Groupoid C*-algebras

Leiden Noncommutative Geometry Seminar

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Groupoids: motivations, definitions and examples

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1.1 Motivations

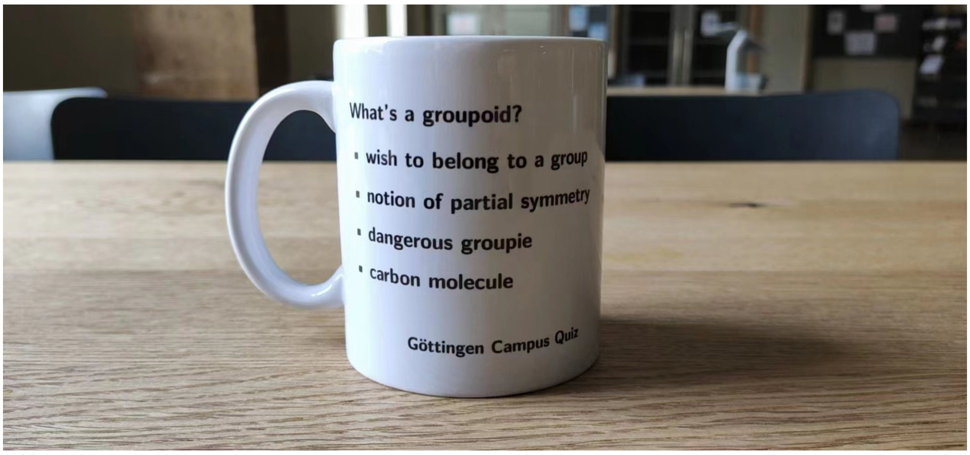

Figure 1: What is a groupoid?

1.1.1 What is a groupoid?

There are at least two answers. One way is to think of a groupoid as '*a generalised group, but the multiplication is only partially defined.* Another is to view a groupoid as *a group with more than one units.* For the second viewpoint, recall that a group can be understood as a category with only one object, and all of its arrows are invertible. Then the group elements correspond to the arrows, and the product of group elements correspond to the composition of arrows. A groupoid, on the other hand, has a set of objects and arrows, each object corresponding to a unit by identifying this object with the identity arrow associated to it.

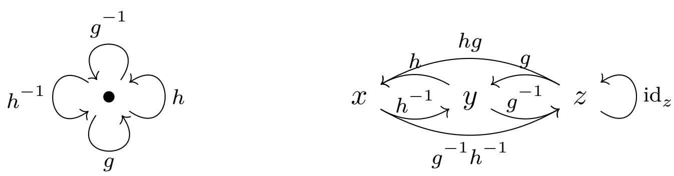

Figure 2: A group is a category with one object and all arrows are invertible. A groupoid is a small category whose all arrows are invertible.

Our main reference [Wil19] adopts the first viewpoint, but I feel that the second is more concise, and even more useful in many cases, too. The source and range maps s and r have explicit geometric meanings in the second picture. This is not only for convenience, but also crucial while working with Lie groupoids: in the Lie groupoid the source and range maps are required to be surjective submersions, and the unit space is required to be a smooth manifold. I also do not quite agree with [Wil19, Remark 1.7] where the author calls the categorical viewpoint an abstract nonsense: this seems to be a misuse of the word. An abstract nonsense is a formal proof based on techniques from category theory, usually not specific to a fixed context. So I would say an abstract nonsense is a *method*, and a definition *cannot* be an abstract nonsense. — Commented by Y. Li

1.1.2 Why do we study groupoids?

Before going into the details, it is worthwhile to explain why we care about groupoids. A short answer is that groupoids interact strongly with other fields like dynamical systems and differential geometry, and they themselves are also vital as geometric models of noncommutative spaces.

Dynamical systems and groupoids . Starting from a dynamical system, you can usually construct a groupoid. Depending on the type of the dynamical system (measurable, topological, smooth, ...) you have different structures of the groupoid (Borel, topological, Lie, ...). This groupoid tells you the information of the original dynamical system. For example, let X be a set and G be a group. It is granted that there is a one-to-one correspondence between

$$\text{transitive } G\text{-actions on } X \qquad \text{and} \qquad \text{conjugate classes of subgroups of } G.$$

Now if we work with a Borel group G and a Borel space X. Can we say something about the ergodic actions of G on X? This shall be more difficult than transitive actions, becase ergodic actions involve the data of both the group and the space. Then one needs to construct a groupoid $G \ltimes X$, the *action* groupoid. This is a Borel groupoid, whose Borel structure comes from those of G and X. The answer is that, an ergodic action of G on X corresponds precisely to an *ergodic groupoid.*

Dynamics, topological groupoids and groupoid C^*-algebras . If we start with a topological dynamical system, say, a locally-compact topological group G acting on a locally-compact space X. Then $G \ltimes X$ is a topological groupoid, and we can construct C^*-algebras $\mathrm{C}^*(G \ltimes X)$ and $\mathrm{C}^*_{\mathrm{r}}(G \ltimes X)$, the full and reduced groupoid C^*-algebra of the groupoid $G \ltimes X$. These C^*-algebras are isomorphic to the full and reduced crossed product C^*-algebras, and encode a lot of data of the dynamics: for example, $\mathrm{C}^*_{\mathrm{r}}(G \ltimes X)$ is simple iff the action of G on X is topological free and minimal.

Groupoids and noncommutative spaces . Another important reason to study groupoids is that they are viewed as geometric models for *noncommutative spaces.* Recall that a locally-compact Hausdorff space X corresponds to a commutative C^*-algebra $\mathrm{C}_0(X)$: this is the well-known Gelfand duality. Then noncommutative C^*-algebras play the role of "non-commutative spaces" in the algebraic setting. But sometimes it is desirable to find geometric models of noncommutative spaces. Of course, they cannot be the usual topological spaces because $\mathrm{C}_0(X)$ is always commutative. One attempt is to seek a topological groupoid $\mathcal{G}$, such that its groupoid C^*-algebra is isomorphic to the noncommutative C^*-algebra that we start with. Then $\mathcal{G}$ is a good geometric model for our noncommutative space. If we view a topological space X as a groupoid (see Example 1.15), then its groupoid C^*-algebra is precisely $\mathrm{C}_0(X)$, complying nicely with the classical Gelfand theory.

Foliation, groupoids and index theory . Index theory studies the connection between indices of (pseudo)differential operators and the topology or geometry of the spaces they live in. One of the most celabrated index theorem is the Atiyah–Singer index theorem. The family index theorem is a variant of the Atiyah–Singer index theorem. The set-up is a fibration $E \to X$ over a compact base, and a family of elliptic operators $\{D_x\}_{x \in X}$ parametrised by X, and such that each D_x acts on the vertical fibre E_x. The family index theorem computes the index of the family $\mathrm{Index}(D_x)$, which is an element in $\mathrm{K}^0(X)$, the K-theory of X.

Fibrations are special cases of *foliations*, and one might wish to generalise the theory to arbitrary foliations. However, foliations can be quite well-behaved in general. For example, consider the *Kronecker foliation* of $\mathbb{T}^2 \cong \mathbb{R}^2/2\pi\mathbb{Z}$ defined by the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \vartheta$. If ϑ is rational. Then every orbit (leaf) is closed and homeomorphic to a circle. However, if ϑ is irrational, then every leaf is dense in $\mathbb{T}^2$, and the leaf space with the quotient topology is homeomorphic to a single point. This makes the family index theorem useless.

The problem arises because the leaf space is badly-behaved. The solution is to replace this space by a "noncommutative space" — the *foliation groupoid.* Thus the K-theory of the C^*-algebra of the foliation groupoid becomes a nice receptacle of the family index. This is the now well-known longitudinal index theorem of Connes and Skandalis [CS84].

1.2 Groupoids

Definition 1.1 (First definition of groupoids)**.** A groupoid is a set $\mathcal{G}$ together with:

- a set $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$ of "composable arrows";
- a "multiplication map" $\mathcal{G}^{(2)} \to \mathcal{G}$, $(a, b) \mapsto ab$;
- an "inverse" map $\mathcal{G} \to \mathcal{G}$, $a \mapsto a^{-1}$,

such that:

1. (Associativity) If $(a, b) \in \mathcal{G}^{(2)}$ and $(b, c) \in \mathcal{G}^{(2)}$. Then (ab, c), $(a, bc) \in \mathcal{G}^{(2)}$ and $(ab)c = a(bc)$.
2. (Involutivity) $(a^{-1})^{-1} = a$.
3. (Unit) For any $a \in \mathcal{G}$, $(a^{-1}, a) \in \mathcal{G}^{(2)}$; if $(a, b) \in \mathcal{G}^{(2)}$, then $abb^{-1} = a$ and $a^{-1}ab = b$.

The unit axiom asserts that, unlike a group, a groupoid can have many (one-sided) units; the units of a groupoid $\mathcal{G}$ forms a subset $\mathcal{G}^{(0)} \subseteq \mathcal{G}$, which comes together with a pair of maps $s, r \colon \mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$.

Definition 1.2. Let $\mathcal{G}$ be a groupoid.

- The *unit space* $\mathcal{G}^{(0)}$ of $\mathcal{G}$ is
$$\mathcal{G}^{(0)} := \{a \in \mathcal{G} \mid a = a^{-1} = a^2\}.$$
- The *source* map of $\mathcal{G}$ is $s \colon \mathcal{G} \to \mathcal{G}^{(0)}$, $a \mapsto a^{-1}a$.
  The *range* map of $\mathcal{G}$ is $r \colon \mathcal{G} \to \mathcal{G}^{(0)}$, $a \mapsto aa^{-1}$.

We have the following:

Lemma 1.3. *Let $\mathcal{G}$ be a groupoid.*

1. $\mathcal{G}^{(0)} = \{aa^{-1} \mid a \in \mathcal{G}\}$.
2. $\mathcal{G}^{(2)} = \{(a, b) \in \mathcal{G} \times \mathcal{G} \mid s(a) = r(b)\}$.
3. *If $a, b \in \mathcal{G}$ and $(a, b) \in \mathcal{G}^{(2)}$. Then:*
$$s(a) = r(a^{-1}), \qquad s(ab) = s(b), \qquad r(ab) = r(a),$$
$$(b^{-1}, a^{-1}) \in \mathcal{G}^{(2)}, \qquad b^{-1}a^{-1} = (ab)^{-1}.$$

Proof.

- Clearly $aa^{-1} \in \mathcal{G}^{(0)}$ for any $a \in \mathcal{G}$. If $a \in \mathcal{G}^{(0)}$, then $a = a^2 = aa^{-1}$.
- If $s(a) = r(b)$, then $a^{-1}a = bb^{-1}$. Since $(a, a^{-1}), (a^{-1}, a), (b^{-1}, b), (b, b^{-1}) \in \mathcal{G}^{(2)}$. We have $(a, a^{-1}a) = (a, bb^{-1}) \in \mathcal{G}^{(2)}$ and $(bb^{-1}, b) \in \mathcal{G}^{(2)}$. Then $(a, bb^{-1}b) = (a, b) \in \mathcal{G}^2$. Conversely, if $(a, b) \in \mathcal{G}^{(2)}$. Since $(a^{-1}, a), (b, b^{-1}) \in \mathcal{G}^{(2)}$, the product $a^{-1}abb^{-1}$ makes sense, which equals both bb^{-1} and $a^{-1}a$.
- The three equations in the first line can be quickly checked. If $(a, b) \in \mathcal{G}^{(2)}$. Then $r(a^{-1}) = s(a) = r(b) = s(b^{-1})$ and hence $(b^{-1}, a^{-1}) \in \mathcal{G}^{(2)}$. The product $b^{-1}a^{-1}ab(ab)^{-1}$ makes sense and equals both $b^{-1}a^{-1}$ and $(ab)^{-1}$. $\square$

Remark 1.4. The previous lemma states that, a groupoid can equivalently be described by the data $(\mathcal{G}, \mathcal{G}^{(0)}, s, r, ^{-1})$. This leads to an alternative definitions of groupoids.

Definition 1.5 (Second definition of groupoids)**.** A groupoid is a set $\mathcal{G}$ together with:

- a distinguished subset $\mathcal{G}^{(0)} \subseteq \mathcal{G}$;
- a pair of maps $r, s\colon \mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$;
- a map $\mathcal{G}^2 \to \mathcal{G}$, $(a,b) \mapsto ab$, where

$$\mathcal{G}^{(2)} := \{(a,b) \in \mathcal{G} \times \mathcal{G} \mid s(a) = r(b)\};$$

- a map $\mathcal{G} \to \mathcal{G}$, $a \mapsto a^{-1}$,

such that

1. $r(x) = x = s(x)$ for all $x \in \mathcal{G}^{(0)}$.
2. $r(a)a = a = as(a)$ for all $a \in \mathcal{G}$.
3. $r(a^{-1}) = s(a)$ for all $a \in \mathcal{G}$.
4. $s(a) = a^{-1}a$ and $r(a) = aa^{-1}$ for all $a \in \mathcal{G}$.
5. $r(ab) = r(a)$ and $s(ab) = s(b)$ for all $(a,b) \in \mathcal{G}^{(2)}$.
6. $(ab)c = a(bc)$ whenever $s(a) = r(b)$ and $s(b) = r(c)$.

In this definition, the roles of the range and source maps are highlighted: this is actually more important if we want to study a topological groupoid or a Lie groupoid. I feel that it is sometimes more convenient to denote a groupoid by a diagram $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} \mathcal{G}^{(0)}$.

Example 1.6.

1. A group G is a groupoid $G \rightrightarrows \mathrm{pt}$.
2. A set X is a groupoid $X \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightrightarrows}} X$ together with the trivial multiplication and inverse maps.
3. Group bundle. A group bundle consists of two sets E, X and a surjective map $\pi\colon E \twoheadrightarrow X$ such that $\pi^{-1}(x)$ is a group for every $x \in X$. A group bundle can be viewed as a groupoid $E \underset{\pi}{\overset{\pi}{\rightrightarrows}} X$. In particular: a vector bundle is a groupoid.
4. *Action groupoid.* Let X be a (left) G-set. That is, G acts on X on the left. The action groupoid $G \ltimes X \rightrightarrows X$ is defined as follows. We set $G \ltimes X := G \times X$ as a set and $(G \ltimes X)^{(0)} := X$. The source, range, multiplication and inverse maps are
$$s(g,x) := x, \qquad r(g,x) := g \cdot x, \qquad (h, gx)(g,x) := (hg, x), \qquad (g,x)^{-1} := (g^{-1}, gx).$$
5. *Pair groupoid.* Let X be a set. The pair groupoid is given by $X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X$. The multiplication map is given by $(x,y)(y,z) := (x,z)$ and the inverse map is $(x,y)^{-1} := (y,x)$.
6. *Equivalence relations.* Let X be a set and $\mathcal{R} \subseteq X \times X$ be an equivalence relation on X. Then $\mathcal{R} \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X$ is a groupoid, with multiplication $(x,y)(y,z) := (x,z)$ and inverse $(x,y)^{-1} := (y,x)$.
   - If we set $\mathcal{R} := X \times X$, then we recover the pair groupoid as a special case. If we set $\mathcal{R} := \emptyset$, then we recover the groupoid $X \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightrightarrows}} X$.

- Let $\mathcal{G}$ be any groupoid. We can define an equivalence relation on $\mathcal{G}^{(0)}$ by claiming two units are equivalent iff they are connected by a groupoid element. Equivalently, this is the subset $\mathcal{R}(\mathcal{G}) := \{(r(a), s(a)) \mid a \in \mathcal{G}\} \subseteq \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$. Thus we obtain a groupoid $\mathcal{R}(\mathcal{G}) \rightrightarrows \mathcal{G}^{(0)}$. We say a groupoid $\mathcal{G}$ is *principal* if $\mathcal{G}$ is isomorphic to $\mathcal{R}(\mathcal{G})$ as a groupoid. Equivalently, this means there exists at least one arrow between two units in $\mathcal{G}$.

7. *Fundamental groupoid.* Let X be a topological space and $x \in X$. An important invariant in algebraic topology is the *fundamental group* of X (with basepoint x), defined as the group of (basepoint-fixed homotopy) equivalence classes of loops in X with basepoint x:

$$\pi_1(X, x) := \frac{\{\text{Loop } \gamma \text{ in } X \mid \gamma(0) = \gamma(1) = x\}}{\gamma \sim \gamma' \text{ iff } \gamma \text{ and } \gamma' \text{ are homotopic with basepoint fixed}}.$$

   This definition is not completely satisfying due to the following issues. If X is not path-connected, given $x, y \in X$, $\pi_1(X, x)$ and $\pi_1(X, y)$ may not be isomorphic if x and y do not lie in the same path component. If X is path-connected, then X has a unique path component and a different basepoint gives rise to an isomorphic fundamental group. However, the isomorphism between these two groups depend on the choice of the basepoints and on the specified path connecting them, hence not canonical.

   It is desirable to obtain a mathematical object similar to the fundamental group but does not depend on the choice of a basepoint. A natural idea is to choose (equivalence classes of) paths instead of loops. Unlike loops which are based at a certain point, paths are not concatenatable, unless the starting point of one coincides with the endpoint of another. This is precisely the axiom of a groupoid. So we may define the fundamental groupoid of X as:

$$\Pi_1(X) := \frac{\{\text{Path } \gamma \text{ in } X\}}{\gamma \sim \gamma' \text{ iff } \gamma \text{ and } \gamma' \text{ are homotopic with basepoints fixed}}.$$

   The fundamental groupoid is an important object in algebraic topology.

8. *Tangent groupoid.* Tangent groupoids were introduced by Alain Connes as an approach to study index theory. We briefly mention his construction. The interplay between tangent groupoids and index theory will be discussed in a future talk.

   Let M be a smooth manifold. The tangent groupoid of M is the groupoid

$$\mathbb{T}M := TM \times \{0\} \sqcup M \times M \times (0, 1] \overset{r}{\underset{s}{\rightrightarrows}} M \times [0, 1],$$

   where

$$\begin{aligned} r(x, v, 0) &= (x, 0), \qquad & s(x, v, 0) &= (x, 0); \\ r(x, y, \epsilon) &= (x, \epsilon), \qquad & s(x, y, \epsilon) &= (y, \epsilon), \qquad \epsilon \in (0, 1]. \end{aligned}$$

Definition 1.7 (Subgroupoids)**.** A subgroupoid of a groupoid is a subset $\mathcal{H} \subseteq \mathcal{G}$ such that, the multiplication and inverse maps of $\mathcal{G}$ restricted to $\mathcal{H}$ turns it into a groupoid.

Definition 1.8 (Groupoid homomorphism)**.** A (strict) groupoid homomorphism $f\colon \mathcal{G} \to \mathcal{H}$ is a map such that $f \times f(\mathcal{G}^{(2)}) \subseteq \mathcal{H}^{(2)}$ and $f(ab) = f(a)f(b)$ for all $(a, b) \in \mathcal{G}^{(2)}$. It is an *isomorphism* if there exists another groupoid homomorphism $g\colon \mathcal{H} \to \mathcal{G}$ such that $f \circ g = \mathrm{id}_{\mathcal{H}}$ and $g \circ f = \mathrm{id}_{\mathcal{G}}$.

Remark 1.9. In later talks we shall define another larger class of morphisms between groupoids called *groupoid correspondences*. For clarity we will frequently refer to groupoid homomorphisms as *strict homomorphisms*.

Definition 1.10. Let $\mathcal{G}$ be a groupoid.

- Let $x, y \in \mathcal{G}^{(0)}$. We define the *source fibre at* y to be $\mathcal{G}_y := s^{-1}(y)$, the *range fibre at* x to be $\mathcal{G}^x := r^{-1}(x)$, and $\mathcal{G}_y^x := \mathcal{G}^x \cap \mathcal{G}_y$.
- Let $A, B \subseteq \mathcal{G}^{(0)}$. We define $\mathcal{G}_B := s^{-1}(\mathcal{G})$, $\mathcal{G}^A := r^{-1}(A)$ and $\mathcal{G}_B^A := \mathcal{G}^A \cap \mathcal{G}_B$.

Definition 1.11. Let $\mathcal{G}$ be a groupoid.

- Let $A \subseteq \mathcal{G}^{(0)}$. Then $\mathcal{G}_A^A \subseteq \mathcal{G}$ is a subgroupoid, called the restriction of $\mathcal{G}$ to A.
- Let $x \in \mathcal{G}^{(0)}$. Then $\mathcal{G}_x^x$ is a group, called the *isotropy group* at x.
- The *isotropy groupoid* is the subgroupoid of $\mathcal{G}$:

$$\mathrm{Iso}(\mathcal{G}) := \bigcup_{x \in \mathcal{G}^{(0)}} \mathcal{G}_x^x \rightrightarrows \mathcal{G}^{(0)}.$$

1.3 Topological groupoids

Now we turn to groupoids with extra structures. Charles Ehresmann was the first person to endow groupoids with extra structures while applying them to the study of foliation. Examples include topological groupoids, Borel groupoids (groupoids with measurable structures) and Lie groupoids.

Topological groupoids are the central objects that we will care about in the seminar talks. Let $\mathcal{G}$ be a groupoid that is also a topological space. Then $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$ inherits the product topology of $\mathcal{G} \times \mathcal{G}$.

Definition 1.12. A groupoid $\mathcal{G}$ is a *topological groupoid* if $\mathcal{G}$ is a topological space, and the multiplication map and the inverse map are continuous.

Remark 1.13. Just as in the case of groups, we usually require that a topological groupoid $\mathcal{G}$ is such that:

1. $\mathcal{G}$ is locally-compact.
2. $\mathcal{G}^{(0)}$ is Hausdorff (in the subspace topology).

However, the groupoid $\mathcal{G}$ itself does not have to be Hausdorff. When $\mathcal{G}$ is Hausdorff, its unit space will be closed, see the following lemma.

In fact, non-Hausdorff groupoids arise naturally from dynamical systems and differenrial geometry (e.g. singular foliations). They give rise to interesting C*-algebras.

Lemma 1.14. *Let $\mathcal{G}$ be a topological groupoid. Then $\mathcal{G}$ is Hausdorff iff $\mathcal{G}^{(0)}$ is closed.*

Proof. If $\mathcal{G}$ is Hausdorff. Then every convergent net in $\mathcal{G}$ converges to a unique point. Let $\{a_i\}_{i \in I}$ be a net in $\mathcal{G}^{(0)}$ which converges to $a \in \mathcal{G}$. We claim that the limit a must lie in $\mathcal{G}^{(0)}$ as well. Since $\mathcal{G}$ is a topological groupoid, the source and range maps $s, r \colon \mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ are continuous. Hence $s(a_i) \to s(a)$ and $r(a_i) \to r(a)$. But $a_i \in \mathcal{G}^{(0)}$ for all i, that is, $s(a_i) = a_i = r(a_i)$. Then we have $a_i \to s(a)$, $a_i \to r(a)$ and $a_i \to a$. The Hausdorffness of $\mathcal{G}$ forces $s(a) = a = r(a)$, that is, $a \in \mathcal{G}^{(0)}$.

Now assume that $\mathcal{G}^{(0)}$ is closed. Let $\{a_i\}_{i \in I}$ be any convergent net in $\mathcal{G}$ which converges to both a and b. We must prove that $a = b$. Since the multiplication and the inverse maps are continuous, we have $a_i^{-1} a_i \to a^{-1} b$. But $a_i^{-1} a_i \in \mathcal{G}^{(0)}$ for all i and $\mathcal{G}^{(0)}$ is closed. Therefore, $a^{-1} b \in \mathcal{G}^{(0)}$, which implies that $a = b$. $\square$

The following examples of topological groupoids are just modifications of Example 1.6.

Example 1.15.

1. A topological group G is a topological groupoid $G \rightrightarrows \mathrm{pt}$.
2. A topological space X is a topological groupoid $X \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightrightarrows}} X$ together with the trivial multiplication and inverse maps.

