Groupoid C*-algebras

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2022 Fall Semester
Last modified: September 18, 2022

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1 Groupoids: motivations, definitions and examples (Yuezhao, Sep 13)

1.1 Motivations

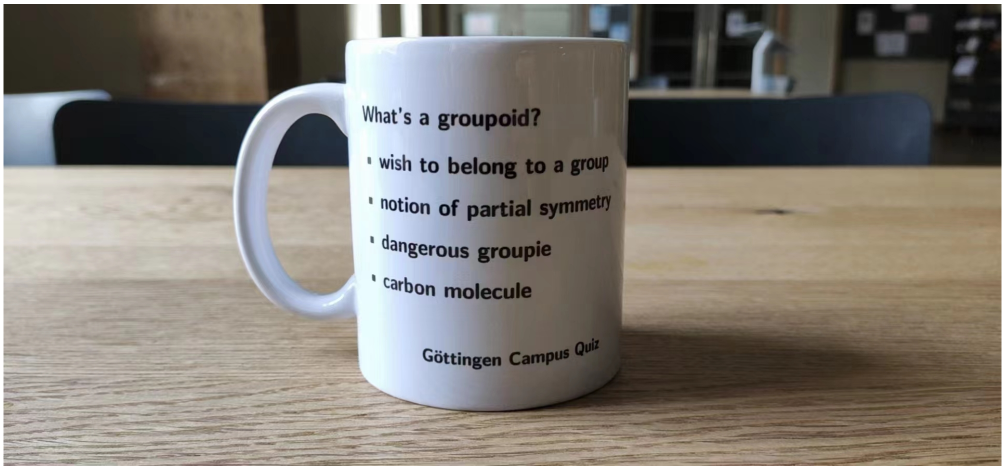

Figure 1: What is a groupoid?

What is a groupoid? There are at least two answers. One way is to think of a groupoid as '*a generalised group, but the multiplication is only partially defined.* Another is to view a groupoid as *a group with more than one units.* For the second viewpoint, recall that a group can be understood as a category with only one object, and all of its arrows are invertible. Then the group elements correspond to the arrows, and the product of group elements correspond to the composition of arrows. A groupoid, on the other hand, has a set of objects and arrows, each object corresponding to a unit by identifying this object with the identity arrow associated to it.

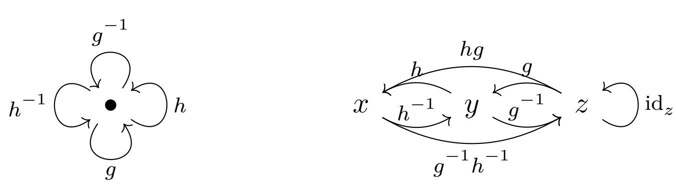

Figure 2: A group is a category with one object and all arrows are invertible. A groupoid is a small category whose all arrows are invertible.

Our main reference [Wil19] adopts the first viewpoint, but I feel that the second is more concise, and even more useful in many cases, too. The source and range maps s and r have explicit geometric meanings in the second picture. This is not only for convenience, but also crucial while working with Lie groupoids: in the Lie groupoid the source and range maps are required to be surjective submersions, and the unit space is required to be a smooth manifold. I also do not quite agree with [Wil19, Remark 1.7] where the author calls the categorical viewpoint an abstract nonsense: this seems to be a misuse of the word. An abstract nonsense is a formal proof based on techniques from category theory, usually not specific to a fixed context. So I would say an abstract nonsense is a *method*, and a definition *cannot* be an abstract nonsense. — Commented by Y. Li

Why do we study groupoids? Before going into the details, it is worthwhile to explain why we care about groupoids. A short answer is that groupoids interact strongly with other fields like dynamical systems and differential geometry, and they themselves are also vital as geometric models of noncommutative spaces.

- **Dynamical systems and groupoids**. Starting from a dynamical system, you can usually construct a groupoid. Depending on the type of the dynamical system (measurable, topological, smooth, ...) you have different structures of the groupoid (Borel, topological, Lie, ...). This groupoid tells you the information of the original dynamical system. For example, let X be a set and G be a group. It is granted that there is a one-to-one correspondence between

  transitive G-actions on X and conjugate classes of subgroups of G.

