Noncommutative geometry of foliations

NCG-Leiden

Notes taken by Yuezhao Li
Mathematical Institute, Leiden University

2023 fall Semester
Last modified: October 2, 2023

Universiteit Leiden
The Netherlands

Contents

September 12, 2023

Foliations: motivations, definitions and examples

Speaker: Yuezhao Li (Leiden University)

1.1 Motivations

1.1.1 From Frobenius theorem to foliations

A *foliation* on a manifold is, roughly speaking, a decomposition of it into immersed submanifolds (called the *leaves* of the foliation), such that the leaves fit together nicely. The theory of foliations is a tremendous component in modern differential geometry, contributed by several big names: Ehresmann, Reeb, Haefliger, Novikov, Thurston, Molino, Sullivan, Connes and many others.

The idea of foliations comes from a much older task: solving differential equations. This can be best by the celebrated Frobenius theorem[1]. Let M be a smooth manifold. Let E be a smooth rank-k distribution over M, that is, a rank-k subbundle of TM. An *integral manifold* of E is an immersed submanifold $N \subseteq M$ such that $T_pN = E_p$ for every $p \in N$. A distribution E is said to be *integrable* if every $x \in M$ is contained in an integral manifold of E. A distribution E is said to be *involutive* if $\Gamma(E)$, the space of smooth sections of E, is a Lie subalgebra of $\Gamma(TM)$ under the Lie bracket.

Theorem 1.1 (Frobenius)**.** *A distribution is integrable iff it is involutive.*

The Frobenius theorem is a vast generalisation of the classical existence theorems in differential equations, that is, the existence (and uniqueness) of solution (integral curve) of linear partial differential equations. The involutivity of a distribution can be viewed as its "local integrability", which by Frobenius theorem implies the existence of an integral manifold at every point. In modern language, the Frobenius theorem says that an involutive distribution gives a foliation of the base manifold, whose leaves are those maximal connected integral submanifolds.

Geometry serves as a big source of foliations. Every submersion of manifolds define a foliation on the total space, whose leaves are the connected components of fibres (Example 1.7). In particular, a fibre bundle defines a foliation of the total space. Another interesting example is the symplectic foliation of a Poisson manifold, which is generated by its Hamiltonian vector fields.

Dynamical systems also supply many interesing instances of folitions. Let G be a Lie group that acts on a smooth manifold M. Suppose that the stabliser groups have constant dimension. Then M is foliated by the connected components of G-orbits (Example 1.8). The dimension condition is necessary here as otherwise the leaves will not have the same dimension. That would be an instance of a singular foliation and an orbifold.

1.1.2 Why noncommutative geometry?

Let $(M, \mathscr{F})$ be a foliated manifold, that is, a manifold together with a foliation thereon. We wish to study the geometry of two "spaces":

1. The geometry of the leaves (generic fibre) L.
2. The geometry of the space of leaves $M/\mathscr{F}$, obtained from the equivalence relation generated by leaves.

However, there is several technical difficulty for us to study them, even in many simplest examples.

- A leaf L of a foliation can be usually non-compact. For example, every leaf of the Kronecker foliation (Example 1.17) is diffeomorphic to $\mathbb{R}$. This non-compactness, in many cases, obstructs us to pass from local (leaves) to global (the space M).

[1] Despite of the name, this theorem is indeed not proven by Frobenius, at whose age the concept of manifolds were not yet established.

- The quotient space $M/\mathscr{F}$ is usually badly behaved as a topological space. In the case of the Kronecker foliation, every leaf is dense in $\mathbb{T}^2$. So the quotient space is necessarily non-Hausdorff.
- We would like to define a measure on the quotient space, so that we can (1) talk about integrals (2) make the word "generic" precise, in the sense of probability. But $M/\mathscr{F}$ is also quite terrible as a measurable space due to ergodicity. That is, every measurable function of $M/\mathscr{F}$ is necessarily constant almost everywhere. Hence $L^p(M/\mathscr{F}) = \mathbb{C}$ for every $p \in [1, \infty]$.

These force us to seek other geometric objects to replace $M/\mathscr{F}$, and leads to Connes' C^*-algebras of foliations and noncommutative geometry. Roughly, these can be summarised as follows:

- The leaves of a foliation generate an equivalence relation of the manifold, and hence a *groupoid* $\mathrm{Hol}(M, \mathscr{F})$, called the holonomy groupoid of the foliation. This is a noncommutative object which encodes the topology of a foliation.
- By some standard techniques introduced by Renault, one can define a C^*-algebra $\mathrm{C}^*(M, \mathscr{F})$ of this groupoid. This is known as Connes' C^*-algebra of foliation.
- A measure on $M/\mathscr{F}$ should be by a measure on M that is holonomy invariant. This is called a transverse measure of the foliation. With such as input, one may get a trace on $\mathrm{C}^*(M, \mathscr{F})$ and hence an "index map" $\mathrm{K}_*(\mathrm{C}^*(M, \mathscr{F})) \to \mathbb{R}$. This reveals the index theory of a foliation.

Thus we have the following nice machine for foliations:

$$\boxed{\begin{array}{c}\text{Foliated manifold}\\ (M, \mathscr{F})\end{array}} \Longrightarrow \boxed{\begin{array}{c}\text{Holonomy groupoid}\\ \mathrm{Hol}(M, \mathscr{F})\end{array}} \Longrightarrow \boxed{\begin{array}{c}\text{Foliation } \mathrm{C}^*\text{-algebra}\\ \mathrm{C}^*(M, \mathscr{F})\end{array}}$$

However, a new question emerges: what are the correct *morphisms*? Note that assigning the groupoid C^*-algebra to a groupoid is not functorial. For example, topological spaces and topological groups are both topological groupoids. Whereas groupoid C^*-algebras for spaces are contravariantly functorial, those for groupois are covariantly functorial. So homomorphisms of groupoids are not satisfying, and we will replace them by a more general concepts of morphisms: the *groupoid correspondences*. These were first introduced in [4] and later referred to as "Hilsum–Skandalis morphisms". A brief account of this is Bram's talk at the groupoid seminar [7].

1.2 Definitions and first examples

1.2.1 By foliation atlas

Definition 1.2 (First definition)**.** Let M be a smooth manifold of dimension n. A (regular) *foliation atltas* of codimension q is an atlas of M, each chart of the form

$$(U_i,\ \varphi_i \colon U_i \to \mathbb{R}^{n-q} \times \mathbb{R}^q),$$

such that the transition functions

$$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} \colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

have the form

$$\varphi_{ij}(x, y) = (g_{ij}(x, y), h_{ij}(y)), \qquad x \in \mathbb{R}^{n-q},\ y \in \mathbb{R}^q$$

for some smooth functions $g_{ij} \colon \mathbb{R}^n \to \mathbb{R}^{n-q}$ and $h_{ij} \colon \mathbb{R}^q \to \mathbb{R}^q$.

A (regular) *foliation* of codimension q of M is a maximal foliation atlas $\mathscr{F}$ of M. We write $\operatorname{codim} \mathscr{F}$ for the codimension of $\mathscr{F}$.

A *foliated manifold* is a pair $(M, \mathscr{F})$ of a manifold M together with a foliation $\mathscr{F}$ of it.

Definition 1.3. A *plaque* of a foliation chart $(U_i,\ \varphi_i\colon U_i \to \mathbb{R}^{n-q} \times \mathbb{R}^q)$ is a connected component of the submanifold $\varphi^{-1}(\mathbb{R}^{n-q} \times \{y\})$ for some $y \in \mathbb{R}^q$. Two points $x, y \in M$ belong to the same *leaf* if there exist a sequence of foliation charts $U_1, \dots, U_k$ and a sequence of points $x = p_1, \dots, p_k = y$ with $p_i \in U_i$ and such that p_{i-1} and p_i belong to the same plaque.

A *leaf* is a collection of points that belong to the same leaf.

The *space of leaves* is the quotient space $M\mathscr{F}$ obtained by the equivalence relation generated by the leaves.

One can easily show that leaves are immersed submanifolds of M:

Proposition 1.4. *Let $(M, \mathscr{F})$ be a codimension-q foliated manifold. Then every leaf is an immersed submanifold of M of dimension $n-q$.*

Proof. Choose $x \in M$ and let L be the leaf of x. Every plaque is a smooth chart of dimension $n-q$. If x belongs to a plaque, then every point of the plaque belongs to L by definition. This holds for other plaques containing a point that belongs to the same leaf with x. Therefore L is covered by these plaques and they are smoothly compatible. The plaques are immersed submanifolds of M, and this holds for L as well. □

Definition 1.5. Let $(M, \mathscr{F})$ and $(N, \mathscr{G})$ be foliated manifolds. A morphism between them is a smooth map $f : M \to N$ which preserves the foliation. That means, every leaf of $\mathscr{F}$ is mapped into a leaf of N. We also say such a map is foliated.

Example 1.6 (Trivial foliation). The space $\mathbb{R}^n$ admits a trivial foliation: the foliated atlas consists of a single chart $\mathrm{id}\colon \mathbb{R}^n \to \mathbb{R}^{n-q} \times \mathbb{R}^q$. Similarly, any linear isomorphism $A\colon \mathbb{R}^n \to \mathbb{R}^n$ determines a foliation. The leaves are given by $A^{-1}(\mathbb{R}^{n-q} \times \{y\})$.

Example 1.7 (Submersions). Let $\pi\colon E \to M$ be a submersion. This defines a foliation of E whose leaves are connected components of the fibres of π. The codimension of the foliation equals $\dim M$. If every fibre is connected, then $E/\mathscr{F} = M$. If some fibres are not connected, then $E/\mathscr{F}$ is a quotient of M which is necessarily non-Hausdorff.

In particular, a rank-k vector bundle $\pi\colon E \to M$ gives a foliation of E whose leaves are given by $\pi^{-1}(x) \simeq \mathbb{R}^k$.

Example 1.8 (Lie group actions). Another important source of interesting foliations is smooth dynamical systems. Let G be a Lie group which acts on a smooth manifold M. We wish to foliate M into the connected components of the orbits under the G-action. But this cannot be always true. For example, consider the canonical action of $\mathrm{GL}(2,\mathbb{R})$, or $\mathrm{SL}(2,\mathbb{R})$, or $\mathbb{C}^*$ (viewed as a submanifold of $\mathbb{R}^2$ equipped with its usual product) on $\mathbb{R}^2$. The orbits are $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$. However, if it were a foliation, then these orbits should have the same dimension.