3. A topological group bundle consists of two topological spaces E and X, and a quotient map $\pi\colon E \twoheadrightarrow X$, such that for any $x \in X$, $\pi^{-1}(x)$ is a topological group. This is a topological groupoid.
4. Let G be a topological group which acts continuously on a space X. Then the action groupoid $G \ltimes X$ is a topological groupoid.
5. Let X be a topological space, and $\mathcal{R} \subseteq X \times X$ be an equivalence relation on X, equipped with the subspace topology. Then $\mathcal{R} \rightrightarrows X$ is a topological groupoid. In particular, the pair groupoid of a topological space is a topological groupoid.
6. The fundamental groupoid $\Pi_1(X)$ of a topological space X is a bit tricky. There are various ways to topologise it, but the "correct" topology is defined only when X satisfy some nice conditions (path-connected, locally path-connected and semi-locally simply connected). Readers who are familiar with algebraic topology shall notice that these conditions are precisely what one needs to obtain a nice classification theory of covering spaces. In such situation, the fundamental groupoid is realised as a quotient of the pair groupoid and has the quotient topology.
7. The tangent groupoid is a topological groupoid. The topology is defined as follows: we require that $M \times M \times (0, 1]$ is open, and require that a sequence $\{(x_n, y_n, \epsilon_n)\}_n$ in $M \times M \times (0, 1]$ converges to $(x, v, 0)$ iff
$$x_n \to x, \qquad y_n \to x, \qquad \frac{x_n - y_n}{\epsilon_n} \to v.$$

Finally, we define a subclass of topological groupoids called étale groupoids. They are analogs of discrete groupoids, and easier to study than general topological groupoids. It is a bit surprising that étale groupoids are already interesting enough, in the sense that many interesting C*-algebras arise as the C*-algebra of an étale groupoid.

Definition 1.16. A topologial groupoid $\mathcal{G}$ is called an *étale groupoid*, if the source and range maps $s, r\colon \mathcal{G} \to \mathcal{G}$ are étale. That is, s and r are local homeomorphisms.

Remark 1.17. Be careful that the maps s and r are étale as maps $\mathcal{G} \to \mathcal{G}$, but not as maps $\mathcal{G} \to \mathcal{G}^{(0)}$. This is a stronger argument: it asserts that the inclusion map $\mathcal{G}^{(0)} \hookrightarrow \mathcal{G}$ is a topological embedding.

Lemma 1.18. *If $\mathcal{G}$ is an étale groupoid. Then $\mathcal{G}^{(0)} \subseteq \mathcal{G}$ is open.*

Proof. $\mathcal{G}$ is étale implies that, for every $a \in \mathcal{G}$, there is an open neighbourhood $U_a \subseteq \mathcal{G}$ of a such that $s|_{U_a}\colon U_a \to s(U_a)$ is a homeomorphism. Then $\mathcal{G}^{(0)} = \bigcup_{a \in \mathcal{G}} s(U_a)$ is open. ☐

The following lemma states that étale groupoids are "fibrewise discrete".

Lemma 1.19. *If $\mathcal{G}$ is étale. Then for any $x \in \mathcal{G}^{(0)}$, $\mathcal{G}_x$ and $\mathcal{G}^x$ are discrete.*

Proof. $\mathcal{G}$ is étale implies that, for every $a \in \mathcal{G}$, there is an open neighbourhood $U_a \subseteq \mathcal{G}$ of a such that $s|_{U_a}\colon U_a \to s(U_a)$ is a homeomorphism. In particular, s is a bijection.

We claim that $\{a\} = \mathcal{G}_x \cap U_a$. Clearly $\{a\} \subseteq \mathcal{G}_x \cap U_a$. Suppose $b \in \mathcal{G}_x \cap U_a$, then $s(b) = x = s(a)$. Since s is bijective on U_a, we must have $b = a$. Therefore, $\{a\} \subseteq \mathcal{G}_x \cap U_a$ is open in $\mathcal{G}_x$ in the relative topology. So $\mathcal{G}_x$ is discrete. The proof for $\mathcal{G}^x$ is essentially the same. ☐

September 27, 2022

Haar systems and groupoid C*-algebras

Speaker: Yufan Ge (Leiden University)

Groupoid C*-algebras were introduced by Renault [Ren80], which consist of a large class of interesting C*-algebras.

In this lecture, all topological groupoids are assumed to be *locally-compact* and *Hausdorff*.

2.1 Haar systems

A Haar system is the analog of a Haar measure of a topological group. Recall that

Definition 2.1. A *Radon* measure is a measure μ on a locally-compact space X, which is

1. Borel: all open subsets are measurable.
2. regular: μ is inner regular and outer regular. That is,

$$\mu(E) = \sup\{\mu(K) \mid K \subseteq E \text{ is compact}\} \qquad \text{and} \qquad \mu(E) = \inf\{\mu(U) \mid U \supseteq E \text{ is open}\}.$$

3. locally finite: for any $x \in X$, there exists a neighbourhood N of x, such that $\mu(N) < \infty$.

Measures on a locally-compact space X are related to linear functionals of $\mathrm{C_c}(X)$ by the following theorem:

Theorem 2.2 (Riesz–Markov–Kakutani representation theorem)**.** *Let X be a locally-compact Hausdorff space, and $\psi\colon \mathrm{C_c}(X) \to \mathbb{C}$ (or $\psi\colon \mathrm{C_c}(X) \to \mathbb{R}$) be a linear functional. There exists a complex (or real) Radon measure μ on X, such that*

$$\psi(f) = \int_X f \,\mathrm{d}\mu.$$

This theorem allows us a construct a Radon measure from a linear functional.

Definition 2.3 (Haar system)**.** A *Haar system* of a locally-compact Hausdorff groupoid $\mathcal{G}$ is a family of Radon measures $\{\mu^u\}_{u\in\mathcal{G}^{(0)}}$ indexed by $\mathcal{G}^{(0)}$, such that

(HS1) $\operatorname{supp}(\lambda^u) = \mathcal{G}^u$;

(HS2) For any $f \in \mathrm{C_c}(\mathcal{G})$, the function

$$\lambda_f\colon \mathcal{G}^{(0)} \to \mathbb{C}, \qquad u \mapsto \int_{\mathcal{G}} f(\gamma)\,\mathrm{d}\lambda^u(\gamma)$$

is continuous (and hence compactly-supported).

(HS3) For all $\eta \in \mathcal{G}$, the following "left-invariance" holds:

$$\int_{\mathcal{G}} f(\gamma)\,\mathrm{d}\lambda^{r(\eta)}(\gamma) = \int_{\mathcal{G}} f(\eta\gamma)\,\mathrm{d}\lambda^{s(\eta)}\gamma.$$

Example 2.4. Let X be a locally-compact space with Radon measure μ, G be a locally-compact group with a Haar measure μ. Define the groupoid $\mathcal{G} := X \times G \times X \underset{s}{\overset{r}{\rightrightarrows}} X$ with

$$r(x,g,y) := x, \qquad s(x,g,y) := y, \qquad (x,g,y)(y,h,z) := (x,gh,z).$$

Then $\{\lambda^x := \delta_x \times \mu \times \nu\}_{x\in X}$ is a Haar system of $\mathcal{G}$.

Example 2.5. Let $\mathcal{G} := G \ltimes X$ be the action groupoid. Define the linear functional

$$\lambda^x\colon \mathrm{C_c}(X) \to \mathbb{C}, \qquad \lambda^x(f) := \int_{\mathcal{G}} f(g, g^{-1}x)\,\mathrm{d}\mu(g).$$

This precribes a Haar system $\{\lambda^x\}_{x\in X}$ for $\mathcal{G}$ by Riesz–Markov–Kakutani representation theorem. It suffices to check conditions (HS1)–(HS3). For the first one, we need the following

Lemma 2.6. *Let $\lambda\colon \mathrm{C_c}(X) \to \mathbb{C}$ be a positive linear functional. Then*

$$x \in \operatorname{supp}\lambda \qquad \textit{iff} \qquad \lambda(f) > 0 \textit{ for all } f \in \mathrm{C_c}(X) \textit{ with } f(x) > 0.$$

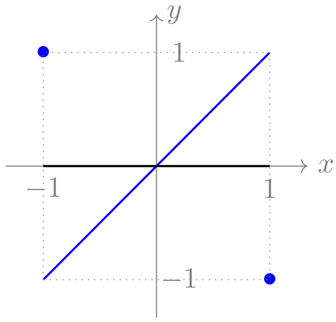

Figure 3: An r-discrete but not étale groupoid: the fibre has one point at each $x \in (-1, 1)$, but has two points at ± 1.

This implies that $\operatorname{supp} \lambda^x = \mathcal{G}^x$.

Remark 2.7. Given a Haar system $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$ and a Radon measure μ on $\mathcal{G}^{(0)}$, we can define a linear functional on $\mathrm{C_c}(\mathcal{G})$:

$$\nu(f) := \int_{\mathcal{G}^{(0)}} \int_{\mathcal{G}} f(\gamma) \, \mathrm{d}\lambda^u(\gamma) \, \mathrm{d}\mu(u).$$

Hence defines a measure on $\mathcal{G}$ by Riesz–Markov–Kakutani representation theorem.

Remark 2.8. $\mathrm{C_c}(\mathcal{G})$ may be viewed as a right $\mathrm{C}_0(\mathcal{G}^{(0)})$-module via

$$f \cdot \psi(\gamma) := f(\gamma)\psi(s(\gamma)), \qquad f \in \mathrm{C_c}(\mathcal{G}), \ \psi \in \mathrm{C_c}(\mathcal{G}^{(0)}).$$

Then the Haar system $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$ allows for a $\mathrm{C}_0(\mathcal{G}^{(0)})$-valued inner product

$$\langle f, g \rangle(u) := \int_{\mathcal{G}} \overline{f(\gamma)} g(\gamma) \, \mathrm{d}\lambda^u(\gamma).$$

The condition (HS2) guarantees that the inner product actually lands in $\mathrm{C}_0(\mathcal{G}^{(0)})$. Endowed with these operations, $\mathrm{C_c}(\mathcal{G})$ becomes a pre Hilbert $\mathrm{C}_0(X)$-module, giving an alternative definition of groupoid C^*-algebras (Remark 2.20)

Definition 2.9. A locally-compact Hausdorff groupoid $\mathcal{G}$ is called *r-discrete* if $\mathcal{G}^{(0)}$ is open.

Lemma 2.10. *If $\mathcal{G}$ is r-discrete. Then $\mathcal{G}^u$ and $\mathcal{G}_u$ are discrete for all $u \in \mathcal{G}^{(0)}$.*

Proof. $\mathcal{G}^{(0)}$ is open, so every singleton $\{u\}$ is open in $\mathcal{G}^u$. Now for any $\gamma \in \mathcal{G}$, let $u = s(\gamma)$ and $v = r(\gamma)$. Then the map

$$\mathcal{G}_v \to \mathcal{G}_u, \qquad \eta \mapsto \eta\gamma$$

is continuous, with $\{\gamma\} = \phi^{-1}(\{u\})$. So $\{\gamma\}$ is open. □

Lemma 2.11. *If $\mathcal{G}$ is r-discrete and r is open. Then r is a local homeomorphism.*

Lemma 2.12. *If $\mathcal{G}$ is r-discrete. Then the counting measure on each fibre form a Haar system iff $\mathcal{G}$ is étale.*

Example 2.13 (An r-discrete but not étale groupoid). Consider the equivalence relation

$$E := \{(-1, 1), (1, -1)\} \cup \{(x, x) \mid -1 \leq x \leq 1\}$$

on $E^0 := [-1, 1]$. The groupoid $E \rightrightarrows E^{(0)}$ is r-discrete, but not étale.

2.2 Groupoid C*-algebras

Let $f, g \in \mathrm{C_c}(\mathcal{G})$. Define

$$f * g(\gamma) := \int_{\mathcal{G}} f(\eta) g(\eta^{-1}\gamma) \, \mathrm{d}\lambda^{r(\eta)}\gamma,$$
$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

Lemma 2.14. *The operations above turn* $\mathrm{C_c}(\mathcal{G})$ *into a* **-algebra.*

Proof. We need to check that $*$ is a convolution product. Notice that $f * g$ is continuous, and has compact support because

$$\mathrm{supp}(f * g) \subseteq \mathrm{supp}(f) \, \mathrm{supp}(g).$$

Now we check associativity. We have

$$\begin{aligned}
((f * g) * h)(\gamma) &= \int_{\mathcal{G}} (f * g)(\eta) h(\eta^{-1}\gamma) \, \mathrm{d}\lambda^{r(\gamma)}\eta \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}} f(\xi) g(\xi^{-1}\eta) h(\eta^{-1}\gamma) \, \mathrm{d}\lambda^{r(\eta)}\xi \, \mathrm{d}\lambda^{r(\gamma)}\eta; \\
(f * (g * h))(\gamma) &= \int_{\mathcal{G}} f(\xi) (g * h)(\xi^{-1}\gamma) \, \mathrm{d}\lambda^{r(\gamma)}\xi \\
&= \int_{\mathcal{G}} f(\xi) \int_{\mathcal{G}} g(\eta) h(\eta^{-1}\xi^{-1}\gamma) \, \mathrm{d}\lambda^{r(\xi^{-1}\gamma)}\eta \, \mathrm{d}\lambda^{r(\gamma)}\xi \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}} f(\xi) g(\eta) h(\eta^{-1}\xi^{-1}\gamma) \, \mathrm{d}\lambda^{s(\xi)}\eta \, \mathrm{d}\lambda^{r(\gamma)}\xi
\end{aligned}$$

Replace η by $\xi^{-1}\eta$:

$$= \int_{\mathcal{G}} \int_{\mathcal{G}} f(\xi) g(\eta^{-1}\eta) h(\eta^{-1}\gamma) \, \mathrm{d}\lambda^{s(\xi)}(\xi^{-1}\eta) \, \mathrm{d}\lambda^{r(\gamma)}\xi$$

Now use (HS3), $\mathrm{d}\lambda^{r(\eta)}(\gamma) = \mathrm{d}\lambda^{s(\eta)}(\eta^{-1}\gamma)$:

$$= \int_{\mathcal{G}} \int_{\mathcal{G}} f(\xi) g(\xi^{-1}\eta) h(\eta^{-1}\gamma) \, \mathrm{d}\lambda^{r(\eta)}\xi \, \mathrm{d}\lambda^{r(\gamma)}\eta. \qquad \square$$

Example 2.15. A Haar system of an étale groupoid $\mathcal{G}$ is given by the fibrewise counting measure. Then for $f, g \in \mathrm{C_c}(\mathcal{G})$:

$$f * g(\gamma) = \sum_{r(\eta) = r(\gamma)} f(\eta) g(\eta^{-1}\gamma) = \sum_{\alpha\beta = \gamma} f(\alpha) g(\beta).$$

We wish to complete $\mathrm{C_c}(\mathcal{G})$ into a C*-algebra. Let $\pi \colon \mathrm{C_c}(\mathcal{G}) \to \mathbb{B}(H_\pi)$ be a *-representation on a Hilbert space. Then $\|f\|_\pi := \|\pi(f)\|_{\mathbb{B}(H_\pi)}$ is a norm for $\mathrm{C_c}(\mathcal{G})$, which might be unbounded. To obtain a C*-norm of $\mathrm{C_c}(\mathcal{G})$, one needs to restrict to representations which have a common upper bound. In the case of group C*-algebras, one considers all possible norms bounded by the L^1-norm, and takes suitable representations to obtain the full and reduced C*-norm. The analog of the L^1-norm in the groupoid case is the I-norm.

Definition 2.16 (***I*-norm**)**.** Let $f \in \mathrm{C_c}(\mathcal{G})$. Define

$$\|f\|_{I,r} := \sup_{u \in \mathcal{G}^{(0)}} \int_{\mathcal{G}} |f(\gamma)| \, \mathrm{d}\lambda^u(\gamma),$$
$$\|f\|_{I,s} := \sup_{u \in \mathcal{G}^{(0)}} \int_{\mathcal{G}} |f(\gamma^{-1})| \, \mathrm{d}\lambda^u(\gamma),$$
$$\|f\|_I := \max\{\|f\|_{I,r}, \|f\|_{I,s}\}.$$

Definition 2.17. A *-homomorphism $\pi\colon \mathrm{C_c}(\mathcal{G}) \to \mathbb{B}(H_\pi)$ is called I-norm bounded, if $\|f\|_\pi \le \|f\|_I$ for all f.

Given a Radon measure μ on $\mathcal{G}^{(0)}$. Recall that we may define a Radon measure on $\mathcal{G}$ as in Remark 2.7:

$$\nu(f) := \int_{\mathcal{G}^{(0)}} \int_{\mathcal{G}} f(\gamma)\, \mathrm{d}\lambda^u(\gamma)\, \mathrm{d}\mu(u).$$

We also define

$$\nu^{-1}(f) := \int_{\mathcal{G}^{(0)}} \int_{\mathcal{G}} f(\gamma^{-1})\, \mathrm{d}\lambda^u(\gamma)\, \mathrm{d}\mu(u).$$

Define

$$\mathrm{Ind}_\mu\colon \mathrm{C_c}(\mathcal{G}) \to \mathbb{B}(L^2(\mathcal{G}, \nu^{-1})), \qquad \mathrm{Ind}_\mu(f)(h) := \int_{\mathcal{G}} f(\eta) h(\eta^{-1}\gamma)\, \mathrm{d}\lambda^{r(\gamma)}(\eta).$$

Proposition 2.18. *Let $\mathcal{G}$ be a locally-compact Hausdorff groupoid with Haar system, and μ be any Radon measure on $\mathcal{G}^{(0)}$. Then Ind_μ is an I-norm bounded representation of $\mathrm{C_c}(\mathcal{G})$.*

Definition 2.19. Let $\mathcal{G}$ be a locally-compact Hausdorff groupoid with Haar system.

- The full norm of $\mathrm{C_c}(\mathcal{G})$ is

  $$\|f\| := \sup\{\|\pi(f)\| \mid \pi \text{ is an } I\text{-norm bounded representation.}\}$$

  The full groupoid C*-algebra $\mathrm{C}^*(\mathcal{G})$ is the completion of $\mathrm{C_c}(\mathcal{G})$ under the full norm.

- The reduced norm of $\mathrm{C_c}(\mathcal{G})$ is

  $$\|f\|_{\mathrm{r}} := \sup\{\|\mathrm{Ind}_{\delta_u}(f)\| \mid u \in \mathcal{G}^{(0)},\ \delta_u \text{ is the Dirac measure supported on } u.\}$$

  The reduced groupoid C*-algebra $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})$ is the completion of $\mathrm{C_c}(\mathcal{G})$ under the reduced norm.

Remark 2.20. There is an (even nicer) construction of reduced groupoid C*-algebras $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})$ as follows. As in Remark 2.8, $\mathrm{C_c}(\mathcal{G})$ is a pre-Hilbert $\mathrm{C}_0(\mathcal{G}^{(0)})$ module. That is, $\langle f, g\rangle$ is the restriction of $f^* * g$ to $\mathcal{G}^{(0)}$ and $f \cdot \psi(\gamma) := f(\gamma)\psi(s(\gamma))$ for $f, g \in \mathrm{C_c}(\mathcal{G})$ and $\psi \in \mathrm{C}_0(\mathcal{G}^{(0)})$. We complete $\mathrm{C_c}(\mathcal{G})$ to a Hilbert $\mathrm{C}_0(\mathcal{G}^{(0)})$-module, denoted by $L^2(\mathcal{G}, \nu)$.

The multiplication action of $\mathrm{C_c}(\mathcal{G})$ on itself extends to a bounded representation $\pi\colon \mathrm{C_c}(\mathcal{G}) \to \mathbb{B}(L^2(\mathcal{G}, \nu))$. Define the reduced norm of $f \in \mathrm{C_c}(\mathcal{G})$ to be $\|f\|_{\mathrm{r}} := \|\pi(f)\|$. One checks that this coincides with the reduced norm defined above, hence refines reduced groupoid C*-algebras via completion.

The case of non-Hausdorff groupoids is more involved: one needs to replace $\mathrm{C}_0(\mathcal{G}^{(0)})$ by a bigger algebra, which contains some Borel non-continuous functions on $\mathcal{G}^{(0)}$. See [KS02] for more details.

Example 2.21. Let X be a locally-compact Hausdorff space, viewed as a groupoid $\mathcal{G} := X \rightrightarrows X$. Then $\mathrm{C_c}(\mathcal{G}) = \mathrm{C_c}(X)$. The I-norm is $\|f\|_I = \sup_{x \in X}|f(x)| = \|f\|_\infty$. This is in fact a C*-norm, achieved by the multiplication representation on $L^2(X, \mu)$ for any Radon measure μ on X. The groupoid C*-algebra $\mathrm{C}^*(\mathcal{G}) = \mathrm{C}_0(X)$.

Example 2.22. Let X be a finite set of n-elements, and $\mathcal{R}$ be the trivial equivalence relation on X. Then the groupoid C*-algebra $\mathrm{C}^*(\mathcal{R}) \cong \mathbb{M}_n(\mathbb{C})$.

Example 2.23. Let G be a locally-compact topological group, viewed as a groupoid $\mathcal{G} := G \rightrightarrows \mathrm{pt}$. If G is unimodular, then the definition of group and groupoid C*-algebras of G coincide. If G is not unimodular, we need tiny modification to build the isomorphism between them two. Let $f \in \mathrm{C_c}(G)$. The involution on $\mathrm{C_c}(G)$ is

$$f^*(x) := \Delta(x)^{-1}\overline{f(x^{-1})}.$$

The involution on $\mathrm{C_c}(\mathcal{G})$ is

$$f^*(x) := \overline{f(x^{-1})}.$$

Define

$$\phi\colon \mathrm{C_c}(G) \to \mathrm{C_c}(\mathcal{G}), \qquad f \mapsto \Delta^{-1/2} f.$$

We claim that this is a *-isomorphism of *-algebras, and extends to an isometric isomorphism of C^*-algebras $\mathrm{C}^*(G) \xrightarrow{\cong} \mathrm{C}^*(\mathcal{G})$. Let π be an I-norm bounded representation of $\mathrm{C_c}(\mathcal{G})$. Then $\pi \circ \phi$ is a representation of $\mathrm{C_c}(G)$. Since $\|\phi(f)\|$ is by definition the supremum over all I-norm bounded representations, we have $\|\phi(f)\| \leq \|f\|$. The other side holds using a disintegration theorem of group representations and we omit here. Therefore, $\|\phi(f)\| = \|f\|$.

October 4, 2022

Groupoid actions and equivalence actions

Speaker: Jack Ekenstam (Leiden University)

Throughout this lecture, $\mathcal{G}$ will denote a groupoid. If $\mathcal{G}$ is a topological groupoid, we will always assume that it is locally-compact and Hausdorff.

3.1 Groupoid actions

Definition 3.1. Let $\mathcal{G}$ be a groupoid. Let X be a set together with a map $r_X\colon X \to \mathcal{G}^{(0)}$ called the *moment map.* A left action of $\mathcal{G}$ on X is a map

$$\mathcal{G} * X \to X, \qquad (\gamma, x) \mapsto \gamma x,$$

where

$$\mathcal{G} * X := \{(\gamma, x) \in \mathcal{G} \times X \mid s(\gamma) = r_X(x)\},$$

such that

- $r_X(x)x = x$ for all $x \in X$.
- If $(\gamma, \eta) \in \mathcal{G}^{(2)}$ and $(\eta, x) \in \mathcal{G} * X$. Then $(\gamma\eta, x) \in \mathcal{G} * X$ and $(r\gamma)x = \gamma(\eta x)$.

A right action is defined similarly, while in that case a moment map is denoted by s_X for consistency. If $\mathcal{G}$ acts on X on the left (resp. right), we write $G \curvearrowright X$ (resp. $X \curvearrowleft G$) and call X a left (resp. right) $\mathcal{G}$-set. Unless specified, groupoid actions are always assumed to be left actions.

