$ is called *r-discrete* if $\mathcal{G}^{(0)}$ is open.

Lemma 2.10. *If $\mathcal{G}$ is r-discrete. Then $\mathcal{G}^u$ and $\mathcal{G}_u$ are discrete for all $u \in \mathcal{G}^{(0)}$.*

Proof. $\mathcal{G}^{(0)}$ is open, so every singleton $\{u\}$ is open in $\mathcal{G}^u$. Now for any $\gamma \in \mathcal{G}$, let $u = s(\gamma)$ and $v = r(\gamma)$. Then the map

$$\mathcal{G}_v \to \mathcal{G}_u, \qquad \eta \mapsto \eta\gamma$$

is continuous, with $\{\gamma\} = \phi^{-1}(\{u\})$. So $\{\gamma\}$ is open. $\square$

Lemma 2.11. *If $\mathcal{G}$ is r-discrete and r is open. Then r is a local homeomorphism.*

Lemma 2.12. *If $\mathcal{G}$ is r-discrete. Then the counting measure on each fibre form a Haar system iff $\mathcal{G}$ is étale.*

Example 2.13 (An r-discrete but not étale groupoid)*.* Consider the equivalence relation

$$E := \{(-1, 1), (1, -1)\} \cup \{(x, x) \mid -1 \le x \le 1\}$$

on $E^0 := [-1, 1]$. The groupoid $E \rightrightarrows E^{(0)}$ is r-discrete, but not étale.

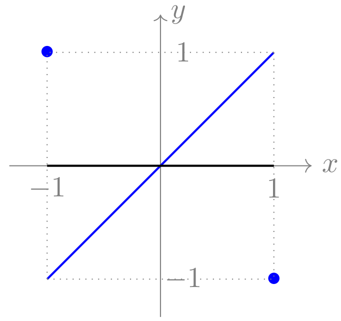

Figure 3: An r-discrete but not étale groupoid: the fibre has one point at each $x \in (-1, 1)$, but has two points at ± 1.

2.2 Groupoid C*-algebras

Let $f, g \in \mathrm{C_c}(\mathcal{G})$. Define

$$f * g(\gamma) := \int_{\mathcal{G}} f(\eta) g(\eta^{-1}\gamma) \,\mathrm{d}\lambda^{r(\eta)}\gamma,$$
$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

Lemma 2.14. *The operations above turn* $\mathrm{C_c}(\mathcal{G})$ *into a* **-algebra.*

Proof. We need to check that $*$ is a convolution product. Notice that $f * g$ is continuous, and has compact support because

$$\operatorname{supp}(f * g) \subseteq \operatorname{supp}(f) \operatorname{supp}(g).$$

Now we check associativity. We have

$$\begin{aligned}
((f * g) * h)(\gamma) &= \int_{\mathcal{G}} (f * g)(\eta) h(\eta^{-1}\gamma) \,\mathrm{d}\lambda^{r(\gamma)}\eta \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}} f(\xi) g(\xi^{-1}\eta) h(\eta^{-1}\gamma) \,\mathrm{d}\lambda^{r(\eta)}\xi \,\mathrm{d}\lambda^{r(\gamma)}\eta; \\
(f * (g * h))(\gamma) &= \int_{\mathcal{G}} f(\xi) (g * h)(\xi^{-1}\gamma) \,\mathrm{d}\lambda^{r(\gamma)}\xi \\
&= \int_{\mathcal{G}} f(\xi) \int_{\mathcal{G}} g(\eta) h(\eta^{-1}\xi^{-1}\gamma) \,\mathrm{d}\lambda^{r(\xi^{-1}\gamma)}\eta \,\mathrm{d}\lambda^{r(\gamma)}\xi \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}} f(\xi) g(\eta) h(\eta^{-1}\xi^{-1}\gamma) \,\mathrm{d}\lambda^{s(\xi)}\eta \,\mathrm{d}\lambda^{r(\gamma)}\xi
\end{aligned}$$

Replace η by $\xi^{-1}\eta$:

$$= \int_{\mathcal{G}} \int_{\mathcal{G}} f(\xi) g(\eta^{-1}\eta) h(\eta^{-1}\gamma) \,\mathrm{d}\lambda^{s(\xi)}(\xi^{-1}\eta) \,\mathrm{d}\lambda^{r(\gamma)}\xi$$

Now use (HS3), $\mathrm{d}\lambda^{r(\eta)}(\gamma) = \mathrm{d}\lambda^{s(\eta)}(\eta^{-1}\gamma)$:

$$= \int_{\mathcal{G}} \int_{\mathcal{G}} f(\xi) g(\xi^{-1}\eta) h(\eta^{-1}\gamma) \,\mathrm{d}\lambda^{r(\eta)}\xi \,\mathrm{d}\lambda^{r(\gamma)}\eta. \qquad \square$$

Example 2.15. A Haar system of an étale groupoid $\mathcal{G}$ is given by the fibrewise counting measure. Then for $f, g \in \mathrm{C_c}(\mathcal{G})$:

$$f * g(\gamma) = \sum_{r(\eta) = r(\gamma)} f(\eta) g(\eta^{-1}\gamma) = \sum_{\alpha\beta = \gamma} f(\alpha) g(\beta).$$

We wish to complete $\mathrm{C_c}(\mathcal{G})$ into a C*-algebra. Let $\pi\colon \mathrm{C_c}(\mathcal{G}) \to \mathbb{B}(H_\pi)$ be a *-representation on a Hilbert space. Then $\|f\|_\pi := \|\pi(f)\|_{\mathbb{B}(H_\pi)}$ is a norm for $\mathrm{C_c}(\mathcal{G})$, which might be unbounded. To obtain a C*-norm of $\mathrm{C_c}(\mathcal{G})$, one needs to restrict to representations which have a common upper bound. In the case of group C*-algebras, one considers all possible norms bounded by the L^1-norm, and takes suitable representations to obtain the full and reduced C*-norm. The analog of the L^1-norm in the groupoid case is the I-norm.

