Noncommutative geometry of foliations

NCG-Leiden

Notes taken by Yuezhao Li
Mathematical Institute, Leiden University

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Foliations: motivations, definitions and examples

Speaker: Yuezhao Li (Leiden University)

1.1 Motivations

1.1.1 From Frobenius theorem to foliations

A *foliation* on a manifold is, roughly speaking, a decomposition of it into immersed submanifolds (called the *leaves* of the foliation), such that the leaves fit together nicely. The theory of foliations is a tremendous component in modern differential geometry, contributed by several big names: Ehresmann, Reeb, Haefliger, Novikov, Thurston, Molino, Sullivan, Connes and many others.

The idea of foliations comes from a much older task: solving differential equations. This can be best by the celebrated Frobenius theorem[1]. Let M be a smooth manifold. Let E be a smooth rank-k distribution over M, that is, a rank-k subbundle of TM. An *integral manifold* of E is an immersed submanifold $N \subseteq M$ such that $T_pN = E_p$ for every $p \in N$. A distribution E is said to be *integrable* if every $x \in M$ is contained in an integral manifold of E. A distribution E is said to be *involutive* if $\Gamma(E)$, the space of smooth sections of E, is a Lie subalgebra of $\Gamma(TM)$ under the Lie bracket.

Theorem 1.1 (Frobenius)**.** *A distribution is integrable iff it is involutive.*

The Frobenius theorem is a vast generalisation of the classical existence theorems in differential equations, that is, the existence (and uniqueness) of solution (integral curve) of linear partial differential equations. The involutivity of a distribution can be viewed as its "local integrability", which by Frobenius theorem implies the existence of an integral manifold at every point. In modern language, the Frobenius theorem says that an involutive distribution gives a foliation of the base manifold, whose leaves are those maximal connected integral submanifolds.

Geometry serves as a big source of foliations. Every submersion of manifolds define a foliation on the total space, whose leaves are the connected components of fibres (Example 1.7). In particular, a fibre bundle defines a foliation of the total space. Another interesting example is the symplectic foliation of a Poisson manifold, which is generated by its Hamiltonian vector fields.

Dynamical systems also supply many interesing instances of folitions. Let G be a Lie group that acts on a smooth manifold M. Suppose that the stabliser groups have constant dimension. Then M is foliated by the connected components of G-orbits (Example 1.8). The dimension condition is necessary here as otherwise the leaves will not have the same dimension. That would be an instance of a singular foliation and an orbifold.

1.1.2 Why noncommutative geometry?

Let $(M, \mathcal{F})$ be a foliated manifold, that is, a manifold together with a foliation thereon. We wish to study the geometry of two "spaces":

1. The geometry of the leaves (generic fibre) F.
2. The geometry of the space of leaves $M/\mathcal{F}$, obtained from the equivalence relation generated by leaves.

However, there is several technical difficulty for us to study them, even in many simplest examples.

- A leaf F of a foliation can be usually non-compact. For example, every leaf of the Kronecker foliation (Example 1.17) is diffeomorphic to $\mathbb{R}$. This non-compactness, in many cases, obstructs us to pass from local (leaves) to global (the space M).

[1] Despite of the name, this theorem is indeed not proven by Frobenius, at whose age the concept of manifolds were not yet established.

- The quotient space $M/\mathcal{F}$ is usually badly behaved as a topological space. In the case of the Kronecker foliation, every leaf is dense in $\mathbb{T}^2$. So the quotient space is necessarily non-Hausdorff.
- We would like to define a measure on the quotient space, so that we can (1) talk about integrals (2) make the word "generic" precise, in the sense of probability. But $M/\mathcal{F}$ is also quite terrible as a measurable space due to ergodicity. That is, every measurable function of $M/\mathcal{F}$ is necessarily constant almost everywhere. Hence $L^p(M/\mathcal{F}) = \mathbb{C}$ for every $p \in [1, \infty]$.