If $\mathcal{G}$ is a topological groupoid and X is a topological space, such that the moment map and the groupoid action are continous. Then we say X is a $\mathcal{G}$-space.

Definition 3.2. Let X be a $\mathcal{G}$-set. We say:

- $\mathcal{G}$ acts *transitively* on X, if for all $x, y \in X$, there exists $\gamma \in \mathcal{G}$ such that $x = \gamma y$.
- $\mathcal{G}$ acts *freely* on X, if $\gamma x = x$ for some x implies that $\gamma = r_X(x)$.

Example 3.3.

- Let $\mathcal{G}$ be a groupoid. Then $\mathcal{G} \curvearrowright \mathcal{G} \curvearrowleft \mathcal{G}$ in an obvious way. Also $\mathcal{G} \curvearrowright \mathcal{G}^{(0)} \curvearrowleft \mathcal{G}$.
- Let G be a group and X be a G-set. Then $G \ltimes X \curvearrowright X \curvearrowleft G \ltimes X$.

We may also define the groupoid version of the action groupoid of a group:

Definition 3.4. Let X be a $\mathcal{G}$-set. The action groupoid $\mathcal{G} \ltimes X$ is defined as

$$\mathcal{G} * X \rightrightarrows X, \qquad s(\gamma, x) = x,\ r(\gamma, x) = \gamma x.$$

Definition 3.5. Let X be a left $\mathcal{G}$-set and $x \in X$. The *orbit* of x is

$$\{\gamma x \mid (\gamma, x) \in \mathcal{G} * X\}.$$

Denote by $\mathcal{G}\backslash X$ the set of orbits. If X is a right $\mathcal{G}$-set, we write $X/\mathcal{G}$ for the set of orbits.

If X is a $\mathcal{G}$-space. Then we may endow $\mathcal{G}\backslash X$ with the quotient topology and call it the orbit space.

Remark 3.6. Recall that in the quotient topology: $U \subseteq \mathcal{G}\backslash X$ is open iff $q^{-1}(U) \subseteq X$ is open, where q is the quotient map. A map $f: \mathcal{G}\backslash X \to Y$ is continuous in the quotient topology iff it lifts to a continuous map $\tilde{f}: X \to Y$.

Definition 3.7. A groupoid $\mathcal{G}$ is called

- *principal*, if $\mathcal{G} \curvearrowright \mathcal{G}^{(0)}$ is free.
- *transitive*, if $\mathcal{G} \curvearrowright \mathcal{G}^{(0)}$ is transitive.

Proposition 3.8. *If $\mathcal{G}$ is a topological groupoid with r, s open. Let X be a $\mathcal{G}$-space. Then $q: X \to \mathcal{G}\backslash X$ is open.*

Corollary 3.9. *Let $\mathcal{G}$ be a locally-compact Hausdorff groupoid with r, s open. Let X be a locally-compact $\mathcal{G}$-space. Then $\mathcal{G}\backslash X$ is locally-compact. If X is second countable, then $\mathcal{G}\backslash X$ is too.*

Definition 3.10. Let $\mathcal{G}$ be a locally-compact Hausdorff groupoid acting on a locally-compact Hausdorff space X. We say the action is *proper*, if the map

$$\Theta: \mathcal{G} * X \to X \times X, \qquad (\gamma, x) \mapsto (\gamma x, x)$$

is proper, i.e. the pre-image of a compact set is compact.

Proposition 3.11. *Let $\mathcal{G}$ be a locally-compact Hausdorff groupoid acting on a locally-compact space X. The followings are equivalent:*

1. *$\mathcal{G}$ acts properly.*
2. *For all compact subsets K and L in X, the set*
$$\mathrm{pr}_1(\Theta^{-1}(K \times L)) = \{\gamma \in \mathcal{G} \mid K \cap \gamma L \neq \emptyset\}$$
is a compact subset in $\mathcal{G}$.
3. *Let $\{x_i\}$ be a net converging to x and $\{\gamma_i x_i\}$ be a net converging to y. Then $\{\gamma_i\}$ has convergent subnet.*

Proposition 3.12. *Let $\mathcal{G}$ be a locally-compact Hausdorff groupoid with r, s open. Let X be a proper $\mathcal{G}$-space. Then $\mathcal{G}\backslash X$ is locally-compact Hausdorff.*

Proof. Let $\{\mathcal{G}x_i\}$ be a net in $\mathcal{G}\backslash X$ which converges to both $\mathcal{G}x$ and $\mathcal{G}y$. We claim that $\mathcal{G}x = \mathcal{G}y$. Since the quotient map $q: X \to \mathcal{G}\backslash X$ is open, we may assume that $x_i \to x$ and $\gamma_i x_i \to y$ by Remark 3.6. Then there is a subnet of $\{\gamma_i\}$ converging to γ by Proposition 3.11. But X is Hausdorff. So $\gamma_i x_i$ converges to γx. We therefore have $y = \gamma x$ and $\mathcal{G}x = \mathcal{G}y$. □

Definition 3.13. Let $\mathcal{G}$ be a locally-compact Hausdorff groupoid acting on a locally-compact Hausdorff space X. We say the action is *Cartan*, if every $x \in X$ has a compact neighbourhood K such that $\Theta^{-1}(K \times K)$ is compact.

3.1.1 Mackey–Glimm–Ramsey dichotomy

Let $\mathcal{G}$ be a second-countable locally-compact Hausdorff groupoid with r, s open, and X be a $\mathcal{G}$-space. The continuous map

$$\phi_x \colon \mathcal{G}_{r(x)} \to \mathcal{G}X, \qquad \gamma \mapsto \gamma x$$

is not open in general. When it is open, the diagram

$$\begin{tikzcd} \mathcal{G}_x \arrow[r, "\phi_x"] \arrow[d, "q"'] & \mathcal{G}X \\ \mathcal{G}_x/H_x \arrow[ur, "\overline{\phi}_x"'] & \end{tikzcd}$$

imples that the map $\overline{\phi}_x \colon \mathcal{G}_x/H_x \to \mathcal{G}X$ is a homeomorphism, where

$$H_x := \{\gamma \in \mathcal{G}_{r(x)} \mid \gamma x = x\}.$$

We wish to know when we are in such "nice" situations. These are called the *Mackey–Glimm–Ramsey dichotomy* and justified by the following theorem.

Theorem 3.14. *Let $\mathcal{G}$ be a second-countable, locally-compact Hausdorff groupoid with r, s open. Let X be a locally-compact Hausdorff $\mathcal{G}$-space. The followings are equivalent:*

1. *$\mathcal{G}\backslash X$ is T_0-space.*
2. *$\mathcal{G}\backslash X$ is almost Hausdorff.*
3. *Each orbit is locally closed in X.*
4. *Each orbit is a G_δ-subset of X.*
5. *For any $x \in X$, the map $\phi_x \colon \mathcal{G}_{r(x)} \to \mathcal{G}X$, $\gamma \mapsto \gamma x$ is open.*

3.2 Equivalence of groupoids

Equivalence of groupoids were introduced by Renault in [Ren82]. One motivation is to define an equivalence relation which is weaker than isomorphisms, but gives Morita–Rieffel equivalence of C*-algebras.

Definition 3.15. Let $\mathcal{G}$ and $\mathcal{H}$ be locally-compact Hausdorff groupoids with r and s open. A locally-compact Hausdorff space Z is called a $(\mathcal{G}, \mathcal{H})$-(Morita) equivalence, if the followings hold:

1. Z is a free and proper left $\mathcal{G}$-space and a free and proper right $\mathcal{H}$-space.
2. The left $\mathcal{G}$-action commutes with the right $\mathcal{H}$-action.
3. The moment map $r \colon Z \to \mathcal{G}^{(0)}$ is open and induces a homeomorphism $Z/\mathcal{H} \to \mathcal{G}^{(0)}$.
   $s \colon Z \to \mathcal{H}^{(0)}$ is open and induces a homeomorphism $\mathcal{G}\backslash Z \to \mathcal{H}^{(0)}$.

Example 3.16. Let $\Gamma \curvearrowright X \curvearrowleft \Gamma'$ be commuting actions which are free and proper. Then $\Gamma\backslash X$ and X/Γ' are locally-compact Hausdorff. Then X is an equivalence between $\Gamma \ltimes (X/\Gamma')$ and $(\Gamma\backslash X) \rtimes \Gamma'$. This works for groupoid actions as well.

As a special case, if we take $X \curvearrowleft \Gamma'$ to be the trivial action. Then X is an equivalence between $\Gamma \ltimes X$ and $\Gamma\backslash X$.

Example 3.17 (Blow-ups). Let $\mathcal{G}$ be a locally-compact Hausdorff groupoid with open range. Let Z be a locally-compact space, $f: Z \to \mathcal{G}^{(0)}$ be an open continuous map. Regard f as a moment map, then Z can be viewed as a left $\mathcal{G}$-space and a right $\mathcal{G}$-space. The blow-up groupoid is defined as

$$\mathcal{G}[Z] := \{(z_1, \gamma, z_2) \in Z \times \mathcal{G} \times Z \mid f(z_1) = r(\gamma),\, s(\gamma) = r(z_2)\}.$$

The groupoid operations are the nature ones. The unit space is $\{(z, f(z), z)\}$ so we can view $\mathcal{G}[Z]$ as a groupoid over Z.

Let

$$Z * \mathcal{G} := \{(z, \gamma) \in Z \times \mathcal{G} \mid f(z) = r(\gamma)\}.$$

We claim that $Z * \mathcal{G}$ actually defines a $(\mathcal{G}[Z], \mathcal{G})$-equivalence. The moment maps are given by

$$s(z, \gamma) := s(\gamma), \qquad r(z, \gamma) := z,$$

and the actions are

$$(z_1, \gamma, z_2)(z_2, \gamma') := (z_1, \gamma\gamma'), \qquad (z, \gamma)\eta := (z, \gamma\eta).$$

Remark 3.18. Let Z be a $(\mathcal{G}, \mathcal{H})$-equivalence. Then the homeomorphism $Z/\mathcal{H} \to \mathcal{G}^{(0)}$ (resp. $\mathcal{G}\backslash Z \to \mathcal{H}^{(0)}$) is $\mathcal{G}$- (resp. $\mathcal{H}$-)equivariant. These induces homeomorphisms

$$\mathcal{G}\backslash\mathcal{G}^{(0)} \xleftarrow{\cong} \mathcal{G}\backslash Z/\mathcal{H} \xrightarrow{\cong} \mathcal{H}^{(0)}/\mathcal{H}.$$

So $\mathcal{G}\backslash\mathcal{G}^{(0)}$ is homeomorphic to $\mathcal{H}^{(0)}/\mathcal{H}$.

Proposition 3.19. *The groupoid equivalence is an equivalence relation.*

Sketch of proof.

- $\mathcal{G}$ is a $(\mathcal{G}, \mathcal{G})$-equivalence.
- If Z is a $(\mathcal{G}, \mathcal{H})$-equivalence, with moment maps $r_Z: Z \to \mathcal{G}^{(0)}$ and $s_Z: Z \to \mathcal{H}^{(0)}$. Define an $(\mathcal{H}, \mathcal{G})$-equivalence Z^{op} as follows:
  - As a space, Z^{op} is homeomorphic to Z. We write $\bar{z} \in Z^{\mathrm{op}}$ for the image of $z \in Z$ in order to distinguish.
  - The left $\mathcal{H}$-action on Z^{op} defined by the followings:
    $$r'_{Z^{\mathrm{op}}}(\bar{z}) := s_Z(z), \qquad \gamma\bar{z} := z\gamma^{-1}.$$
  - The right $\mathcal{G}$-action on Z^{op} defined by the followings:
    $$s'_{Z^{\mathrm{op}}}(\bar{z}) := r_Z(z), \qquad \bar{z}\eta := \eta^{-1}z.$$
- Let Z be a $(\mathcal{G}, \mathcal{H})$-equivalence and Y be an $(\mathcal{H}, \mathcal{K})$-equivalence. Then a $(\mathcal{G}, \mathcal{K})$-equivalence is given by the quotient
  $$\{(z, y) \in Z \times Y \mid s(z) = r(y)\}/\mathcal{H},$$
  where the right $\mathcal{H}$-action is given by
  $$(z, y) \cdot \beta := (z\beta, \beta^{-1}y). \qquad \square$$

Let Z be a $(\mathcal{G}, \mathcal{H})$-equivalence. Then

$$Z *_s Z := \{(x, y) \in Z \times Z \mid s(x) = s(y)\}$$

is closed and locally-compact. It is a free and proper right $\mathcal{H}$-space with diagonal action. In particular, we have:

Lemma 3.20. *Given $(x, y) \in Z *_s Z$. There exists a unique element $\tau_{x,y} \in \mathcal{G}$ such that $\tau_{x,y}y = x$.*

The map

$$Z *_s Z \to \mathcal{G}, \qquad (x, y) \mapsto \tau_{x,y}$$

is continuous and open. It factors through the homeomorphism $Z *_s Z/\mathcal{H} \to \mathcal{G}$.

With the help of this lemma we are able to prove that

Theorem 3.21. *Let $\mathcal{G}$ and $\mathcal{H}$ be locally-compact Hausdorff groupoids. Then $\mathcal{G}$ is equivalent to $\mathcal{H}$ iff there exists a space Z such that the blow-up groupoids $\mathcal{G}[Z]$ and $\mathcal{H}[Z]$ are isomorphic.*

October 11, 2022

Groupoid correspondences

Speaker: Bram Mesland (Leiden University)

In this talk, all groupoids are locally-compact and Hausdorff and equipped with a Haar system. As a consequence, their source and range maps are open. We will refer to them simply as *groupoids*.

Some main references of this talk are [HS87, MSO99, Mrč99, Lan01, Tu04].

4.1 Groupoid correspondences

We wish to construct a nice category of groupoids, such that taking the groupoid C*-algebra is a functor mapping to a suitable category of C*-algebras: the category of C*-correspondence Corr. We also request that Morita equivalent groupoids are mapped to Morita–Rieffel equivalent C*-algebras. This requires a suitable notion of "generalised homomorphisms" between groupoids. These, as we will define in the following, are *groupoid correspondences*.

Definition 4.1. Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids. A *(groupoid) correspondence*, or a *generalised homomorphism* from $\mathcal{G}$ to $\mathcal{H}$, is a space Z with commuting left $\mathcal{G}$-action and right $\mathcal{H}$-action

$$\mathcal{G} \supseteq \mathcal{G}^{(0)} \xleftarrow{r_Z} Z \xrightarrow{s_Z} \mathcal{H}^{(0)} \subseteq \mathcal{H},$$

or briefly,

$$\mathcal{G} \leftarrow Z \rightarrow \mathcal{H},$$

such that:

- $\mathcal{G} \curvearrowright Z$ is free and proper.
- $Z \curvearrowleft \mathcal{H}$ is proper.
- The moment map $Z \xrightarrow{s_Z} \mathcal{H}^{(0)}$ factors through the homeomorphism $\mathcal{G}\backslash Z \xrightarrow{\cong} \mathcal{H}^{(0)}$. That is, the following diagram commutes:

$$\begin{tikzcd} Z \arrow[rr, "s_Z"] \arrow[dr, "q"'] & & \mathcal{H}^{(0)} \\ & \mathcal{G}\backslash Z \arrow[ur, "\cong"'] & \end{tikzcd}$$

Remark 4.2. Our definition of groupoid correspondences is slightly different from that from [Wil19], wherein the action $Z \curvearrowleft \mathcal{H}$ is not assumed to be proper. However, the properness condition is essential to give a C*-correspondence.

Example 4.3. Let $\varphi\colon \mathcal{H} \to \mathcal{G}$ be a strict homomorphism. Do we need φ to be proper? Define

$$\operatorname{Graph}(\varphi) := \{(\gamma, u) \in \mathcal{G} \times \mathcal{H}^{(0)} \mid \phi(u) = s(\gamma)\}.$$

It admits a left $\mathcal{G}$-action obviously, and a right $\mathcal{H}$-actions via

$$(\gamma, u)h := (\gamma\varphi(h), s(h)).$$

This gives a correspondence.

As a special case, if $\mathcal{H} \subseteq \mathcal{G}$ is a subgroupoid. Then the inclusion $\mathcal{H} \hookrightarrow \mathcal{G}$ gives a correspondence.

4.1.1 Composition of correspondences

Now we define the composition of correspondences. Given two correspondences

$$\mathcal{G} \xleftarrow{r_Z} Z \xrightarrow{s_Z} \mathcal{H}, \qquad \mathcal{H} \xleftarrow{r_W} W \xrightarrow{s_W} \mathcal{K},$$

Let

$$Z * W := \{(z, w) \mid Z \times W \mid s_Z(z) = r_W(w)\}.$$

$\mathcal{H}$ acts on $Z * W$ on the right via the diagonal action. Consider the quotient

$$Z *_{\mathcal{H}} W := \{(z, w) \in Z \times W \mid s_Z(z) = r_W(w)\}/\mathcal{H}.$$

Since $\mathcal{H} \curvearrowright W$ freely and properly, the space $Z *_{\mathcal{H}} W$ is locally-compact Hausdorff ([Wil19, Proposition 2.18]). We equip it with a free and proper left $\mathcal{G}$-action, and a proper right $\mathcal{K}$-action

$$g[z, w] := [gz, w], \quad [z, w]k := [z, wk], \qquad \text{for } g \in \mathcal{G},\ k \in \mathcal{K},\ [z, w] \in Z *_{\mathcal{H}} W.$$

Definition and Lemma 4.4. If $\mathcal{H}$ has open range and source maps. Then $\mathcal{G} \leftarrow Z *_{\mathcal{H}} W \rightarrow \mathcal{K}$ is a correspondence. We define it to be the composition of

$$\mathcal{G} \xleftarrow{r_Z} Z \xrightarrow{s_Z} \mathcal{H} \quad \text{and} \quad \mathcal{H} \xleftarrow{r_W} W \xrightarrow{s_W} \mathcal{K}.$$

Thanks to the composition, we can now define a category of groupoid correspondences.

Definition 4.5. The category Gr of groupoid correspondences consist of the following data:

- Objects are groupoid.
- An arrow from $\mathcal{G}$ to $\mathcal{H}$ is a correspondence $\mathcal{G} \leftarrow Z \rightarrow \mathcal{H}$.
- Composition of arrows is given by Definition and Lemma 4.4.

Example 4.6. The correspondence $\mathcal{G} \xleftarrow{r} \mathcal{G} \xrightarrow{s} \mathcal{G}$ is the identity arrow of $\mathcal{G}$ in Gr. In fact, we have

$$\mathcal{G} *_{\mathcal{G}} Z \cong Z, \qquad W *_{\mathcal{G}} \mathcal{G} \cong W$$

for a left $\mathcal{G}$-space Z and a right $\mathcal{G}$-space W.

A correspondence $\mathcal{G} \leftarrow Z \rightarrow \mathcal{H}$ is invertible, if there exists another correspondence $\mathcal{H} \leftarrow W \rightarrow \mathcal{G}$, such that

$$Z *_{\mathcal{H}} W \cong \mathcal{G}, \qquad W *_{\mathcal{G}} Z \cong \mathcal{H}.$$

4.1.2 Morita equivalences revisited

Proposition 4.7. *A correspondence $\mathcal{G} \leftarrow Z \rightarrow \mathcal{H}$ is invertible iff Z is an $(\mathcal{G}, \mathcal{H})$-equivalence.*

Remark 4.8.

- Let $\mathcal{G}$ and $\mathcal{H}$ be groups. Then they are Morita equivalent as groupoids iff they are isomorphic as groups.
- Let $\mathcal{G}$ and $\mathcal{H}$ be spaces. Then they are Morita equivalent as groupoids iff they are homeomorphic as spaces.

Now we revisit the blow-up construction in the previous talk, and show that a Morita equivalence can be lifted to an isomorphism of groupoids.

Let $\mathcal{G} \leftarrow Z \rightarrow \mathcal{H}$ be a correspondence. Recall that (Example 3.17) the blow-up groupoid is defined as

$$\mathcal{H}[Z] := \{(z_1, h, z_2) \in Z \times \mathcal{H} \times Z \mid s(z_1) = r(h),\ r(z_2) = s(h)\},$$

and the space

$$W := \{(z, h) \in Z \times \mathcal{H} \mid s(z) = r(h)\}$$

implements a $(\mathcal{H}[Z], \mathcal{H})$-equivalence.

Define

$$\psi\colon \mathcal{H}[Z] \to \mathcal{G}, \qquad (z_1, g, z_2) \mapsto (z_1, gz_2) \in Z *_{\mathcal{H}} Z.$$

Since $\mathcal{G}$ acts on Z freely, the object (z_1, gz_2) uniquely determines an element in $\mathcal{G}$. The composition yields a map $\psi\colon \mathcal{H}[Z] \to \mathcal{G}$.

Proposition 4.9. *Let $\mathcal{G} \leftarrow Z \to \mathcal{H}$ be a correspondence. Then $Z \cong W^{\mathrm{op}} *_{\mathcal{H}[Z]} \mathrm{Graph}(\psi)$. So Z is a composition of a blow-up groupoid with a strict homomorphism.*

4.2 From groupoid correspondences to C*-correspondences

Now we pass to C*-correspondences. As mentioned before, we wish to furnish a functor, which on the object level maps groupoids to their groupoid C*-algebras. The suitable target category is the category of C*-correspondences.

Definition 4.10. The category Corr of C*-correspondences consist of the following data:

- Objects are C*-algebras.
- An arrow from A to B is a (A, B)-correspondence ${}_AX_B$. That is, a right Hilbert B-module X and a *-homomorphism $A \to \mathbb{B}_B(X)$ to the bounded adjointable operators on the Hilbert B-module X.
- Composition of arrows is given by the tensor product of Hilbert C*-modules

$${}_AX_B \circ {}_BY_C := {}_A(X \otimes_B Y)_C.$$

Recall that two C*-algebras A and B are Morita–Rieffel equivalent iff there is an imprimitivity (or Morita equivalence) A, B-bimodule ${}_AE_B$. That is, a Hilbert B-module E together with a *-isomorphism $A \to \mathbb{K}_B(E)$. We write $A \sim_{\mathrm{Morita}} B$ if A and B are Morita–Rieffel equivalent.

The following propositions (sometimes used as definitions) are well-known to C*-algebraists:

Proposition 4.11.
- *$A \sim_{\mathrm{Morita}} B$ iff $A \cong B$ in Corr.*
- *If A and B are separable. Then $A \sim_{\mathrm{Morita}} B$ iff $\mathbb{K} \otimes A \cong \mathbb{K} \otimes B$ as C*-algebras.*

We wish to construct a functor $\mathsf{Gr} \to \mathsf{Corr}$. On the arrow level this means we need to construct a C*-correspondence out of a groupoid correspondence.

Let $\mathcal{G} \leftarrow Z \to \mathcal{H}$ be a groupoid correspondence. Pick any $z \in Z$ with $s(z) = r(\eta)$. Then $\mathrm{C_c}(Z)$ carries a $\mathrm{C_c}(\mathcal{H})$-valued inner product given by

$$\langle \psi, \phi \rangle(\eta) := \int_{\mathcal{G}} \overline{\psi(g^{-1}z)} \phi(g^{-1}z\eta)\, \mathrm{d}\lambda^{r(z)} g, \qquad \psi, \phi \in \mathrm{C_c}(Z).$$

This is independent of the choice of z because $\mathcal{G}\backslash Z \cong \mathcal{H}^{(0)}$. The integral converges because $\mathcal{G}\backslash Z \cong \mathcal{G}^{(0)}$.