In order to exclude such singular cases, consider the isotropy subgroups at x:

$$G_x := \{g \in G \mid gx = x\}$$

This is a closed subgroup, hence a Lie subgroup. The orbit of x is identified with the homogeneous space G/G_x, and immersed into M. We say that the action of G on M is *foliated*, if G_x has constant dimension. Then all orbits have the same dimension, and the orbits form a foliation of M.

For example, let $G = \mathbb{R}$. Then an $\mathbb{R}$-action on M

$$\mu\colon \mathbb{R} \times M \to M$$

is also called a *flow*. The vector field associated to the flow is

$$X(x) := \left.\frac{\partial \mu(t,x)}{\partial t}\right|_{t=0}.$$

An $\mathbb{R}$-action is foliated, iff its associated vector field X is nowhere vanishing. Then the leaves of the foliation are given by the integral curves of X.

1.2.2 By Haefliger structure

A foliation can be equivalently described in several equivalent ways. The following definition is based on a "Haefliger structure".

Definition 1.9 (Second definition)**.** A codimension-q foliation of a manifold M is given by an open cover $\{U_i\}$ of M together with submersions

$$s_i\colon U_i \to \mathbb{R}^q,$$

such that there are (necessarily unique) diffeomorphisms

$$\gamma_{ij}\colon s_j(U_i \cap U_j) \to s_i(U_i \cap U_j)$$

satisfying:

$$\gamma_{ij} \circ s_j = s_i, \quad \gamma_{ij} \circ \gamma_{jk} = \gamma_{ik}.$$

These maps γ_{ij} are called the *Haefliger cocycles* of the foliation. We briefly call the data $\{U_i, s_i, \gamma_{ij}\}_{ij}$ a *Haefliger structure*.

Equivalence with first definition. Suppose we are given a Haefliger structure $\{U_i, s_i, \gamma_{ij}\}_{ij}$. Let $\{(V_k, \varphi_k)\}_k$ be an atlas of M. Up to refinement, we may assume that each V_k is contained in a single U_{i_k}. Then $s_{i_k}|_{V_k}$ is also a submersion, hence there is a diffeomorphism $\kappa_k\colon s_{i_k}(V_k) \to \mathbb{R}^q$ such that $\kappa_k \circ s_{i_k}$ retricts to the projection $\mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q \to \mathbb{R}^q$ on local charts. That is, $\kappa_k \circ s_{i_k} \circ \varphi_k^{-1} = \mathrm{pr}_{\mathbb{R}^q}$. We claim that $\{V_k, \varphi_k\}$ is a foliation chart in the sense of Definition 1.2: we have

$$\begin{aligned}
\mathrm{pr}_{\mathbb{R}^q} \circ \varphi_{kl}(x,y) &= \mathrm{pr}_{\mathbb{R}^q} \circ \varphi_k \circ \varphi_l^{-1}(x,y)\\
&= \kappa_k \circ s_{i_k} \circ \varphi_l^{-1}(x,y)\\
&= \kappa_k \circ \gamma_{i_k i_l} \circ s_{i_l} \circ \varphi_l^{-1}(x,y)\\
&= \kappa_k \circ \gamma_{i_k i_l} \circ \kappa_l^{-1}(y).
\end{aligned}$$

Conversely, given a foliation atlas $\{U_i, \varphi_i\colon U_i \to \mathbb{R}^{n-q} \times \mathbb{R}^q\}$ such that $\varphi_{ij}(x,y) = (g_{ij}(x,y), h_{ij}(y))$. Set $s_i := \mathrm{pr}_{\mathbb{R}^q} \circ \varphi_i$ and $\gamma_{ij} := h_{ij}$. These render the desired Haefliger cocycles. □

Example 1.10. Let $\pi\colon E \to M$ be a rank-$(n-q)$ vector bundle and $\dim M = q$. This gives a codimension-q foliation on M whose leaves are $\pi^{-1}(x)$ for $x \in M$. Let $\{U_i, \varphi_i\colon U_i \to \mathbb{R}^q\}$ be an atlas of M, and $\{\Phi_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{R}^{n-q}\}$ be the associated local trivialisations of E. Then the foliation chart on $\pi^{-1}(U_i)$ is just the composition of the local trivialisations Φ_i, with the chart of U_i. The Haefliger cocycles are just the transition functions $h_{ij}\colon U_i \cap U_j \to \mathrm{GL}(q, \mathbb{R})$ of the base manifold M.

1.2.3 By involutive distribution

The alternative definition below justifies the connection between foliations and the Frobenius theorem:

Definition 1.11 (Third definition)**.** A codimension-q foliation $\mathscr{F}$ of a manifold M is given by an involutive distribution.

The involutive distribution is called the *tangent bundle* of the foliation $\mathscr{F}$ and denoted by $T\mathscr{F}$.

The equivalence with the first definition is heavily based on the Frobenius theorem. We will omit the proof and only sketch how to derive these definitions from each other. The proof can be found in [6, Section 1.2]; see also [5, Theorem 19.21].

Let $E \subseteq TM$ be an involutive distribution. By Frobenius theorem, it is integrable: every point $x \in M$ belongs to an integral manifold. These manifolds form a foliation of M.

Conversely, if $(M, \mathscr{F})$ is a foliated manifold. For every $x \in M$, let L be the leaf of x. Then the collection $\{T_x L\}_{x \in M}$ is a vector subbundle of TM which is involutive. We denote this bundle by $T_x\mathscr{F}$ and also write $T_x\mathscr{F}$ for $T_x L$.

Example 1.12. A vector bundle $E \to M$ gives a foliation $\mathscr{F}$ on the total space E. The tangent bundle of the foliation $T\mathscr{F}$ is just the vertical bundle of $E \to M$: $VE := \ker T\pi$. Note that every leaf is vertical in the sense that $T\mathscr{F} \subseteq \ker T\pi$.

Example 1.13. Let G be a Lie group which has a foliated action on M. Then $T\mathscr{F}$ is spanned by the fundamental vector fields of this group action. Namely, the image of the bundle homomorphism

$$\mathfrak{g} \times M \to TM, \qquad (\xi, x) \mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} x \exp(t\xi).$$

Remark 1.14. We have seen so far that a Lie group action generates a foliation necessarily when the dimensions of the stabliser groups are constant. If this is not the case, then we speak of a *singular foliation*. The following definition (due to Peter Stefan, Iakovos Androulidakis and Georges Skandalis) is modelled on the third definition that we have introduced above.

Definition 1.15. A *(singular) foliation* is a locally finitely generated, involutive $\mathrm{C}^\infty(M)$-submodule of $\Gamma_\mathrm{c}(M, TM)$.

Singular foliations are much more complicated to deal with as opposed to those regular ones. For example, the holonomy groupoid of a singular foliation is hard to define. It seems that the most general definition so far is the one given in [1]. We might talk about this in a later talk.

1.3 Constructions and more examples

1.3.1 Product foliations

Let $(M, \mathscr{F})$ and $(N, \mathscr{G})$ be foliated manifolds, given by a Haefliger structure $(U_i, s_i, \gamma_{ij})_{ij}$ on M and $(U'_k, s'_k, \gamma'_{kl})$ on N. There is a *product foliation* $\mathscr{F} \times \mathscr{G}$ on $M \times N$, given by the Haefliger structure

$$(U_i \times U'_k, s_i \times s'_k, \gamma_{ij} \times \gamma'_{kl})_{ijkl}.$$

The product foliation $\mathscr{F} \times \mathscr{G}$ has codimension $\operatorname{codim}\mathscr{F} + \operatorname{codim}\mathscr{G}$. The tangent bundle $T(\mathscr{F} \times \mathscr{G}) \simeq T\mathscr{F} \oplus T\mathscr{G} \subseteq TM \oplus TN \simeq T(M \times N)$.

1.3.2 Pullback foliations along transverse maps

Definition 1.16. Let $(M, \mathscr{F})$ be a foliated manifold. A smooth map $f : N \to M$ is *transverse* to $\mathscr{F}$, if f is transverse to all the leaves of $\mathscr{F}$. That is, for every $x \in N$,

$$T_{f(x)}M = T_{f(x)}\mathscr{F} + T_x f(T_x N).$$

Let $\mathscr{F}$ be given by the Haefliger structure $(U_i, s_i, \gamma_{ij})_{ij}$. We claim that the data

$$\big(V_i := f^{-1}(U_i), \quad s'_i := s_i \circ f|_{V_i}, \quad \gamma_{ij}\big)_{ij}$$

defines a Haefliger structure. Its determined foliation is called the *pullback foliation* on N along f, denoted by $f^*\mathscr{F}$.

Proof. We must show that the maps $s'_i \colon V_i \to \mathbb{R}^q$ are submersion, that is, $T_x s'_i = T_{f(x)}s_i \circ T_x f$ is surjective for every $x \in V_i$. Since f is transverse to $\mathscr{F}$, we have that the map

$$\tilde{f} : T_x V_i \xrightarrow{T_x f} T_{f(x)}U_i \to T_{f(x)}U_i / T_{f(x)}\mathscr{F}$$

is surjective. The submersion s_i is trivial along the leaves, that is, $T_{f(x)}s_i = 0$ on $T_{f(x)}\mathscr{F} \subseteq T_{f(x)}U_i$. So $T_x s_i$ factors through the quotient map $T_{f(x)}U_i \to T_{f(x)}U_i / T_{f(x)}\mathscr{F}$. Thus the diagram

$$\begin{tikzcd}
T_x V_i \arrow[r, "T_x f"] \arrow[dr, two heads, "\tilde{f}"'] & T_{f(x)}U_i \arrow[r, two heads, "T_{f(x)}s_i"] \arrow[d, two heads] & \mathbb{R}^q \\
& T_{f(x)}U_i / T_{f(x)}\mathscr{F} \arrow[ur, "\tilde{s}_i"'] &
\end{tikzcd}$$

commutes. So $T_{f(x)}s_i \circ T_x f = \tilde{s}_i \circ \tilde{f}$ is the composition of two surjective maps, hence surjective. $\square$

We have $\operatorname{codim} f^*\mathscr{F} = \operatorname{codim}\mathscr{F}$ and $Tf^*\mathscr{F} = (Tf)^{-1}(T\mathscr{F})$.