Definition 2.16 (***I*-norm**)**.** Let $f \in \mathrm{C_c}(\mathcal{G})$. Define

$$\|f\|_{I,r} := \sup_{u\in\mathcal{G}^{(0)}} \int_{\mathcal{G}} |f(\gamma)| \,\mathrm{d}\lambda^u(\gamma),$$
$$\|f\|_{I,s} := \sup_{u\in\mathcal{G}^{(0)}} \int_{\mathcal{G}} |f(\gamma^{-1})| \,\mathrm{d}\lambda^u(\gamma),$$
$$\|f\|_I := \max\{\|f\|_{I,r}, \|f\|_{I,s}\}.$$

Definition 2.17. A *-homomorphism $\pi\colon \mathrm{C_c}(\mathcal{G}) \to \mathbb{B}(H_\pi)$ is called I-norm bounded, if $\|f\|_\pi \leq \|f\|_I$ for all f.

Given a Radon measure μ on $\mathcal{G}^{(0)}$. Recall that we may define a Radon measure on $\mathcal{G}$ as in Remark 2.7:

$$\nu(f) := \int_{\mathcal{G}^{(0)}} \int_{\mathcal{G}} f(\gamma) \,\mathrm{d}\lambda^u(\gamma) \,\mathrm{d}\mu(u).$$

We also define

$$\nu^{-1}(f) := \int_{\mathcal{G}^{(0)}} \int_{\mathcal{G}} f(\gamma^{-1}) \,\mathrm{d}\lambda^u(\gamma) \,\mathrm{d}\mu(u).$$

Define

$$\mathrm{ind}_\mu\colon \mathrm{C_c}(\mathcal{G}) \to \mathbb{B}(L^2(\mathcal{G},\nu^{-1})), \qquad \mathrm{ind}_\mu(f)(h) := \int_{\mathcal{G}} f(\eta)h(\eta^{-1}\gamma) \,\mathrm{d}\lambda^{r(\gamma)}(\eta).$$

Proposition 2.18. *Let $\mathcal{G}$ be a locally-compact Hausdorff groupoid with Haar system, and μ be any Radon measure on $\mathcal{G}^{(0)}$. Then ind_μ is an I-norm bounded representation of $\mathrm{C_c}(\mathcal{G})$.*

Definition 2.19.

- The full norm of $\mathrm{C_c}(\mathcal{G})$ is

$$\|f\| := \sup\{\|\pi(f)\| \mid \pi \text{ is an } I\text{-norm bounded representation.}\}$$

  The full groupoid C*-algebra $\mathrm{C}^*(\mathcal{G})$ is the completion of $\mathrm{C_c}(\mathcal{G})$ under the full norm.

- The reduced norm of $\mathrm{C_c}(\mathcal{G})$ is

$$\|f\|_{\mathrm{r}} := \sup\{\|\mathrm{ind}_{\delta_u}(f)\| \mid u \in \mathcal{G}^{(0)},\ \delta_u \text{ is the Dirac measure supported on } u.\}$$

  The reduced groupoid C*-algebra $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})$ is the completion of $\mathrm{C_c}(\mathcal{G})$ under the reduced norm.

Remark 2.20. There is an (even nicer) construction of reduced groupoid C*-algebras $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})$ as follows. As in Remark 2.8, $\mathrm{C_c}(\mathcal{G})$ is a pre-Hilbert $\mathrm{C}_0(\mathcal{G}^{(0)})$ module. That is, $\langle f, g\rangle$ is the restriction of $f^* * g$ to $\mathcal{G}^{(0)}$ and $f \cdot \psi(\gamma) := f(\gamma)\psi(s(\gamma))$ for $f, g \in \mathrm{C_c}(\mathcal{G})$ and $\psi \in \mathrm{C}_0(\mathcal{G}^{(0)})$. We complete $\mathrm{C_c}(\mathcal{G})$ to a Hilbert $\mathrm{C}_0(\mathcal{G}^{(0)})$-module, denoted by $L^2(\mathcal{G},\nu)$.

The multiplication action of $\mathrm{C_c}(\mathcal{G})$ on itself extends to a bounded representation $\pi\colon \mathrm{C_c}(\mathcal{G}) \to \mathbb{B}(L^2(\mathcal{G},\nu))$. Define the reduced norm of $f \in \mathrm{C_c}(\mathcal{G})$ to be $\|f\|_{\mathrm{r}} := \|\pi(f)\|$. One checks that this coincides with the reduced norm defined above, hence refines reduced groupoid C*-algebras via completion.

The case of non-Hausdorff groupoids is more involved: one needs to replace $\mathrm{C}_0(\mathcal{G}^{(0)})$ by a bigger algebra, which contains some Borel non-continuous functions on $\mathcal{G}^{(0)}$. See [KS02] for more details.