These force us to seek other geometric objects to replace $M/\mathcal{F}$, and leads to Connes' C^*-algebras of foliations and noncommutative geometry. Roughly, these can be summarised as follows:

- The leaves of a foliation generate an equivalence relation of the manifold, and hence a *groupoid* $\mathrm{Hol}(M, \mathcal{F})$, called the holonomy groupoid of the foliation. This is a noncommutative object which encodes the topology of a foliation.
- By some standard techniques introduced by Renault, one can define a C^*-algebra $\mathrm{C}^*(M, \mathcal{F})$ of this groupoid. This is known as Connes' C^*-algebra of foliation.
- A measure on $M/\mathcal{F}$ should be by a measure on M that is holonomy invariant. This is called a transverse measure of the foliation. With such as input, one may get a trace on $\mathrm{C}^*(M, \mathcal{F})$ and hence an "index map" $\mathrm{K}_*(\mathrm{C}^*(M, \mathcal{F})) \to \mathbb{R}$. This reveals the index theory of a foliation.

Thus we have the following nice machine for foliations:

$$\boxed{\begin{array}{c}\text{Foliated manifold}\\ (M,\mathcal{F})\end{array}} \Longrightarrow \boxed{\begin{array}{c}\text{Holonomy groupoid}\\ \mathrm{Hol}(M,\mathcal{F})\end{array}} \Longrightarrow \boxed{\begin{array}{c}\text{Foliation } \mathrm{C}^*\text{-algebra}\\ \mathrm{C}^*(M,\mathcal{F})\end{array}}$$

However, a new question emerges: what are the correct *morphisms*? Note that assigning the groupoid C^*-algebra to a groupoid is not functorial. For example, topological spaces and topological groups are both topological groupoids. Whereas groupoid C^*-algebras for spaces are contravariantly functorial, those for groupois are covariantly functorial. So homomorphisms of groupoids are not satisfying, and we will replace them by a more general concepts of morphisms: the *groupoid correspondences*. These were first introduced in [3] and later referred to as "Hilsum–Skandalis morphisms". A brief account of this is Bram's talk at the groupoid seminar [6].

1.2 Definitions and first examples

1.2.1 By foliation atlas

Definition 1.2 (first definition)**.** Let M be a smooth manifold of dimension n. A (regular) *foliation atltas* of codimension q is an atlas of M, each chart of the form

$$(U_i,\ \varphi_i \colon U_i \to \mathbb{R}^{n-q} \times \mathbb{R}^q),$$

such that the transition functions

$$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} \colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

have the form

$$\varphi_{ij}(x, y) = (g_{ij}(x, y), h_{ij}(y)), \qquad x \in \mathbb{R}^{n-q},\ y \in \mathbb{R}^q$$

for some smooth functions $g_{ij} \colon \mathbb{R}^n \to \mathbb{R}^{n-q}$ and $h_{ij} \colon \mathbb{R}^q \to \mathbb{R}^q$.

A (regular) *foliation* of codimension q of M is a maximal foliation atlas $\mathcal{F}$ of M. We write $\operatorname{codim} \mathcal{F}$ for the codimension of $\mathcal{F}$.

A *foliated manifold* is a pair $(M, \mathcal{F})$ of a manifold M together with a foliation $\mathcal{F}$ of it.

Definition 1.3. A *plaque* of a foliation chart $(U_i,\ \varphi_i\colon U_i \to \mathbb{R}^{n-q} \times \mathbb{R}^q)$ is a connected component of the submanifold $\varphi^{-1}(\mathbb{R}^{n-q} \times \{y\})$ for some $y \in \mathbb{R}^q$. Two points $x, y \in M$ belong to the same *leaf* if there exist a sequence of foliation charts $U_1, \ldots, U_k$ and a sequence of points $x = p_1, \ldots, p_k = y$ with $p_i \in U_i$ and such that p_{i-1} and p_i belong to the same plaque.

A *leaf* is a collection of points that belong to the same leaf.

The *space of leaves* is the quotient space $M\mathcal{F}$ obtained by the equivalence relation generated by the leaves.