1.3.3 Quotient foliations of covering space actions

Let $(M, \mathscr{F})$ be a foliated manifold such that M carries a free, properly discontinuous action of a discrete Lie group G. Then the quotient manifold M/G is Hausdorff. Assume that $\mathscr{F}$ is *invariant* under the G-action, that is, every $g \in G$ is an automorphism of the foliation $(M, \mathscr{F})$. Then there is an induced foliation $\mathscr{F}/G$ on M/G as follows.

Since G acts freely and properly discontinuously, the quotient map $\pi\colon M \to M/G$ defines a covering space. Let $\{(U_i, \varphi_i\colon U_i \to \mathbb{R}^{n-q} \times \mathbb{R}^q)\}_i$ be a foliation atlas of $\mathscr{F}$. Then π is a local diffeomorphism. We may assume that $\pi|_{U_i}$ is a diffeomorphism by replacing U_i by its refinement. Then it has an inverse section s_i and

$$\{(\varphi_i \circ s_i\colon \pi(U_i) \to \mathbb{R}^{n-q} \times \mathbb{R}^q)\}_i$$

renders a foliation $\mathscr{F}/G$ for M/G. The leaves of $\mathscr{F}/G$ are quotient manifolds L/G_L, where L is a leaf of $\mathscr{F}$ and G_L is the isotropy subgroup of the leaf L. The latter is well-defined because every $g \in G$ maps a leaf into a leaf.

We have $\operatorname{codim}(\mathscr{F}/G) = \operatorname{codim}(\mathscr{F})$ and $T(\mathscr{F}/G) = T\pi(T\mathscr{F})$.

Example 1.17 (Kronecker foliation). Let $\vartheta \in \mathbb{R} \setminus \mathbb{Q}$ and define the submersion

$$s\colon \mathbb{R}^2 \to \mathbb{R}, \qquad s(x,y) = x - \vartheta y.$$

Then it defines a foliation on $\mathbb{R}^2$ following Example 1.6. The group $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ freely and properly discontinuously, and the aforementioned foliation on $\mathbb{R}^2$ is $\mathbb{Z}^2$-invariant. So there is an induced foliation $\mathscr{F}$ on the quotient manifold $\mathbb{T}^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$. This is known as the *Kronecker foliation* on $\mathbb{T}^2$. Equivalently, one can consider the integral curves generated by the differential equation $\frac{\mathrm{d}x}{\mathrm{d}y} = \vartheta$.

Example 1.18 (Foliation of the Möbius band). Recall that the Möbius bundle is the quotient space of $\mathbb{R}^2$:

$$\text{Möb} := \mathbb{R}^2/\!\sim, \qquad (x,y) \sim (x+1, -y).$$

The projection $\mathbb{R}^2 \to \text{Möb}$ is a two-fold covering space. The trivial codimension-1 foliation of $\mathbb{R}^2$ is invariant under this group action, hence there is a quotient foliation of Möb. The interesting property of this foliation is that every leaf (which is diffeomorphic to $\mathbb{T}$) will wrap around the whole band twice except for the middle one generated by the plaques with $y = 0$.

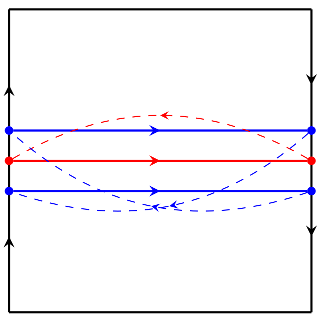

Figure 1.1: A foliation of the Möbius band

1.3.4 Suspensions

Suspensions of diffeomorphism groups are special cases of the quotient foliation. They consititute a large class of interesting foliations.

Let F be a smooth manifold and $f : F \to F$ be an automorphism. Consider the trivial 1-dimensional foliation on $M = \mathbb{R} \times F$, i.e. the leaves are $\mathbb{R} \times \{x\}$ for $x \in F$. There is a $\mathbb{Z}$-action on M generated by the diffeomorphism

$$(t, x) \mapsto (t + 1, f(x)).$$

The action is free and proper, and leaves the foliation on M invariant. So there is a quotient foliation on the quotient manifold $\mathbb{R} \times_{\mathbb{Z}} F$. This is called the *suspension* of the diffeomorphism f, or the suspension of the group $\mathbb{Z}$ generated by f.

1.3.5 Flat bundles

The suspensions are a special case of flat bundles. Let M be a manifold and $\widetilde{M}$ be the universal cover of M. Then $\widetilde{M} \to M$ is a (right) principal $\pi_1(M)$-bundle. Assume that $\pi_1(M)$ acts on another manifold F. Then there is an associated fibre bundle

$$\widetilde{M} \times_{\pi_1(M)} F \to M.$$

The space $\widetilde{M} \times_{\pi_1(M)} F$ is the quotient of $\widetilde{M} \times F$ under the equivalence relation $(x, y) \sim (xg^{-1}, gy)$.

The submersion $\widetilde{M} \times F \to F$ generates a foliation of $\widetilde{M} \times F$ which is invariant under the action of $\pi_1(M)$. So there is a quotient foliation on $\widetilde{M} \times_{\pi_1(M)} F$. This is called a *flat bundle*. In contrast to the foliation coming from a submersion, in which every leaf is vertical in the sense that $T\mathscr{F} \subseteq \ker T\pi$, in a flat bundle all leaves are horizontal, that is, $T_x\mathscr{F} \simeq T_{\pi(x)}M$.

The suspension is the special case where $M = \mathbb{T}$ and the action of $\pi_1(M) = \mathbb{Z}$ on F is generated by the diffeomorphism f.

1.3.6 Reeb foliation

Foliations can be defined on manifolds with boundary as well. But this requires that the foliation behaves nicely near the boundary, that is, the foliation is either *transverse* to the boundary or *tangent* to the boundary.

Definition 1.19 ([3, Definition 1.1.11]). Let $(M, \mathscr{F})$ be a foliated manifold, and $N \subseteq M$ be a submanifold. We say that:

- $\mathscr{F}$ is *transverse* to N, denoted by $\mathscr{F} \pitchfork N$, if every leaf of $\mathscr{F}$ is transverse to N. That is,

  $$T_xM = T_x\mathscr{F} + T_xN$$

  for every $x \in N$.
- $\mathscr{F}$ is *tangent* to N, if $T_x\mathscr{F} \subseteq T_xN$ for every $x \in N$. Equivalently this means every leaf L of $\mathscr{F}$ is either disjoint from N, or contained in N.

The *Reeb foliation* of the solid torus $X = \mathbb{D}^2 \times \mathbb{T}$ is a foliation on a manifold with boundary, of the second sort. Let

$$\mathbb{D}^2 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

be the unit disk. Foliate the solid cylinder $\mathbb{D}^2 \times \mathbb{R}$ by the submersion

$$f : \operatorname{Int}(\mathbb{D}^2) \times \mathbb{R} \to \mathbb{R}, \qquad f(x, y, t) = \exp\left(\frac{1}{1 - x^2 - y^2}\right) - t$$

and require that the boundary of the solid cylinder is another leaf. This gives a foliation of the solid cylinder.

The translation action of $\mathbb{Z}$ on the second entry of $\mathbb{D}^2 \times \mathbb{R}$ preserves this foliation. So there is a quotient foliation on solid torus $X = \mathbb{D}^2 \times \mathbb{R}/\mathbb{Z} \simeq \mathbb{D}^2 \times \mathbb{T}$. This is the *Reeb foliation* on the solid torus. Note that every leaf is diffeomorphic to $\mathbb{R}^2$ except for the boundary leaf $\mathbb{T}^2$.

Now we describe the Reeb foliation on $\mathbb{S}^3$. A statement from topology says that $\mathbb{S}^3$ can be obtained by gluing together two copies of the solid torus along the boundary $\mathbb{T}^2$:

$$\mathbb{S}^3 = X \cup_{\partial X} X, \qquad X = \mathbb{D}^2 \times \mathbb{T}, \qquad \partial X = \mathbb{T}^2.$$

The Reeb foliation can be glued together to form a foliation on $\mathbb{S}^3$ as well. This does happen in general: one needs certain "tameness" conditions in order to glue two foliations together along the boundary; we shall, however, not talk about this in detail. Then we obtain the *Reeb foliation* on $\mathbb{S}^3$. It has all leaves diffeomorphic to $\mathbb{R}^2$ except for the "boundary" leaf of solid cylinder, which is compact.

The importance of the Reeb foliation is illustrated by the following celebrated theorem:

Theorem 1.20 (Novikov's compact leaf theorem)**.** *Every codimension-*1 *foliation of* $\mathbb{S}^3$ *has a compact leaf, bounding a solid torus with the Reeb foliation.*

So the Reeb foliation plays an important role in the foliation theory of $\mathbb{S}^3$.

September 19, 2023

Holonomy and stability

Speaker: Yufan Ge (Leiden University)

2.1 Holonomy

2.1.1 Motivation: Poincaré return map

The idea of holonomy goes back to the Poincaré return map. Let $(\phi, M, \mathbb{R})$ be a global dynamical system, where

$$\phi : \mathbb{R} \times M \to M$$

is the flow. Suppose that γ is a periodic orbit, that is, there exists a time T such that $\phi(T, x) = x$ for every $x \in \gamma$. The orbit is thus a circle. One is then motivated to study the property of this closed orbit. The idea of Poincaré is to look at how the points near this closed orbit behave with time. This is done through the Poincaré section and the Poincaré return map.

Let $p \in \gamma$ and let S be a local smooth section through p which is transverse to the flows of ϕ. Such an S is called a *Poincaré section*. A function $P : U \to S$ from an open and connected neighbourhood $U \subseteq S$ of p to S is called a *Poincaré map*. if the followings hold:

- $P(p) = p$.
- $P(U)$ is diffeomorphism to U.
- For every $x \in U$, $P(x) = \phi(t_0, x)$, where $t_0 = \min\{t \in \mathbb{R} \mid \phi(t, x) \in S\}$. Namely, t_0 is the first time that x returns to S.