Example 2.21. Let X be a locally-compact Hausdorff space, viewed as a groupoid $\mathcal{G} := X \rightrightarrows X$. Then $\mathrm{C_c}(\mathcal{G}) = \mathrm{C_c}(X)$. The I-norm is $\|f\|_I = \sup_{x\in X}|f(x)| = \|f\|_\infty$. This is in fact a C^*-norm, achieved by the multiplication representation on $L^2(X,\mu)$ for any Radon measure μ on X. The groupoid C^*-algebra $\mathrm{C}^*(\mathcal{G}) = \mathrm{C}_0(X)$.

Example 2.22. Let X be a finite set of n-elements, and $\mathcal{R}$ be the trivial equivalence relation on X. Then the groupoid C^*-algebra $\mathrm{C}^*(\mathcal{R}) \cong \mathbb{M}_n(\mathbb{C})$.

Example 2.23. Let G be a locally-compact topological group, viewed as a groupoid $\mathcal{G} := G \rightrightarrows \mathrm{pt}$. If G is unimodular, then the definition of group and groupoid C^*-algebras of G coincide. If G is not unimodular, we need tiny modification to build the isomorphism between them two. Let $f \in \mathrm{C_c}(G)$. The involution on $\mathrm{C_c}(G)$ is

$$f^*(x) := \Delta(x)^{-1}\overline{f(x^{-1})}.$$

The involution on $\mathrm{C_c}(\mathcal{G})$ is

$$f^*(x) := \overline{f(x^{-1})}.$$

Define

$$\phi\colon \mathrm{C_c}(G) \to \mathrm{C_c}(\mathcal{G}), \qquad f \mapsto \Delta^{-1/2}f.$$

We claim that this is a *-isomorphism of *-algebras, and extends to an isometric isomorphism of C^*-algebras $\mathrm{C}^*(G) \xrightarrow{\cong} \mathrm{C}^*(\mathcal{G})$. Let π be an I-norm bounded representation of $\mathrm{C_c}(\mathcal{G})$. Then $\pi\circ\phi$ is a representation of $\mathrm{C_c}(G)$. Since $\|\phi(f)\|$ is by definition the supremum over all I-norm bounded representations, we have $\|\phi(f)\| \leq \|f\|$. The other side holds using a disintegration theorem of group representations and we omit here. Therefore, $\|\phi(f)\| = \|f\|$.

3 Groupoid actions and equivalence actions (Jack, Oct 4)

Throughout this lecture, $\mathcal{G}$ will denote a groupoid. If $\mathcal{G}$ is a topological groupoid, we will always assume that it is locally-compact and Hausdorff.

3.1 Groupoid actions

Definition 3.1. Let $\mathcal{G}$ be a groupoid. Let X be a set together with a map $r_X\colon X \to \mathcal{G}^{(0)}$ called the *moment map.* A left action of $\mathcal{G}$ on X is a map

$$\mathcal{G} * X \to X, \qquad (\gamma, x) \mapsto \gamma x,$$

where

$$\mathcal{G} * X := \{(\gamma, x) \in \mathcal{G}\times X \mid s(\gamma) = r_X(x)\},$$

such that

- $r_X(x)x = x$ for all $x \in X$.
- If $(\gamma,\eta) \in \mathcal{G}^{(2)}$ and $(\eta, x) \in \mathcal{G} * X$. Then $(\gamma\eta, x) \in \mathcal{G} * X$ and $(r\gamma)x = \gamma(\eta x)$.

A right action is defined similarly, while in that case a moment map is denoted by s_X for consistency. If $\mathcal{G}$ acts on X on the left (resp. right), we write $G \curvearrowright X$ (resp. $X \curvearrowleft G$) and call X a left (resp. right) $\mathcal{G}$-set. Unless specified, groupoid actions are always assumed to be left actions.

If $\mathcal{G}$ is a topological groupoid and X is a topological space, such that the moment map and the groupoid action are continous. Then we say X is a $\mathcal{G}$-space.

Definition 3.2. Let X be a $\mathcal{G}$-set. We say:

- $\mathcal{G}$ acts *transitively* on X, if for all $x, y \in X$, there exists $\gamma \in \mathcal{G}$ such that $x = \gamma y$.
- $\mathcal{G}$ acts *freely* on X, if $\gamma x = x$ for some x implies that $\gamma = r_X(x)$.

Example 3.3.

- Let $\mathcal{G}$ be a groupoid. Then $\mathcal{G} \curvearrowright \mathcal{G} \curvearrowleft \mathcal{G}$ in an obvious way. Also $\mathcal{G} \curvearrowright \mathcal{G}^{(0)} \curvearrowleft \mathcal{G}$.
- Let G be a group and X be a G-set. Then $G \ltimes X \curvearrowright X \curvearrowleft G \ltimes X$.

We may also define the groupoid version of the action groupoid of a group:

Definition 3.4. Let X be a $\mathcal{G}$-set. The action groupoid $\mathcal{G} \ltimes X$ is defined as

$$\mathcal{G} * X \rightrightarrows X, \qquad s(\gamma, x) = x, \; r(\gamma, x) = \gamma x.$$

Definition 3.5. Let X be a left $\mathcal{G}$-set and $x \in X$. The *orbit* of x is

$$\{\gamma x \mid (\gamma, x) \in \mathcal{G} * X\}.$$

Denote by $\mathcal{G}\backslash X$ the set of orbits. If X is a right $\mathcal{G}$-set, we write $X/\mathcal{G}$ for the set of orbits.

If X is a $\mathcal{G}$-space. Then we may endow $\mathcal{G}\backslash X$ with the quotient topology and call it the orbit space.

