INTRODUCTION TO ALGORITHMS

Lecture 10: Minimum Spanning Tree Algorithms

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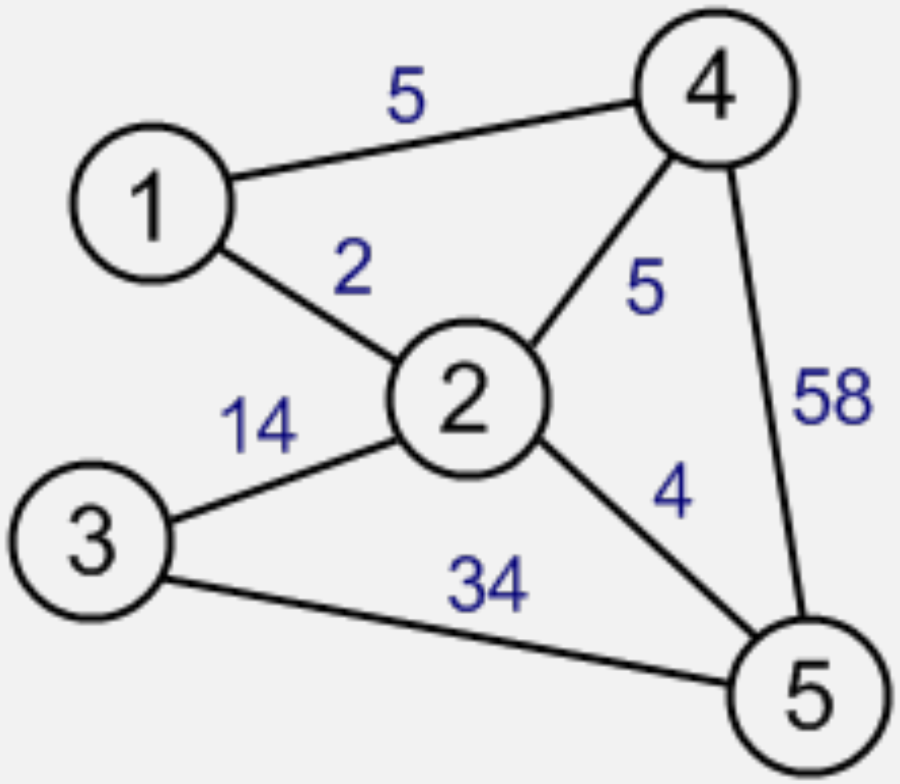
MINIMUM SPANNING TREES

- *Introduction*
- *edge-weighted graph API*
- *Greedy Strategy*
- *Kruskal's algorithm*
- *Prim's algorithm*
- *context*

Edge-Weighted Graph



Weighted edge API

Edge abstraction needed for weighted edges.

	public class Edge implements Comparable<Edge>	
	Edge(int v, int w, double weight)	*create a weighted edge v-w*
int	either()	*either endpoint*
int	other(int v)	*the endpoint that's not v*
int	compareTo(Edge that)	*compare this edge to that edge*
double	weight()	*the weight*
String	toString()	*string representation*

Idiom for processing an edge e: `int v = e.either(), w = e.other(v);`

Weighted edge: Java implementation

```
public class Edge implements Comparable<Edge>
{
   private final int v, w;
   private final double weight;

   public Edge(int v, int w, double weight)
   {
      this.v = v;
      this.w = w;
      this.weight = weight;
   }

   public int either()
   {  return v;  }

   public int other(int vertex)
   {
      if (vertex == v) return w;
      else return v;
   }

   public int compareTo(Edge that)
   {
      if      (this.weight < that.weight) return -1;
      else if (this.weight > that.weight) return +1;
      else                                return  0;
   }
}
```