The idea of holonomy is reminiscent of that of the Poincaré return maps, which describes the behaviour of all the nearby leaves of a given leaf.

2.1.2 The holonomy map over a path

Let $(M, \mathscr{F})$ be a foliated manifold. Let α be a smooth path in a leaf L of $\mathscr{F}$ from x to y. That is,

$$\alpha : [0,1] \to L, \qquad \alpha(0) = x, \quad \alpha(1) = y.$$

We wish to define the holonomy map as a local map "along" the leaves, "over" the path α, which takes x to y. More precisely, it is defined as an equivalence class of smooth maps called a germ:

Definition 2.1. Let M and N be smooth manifolds. Let $x \in M$ and $y \in N$. A *germ of smooth maps* from x to y is an equivalence class of smooth maps

$$f : U \to V, \quad \text{satisfying } f(x) = y$$

where U is an open neighbourhood of x and y is an open neighbourhood of y. Two smooth maps $f : U \to V$ and $f' : U' \to V'$ are equivalent as germs if there exists an open neighbourhood $W \subseteq U \cap U'$ of x such that $f|_W = f'|_W$. We write germ(f) for the germ of f.

Similarly, one can define a *germ of local diffeomorphisms* from x to y, by requiring in addition that f is a local diffeomorphism, that is, a diffeomorphism from an open neighbourhood of x to its image. Clearly, if f is a local diffeomorphism, then the germ (of diffeomorphisms) of f is the same thing as the germ of smooth maps of f.

Let $x \in M$. Denote the set of germs of local diffeomorphisms $f : (M, x) \to (M, x)$ by $\mathrm{Diff}_x(M)$. Clearly this is a group under the composition of smooth maps.

The holonomy of a leaf L along a path $\alpha\colon [0,1] \to L$ will be defined to be a germ of smooth maps from $x = \alpha(0)$ to $y = \alpha(1)$. In order to do that, a choice of transverse section across x, resp. y, is needed.

Definition 2.2. Let $(M, \mathscr{F})$ be a foliated manifold of codimension q. A *transverse section*, or a *transversal*, is a q-dimensional submanifold of M which is transverse to $\mathscr{F}$ (Definition 1.19).

Definition and Lemma 2.3. Let L be a leaf of $(M, \mathscr{F})$ and $\alpha\colon [0,1] \to L$ be a smooth path in L. Let $x = \alpha(0)$ and $y = \alpha(1)$. Let T and S be transverse sections of $(M, \mathscr{F})$ such that $x \in T$ and $y \in S$.

The *holonomy map* $\mathrm{hol}^{S,T}(\alpha)$ of the leaf L along the path α, with respect to the transversals T and S, is defined as follows.

1. Assume first that $\alpha([0,1])$ is contained in a single foliation chart. Then x and y belong to the same plaque. *There exists a smooth map $f : A \to S$, where $A \subseteq U$ is an open subset containing x, such that:*

   - *$f(x) = y$;*
   - *For any $x' \in A$, $f(x')$ and x' belongs to the same plaque.*
   - *f is a local diffeomorphism at x.*

   In particular, the germ of f is uniquely determined. Then we define the holonomy map as the germ of f:
   $$\mathrm{hol}^{S,T}(\alpha) := \mathrm{germ}(f).$$

2. In the general case, *there exists a finite sequence of foliation charts $\{U_0, \dots, U_k\}$ such that* $\alpha([\frac{i}{k}, \frac{i+1}{k}]) \subseteq U_i$. Set $x_i = \alpha(\frac{i}{k})$ and α_i to be the restriction of α to $[\frac{i}{k}, \frac{i+1}{k}]$. Choose transversals $T = T_0, T_1, \dots, T_n = S$ of $\mathscr{F}$ at $x = x_0, x_1, \dots, x_n = y$. Define
   $$\mathrm{hol}^{S,T}(\alpha) := \mathrm{hol}^{T_k, T_{k-1}}(\alpha_{k-1}) \circ \mathrm{hol}^{T_{k-1}, T_{k-2}}(\alpha_{k-1}) \circ \cdots \circ \mathrm{hol}^{T_1, T_0}(\alpha_0).$$

   In particular, $\mathrm{hol}^{S,T}(\alpha)$ *does not depend on the intermediate transversals $T_1, \dots, T_{k-1}$.*

Detailed proofs of the stataments above can be found in [2, Chapter IV], which we omit here. To understand this construction, let us work in a foliation chart $(U, \varphi\colon U \to \mathbb{R}^{n-q} \times \mathbb{R}^q)$ centered at x. Then x belongs to the plaque $\varphi^{-1}(\mathbb{R}^{n-q} \times \{0\})$. A transverse section S at x has dimension q and intersects $T_x\mathscr{F}$ transversely, which means that

$$T_x M = T_x S \oplus T_x \mathscr{F}.$$

Therefore, locally S intersects the leaf of x, and *every leaf* in a neighbourhood of x, at a single point. We may assume that U is small enough so that S intersects every leaf in U at a unique point. Then S is given by $\varphi^{-1}(\{0\} \times \mathbb{R}^q)$ up to a coordinate change.

Now assume that $\alpha: [0,1] \to L$ is contained in a single chart U. Assume without loss of generality that the transverse sections T at $x = \alpha(0)$ and S at $y = \alpha(1)$ intersect the every leaf in U at a unique point. Then the holonomy map $\mathrm{hol}^{S,T}(\alpha)$ is defined as follows: for any $x' \in T$, which belongs to a leaf L'. Then $\mathrm{hol}^{S,T}(\alpha)(x')$ is the unique point that belongs to the intersection $S \cap L'$.

These process can be done for paths which do not lie in a unique foliation chart as well, whereas the *holonomy groups* become significant as they might be non-trivial.

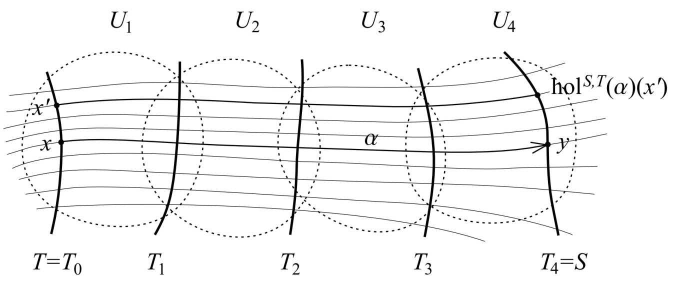

Figure 2.1: Holonomy maps. This figure is taken from [6, Fig 2.1].

Proposition 2.4. *The holonomy maps have the following properties:*

1. *If α is a path in L from x to y, and β is a path in L from y to z. Let T, S, R be transverse sections of $\mathscr{F}$ at x, y, z. Then*
$$\mathrm{hol}^{R,T}(\beta\alpha) = \mathrm{hol}^{R,S}(\beta) \circ \mathrm{hol}^{S,T}(\alpha).$$
2. Homotopy invariance. *If α and α' are paths which are homotopic in a leaf with basepoints fixed. Then* $\mathrm{hol}^{S,T}(\alpha) = \mathrm{hol}^{S,T}(\alpha')$.
3. *Let α be a path in L from x to y. Let T, T' be transverse sections at x and S, S' be transverse sections at y. Then*
$$\mathrm{hol}^{S',T'}(\alpha) = \mathrm{hol}^{S',S}(\bar{y}) \circ \mathrm{hol}^{S,T}(\alpha) \circ \mathrm{hol}^{T,T'}(\bar{x}),$$
where $\bar{x}$ (resp. $\bar{y}$) stands for the identity path at x (resp. at y).

2.1.3 The honolomy group

Definition 2.5. Let $(M, \mathscr{F})$ be a foliated manifold and L a leaf of $\mathscr{F}$. The *holonomy group* of L at x with respect to the transverse section T, denoted by $\mathrm{Hol}(L, T, x)$, is defined as the image of the map
$$\mathrm{hol}^{T,T} : \pi_1(L, x) \to \mathrm{Diff}_x(T) \simeq \mathrm{Diff}_0(\mathbb{R}^q).$$

It is clear (indeed, from 3 of the previous proposition) that a different choice of T, or a different choice of $x \in L$, results in a conjugacy of the holonomy map by a germ of diffeomorphism. Therefore we may define $\mathrm{Hol}(L)$ to be the conjugacy class of the holonomy maps in all of those $\mathrm{Hol}(L, T, x)$ with $x \in L$ and T a transverse section at x.

The following result guarantees that it is sufficient to consider only $\mathrm{Im}(\mathrm{hol}^{T,T})$:

Theorem 2.6 (Transverse uniformity of leaves, [2, §III.2, Theorem 3])**.** *Let L be a leaf of $\mathscr{F}$ and $x, y \in L$. Then there exists transversals T at x and S at y, together with a local diffeomorphism f from an open subset $U \subseteq T$ to S, such that for any leaf L':*
$$f(L' \cap T) = L' \cap S.$$

Therefore, we only need to consider the first return map.

Example 2.7. If L is simply connected. Then $\pi_1(L, x) = 0$ and hence $\mathrm{Hol}(L) = 0$.

Example 2.8 (Foliation of the Möbius band). A foliation of the Möbius band, as in Example 1.18, has two distinct classes of leaves: the middle leaf "wraps around" the base only once, whereas every other leaf wraps around the base twice. This can be made precise with holonomy.

Identify Möbius bundle with the quotient space of $\mathbb{R}^2$:

$$\text{Möb} := \mathbb{R}^2/\sim, \qquad (x,y) \sim (x+1,-y).$$

Every leaf is diffeomorphic to $\mathbb{S}^{(1)}$, whose fundamental group $\pi_1(\mathbb{S}^{(1)}) = \mathbb{Z}$ is generated by a loop of it with winding number 1. The holonomy group of a leaf is thus a quotient of $\mathbb{Z}$. For every other leaf L except for the middle one and choose $x \in L$. There exists a transverse section at x which does not cross the middle leaf L_{mid}. Those leaves which intersect the transverse section are leaves of a trivial foliation of $\mathbb{S}^{(1)} \times I$ for some open interval $I \subseteq \mathbb{R}$ describing this transverse section. Thus $\text{Hol}(L) = 0$.