Remark 3.6. Recall that in the quotient topology: $U \subseteq \mathcal{G}\backslash X$ is open iff $q^{-1}(U) \subseteq X$ is open, where q is the quotient map. A map $f\colon \mathcal{G}\backslash X \to Y$ is continuous in the quotient topology iff it lifts to a continuous map $\tilde{f}\colon X \to Y$.

Definition 3.7. A groupoid $\mathcal{G}$ is called

- *principal*, if $\mathcal{G} \curvearrowright \mathcal{G}^{(0)}$ is free.
- *transitive*, if $\mathcal{G} \curvearrowright \mathcal{G}^{(0)}$ is transitive.

Proposition 3.8. *If $\mathcal{G}$ is a topological groupoid with r, s open. Let X be a $\mathcal{G}$-space. Then $q\colon X \to \mathcal{G}\backslash X$ is open.*

Proof. Let $U \subseteq X$. Then $\mathcal{G}U = q^{-1}(q(U))$ is open. Then use Fell's creterion. TBAI will complete this proof soon. ☐

Corollary 3.9. *Let $\mathcal{G}$ be a locally-compact Hausdorff groupoid with r, s open. Let X be a locally-compact $\mathcal{G}$-space. Then $\mathcal{G}\backslash X$ is locally-compact. If X is second countable, then $\mathcal{G}\backslash X$ is too.*

Definition 3.10. Let $\mathcal{G}$ be a locally-compact Hausdorff groupoid acting on a locally-compact Hausdorff space X. We say the action is *proper*, if the map

$$\Theta\colon \mathcal{G} * X \to X \times X, \qquad (\gamma, x) \mapsto (\gamma x, x)$$

is proper, i.e. the pre-image of a compact set is compact.

Proposition 3.11. *Let $\mathcal{G}$ be a locally-compact Hausdorff groupoid acting on a locally-compact space X. The followings are equivalent:*

1. *$\mathcal{G}$ acts properly.*

2. *For all compact subsets K and L in X, the set*

$$\mathrm{pr}_1(\Theta^{-1}(K \times L)) = \{\gamma \in \mathcal{G} \mid K \cap \gamma L \neq \emptyset\}$$

is a compact subset in $\mathcal{G}$.

3. *Let $\{x_i\}$ be a net converging to x and $\{\gamma_i x_i\}$ be a net converging to y. Then $\{\gamma_i\}$ has convergent subnet.*

Proposition 3.12. *Let $\mathcal{G}$ be a locally-compact Hausdorff groupoid with r, s open. Let X be a proper $\mathcal{G}$-space. Then $\mathcal{G}\backslash X$ is locally-compact Hausdorff.*

Proof. Let $\{\mathcal{G}x_i\}$ be a net in $\mathcal{G}\backslash X$ which converges to both $\mathcal{G}x$ and $\mathcal{G}y$. We claim that $\mathcal{G}x = \mathcal{G}y$. Since the quotient map $q\colon X \to \mathcal{G}\backslash X$ is open, we may assume that $x_i \to x$ and $\gamma_i x_i \to y$ by Remark 3.6. Then there is a subnet of $\{\gamma_i\}$ converging to γ by Proposition 3.11. But X is Hausdorff. So $\gamma_i x_i$ converges to γx. We therefore have $y = \gamma x$ and $\mathcal{G}x = \mathcal{G}y$. □

Definition 3.13. Let $\mathcal{G}$ be a locally-compact Hausdorff groupoid acting on a locally-compact Hausdorff space X. We say the action is *Cartan*, if every $x \in X$ has a compact neighbourhood K such that $\Theta^{-1}(K \times K)$ is compact.

3.1.1 Mackey–Glimm–Ramsey dichotomy

There is a short part on Mackey–Glimm–Ramsey dichotomy, and Jack mentioned this in his talk. I need some more time to finish it and have to leave it blank for now. This will be covered soon. I am very sorry for this (and especially for Jack, for a delay in the reflextion of your nice talk). — Y. Li

3.2 Equivalence of groupoids

Equivalence of groupoids were introduced by Renault in [Ren82]. One motivation is to define an equivalence relation which is weaker than isomorphisms, but gives Morita–Rieffel equivalence of C*-algebras.

Definition 3.14. Let $\mathcal{G}$ and $\mathcal{H}$ be locally-compact Hausdorff groupoids with r and s open. A locally-compact Hausdorff space Z is called a $(\mathcal{G}, \mathcal{H})$-(Morita) equivalence, if the followings hold:

1. Z is a free and proper left $\mathcal{G}$-space and a free and proper right $\mathcal{H}$-space.

2. The left $\mathcal{G}$-action commutes with the right $\mathcal{H}$-action.

3. The moment map $r\colon Z \to \mathcal{G}^{(0)}$ is open and induces a homeomorphism $Z/\mathcal{H} \to \mathcal{G}^{(0)}$.
   $r\colon Z \to \mathcal{H}^{(0)}$ is open and induces a homeomorphism $\mathcal{G}\backslash Z \to \mathcal{H}^{(0)}$.

Example 3.15. Let $\Gamma \curvearrowright X \curvearrowleft \Gamma'$ be commuting actions which are free and proper. Then $\Gamma\backslash X$ and X/Γ' are locally-compact Hausdorff. Then X is an equivalence between $\Gamma \ltimes (X/\Gamma')$ and $(\Gamma\backslash X) \rtimes \Gamma'$. This works for groupoid actions as well.