constructor

either endpoint

other endpoint

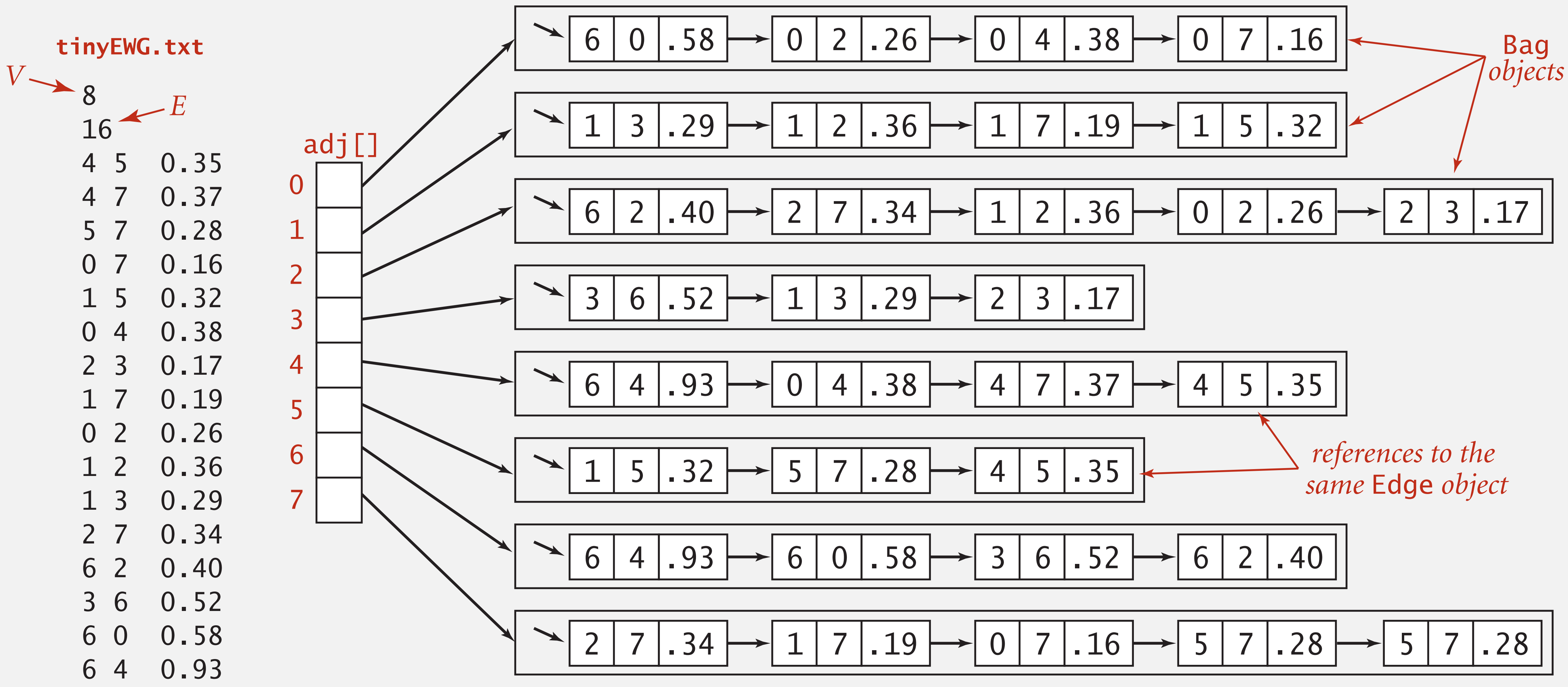
compare edges by weight

Edge-weighted graph API

public class EdgeWeightedGraph		
	EdgeWeightedGraph(int V)	*create an empty graph with V vertices*
	EdgeWeightedGraph(In in)	*create a graph from input stream*
void	addEdge(Edge e)	*add weighted edge e to this graph*
Iterable<Edge>	adj(int v)	*edges incident to v*
Iterable<Edge>	edges()	*all edges in this graph*
int	V()	*number of vertices*
int	E()	*number of edges*
String	toString()	*string representation*

Edge-weighted graph: adjacency-lists representation

Maintain vertex-indexed array of Edge lists.



Edge-weighted graph: adjacency-lists implementation

```
public class EdgeWeightedGraph
{
   private final int V;
   private final Bag<Edge>[] adj;

   public EdgeWeightedGraph(int V)
   {
      this.V = V;
      adj = (Bag<Edge>[]) new Bag[V];
      for (int v = 0; v < V; v++)
         adj[v] = new Bag<Edge>();
   }

   public void addEdge(Edge e)
   {
      int v = e.either(), w = e.other(v);
      adj[v].add(e);
      adj[w].add(e);
   }

   public Iterable<Edge> adj(int v)
   {  return adj[v];  }
}
```