The middle leaf L_{mid} is the image of the line $y = 0$ under the quotient map $\mathbb{R}^2 \to \text{Möb}$. Choose the basepoint to be $(-1,0) \in L_{\text{mid}}$ and the transverse section be the line $x = -1$. Then $\pi_1(L_{\text{mid}})$ is generated by the straightline path α connecting x and $x' = (1,0)$, which becomes a loop of winding number 1 in Möb. Now for any other point $(-1,-\varepsilon)$ in an open neighbourhood of $(-1,0)$, the holonomy map over α sends $(-1,-\varepsilon)$ to $(1,-\varepsilon) \sim (-1,\varepsilon)$. This means that the generator of $\pi_1(L_{\text{mid}})$ is sent to a generator of degree 2. This implies that $\text{Hol}(L_{\text{mid}}) = \mathbb{Z}/2$.

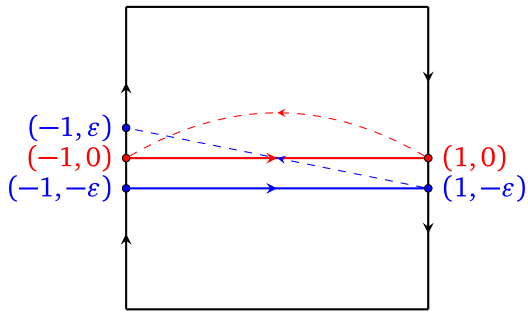

Figure 2.2: Holonomy of the middle leaf of the Möbius band

Example 2.9 (Reeb foliation). Consider the Reeb foliation of the solid torus as in Example 1.3.6. All non-compact leaves (which are diffeomorphic to $\mathbb{R}^2$) are contractible and have trivial holonomy group. The interesting case is the unique compact leaf $L = \mathbb{T}^2$. Identify a transverse section at $x \in L$ with $[0,+\infty)$ where x is identified with 0. Then $\text{Hol}(L)$ is generated by $f : [0,+\infty) \to [0,+\infty)$ with $f(0) = 0$ and $f(t) < t$ for all $t > 0$.

Likewise, the Reeb foliation of the unique compact leaf L of $\mathbb{S}^3$ has holonomy group $\mathbb{Z}^2$ generated by

$$f : \mathbb{R} \to \mathbb{R}, \qquad \begin{cases} f(t) = t, & t \leq 0; \\ f(t) < t, & t > 0. \end{cases}$$

$$h : \mathbb{R} \to \mathbb{R}, \qquad \begin{cases} h(t) = t, & t \geq 0; \\ h(t) > t, & t < 0. \end{cases}$$

Here the transverse section at $x \in L$ is identified with $\mathbb{R}$.

2.2 Stability

Definition 2.10 ([8, Definition 4.2]). Let $(M, \mathscr{F})$ be a foliated manifold. A subset $B \subseteq M$ is called *stable*, if for every open neighbourhood W of B in M, there exists an open neighbourhood $W' \subseteq W$ of B in M, such that every leaf intersecting W' is contained in W.

One of the classical problems in foliation theory is to find necessary and sufficient conditions for a leaf $L \subseteq M$ to be stable. These are given by stability theorems. First of all, there is a close relation between stability and finiteness of the holonomy group.

Example 2.11. Consider the foliation of $\mathbb{R}^2 \setminus \{(0,0)\}$ by cocentred circles $\{(x,y) \mid x^2+y^2=r\}$. Then each leaf is stable and compact, and each leaf has trivial holonomy group.

Example 2.12. The Reeb foliation of either the solid torus or of $\mathbb{S}^3$ has a unique compact leaf. It is unstable, and has infinite holonomy group.

Indeed, we have the following sufficient condition.

Theorem 2.13 ([8, Theorem 4.2])**.** *A compact leaf with finite holonomy group is stable.*

Idea of proof. Let L be a compact leaf with finite holonomy group $\mathrm{Hol}(L)$. Choose a basepoint $x \in L$ and a transverse section Σ at x. The following proof is an excerpt from the reference. I might complete it at some point.

- By compactness, one can find a finite set of foliation charts $\{U_1, \ldots, U_k\}$ covering the leaf.
- Since $\mathrm{Hol}(L,x)$ is finite and $\{U_1, \ldots, U_k\}$ is a finite cover, we are able to shrink the domain, so that all holonomy maps h_j are defined on a common open subset of the transverse section Σ.
- Take the union of all plaques of the form

$$C_y^* = \{\pi_j^{-1}(h_j(f_{\beta_i}(y))) \mid 1 \le i \le k,\ 1 \le j \le r\}$$

  Then one can show that C_y^* is the leaf containing y.
- Finally, set

$$W' = \bigcup_{y \in D'} F_y \subseteq W.$$

  This is the set required in the definition of stability. □

2.2.1 Riemannian foliation

Let M be an n-dimensional manifold. Denote by $\mathfrak{X}(M)$ the space of smooth vector fields over M. A symmetric, $\mathrm{C}^\infty(M)$-bilinear form

$$g : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathrm{C}^\infty(M)$$

is said to be *positive*, if

$$g(X,X) \ge 0, \qquad \text{for all } X \in \mathfrak{X}(M).$$

Let $x \in M$. Set

$$\ker g_x := \{\xi \in T_xM \mid g_x(\xi, T_xM) = 0\}.$$

The *Lie derivative* $\mathscr{L}_X g$ of g in the direction of $X \in \mathfrak{X}(M)$ is the symmetrm C^∞-bilinear form given by

$$\mathscr{L}_X g(Y,Z) := Xg(Y,Z) - g([X,Y],Z) - g(Y,[X,Z]).$$

Definition 2.14. Let $(M, \mathscr{F})$ be a foliated manifold. A *transverse metric* on M is a positive, symmetric $\mathrm{C}^\infty(M)$-bilinear form g such that:

1. $\ker g_x = T_x\mathscr{F}$ for all $x \in M$.
2. $\mathscr{L}_X g = 0$ for any vector field X on M which is tangent to $\mathscr{F}$.

A *Riemannian foliation* $(\mathscr{F}, g)$ of a manifold M is a foliation $\mathscr{F}$ of M together with a transverse metric g.

Remark 2.15. Note that

1. 1 in the previous definition implies thast g is the pullback of a Riemannian structure on the normal bundle $N\mathscr{F}$, along the projection $TM \to N\mathscr{F}$.

2. 2 in the previous definition is a local condition: consider a foliation chart $(U, \varphi\colon U \to \mathbb{R}^{n-q} \times \mathbb{R}^q)$ and set $\varphi(x) = (x_1, \dots, x_{n-q}, y_1, \dots, y_q)$. From 1 we know that g is determined by $g_{ij} := g(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j})$. One can show, after some computation, that 2 holds iff $\frac{\partial g_{ij}}{\partial x_k} = 0$ for all k, i, j. Equivalently, $g|_U$ is the pullback of a Riemannian metric on $\mathbb{R}^2$ along the projection $\mathrm{pr}_{\mathbb{R}^q} \circ \varphi\colon U \to \mathbb{R}^q$.

In fact, we have:

Theorem 2.16. *Let $(M, \mathscr{F})$ be a foliated manifold. Then a Riemannian structure on $N\mathscr{F}$ determines a transverse metric iff it is holonomy invariant.*

Theorem 2.17 ([6, Theorem 2.6])**.** *Let $(\mathscr{F}, g)$ be a Riemannian foliation of M and assume that all leaves are compact. Then each leaf has finite holonomy group.*

Idea of proof. Let L be a leaf of $\mathscr{F}$, $x \in L$, and S a transverse section at x. The induced Riemannian structure of S yields an exponential map

$$\exp_x\colon \mathrm{B}(0, \varepsilon) \to S, \quad \text{such that } \exp_x(0) = x,$$

where $\mathrm{B}(0, \varepsilon)$ is the ε-ball centred at 0 of $T_x M$. Denote the image of $\exp_x$ by U.

Since L is compact, $\pi_1(L)$ is finitely generated, and hence $\mathrm{Hol}(L)$ is finitely generated, say, by holonomy maps $\{h_1, \dots, h_n\}$ where $h_i\colon V_i \to U$ is a diffeomorphism onto the image. We may replace all V_i's by a common domain $V := \cap_{i=1}^n V_i$. By Theorem 2.16, every h_i is an isometry. Thus we may assume that $V = \exp_x(\mathrm{B}(0, \delta))$ for some $\delta \le \varepsilon$ and that $h_i(V) = V$ for all i. Then we have represented $\mathrm{Hol}(L)$ as a group of isometry on V.

Since L is compact, $L \cap S$ is a discrete set. But U, and hence V, is relatively compact in S. Thus $L \cap V \subseteq L \cap S$ is finite. Therefore the orbit of any element in $\mathrm{Hol}(L)$, viewed as an action of a group of isometry, is finite. The differential of this representation at x is a group of orthogonal linear transformations on $T_x V$, such that every orbit of the action is finite. This forces the group $\mathrm{Hol}(L)$ to be finite. $\square$

Corollary 2.18. *Let $(\mathscr{F}, g)$ be a Riemannian foliation of M. If all leaves are compact, then all leaves are stable.*

2.2.2 Reeb stability

Finally, we have the following Reeb stability theorems as consequences of Theorem 2.13.

Theorem 2.19 (Local Reeb stability)**.** *Let $(M, \mathscr{F})$ be a foliated manifold. Let L be a compact leaf with finite holonomy group. Then for each neighbourhood W of L, there is an $\mathscr{F}$-invariant tubular neighbourhood $W' \subseteq W$ with*

$$\pi\colon W' \to L$$

satisfying:

1. *Every leaf $L' \subseteq W'$ is compact with finite holonomy group, hence stable.*
2. *If $L' \subseteq W'$ is a leaf. Then the restriction $\pi|_{L'}\colon L' \to L$ is a finite covering map.*

Theorem 2.20 (Global Reeb stability)**.** *Let $\mathscr{F}$ be a codimension-1 foliation of a compact connected manifold M. If $\mathscr{F}$ admits a compact leaf L with finite fundamental group, then all leaves of $\mathscr{F}$ are compact with finite fundamental group. So all leaves have finite holonomy group and hence stable.*

If $\mathscr{F}$ is transversely orientable, that is, $N\mathscr{F} = TM/T\mathscr{F}$ is orientable. Then every leaf of $\mathscr{F}$ is diffeomorphic to L, and M is the total space of a fibration $f\colon M \to \mathbb{S}^{(1)}$ with fibre L.