As a special case, if we take $X \curvearrowleft \Gamma'$ to be the trivial action. Then X is an equivalence between $\Gamma \ltimes X$ and $\Gamma\backslash X$.

Example 3.16 (Blow-ups). Let $\mathcal{G}$ be a locally-compact Hausdorff groupoid with open range. Let Z be a locally-compact space, $f\colon Z \to \mathcal{G}^{(0)}$ be an open continuous map. Regard f as a moment map, then Z can be viewed as a left $\mathcal{G}$-space and a right $\mathcal{G}$-space. The blow-up groupoid is defined as

$$\mathcal{G}[Z] := \{(z_1, \gamma, z_2) \in \mathcal{G} \times Z \times \mathcal{G} \mid f(z_1) = r(\gamma),\, s(\gamma) = r(z_2)\}.$$

The groupoid operations are the nature ones. The unit space is $\{(z, f(z), z)\}$ so we can view $\mathcal{G}[Z]$ as a groupoid over Z.

Let

$$Z * \mathcal{G} := \{(z, \gamma) \in Z \times \mathcal{G} \mid f(z) = r(\gamma)\}.$$

We claim that $Z * \mathcal{G}$ actually defines a $(\mathcal{G}[Z], \mathcal{G})$-equivalence. The moment maps are given by

$$s(z, \gamma) := s(\gamma), \qquad r(z, \gamma) := z,$$

and the actions are

$$(z_1, \gamma, z_2)(z_2, \gamma') := (z_1, \gamma\gamma'), \qquad (z, \gamma)\eta := (z, \gamma\eta).$$

Remark 3.17. Let Z be a $(\mathcal{G}, \mathcal{H})$-equivalence. Then the homeomorphism $Z/\mathcal{H} \to \mathcal{G}^{(0)}$ (resp. $\mathcal{G}\backslash Z \to \mathcal{H}^{(0)}$) is $\mathcal{G}$- (resp. $\mathcal{H}$-)equivariant. These induces homeomorphisms

$$\mathcal{G}\backslash\mathcal{G}^{(0)} \xleftarrow{\cong} \mathcal{G}\backslash Z/\mathcal{H} \xrightarrow{\cong} \mathcal{H}^{(0)}/\mathcal{H}.$$

So $\mathcal{G}\backslash\mathcal{G}^0$ is homeomorphic to $\mathcal{H}^{(0)}/\mathcal{H}$.

Proposition 3.18. *The groupoid equivalence is an equivalence relation.*

Sketch of proof.

- $\mathcal{G}$ is a $(\mathcal{G}, \mathcal{G})$-equivalence.
- If Z is a $(\mathcal{G}, \mathcal{H})$-equivalence, with moment maps $r_Z\colon Z \to \mathcal{G}^{(0)}$ and $s_Z\colon Z \to \mathcal{H}^{(0)}$. Define an $(\mathcal{H}, \mathcal{G})$-equivalence Z^{op} as follows:
  - As a space, Z^{op} is homeomorphic to Z. We write $\bar{z} \in Z^{\mathrm{op}}$ for the image of $z \in Z$ in order to distinguish.
  - The left $\mathcal{H}$-action on Z^{op} defined by the followings:
  $$r'_{Z^{\mathrm{op}}}(\bar{z}) := s_Z(z), \qquad \gamma\bar{z} := z\gamma^{-1}.$$
  - The right $\mathcal{G}$-action on Z^{op} defined by the followings:
  $$s'_{Z^{\mathrm{op}}}(\bar{z}) := r_Z(z), \qquad \bar{z}\eta := \eta^{-1}z.$$
- Let Z be a $(\mathcal{G}, \mathcal{H})$-equivalence and Y be an $(\mathcal{H}, \mathcal{K})$-equivalence. Then a $(\mathcal{G}, \mathcal{K})$-equivalence is given by the quotient
  $$\{(z, y) \in Z \times Y \mid s(z) = r(y)\}/\mathcal{H},$$
  where the right $\mathcal{H}$-action is given by
  $$(z, y) \cdot \beta := (z\beta, \beta^{-1}y).$$ □

Let Z be a $(\mathcal{G}, \mathcal{H})$-equivalence. Then

$$Z *_s Z := \{(x, y) \in Z \times Z \mid s(x) = s(y)\}$$

is closed and locally-compact. It is a free and proper right $\mathcal{H}$-space with diagonal action. In particular, we have:

Lemma 3.19. *Given $(x, y) \in Z *_s Z$. There exists a unique element $\tau_{x,y} \in \mathcal{G}$ such that $\tau_{x,y}y = x$.*

The map

$$Z *_s Z \to \mathcal{G}, \qquad (x, y) \mapsto \tau_{x,y}$$

is continuus and open. It factors through the homeomorphism $Z *_s Z/\mathcal{H} \to \mathcal{G}$.

With the help of this lemma we are able to prove that

Theorem 3.20. *Let $\mathcal{G}$ and $\mathcal{H}$ be locally-compact Hausdorff groupoids. Then $\mathcal{G}$ is equivalent to $\mathcal{H}$ iff there exists a space Z such that the blow-up groupoids $\mathcal{G}[Z]$ and $\mathcal{H}[Z]$ are isomorphic.*

4 Groupoid correspondences (Bram, Oct 11)

In this talk, all groupoids are locally-compact and Hausdorff and equipped with a Haar system. As a consequence, their source and range maps are open. We will refer to them simply as *groupoids*.