same as `Graph`, but adjacency lists of `Edge`s instead of integers

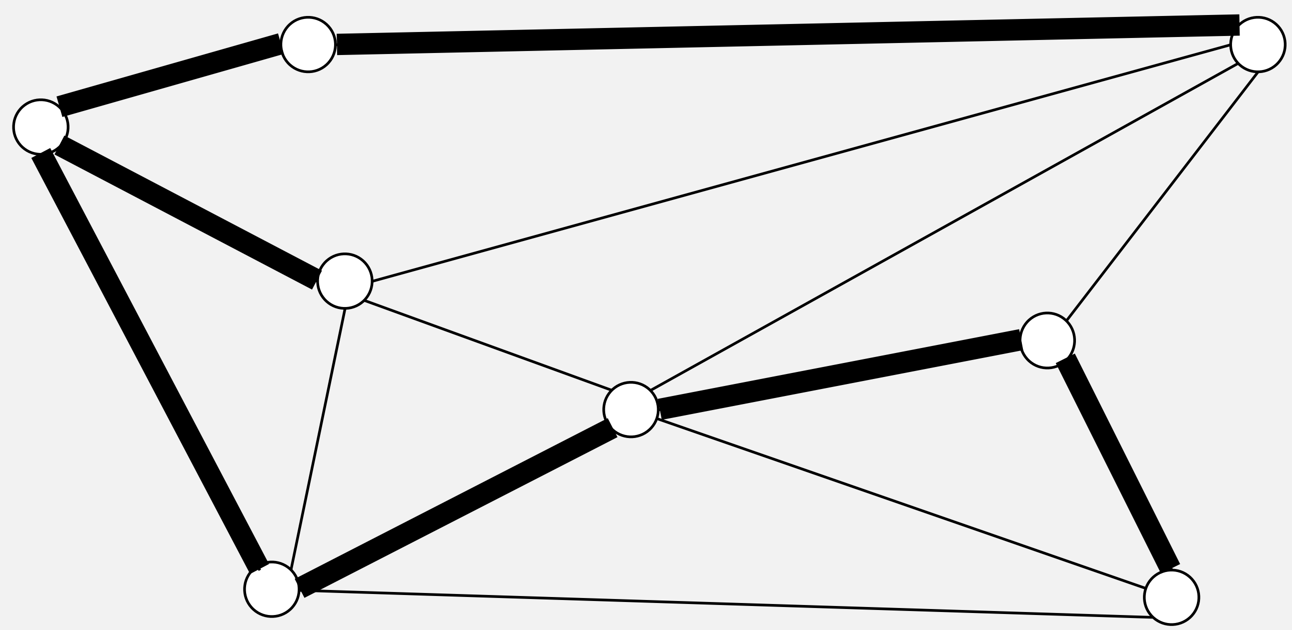
constructor

add edge to both adjacency lists

Minimum spanning tree

Def. A spanning tree of G is a subgraph T that is:

- Connected.
- Acyclic.
- Includes all of the vertices.



graph G

Minimum spanning tree

Def. A spanning tree of G is a subgraph T that is:

- Connected.
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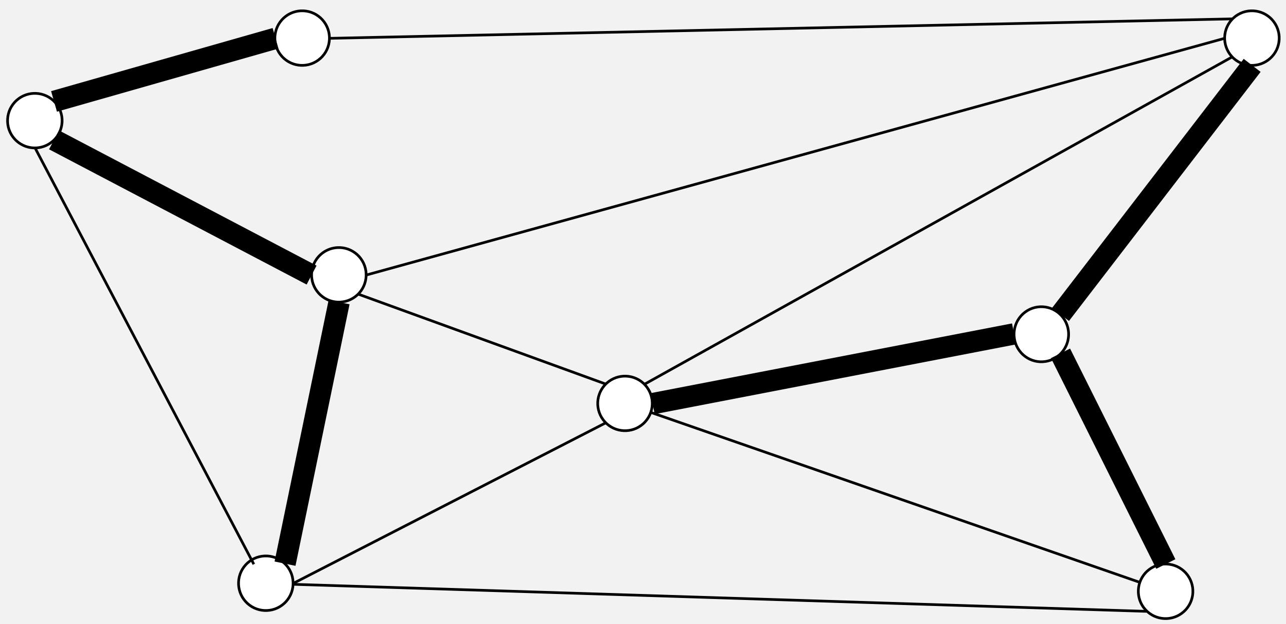