September 26, 2023

Lie groupoids

Speaker: Jack Ekenstam (Leiden University)

A Lie groupoid is a topological groupoid whose arrow space and unit space are equipped with smooth structures. A lot of constructions of topological groupoids (c.f. [7]) are therefore available, too. In this lecture, we will review these constructions and specify them to Lie groupoids. We will also introduce two Lie groupoids constructed from a foliation: the monodromy groupoid and the holonomy groupoid.

3.1 Definition of Lie groupoids

Recall that a groupoid can be defined as a small category whose arrows are invertible. We may specify the two sets $\mathscr{G}^{(1)}$ ("the set of arrows") and $\mathscr{G}^{(0)}$ ("the set of units"), together with several structure maps, to give the structure of a groupoid.

Definition 3.1 (Groupoids)**.** A groupoid $\mathscr{G}$ is given by two sets $\mathscr{G}^{(1)}$ (the set of arrows) and $\mathscr{G}^{(0)}$ (the set of units), together with the following five structure maps:

- a pair of maps $r, s\colon \mathscr{G}^{(1)} \rightrightarrows \mathscr{G}^{(0)}$ called the *range* map and the *source* map;
- a map $\cdot\colon \mathscr{G}^2 \to \mathscr{G}^{(1)}$, $(g,h) \mapsto gh$ called the *multiplication* map, where
$$\mathscr{G}^{(2)} := \mathscr{G}^{(1)} \underset{s,\mathscr{G}^{(0)},r}{\times} \mathscr{G}^{(1)} = \{(g,h) \in \mathscr{G}^{(1)} \times \mathscr{G}^{(1)} \mid s(g) = r(h)\};$$
- a map $()^{-1}\colon \mathscr{G}^{(1)} \to \mathscr{G}^{(1)}$, $g \mapsto g^{-1}$, called the *inverse* map;
- a map $1\colon \mathscr{G}^{(0)} \to \mathscr{G}^{(1)}$, $x \mapsto 1_x$ called the *unit* map;

such that:

1. $s(hg) = s(g), \quad r(hg) = r(h);$
2. $k(hg) = k(hg);$
3. $1_{r(g)}g = g = g1_{s(g)};$
4. $s(g^{-1}) = r(g), \quad t(g^{-1}) = s(g), \quad g^{-1}g = 1_{s(g)}, \quad gg^{-1} = 1_{r(g)}.$

In most cases, we will only specify the range and source maps r and s, as all other structure maps are easy to define. It is usually convenient to write $\mathscr{G}$ for $\mathscr{G}^{(1)}$, and identify $\mathscr{G}^{(0)}$ with the subset $1(\mathscr{G}^{(0)})$ of $\mathscr{G}$.

The following notations are more widely used in the operator algebra community in contrast to differential geometers:

Definition 3.2. Let $\mathscr{G}$ be a groupoid.

- Let $x, y \in \mathscr{G}^{(0)}$. We define the *source fibre at* y to be $\mathscr{G}_y := s^{-1}(y)$, the *range fibre at* x to be $\mathscr{G}^x := r^{-1}(x)$, and $\mathscr{G}^x_y := \mathscr{G}^x \cap \mathscr{G}_y$.
- Let $A, B \subseteq \mathscr{G}^{(0)}$. We define $\mathscr{G}_B := s^{-1}(\mathscr{G})$, $\mathscr{G}^A := r^{-1}(A)$ and $\mathscr{G}^A_B := \mathscr{G}^A \cap \mathscr{G}_B$.

Definition 3.3. Let $\mathscr{G}$ be a groupoid.

- Let $A \subseteq \mathscr{G}^{(0)}$. Then $\mathscr{G}^A_A \subseteq \mathscr{G}$ is a subgroupoid, called the restriction of $\mathscr{G}$ to A.
- Let $x \in \mathscr{G}^{(0)}$. Then $\mathscr{G}^x_x$ is a group, called the *isotropy group* at x.

A groupoid homomorphism $\mathscr{G} \to \mathscr{H}$ is a functor between these categories. More concretely,

Definition 3.4 (Groupoid homomorphisms)**.** A (strict) groupoid homomorphism $\phi: \mathscr{G} \to \mathscr{H}$ is a map such that $\phi \times \phi(\mathscr{G}^{(2)}) \subseteq \mathscr{H}^{(2)}$ and $\phi(gh) = \phi(g)\phi(h)$ for all $(g,h) \in \mathscr{G}^{(2)}$. It is an *isomorphism* if there exists another groupoid homomorphism $\psi: \mathscr{H} \to \mathscr{G}$ such that $\psi \circ \phi = \mathrm{id}_{\mathscr{H}}$ and $\phi \circ \psi = \mathrm{id}_{\mathscr{G}}$.

A groupoid homomorphism $\phi: \mathscr{G} \to \mathscr{H}$ defines a pair of maps $\phi^{(1)}: \mathscr{G}^{(1)} \to \mathscr{H}^{(1)}$ and $\phi^{(0)}: \mathscr{G}^{(0)} \to \mathscr{H}^{(0)}$ in an obvious way.

Definition 3.5 (Lie groupoids)**.** A *Lie groupoid* is a groupoid $\mathscr{G}$, such that:

- $\mathscr{G}^{(0)}$ is a smooth, Hausdorff and second countable manifold.
- $\mathscr{G}^{(1)}$ is a smooth, not necessarily Hausdorff, not necessarily second countable manifold.
- $s: \mathscr{G}^{(1)} \to \mathscr{G}^{(0)}$ is a submersion with Hausdorff fibres.
- All the structure maps are smooth.

Remark 3.6. The definition above has the following immediate consequences:

1. The map $r: \mathscr{G}^{(1)} \to \mathscr{G}^{(0)}$ is also a submersion, because $r = s \circ ()^{-1}$, and $()^{-1}$ is a diffeomorphism as $(()^{-1})^2 = \mathrm{id}$.
2. $\mathscr{G}^{(2)}$ is also a smooth manifold because s is a submersion.

Definition 3.7 (Lie groupoid homomorphisms)**.** A homomorphism $\phi: \mathscr{G} \to \mathscr{H}$ between Lie groupoids is a groupoid homomorphism such that that $\phi^{(1)}: \mathscr{G}^{(1)} \to \mathscr{H}^{(1)}$ and $\phi^{(0)}: \mathscr{G}^{(0)} \to \mathscr{H}^{(0)}$ are smooth.

Example 3.8.

1. A *smooth manifold* M can be viewed as a Lie groupoid with $\mathscr{G}^{(1)} = \mathscr{G}^{(0)} = M$, $r = s = \mathrm{id}$.
2. A *Lie group* G can be viewed as a Lie groupoid with $\mathscr{G}^{(1)} = G$ and $\mathscr{G}^{(0)} = \mathrm{pt}$, r and s being the unique smooth map to a point.
3. The *pair groupoid* of a smooth manifold M is the groupoid $\mathrm{Pair}(M)$ with $\mathrm{Pair}(M)^{(1)} = M \times M$, $\mathrm{Pair}(M)^{(0)} = M$, $r = \mathrm{pr}_1$, $s = \mathrm{pr}_2$, $(x,y)\cdot(y,z) = (x,z)$.
4. *Equivalence relations*. Let $\mathscr{R}$ be an immersed submanifold of $M \times M$ that defines an equivalence relation. Then $\mathscr{R}$ defines a Lie groupoid with $\mathscr{G}^{(1)} = \mathscr{R}$, $\mathscr{G}^{(0)} = M$, $r = \mathrm{pr}_1$, $s = \mathrm{pr}_2$, $(x,y)\cdot(y,z) = (x,z)$. This is a Lie subgroupoid of $\mathrm{Pair}(M)$, meaning that there is an injective Lie groupoid homomorphism $\mathscr{R} \hookrightarrow \mathrm{Pair}(M)$.
5. *Action groupoid*. Let G be a Lie group acting smoothly on a manifold M (on the left), then the associated action groupoid $G \ltimes M$ is defined by $(G \ltimes M)^{(1)} = G \times M$, $(G \ltimes M)^{(0)} = M$, $s(g,x) = x$, $r(g,x) = gx$, $(g,hx)\cdot(h,x) = (gh,x)$.
6. *Kernel groupoid* or *submersion groupoid*. Any smooth map $p: N \to M$ induces a Lie groupoid homomorphism $p: \mathrm{Pair}(N) \to \mathrm{Pair}(M)$ in an obvious way. If p is moreover a submersion, then set $\ker p := \{(y,y') \in N \times N \mid p(y) = p(y')\}$. This defines a Lie groupoid with $\mathscr{G}^{(1)} = \ker p$, $\mathscr{G}^{(0)} = N$, $r = \mathrm{pr}_1$ and $s = \mathrm{pr}_2$. Note that $\ker p$ is a smooth manifold because p is a submersion.
7. *Fundamental groupoid*. Let M be a manifold. Its fundamental groupoid $\Pi_1(M)$ is defined as follows. The unit space of $\Pi_1(M)$ is M. An arrow from x to y is a homotopy class of paths in M, relative to endpoints, from x to y. The range (resp. source) map send a path to its endpoint (resp. starting point), and the multiplication is defined by concatenation of paths. The smooth structure of $\Pi_1(M)$ is a bit tricky, and we leave that to Section 3.2 of monodromy and holonomy groupoids.

Definition 3.9. Let $\mathscr{G}$ be a Lie groupoid.

1. A *global bisection* is a section $\sigma: \mathscr{G}^{(0)} \to \mathscr{G}^{(1)}$ of $s: \mathscr{G}^{(1)} \to \mathscr{G}^{(0)}$, such that $r \circ \sigma: \mathscr{G}^{(0)} \to \mathscr{G}^{(0)}$ is a diffeomorphism.

   The global bisections form a group called the *gauge group* of $\mathscr{G}$: the group laws are given by

$$\sigma\sigma'(x) := \sigma(r(\sigma'(x)))\sigma'(x), \qquad \sigma^{-1}(x) := \sigma(x)^{-1}.$$

2. A *local bisection* is a *local* section $\sigma\colon U \to \mathscr{G}^{(1)}$ of $s\colon \mathscr{G}^{(1)} \to \mathscr{G}^{(0)}$, where $U \subseteq \mathscr{G}^{(0)}$ is an open subset, such that $r \circ \sigma\colon U \to \mathscr{G}^{(0)}$ is an open embedding.