One can easily show that leaves are immersed submanifolds of M:

Proposition 1.4. *Let $(M, \mathcal{F})$ be a codimension-q foliated manifold. Then every leaf is an immersed submanifold of M of dimension $n - q$.*

Proof. Choose $x \in M$ and let F be the leaf of x. Every plaque is a smooth chart of dimension $n - q$. If x belongs to a plaque, then every point of the plaque belongs to F by definition. This holds for other plaques containing a point that belongs to the same leaf with x. Therefore F is covered by these plaques and they are smoothly compatible. The plaques are immersed submanifolds of M, and this holds for F as well. □

Definition 1.5. Let $(M, \mathcal{F})$ and $(N, \mathcal{G})$ be foliated manifolds. A morphism between them is a smooth map $f\colon M \to N$ which preserves the foliation. That means, every leaf of $\mathcal{F}$ is mapped into a leaf of N. We also say such a map is foliated.

Example 1.6 (trivial foliation). The space $\mathbb{R}^n$ admits a trivial foliation: the foliated atlas consists of a single chart $\mathrm{id}\colon \mathbb{R}^n \to \mathbb{R}^{n-q} \times \mathbb{R}^q$. Similarly, any linear isomorphism $A\colon \mathbb{R}^n \to \mathbb{R}^n$ determines a foliation. The leaves are given by $A^{-1}(\mathbb{R}^{n-q} \times \{y\})$.

Example 1.7 (submersions). Let $\pi\colon E \to M$ be a submersion. This defines a foliation of E whose leaves are connected components of the fibres of π. The codimension of the foliation equals $\dim M$. If every fibre is connected, then $E/\mathcal{F} = M$. If some fibres are not connected, then $E/\mathcal{F}$ is a quotient of M which is necessarily non-Hausdorff.

In particular, a rank-k vector bundle $\pi\colon E \to M$ gives a foliation of E whose leaves are given by $\pi^{-1}(x) \simeq \mathbb{R}^k$.

Example 1.8 (Lie group actions). Another important source of interesting foliations is smooth dynamical systems. Let G be a Lie group which acts on a smooth manifold M. We wish to foliate M into the connected components of the orbits under the G-action. But this cannot be always true. For example, consider the canonical action of $\mathrm{GL}(2, \mathbb{R})$, or $\mathrm{SL}(2, \mathbb{R})$, or $\mathbb{C}^*$ (viewed as a submanifold of $\mathbb{R}^2$ equipped with its usual product) on $\mathbb{R}^2$. The orbits are $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$. However, if it were a foliation, then these orbits should have the same dimension.

In order to exclude such singular cases, consider the isotropy subgroups at x:

$$G_x := \{g \in G \mid gx = x\}$$

This is a closed subgroup, hence a Lie subgroup. The orbit of x is identified with the homogeneous space G/G_x, and immersed into M. We say that the action of G on M is *foliated*, if G_x has constant dimension. Then all orbits have the same dimension, and the orbits form a foliation of M.

For example, let $G = \mathbb{R}$. Then an $\mathbb{R}$-action on M

$$\mu\colon \mathbb{R} \times M \to M$$

is also called a *flow*. The vector field associated to the flow is

$$X(x) := \left.\frac{\partial \mu(t, x)}{\partial t}\right|_{t=0}.$$

An $\mathbb{R}$-action is foliated, iff its associated vector field X is nowhere vanishing. Then the leaves of the foliation are given by the integral curves of X.

1.2.2 By Haefliger structure

A foliation can be equivalently described in several equivalent ways. The following definition is based on a "Haefliger structure".

Definition 1.9 (second definition)**.** A codimension-q foliation of a manifold M is given by an open cover $\{U_i\}$ of M together with submersions

$$s_i\colon U_i \to \mathbb{R}^q,$$

such that there are (necessarily unique) diffeomorphisms

$$\gamma_{ij}\colon s_j(U_i \cap U_j) \to s_i(U_i \cap U_j)$$

satisfying:

$$\gamma_{ij} \circ s_j = s_i, \quad \gamma_{ij} \circ \gamma_{jk} = \gamma_{ik}.$$

These maps γ_{ij} are called the *Haefliger cocycles* of the foliation. We briefly call the data $\{U_i, s_i, \gamma_{ij}\}_{ij}$ a *Haefliger structure*.