  Now if we work with a Borel group G and a Borel space X. Can we say something about the ergodic actions of G on X? This shall be more difficult than transitive actions, becase ergodic actions involve the data of both the group and the space. Then one needs to construct a groupoid $G \ltimes X$, the *action* groupoid. This is a Borel groupoid, whose Borel structure comes from those of G and X. The answer is that, an ergodic action of G on X corresponds precisely to an *ergodic groupoid*.

- **Dynamics, topological groupoids and groupoid C^*-algebras**. If we start with a topological dynamical system, say, a locally compact topological group G acting on a locally compact space X. Then $G \ltimes X$ is a topological groupoid, and we can construct C^*-algebras $\mathrm{C}^*_{\mathrm{full}}(G \ltimes X)$ and $\mathrm{C}^*_{\mathrm{red}}(G \ltimes X)$, the full and reduced groupoid C^*-algebra of the groupoid $G \ltimes X$. These C^*-algebras are isomorphic to the full and reduced crossed product C^*-algebras, and encode a lot of data of the dynamics: for example, $\mathrm{C}^*_{\mathrm{red}}(G \ltimes X)$ is simple iff the action of G on X is topological free and minimal.

- **Groupoids and noncommutative spaces**. Another important reason to study groupoids is that they are viewed as geometric models for *noncommutative spaces*. Recall that a locally compact Hausdorff space X corresponds to a commutative C^*-algebra $\mathrm{C}_0(X)$: this is the well-known Gelfand duality. Then noncommutative C^*-algebras play the role of "non-commutative spaces" in the algebraic setting. But sometimes it is desirable to find geometric models of noncommutative spaces. Of course, they cannot be the usual topological spaces because $\mathrm{C}_0(X)$ is always commutative. One attempt is to seek a topological groupoid $\mathcal{G}$, such that its groupoid C^*-algebra is isomorphic to the noncommutative C^*-algebra that we start with. Then $\mathcal{G}$ is a good geometric model for our noncommutative space. If we view a topological space X as a groupoid (see Example 1.14), then its groupoid C^*-algebra is precisely $\mathrm{C}_0(X)$, complying nicely with the classical Gelfand theory.

- **Foliation, groupoids and index theory**. Index theory studies the connection between indices of (pseudo)differential operators and the topology or geometry of the spaces they live in. One of the most celabrated index theorem is the Atiyah–Singer index theorem. The family index theorem is a variant of the Atiyah–Singer index theorem. The set-up is a fibration $E \to X$ over a compact base, and a family of elliptic operators $\{D_x\}_{x \in X}$ parametrised by X, and such that each D_x acts on the vertical fibre E_x. The family index theorem computes the index of the family $\mathrm{Index}(D_x)$, which is an element in $\mathrm{K}^0(X)$, the K-theory of X.

  Fibrations are special cases of *foliations*, and one might wish to generalise the theory to arbitrary foliations. However, foliations can be quite well-behaved in general. For example, consider the *Kronecker foliation* of $\mathbb{T}^2 \cong \mathbb{R}^2/2\pi\mathbb{Z}$ defined by the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \theta$. If θ is rational. Then every orbit (leaf) is closed and homeomorphic to a circle. However, if θ is irrational, then every leaf is dense in $\mathbb{T}^2$, and the leaf space with the quotient topology is homeomorphic to a single point. This makes the family index theorem useless.

  The problem arises because the leaf space is badly-behaved. The solution is to replace this space by a "noncommutative space" — the *foliation groupoid*. Thus the K-theory of the C^*-algebra of the foliation groupoid becomes a nice receptacle of the family index. This is the now well-known longitudinal index theorem of Connes and Skandalis [CS84].