Now $\mathrm{C_c}(Z)$ is a $(\mathrm{C_c}(\mathcal{G}), \mathrm{C_c}(\mathcal{H}))$-bimodule: the left $\mathrm{C_c}(\mathcal{G})$-module structure is

$$f \cdot \phi(z) := \int_{\mathcal{G}} f(g)\phi(g^{-1}z)\, \mathrm{d}\lambda^{r(z)} g;$$

and the right $\mathrm{C_c}(\mathcal{H})$-module structure is given by

$$\phi \cdot g(z) := \int_{\mathcal{G}} \phi(zh)g(h^{-1})\, \mathrm{d}\lambda^{s(z)} h.$$

One checks that the followings are satisfied:

$$\langle f^*\psi, \phi \rangle = \langle \psi, f\phi \rangle, \qquad \langle \psi, \phi \cdot g \rangle = \langle \psi, \phi \rangle * g, \qquad \langle f\phi, f\phi \rangle \leq \|f\|_{\mathrm{C}^*(\mathcal{G})} \langle \phi, \phi \rangle.$$

The last equality guarantees that

$$\|\phi\|^2 := \|\langle\phi,\phi\rangle\|_{\mathrm{C}^*(\mathcal{H})}$$

defines a norm on $\mathrm{C_c}(Z)$, making it into a pre-Hilbert $\mathrm{C}^*(\mathcal{H})$-module.

Denote by $X(Z)$ the right $\mathrm{C}^*(\mathcal{H})$-module completion of $\mathrm{C_c}(Z)$. Upon replacing all norms on convolution algebras by the reduced norms we obtain a reduced version $X_\mathrm{r}(Z)$ as a right $\mathrm{C}^*_\mathrm{r}(\mathcal{H})$-module.

Theorem 4.12 ([MSO99, Tu04]). *${}_{\mathrm{C}^*(\mathcal{G})}X(Z)_{\mathrm{C}^*(\mathcal{H})}$ and ${}_{\mathrm{C}^*_\mathrm{r}(\mathcal{G})}X_\mathrm{r}(Z)_{\mathrm{C}^*_\mathrm{r}(\mathcal{H})}$ are C^*-correspondences.*

Theorem 4.13 ([MRW87]). *Let $G \leftarrow Z \rightarrow W$ be a $(\mathcal{G},\mathcal{H})$-equivalence. Then $X(Z)$ and $X_\mathrm{r}(Z)$ are imprimitivity bimodules.*

Theorem 4.14. *There is a functor $\mathsf{Gr} \to \mathsf{Corr}$, which:*

- *on the object level, sends a groupoid $\mathcal{G}$ to its groupoid C^*-algebra $\mathrm{C}^*(\mathcal{G})$.*
- *on the arrow level, sends a groupoid correspondence $\mathcal{G} \leftarrow Z \rightarrow \mathcal{H}$ to a C^*-correspondence ${}_{\mathrm{C}^*(\mathcal{G})}X(Z)_{\mathrm{C}^*(\mathcal{H})}$.*

As a corollary, Morita equivalent groupoids have Morita–Rieffel equivalent C^*-algebras.

Remark 4.15. The most difficult part of the proof is to show that $\mathrm{C_c}(Z *_\mathcal{H} W)$ is dense in $X(Z) \otimes_{\mathrm{C}^*(\mathcal{H})} X(W)$, so that the composition of groupoid correspondences is sent to the composition of C^*-correspondences. For this to be true, we have different choices of axioms for a groupoid correspondence. One option is in [Hol17].

Remark 4.16. An alternative proof of the equivalence theorem of reduced groupoid C^*-algebras (Theorem 4.14) is given in [SW12]. This is explained in more details in Lecture 5 of Dimitris.

The main ingredient is the *linking groupoid*. Let $\mathcal{G} \leftarrow Z \rightarrow \mathcal{H}$ be a $(\mathcal{G},\mathcal{H})$-equivalence. The linking groupoid is defined as

$$\mathcal{L} := \mathcal{G} \sqcup Z \sqcup Z^{\mathrm{op}} \sqcup \mathcal{H} \rightrightarrows \mathcal{G}^{(0)} \sqcup \mathcal{H}^{(0)}.$$

The source and range maps are inherited from the source and range maps of $\mathcal{G}$, Z, Z^{op} and $\mathcal{H}$. The multiplication of $\mathcal{L}$ restricts to the multiplication of $\mathcal{G}$ and $\mathcal{H}$, and the groupoid actions of $\mathcal{G}$ and $\mathcal{H}$ on Z and Z^{op}. The inverse of $\mathcal{L}$ restricts to the inverse of $\mathcal{G}$ and $\mathcal{H}$, and the identity homemorphism $Z \to Z^{\mathrm{op}}$.

The data above define a groupoid; in particular, it accommodates a Haar system if so do $\mathcal{G}$ and $\mathcal{H}$. This is because the actions of $\mathcal{G}$ and $\mathcal{H}$ on Z induce homeomorphisms $Z/\mathcal{H} \to \mathcal{G}^{(0)}$ and $\mathcal{G}\backslash Z \to \mathcal{H}^{(0)}$. Thus, given any $u \in \mathcal{G}^{(0)}$, pick any $z \in Z$ with $r(z) = u$. There is a Radon measure ν^u on Z determined by the linear functional

$$\mathrm{C_c}(Z) \to \mathbb{C}, \qquad \phi \mapsto \int_\mathcal{H} \phi(z\eta)\,\mathrm{d}\lambda^{s(z)}\eta,$$

which is supported on the orbit of z under the $\mathcal{H}$-action. It is independent of the choice of z in the source fibre due to the left-invariance of λ.

Therefore,

$$\int_Z \phi(z)\,\mathrm{d}\lambda^{s(z)}z := \int_\mathcal{H} \phi(z\eta)\,\mathrm{d}\nu^u\eta$$

determines a set of Haar measure $\{\nu^u\}_{u\in\mathcal{G}^{(0)}}$, each ν^u supported on $r^{-1}(u)$. A similar construction applies to Z^{op} and yield a set of Haar measure indexed by $\mathcal{H}^0$. These, together with the Haar system of $\mathcal{G}$ and $\mathcal{H}$, furnish a Haar system of L (see [SW12, Lemma 4]).

Eventually, Sims and Williams showed that the full (reduced) groupoid C^*-algebra $\mathrm{C}^*(\mathcal{L})$ ($\mathrm{C}^*_\mathrm{r}(\mathcal{L})$) are isomorphic to the *linking algebra* of the imprimitive bimodule $X(Z)$ ($X_\mathrm{r}(Z)$). This provides a uniform and structral setup for equivalence theorems of groupoids and their C^*-algebras.

October 18, 2022

Morita equivalence of groupoids and their C*-algebras

Speaker: Dimitris Gerontogiannis (Leiden University)

Recall from Bram's talk that we are able to construct a functor

$$\mathsf{Gr} \to \mathsf{Corr}$$

from the category of groupoid correspondences to the category of C^*-correspondences. In particular, we have a notion of *correspondences* as morphisms in Gr which are strictly weaker than groupoid homomorphisms, and such that the isomorphisms in both categories are implemented by Morita(–Rieffel) equivalences. In this talk, we provide more details of the proof and the underlying constructions of linking groupoids.

5.1 Morita equivalences

Recall the Morita–Rieffel equivalence of C^*-algebras. They are implemented by imprimitivity bimodules.

Definition 5.1. Let A and B be C^*-algebras. An *imprimitivity A, B-bimodule* is given by a right Hilbert B-module E which is simultaneously a left Hilbert A-module , and such that the B-valued inner product $\langle \cdot, \cdot \rangle_B$ is compatible with the A-valued inner product ${}_A\langle \cdot, \cdot \rangle$. That is,

$$\langle x, ay \rangle_B = \langle a^* x, y \rangle_B, \quad {}_A\langle xb^*, y \rangle = {}_A\langle x, yb \rangle, \quad {}_A\langle x, y \rangle z = x \langle y, z \rangle_B$$

for all $x, y, z \in E$, $a \in A$ and $b \in B$.

We say A and B are *Morita–Rieffel* equivalent, if there exists an imprimitivity bimodule between them.

5.1.1 Kronecker flow of irrational angle ϑ

Consider the action $\mathbb{R} \curvearrowright \mathbb{T}^2$ by

$$t \cdot (z_1, z_2) := (\mathrm{e}^{2\pi i t \vartheta} z_1, \mathrm{e}^{2\pi i t} z_2).$$

We form the action groupoid $\mathcal{G} := \mathbb{T}^2 \rtimes \mathbb{R}$. Its elements are of the form $(z_1, z_2, t) \in \mathbb{T}^2 \times \mathbb{R}$, with multiplication given by

$$(\mathrm{e}^{2\pi i t \vartheta z_1}, \mathrm{e}^{2\pi i t} z_2, s)(z_1, z_2, t) = (z_1, z_2, s + t).$$

The groupoid C^*-algebra $\mathrm{C}^*(\mathcal{G})$ is very large. We may, however, construct a Morita–Rieffel equivalent C^*-algebra to it, using *transversal*.

A transversal is a subspace in the object space which intersects every orbit. In this example, we restrict to a single copy of $\mathbb{T}$:

$$T := \mathbb{T} \times \{1\} \times \{0\} \subseteq \mathcal{G}^{(0)}$$

Restricting $\mathcal{G}$ to T (Definition 1.11) yields a subgroupoid $\mathcal{G}_T^T$, which is naturally isomorphic to $\mathbb{T} \rtimes \mathbb{Z}$, with multiplication given by

$$(\mathrm{e}^{2\pi i k \vartheta}, m)(z, n) := (z, m + n).$$

Notice that we have replaced $\mathbb{R}$ by $\mathbb{Z}$: this is because by restricting to T, the action given by $\mathbb{R}$ are required to be an integer multiple of 2π.

We leave the following as an

Exercise 5.2. $\mathcal{G}_T^T$ is an étale groupoid.

Now notice that we have a groupoid correspondence

$$\mathcal{G} \xleftarrow{r} \mathcal{G}_T^T \xrightarrow{s} \mathcal{G}_T^T$$

with two commuting free and proper actions. These actions factor through homeomorphisms

$$\mathcal{G}/\mathcal{G}_T^T \cong \mathcal{G}^{(0)} \qquad \text{and} \qquad \mathcal{G}\backslash\mathcal{G}_T^T \cong T,$$

hence give a Morita equivalence. Therefore $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})$ is Morita–Rieffel equivalent to $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G}_T^T) = A_\vartheta$, the irrational rotation algebra (or noncommutative torus). Moreover, $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})$ is stable and separable. Hence

$$\mathrm{C}^*_{\mathrm{r}}(\mathcal{G}) \cong \mathrm{C}^*_{\mathrm{r}}(\mathcal{G}) \otimes \mathbb{K} \cong A_\vartheta \otimes \mathbb{K}.$$

5.1.2 Brown–Green–Rieffel theorem

The notion of Morita–Rieffel equivalence is an analog of Morita equivalence of rings. Two rings A and B are Morita equivalent iff they have the same category of modules. One might wish to translate this notion to C*-algebras as well, replacing modules over rings by Hilbert spaces. This will yield a weaker notion than Morita–Rieffel equivalence.

For now let us write $\mathrm{Rep}(A)$ for the category of Hilbert spaces admitting non-degenerate left actions of A, with arrows unitary intertwiners. We say two A and B are weakly Morita equivalent, if $\mathrm{Rep}(A) \cong \mathrm{Rep}(B)$. Then we have:

Proposition 5.3. *If A and B are Morita–Rieffel equivalent through imprimitivity bimodule E. Then the functor*

$$E \otimes_B - \colon \mathrm{Rep}(B) \to \mathrm{Rep}(A)$$

is an equivalence of category. Hence Morita–Rieffel equivalent C**-algebras are weakly Morita equivalent.*

Remark 5.4. The fullness of E as a Hilbert B-module assures that $E \otimes_B -$ sends a non-degenerate representation to another non-degenerate representation.

The following colourful theorem illustrates equivalent characterisations of Morita–Rieffel equivalence:

Theorem 5.5 (Brown–Green–Rieffel)**.** *Let A and B be* C**-algebras. The followings are equivalent:*

1. *A is Morita–Rieffel equivalent to B.*
2. *There exists a full Hilbert B-module E such that $A \cong \mathbb{K}_B(E)$.*

If moreover A and B are σ-unital. Then both 1 and 2 are also equivalent to:

3. *$A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$.*

5.2 Equivalence theorem of groupoid C*-algebras

In the following, we will prove the equivalence theorem (Theorem 4.14) following [SW12]. Let us rephrase it here:

Theorem 5.6. *Morita equivalent groupoids have Morita–Rieffel equivalent* C**-algebras.*

5.2.1 Linking algebra

A key tool in the proof is the linking algebra. Let us recall:

Definition 5.7. Let A be a C*-algebra.

- A *corner* of A is a subalgebra of the form pAp, where p is a projection in the multiplier of A. It is a *full* corner if $\overline{ApA} = A$.

- Two corners pAp and qAq are *complementary*, if $p+q=1$.

Theorem 5.8 (Brown–Green–Rieffel)**.** *$A \sim_{\text{Morita}} B$ iff there exists a C^*-algebra C such that A and B are full, complementary corners in a C^*-algebra C.*

The C^*-algebra C is called a linking algebra *of A and B.*

Proof. If $A = pCp$. Then Cp is a imprimitivity C, A-bimodule. So $C \sim_{\text{Morita}} A$. Similarly $B \sim_{\text{Morita}} C$. Therefore $A \sim_{\text{Morita}} B$.

Conversely. Let ${}_AE_B$ be an imprimitivity A, B-bimodule. Let

$$\mathrm{C}_0 := \left\{ \begin{pmatrix} a & x \\ \overline{y} & b \end{pmatrix} \middle| \begin{array}{c} a \in A,\, b \in B \\ x \in E,\, \overline{y} \in E^{\mathrm{op}} \end{array} \right\},$$

where E^{op} denotes the dual module of E, which is an imprimitivity B, A-bimodule. The image of $y \in E$ under the anti-isomorphism $E \xrightarrow{\cong} E^{\mathrm{op}}$ is denoted by $\overline{y}$.

We turn C_0 into a *-algebra by setting

$$\begin{pmatrix} a_1 & x_1 \\ \overline{y_1} & b_1 \end{pmatrix} \begin{pmatrix} a_2 & x_2 \\ \overline{y_2} & b_2 \end{pmatrix} := \begin{pmatrix} a_1a_2 + {}_A\langle x_1, y_2\rangle & a_1x_2 + x_1b_2 \\ \overline{y_1}a_2 + b_1\overline{y_2} & \langle y_1, x_2\rangle_B + b_1b_2 \end{pmatrix},$$

$$\begin{pmatrix} a & x \\ \overline{y} & b \end{pmatrix}^* := \begin{pmatrix} a^* & y \\ \overline{x} & b^* \end{pmatrix}.$$

View B as a Hilbert B-module. Consider the direct sum Hilbert B-module $E \oplus B$. Then C_0 acts on $E \oplus B$ as bounded adjointable operators:

$$\begin{pmatrix} a & x \\ \overline{y} & b \end{pmatrix} \begin{pmatrix} z \\ b' \end{pmatrix} := \begin{pmatrix} az + xb' \\ \langle y, z\rangle_B + bb' \end{pmatrix}.$$

Let C be the completion of C_0 in $\mathbb{B}_B(E \oplus B)$. Then A (or B) embeds as the top-left (or bottom-right) corner of C. Notice that here we uses E is full, because $aE = 0$ implies $0 = a\langle E, E\rangle = aA$ and hence $a = 0$.

Now we let

$$p := \begin{pmatrix} \mathrm{id}_E & \\ & \end{pmatrix}, \qquad q := \begin{pmatrix} & \\ & \mathrm{id}_B \end{pmatrix}.$$

They satisfy $p, q \in \mathcal{M}(C)$ and $p + q = 1$. Hence $A = pCp$ and $B = qCq$ are complementary corners in C. ☐

5.2.2 Linking groupoid

Now we construct linking groupoids. Let $\mathcal{G}$ be a groupoid with Haar system λ, $\mathcal{H}$ be a groupoid with Haar system β. Let $\mathcal{G} \xleftarrow{r} Z \xrightarrow{s} \mathcal{H}$ be an equivalence. The linking groupoid is

$$\mathcal{L} := \mathcal{G} \sqcup Z \sqcup Z^{\mathrm{op}} \sqcup \mathcal{H} \rightrightarrows \mathcal{G}^{(0)} \sqcup \mathcal{H}^{(0)}.$$

We need to construct a Haar system for $\mathcal{L}$, which comes from the Haar systems of $\mathcal{G}$ and $\mathcal{H}$. Since $r_Z\colon Z \to \mathcal{G}^{(0)}$ factor through the homeomorphism $Z/\mathcal{H} \xrightarrow{\cong} \mathcal{G}^{(0)}$. For any $u \in \mathcal{G}^{(0)}$, pick $z \in r_Z^{-1}(u)$. Associate to it a Radon measure σ_Z^u on Z:

$$\sigma_Z^u(\phi) := \int_{\mathcal{H}} \phi(z, \eta)\, \mathrm{d}\beta^{s(z)\eta}, \qquad \phi \in \mathrm{C_c}(Z)$$

which is supported on the orbit $z \cdot \mathcal{H}$ and is *independent* of z. Likewise we define a Radon measure $\sigma_{Z^{\mathrm{op}}}^v$ on Z^{op} supported on $\overline{z} \cdot \mathcal{G}$ for $v \in \mathcal{H}^{(0)}$, $\overline{z} \in r_{Z^{\mathrm{op}}}(v)$. We also have

Proposition 5.9. *The map*

$$u \mapsto \int_Z \phi(z)\,\mathrm{d}\sigma_Z^u(z)$$

is continuous on $\mathrm{C_c}(\mathcal{G}^{(0)})$.

Then we have

Proposition 5.10. *For every* $w \in \mathcal{L}^{(0)} = \mathcal{G}^{(0)} \sqcup \mathcal{H}^{(0)}$. *For each* $F \in \mathrm{C_c}(L)$, *define*

$$k^w(F) := \begin{cases} \lambda^w(F|_{\mathcal{G}}) + \sigma_Z^w(F|_Z) & \text{if } w \in \mathcal{G}^{(0)}; \\ \sigma_{Z^{\mathrm{op}}}^w(F|_{Z^{\mathrm{op}}}) + \beta^w(F|_{\mathcal{H}}) & \text{if } w \in \mathcal{H}^0. \end{cases}$$

Then $\{k^w\}_{w\in\mathcal{L}^{(0)}}$ *defines a Haar system.*

Proposition 5.11. *We have Morita equivalences* $\mathrm{C}^*(\mathcal{G},\lambda) \sim_{\mathrm{Morita}} \mathrm{C}^*(\mathcal{H},\beta)$ *and* $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G},\lambda) \sim_{\mathrm{Morita}} \mathrm{C}^*_{\mathrm{r}}(\mathcal{H},\beta)$.

Proof. Define the *-homomorphism $v\colon \mathrm{C_b}(\mathcal{L}^{(0)}) \to \mathcal{M}(\mathrm{C}^*(\mathcal{L}))$ by

$$(v(\phi)f)(\gamma) := \phi(r(\gamma))f(\gamma), \qquad (fv(\phi))(\gamma) := f(\gamma)\phi(s(\gamma)).$$

The characteristic functions $\chi_{\mathcal{G}^{(0)}}, \chi_{\mathcal{H}^{(0)}} \in \mathrm{C_b}(\mathcal{L}^{(0)})$. Their images in $\mathcal{M}(\mathrm{C}^*(\mathcal{L}))$ are projections; let us call them $P_{\mathcal{G}}$ and $P_{\mathcal{H}}$. If $P_{\mathcal{G}}$ and $P_{\mathcal{H}}$ are full, then $P_{\mathcal{G}}\mathrm{C}^*(\mathcal{L})P_{\mathcal{G}} \sim_{\mathrm{Morita}} P_{\mathcal{H}}\mathrm{C}^*(\mathcal{L})P_{\mathcal{H}}$ via $P_{\mathcal{G}}\mathrm{C}^*(\mathcal{L})P_{\mathcal{H}}$. We also have $\mathrm{C_c}(\mathcal{G}) \subseteq P_{\mathcal{G}}\mathrm{C}^*(\mathcal{L})P_{\mathcal{G}}$ and $\mathrm{C_c}(\mathcal{H}) \subseteq P_{\mathcal{H}}\mathrm{C}^*(\mathcal{L})P_{\mathcal{H}}$ as dense subalgebras. This will finish the proof that $\mathrm{C}^*(\mathcal{G}) \sim_{\mathrm{Morita}} \mathrm{C}^*(\mathcal{H})$. A similar result holds for the reduced counterpart $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G}) \sim_{\mathrm{Morita}} \mathrm{C}^*_{\mathrm{r}}(\mathcal{H})$.

Henceforth we need to prove that $P_{\mathcal{G}}$ (and by symmetry, $P_{\mathcal{H}}$) is full. We use the following

Lemma 5.12 ([MRW87, Proposition 2.10])**.** *There is a net* $\{e_\alpha\}$ *in* $\mathrm{C_c}(\mathcal{G})$ *of the form*

$$e_\alpha = \sum_{i=1}^{n_\alpha} \langle \phi_i^\alpha, \phi_i^\alpha \rangle_{\mathrm{C}^*(\mathcal{G})}$$

for a finite number n_α *and* $\phi_i^\alpha \in \mathrm{C_c}(Z)$. *This is an approximate identity with respect to the inductive limit topology for the action* $\mathrm{C_c}(\mathcal{G}) \curvearrowright \mathrm{C_c}(\mathcal{G})$ *and* $\mathrm{C_c}(\mathcal{G}) \curvearrowright \mathrm{C_c}(Z)$. *That is,*

$$e_\alpha * \phi \to \phi, \quad \psi * e_\alpha \to \psi,$$

for $\phi \in \mathrm{C_c}(\mathcal{G})$ *and* $\psi \in \mathrm{C_c}(\mathcal{G})$ *or* $\psi \in \mathrm{C_c}(Z)$.

Now for $F, K \in \mathrm{C_c}(\mathcal{L})$. Write them as block matrices

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

where $F_{11} := F|_G$, $F_{12} := F|_Z$, $F_{21} := F|_{Z^{\mathrm{op}}}$, $F_{22} := F|_{\mathcal{H}}$ and similar for K.

We compute the operator $F * p_{\mathcal{G}} * K$, which is of the form

$$\begin{pmatrix} F_{11} * K_{11} & F_{11} \cdot K_{22} \\ F_{21} \cdot K_{11} & \langle F_{21}^*, K_{12} \rangle_{\mathrm{C}^*(\mathcal{G})} \end{pmatrix}.$$

Lemma 5.12 imples that the elements of the form $F_{11} * K_{11}$ are dense in $\mathrm{C_c}(\mathcal{G})$ and that $F_{12} \cdot K_{22}$ are dense in $\mathrm{C_c}(Z)$. A similar statement for the right $\mathcal{H}$-action on Z implies that the elements of the form $F_{21} \cdot K_{11}$ are also dense in $\mathrm{C_c}(Z^{\mathrm{op}})$. It is left to show that elements of the form $\langle F_{21}^*, K_{12} \rangle_{\mathrm{C}^*(\mathcal{G})}$ are also dense in $\mathrm{C_c}(\mathcal{H})$. This is due to a standard technique from [Wil07, Page 115].

Hence operators of the form $F * p_{\mathcal{G}} * K$ are dense in $\mathrm{C}^*(\mathcal{L})$. We conclude that $p_{\mathcal{G}}$ is full. ☐

Remark 5.13. As a corollary. Up to Morita equivalence, the groupoid C*-algebra is independent of the underlying Haar system.

Example 5.14. As an example. Consider $G \curvearrowright X \curvearrowleft H$ where G and H are groups acting freely and properly on X. The freeness and properness assures that both $G\backslash X$ and X/H are locally-compact and Hausdorff. Then we have a Morita equivalence given by

$$G \ltimes X/H \leftarrow X \rightarrow G\backslash X \rtimes H.$$

As a special case, if H is trivial. Then we obtain the Morita equivalence $G \ltimes X \sim_{\text{Morita}} G\backslash X$ implemented by the space X.