Equivalence with first definition. Suppose we are given a Haefliger structure $\{U_i, s_i, \gamma_{ij}\}_{ij}$. Let $\{(V_k, \varphi_k)\}_k$ be an atlas of M. Up to refinement, we may assume that each V_k is contained in a single U_{i_k}. Then $s_{i_k}|_{V_k}$ is also a submersion, hence there is a diffeomorphism $\kappa_k\colon s_{i_k}(V_k) \to \mathbb{R}^q$ such that $\kappa_k \circ s_{i_k}$ retricts to the projection $\mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q \to \mathbb{R}^q$ on local charts. That is, $\kappa_k \circ s_{i_k} \circ \varphi_k^{-1} = \mathrm{pr}_{\mathbb{R}^q}$. We claim that $\{V_k, \varphi_k\}$ is a foliation chart in the sense of Definition 1.2: we have

$$\begin{aligned}
\mathrm{pr}_{\mathbb{R}^q} \circ \varphi_{kl}(x,y) &= \mathrm{pr}_{\mathbb{R}^q} \circ \varphi_k \circ \varphi_l^{-1}(x,y) \\
&= \kappa_k \circ s_{i_k} \circ \varphi_l^{-1}(x,y) \\
&= \kappa_k \circ \gamma_{i_k i_l} \circ s_{i_l} \circ \varphi_l^{-1}(x,y) \\
&= \kappa_k \circ \gamma_{i_k i_l} \circ \kappa_l^{-1}(y).
\end{aligned}$$

Conversely, given a foliation atlas $\{U_i, \varphi_i\colon U_i \to \mathbb{R}^{n-q} \times \mathbb{R}^q\}$ such that $\varphi_{ij}(x,y) = (g_{ij}(x,y), h_{ij}(y))$. Set $s_i := \mathrm{pr}_{\mathbb{R}^q} \circ \varphi_i$ and $\gamma_{ij} := h_{ij}$. These render the desired Haefliger cocycles. ☐

Example 1.10. Let $\pi\colon E \to M$ be a rank-$(n-q)$ vector bundle and $\dim M = q$. This gives a codimension-q foliation on M whose leaves are $\pi^{-1}(x)$ for $x \in M$. Let $\{U_i, \varphi_i\colon U_i \to \mathbb{R}^q\}$ be an atlas of M, and $\{\Phi_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{R}^{n-q}\}$ be the associated local trivialisations of E. Then the foliation chart on $\pi^{-1}(U_i)$ is just the composition of the local trivialisations Φ_i, with the chart of U_i. The Haefliger cocycles are just the transition functions $h_{ij}\colon U_i \cap U_j \to \mathrm{GL}(q, \mathbb{R})$ of the base manifold M.

1.2.3 By involutive distribution

The alternative definition below justifies the connection between foliations and the Frobenius theorem:

Definition 1.11 (third definition)**.** A codimension-q foliation $\mathcal{F}$ of a manifold M is given by an involutive distribution.

The involutive distribution is called the *tangent bundle* of the foliation $\mathcal{F}$ and denoted by $T\mathcal{F}$.

The equivalence with the first definition is heavily based on the Frobenius theorem. We will omit the proof and only sketch how to derive these definitions from each other. The proof can be found in [5, Section 1.2]; see also [4, Theorem 19.21].

Let $E \subseteq TM$ be an involutive distribution. By Frobenius theorem, it is integrable: every point $x \in M$ belongs to an integral manifold. These manifolds form a foliation of M.

Conversely, if $(M, \mathcal{F})$ is a foliated manifold. For every $x \in M$, let F be the leaf of x. Then the collection $\{T_x F\}_{x \in M}$ is a vector subbundle of TM which is involutive. We denote this bundle by $T_x\mathcal{F}$ and also write $T_x\mathcal{F}$ for $T_x F$.

Example 1.12. A vector bundle $E \to M$ gives a foliation $\mathcal{F}$ on the total space E. The tangent bundle $T\mathcal{F}$ is the vertical bundle $VE := \ker T\pi$.

Example 1.13. Let G be a Lie group which has a foliated action on M. Then $T\mathcal{F}$ is spanned by the fundamental vector fields of this group action. Namely, the image of the bundle homomorphism

$$\mathfrak{g} \times M \to TM, \qquad (\xi, x) \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} x \exp(t\xi).$$

Remark 1.14. We have seen so far that a Lie group action generates a foliation necessarily when the dimensions of the stabliser groups are constant. If this is not the case, then we speak of a *singular foliation*. The following definition (due to Peter Stefan, Iakovos Androulidakis and Georges Skandalis) is modelled on the third definition that we have introduced above.