4.1 Groupoid correspondences

We wish to construct a nice category of groupoids, such that taking the groupoid C^*-algebra is a functor mapping to a suitable category of C^*-algebras: the category of C^*-correspondence Corr. We also request that Morita equivalent groupoids are mapped to Morita–Rieffel equivalent C^*-algebras. This requires a suitable notion of "generalised homomorphisms" between groupoids. These, as we will define in the following, are *groupoid correspondences*.

Definition 4.1. Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids. A *(groupoid) correspondence*, or a *generalised homomorphism* from $\mathcal{G}$ to $\mathcal{H}$, is a space Z with commuting left $\mathcal{G}$-action and right $\mathcal{H}$-action

$$\mathcal{G} \supseteq \mathcal{G}^{(0)} \xleftarrow{r_Z} Z \xrightarrow{s_Z} \mathcal{H}^{(0)} \subseteq \mathcal{H},$$

or briefly,

$$\mathcal{G} \leftarrow Z \rightarrow \mathcal{H},$$

such that:

- $\mathcal{G} \curvearrowright Z$ is free and proper.
- $Z \curvearrowleft \mathcal{H}$ is proper.
- The moment map $Z \xrightarrow{s_Z} \mathcal{H}^{(0)}$ factors through the homeomorphism $\mathcal{G}\backslash Z \xrightarrow{\cong} \mathcal{H}^{(0)}$. That is, the following diagram commutes:

$$\begin{tikzcd} Z \arrow[rr, "s_Z"] \arrow[dr, "q"'] & & \mathcal{H}^{(0)} \\ & \mathcal{G}\backslash Z \arrow[ur, "\cong"'] & \end{tikzcd}$$

Remark 4.2. Our definition of groupoid correspondences is slightly different from that from [Wil19], wherein the action $Z \curvearrowleft \mathcal{H}$ is not assumed to be proper. However, the properness condition is essential to give a C^*-correspondence.

Example 4.3. Let $\varphi\colon \mathcal{H} \to \mathcal{G}$ be a strict homomorphism. Do we need φ to be proper? Define

$$\operatorname{Graph}(\varphi) := \{(\gamma.u) \in \mathcal{G} \times \mathcal{H}^{(0)} \mid \phi(u) = s(\gamma)\}.$$

It admits a left $\mathcal{G}$-action obviously, and a right $\mathcal{H}$-actions via

$$(\gamma, u)h := (\gamma\varphi(h), s(h)).$$

This gives a correspondence.

As a special case, if $\mathcal{H} \subseteq \mathcal{G}$ is a subgroupoid. Then the inclusion $\mathcal{H} \hookrightarrow \mathcal{G}$ gives a correspondence.

4.1.1 Composition of correspondences

Now we define the composition of correspondences. Given two correspondences

$$\mathcal{G} \xleftarrow{r_Z} Z \xrightarrow{s_Z} \mathcal{H}, \qquad \mathcal{H} \xleftarrow{r_W} W \xrightarrow{s_W} \mathcal{K},$$

Let

$$Z * W := \{(z, w) \mid Z \times W \mid s_Z(z) = r_W(w)\}.$$

$\mathcal{H}$ acts on $Z * W$ on the right via the diagonal action. Consider the quotient

$$Z *_{\mathcal{H}} W := \{(z, w) \in Z \times W \mid s_Z(z) = r_W(w)\}/\mathcal{H}.$$

Since $\mathcal{H} \curvearrowright W$ freely and properly, the space $Z *_{\mathcal{H}} W$ is locally-compact Hausdorff ([Wil19, Proposition 2.18]). We equip it with a free and proper left $\mathcal{G}$-action, and a proper right $\mathcal{K}$-action

$$g[z, w] := [gz, w], \quad [z, w]k := [z, wk], \qquad \text{for } g \in \mathcal{G},\ k \in \mathcal{K},\ [z, w] \in Z *_{\mathcal{H}} W.$$

Definition and Lemma 4.4. If $\mathcal{H}$ has open range and source maps. Then $\mathcal{G} \leftarrow Z *_{\mathcal{H}} W \to \mathcal{K}$ is a correspondence. We define it to be the composition of

$$\mathcal{G} \xleftarrow{r_Z} Z \xrightarrow{s_Z} \mathcal{H} \quad \text{and} \quad \mathcal{H} \xleftarrow{r_W} W \xrightarrow{s_W} \mathcal{K}.$$

Thanks to the composition, we can now define a category of groupoid correspondences.

Definition 4.5. The category Gr of groupoid correspondences consist of the following data:

- Objects are groupoid.
- An arrow from $\mathcal{G}$ to $\mathcal{H}$ is a correspondence $\mathcal{G} \leftarrow Z \to \mathcal{H}$.
- Composition of arrows is given by Definition and Lemma 4.4.

Example 4.6. The correspondence $\mathcal{G} \xleftarrow{r} \mathcal{G} \xrightarrow{s} \mathcal{G}$ is the identity arrow of $\mathcal{G}$ in Gr. In fact, we have

$$\mathcal{G} *_{\mathcal{G}} Z \cong Z, \qquad W *_{\mathcal{G}} \mathcal{G} \cong W$$

for a left $\mathcal{G}$-space Z and a right $\mathcal{G}$-space W.

A correspondence $\mathcal{G} \leftarrow Z \to \mathcal{H}$ is invertible, if there exists another correspondence $\mathcal{H} \leftarrow W \to \mathcal{G}$, such that

$$Z *_{\mathcal{H}} W \cong \mathcal{G}, \qquad W *_{\mathcal{G}} Z \cong \mathcal{H}.$$

4.1.2 Morita equivalences revisited

Proposition 4.7. *A correspondence $\mathcal{G} \leftarrow Z \to \mathcal{H}$ is invertible iff Z is an $(\mathcal{G}, \mathcal{H})$-equivalence.*

Remark 4.8.