1.2 Groupoid

Definition 1.1 (First definition of groupoids)**.** A groupoid is a set $\mathcal{G}$ together with:

- a set $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$ of "composable arrows";
- a "multiplication map" $\mathcal{G}^{(2)} \to \mathcal{G}$, $(a, b) \mapsto ab$;
- an "inverse" map $\mathcal{G} \to \mathcal{G}$, $a \mapsto a^{-1}$,

such that:

1. (Associativity) If $(a, b) \in \mathcal{G}^{(2)}$ and $(b, c) \in \mathcal{G}^{(2)}$. Then (ab, c), $(a, bc) \in \mathcal{G}^{(2)}$ and $(ab)c = a(bc)$.
2. (Involutivity) $(a^{-1})^{-1} = a$.
3. (Unit) For any $a \in \mathcal{G}$, $(a^{-1}, a) \in \mathcal{G}^{(2)}$; if $(a, b) \in \mathcal{G}^{(2)}$, then $abb^{-1} = a$ and $a^{-1}ab = b$.

The unit axiom asserts that, unlike a group, a groupoid can have many (one-sided) units; the units of a groupoid $\mathcal{G}$ forms a subset $\mathcal{G}^{(0)} \subseteq \mathcal{G}$, which comes together with a pair of maps $s, r \colon \mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$.

Definition 1.2. Let $\mathcal{G}$ be a groupoid.

- The *unit space* $\mathcal{G}^{(0)}$ of $\mathcal{G}$ is
$$\mathcal{G}^{(0)} := \{a \in \mathcal{G} \mid a = a^{-1} = a^2\}.$$
- The *source* map of $\mathcal{G}$ is $s \colon \mathcal{G} \to \mathcal{G}^{(0)}$, $a \mapsto a^{-1}a$.
  The *range* map of $\mathcal{G}$ is $r \colon \mathcal{G} \to \mathcal{G}^{(0)}$, $a \mapsto aa^{-1}$.

We have the following:

Lemma 1.3. *Let $\mathcal{G}$ be a groupoid.*

1. $\mathcal{G}^{(0)} = \{aa^{-1} \mid a \in \mathcal{G}\}$.
2. $\mathcal{G}^{(2)} = \{(a, b) \in \mathcal{G} \times \mathcal{G} \mid s(a) = r(b)\}$.
3. *If $a, b \in \mathcal{G}$ and $(a, b) \in \mathcal{G}^{(2)}$. Then:*
$$s(a) = r(a^{-1}), \qquad s(ab) = s(b), \qquad r(ab) = r(a),$$
$$(b^{-1}, a^{-1}) \in \mathcal{G}^{(2)}, \qquad b^{-1}a^{-1} = (ab)^{-1}.$$

Proof.

- Clearly $aa^{-1} \in \mathcal{G}^{(0)}$ for any $a \in \mathcal{G}$. If $a \in \mathcal{G}^{(0)}$, then $a = a^2 = aa^{-1}$.
- If $s(a) = r(b)$, then $a^{-1}a = bb^{-1}$. Since $(a, a^{-1}), (a^{-1}, a), (b^{-1}, b), (b, b^{-1}) \in \mathcal{G}^{(2)}$. We have $(a, a^{-1}a) = (a, bb^{-1}) \in \mathcal{G}^{(2)}$ and $(bb^{-1}, b) \in \mathcal{G}^{(2)}$. Then $(a, bb^{-1}b) = (a, b) \in \mathcal{G}^2$. Conversely, if $(a, b) \in \mathcal{G}^{(2)}$. Since $(a^{-1}, a), (b, b^{-1}) \in \mathcal{G}^{(2)}$, the product $a^{-1}abb^{-1}$ makes sense, which equals both bb^{-1} and $a^{-1}a$.
- The three equations in the first line can be quickly checked. If $(a, b) \in \mathcal{G}^{(2)}$. Then $r(a^{-1}) = s(a) = r(b) = s(b^{-1})$ and hence $(b^{-1}, a^{-1}) \in \mathcal{G}^{(2)}$. The product $b^{-1}a^{-1}ab(ab)^{-1}$ makes sense and equals both $b^{-1}a^{-1}$ and $(ab)^{-1}$. □

Remark 1.4. The previous lemma states that, a groupoid can equivalently be described by the data $(\mathcal{G}, \mathcal{G}^{(0)}, s, r, ^{-1})$. This leads to an alternative definitions of groupoids.