Definition 1.15. A *(singular) foliation* is a locally finitely generated, involutive $\mathrm{C}^\infty(M)$-submodule of $\Gamma_\mathrm{c}(M, TM)$.

Singular foliations are much more complicated to deal with as opposed to those regular ones. For example, the holonomy groupoid of a singular foliation is hard to define. It seems that the most general definition so far is the one given in [1]. We might talk about this in a later talk.

1.3 Constructions and more examples

1.3.1 Product foliations

Let $(M, \mathcal{F})$ and $(N, \mathcal{G})$ be foliated manifolds, given by a Haefliger structure $(U_i, s_i, \gamma_{ij})_{ij}$ on M and $(U'_k, s'_k, \gamma'_{kl})$ on N. There is a *product foliation* $\mathcal{F} \times \mathcal{G}$ on $M \times N$, given by the Haefliger structure

$$(U_i \times U'_k, s_i \times s'_k, \gamma_{ij} \times \gamma'_{kl})_{ijkl}.$$

The product foliation $\mathcal{F} \times \mathcal{G}$ has codimension $\operatorname{codim} \mathcal{F} + \operatorname{codim} \mathcal{G}$. The tangent bundle $T(\mathcal{F} \times \mathcal{G}) \simeq T\mathcal{F} \oplus T\mathcal{G} \subseteq TM \oplus TN \simeq T(M \times N)$.

1.3.2 Pullback foliations along transverse maps

Definition 1.16. Let $(M, \mathcal{F})$ be a foliated manifold. A smooth map $f\colon N \to M$ is *transverse* to $\mathcal{F}$, if f is transverse to all the leaves of $\mathcal{F}$. That is, for every $x \in N$,

$$T_{f(x)}M = T_{f(x)}\mathcal{F} + T_x f(T_x N).$$

Let $\mathcal{F}$ be given by the Haefliger structure $(U_i, s_i, \gamma_{ij})_{ij}$. We claim that the data

$$\left(V_i := f^{-1}(U_i), \quad s'_i := s_i \circ f|_{V_i}, \quad \gamma_{ij}\right)_{ij}$$

defines a Haefliger structure. Its determined foliation is called the *pullback foliation* on N along f, denoted by $f^*\mathcal{F}$.

Proof. We must show that the maps $s'_i\colon V_i \to \mathbb{R}^q$ are submersion, that is, $T_x s'_i = T_{f(x)} s_i \circ T_x f$ is surjective for every $x \in V_i$. Since f is transverse to $\mathcal{F}$, we have that the map

$$\tilde{f}\colon T_x V_i \xrightarrow{T_x f} T_{f(x)} U_i \to T_{f(x)} U_i / T_{f(x)}\mathcal{F}$$

is surjective. The submersion s_i is trivial along the leaves, that is, $T_{f(x)} s_i = 0$ on $T_{f(x)}\mathcal{F} \subseteq T_{f(x)} U_i$. So $T_x s_i$ factors through the quotient map $T_{f(x)} U_i \to T_{f(x)} U_i / T_{f(x)}\mathcal{F}$. Thus the diagram

$$\begin{tikzcd}
T_x V_i \arrow[r, "T_x f"] \arrow[dr, two heads, "\tilde{f}"'] & T_{f(x)} U_i \arrow[r, two heads, "T_{f(x)} s_i"] \arrow[d, two heads] & \mathbb{R}^q \\
& T_{f(x)} U_i / T_{f(x)}\mathcal{F} \arrow[ur, two heads, "\tilde{s}_i"'] &
\end{tikzcd}$$

commutes. So $T_{f(x)} s_i \circ T_x f = \tilde{s}_i \circ \tilde{f}$ is the composition of two surjective maps, hence surjective. $\square$

We have $\operatorname{codim} f^*(\mathcal{F}) = \operatorname{codim} \mathcal{F}$ and $T f^*(\mathcal{F}) = (T_x f)^{-1}(T\mathcal{F})$.