- Let $\mathcal{G}$ and $\mathcal{H}$ be groups. Then they are Morita equivalent as groupoids iff they are isomorphic as groups.
- Let $\mathcal{G}$ and $\mathcal{H}$ be spaces. Then they are Morita equivalent as groupoids iff they are homeomorphic as spaces.

Now we revisit the blow-up construction in the previous talk, and show that a Morita equivalence can be lifted to an isomorphism of groupoids.

Let $\mathcal{G} \leftarrow Z \to \mathcal{H}$ be a correspondence. Recall that (Example 3.16) the blow-up groupoid is defined as

$$\mathcal{H}[Z] := \{(z_1, h, z_2) \in Z \times \mathcal{H} \times Z \mid s(z_1) = r(h),\ r(z_2) = s(h)\},$$

and the space

$$W := \{(z, h) \in Z \times \mathcal{H} \mid s(z) = r(h)\}$$

implements a $(\mathcal{H}[Z], \mathcal{H})$-equivalence.

Define

$$\psi \colon \mathcal{H}[Z] \to \mathcal{G}, \qquad (z_1, g, z_2) \mapsto (z_1, gz_2) \in Z *_{\mathcal{H}} Z.$$

Since $\mathcal{G}$ acts on Z freely, the object (z_1, gz_2) uniquely determines an element in $\mathcal{G}$. The composition yields a map $\psi \colon \mathcal{H}[Z] \to \mathcal{G}$.

Proposition 4.9. *Let $\mathcal{G} \leftarrow Z \to \mathcal{H}$ be a correspondence. Then $Z \cong W^{\mathrm{op}} *_{\mathcal{H}[Z]} \mathrm{Graph}(\psi)$. So Z is a composition of a blow-up groupoid with a strict homomorphism.*

4.2 From groupoid correspondences to C*-correspondences

Now we pass to C*-correspondences. As mentioned before, we wish to furnish a functor, which on the object level maps groupoids to their groupoid C*-algebras. The suitable target category is the category of C*-correspondences.

Definition 4.10. The category Corr of C*-correspondences consist of the following data:

- Objects are C*-algebras.
- An arrow from A to B is a (A, B)-correspondence ${}_A X_B$. That is, a right Hilbert B-module X and a *-homomorphism $A \to \mathbb{B}_B(X)$ to the bounded adjointable operators on the Hilbert B-module X.
- Composition of arrows is given by the tensor product of Hilbert C*-modules

$$ {}_A X_B \circ {}_B Y_C := {}_A(X \otimes_B Y)_C.$$

Recall that two C*-algebras A and B are Morita–Rieffel equivalent iff there is an imprimitivity (or Morita equivalence) A, B-bimodule ${}_A E_B$. That is, a Hilbert B-module E together with a *-isomorphism $A \to \mathbb{K}_B(E)$. We write $A \sim_{\mathrm{Morita}} B$ if A and B are Morita–Rieffel equivalent.

The following propositions (sometimes used as definitions) are well-known to C*-algebraists:

Proposition 4.11.

- *$A \sim_{\mathrm{Morita}} B$ iff $A \cong B$ in Corr.*
- *If A and B are separable. Then $A \sim_{\mathrm{Morita}} B$ iff $\mathbb{K} \otimes A \cong \mathbb{K} \otimes B$ as C*-algebras.*

We wish to construct a functor $\mathsf{Gr} \to \mathsf{Corr}$. On the arrow level this means we need to construct a C*-correspondence out of a groupoid correspondence.

Let $\mathcal{G} \leftarrow Z \to \mathcal{H}$ be a groupoid correspondence. Pick any $z \in Z$ with $s(z) = r(\eta)$. Then $\mathrm{C_c}(Z)$ carries a $\mathrm{C_c}(\mathcal{H})$)-valued inner product given by

$$\langle \psi, \phi \rangle(\eta) := \int_{\mathcal{G}} \overline{\psi(g^{-1}z)} \phi(g^{-1}z\eta) \, \mathrm{d}\lambda^{r(z)} g, \qquad \psi, \phi \in \mathrm{C_c}(Z).$$

This is independent of the choice of z because $\mathcal{G} \backslash Z \cong \mathcal{H}^{(0)}$. The integral converges because $\mathcal{G} \backslash Z \cong \mathcal{G}^{(0)}$.

Now $\mathrm{C_c}(Z)$ is a $(\mathrm{C_c}(\mathcal{G}), \mathrm{C_c}(\mathcal{H}))$-bimodule: the left $\mathrm{C_c}(\mathcal{G})$-module structure is

$$f \cdot \phi(z) := \int_{\mathcal{G}} f(g)\phi(g^{-1}z)\,\mathrm{d}\lambda^{r(z)}g;$$

and the right $\mathrm{C_c}(\mathcal{H})$-module structure is given by

$$\phi \cdot g(z) := \int_{\mathcal{G}} \phi(zh)g(h^{-1})\,\mathrm{d}\lambda^{s(z)}h.$$

One checks that the followings are satisfied:

$$\langle f^*\psi, \phi\rangle = \langle \psi, f\phi\rangle, \qquad \langle \psi, \phi \cdot g\rangle = \langle \psi, \phi\rangle * g, \qquad \langle f\phi, f\phi\rangle \leq \|f\|_{\mathrm{C}^*(\mathcal{G})}\langle \phi, \phi\rangle.$$

The last equality guarantees that

$$\|\phi\|^2 := \|\langle \phi, \phi\rangle\|_{\mathrm{C}^*(\mathcal{H})}$$

defines a C^*-norm on $\mathrm{C_c}(Z)$.