October 25, 2022

Morita equivalence of Fell bundles

Speaker: Bram Mesland (Leiden University)

6.1 Motivation

Let H be a locally-compact group. Let Z be a free and proper right H-space. Then H acts freely and properly on $Z \times Z$ via the diagonal action

$$(z_1, z_2) \cdot h := (z_1 h, z_2 h).$$

Write $Z \times_H Z := Z \times Z/H$ for the quotient space. It is a groupoid over the Hausdorff space Z/H with structure maps given by

$$[z_1, z_2] \cdot [z_2, z_3] := [z_1, z_3], \qquad r([z_1, z_2]) = z_1, \qquad s([z_1, z_2]) = z_2,$$
$$\iota(z) = [z, z], \qquad [z_1, z_2]^{-1} = [z_2, z_1].$$

Namely, it is the reduction of the pair groupoid $Z \times Z \rightarrow Z$. It is called the *Atiyah groupoid* of the principal H-bundle $Z \rightarrow Z/H$. In particular, $Z \times_H Z$ is Morita equivalent to H. The Morita equivalence is implemented by Z with left $Z \times_H Z$-action

$$[z_1, z_2] \cdot z := z_1 \cdot h, \qquad \text{where } h \in H \text{ is uniquely decided by } z_2 h = z.$$

Then the completion $X(Z)$ of $\mathrm{C_c}(Z)$ is a imprimitivity bimodule between $\mathrm{C}^*(Z \times_H Z)$ and $\mathrm{C}^*(H)$. We wish to understand the C*-algebra $\mathrm{C}^*(Z \times_H Z)$. First notice that it contains a dense subalgebra $\mathrm{C}_\mathrm{c}^H(Z \times Z)$ consisting of H-equivariant maps on $Z \times Z$ with compact support. On this subalgebra, the $\mathrm{C_c}(H)$-valued inner product is given by

$$\langle f_1, f_2 \rangle(h) := \int \overline{f_1(z)} f_2(zh)\, \mathrm{d}\mu(z) =: \langle f_1, \alpha(h) f_2 \rangle_{L^2(Z)}.$$

We call the function $h \mapsto \langle f_1, f_2 \rangle(h)$ the *matrix coefficient* of $\alpha(h)$. We need to assume that it lies in $L^1(H)$.

Now let (ρ, V_ρ) be a unitary representation. Consider the pre-Hilbert $\mathrm{C_c}(H)$-module $\mathrm{C_c}(Z) \otimes_H V_\rho$, whose inner product is given by

$$\langle f_1 \otimes v_1, f_2 \otimes v_2 \rangle(h) := \langle f_1 \otimes v_1, (\alpha \otimes \rho)(h) f_2 \otimes v_2 \rangle = \langle f_1, \alpha(h) f_2 \rangle_{L^2(Z)} \cdot \langle v_1, \rho(h) v_2 \rangle_{V_\rho}.$$

Denote the completion of this pre-Hilbert module by $X(Z, V_\rho)$.

Notice that already in the case where ρ is the trivial representation, $X(Z)$ implements a Morita–Rieffel equivalence between $\mathrm{C}^*(H)$ and $\mathrm{C}^*(H \times_Z H)$. Does $X(Z, V_\rho)$ also implement a Morita–Rieffel equivalence between suitable C^*-algebras? For this we need to understand the C^*-algebra of compact operators on $X(Z, V_\rho)$.

Denote by $\mathrm{C}_\mathrm{c}^H(Z \times Z, \mathbb{K}(V_\rho))$ the space of H-equivariant continuous functions $\phi\colon Z \times Z \to \mathbb{K}(V_\rho)$ such that the map

$$Z \times Z \to \mathbb{C}, \qquad \xi \mapsto \|\phi(\xi)\|$$

has compact support on $Z \times_H Z$. It carries a *-algebra structure

$$\phi_1 * \phi_2(\eta) := \int_{Z\times Z} \phi_1(\xi)\phi_2(\xi^{-1}\eta)\, \mathrm{d}\nu^{r(\eta)}\xi, \qquad \phi^*(\xi) := \phi(\xi^{-1})^*.$$

With this *-algebra structure, $\mathrm{C}_\mathrm{c}^H(Z \times Z, \mathbb{K}(V_\rho))$ is a *-subalgebra of $\mathbb{K}(X(Z, V_\rho))$.

Consider the following *associated algebra bundle*

$$(Z \times Z) \times_H \mathbb{K}(V_\rho) \to Z \times_H Z. \tag{1}$$

This is an upper-semi-continous bundle of Banach spaces because $\xi \mapsto \|\xi\|$ is upper-semi-continuous. The sections of this upper-semi-continous Banach bundle, as is well-known in the theory of principal bundles, are in natural bijection with such H-equivariant functions. That is, we have

$$\Gamma_\mathrm{c}(Z \times_H Z, (Z \times Z) \times_H \mathbb{K}(V_\rho)) \cong \mathrm{C}_\mathrm{c}^H(Z \times Z, \mathbb{K}(V_\rho)), \tag{2}$$

where $\Gamma_\mathrm{c}(Z \times_H Z, (Z \times Z) \times_H \mathbb{K}(V_\rho))$ is the *-algebra of compactly-supported sections of the algebra bundle (1). This is a pre C^*-algebra. It turns out that $(Z \times Z) \times_H \mathbb{K}(V_\rho)$ is indeed a Fell bundle over the Atiyah groupoid $Z \times_H Z \to Z/H$, and the completion of $\Gamma_\mathrm{c}(Z \times_H Z, (Z \times Z) \times_H \mathbb{K}(V_\rho))$ is the *section* C^**-algebra* of this Fell bundle.

6.2 Fell bundles

Let $\mathcal{G}$ be a groupoid. Let $\pi\colon \mathcal{A} \to \mathcal{G}$ be an upper-semi-continuous bundle of Banach spaces. Set

$$\mathcal{A}^{(2)} := \{(a_1, a_2) \in \mathcal{A} \times \mathcal{A} \mid (\pi(a_1), \pi(a_2)) \in \mathcal{G}^{(2)}\}.$$

Definition 6.1. A *Fell bundle* is an upper-semi-continuous bundle of Banach spaces $\mathcal{A} \to \mathcal{G}$ together with:

- A continous, bilinear, associative "multiplication map" $\mathcal{A}^{(2)} \to \mathcal{A}$, $(a, b) \mapsto ab$;
- An anti-linear "involution map" $^*\colon \mathcal{A} \to \mathcal{A}$, $a \mapsto a^*$,

such that:

1. $\pi(ab) = \pi(a)\pi(b)$.
2. $\pi(a^*) = \pi(a)^{-1}$.
3. $(ab)^* = b^*a^*$.
4. For all $x \in \mathcal{G}^{(0)}$, $\pi^{-1}(x)$ is a C^*-algebra.
5. For all $\xi \in \mathcal{G}$, $\pi^{-1}(\xi)$ is a $(\pi^{-1}(r(\xi)), \pi^{-1}(s(\xi)))$-Morita equivalence.

Remark 6.2. $\mathcal{A}$ may be viewed as a groupoid in Corr, and π is a *-functor from Corr to $\mathcal{G}$ considered as a category.

Let $\pi\colon \mathcal{A} \to \mathcal{G}$ be a Fell bundle. The space of compactly-supported sections $\Gamma_c(\mathcal{G}, A)$ is a *-algebra: the convolution product and the involution are given by

$$s_1 * s_2(\eta) := \int_{\mathcal{G}} s_1(\xi) s_2(\xi^{-1}\eta) \, \mathrm{d}\lambda^{r(\eta)}\xi,$$
$$s^*(\xi) := s(\xi^{-1})^*.$$

It may be equipped with a C^*-norm and completed into a C^*-algebra, called the *section* C^**-algebra* of this Fell bundle.

Example 6.3. Let Γ be a discrete group. A strongly Γ-graded C^*-algebra is a C^*-algebra A together with subspaces $\{A_\gamma\}_{\gamma\in\Gamma}$ indexed by Γ, and such that

$$A = \overline{\bigoplus_{\gamma\in\Gamma} A_\gamma}, \quad A_\gamma A_\delta \subseteq A_{\gamma\delta}, \quad A_\gamma^* = A_{\gamma^{-1}}, \quad A_{\gamma^{-1}} A_\gamma = A_e = A_\gamma A_{\gamma^{-1}}.$$

A strongly Γ-graded C^*-algebra as above gives a Fell bundle

$$\coprod_{\gamma\in\Gamma} A_\gamma$$

over Γ. The section C^*-algebra recovers A.

Proposition 6.4. (1) *is a Fell bundle, with structure maps*

$$(z_1, z_2, k_1)(z_2', z_3, k_2) = (z_1, z_3 h, k_1 \rho(h^{-1}) k_2 \rho(h)) \quad \textit{for } z_2 = z_2' h;$$
$$(z_1, z_2, k)^* = (z_2, z_1, k^*).$$

Under the structure maps, the **-algebra of compactly-supported sections of this Fell bundle is isomorphism to the* **-algebra of compact-supported kernels. That is, the bijective correspondence of sets* (2) *is an isomorphism of* **-algebras.*

6.3 Equivalence of Fell bundles

Now we describe actions of Fell bundles.

Definition 6.5. Let $q\colon E \to Z$ be a upper-semi-continuous Banach bundle. Let Z be a left $\mathcal{G}$-space. Let $\pi\colon \mathcal{A} \to \mathcal{G}$ be a Fell bundle. Set

$$\mathcal{A} * E := \{(a, e) \mid s(\pi(b)) = r(q(e))\}.$$

A left action of $\mathcal{A}$ on E is a map

$$\mathcal{A} * E \to E, \qquad (a, e) \mapsto ae,$$

such that:

1. $q(ae) = \pi(a) q(e)$.
2. $a(be) = (ab)e$.
3. $\|be\| = \|b\|\|e\|$.

Right actions are defined similarly.

Definition 6.6. Let $\mathcal{G} \leftarrow Z \rightarrow \mathcal{H}$ be a Morita equivalence of groupoids. Let $\pi_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{G}$ and $\pi_{\mathcal{B}}\colon \mathcal{B} \to \mathcal{H}$ be Fell bundles.

An equivalence of Fell bundles $\mathcal{A}$ and $\mathcal{B}$ is an upper-semi-continuous Banach bundle $q\colon E \to Z$ such that there are sesquilinear maps

$$\begin{aligned} E \times_s E &\to \mathcal{A}, & (e_1, e_2) &\mapsto {}_{\mathcal{A}}\langle e_1, e_2\rangle, \\ E \times_r E &\to \mathcal{B}, & (e_1, e_2) &\mapsto \langle e_1, e_2\rangle_{\mathcal{B}}, \end{aligned}$$

where

$$\begin{aligned} E \times_s E &:= \{(e_1, e_2) \mid s(q(e_1)) = s(q(e_2))\}, \\ E \times_r E &:= \{(e_1, e_2) \mid r(q(e_1)) = r(q(e_2))\}. \end{aligned}$$

satisfying:

1. $\pi_{\mathcal{G}}({}_{\mathcal{A}}\langle e_1, e_2\rangle) = [q(e_1), q(e_2)]_{\mathcal{G}} \in Z \times_s Z/\mathcal{H}$.
2. $\pi_{\mathcal{H}}(\langle e_1, e_2\rangle_{\mathcal{B}}) = [q(e_1), q(e_2)]_{\mathcal{H}} \in \mathcal{G}\backslash Z \times_r Z$.
3. $\langle e_1, e_2\rangle^* = \langle e_2, e_1\rangle$.
4. ${}_{\mathcal{A}}\langle ae_1, e_2\rangle = a\,{}_{A}e_1, e_2$ and $\langle e_1, e_2 b\rangle_{\mathcal{B}} = \langle e_1, e_2\rangle_{\mathcal{B}} b$.
5. $e_1\langle e_2, e_3\rangle_{\mathcal{B}} = {}_{\mathcal{A}}\langle e_1, e_2\rangle e_3$.
6. Each $q^{-1}(z)$ is a Morita–Rieffel equivalence between $(\pi_{\mathcal{G}}^{-1}(r(Z)), \pi_{\mathcal{H}}^{-1}(s(Z)))$.

Let us write

$$\begin{tikzcd} \mathcal{A} \arrow[d] & E \arrow[l] \arrow[r] \arrow[d] & \mathcal{B} \arrow[d] \\ \mathcal{G} & Z \arrow[l] \arrow[r] & \mathcal{H} \end{tikzcd}$$

for such an equivalence of Fell bundles.

Theorem 6.7. *Let $q\colon E \to Z$ be an equivalence of Fell bundles $\mathcal{A} \to \mathcal{G}$ and $\mathcal{B} \to \mathcal{H}$. Then the space of compactly-supoorted sections $\Gamma_{\mathrm{c}}(Z, E)$ admits a pre-Hilbert $(\Gamma_{\mathrm{c}}(\mathcal{G}, \mathcal{A}), \Gamma_{\mathrm{c}}(\mathcal{H}, \mathcal{B}))$-bimodule structure. Its completion implements a Morita–Rieffel equivalence of the corresponding* C^**-algebras.*

Now we are at the place to state the equivalence between the C^*-algebras

$$\mathrm{C}^*(H) \qquad \text{and} \qquad \mathrm{C}^*((Z \times Z) \times_H \mathbb{K}(V_\rho))$$

implemented by $X(Z, V_\rho)$. Notice that both sides are section C^*-algebras of a Fell bundle

$$(Z \times Z) \times_H \mathbb{K}(V_\rho) \qquad \text{and} \qquad H \times \mathbb{C}.$$

The goal is to show the following equivalence of Fell bundles:

$$\begin{tikzcd} (Z \times Z) \times_H \mathbb{K}(V_\rho) \arrow[d] & Z \times V_\rho \arrow[l] \arrow[r] \arrow[d] & H \times \mathbb{C} \arrow[d] \\ Z \times_H Z & Z \arrow[l] \arrow[r] & H. \end{tikzcd}$$

The structure maps are given as follows:

- $(z, v)(h, \lambda) := (zh, \lambda\rho(h^{-1})v)$, for $(z, v) \in Z \times V_\rho$ and $(h, \lambda) \in H \times \mathbb{C}$.

- $(z_1, z_2, k) \cdot (z, v) := (z_1 h, \rho(h^{-1}) k \rho(h) v)$, for $(z_1, z_2, k) \in (\mathbb{Z} \times Z) \times_H \mathbb{K}(V_\rho)$ and $(z, v) \in Z \times V_\rho$ with $z_2 h = z$.
- ${}_{(Z\times Z)\times_H \mathbb{K}(V_\rho)}\langle (z_1, v_1), (z_2, v_2) \rangle := (z_1, z_2, |v_1\rangle \langle v_2|)$, for $(z_1, v_1), (z_2, v_2) \in Z \times V_\rho$.
- $\langle (z_1, v_1), (z_2, v_2) \rangle_{H\times\mathbb{C}} := (h^{-1}, \langle v_1, \rho(h) v_2 \rangle)$, for $(z_1, v_1), (z_2, v_2) \in Z \times V_\rho$ with $z_2 h = z_1$.

Theorem 6.8 (Mesland–Şengün)**.** *If (ρ, V_ρ) has L^1-matrix coefficients. Then for any free and proper H-space Z, there is a Morita equivalence between the section* C^**-algebra of the Fell bundle $(Z \times Z) \times_H \mathbb{K}(V_\rho)$ and* $\mathrm{C}^*(H)$*.*

November 1, 2022

Purely-infinite C*-algebras from dynamical systems

Speaker: Francesca Arici (Leiden University)

The main reference of this talk is [AD97]; relavent results (which are unfortunately not covered due to time issues) are in [BL20, ADS19].

Let A be a C^*-algebra.

Definition 7.1. A projection $p \in A$ is *infinite* if it is equivalent to a proper subprojection of itself. That is, there exists an isometry v such that

$$v^* v = p, \qquad vv^* \leq p \qquad \text{but } vv^* \neq p.$$

We say p is *properly infinite*, if it is infinite and has two mutually orthogonal subprojections, which are both equivalent to p.

Definition 7.2. An AF-algebra is a C^*-algebra A such that there is an increasing sequence $A_1 \subseteq A_2 \subseteq \cdots$ of finite-dimensional C^*-subalgebras of A, such that $\overline{\cup_i A_i} = A$.

Theorem 7.3. *Any projection in an AF-algebra is finite.*

Pure infiniteness was first introduced for von Neumann algebras, then defined by Cuntz for *simple* C^*-algebras and by Kirchberg and Rørdam for non-simple C^*-algebras. In this talk, we will only look at simple purely-infinite C^*-algebras.

Definition 7.4. A C^*-subalgebra $B \subseteq A$ is called *hereditary* if $a \in A$ and $b \in B$ satisfy $0 \leq a \leq b$, then $a \in B$.

Example 7.5. Every ideal in A is hereditary.

Definition 7.6. A simple C^*-algebra is *purely-infinite* if every non-zero hereditary C^*-subalgebra contains an infinite projection.

Why do we care about these C^*-algebras? Kirchberg proved that simple, nuclear, purely-infinite separable stable C^*-algebras that satisfy the UCT are classified by their K-theory.

We want to study sufficient conditions for the C^*-algebra $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})$ of an r-discrete groupoid to be purely-infinite. For these, we first define another concept of freeness or principalness for topological groupoids.

Definition 7.7. A topological groupoid $\mathcal{G}$ is *topologically principal* or *essentially free*, if the set

$$\{x \in \mathcal{G}^{(0)} \mid \mathcal{G}^x_x = \{x\}\}$$

is dense in $\mathcal{G}$.

Recall that (c.f. Definition 2.19 and Remark 2.20) the reduced groupoid C*-algebra $\mathrm{C}^*_\mathrm{r}(\mathcal{G})$ for second-countable, r-discrete groupoids is the Hilbert $\mathrm{C}_0(\mathcal{G}^{(0)})$-module completion of $\mathrm{C}_\mathrm{c}(\mathcal{G})$ under the norm given by the left regular representation. In particular, we have that (c.f. [Ren80]) $\mathrm{C}^*_\mathrm{r}(\mathcal{G})$ has a tracial state iff there exists a finite $\mathcal{G}$-invariant probability measure μ on $\mathcal{G}^{(0)}$.

Proposition 7.8 ([Ren80, Chapter II, Proposition 4.6])**.** *If $\mathcal{G}$ is essentially free and r-discrete. Then*

$$\mathrm{C}^*_\mathrm{r}(\mathcal{G}) \textit{ is simple} \quad \textit{iff} \quad \mathcal{G} \textit{ is minimal.}$$

We need one more definition to give a sufficient condition for pure infiniteness of $\mathrm{C}^*_\mathrm{r}(\mathcal{G})$.

Definition 7.9. A topological groupoid $\mathcal{G}$ is *locally-contracting* if for every non-empty open subset $U \subseteq \mathcal{G}^{(0)}$, there exists an open subset $V \subseteq U$ and an open bisection S such that $\overline{V} \subseteq s(S)$ and $\alpha_{S^{-1}}(\overline{V}) \subsetneqq V$. Here the map $\alpha_{S^{-1}}$ is defined as

$$\alpha_{S^{-1}} \colon s(S) \to r(S), \qquad x \mapsto s(xS^{-1}).$$

Proposition 7.10. *Let $\mathcal{G}$ be a second-countable, locally-compact, r-discrete, essentially free, minimal and locally-contracting groupoid. Then $\mathrm{C}^*_\mathrm{r}(\mathcal{G})$ is purely-infinite.*

November 8, 2022

Induced representations of groupoids

Speaker: Torstein Ulsnæs (SISSA & Leiden University)

The following notes come almost entirely from Torstein's blog post. I wish to thank Torstein for kindly sharing me with the source codes. I also apologise for making some modifications to fit his hard work into this document, and for changing some symbols for consistency. — Y.Li

In this post, we collect some of the basic properties of induced representations of groupoid C*-algebras. The main focus will be the full groupoid C*-algebra, and we barely mention the definition of unitary representations of groupoids. In the sequel, unless stated otherwise, G will denote a second countable locally compact Hausdorff groupoid with a Haar system λ.

8.1 Induced representations finite groups

Given a subgroup of a finite group $H \subseteq G$, the group algebra $\mathbb{C}G$ has a natural right $\mathbb{C}H$-action and a left $\mathbb{C}G$ action, both by convolution. Any unitary representation $u \colon H \to \mathbb{B}(V_u)$ can "induce" a unitary representation of G by the following three step process

1. Extend u to a representation π_u of $\mathbb{C}H$ (given by the "integrated form"

$$\pi_u(f) = \int_G f(g)u(g)\,\mathrm{d}\lambda(G)$$

for all $f \in \mathrm{C}_\mathrm{c}(G)$).

2. Extend π_u to a representation of G on $\mathbb{C}G \otimes_{\mathbb{C}H} V_u$, endowed with the inner product

$$\langle f \otimes v, f' \otimes v' \rangle := \langle \pi_u(f^* * f)v, v' \rangle$$

given by left multiplication of $\mathbb{C}G$ on $\mathbb{C}G$.

3. Restrict this representation to G.

This is not the only way to induce representations from a subgroup to G. Parabolic inductions from parabolic subgroups of reductive algebraic groups are one example, but also one could very well have chosen any finite vector space with a $(\mathbb{C}G, \mathbb{C}H)$-bimodule structure in place of $\mathbb{C}G$ in the above process and got a finite representation of G.

The choice of $\mathbb{C}G$ has many advantages though. To list a few - we know what the induced rep resentations look like, we have a characterization of which of them are irreducible (by Machey's machinery) and we know that this particular choice of bimodule yields a functor adjoint to the very natural "restriction" functor which sends a representation π of G to its restriction to H (Frobenius reciprocity theorem).

8.2 Unitary representations of groupoids

For groupoids howerver, the notion of unitary representations is somewhat convoluted at first glance. A unitary representation of a groupoid $\mathcal{G}$ is defined to be a triple $(\mu, \mathcal{G}^{(0)} * H, L)$ where

- μ is a quasi-invariant measure on $\mathcal{G}^{(0)}$, meaning $\lambda_u \times \mu$ and $\lambda^u \times \mu$ are equivalent measures on $\mathcal{G}$, where $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$ is the Haar system of $\mathcal{G}$ (recall that $\operatorname{supp}(\lambda^u) = \mathcal{G}^u = \{\gamma \in \mathcal{G} \mid r(\gamma) = u\}$) and λ_u is the pushforward of λ^u under inversion (so we have $\operatorname{supp} \lambda_u = \mathcal{G}_u$).
- $\mathcal{G}^{(0)} * H$ is a Borel bundle, which is a type of measurable bundle of Hilbert spaces over $\mathcal{G}^{(0)}$. See [Wil19, Definition 3.32] for the precise definition.
- L is a Borel morphism of groupoids
$$L\colon \mathcal{G} \to \operatorname{Iso}(\mathcal{G}^{(0)} * H), \qquad L(\gamma) = (r(\gamma), L_\gamma, s(\gamma)),$$
where $\operatorname{Iso}(\mathcal{G}^{(0)} * H)$ is the isomorphism groupoid of the Borel bundle, given by
$$\operatorname{Iso}(\mathcal{G}^{(0)} * H) := \{(x, V, y) \mid V\colon H_x \to H_y \text{ is unitary}\}.$$

Just like for ordinary groups, any representation of a groupoid C*-algebra can be written as the integrated form of a unitary representation (see the Integration and Disintegration theorems of Renault). Similar statements also hold for crossed products by groupoid actions.

8.3 Induced representations of locally compact groups

Let $H \subseteq G$ be a closed subgroup of a locally compact group G assumed (for ease of notation) to be unimodular. The general case can be found in, for instance, [Ech17]. The space $X_0 = \mathrm{C_c}(G)$ can be equipped with a right pre-Hilbert $\mathrm{C_c}(H)$-module structure with respect to the inner product
$$\langle f, g\rangle_H(h) = \int_G \overline{f(s^{-1})} g(s^{-1}h)\, \mathrm{d}s.$$

The completion X_H^G with respect to this inner product is a Hilbert $\mathrm{C}^*(H)$-module on which $\mathrm{C}^*(G)$ acts on the left by convolution.