Definition 1.5 (Second definition of groupoids)**.** A groupoid is a set $\mathcal{G}$ together with:

- a distinguished subset $\mathcal{G}^{(0)} \subseteq \mathcal{G}$;
- a pair of maps $r, s \colon \mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$;
- a map $\mathcal{G}^2 \to \mathcal{G}$, $(a, b) \mapsto ab$, where

$$\mathcal{G}^{(2)} := \{(a, b) \in \mathcal{G} \times \mathcal{G} \mid s(a) = r(b)\};$$

- a map $\mathcal{G} \to \mathcal{G}$, $a \mapsto a^{-1}$,

such that

1. $r(x) = x = s(x)$ for all $x \in \mathcal{G}^{(0)}$.
2. $r(a)a = a = as(a)$ for all $a \in \mathcal{G}$.
3. $r(a^{-1}) = s(a)$ for all $a \in \mathcal{G}$.
4. $s(a) = a^{-1}a$ and $r(a) = aa^{-1}$ for all $a \in \mathcal{G}$.
5. $r(ab) = r(a)$ and $s(ab) = s(b)$ for all $(a, b) \in \mathcal{G}^{(2)}$.
6. $(ab)c = a(bc)$ whenever $s(a) = r(b)$ and $s(b) = r(c)$.

In this definition, the roles of the range and source maps are highlighted: this is actually more important if we want to study a topological groupoid or a Lie groupoid. I feel that it is sometimes more convenient to denote a groupoid by a diagram $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} \mathcal{G}^{(0)}$.

Example 1.6.

1. A group G is a groupoid $G \rightrightarrows \mathrm{pt}$.
2. A set X is a groupoid $X \overset{\mathrm{id}}{\underset{\mathrm{id}}{\rightrightarrows}} X$ together with the trivial multiplication and inverse maps.
3. Group bundle. A group bundle consists of two sets E, X and a surjective map $\pi \colon E \twoheadrightarrow X$ such that $\pi^{-1}(x)$ is a group for every $x \in X$. A group bundle can be viewed as a groupoid $E \overset{\pi}{\underset{\pi}{\rightrightarrows}} X$. In particular: a vector bundle is a groupoid.
4. *Action groupoid.* Let X be a (left) G-set. That is, G acts on X on the left. The action groupoid $G \ltimes X \rightrightarrows X$ is defined as follows. We set $G \ltimes X := G \times X$ as a set and $(G \ltimes X)^{(0)} := X$. The source, range, multiplication and inverse maps are

$$s(g, x) := x, \qquad r(g, x) := g \cdot x, \qquad (h, gx)(g, x) := (hg, x), \qquad (g, x)^{-1} := (g^{-1}, gx).$$

5. *Pair groupoid.* Let X be a set. The pair groupoid is given by $X \times X \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} X$. The multiplication map is given by $(x, y)(y, z) := (x, z)$ and the inverse map is $(x, y)^{-1} := (y, x)$.
6. *Equivalence relations.* Let X be a set and $\mathcal{R} \subseteq X \times X$ be an equivalence relation on X. Then $\mathcal{R} \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} X$ is a groupoid, with multiplication $(x, y)(y, z) := (x, z)$ and inverse $(x, y)^{-1} := (y, x)$.

- If we set $\mathcal{R} := X \times X$, then we recover the pair groupoid as a special case. If we set $\mathcal{R} := \emptyset$, then we recover the groupoid $X \overset{\mathrm{id}}{\underset{\mathrm{id}}{\rightrightarrows}} X$.
- Let $\mathcal{G}$ be any groupoid. We can define an equivalence relation on $\mathcal{G}^{(0)}$ by claiming two units are equivalent iff they are connected by a groupoid element. Equivalently, this is the subset $\mathcal{R}(\mathcal{G}) := \{(r(a), s(a)) \mid a \in \mathcal{G}\} \subseteq \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$. Thus we obtain a groupoid $\mathcal{R}(\mathcal{G}) \rightrightarrows \mathcal{G}^{(0)}$. We say a groupoid $\mathcal{G}$ is *principal* if $\mathcal{G}$ is isomorphic to $\mathcal{R}(\mathcal{G})$ as a groupoid. Equivalently, this means there exists at least one arrow between two units in $\mathcal{G}$.