not connected

Minimum spanning tree

Def. A spanning tree of G is a subgraph T that is:

- Connected.
- Acyclic.
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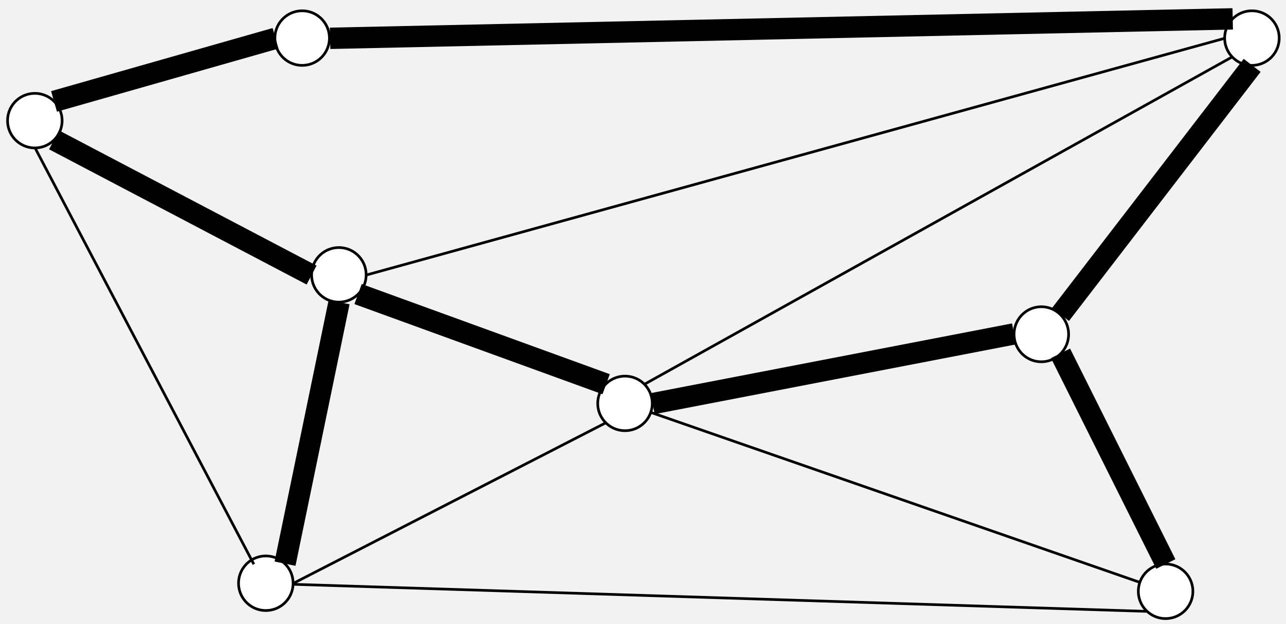

not acyclic

Minimum spanning tree

Def. A spanning tree of G is a subgraph T that is:

- Connected.
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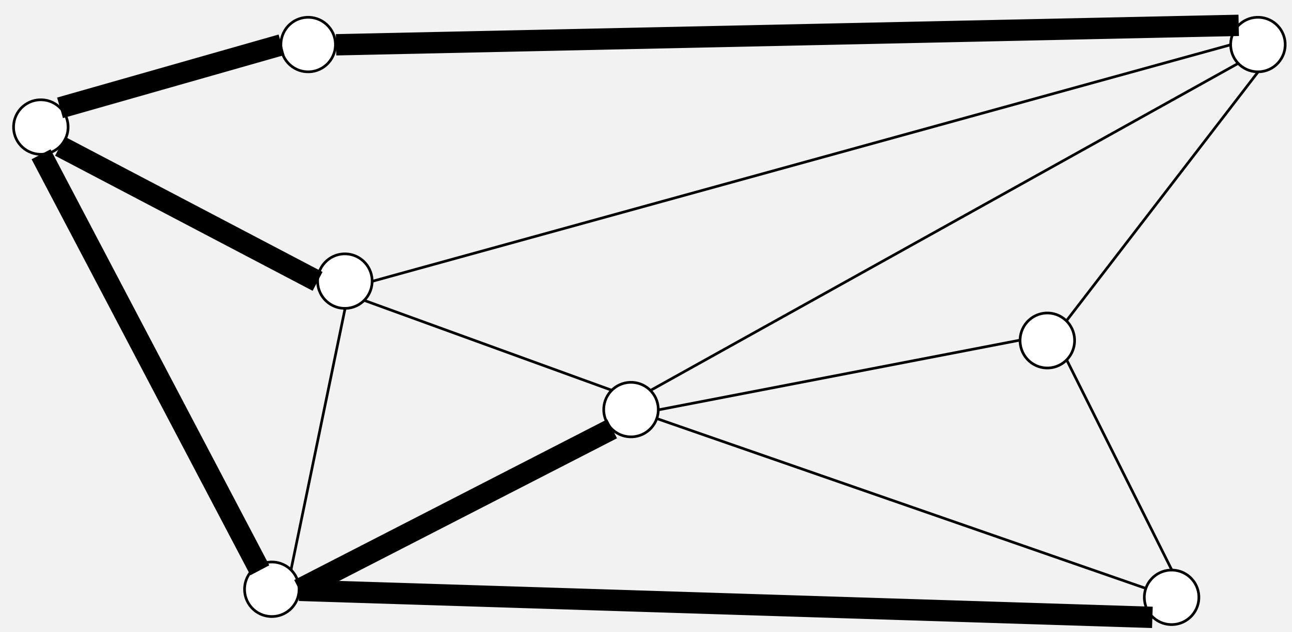
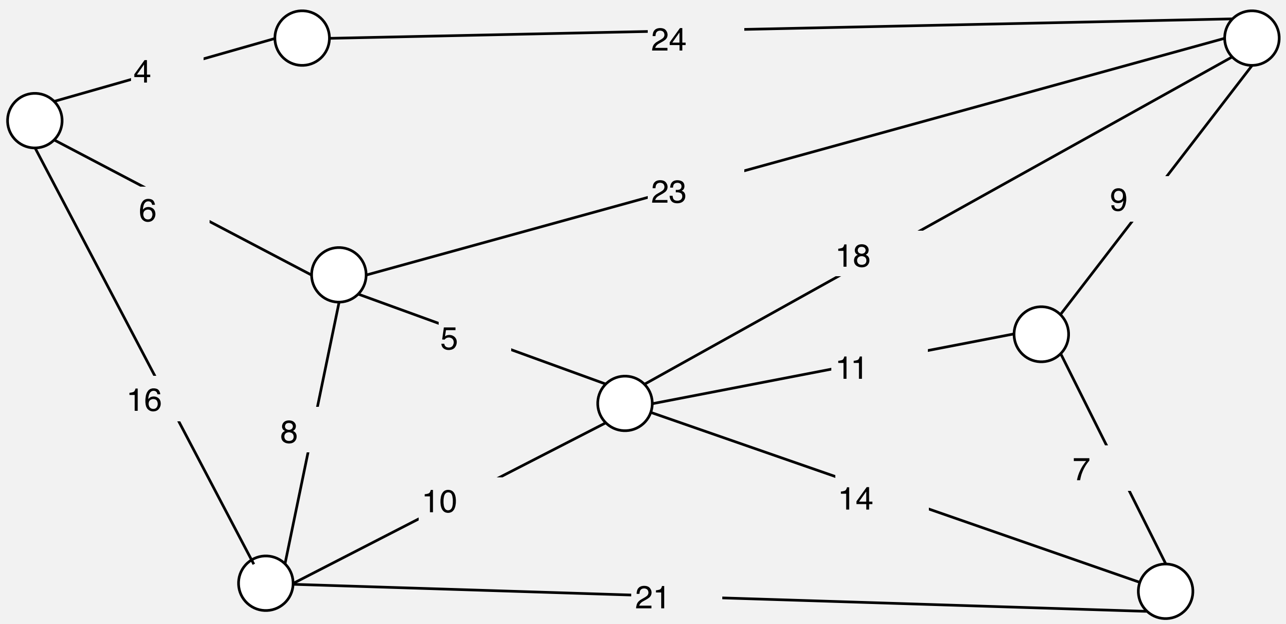


not spanning

Minimum spanning tree problem

Input. Connected, undirected graph G with positive edge weights.