Any representation $\pi\colon \mathrm{C}^*(H) \to \mathbb{B}(V_\pi)$ now induces a representation of $\mathrm{C}^*(G)$ via
$$\mathrm{C}^*(G) \to X_H^G \otimes_\pi V_\pi$$
in the same way as above. Note that we have used the internal tensor products of Hilbert C*-modules, which means $X_H^G \otimes_\pi V_\pi$ is a Hilbert space completion of $\mathrm{C_c}(G) \otimes V_\pi$ with respect to the (possibly degenerate) inner product
$$\langle f \otimes v, f' \otimes v'\rangle = \langle \pi(\langle f, f'\rangle_H)v, v'\rangle_\pi,$$
where $\langle -, -\rangle_\pi$ the inner product on V_π.

When X_H^G can be chose to be an imprimitivity bimodule (i.e. $\mathrm{C}^*(G) \cong \mathbb{K}(X_H^G)$), all unitary representations of G can be induced from those of H. The imprimitivity theorem tells us which unitary representations of G are induced from G. The short version of this theorem goes as follows. A unitary representation $u\colon G \to \mathrm{U}(\mathbb{B}(V_\sigma))$ is induced from a unitary representation of H if and only if there is a non-degenerate representation $\pi\colon \mathrm{C}(G/H) \to \mathbb{B}(V_\sigma)$ such that (π, σ) is a covariant representation of the dynamical system $(\mathrm{C}(G/H), G, U)$, meaning that

$$\pi(U_g(f)) = u_g \pi(f) u_g^*$$

with $U_g(f)(x) := f(g^{-1}x)$.

8.4 Induced representations of Groupoids

Let $\mathcal{H} \subseteq \mathcal{G}$ be a closed subgroupoid of $\mathcal{G}$ with Haar systems $\{\alpha^u\}_{u\in\mathcal{H}^{(0)}}$ and $\{\lambda^v\}_{v\in\mathcal{G}^{(0)}}$ respectively.

For our groupoid $\mathcal{G}$, however, we will replace $\mathrm{C_c}(\mathcal{G})$ by $\mathrm{C_c}(\mathcal{G}_{\mathcal{H}^{(0)}})$, where

$$\mathcal{G}_{\mathcal{H}^{(0)}} = s^{-1}(\mathcal{H}^{(0)}) = \{\gamma \in \mathcal{G} \mid s(\gamma) \in \mathcal{H}^{(0)}\}.$$

Note that if $\mathcal{G}$ is a group, then $\mathcal{G}^{(0)}$ is a single point, hence $\mathcal{G}_{\mathcal{H}^{(0)}} = \mathcal{G}$. This is a closed subspace of $\mathcal{G}$ containing $\mathcal{H}$. The function algebra $\mathrm{C_c}(\mathcal{G}_{\mathcal{H}^{(0)}})$ carries a right $\mathrm{C_c}(\mathcal{H})$-action, a $\mathrm{C_c}(\mathcal{H})$-valued inner product and a left $\mathrm{C_c}(\mathcal{G})$-action as follows. For $f \in \mathrm{C_c}(\mathcal{G})$, $\phi, \psi \in \mathrm{C_c}(\mathcal{G}_{\mathcal{H}^{(0)}})$, $f' \in \mathrm{C_c}(\mathcal{H})$, $h \in \mathcal{H}$:

$$\begin{aligned}
(f \cdot \phi)(\xi) &= \int_{\mathcal{G}} f(\xi)\phi(\gamma^{-1}\xi)\,\mathrm{d}\lambda^{r(\xi)}(\gamma), \\
\phi f'(\xi) &= \int_{\mathcal{H}} \phi(\xi h) f'(h^{-1})\,\mathrm{d}\alpha^{s(\xi)}(h), \\
\langle \phi, \psi \rangle_*(h) &= \int_{\mathcal{G}} \overline{\phi(\gamma)}\psi(\gamma h)\,\mathrm{d}\lambda^{r(h)}(\gamma).
\end{aligned}$$

Given a representation $\pi\colon \mathrm{C}^*(\mathcal{H}) \to \mathbb{B}(V_\pi)$, we can form the Hilbert space $V_{\mathrm{Ind},\pi}$ as the completion of the pre-Hilbert space $\mathrm{C_c}(\mathcal{G}_{\mathcal{H}^{(0)}}) \otimes V_\pi$ with respect to the (possibly degenerate) inner product

$$\langle \phi \otimes h, \psi \otimes k \rangle := \langle \pi(\langle \phi, \psi \rangle) h, k \rangle.$$

Now the induced representation

$$\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} \pi\colon \mathrm{C}^*(\mathcal{G}) \to \mathbb{B}(V_{\mathrm{Ind},\pi})$$

is determined by sending an $f \in \mathrm{C_c}(\mathcal{G})$ to the operator acting on $\xi \otimes v \in \mathrm{C_c}(\mathcal{G}_{\mathcal{H}^{(0)}}) \otimes V_\pi \subseteq V_{\mathrm{Ind},\pi}$ by

$$(\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} \pi)(f)(\xi \otimes_\pi v) = f \cdot \xi \otimes_\pi v.$$

In summary, other than the explicit realization of the imprimitivity correspondence $X_{\mathcal{H}}^{\mathcal{G}}$, the process of inducing representations from closed subgroupoids runs perfectly parallel to that of ordinary group theory.

Recall that a full Hilbert $\mathrm{C}^*(\mathcal{H})$-module X together with a non-degenerate *-homomorphism $\phi\colon \mathrm{C}^*(\mathcal{G}) \to \mathbb{B}(X)$ is called a $(\mathrm{C}^*(\mathcal{G}), \mathrm{C}^*(\mathcal{H}))$-($\mathrm{C}^*$)-correspondence. It is customary to think of C^*-correspondences are "generalized morphisms" $\mathrm{C}^*(\mathcal{G}) \to \mathrm{C}^*(\mathcal{H})$, and construct a category Corr whose objects are C^*-algebras and whose morphisms are (isomorphism classes of) correspondences. This is due to the fact that many "rigidity results" and equivalences about groupoids translate only to assertions of Morita equivalence of their corresponding C^*-algebras (Morita equivalences are nothing but the isomorphisms in both categories). See for instance Renault's equivalence theorem [Ren80].

With this setup, one can see that if $\mathrm{Rep}(A)$ denotes the collection of unitary equivalence classes of non-degenerate representations of a C^*-algebra A, then

$$\mathrm{Rep}(A) = \mathrm{Hom}_{\mathsf{Corr}}(A, \mathbb{C}).$$

A C^*-correspondence $[X, \phi] \in \mathrm{Hom}_{\mathsf{Corr}}(A, B)$ gives a map

$$X\text{-Ind}\colon \mathrm{Rep}(B) \to \mathrm{Rep}(A).$$

The composition is the internal tensor product of Hilbert C^*-modules.

For more on induction of groups/ideals/C^*-algebras and the correspondence category I highly recommend [CELY17, Chapter 2]. For more on internal tensor products and Hilbert modules the standard reference is [Lan95].

We shall usually omit the C^*-correspondence X when it is clear from the context and replace $X\text{-Ind}_{\mathcal{H}}^{\mathcal{G}}$ by $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}$, or even by Ind when the groups are also clear.

Since the process of groupoid induction is so similar to that of groups, it should come as no surprise that many properties translate word for word from group theory. Here we list some of the most fundamental, all of which can be found in [Wil19] and proofs in the cited reference therein.

8.4.1 Direct sums

Let $\pi_i\colon \mathrm{C}^*(\mathcal{H}) \to \mathbb{B}(V_i)$ be a collection of representations. Denote by $\oplus_i \pi_i$ the direct sum representation. Then for any C^*-correspondence $[X, \psi]\colon \mathrm{C}^*(\mathcal{G}) \to \mathrm{C}^*(\mathcal{H})$, we have

$$X\text{-Ind}_{\mathcal{H}}^{\mathcal{G}} \left(\bigoplus_i \pi_i \right) = \bigoplus_i X\text{-Ind}_{\mathcal{H}}^{\mathcal{G}} \pi_i.$$

8.4.2 Kernels

Let $[X, \phi]\colon \mathrm{C}^*(\mathcal{G}) \to \mathrm{C}^*(\mathcal{H})$ be a C^*-correspondence. We can also induce ideals from $\mathrm{C}^*(\mathcal{G})$ to $\mathrm{C}^*(\mathcal{H})$ as follows. If $J \subseteq \mathrm{C}^*(\mathcal{H})$ is an ideal then define

$$X\text{-Ind}_{\mathcal{H}}^{\mathcal{G}} J := \{a \in \mathrm{C}^*(\mathcal{G}) \mid \langle ax, y\rangle_{\mathcal{H}} \in J, \text{ for all } x, y \in X\},$$

which is closed by Cohens factorization theorem. This turns out to be an ideal of $\mathrm{C}^*(\mathcal{G})$. With this asignment we get the formula for any representation $\pi\colon \mathrm{C}^*(\mathcal{H}) \to B(V_\pi)$

$$\ker \mathrm{Ind}(\pi) = \mathrm{Ind}(\ker(\pi)).$$

8.4.3 Induction in stages

If $\mathcal{H} \subseteq \mathcal{K} \subseteq \mathcal{G}$ be closed groupoids. Let $[Y_{\mathcal{H}}^{\mathcal{K}}, \phi_{\mathcal{K}}^{\mathcal{H}}]$ and $[Y_{\mathcal{K}}^{\mathcal{G}}, \phi_{\mathcal{K}}^{\mathcal{G}}]$ be $(\mathrm{C}^*(\mathcal{K}), \mathrm{C}^*(\mathcal{H}))$- and $(\mathrm{C}^*(\mathcal{G}), \mathrm{C}^*(\mathcal{H}))$-correspondences. Then with

$$[Y_{\mathcal{H}}^{\mathcal{G}}, \psi] = [Y_{\mathcal{H}}^{\mathcal{K}}, \phi_{\mathcal{K}}^{\mathcal{G}}] \circ [Y_{\mathcal{K}}^{\mathcal{G}}, \phi_{\mathcal{K}}^{\mathcal{H}}] = [Y_{\mathcal{H}}^{\mathcal{K}} \otimes_{\phi_{\mathcal{H}}^{\mathcal{K}}} Y_{\mathcal{H}}^{\mathcal{K}}, \phi_{\mathcal{K}}^{\mathcal{G}} \otimes 1],$$

together with a representation

$$\pi\colon \mathrm{C}^*(\mathcal{H}) \to B(V_\pi).$$

The following holds:

$$Y_{\mathcal{H}}^{\mathcal{K}}\text{-Ind}(Y_{\mathcal{K}}^{\mathcal{G}}\text{-Ind}\, \pi) = Y_{\mathcal{H}}^{\mathcal{G}}\text{-Ind}\, \pi.$$

In particular,

$$\mathrm{Ind}_{\mathcal{H}}^{\mathcal{K}}(\mathrm{Ind}_{\mathcal{K}}^{\mathcal{G}}\, \pi) = \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}\, \pi.$$

8.5 Some examples

8.5.1 Locally closed orbits

Let $\mathcal{G}$ be a transitive groupoid, i.e. acts transitively on its unit space $\mathcal{G}^{(0)}$. This is quite similar to the case of a group. Namely, let

$$\mathcal{G}_u^u = \{\gamma \in \mathcal{G} \mid r(\gamma) = s(\gamma) = u\}$$

be the isotropy group of $u \in \mathcal{G}^{(0)}$. Since the action is transitive, the imprimitivity theorem gives us a Morita equivalence

$$\mathrm{C}^*(\mathcal{G}_U^U) \sim_{\mathrm{Morita}} \mathrm{C}^*(\mathcal{G}).$$

In particular, all representations of $\mathrm{C}^*(\mathcal{G})$ are induced from an(y fixed) isotropy group. For instance, if the action of $\mathcal{G}$ on $\mathcal{G}^{(0)}$ has a trivial isotropy group, then $\mathrm{Prim}(\mathrm{C}^*(\mathcal{G}))$ is a point.

The representation theory of transitive groupoid C^*-algebras hence often reduces to that of group C^*-algebras.

The above theorem holds if all orbits of $\mathcal{G}$ in $\mathcal{G}^{(0)}$ are locally closed (meaning they are open in their closure). Though, as we will see, one would have to pick at least one isotropy group from each orbit to ensure all representations are reached.

8.5.2 Regular representations and the reduced C^*-norm

The second examples are the "induced" regular representations

$$\mathrm{Ind}_\mu \colon \mathrm{C}^*(\mathcal{G}) \to \mathbb{B}(L^2(\mathcal{G}^{(0)}, \nu^{-1}))$$

(see Definition 2.19). They are just the induced representations given by the regular representation

$$\pi \colon \mathrm{C}^*(\mathcal{H}) \to \mathbb{B}(V_\pi)$$

for $\mathcal{H} = \mathcal{G}^{(0)}$ and $V_\pi = L^2(\mathcal{G}^{(0)}, \mu)$. In particular, if $\mu = \delta_u$ is the Dirac measure supported on a fixed $u \in \mathcal{G}^{(0)}$. Then the inner product on V_π is just

$$\langle f, f' \rangle = \int_{\mathcal{G}} \overline{f(\gamma)} f(\gamma u) \,\mathrm{d}\lambda^{r(u)}(\gamma) = \int_{\mathcal{G}} |f(\gamma)|^2 \,\mathrm{d}\lambda^u(\gamma).$$

So $V_{\mathrm{Ind},\pi}$ is the completion of

$$\mathrm{C_c}(\mathcal{G}) \otimes L^2(\mathcal{G}^{(0)}, \delta_u) \cong \mathrm{C_c}(\mathcal{G})$$

with respect to the inner product

$$\langle f \otimes \phi, f' \otimes \phi' \rangle \colon = \langle \pi(\langle f, f' \rangle_*)\phi, \phi' \rangle.$$

Then $V_{\mathrm{Ind},\pi}$ can be identified with $L^2(\mathcal{G}_u, \lambda_u)$.

Now any representation of C^*-algebra can be written as a (possibly infinite) direct sum of irreducible representations. Using the fact that induction preserves direct sums, one can conclude that if $\rho \colon \mathrm{C}_0(\mathcal{G}^{(0)}) \to \mathbb{B}(V)$ is a faithful representation, then

$$|| \operatorname{Ind}_{\mathcal{G}}^{\mathcal{G}^{(0)}} \rho(f)|| \le ||f||_{\mathrm{r}}.$$

If ρ is the regular representation described above, given by a Radon measure with full support on $\mathcal{G}^{(0)}$. Then

$$\|f\|_{\mathrm{r}} = \|\operatorname{Ind}_{\mathcal{G}}^{\mathcal{G}^{(0)}} \rho\|.$$

8.5.3 Induction from groupoids with closed orbits

Assume now that $F \subseteq \mathcal{G}^{(0)}$ is a closed $\mathcal{G}^{(0)}$-invariant subset of the unit space. One easily checks that

$$\mathcal{G}_F^F = \{\gamma \in \mathcal{G} \mid r(\gamma), s(\gamma) \in F\}$$

is a subgroupoid of $\mathcal{G}$. We will show that not all representations of $\mathcal{G}$ are of the form $\operatorname{Ind}_{\mathcal{G}_F^F}^{\mathcal{G}} \pi$ for some representation of $\mathrm{C}^*(\mathcal{G}_F^F)$, which (by induction in stages) implies not all are of the form $\operatorname{Ind}_{\mathcal{G}_U^U}^{\mathcal{G}} \pi$ for $u \in F$).

To do this however, we will need some propositions. Let π be as above, then we define the associated M-representation of π to be the representation M_π satisfying, for every $\phi \in \mathrm{C}_0(\mathcal{H}^{(0)})$ and $f \in \mathrm{C_c}(\mathcal{H})$

$$M(\phi)\pi(f) = \pi((\phi \circ r) \cdot f),$$

where r is the range map of $\mathcal{H}$ and $\cdot$ is just pointwise multiplication. Explicitly we have

$$M_\pi = \overline{\pi} \circ V$$

where

$$V \colon \mathrm{C}_0(\mathcal{G}^{(0)}) \to \mathcal{M}(\mathrm{C}^*(\mathcal{G})), \qquad \phi \mapsto (||\phi||_\infty - |\phi|^2)^{1/2}.$$

(See [Wil19, Lemma 1.48]) and $\overline{\pi}$ is the extension of π to $\mathcal{M}(\mathrm{C}^*(\mathcal{G}))$ (which is still non-degenerate!)

Having this at our disposal, one can define the support of π, to the closed subset of $\mathcal{G}^{(0)}$ corresponding to the ideal $\ker(M_\pi)$, that is, support of π is the largest set C for which

$$\{f \in \mathrm{C}_0(\mathcal{G}^{(0)}) \mid f_C^C = 0\} = \ker(M_\pi).$$

If M_π is the usual multiplication representation on some measure space $L^2(\mathcal{G}^{(0)}, \mu)$, then the support of M_π is simply the support of μ, hence the name.

One should really think of the support map as something that determines the essential domain of π as the following proposition of [Wil19] shows:

Theorem 8.1. *Let $\pi \colon \mathrm{C}^*(\mathcal{G}) \to \mathbb{B}(V_\pi)$ be any non-degenerate representation with $\operatorname{supp}(\pi) = F \subsetneq \mathcal{G}^{(0)}$. Then the subset F is closed and $\mathcal{G}$-invariant, and the representation π factors through the (surjective) map*

$$j_F^{\mathcal{G}} \colon \mathrm{C}^*(\mathcal{G}) \to \mathrm{C}^*(\mathcal{G}_F^F)$$

induced by the inclusion $\mathcal{G}_F^F \hookrightarrow \mathcal{G}$.

Proof. The proof is rather obvious if one believes that if $U = \mathcal{G}^{(0)} \backslash F$ is the complement of F then $\mathrm{C_c}(\mathcal{G}_U^U)$ is dense in $\ker(j_F^{\mathcal{G}})$. But this is always the case, as the inclusion $\mathrm{C_c}(\mathcal{G}|_U) \to \mathrm{C_c}(\mathcal{G})$ sits in a short exact sequence

$$0 \to \mathrm{C}^*(\mathcal{G}_U^U) \to \mathrm{C}^*(\mathcal{G}) \to \mathrm{C}^*(\mathcal{G}_F^F) \to 0.$$

So it suffices to check that $\mathrm{C_c}(\mathcal{G}_U^U) \subseteq \ker(j_F^{\mathcal{G}})$. To show this, pick a $\phi \in \mathrm{C_c}(U)$ such that $(\phi \circ r)f = f$ (i.e. a bump function which is one on all $x \in \mathcal{G}^{(0)}$ where $f(\gamma) \neq 0$ for some γ with $r(\gamma) = 1$). Then using the formula for the associated M-representation above we have

$$M(\phi)\pi(f) = \pi((\phi \circ r)f) = \pi(f).$$

but $M(\phi) = 0$ since the support is F. $\square$

The converse of the above proposition is also true, meaning, if π factors through $j_F^{\mathcal{G}}$, then the support of π must be contained in F. Similarly one can say something about the support of an induced representation, as the following proposition from [Wil19] and its corollaries show

Proposition 8.2. *Let $\mathcal{G}$ be a groupoid. Let $\mathcal{H} \subseteq \mathcal{G}$ be a closed subgroupoid. Let $F \subseteq \mathcal{H}^{(0)}$ be a closed $\mathcal{H}$-invariant subset. Let $\pi\colon \mathrm{C}^*(\mathcal{H}) \to B(V_\pi)$ be a non-degenerate representation with* $\operatorname{supp}(\pi) \subseteq F$*. Let E is a closed $\mathcal{G}$-invariant set such that $F \subseteq E \subseteq \mathcal{G}^{(0)}$. Then as in the previous proposition π factors as $\overline{\pi} \circ j_F^{\mathcal{H}}$ and we have*

$$\operatorname{Ind}_{\mathcal{H}}^{\mathcal{G}} \pi = \operatorname{Ind}_{\mathcal{H}}^{\mathcal{G}}(\overline{\pi} \circ j_F^{\mathcal{H}}) = \operatorname{Ind}_{\mathcal{H}_F^F}^{\mathcal{G}_E^E}(\overline{\pi}) \circ j_E^{\mathcal{G}}.$$

Note that the above proposition simplys says, if a representation is induced from a representation with support $F \subseteq \mathcal{H}^{(0)}$ then the support of the induced representation must be contained in the $\mathcal{G}$-orbit of $\overline{\mathcal{G}F} \subseteq \mathcal{G}^{(0)}$.

The proof is not very glamorous so we opt to leave it out, but just mention that the map implementing the isomorphism is the restriction map

$$r\colon \mathrm{C_c}(\mathcal{G}_{\mathcal{H}^{(0)}}) \to \mathrm{C_c}(\mathcal{G}_E^F).$$

Interested readers can consult [Wil19, Chapter 5.6] for the proof.

8.5.4 Amenable groups

There is a beautiful theorem which states that given a surjective morphism $f\colon G \to G'$ of locally compact groups such that $N = \ker(f)$ is amenable, then f induces a morphism of the C^*-algebras

$$f\colon \mathrm{C}^*_\mathrm{r}(G) \to \mathrm{C}^*_\mathrm{r}(G').$$

In general one cannot assume the range of $f(\mathrm{C}^*_\mathrm{r}(G))$ is contained in $\mathrm{C}^*_\mathrm{r}(G)$ (or in $\mathrm{C}^*(G)$).

The reason this holds for amenable kernels is the following. Since N is amenable, $\mathrm{C}^*_\mathrm{r}(N) = \mathrm{C}^*(N)$. So every unitary representation lifts to $\mathrm{C}^*_\mathrm{r}(G)$, hence also the trivial representation $1_N\colon N \to \mathbb{C}$. Note that we have a unitary equivalence between the representations

$$\operatorname{Ind}_N^G 1_N \sim \lambda_{G/N},$$

where $\lambda_{G/N}$ denotes the regular representation of G/N.

Since induced representations preserve weak containment, we have $\ker(\lambda_{N/G}) \subseteq \ker(\lambda G)$ hence f^* induces a map

$$f^*\colon \mathrm{C}^*_\mathrm{r}(G) \to \mathrm{C}^*_\mathrm{r}(G'),$$

since $G' = G/N$.

November 15, 2022

Existence and uniqueness of Haar systems

Speaker: Malte Leimbach (Radboud University Nijmegen)

The main reference for this talk is [Wil19, Chapter 6].

Throughout this talk, a groupoid always refers to a locally-compact Hausdorff topological groupoid.

9.1 Existence of Haar systems on second-countable groupoids

We start with a slight generalisation of Haar systems called *π-systems.*

Definition 9.1. Let $\pi\colon Y \to X$ be a continuous surjective maps between locally-compact spaces.

- A *π-system* is a family $\{\beta^x\}_{x\in X}$ of positive Radon measures on Y, such that

  ($\boldsymbol{\pi}$S0) $\operatorname{supp}(\beta^x) \subseteq \pi^{-1}(x)$.

(πS2) For any $f \in \mathrm{C_c}(Y)$, the map

$$x \mapsto \beta(f)(x) := \int_Y f(y)\,\mathrm{d}\beta^x(y)$$

is continuous.

- We say a π-system is *full*, if

  (πS1) $\mathrm{supp}(\beta^x) = \pi^{-1}(x)$.