   The germs of local bisections form a groupoid $\mathrm{Bis}(\mathscr{G})$. Its multiplication is, locally, given by the same as the gauge group of global bisections.

Remark 3.10. It is not clear to me what the Lie groupoid structure $\mathrm{Bis}(\mathscr{G})$ is. Here is a possible construction. The collection $\mathrm{Diff}(M)$ of local diffeomorphisms of a manifold has the structure of a pseudogroup (a group-like object, satisfying some sheaf-like conditions; the prototype is precisely the set of local diffeomorphisms). Pseudogroups have a close relation with groupoids, namely, the germs of $\mathrm{Diff}(M)$ form naturally a groupoid over M: let f be a local diffeomorphism with $f(x) = y$. Then its representing germ of local diffeomorphisms from x to y is an element in this groupoid with source x and range y.

Let $\mathscr{G}$ be a Lie groupoid and σ be a local bisection. Since $r \circ \sigma$ is an open embedding, it is a local diffeomorphism of $\mathscr{G}^{(0)}$. The collection of all $r \circ \sigma$ with σ a local bisection is a subpseudogroup of $\mathrm{Diff}(\mathscr{G}^{(0)})$, and the germs define a groupoid over $\mathscr{G}^{(0)}$. What is the Lie groupoid structure? This is done by assuming the range and source maps are local diffeomorphisms. Since $\mathscr{G}^{(0)}$ carries a smooth structure, the charts pull back to charts of $\mathrm{Bis}(\mathscr{G})$ and in turn give a Lie groupoid.

Proposition 3.11. *The natural groupoid homomorphism* $\mathrm{Bis}(\mathscr{G}) \to \mathscr{G}$ *is smooth.*

Proposition 3.12. *Let $\mathscr{G}$ be a Lie groupoid. Then for every $g \in \mathscr{G}^{(1)}$, there exists a local bisection $\sigma\colon U \to \mathscr{G}^{(1)}$ such that $g \in \mathrm{Im}(\sigma)$.*

Proof. See [6, Proposition 5.3]. □

Definition 3.13 (Étale Lie groupoids)**.** A Lie groupoid $\mathscr{G}$ is called *étale* if $\dim \mathscr{G}^{(1)} = \dim \mathscr{G}^{(0)}$.

The word "étale" comes from the fact that if $\dim \mathscr{G}^{(1)} = \dim \mathscr{G}^{(0)}$, then the range and source maps s and r are local diffeomorphisms (because they are submersions of equal rank), and a local diffeomorphism is also called an étale map.

Example 3.14. It is is tautological that the groupoid of bisections $\mathrm{Bis}(\mathscr{G})$ is étale.

3.2 Groupoids of foliations

Definition 3.15. Let $(M, \mathscr{F})$ be a foliated manifold.

1. The *monodromy groupoid* $\mathrm{Mon}(M, \mathscr{F})$ is given by:

   - The unit space is M.
   - If x and y are two points on the same leaf L, then an arrow from x to y is a *homotopy* class of paths from x to y in L relative to the basepoints. That is, a homotopy of paths in L with basepoints fixed.

     If x and y are on different leaves, then there is no arrow from x to y.

2. The *holonomy groupoid* $\mathrm{Hol}(M, \mathscr{F})$ is defined in a similar fashion with $\mathrm{Mon}(M, \mathscr{F})$. The only difference is to replace the *homotopy* classes of paths by *holonomy* classes of paths.

Proposition 3.16. *Both* $\mathrm{Mon}(M, \mathscr{F})$ *and* $\mathrm{Hol}(M, \mathscr{F})$ *are étale Lie groupoids.*

Proof. See [6, Proposition 5.6]. □

In particular, let $\mathscr{F}$ be the trivial foliation of codimension 0 of a manifold M. Then $\mathrm{Mon}(M, \mathscr{F}) = \Pi_1(M)$. So $\Pi_1(M)$ is an étale Lie groupoid as well.

Proposition 3.17. *Let $(M, \mathscr{F})$ be a foliated manifold, with monodromy groupoid $\mathrm{Mon}(M, \mathscr{F})$ and holonomy groupoid $\mathrm{Hol}(M, \mathscr{F})$. We have the following immediate results:*

1. *There is a natural map* $\mathrm{Mon}(M, \mathscr{F}) \to \mathrm{Hol}(M, \mathscr{F})$*, by sending the homotopy class of a path to its holonomy class.*

2. *The orbits of* $\mathrm{Mon}(M,\mathscr{F})$ *and* $\mathrm{Hol}(M,\mathscr{F})$*, namely, the equivalence classes of points which are connected by an element in these groupoids, are the leaves of* $\mathscr{F}$*.*
3. *The isotropy groups of both of* $\mathrm{Mon}(M,\mathscr{F})$ *and* $\mathrm{Hol}(M,\mathscr{F})$ *are discrete. This is because the* M *is a manifold, whose fundamental group is discrete.*
4. *Let* $x \in L$*. The the range map of* $\mathrm{Mon}(M,\mathscr{F})$ *restricts to a universal cover* $\mathrm{Mon}(M,\mathscr{F})_x \to L$*.*

3.3 Constructing new Lie groupoids from old ones

3.3.1 Induced groupoids

Let $\mathscr{G}$ be a Lie groupoid, and $\phi : M \to \mathscr{G}^{(0)}$ be a smooth map. We are going to define a groupoid $\phi^*\mathscr{G}$ over M called the *induced groupoid* or *pullback groupoid*. An arrow in this groupoid from x to y should be identified with an arrow in $\mathscr{G}$ from $\phi(x)$ to $\phi(y)$. That means, we define

$$(\phi^*\mathscr{G})^{(1)} := M \underset{\phi,\mathscr{G}^{(0)},r}{\times} \mathscr{G}^{(1)} \underset{s,\mathscr{G}^{(0)},\phi}{\times} M$$

and equip it with the multiplication in $\mathscr{G}$. In other words, we define $(\phi^*\mathscr{G})^{(1)}$ to be the object constructed from two pullback diagrams:

$$\begin{tikzcd}
(\phi^*\mathscr{G})^{(1)} \arrow[rr] \arrow[d] & & M \arrow[d, "\phi"] \\
\mathscr{G}^{(1)} \underset{s,\mathscr{G}^{(0)},\phi}{\times} M \arrow[r, "\mathrm{pr}_1"] \arrow[d] & \mathscr{G}^{(1)} \arrow[r, "r"] \arrow[d, "s"] & \mathscr{G}^{(0)} \\
M \arrow[r, "\phi"'] & \mathscr{G}^{(0)} &
\end{tikzcd}$$

Note that the fibred product

$$\mathscr{G}^{(1)} \underset{s,\mathscr{G}^{(0)},\phi}{\times} M := \{(g,x) \in \mathscr{G}^{(1)} \times M \mid s(g) = \phi(x)\}$$

has a smooth structure because s is a submersion. Suppose that in addition $r \circ \mathrm{pr}_1$ is also a submersion, then $(\phi^*\mathscr{G})^{(1)}$ has a smooth structure as well. This construction is referred to as the *blow-up groupoid* in [7].

3.3.2 Smooth transformations

If we view a groupoid as a category, then a groupoid homomorphism $\mathscr{G} \to \mathscr{H}$ is just a functor. Then we may also speak about natural transformations, which are morphisms between functors. In the category of Lie groupoids, those transformations are required to be smooth as well.

More precisely, let $\phi, \psi : \mathscr{G} \rightrightarrows \mathscr{G}$ be Lie groupoid homomorphisms. A *smooth (natural) transformation* $T : \phi \Rightarrow \psi$ is a smooth map

$$T : \mathscr{G}^{(0)} \to \mathscr{H}^{(1)},$$

such that:

- for every $x \in \mathscr{G}^{(0)}$, $T(x)$ is an arrow from $\phi(x)$ to $\psi(x)$;
- for every arrow g from x to y, the diagram

$$\begin{tikzcd}
\phi(x) \arrow[r, "T(x)"] \arrow[d, "\phi(g)"'] & \psi(x) \arrow[d, "\psi(g)"] \\
\phi(y) \arrow[r, "T(y)"'] & \psi(y)
\end{tikzcd}$$

which in other words means that T is a natural transformation if we forget the smooth structures of $\mathscr{G}$ and $\mathscr{H}$.

In particular, given Lie groupoids $\mathscr{G}$ and $\mathscr{H}$, we may build a new groupoid, whose:

- objects are homomorphisms $\mathscr{G} \to \mathscr{H}$;
- an arrow from ϕ to ψ is a smooth natural transformation $T\colon \phi \Rightarrow \psi$.

In fact, Lie groupoids, homomorphisms and transformations form a 2-category.

3.3.3 Products and sums

In the 2-category of Lie groupoids described above, the 2-product and 2-coproduct (i.e. the corresponding universal objects in a 2-category) can be explicitly constructed as the product Lie groupoid and the direct sum Lie groupoid. Given Lie groupoids $\mathscr{G}$ and $\mathscr{H}$, we may define:

- The product groupoid $\mathscr{G} \times \mathscr{H}$, with $(\mathscr{G} \times \mathscr{H})^{(1)} = \mathscr{G}^{(1)} \times \mathscr{H}^{(1)}$ and $(\mathscr{G} \times \mathscr{H})^{(0)} = \mathscr{G}^{(0)} \times \mathscr{H}^{(0)}$. The structure maps are inherited from $\mathscr{G}$ and $\mathscr{H}$.
- The direct sum groupoid $\mathscr{G} \oplus \mathscr{H}$, with $(\mathscr{G} \oplus \mathscr{H})^{(1)} = \mathscr{G}^{(1)} \coprod \mathscr{H}^{(1)}$ and $(\mathscr{G} \times \mathscr{H})^{(0)} = \mathscr{G}^{(0)} \coprod \mathscr{H}^{(0)}$. The structure maps are inherited from $\mathscr{G}$ and $\mathscr{H}$.