1.3.3 Quotient foliations of covering space actions

Let $(M, \mathcal{F})$ be a foliated manifold such that M carries a free, properly discontinuous action of a discrete Lie group G. Then the quotient manifold M/G is Hausdorff. Assume that $\mathcal{F}$ is *invariant* under the G-action, that is, every $g \in G$ is an automorphism of the foliation $(M, \mathcal{F})$. Then there is an induced foliation $\mathcal{F}/G$ on M/G as follows.

Since G acts freely and properly discontinuously, the quotient map $\pi \colon M \to M/G$ defines a covering space. Let $\{(U_i, \varphi_i \colon U_i \to \mathbb{R}^{n-q} \times \mathbb{R}^q)\}_i$ be a foliation atlas of $\mathcal{F}$. Then π is a local diffeomorphism. We may assume that $\pi|_{U_i}$ is a diffeomorphism by replacing U_i by its refinement. Then it has an inverse section s_i and

$$\{(\varphi_i \circ s_i \colon \pi(U_i) \to \mathbb{R}^{n-q} \times \mathbb{R}^q)\}_i$$

renders a foliation $\mathcal{F}/G$ for M/G. The leaves of $\mathcal{F}/G$ are quotient manifolds L/G_L, where L is a leaf of $\mathcal{F}$ and G_L is the isotropy subgroup of the leaf L. The latter is well-defined because every $g \in G$ maps a leaf into a leaf.

We have $\operatorname{codim}(\mathcal{F}/G) = \operatorname{codim}(\mathcal{F})$ and $T(\mathcal{F}/G) = T\pi(T\mathcal{F})$.

Example 1.17 (Kronecker foliation). Let $\vartheta \in \mathbb{R} \setminus \mathbb{Q}$ and define the submersion

$$s \colon \mathbb{R}^2 \to \mathbb{R}, \qquad s(x, y) = x - \vartheta y.$$

Then it defines a foliation on $\mathbb{R}^2$ following Example 1.6. The group $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ freely and properly discontinuously, and the aforementioned foliation on $\mathbb{R}^2$ is $\mathbb{Z}^2$-invariant. So there is an induced foliation $\mathcal{F}$ on the quotient manifold $\mathbb{T}^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$. This is known as the *Kronecker foliation* on $\mathbb{T}^2$. Equivalently, one can consider the integral curves generated by the differential equation $\frac{\mathrm{d}x}{\mathrm{d}y} = \vartheta$.

Example 1.18 (foliation of the Möbius band). Recall that the Möbius bundle is the quotient space of $\mathbb{R}^2$:

$$\text{Möb} := \mathbb{R}^2/\sim, \qquad (x, y) \sim (x + 1, -y).$$

The projection $\mathbb{R}^2 \to$ Möb is a two-fold covering space. The trivial codimension-1 foliation of $\mathbb{R}^2$ is invariant under this group action, hence there is a quotient foliation of Möb. The interesting property of this foliation is that every leaf (which is diffeomorphic to $\mathbb{T}$) will wrap around the whole band twice except for the middle one generated by the plaques with $y = 0$.

1.3.4 Suspensions

Suspensions of diffeomorphism groups are special cases of the quotient foliation. They consititute a large class of interesting foliations.

Let F be a smooth manifold and $f \colon F \to F$ be an automorphism. Consider the trivial 1-dimensional foliation on $M = \mathbb{R} \times F$, i.e. the leaves are $\mathbb{R} \times \{x\}$ for $x \in F$. There is a $\mathbb{Z}$-action on M generated by the diffeomorphism

$$(t, x) \mapsto (t + 1, f(x)).$$

The action is free and proper, and leaves the foliation on M invariant. So there is a quotient foliation on the quotient manifold $\mathbb{R} \times_{\mathbb{Z}} F$. This is called the *suspension* of the diffeomorphism f, or the suspension of the group $\mathbb{Z}$ generated by f.