7. *Fundamental groupoid.* Let X be a topological space and $x \in X$. An important invariant in algebraic topology is the *fundamental group* of X (with basepoint x), defined as the group of (basepoint-fixed homotopy) equivalence classes of loops in X with basepoint x:

$$\pi_1(X, x) := \frac{\{\text{Loop } \gamma \text{ in } X \mid \gamma(0) = \gamma(1) = x\}}{\gamma \sim \gamma' \text{ iff } \gamma \text{ and } \gamma' \text{ are homotopic with basepoint fixed}}.$$

This definition is not completely satisfying due to the following issues. If X is not path-connected, given $x, y \in X$, $\pi_1(X, x)$ and $\pi_1(X, y)$ may not be isomorphic if x and y do not lie in the same path component. If X is path-connected, then X has a unique path component and a different basepoint gives rise to an isomorphic fundamental group. However, the isomorphism between these two groups depend on the choice of the basepoints and on the specified path connecting them, hence not canonical.

It is desirable to obtain a mathematical object similar to the fundamental group but does not depend on the choice of a basepoint. A natural idea is to choose (equivalence classes of) paths instead of loops. Unlike loops which are based at a certain point, paths are not concatenatable, unless the starting point of one coincides with the endpoint of another. This is precisely the axiom of a groupoid. So we may define the fundamental groupoid of X as:

$$\Pi_1(X) := \frac{\{\text{Path } \gamma \text{ in } X\}}{\gamma \sim \gamma' \text{ iff } \gamma \text{ and } \gamma' \text{ are homotopic with basepoints fixed}}.$$

The fundamental groupoid is an important object in algebraic topology.

8. *Tangent groupoid.* Tangent groupoids were introduced by Alain Connes as an approach to study index theory. We briefly mention his construction. The interplay between tangent groupoids and index theory will be discussed in a future talk.

Let M be a smooth manifold. The tangent groupoid of M is the groupoid

$$\mathbb{T}M := TM \times \{0\} \coprod M \times M \times (0, 1] \overset{r}{\underset{s}{\rightrightarrows}} M \times [0, 1],$$

where

$$\begin{aligned} r(x, v, 0) &= (x, 0), \qquad & s(x, v, 0) &= (x, 0); \\ r(x, y, \epsilon) &= (x, \epsilon), \qquad & s(x, y, \epsilon) &= (y, \epsilon), \qquad \epsilon \in (0, 1]. \end{aligned}$$

Definition 1.7 (Subgroupoids)**.** A subgroupoid of a groupoid is a subset $\mathcal{H} \subseteq \mathcal{G}$ such that, the multiplication and inverse maps of $\mathcal{G}$ restricted to $\mathcal{H}$ turns it into a groupoid.

Definition 1.8 (Groupoid homomorphism)**.** A groupoid homomorphism $f \colon \mathcal{G} \to \mathcal{H}$ is a map such that $f \times f(\mathcal{G}^{(2)}) \subseteq \mathcal{H}^{(2)}$ and $f(ab) = f(a)f(b)$ for all $(a, b) \in \mathcal{G}^{(2)}$. It is an *isomorphism* if there exists another groupoid homomorphism $g \colon \mathcal{H} \to \mathcal{G}$ such that $f \circ g = \mathrm{id}_{\mathcal{H}}$ and $g \circ f = \mathrm{id}_{\mathcal{G}}$.

Definition 1.9.
- Let $x, y \in \mathcal{G}^0$. We define the *source fibre at y* to be $\mathcal{G}_y := s^{-1}(y)$, the *range fibre at x* to be $\mathcal{G}^x := r^{-1}(x)$, and $\mathcal{G}_y^x := \mathcal{G}^x \cap \mathcal{G}_y$.
- Let $A, B \subseteq \mathcal{G}^{(0)}$. We define $\mathcal{G}_B := s^{-1}(\mathcal{G})$, $\mathcal{G}^A := r^{-1}(A)$ and $\mathcal{G}_B^A := \mathcal{G}^A \cap \mathcal{G}_B$.