Denote by $X(Z)$ the right $\mathrm{C}^*(\mathcal{H})$-module completion of $\mathrm{C_c}(Z)$. Upon replacing all norms on convolution algebras by the reduced norms we obtain a reduced version $X_{\mathrm{r}}(Z)$ as a right $\mathrm{C}^*_{\mathrm{r}}(\mathcal{H})$-module.

Theorem 4.12 ([MSO99, Tu04])**.** ${}_{\mathrm{C}^*(\mathcal{G})}X(Z)_{\mathrm{C}^*(\mathcal{H})}$ *and* ${}_{\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})}X_{\mathrm{r}}(Z)_{\mathrm{C}^*_{\mathrm{r}}(\mathcal{H})}$ *are* C^**-correspondences.*

Theorem 4.13 ([MRW87])**.** *Let $G \leftarrow Z \rightarrow W$ be a $(\mathcal{G}, \mathcal{H})$-equivalence. Then $X(Z)$ and $X_{\mathrm{r}}(Z)$ are imprimitivity bimodules.*

Theorem 4.14. *There is a functor* $\mathsf{Gr} \to \mathsf{Corr}$*, which:*

- *on the object level, sends a groupoid $\mathcal{G}$ to its groupoid* C^**-algebra* $\mathrm{C}^*(\mathcal{G})$*.*
- *on the arrow level, sends a groupoid correspondence $\mathcal{G} \leftarrow Z \rightarrow \mathcal{H}$ to a* C^**-correspondence* ${}_{\mathrm{C}^*(\mathcal{G})}X(Z)^*_{\mathrm{C}}(\mathcal{H})$*.*

Corollary 4.15. *Morita equivalent groupoids have Morita–Rieffel equivalent* C^**-algebras.*

Remark 4.16. The most difficult part of the proof is to show that $\mathrm{C_c}(Z *_{\mathcal{H}} W)$ is dense in $X(Z) \otimes_{\mathrm{C}^*(\mathcal{H})} X(W)$, so that the composition of groupoid correspondences is sent to the composition of C^*-correspondences. For this to be true, we have different choices of axioms for a groupoid correspondence. One option is in [Hol17].

Remark 4.17. An alternative proof of the equivalence theorem of reduced groupoid C^*-algebras (Corollary 4.15) is given in [SW12]. The main ingredient is the *linking groupoid.* Let $\mathcal{G} \leftarrow Z \rightarrow \mathcal{H}$ be a $(\mathcal{G}, \mathcal{H})$-equivalence. The linking groupoid is defined as

$$L := \mathcal{G} \sqcup Z \sqcup Z^{\mathrm{op}} \sqcup \mathcal{H} \rightrightarrows \mathcal{G}^{(0)} \sqcup \mathcal{H}^{(0)}.$$

The source and range maps are inherited from the source and range maps of $\mathcal{G}$, Z, Z^{op} and $\mathcal{H}$. The multiplication of L restricts to the multiplication of $\mathcal{G}$ and $\mathcal{H}$, and the groupoid actions of $\mathcal{G}$ and $\mathcal{H}$ on Z and Z^{op}. The inverse of L restricts to the inverse of $\mathcal{G}$ and $\mathcal{H}$, and the identity homemorphism $Z \to Z^{\mathrm{op}}$.

The data above define a groupoid; in particular, it accommodates a Haar system if so do $\mathcal{G}$ and $\mathcal{H}$. This is because the actions of $\mathcal{G}$ and $\mathcal{H}$ on Z induce homeomorphisms $Z/\mathcal{H} \to \mathcal{G}^{(0)}$ and $\mathcal{G}\backslash Z \to \mathcal{H}^{(0)}$. Thus, given any $u \in \mathcal{G}^0$, pick any $z \in Z$ with $r(z) = u$. There is a Radon measure ν^u on Z determined by the linear functional

$$\mathrm{C_c}(Z) \to \mathbb{C}, \qquad \phi \mapsto \int_{\mathcal{H}} \phi(z\eta)\,\mathrm{d}\lambda^{s(z)}\eta,$$

which is supported on the orbit of z under the $\mathcal{H}$-action. It is independent of the choice of z in the source fibre due to the left-invariance of λ.

Therefore,

$$\int_Z \phi(z)\,\mathrm{d}\lambda^{s(z)} := \int_{\mathcal{H}} \phi(z\eta)\,\mathrm{d}\nu^u\eta$$

determines a Haar system $\{\nu^u\}_{u\in\mathcal{G}^0}$. A similar construction applies to Z^{op} and yield a Haar system indexed by $\mathcal{H}^0$. These, together with the Haar system of $\mathcal{G}$ and $\mathcal{H}$, furnish a Haar system of L.

Eventually, Sims and Williams showed that the full (reduced) groupoid C^*-algebra $\mathrm{C}^*(L)$ $(\mathrm{C}^*_{\mathrm{r}}(L))$ are isomorphic to the *linking algebra* of the imprimitive bimodule $X(Z)$ $(X_{\mathrm{r}}(Z))$. This provides a uniform and structral setup for equivalence theorems of groupoids and their C^*-algebras.

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