SEMINAR ON THE (NONCOMMUTATIVE) GEOMETRY OF FOLIATIONS

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In 2023 fall semester, the noncommutative geometry group of Leiden university will be running a seminar on foliations. A *foliation* on a manifold is, roughly speaking, a decomposition of it into immersed submanifolds (called the *leaves* of the foliation), such that the leaves fit together nicely. Foliations arise naturally in various geometric contexts, e.g. solutions of differential equations and integrable systems, and in symplectic geometry.

The theory of foliations is too rich on its own, and we would rather focus on their noncommutative geometry. The noncommutative geometry of foliations was initiated by Alain Connes in the late 1970s, when he realised that the geometry of a foliation can be encoded in a noncommutative C^*-algebra. The construction is in two steps: one first builds the *holonomy groupoid* of the foliation, then takes its reduced groupoid C^*-algebra. This C^*-algebra might be viewed as the "noncommutative space" of leaves, and is the desired receptacle for the index theory of longitudinal elliptic operators. This was achieved in the seminal work of Connes and Skandalis. We will reach this topic at the end of this seminar.

Another focus of the seminar is to understand certain foliations arising from dynamical systems. Lindsey and Treviño have constructed translation surfaces from bi-infinite Bratteli diagrams. Their construction is essentially combinatorial. Putnam and Treviño investigated on these surfaces, based on operator algebraic techniques. Slightly more precisely, they build explicit relations between the groupoid C^*-algebras constructed from the Bratteli diagram, and the C^*-algebras constructed from suitable foliations on the translation surface. This also allows them to compute the K-theory of the aforementioned C^*-algebras.

1. Preliminary topics of the talks

A preceeding reading seminar on groupoids in 2022 fall semester consists of 12 talks in total, so we will have roughly as many talks. I would like to divide the seminar into three parts by their topics:

Part I: Introduction to foliations. We shall begin our seminar with the preliminaries of foliations, following [1, Chapter 1–3]. The prerequisite for this part is the basic knowledge on differentiable manifold, covered by e.g. the undergraduate course "Differentiable Manifolds II" in Leiden. We need roughly three talks to cover the essentials.

Possible topics.

- *Definitions and examples of foliations*, following [1, Chapter 1]. Present the definition and first examples of foliations on manifold.

- *Holonomy and stability*, following [1, Chapter 2]. Present the definition of holonomy and some stability theorems. Discuss how these concepts are related to dynamics.
- *Groupoids of foliations*, following [1, Chapter 5]. Define the holonomy groupoid and the monodromy groupoid of a foliation. Notes generated from the previous groupoid seminar might also be useful.

Part II: C^***-algebras and dynamics of foliations.** We will introduce the C^*-algebras of foliations, which are special cases of groupoid C^*-algebras. Then we will focus on the article [7] of Putnam and Treviño, investigating the ideas and constructions therein.

Possible topics.

- C^**-algebras of foliations*, following [6, Chapter VI] or [5, Chapter 5–7]. The speaker should help the audiences recall the construction of groupoid C^*-algebras.
- *Translation surfaces from Bratteli diagrams* (2–3 talks). Present the construction of translation surfaces from bi-infinite Bratteli diagrams following [8]. Define the C^*-algebra of right and left tail equivalences and show that they are AF-algebras. Define the foliation on the translation surface and its corresponding C^*-algebra. Explain their relations and compute their K-theory. The main resource is [7].

Part III: Longitudinal index theorem. We will study the longitudinal index theorem of Connes and Skandalis [2]. A more general result, concerning K-oriented maps between foliations and their associated "shriek classes" in KK-theory, was elaborated in [3]. The latter article was suggested to me by Vito Zenobi in May when he visited our group in Leiden. Unbounded KK-theory is popular in our community in the recent decade. Time permitting, it will be of (my own) interest to investigate the applications of unbounded KK-theory in Riemannian foliations. To which extent is the construction of Connes, Skandalis and Hilsum available on the unbounded level? A possible reference I have in mind is [4].

Possible topics.

- *Longitudinal index theorem* (2 talks). The first talk should be a refreshing of Kasparov theory, including the picture using *correspondences* due to Connes and Skandalis (and later by Emerson and Meyer [9]). The second talk is devoted to the proof of the longitudinal index theorem. Notes generated from the previous KK-theory seminar might be useful.
- *K-oriented maps of leaf spaces and functoriality in Kasparov theory.* Define the shriek class of a K-oriented map between foliations in KK-theory following [3]. The longitudinal index theorem can be viewed as a special case of this result.
- *Riemannian foliations in unbounded KK-theory.* Define Riemannian foliations and Riemannian foliated bundles. Construct the unbounded Kasparov modules in "nice" cases following [4]. Some unbounded KK-theoretic computation should be covered.

2. Practical information

Date and time. Preferably start from the week September 11–15. Time to be decided according to the participants' preference.

Location. Leiden university. Room to be decided.

Contact. `mailto:y.li@math.leidenuniv.nl`

References

[1] I. Moerdijk and J. Mrčun, *Introduction to foliations and Lie groupoids*, Cambridge Studies in Advanced Mathematics, vol. 91, Cambridge University Press, Cambridge, 2003. MR **2012261**

[2] A. Connes and G. Skandalis, *The longitudinal index theorem for foliations*, Publ. Res. Inst. Math. Sci. **20** (1984), no. 6, 1139–1183, doi: 10.2977/prims/1195180375. MR **0775126**

[3] Michel Hilsum and Georges Skandalis, *Morphismes K-orientés d'espaces de feuilles et fonctorialité en théorie de Kasparov (d'après une conjecture d'A. Connes)*, Ann. Sci. École Norm. Sup. (4) **20** (1987), no. 3, 325–390 (French, with English summary). MR **0925720**

[4] Vito Felice Zenobi, *Interior Kasparov products for ϱ-classes on Riemannian foliated bundles*, J. Funct. Anal. **284** (2023), no. 9, Paper No. 109863, 35, doi: 10.1016/j.jfa.2023.109863. MR **4549581**

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[6] Calvin C. Moore and Claude L. Schochet, *Global analysis on foliated spaces*, 2nd ed., Mathematical Sciences Research Institute Publications, vol. 9, Cambridge University Press, New York, 2006. MR **2202625**

[7] Ian F. Putnam and Rodrigo Treviño, *Bratteli diagrams, translation flows and their C^*-algebras* (2022), eprint. arXiv: 2205.01537.

[8] Kathryn Lindsey and Rodrigo Treviño, *Infinite type flat surface models of ergodic systems*, Discrete Contin. Dyn. Syst. **36** (2016), no. 10, 5509–5553, doi: 10.3934/dcds.2016043. MR **3543559**

[9] Heath Emerson and Ralf Meyer, *Bivariant K-theory via correspondences*, Adv. Math. **225** (2010), no. 5, 2883–2919, doi: 10.1016/j.aim.2010.04.024. MR **2680187**