Definition 1.10.
- Let $A \subseteq \mathcal{G}^{(0)}$. Then $\mathcal{G}_A^A \subseteq \mathcal{G}$ is a subgroupoid, called the restriction of $\mathcal{G}$ to A.
- Let $x \in \mathcal{G}^0$. Then $\mathcal{G}_x^x$ is a group, called the *isotropy group* at x.
- The *isotropy groupoid* is the subgroupoid of $\mathcal{G}$:

$$\operatorname{Iso}(\mathcal{G}) := \bigcup_{x \in \mathcal{G}^{(0)}} \mathcal{G}_x^x \rightrightarrows \mathcal{G}^{(0)}.$$

1.3 Topological groupoid

Now we turn to groupoids with extra structures. Charles Ehresmann was the first person to endow groupoids with extra structures while applying them to the study of foliation. Examples include topological groupoids, Borel groupoids (groupoids with measurable structures) and Lie groupoids.

Topological groupoids are the central objects that we will care about in the seminar talks. Let $\mathcal{G}$ be a groupoid that is also a topological space. Then $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$ inherits the product topology of $\mathcal{G} \times \mathcal{G}$.

Definition 1.11. A groupoid $\mathcal{G}$ is a *topological groupoid* if $\mathcal{G}$ is a topological space, and the multiplication map and the inverse map are continuous.

Remark 1.12. Just as in the case of groups, we usually require that a topological groupoid $\mathcal{G}$ is such that:

1. $\mathcal{G}$ is locally compact.
2. $\mathcal{G}^{(0)}$ is Hausdorff (in the subspace topology).

However, the groupoid $\mathcal{G}$ itself does not have to be Hausdorff. When $\mathcal{G}$ is Hausdorff, its unit space will be closed, see the following lemma.

In fact, non-Hausdorff groupoids arise naturally from dynamical systems and differenrial geometry (e.g. singular foliations). They give rise to interesting C*-algebras.

Lemma 1.13. *Let $\mathcal{G}$ be a topological groupoid. Then $\mathcal{G}$ is Hausdorff iff $\mathcal{G}^{(0)}$ is closed.*

Proof. If $\mathcal{G}$ is Hausdorff. Then every convergent net in $\mathcal{G}$ converges to a unique point. Let $\{a_i\}_{i \in I}$ be a net in $\mathcal{G}^{(0)}$ which converges to $a \in \mathcal{G}$. We claim that the limit a must lie in $\mathcal{G}^{(0)}$ as well. Since $\mathcal{G}$ is a topological groupoid, the source and range maps $s, r \colon \mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ are continuous. Hence $s(a_i) \to s(a)$ and $r(a_i) \to r(a)$. But $a_i \in \mathcal{G}^{(0)}$ for all i, that is, $s(a_i) = a_i = r(a_i)$. Then we have $a_i \to s(a)$, $a_i \to r(a)$ and $a_i \to a$. The Hausdorffness of $\mathcal{G}$ forces $s(a) = a = r(a)$, that is, $a \in \mathcal{G}^{(0)}$.

Now assume that $\mathcal{G}^{(0)}$ is closed. Let $\{a_i\}_{i \in I}$ be any convergent net in $\mathcal{G}$ which converges to both a and b. We must prove that $a = b$. Since the multiplication and the inverse maps are continuous, we have $a_i^{-1} a_i \to a^{-1} b$. But $a_i^{-1} a_i \in \mathcal{G}^{(0)}$ for all i and $\mathcal{G}^{(0)}$ is closed. Therefore, $a^{-1} b \in \mathcal{G}^{(0)}$, which implies that $a = b$. $\square$

The following examples of topological groupoids are just modifications of Example 1.6.

Example 1.14.
1. A topological group G is a topological groupoid $G \rightrightarrows \mathrm{pt}$.

2. A topological space X is a topological groupoid $X \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightrightarrows}} X$ together with the trivial multiplication and inverse maps.