3.3.4 Strong fibred products

Let $\phi\colon \mathscr{G} \to \mathscr{K}$ and $\psi\colon \mathscr{H} \to \mathscr{K}$ be Lie groupoid homomorphisms. We may define a groupoid $\mathscr{G} \underset{\phi,\mathscr{K},\psi}{\times} \mathscr{H}$, called the *strong fibred product*, by

$$(\mathscr{G} \underset{\phi,\mathscr{K},\psi}{\times} \mathscr{H})^{(1)} := \mathscr{G}^{(1)} \underset{\phi^{(1)},\mathscr{K}^{(1)},\psi^{(1)}}{\times} \mathscr{H}^{(1)} = \{(g,h) \in \mathscr{G}^{(1)} \times \mathscr{H}^{(1)} \mid \phi^{(1)}(g) = \psi^{(1)}(h)\};$$

$$(\mathscr{G} \underset{\phi,\mathscr{K},\psi}{\times} \mathscr{H})^{(0)} := \mathscr{G}^{(0)} \underset{\phi^{(0)},\mathscr{K}^{(0)},\psi^{(0)}}{\times} \mathscr{H}^{(0)} = \{(x,y) \in \mathscr{G}^{(0)} \times \mathscr{H}^{(0)} \mid \phi^{(0)}(x) = \psi^{(0)}(y)\}.$$

The structure maps are obvious. This gives a groupoid, which is not always a Lie groupoid: it is, if the map $\phi^{(0)} \times \psi^{(0)}\colon \mathscr{G}^{(0)} \times \mathscr{H}^{(0)} \to \mathscr{K}^{(0)} \times \mathscr{K}^{(0)}$ is transverse to the diagonal

$$\Delta\mathscr{K}^{(0)} = \{(x,x) \mid x \in \mathscr{K}^{(0)}\}.$$

In that case we have that

$$(\mathscr{G} \underset{\phi,\mathscr{K},\psi}{\times} \mathscr{H})^{(0)} = (\phi^{(0)} \times \psi^{(0)})^{-1}(\Delta\mathscr{K}^{(0)})$$

is indeed a manifold.

3.3.5 Weak fibred products

As opposed to strong fibred products, a “larger” and in fact universal fibred product can be constructed as follows. Let $\phi\colon \mathscr{G} \to \mathscr{K}$ and $\psi\colon \mathscr{H} \to \mathscr{K}$ be Lie groupoid homomorphisms. The *weak fibred product P* is defined as follows:

- Objects are triples (x,k,y) with $x \in \mathscr{G}^{(0)}$, $y \in \mathscr{H}^{(0)}$ and $k \in \mathscr{K}_{\phi(x)}^{\psi(y)}$.
- An arrow from (x,k,y) to (x',k',y') is a pair (g,h) with $g \in \mathscr{G}^{(1)}$, $h \in \mathscr{H}^{(1)}$ such that the diagram

$$\begin{tikzcd} \phi(x) \arrow[r, "k"] \arrow[d, "\phi(g)"'] & \psi(y) \arrow[d, "\psi(h)"] \\ \phi(x') \arrow[r, "k'"'] & \psi(y') \end{tikzcd}$$

commutes in $\mathscr{K}$ (viewed as a category).

Remark: To my knowledge, P is usually called a comma category, denoted by $\phi \downarrow \psi$. A comma category can be understood as a 1-categorical fibred product or even a 2-categorical limit, in the 2-category of categories or groupoids. This explains why this construction is indeed universal.

The unit space of P can be also written as

$$P^{(0)} = \mathscr{G}^{(0)} \underset{\phi^{(0)},\mathscr{K}^{(0)},r}{\times} \mathscr{K}^{(1)} \underset{s,\mathscr{K}^{(0)},\psi}{\times} \mathscr{H}^{(0)},$$

which is not always a manifold. But if either $\phi^{(0)} \colon \mathscr{G}^{(0)} \to \mathscr{K}^{(0)}$ or $\psi^{(0)} \colon \mathscr{H}^{(0)} \to \mathscr{K}^{(0)}$ is a submersion, then $P^{(0)}$ has a manifold structure. And one can show that $P^{(1)}$ has a manifold structure as well. These render a Lie groupoid structure for P.

3.4 Equivalence of Lie groupoids

Recall from category theory that two categories $\mathscr{C}$ and $\mathscr{D}$ are *strongly equivalent*, if there exist a pair of functors

$$F \colon \mathscr{C} \to \mathscr{D}, \qquad G \colon \mathscr{D} \to \mathscr{C},$$

such that there are natural isomorphisms

$$F \circ G \simeq \mathrm{id}_{\mathscr{D}}, \qquad G \circ F \simeq \mathrm{id}_{\mathscr{C}}.$$

The functors F and G are called strong equivalences between the categories $\mathscr{C}$ and $\mathscr{D}$.

A well-known result in category theory states that, assuming the axiom of choice, then a functor $F \colon \mathscr{C} \to \mathscr{D}$ is a strong equivalence iff it is a *weak equivalence*, which means:

- F is *essentially surjective*, that is, for every $y \in \mathscr{D}$, there exists $x \in \mathscr{C}$ such that $Fx \simeq y$ in $\mathscr{D}$.
- F is *fully faithful*, that is, for every $x, y \in \mathscr{C}$, there is a natural bijection $\mathscr{D}(Fx, Fy) \simeq \mathscr{C}(x, y)$.

This section aims at adapting all these to Lie groupoid. This means we need to add extra smoothness conditions to the functors as well as the natural transformations, as follows.

Definition 3.18 (Strong equivalences)**.** A Lie groupoid homomorphism $\phi \colon \mathscr{G} \to \mathscr{H}$ is called a *strong equivalence*, if there is another Lie groupoid homomorphism $\psi \colon \mathscr{H} \to \mathscr{G}$, together with *smooth* transformations

$$T \colon \phi \circ \psi \Rightarrow \mathrm{id}_{\mathscr{H}}, \qquad S \colon \psi \circ \phi \Rightarrow \mathrm{id}_{\mathscr{G}}.$$

Definition 3.19 (Weak equivalences)**.** A Lie groupoid homomorphism $\phi \colon \mathscr{G} \to \mathscr{H}$ is called a *weak equivalence*, if:

- ϕ is *essentially surjective*: the map
$$\mathscr{H}^{(1)} \underset{s,\mathscr{H}^{(0)},\phi}{\times} \mathscr{G}^{(0)} \xrightarrow{\mathrm{pr}_1} \mathscr{H}^{(1)} \xrightarrow{r} \mathscr{H}^{(0)}$$
is a surjective submersion. Here
$$\mathscr{H}^{(1)} \underset{s,\mathscr{H}^{(0)},\phi}{\times} \mathscr{G}^{(0)} := \{(h, x) \in \mathscr{H}^{(1)} \times \mathscr{G}^{(0)} \mid s(h) = \phi(x)\}.$$

- ϕ is *fully faithful*: the following diagram
$$\begin{tikzcd}
\mathscr{G}^{(1)} \arrow[r, "\phi^{(1)}"] \arrow[d, "s\times t"'] & \mathscr{H}^{(1)} \arrow[d, "s\times t"] \\
\mathscr{G}^{(0)} \times \mathscr{G}^{(0)} \arrow[r, "\phi^{(0)}\times\phi^{(0)}"'] & \mathscr{H}^{(0)} \times \mathscr{H}^{(0)}
\end{tikzcd}$$
is a fibred product (pullback diagram) in the category of manifolds.

Proposition 3.20. *Every strong equivalence of Lie groupoids is a weak equivalence.*

Remark 3.21. It is important to remark that, the existence of a weak equivalence $\phi : \mathscr{G} \to \mathscr{H}$ does *not* guarantee the existence of a weak equivalence $\psi : \mathscr{H} \to \mathscr{G}$. So when we say two Lie groupoids $\mathscr{G}$ and $\mathscr{H}$ are *weakly equivalent*, or *Morita equivalent*, we shall mean that they are equivalent in the equivalence relation *generated by* weak equivalences. That means, if there are weak equivalences

$$\phi : \mathscr{G} \to \mathscr{K}, \qquad \psi : \mathscr{H} \to \mathscr{K}.$$

Proposition 3.22. *A Lie groupoid is transitive iff it is Morita equivalent to a Lie group.*

References

[1] Iakovos Androulidakis and Georges Skandalis. The holonomy groupoid of a singular foliation. *J. Reine Angew. Math.*, **626**: (2009), 1–37. URL: https://doi.org/10.1515/CRELLE.2009.001 (p. 6)

[2] César Camacho and Alcides Lins Neto. *Geometric theory of foliations*. Translated from the Portuguese by Sue E. Goodman. Birkhäuser Boston, Inc., Boston, MA, 1985, vi+205. URL: https://doi.org/10.1007/978-1-4612-5292-4 (pp. 10, 11)

[3] Alberto Candel and Lawrence Conlon. *Foliations. I*. Vol. 23. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2000, xiv+402. URL: https://doi.org/10.1090/gsm/023 (p. 8)

[4] Michel Hilsum and Georges Skandalis. Morphismes K-orientés d'espaces de feuilles et fonctorialité en théorie de Kasparov (d'après une conjecture d'A. Connes). *Ann. Sci. École Norm. Sup. (4)*, **20**:3 (1987), 325–390. URL: http://www.numdam.org/item?id=ASENS_1987_4_20_3_325_0 (p. 3)

[5] John M. Lee. *Introduction to smooth manifolds*. Vol. 218. Graduate Texts in Mathematics. Springer-Verlag, New York, 2003, xviii+628. URL: https://doi.org/10.1007/978-0-387-21752-9 (p. 5)

[6] I. Moerdijk and J. Mrčun. *Introduction to foliations and Lie groupoids*. Vol. 91. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2003, x+173. URL: https://doi.org/10.1017/CBO9780511615450 (pp. 5, 11, 14, 17)

[7] NCG-Leiden. Seminar notes on groupoid C^*-algebras (2022). URL: https://ncg-leiden.github.io/groupoid2022/groupoid_notes.pdf (pp. 3, 15, 18)

[8] Bruno Scardua and Carlos Arnoldo Morales Rojas. *Geometry, dynamics and topology of foliations*. A first course. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017, xiii+179. URL: https://doi.org/10.1142/10366 (pp. 12, 13)