- If in addition X and Y are $\mathcal{G}$-spaces for a groupoid $\mathcal{G}$. We say a π-system is $\mathcal{G}$*-equivariant*, if

  (πS3) For all $f \in \mathrm{C_c}(Y)$ and $\gamma \in \mathcal{G}$, the following holds:

$$\int_Y f(\gamma y)\,\mathrm{d}\beta^x(y) = \int_Y f(y)\,\mathrm{d}\beta^{\gamma x} y.$$

Example 9.2. A Haar system of a groupoid $\mathcal{G}$ is a full, $\mathcal{G}$-equivariant π-system for $\pi := r\colon \mathcal{G} \to \mathcal{G}^{(0)}$ and the obvious actions $\mathcal{G} \curvearrowright \mathcal{G}$ and $\mathcal{G} \curvearrowright \mathcal{G}^{(0)}$. Compare Definition 2.3.

If a groupoid has a Haar system. Then the range and source maps are open. There are groupoids whose source and range maps are not open (e.g. Example 2.13), this means that not all groupoids admit a Haar system. In particular, an r-discrete groupoid admits a Haar system iff it is étale. Then we may ask if the converse statement holds. That is:

Question 9.3. If $\mathcal{G}$ is a groupoid with open range and source maps. Does $\mathcal{G}$ always admit a Haar system?

Anton Deitmar proposed the following counterexample.

Lemma 9.4. *Let X be a locally-compact Hausdorff space. Then the pair groupoid $X \times X \rightrightarrows X$ admits a Haar system iff there is a full Radon measure on X.*

Example 9.5 ([Dei18]). Let X be the one-point compactification of an uncountable discrete set D. Suppose X possessed a full Radon measure μ. Then $\mu(\{d\}) > 0$ for any $d \in D$. This implies $\mu(X) = +\infty$, which is a contradiction. Therefore, X does not have a full Radon measure and $X \times X \rightrightarrows X$ does not admit a Haar system.

The space X in the counterexample above is quite pathological: it is not second-countable. If X is second-countable, then there are always full Radon measures on X. Anton Deitmar proposed the following more realistic conjecture.

Conjecture 9.6 ([Dei18]). If $\mathcal{G}$ is a second-countable groupoid with open range and source maps. Then $\mathcal{G}$ has a Haar system.

We do not have an affirmative answer to the existence of a Haar system on a second-countable groupoid in general. However, in the special case of *groupoid group bundles*, Renault proved that when they are second-countable with open range and source maps, then they admit Haar systems.

Definition 9.7. A *groupoid group bundle* is a groupoid $\mathcal{G}$ whose range map r and source map s coincides.

Theorem 9.8 ([Ren91, Lemma 1.3])**.** *A second-countable groupoid group bundle with open range and source maps admits a Haar system.*

For the proof, we need the following technical definitions and lemma.

Definition 9.9.
- Let $\mathcal{G}$ be a groupoid. A subset $D \subseteq \mathcal{G}$ is *diagonally compact*, if for any compact subset $K \subseteq \mathcal{G}^{(0)}$, the orbit of K in $\mathcal{G}^{(0)}$ under the left and right actions $\mathcal{G} \curvearrowright \mathcal{G}^{(0)}$ and $\mathcal{G}^{(0)} \curvearrowleft \mathcal{G}$ are compact.

- Let $\pi\colon Y \to X$ be a continuous surjection. A subset $A \subseteq Y$ is *π-compact*, if $A \cap \pi^{-1}(K)$ is compact for all compact subsets $K \subseteq X$.
- Denote by $\mathrm{c}(\mathbb{N})$ the subspace of Cauchy sequences in $\ell^\infty(\mathbb{N})$. A *generalised limit* is a state on $\ell^\infty(\mathbb{N})$ which extends the limit functional

$$\lim\colon \mathrm{c}(\mathbb{N}) \to \mathbb{C}, \qquad (x_n)_{n\in\mathbb{N}} \mapsto \lim x_n.$$

Lemma 9.10. *Let $\mathcal{G}$ be a second-countable groupoid. There exists a diagonally compact neighbourhood of $\mathcal{G}^{(0)}$ in $\mathcal{G}$.*

Lemma 9.11. *Let $(x_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$. If there exists $x \in \mathbb{C}$ such that $\Lambda((x_n)_{n\in\mathbb{N}}) = x$ for all generalised limit Λ. Then $(x_n)_{n\in\mathbb{N}}$ is Cauchy and $x_n \to x$.*

Idea of the proof. Let $\mathcal{G}$ be a groupoid group bundle. Let N be a diagonally compact neighbourhood of $\mathcal{G}^{(0)}$. Let f_0 be a non-negative continuous function on $\mathcal{G}$ such that

$$f_0|_{\mathcal{G}^{(0)}} = 1 \qquad \text{and} \qquad \operatorname{supp}(f_0) \subseteq N.$$

Notice that $\operatorname{supp}(f_0)$ is r-compact.

For any $u \in \mathcal{G}^{(0)}$, let λ^u be the Haar measure on $\mathcal{G}^u_n = \mathcal{G}_u = \mathcal{G}^u$ such that

$$\int_{\mathcal{G}^u_u} f_0(t)\,\mathrm{d}\lambda^u t = 1.$$

We claim that $\{\lambda^u\}_{u\in\mathcal{G}^{(0)}}$ is a Haar system on $\mathcal{G}$. (HS1) and (HS3) of Definition 2.3 are easy. The difficult part is (HS2). That is, the map

$$u \mapsto \int_{\mathcal{G}^u_u} f(t)\,\mathrm{d}\lambda^u(t)$$

is continuous, for all $f \in \mathrm{C_c}(\mathcal{G})$. We prove in steps.

1. Show that the map $u \mapsto \lambda^u(K)$ is bounded for any compact subset $K \subseteq \mathcal{G}^u_u$.
2. Since $\mathcal{G}$ is second-countable. Choose a sequence $\{u_n\}_{n\in\mathbb{N}}$ approaching u. For any $f \in \mathrm{C_c}(\mathcal{G})$, define the map
$$\tilde{f}\colon \mathbb{N} \to \mathbb{C}, \qquad \tilde{f}(n) := \int_{\mathcal{G}^{u_n}_{u_n}} f(t)\,\mathrm{d}\lambda^{u_n}(t).$$
Since $u \mapsto \lambda^u(K)$ is bounded, we have $\tilde{f} \in \ell^\infty(\mathbb{N})$.
3. Let Λ be a generalised limit. Define
$$\mu(f) := \Lambda(\tilde{f}).$$
Then μ is a positive linear functional on $\mathrm{C_c}(\mathcal{G})$ which is supported on $\mathcal{G}^u_u$. So we may view μ as a positive linear functional on $\mathrm{C_c}(\mathcal{G}^u_u)$.
4. Show that μ is left-invariant, hence a Haar measure on $\mathcal{G}$.
5. Show that $\mu = \lambda^u$. $\square$

9.2 Haar systems on equivalent groupoids

Until the end of this lecture, all spaces are assumed to be second-countable.

We will prove the following theorem:

Theorem 9.12 ([Wil16])**.** *Every second-countable groupoid, which is Morita equivalent to a second-countable groupoid with a Haar system, admits a Haar system.*

Lemma 9.13 (Blanchard)**.** *Let $\pi\colon Y \to X$ be a continuous surjection. Then π is open iff Y admits a full π-system.*

Definition and Lemma 9.14. Let $\pi\colon Y \to X$ be a continuous open surjection. Then there is a non-negative function $\phi \in \mathrm{C}(Y)$ such that $\operatorname{supp}(\phi)$ is π-compact and $\pi(\{y \in Y \mid \phi(y) > 0\}) = X$. Such a function ϕ is called a *Bruhat section* of $\pi\colon Y \to X$.

Recall that

Definition 9.15. Let $\mathcal{G}$ be a groupoid. Let Z be a free and proper left $\mathcal{G}$-space. The *imprimitivity groupoid* of Z is defined as

$$\mathcal{G}^Z := \mathcal{G}\backslash Z *_r Z \rightrightarrows \mathcal{G}\backslash Z.$$

Lemma 9.16 ([Wil19, Lemma 2.44])**.** *If Z is a $(\mathcal{G},\mathcal{H})$-equivalence. Then $\mathcal{H} \cong \mathcal{G}^Z$.*

Lemma 9.17 (Kumjian–Muhly–Renault–Williams)**.** *Let Z be a $(\mathcal{G},\mathcal{H})$-equivalence. Then $\mathcal{H}$ admits a Haar system iff Z admits a full, $\mathcal{G}$-equivariant r_Z-system, where $r_Z\colon Z \to \mathcal{G}^{(0)}$ is the moment map.*

Proof. Use $\mathcal{H} \cong \mathcal{G}^Z$ and [Wil19, Lemma 3.16]. □

Proposition 9.18. *Let $\mathcal{G}$ be a second-countable, locally-compact Hausdorff groupoid with a Haar system $\{\lambda^u\}_{u\in\mathcal{G}^{(0)}}$. Let Z be a proper $\mathcal{G}$-space, not necessarily free. Assume that the moment map $r_Z\colon Z \to \mathcal{G}^{(0)}$ is open. Then there is a full $\mathcal{G}$-equivariant r_Z-system for Z.*

Proof. Let $\pi\colon Z \to \mathcal{G}\backslash Z$ be the orbit projection. It is a continuous, surjective open map. Let $f \in \mathrm{C}(Z)$ be a Bruhat section for π. Let $\{\beta^u\}_{u\in\mathcal{G}^{(0)}}$ be a full r_Z-system. Define a collection of Radon measures $\{\nu^u\}_{u\in\mathcal{G}^{(0)}}$ via

$$\nu^u\colon \mathrm{C_c}(Z) \to \mathbb{C}, \qquad \nu^u(f) := \int_{\mathcal{G}}\int_Z f(\gamma z)\phi(z)\,\mathrm{d}\beta^{s(\gamma)}(z)\,\mathrm{d}\lambda^u(\gamma).$$

We check that $\{\nu^u\}_{u\in\mathcal{G}^{(0)}}$ is a Haar system. (πS1) and (πS2) are clear. For the equivariance (πS3), we have

$$\begin{aligned}\int_Z f(\eta z)\,\mathrm{d}\nu^{s(\eta)}(z) &= \int_{\mathcal{G}}\int_Z f(\eta\gamma z)\,\mathrm{d}\beta^{s(\gamma)}(z)\,\mathrm{d}\lambda^{s(\eta)}(\gamma)\\ &= \int_{\mathcal{G}}\int_Z f(\gamma z)\phi(z)\,\mathrm{d}\beta^{s(\gamma)}(z)\,\mathrm{d}\lambda^{r(\eta)}(\gamma)\\ &= \int_Z f(z)\,\mathrm{d}\nu^{r(\eta)}(z).\end{aligned}$$

For the second equality we use the fact that $\{\lambda^u\}_{u\in\mathcal{G}^{(0)}}$ is a Haar system, so (HS3) follows. This finishes the proof that $\{\nu^u\}_{u\in\mathcal{G}^{(0)}}$ is a $\mathcal{G}$-equivariant r_Z-system. □

9.2.1 Examples

Example 9.19. Every second-countable, proper (i.e. $\mathcal{G} \curvearrowright \mathcal{G}^{(0)}$ properly) and principal groupoid admits a Haar system. In fact, $\mathcal{G}^{(0)}$ implements an equivalence between $\mathcal{G} \ltimes (\mathcal{G}^{(0)}/\mathcal{G}^{(0)}) \cong \mathcal{G}$ and $(\mathcal{G}\backslash\mathcal{G}^{(0)}) \rtimes \mathcal{G}^{(0)} \cong \mathcal{G}\backslash\mathcal{G}^{(0)}$. Since the orbit space $\mathcal{G}\backslash\mathcal{G}^{(0)}$ admits a Haar system trivially (i.e. the pointwise Dirac measure), $\mathcal{G}$ admits a Haar system as well.

Example 9.20. Every transitive (i.e. $\mathcal{G} \curvearrowright \mathcal{G}^{(0)}$ transitively) groupoid admits a Haar system because $\mathcal{G} \sim_{\text{Morita}} \mathcal{G}_u^u$ for any $u \in \mathcal{G}^{(0)}$.

Example 9.21. Recall that $\mathcal{G}[Z] \sim_{\text{Morita}} \mathcal{G}$. So $\mathcal{G}[Z]$ allows for a Haar system if $\mathcal{G}$ does.

Example 9.22. Let $\mathcal{G}$ be a groupoid. A subgroupoid $\mathcal{H} \subseteq \mathcal{G}$ is *wide* if $\mathcal{H}^{(0)} = \mathcal{G}^{(0)}$. If $\mathcal{G}$ has a Haar system and $r_{\mathcal{H}}$ is open, then $\mathcal{H}$ has a Haar system: notice that $\mathcal{H} \cong \mathcal{G} \ltimes \mathcal{G}/\mathcal{H}$ ([Wil19, Corollary 2.50]) and that if $\mathcal{G}$ has a Haar system, then $\mathcal{G} \ltimes \mathcal{G}/\mathcal{H}$ also has ([Wil19, Exercise 2.1.7]).

Example 9.23. Let $\mathcal{G}$ be a groupoid and Z be a free and proper $\mathcal{G}$-space. Assume that r_Z is open. Then Z implements a $(\mathcal{G}, \mathcal{G}^Z)$-equivalence. If $\mathcal{G}$ has a Haar system. Then $\mathcal{G}^Z$ has a Haar system, too.

November 22, 2022

Inverse semigroups and groupoids

Speaker: Jack Ekenstam (Leiden University)

The main references of this talk are [Pat99, Kum84].

Our main target is to show that an étale groupoid can be viewed as an action groupoid of an inverse semigroup on a space.

Definition 10.1. A *semigroup* is a set S together with an associative binary operation. A semigroup S is called an *inverse semigroup* if for all $s \in S$, there exists a unique element $s^* \in S$ with

$$s^*ss^* = s^*, \qquad ss^*s = s.$$

Example 10.2. Let X be a locally-compact Hausdorff space. The set of *partial homeomorphisms* — that is, homeomorphisms $U \xrightarrow{\cong} V$ between open sets $U, V \subseteq X$ — is an inverse semigroup.

Definition 10.3. A *localisation* is a pair (X, S) where X is a locally-compact Hausdorff space, and S is a countable inverse semigroup acting on X by partial homeomorphisms $\{\alpha_s\}_{s \in S}$, such that the domains of α_s' form a topological basis for X.

Example 10.4. A localisation is given as follows. Let $Y \rightrightarrows X$ be a covering space, but with two covering maps π_1 abnd π_2. Take the trivialising covers $\{U_i\}$ and $\{V_j\}$ for π_1 and π_2, and then set $W_ij := U_i \cap V_j \subseteq Y$. Define $\alpha_{ij} := \pi_1(W_{ij}) \to \pi_2(W_{ij})$ by first lifting to Y using π_1, then projects back to X using π_2.

Example 10.5. Let $\mathcal{G}$ be an étale groupoid. Denote by $\operatorname{Bis}(\mathcal{G})$ the set of local bisections of $\mathcal{G}$. This is an inverse semigroup. Let S be a countable inverse semigroup of $\operatorname{Bis}(\mathcal{G})$. Then S acts on $\mathcal{G}^{(0)}$ via $x \cdot s := s^*xs$.[1] In particular, the idempotents of S form a basis for the topology of $\mathcal{G}^{(0)}$.

Let (X, S) be a localisation. Let $U \subseteq X$ be an open subset. Then $C_0(U) \subseteq C_0(X)$ is an ideal. In particular, for each $s \in S$ with domain $\operatorname{Dom} s$, $C_0(\operatorname{Dom} s)$ is an ideal of $C_0(X)$. Define the map

$$\gamma_s \colon C_0(\operatorname{Dom} s^*) \to C_0(\operatorname{Dom} s), \qquad \gamma_s(f)(x) := f(x \cdot s).$$

[1] This is actually an partial anti-homeomorphism. Since the duality between spaces and C*-algebras is contravariant, this allows us to work with local *-isomorphisms of C*-algebras instead of anti-isomorphisms.

Now we may proceed with the noncommutative setting. Assume for simplicity that A is a *unital* C^*-algebra. An inverse semigroup S acts on A by local *-isomorphisms. That is, *-isomorphisms $\alpha_s\colon E_s \to E_{s^*}$ where E_s, E_{s^*} are closed ideals in A. We call (A, S) an *covariant system.*

We wish to describe a crossed product "$A \rtimes S$". Consider the space

$$\mathrm{C}(A,S) := \{\theta\colon S \to A \mid \theta(s) \in E_s \text{ for all } s \text{ and } \theta \text{ has finite support}\}.$$

It is spanned by the space

$$\{(a,s) \in \mathrm{C}(A,S) \mid (a,s)(t) = a\delta_{s,t}\}.$$

We may a *-algebra structure via

$$(a,s)(b,t) := (s[(s^*a)b], st), \qquad (a,x)^* := (s^*a^*, s^*),$$

and extends to the whole of $\mathrm{C}(A,S)$.

It suffices to define a C^*-norm for $\mathrm{C}(A,S)$ and complete it into a C^*-algebra. How to do that?

10.1 Representation of covariant systems

Definition 10.6. A covariant representation of a covariant system (A,S) is a pair (ϕ,π), where:

- ϕ is a *-representation $A \to \mathbb{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$;
- π is a representation of S on H by *partial isometries*,

such that:

- For all $s \in S$, $\operatorname{Dom}\pi(s) = \phi(E_{s^*})\mathcal{H}$.
- For all $a \in E_{s^*}$, $\pi(s^*) = \phi(\gamma_s(a))$.

Definition 10.7. A representation of $\mathrm{C}(A,S)$ is a non-degenerate *-representation Φ of $\mathrm{C}(A,S)$ such that $\Phi(a,e_1) = \Phi(a,e_2)$ for all idempotents e_1, e_2.

Proposition 10.8. *There is a 1-1 correspondence between*

$$\textit{Covariant representations of } (A,S) \qquad \textit{and} \qquad \textit{representations of } \mathrm{C}(A,S).$$

Proof. If (ϕ,π) is a covariant representation of (A,S) on $\mathcal{H}$. Define

$$\Phi\colon \mathrm{C}(A,S) \to \mathbb{B}(\mathcal{H}), \qquad \Phi(b,s) := \phi(b)\pi(s)$$

for $(b,s)(t) := b\delta_{s,t}$. This is a representation of $\mathrm{C}(A,S)$.

Conversely. Let Φ be a representation of $\mathrm{C}(A,S)$. We claim that there exists a covariant representation (ϕ,π) of (A,S) such that $\Phi(b,s) = \phi(b)\pi(s)$. This is given by $\phi(a) := \Phi(a,e)$ and $\pi(s) := \Phi(1,s)$ for any idempotent e. We omit the proof. $\square$

Let $\theta \in \mathrm{C}(A,S)$. Define

$$\|\theta\| := \sup\{\Phi(\theta) \mid \Phi \text{ is a representation of } \mathrm{C}(A,S)\}.$$

This might only be a semi-norm. So we need to divide by $N := \{\theta \in \mathrm{C}(A,S) \mid \|\theta\| = 0\}$ and set

$$A \rtimes S := \overline{\mathrm{C}(A,S)/N}^{\|\cdot\|}.$$

There is still something to be careful with. Namely, we need to have enough representations for $\mathrm{C}(A,S)$, which is not assured in general. But in the case of étale groupoids, such a representation is obtained by the induced representations of groupoids, and we do have enough representations to obtain a C^*-norm and a C^*-algebra.

10.2 Étale groupoid C*-algebras as inverse semigroup crossed products

Theorem 10.9. *Let $\mathcal{G}$ be an étale groupoid. Let S be a countable "additive" inverse subsemigroup of* $\mathrm{Bis}(\mathcal{G})$*. Additivity means*

$$\text{If } A, B \in S \text{ satisfy } A \cup B \in \mathrm{Bis}(\mathcal{G}). \text{ Then } A \cup B \in S.$$

Then $\mathrm{C}_0(\mathcal{G}^{(0)}, S)$ *is a covariant system, and* $\mathrm{C}^*(\mathcal{G}) \cong \mathrm{C}_0(\mathcal{G}^{(0)}) \rtimes S$.

Conversely. Let X be a space and S be an inverse semigroup acting on X by local (anti)-homeomorphisms. We want to construct a groupoid such that its C*-algebra is isomorphic to $\mathrm{C}_0(X) \rtimes S$. Obvious attempt is

$$\Xi := \{(x, s) \in X \times S \mid x \in \operatorname{Dom} s,\ s \in S\} \rightrightarrows \Xi^{(0)} := X$$

with

$$s(x, s) := x, \qquad r(x, s) := x \cdot s, \qquad (x, s)^{-1} := (x \cdot s, s^*).$$

But this forces $(x, s) = (x, stt^*)$ which is not true in general.

This happens precisely because composing local homeomorphisms will contract the domains. To solve this, we need to replace Ξ by a quotient of it.

Define the equivalence relation $\sim$ on Ξ:

$$(x, s) \sim (y, t) \quad \text{iff} \quad x = y \text{ and there exists an idempotent } e \in S \text{ such that } x \in \operatorname{Dom} e \text{ and } es = et.$$

Set

$$\mathcal{G}(X, S) := \Xi/\!\sim .$$

Easily one shows that this is a groupoid.

Let (X, S) be a localisation. Define an equivalence relation on S:

$$s \sim t \quad \text{iff} \quad ss^* = tt^*, \text{ and for all } x \in \operatorname{Dom} ss^*, \text{ there exists an idempotent } e \in S \text{ such that } x \in \operatorname{Dom} e \text{ and } es = et.$$

Definition 10.10. A localisation (X, S) is *additive* if the followings hold:

1. $s \sim t$ implies $s = t$.

2. If s and t are compatible (that is, $\operatorname{Dom} s \cap \operatorname{Dom} t \neq \emptyset$ and $s = t$ on $\operatorname{Dom} s \cap \operatorname{Dom} t$), and $f = ss^* \cup tt^* \subseteq X$. Then there exists $w \in S$ such that $f = ww^*$ and $x \in \operatorname{Dom} ss^*$, such that there exists an idempotent $e \in S$ with $x \in \operatorname{Dom} e$ and $es = ew$.

Theorem 10.11. *If (X, S) is an additive localisation. Then* $\mathrm{C}^*(\mathcal{G}(X, S)) \cong \mathrm{C}_0(X) \rtimes S$.

November 29, 2022

Tangent groupoids and index theory

Speaker: Yuezhao Li (Leiden University)

The main references are [Con94, Lan03]. Some useful computation is carried over in [Hig10].

11.1 Overview

11.1.1 What is index theory?

An index theorem, roughly speaking, is an equation of the form:

$$\text{Analytic index} = \text{Topological index}.$$

Index theory is diverse in scope now, but at the very beginning people were concerned with the index of an elliptic operator on a manifold. An elliptic operator is, roughly speaking, a Fredholm operator. The *analytic index* of an elliptic operator P is just its Fredholm index

$$\mathrm{Index}(P) := \dim\ker P - \dim\operatorname{coker} P.$$

It is known for long that the Fredholm index of a bounded Fredholm operator is in some sense "topological": it is invariant under homotopies inside the space of Fredholm operators (equipped with the subspace topology from bounded operators). Then Gelfand questioned whether there is a certain "topological" formula for the index of an elliptic operator. The answer is the well-known Atiyah–Singer index theorem.

Theorem 11.1 (Atiyah–Singer)**.** *Let P be an elliptic operator on a closed manifold M. Then*

$$\text{a-ind}(P) = \text{t-ind}(P),$$

where

$$\text{a-ind}(P) := \mathrm{Index}(P);$$
$$\text{t-ind}(P) := \{\mathrm{Ch}(\sigma(P)) \cup \mathrm{Td}(\pi^* TX \otimes \mathbb{C})\}[T^*M].$$

Here $\sigma(P)$ is the principal symbol of P, $\mathrm{Ch}(\sigma(P)) \in \mathrm{H}^{\mathrm{even}}(T^*X)$ is the class in even cohomology associated to the Chern character of $\sigma(P)$, $\mathrm{Td}(\pi^*TM \otimes \mathbb{C})$ is the Todd class of the complex vector bundle $\pi^*TM \otimes \mathbb{C}$, and $[T^*M]$ is the fundamental class in the top cohomology of T^*M.