3. A topological group bundle consists of two topological spaces E and X, and a quotient map $\pi \colon E \twoheadrightarrow X$, such that for any $x \in X$, $\pi^{-1}(x)$ is a topological group. This is a topological groupoid.

4. Let G be a topological group which acts continuously on a space X. Then the action groupoid $G \ltimes X$ is a topological groupoid.

5. Let X be a topological space, and $\mathcal{R} \subseteq X \times X$ be an equivalence relation on X, equipped with the subspace topology. Then $\mathcal{R} \rightrightarrows X$ is a topological groupoid. In particular, the pair groupoid of a topological space is a topological groupoid.

6. The fundamental groupoid $\Pi_1(X)$ of a topological space X is a bit tricky. There are various ways to topologise it, but the "correct" topology is defined only when X satisfy some nice conditions (path-connected, locally path-connected and semi-locally simply connected). Readers who are familiar with algebraic topology shall notice that these conditions are precisely what one needs to obtain a nice classification theory of covering spaces. In such situation, the fundamental groupoid is realised as a quotient of the pair groupoid and has the quotient topology.

7. The tangent groupoid is a topological groupoid. The topology is defined as follows: we require that $M \times M \times (0, 1]$ is open, and require that a sequence $\{(x_n, y_n, \epsilon_n)\}_n$ in $M \times M \times (0, 1]$ converges to $(x, v, 0)$ iff
$$x_n \to x, \qquad y_n \to x, \qquad \frac{x_n - y_n}{\epsilon_n} \to v.$$

Finally, we define a subclass of topological groupoids called étale groupoids. They are analogs of discrete groupoids, and easier to study than general topological groupoids. It is a bit surprising that étale groupoids are already interesting enough, in the sense that many interesting C*-algebras arise as the C*-algebra of an étale groupoid.

Definition 1.15. A topologial groupoid $\mathcal{G}$ is called an *étale groupoid*, if the source and range maps $s, r \colon \mathcal{G} \to \mathcal{G}$ are étale. That is, s and r are local homeomorphisms.

Remark 1.16. Be careful that the maps s and r are étale as maps $\mathcal{G} \to \mathcal{G}$, but not as maps $\mathcal{G} \to \mathcal{G}^{(0)}$. This is a stronger argument: it asserts that the inclusion map $\mathcal{G}^{(0)} \hookrightarrow \mathcal{G}$ is a topological embedding.

Lemma 1.17. *If $\mathcal{G}$ is an étale groupoid. Then $\mathcal{G}^{(0)} \subseteq \mathcal{G}$ is open.*

Proof. $\mathcal{G}$ is étale implies that, for every $a \in \mathcal{G}$, there is an open neighbourhood $U_a \subseteq \mathcal{G}$ of a such that $s|_{U_a} \colon U_a \to s(U_a)$ is a homeomorphism. Then $\mathcal{G}^{(0)} = \bigcup_{a \in \mathcal{G}} s(U_a)$ is open. □

The following lemma states that étale groupoids are "fibrewise discrete".

Lemma 1.18. *If $\mathcal{G}$ is étale. Then for any $x \in \mathcal{G}^{(0)}$, $\mathcal{G}_x$ and $\mathcal{G}^x$ are discrete.*

Proof. $\mathcal{G}$ is étale implies that, for every $a \in \mathcal{G}$, there is an open neighbourhood $U_a \subseteq \mathcal{G}$ of a such that $s|_{U_a} \colon U_a \to s(U_a)$ is a homeomorphism. In particular, s is a bijection.

We claim that $\{a\} = \mathcal{G}_x \cap U_a$. Clearly $\{a\} \subseteq \mathcal{G}_x \cap U_a$. Suppose $b \in \mathcal{G}_x \cap U_a$, then $s(b) = x = s(a)$. Since s is bijective on U_a, we must have $b = a$. Therefore, $\{a\} \subseteq \mathcal{G}_x \cap U_a$ is open in $\mathcal{G}_x$ in the relative topology. So $\mathcal{G}_x$ is discrete. The proof for $\mathcal{G}^x$ is essentially the same. □

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