1.3.5 Flat bundles

The suspensions are a special case of flat bundles. Let M be a manifold and $\widetilde{M}$ be the universal cover of M. Then $\widetilde{M} \to M$ is a (right) principal $\pi_1(M)$-bundle. Assume that $\pi_1(M)$ acts on another manifold F. Then there is an associated fibre bundle $\widetilde{M} \times_{\pi_1(M)} F \to M$. The space $\widetilde{M} \times_{\pi_1(M)} F$ is the quotient of $\widetilde{M} \times F$ under the equivalence relation $(x, y) \sim (xg^{-1}, gy)$.

The submersion $\widetilde{M} \times F \to F$ generates a foliation of $\widetilde{M} \times F$ which is invariant under the action of $\pi_1(M)$. So there is a quotient foliation on $\widetilde{M} \times_{\pi_1(M)} F$. This is called a *flat bundle*. In contrast to the foliation coming from a submersion, in which every leaf is vertical in the sense that $T\mathcal{F} \subseteq \ker T\pi$, in a flat bundle the leaves are all horizontal, that is, $T_x\mathcal{F} \simeq T_{\pi(x)}M$.

The suspension is the special case where $M = \mathbb{T}$ and the action of $\pi_1(M) = \mathbb{Z}$ on F is generated by the diffeomorphism f.

1.3.6 Reeb foliation

Foliations can be defined on manifolds with boundary as well. But this requires that the foliation behaves nicely near the boundary, that is, the foliation is either *transverse* to the boundary or *tangent* to the boundary.

Definition 1.19 ([2, Definition 1.1.11])**.** Let $(M, \mathcal{F})$ be a foliated manifold, and $N \subseteq M$ be a submanifold. We say that:

- $\mathcal{F}$ is *transverse* to N, denoted by $\mathcal{F} \pitchfork N$, if every leaf of $\mathcal{F}$ is transverse to N. That is,

$$T_xM = T_x\mathcal{F} + T_xN$$

  for every $x \in N$.
- $\mathcal{F}$ is *tangent* to N, if $T_x\mathcal{F} = T_xN$ for every $x \in N$.

The *Reeb foliation* of the solid torus $X = \mathbb{D}^2 \times \mathbb{T}$ is a foliation on a manifold with boundary, of the second sort. Let

$$\mathbb{D}^2 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

be the unit disk. Foliate the solid cylinder $\mathbb{D}^2 \times \mathbb{R}$ by the submersion

$$f\colon \operatorname{Int}(\mathbb{D}^2) \times \mathbb{R} \to \mathbb{R}, \qquad f(x, y, t) = \exp\left(\frac{1}{1 - x^2 - y^2}\right) - z$$

and require that the boundary of the solid cylinder is another leaf. This gives a foliation of the solid cylinder.

The translation action of $\mathbb{Z}$ on the second entry of $\mathbb{D}^2 \times \mathbb{R}$ preserves this foliation. So there is a quotient foliation on solid torus $X = \mathbb{D}^2 \times \mathbb{R}/\mathbb{Z} \simeq \mathbb{D}^2 \times \mathbb{T}$. This is the *Reeb foliation* on the solid torus. Note that every leaf is diffeomorphic to $\mathbb{R}^2$ except for the boundary leaf $\mathbb{T}^2$.

Now we describe the Reeb foliation on $\mathbb{S}^3$. A statement from topology says that $\mathbb{S}^3$ can be obtained by gluing together two copies of the solid torus along the boundary $\mathbb{T}^2$:

$$\mathbb{S}^3 = X \cup_{\partial X} X, \qquad X = \mathbb{D}^2 \times \mathbb{T}, \qquad \partial X = \mathbb{T}^2.$$

The Reeb foliation can be glued together to form a foliation on $\mathbb{S}^3$ as well. This does happen in general: one needs some "tameness" conditions in order to glue two foliations along the boundary; we will, however, not talk about this in detail. Then we obtain the *Reeb foliation* on $\mathbb{S}^3$. It has all leaves diffeomorphic to $\mathbb{R}^2$ except for the "boundary" leaf of solid cylinder, which is compact.

The importance of the Reeb foliation is illustrated by the following celebrated theorem:

Theorem 1.20 (Novikov's compact leaf theorem)**.** *Every codimension-1 foliation of $\mathbb{S}^3$ has a compact leaf, bounding a solid torus with the Reeb foliation.*

So the Reeb foliation plays an important role in the foliation theory of $\mathbb{S}^3$.

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