Example 11.2.

1. If $P = \mathrm{d} + \mathrm{d}^* \colon \mathrm{C}^\infty(M, \Lambda^{\mathrm{even}} T^*M) \to \mathrm{C}^\infty(M, \Lambda^{\mathrm{odd}} T^*M)$. Then Atiyah–Singer index theorem recovers Gauß–Bonnet theorem.
2. If $P = \overline{\partial} + \overline{\partial}^* \colon \mathrm{C}^\infty(M, \Lambda^{\mathrm{even}} T^{1,0}M) \to \mathrm{C}^\infty(M, \Lambda^{\mathrm{odd}} T^{1,0}M)$. Then Atiyah–Singer index theorem recovers Riemann–Roch theorem.

A complete statement of this whole story will consume several hours. So I only sketch the essential ingredients for understanding this result. Atiyah and Singer already realised that both the analytic and topological indices ought to be understood as a group homomorphism

$$\mathrm{K}(T^*M) \to \mathbb{Z}$$

from the K-theory of the topological space T^*M to the integers.

Let P be an elliptic operator on a closed manifold M. This means, P is a linear map

$$P \colon \mathrm{C}^\infty(M, E) \to \mathrm{C}^\infty(M, E)$$

acting on the smooth sections of a vector bundle $E \to M$, satisfying some extra conditions. With these extra conditions, P is a *pseudo-differential operator*. Denote the set of pseudo-differential operators by $\Psi(M, E)$. The symbol of P is an element $\sigma(P) \in \mathrm{C}(T^*M, \mathbb{M}_n(\mathbb{C}))$. Being elliptic means that $\sigma(P)$ is invertible outside the zero section of T^*M. Then $\sigma(P)$ represents a class in $\mathrm{K}(T^*M)$. Atiyah and Singer showed that the Fredholm index $\mathrm{Index}(P)$ depends only on the class of $\sigma(P)$ inside $\mathrm{K}(T^*M)$. So the analytic index is a map

$$\text{a-ind} \colon \mathrm{K}(T^*M) \to \mathbb{Z}, \quad [\sigma(P)] \mapsto \mathrm{Index}(P). \tag{3}$$

The topological index needs a another machineary called *Thom isomorphism.* If $V \to X$ is a complex vector bundle, then there is an isomorphism

$$\mathrm{K}(X) \cong \mathrm{K}(V).$$

Consider a proper embedding $M \hookrightarrow \mathbb{R}^n$. The tubular neighbourhood theorem claims that there is a tubular neighbourhood N of M in $\mathbb{R}^n$, such that N is homeomorphic to the normal bundle of M in $\mathbb{R}^n$. Identify N with the normal bundle of M in $\mathbb{R}^n$. Then T^*N is a vector bundle over T^*M which allows a complex structure. So the Thom isomorphism follows and we have

$$\mathrm{K}(T^*M) \cong \mathrm{K}(T^*N).$$

Since T^*N is an open subset in $\mathbb{R}^n$. There is an "extension by 0" (or excision) map $T^*N \hookrightarrow T^*\mathbb{R}^n$ which induces a map $\mathrm{K}(T^*N) \to \mathrm{K}(T^*\mathbb{R}^n)$ in K-theory. Composing with the Thom isomorphism and Bott periodicity $\mathrm{K}(T^*\mathbb{R}^n) \cong \mathrm{K}(\mathrm{pt}) \cong \mathbb{Z}$ yields the topological index map

$$\text{t-ind}\colon\ \mathrm{K}(T^*M) \xrightarrow[\cong]{\text{Thom isomorphism}} \mathrm{K}(T^*N) \xrightarrow{\text{excision}} \mathrm{K}(T^*\mathbb{R}^n) \xrightarrow[\cong]{\text{Bott periodicity}} \mathrm{K}(\mathrm{pt}) \cong \mathbb{Z}. \tag{4}$$

Atiyah–Singer index theorem can be rephrased as follows:

Theorem 11.3. *The following diagram commutes:*

$$\begin{tikzcd}
\mathrm{K}(T^*\mathbb{R}^{2n}) \arrow[dr, "\cong"] & \\
\mathrm{K}(T^*M) \arrow[u, "\text{Thom}"] \arrow[r] & \mathbb{Z} \\
\Psi(M,E) \arrow[u, "\sigma"] \arrow[ur, "\text{Index}"'] &
\end{tikzcd}$$

The proof due to Atiyah and Singer is based on the strategy that there is a unique such map satisfying the properties of both indices. This is an elegant proof, but have some tiny drawbacks.

- It is not *conceptually* clear why the analytic index should also be viewed as a map $\mathrm{K}(T^*M) \to \mathbb{Z}$. To show this one has to prove that the analytic index depends only on the symbol, and that determines a class in $\mathrm{K}(T^*M)$.
- The topological index depends on an embedding of M into $\mathbb{R}^n$ and on the Thom isomorphism. Both are inevitable even in the modern proofs of Atiyah–Singer index theorem, but this original construction is not *natural.*

Question 11.4. Is there a natural way to realise both the analytic and the topological indices?

I believe that Alain Connes introduced his tangent groupoids with these doubts in mind. He proposed a very elegant proof (c.f. [Con94, Section II.5]) of the Atiyah–Singer index theorem based on this beautiful construction.

11.1.2 Quantisation

Another interesting motivation for the tangent groupoid comes from physics: the *quantisation.* The subject of C^*-*algebraic deformation quantisation* (also called *strict deformation quantisation*) started after Rieffel's work [Rie89]. The relation between quantisation and the tangent groupoids is surveyed in [Lan03].

Let M be a manifold describing the physical space. Classical physics and quantum physics describe the physics on M using different mathematical frameworks, described in the table below.

A quantisation is a certain way to bridge the two mathematical frameworks. A C^*-algebraic deformation quantisation is a choice of quantisation. In this approach, we try to find a "deformation" from the C^*-algebras $\mathrm{C}_0(T^*M)$ to a suitable C^*-subalgebra of $\mathbb{B}(L^2(M))$. The following well-structured definitions comes from [Ger16].

	State space	Observables	Physical quantities
Classical physics	Poisson manifold T^*M	$f \in \mathrm{C}_0(T^*M, \mathbb{R})$	$f(x)$
Quantum physics	Hilbert space $L^2(M)$	Self-adjoint operator T on $L^2(M)$	$\langle \psi, T\psi \rangle$

Table 1: Comparision between classical physics and quantum physics

Definition 11.5. Let X be a locally-compact Hausdorff space. A continuous field of C^*-algebras over X consists of:

- a C^*-algebra A;
- a family of C^*-algebras $\{A_x\}_{x\in X}$ indexed by X;
- a family of *-homomorphisms $\{\pi_x \colon A \to A_x\}_{x\in X}$ indexed by X,

such that for all $a \in A$:

1. the map $x \mapsto \|\pi_x(a)\|$ is in $\mathrm{C}_0(X)$;
2. $\|a\| = \sup_{x\in X} \|\pi_x(a)\|$;
3. for all $f \in \mathrm{C}_0(X)$, there exists an element $a' \in A$ such that $\pi_x(a') = f(x)\pi_x(a)$ for all $x \in X$.

Definition 11.6 (Rieffel, Landsman)**.** A C^*-algebraic quantisation of a Poisson manifold P consists of the following data:

1. A continuous field of C^*-algebras $A = (A_t)_{t\in[0,1]}$ over $[0,1]$ with $A_0 = \mathrm{C}_0(P)$.
2. A Poisson algebra $\mathcal{A}_0$ dense in A_0.
3. A section $Q \colon \mathcal{A}_0 \to A$ satisfying

$$Q_0(f) = f, \quad Q_t(f^*) = Q_t(f)^*, \quad \lim_{t\to 0} \left\| \frac{i}{t}[Q_t(f), Q_t(g)] - Q_t(\{f,g\}) \right\| = 0$$

   for all $f, g \in \mathcal{A}_0$. Here $Q_t := \pi_t \circ Q$.

Recall that an elliptic operator P on a closed manifold M generates a symbol $\sigma(P) \in \mathrm{C}_0(T^*M, \mathbb{M}_n(\mathbb{C}))$. It is attempting to regard $\sigma(P)$ as a classical observable. Then we are motivated to study a suitable deformation quantisation for the manifold T^*M. The Weyl–Moyal quantisation aims at quantising $\mathrm{C}_0(T^*M)$ with a continuous field of C^*-algebra $(A_t)_{t\in[0,1]}$ such that $A_0 = \mathrm{C}_0(T^*M)$ and $A_t = \mathbb{K}(L^2(M))$ for $t > 0$. Since $\mathrm{K}_0(\mathbb{K}(L^2(M))) \cong \mathbb{Z}$. We would wish to have a map $\mathrm{K}_0(A_0) \to \mathrm{K}_0(A_1)$ which recovers the Atiyah–Singer index map.

Question 11.7*.* What is C^*-algebraic quantisation involved here? Namely, what is the C^*-algebra A? How could we recover the map $\mathrm{K}_0(A_0) \to \mathrm{K}_0(A_1)$?

It turns out that:

- The C^*-algebra A in this quantisation is the groupoid C^*-algebra of the tangent groupoid $\mathbb{T}M$.
- The tangent groupoid $\mathbb{T}M$ is a special case of a deformation groupoid. A deformation groupoid has a *deformation index*, which in the case of tangent groupoids recovers the Atiyah–Singer analytic index.

11.1.3 Tangent groupoids

Definition 11.8. A *Lie groupoid* is a groupoid $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} \mathcal{G}^{(0)}$ such that $\mathcal{G}$ and $\mathcal{G}^{(0)}$ are smooth manifolds, the multiplication and inverse maps are smooth, and the range map r and the source map s are submersions.

Definition 11.9. A *deformation groupoid* is a Lie groupoid of the form

$$\mathcal{G}_1 \times \{0\} \sqcup \mathcal{G}_2 \times (0,1] \rightrightarrows \mathcal{G}^{(0)} \times [0,1],$$

where $\mathcal{G}_1 \rightrightarrows \mathcal{G}^{(0)}$ and $\mathcal{G}_2 \rightrightarrows \mathcal{G}^{(0)}$ are Lie groupoids over the same unit space $\mathcal{G}^{(0)}$.

Roughly speaking, a deformation groupoid should be thought of as glueing two Lie groupoids together along the interval $[0,1]$, in a smooth way.

Definition 11.10. Let M be a closed manifold. The tangent groupoid $\mathbb{T}M$ is the deformation groupoid of the tangent bundle $TM \rightrightarrows M$ and the pair groupoid $M \times M \rightrightarrows M$. That is,

$$\mathbb{T}M := TM \times \{0\} \sqcup M \times M \times (0,1] \rightrightarrows M \times [0,1].$$

Clearly TM and $M \times M$ are Lie groupoids. We need to make sure that $\mathbb{T}M$ is a Lie groupoid. The smooth structure of $\mathbb{T}M$ is obtained as follows. Choose any Riemannian metric on M. Require that the map

$$\begin{array}{rcll} TM \times [0,1] & \to & \mathbb{T}M & \\ (x,v,0) & \mapsto & (x,v,0) & \\ (x,v,t) & \mapsto & (x, \exp_x(-tv), t), & \text{for } t > 0 \end{array}$$

is a local diffeomorphism. This also gives a topology to $\mathbb{T}M$.

11.2 The index theorem in tangent groupoids

11.2.1 The analytic index

Now we reformulate the indices using the language of the tangent groupoids, and prove the index theorem in this framework.

Definition and Lemma 11.11. Let $\mathcal{G} := \mathcal{G}_1 \times \{(0)\} \sqcup \mathcal{G}_2 \times \{(0,1]\} \rightrightarrows \mathcal{G}^{(0)} \times [0,1]$ be a deformation groupoid. Then there is a short exact sequence of C*-algebras

$$\mathrm{C}^*(\mathcal{G}_2 \times (0,1]) \rightarrowtail \mathrm{C}^*(\mathcal{G}) \overset{\mathrm{ev}_0}{\twoheadrightarrow} \mathrm{C}^*(\mathcal{G}_1).$$

In particular, the C*-algebra $\mathrm{C}^*(\mathcal{G}_2 \times (0,1]) \cong \mathrm{C}^*(\mathcal{G}_2) \otimes \mathrm{C}_0(0,1]$ is contractible. So there is a KK-equivalence $[\mathrm{ev}_0] \in \mathrm{KK}(\mathrm{C}^*(\mathcal{G}), \mathrm{C}^*(\mathcal{G}_1))$ and an isomorphism $[\mathrm{ev}_0]\colon \mathrm{K}_*(\mathrm{C}^*(\mathcal{G})) \xrightarrow{\cong} \mathrm{K}_*(\mathrm{C}^*(\mathcal{G}_1))$.

The *deformation index* associated to the deformation groupoid $\mathcal{G}$ is the element

$$[\mathrm{ev}_0]^{-1} \otimes [\mathrm{ev}_1] \in \mathrm{KK}(\mathrm{C}^*(\mathcal{G}_1), \mathrm{C}^*(\mathcal{G}_2)),$$

or the map

$$\mathrm{K}_*(\mathrm{C}^*(\mathcal{G}_1)) \xrightarrow{[\mathrm{ev}_0]^{-1}} \mathrm{K}_*(\mathrm{C}^*(\mathcal{G})) \xrightarrow{[\mathrm{ev}_1]} \mathrm{K}_*(\mathrm{C}^*(\mathcal{G}_2)).$$

In the case of tangent groupoids. We have a map

$$\text{a-ind}\colon \mathrm{K}_0(\mathrm{C}^*(TM)) \xrightarrow{[\mathrm{ev}_0]^{-1}} \mathrm{K}_0(\mathrm{C}^*(\mathbb{T}M)) \xrightarrow{[\mathrm{ev}_1]} \mathrm{K}_0(\mathrm{C}^*(M \times M)). \tag{5}$$

Notice that $\mathrm{C}^*(TM) \cong \mathrm{C}_0(T^*M)$ (this is Poincaré duality in KK-theory), and $\mathrm{C}^*(M \times M) \cong \mathbb{K}(L^2(M))$. The map (5) can thus be identified with a map $\mathrm{K}(T^*M) = \mathrm{K}_0(\mathrm{C}_0(T^*M)) \to \mathrm{K}_0(\mathbb{K}(L^2(M))) \cong \mathbb{Z}$.

Proposition 11.12 ([MP97])**.** *The map* (5) *coincides with the Atiyah–Singer analytic index* (3)*.*

11.2.2 The topological index

How about the topological index? We need to utilise Thom isomorphism $\mathrm{K}(T^*M) \cong \mathrm{K}(T^*N)$ in an essential fashion, so as to send $\mathrm{K}(T^*M)$ to $\mathrm{K}(T^*\mathbb{R}^n)$. For this purpose let us have a closer look at T^*N.

Since N is an open subset of $\mathbb{R}^n$. We have $T^*N \cong N \times \mathbb{R}^n$. Identify N with the normal bundle of M. Then

$$N = \sqcup_{x\in M} N_x \cong \sqcup_{x\in M} T_x\mathbb{R}^n/T_xM = \sqcup_{x\in M}\mathbb{R}^n/T_xM.$$

(Here $\sqcup$ only denotes the set-theoretic union).

Then

$$T^*N \cong N \times \mathbb{R}^n \cong \sqcup_{x\in M}\mathbb{R}^n/T_xM \times \mathbb{R}^n \cong \sqcup_{x\in M}\mathbb{R}^{2n}/T_xM.$$

This indicates us that, we should view T^*N as the quotient of $\mathbb{R}^{2n}$ under a suitable action of the groupoid TM, or rather $\mathbb{T}M$. We make this precise.

Let $j\colon M \hookrightarrow \mathbb{R}^n$ be a proper embedding. Extend to $j\colon M \hookrightarrow T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ via $x \mapsto (j(x), 0)$. This induces a groupoid homomorphism

$$\begin{aligned} h\colon \mathbb{T}M &\to \mathbb{R}^{2n} \\ h(x,v,0) &:= j_*(v) \\ h(x,y,t) &:= \frac{1}{t}\left(j(x)-j(y)\right) \quad \text{for } t>0. \end{aligned}$$

We want to define an action of $\mathbb{T}M$ on $\mathbb{R}^{2n}$ using h. Groupoid actions are fibred. So we need to replace $\mathbb{R}^{2n}$ by the fibred space $\mathbb{T}M^{(0)} \times \mathbb{R}^{2n} = M \times [0,1] \times \mathbb{R}^{2n}$. The tangent groupoid $\mathbb{T}M$ acts on $\mathbb{T}M^{(0)} \times \mathbb{R}^{2n}$ by

$$\gamma \cdot (s(\gamma), v) := (r(\gamma), v + h(\gamma)).$$

This is a free and proper action. Hence the action groupoid $\mathbb{T}M \ltimes (\mathbb{T}M^{(0)} \times \mathbb{R}^{2n})$ is Morita equivalent to the orbit space

$$\mathbb{T}M\backslash\mathbb{T}M^{(0)} \times \mathbb{R}^{2n}.$$

What is the orbit space?

- At $t = 0$. The action is given by

$$TM \curvearrowright M \times \mathbb{R}^{2n}, \qquad (x,a) \xmapsto{v\in T_xM} (x, j_*(x)+a).$$

  The fibre orbit space at x is just $N_x \times \mathbb{R}^n$. So the total space is $N \times \mathbb{R}^n \cong TN$.

- At $t > 0$. The action is given by

$$M \times M \curvearrowright M \times \mathbb{R}^{2n}, \qquad (x,a) \xmapsto{(y,x)\in M\times M} (y,a).$$

  The action is transitive on the M entry, and leaves the $\mathbb{R}^{2n}$ entry unchanged. So the orbit space is $\mathbb{R}^{2n}$.

Hence the orbit space is $TN \times \{0\} \sqcup \mathbb{R}^{2n} \times (0,1]$. This is a special case of the *deformation of normal cone.*

The groupoids $\mathbb{T}M \ltimes (\mathbb{T}M^{(0)} \times \mathbb{R}^{2n})$ and $\mathbb{T}M\backslash\mathbb{T}M^{(0)} \times \mathbb{R}^{2n}$ are Morita equivalent. So they have Morita–Rieffel equivalent C*-algebras and isomorphic K-theory

$$\mathrm{K}_0(\mathbb{T}M \ltimes (\mathbb{T}M^{(0)} \times \mathbb{R}^{2n})) \cong \mathrm{K}_0(\mathbb{T}M\backslash\mathbb{T}M^{(0)} \times \mathbb{R}^{2n}).$$

Connes observed that:

Lemma 11.13 (Connes)**.** *Let α be the action $\mathbb{R}^n \curvearrowright \mathrm{C}^*(\mathbb{T}M)$ given by*

$$\alpha_X f(\gamma) := \mathrm{e}^{i(X\cdot h(\gamma))} f(\gamma).$$

Then the crossed product $\mathrm{C}^*(\mathbb{T}M) \rtimes_\alpha \mathbb{R}^{2n}$ is isomorphic to $\mathrm{C}^*(\mathbb{T}M \ltimes (\mathbb{T}M^{(0)} \times \mathbb{R}^{2n}))$.

The lemma, together with Connes–Thom isomorphism

$$\mathrm{K}_0(\mathrm{C}^*(\mathbb{T}M)) \rtimes_\alpha \mathbb{R}^{2n} \cong \mathrm{K}_0(\mathrm{C}^*(\mathbb{T}M)),$$

provides an isomorphism

$$\mathrm{K}_0(\mathrm{C}^*(\mathbb{T}M)) \cong \mathrm{K}_0(TN \times \{0\} \sqcup \mathbb{R}^{2n} \times (0,1]). \tag{6}$$

This isomorphism will be used in the proof of the index theorem.

Now we are able to describe the topological index as follows.

- Since the space $\mathbb{R}^{2n} \times (0,1]$ is properly contractible. We have an isomorphism in K-theory $\mathrm{K}_0(TN) \cong \mathrm{K}_0(TN \times \{0\} \sqcup \mathbb{R}^{2n} \times (0,1])$.
- The excision $\mathrm{K}_0(\mathrm{C}_0(TN)) \to \mathrm{K}_0(\mathrm{C}_0(\mathbb{R}^{2n}))$ then defines a map $\mathrm{K}_0(TN \times \{0\} \sqcup \mathbb{R}^{2n} \times (0,1]) \to \mathrm{K}_0(\mathrm{C}_0(\mathbb{R}^{2n}))$. Let us denote this map by $[\mathrm{ev}_1]$.
- The isomorphism (6) composed with $[\mathrm{ev}_1]$ yields a map $\mathrm{K}_0(\mathrm{C}^*(\mathbb{T}M)) \to \mathrm{K}_0(\mathrm{C}_0(\mathbb{R}^{2n}))$.
- Composed with the isomorphism $[\mathrm{ev}_0]\colon \mathrm{K}_0(\mathrm{C}_0(T^*M)) \cong \mathrm{K}_0(\mathrm{C}^*(TM)) \xrightarrow{\cong} \mathrm{K}_0(\mathrm{C}^*(\mathbb{T}M))$, we obtain the following map

$$\begin{aligned}\text{t-ind} := \mathrm{K}_0(\mathrm{C}_0(T^*M)) \cong \mathrm{K}_0(\mathrm{C}^*(TM)) &\xrightarrow[\cong]{[\mathrm{ev}_0]^{-1}} \mathrm{K}_0(\mathrm{C}^*(\mathbb{T}M)) \\ &\xrightarrow[\cong]{(6)} \mathrm{K}_0(TN \times \{0\} \sqcup \mathbb{R}^{2n} \times (0,1]) \xrightarrow{[\mathrm{ev}_1]} \mathrm{K}_0(\mathrm{C}_0(\mathbb{R}^{2n})) \cong \mathbb{Z}. \end{aligned} \tag{7}$$

Proposition 11.14. *The Atiyah–Singer topological index (4) coincides with the map (7).*

11.2.3 Proof of the index theorem

The proof of the index theorem is therefore quite tautological, once we have identified the analytic and topological indices with the correct maps in the tangent groupoid.

Proposition 11.15. *The following diagram commutes:*

$$\begin{tikzcd} \mathrm{K}_0(\mathrm{C}^*(TM)) \arrow[r, "{[\mathrm{ev}_0]^{-1}}"] \arrow[d, "\cong"] & \mathrm{K}_0(\mathrm{C}^*(\mathbb{T}M)) \arrow[r, "{[\mathrm{ev}_1]}"] \arrow[d, "\cong"] & \mathrm{K}_0(\mathrm{C}^*(M \times M)) \arrow[d, "\cong"] \\ \mathrm{K}_0(\mathrm{C}_0(TN)) \arrow[r, "{[\mathrm{ev}_0]^{-1}}"] & \mathrm{K}_0(\mathrm{C}_0(TN \times \{0\} \sqcup \mathbb{R}^{2n} \times (0,1]) \arrow[r, "{[\mathrm{ev}_1]}"] & \mathrm{K}_0(\mathrm{C}_0(\mathbb{R}^{2n})) \end{tikzcd} \tag{8}$$

The vertical isomorphisms are: Thom isomorphism, the isomorphism in (6) and the isomorphism $\mathrm{K}_0(\mathrm{C}^*(M \times M)) = \mathrm{K}_0(\mathbb{K}(L^2(M))) \xrightarrow{\cong} \mathrm{K}_0(\mathbb{C}) \xrightarrow{\cong} \mathrm{K}_0(\mathrm{C}_0(\mathbb{R}^{2n}))$.

Finally, the Atiyah–Singer index theorem is proved by noticing that the analytic index is just the composition of maps on the first row, while the topological index is the map $\mathrm{K}_0(\mathrm{C}^*(TM)) \xrightarrow{\cong} \mathrm{K}_0(\mathrm{C}_0(TN))$ followed by the maps on the second row.

Theorem 11.16. a-ind = t-ind*, where* a-ind *is defined as in* (5) *and* t-ind *is defined as in* (7)*.*

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