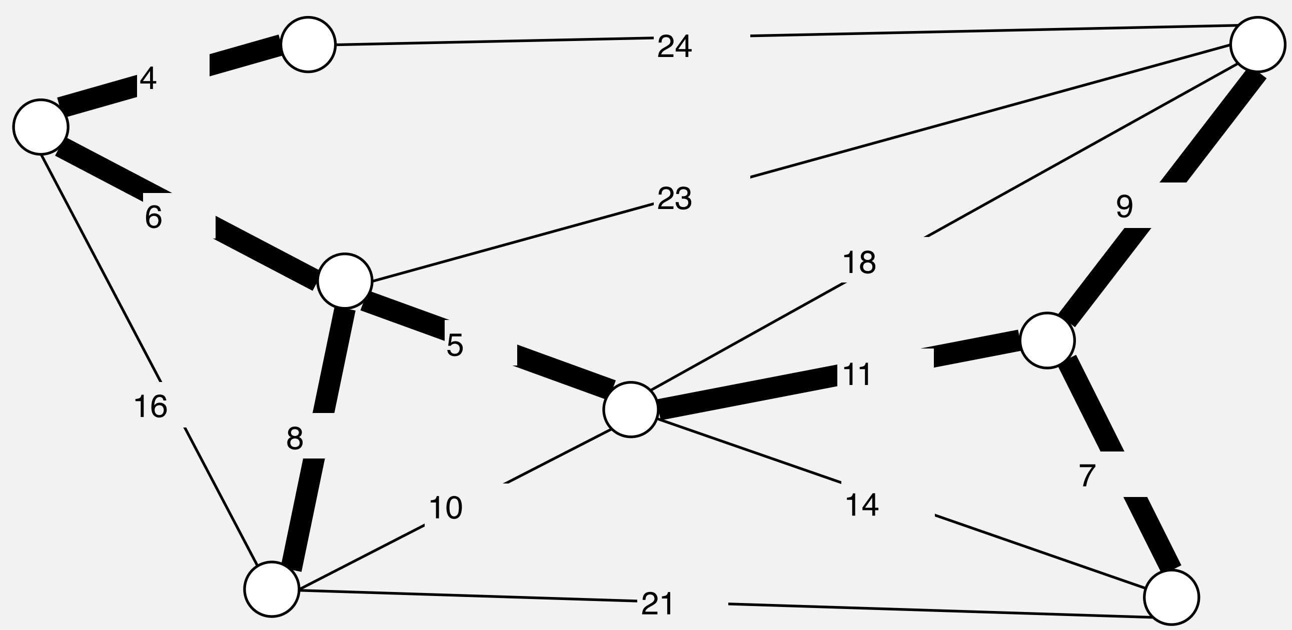


edge-weighted graph G

Minimum spanning tree problem

Input. Connected, undirected graph G with positive edge weights.

Output. A min weight spanning tree.



minimum spanning tree T
(weight = 50 = 4 + 6 + 8 + 5 + 11 + 9 + 7)

Brute force. Try all spanning trees?

Minimum spanning tree API

Q. How to represent the MST?

public class MST		
	MST(EdgeWeightedGraph G)	*constructor*
Iterable<Edge>	edges()	*edges in MST*
double	weight()	*weight of MST*

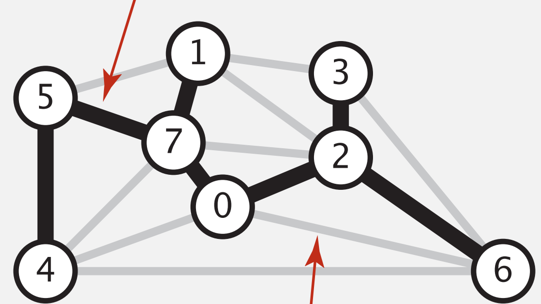
tinyEWG.txt

V → 8

E → 16

```
4 5  0.35
4 7  0.37
5 7  0.28
0 7  0.16
1 5  0.32
0 4  0.38
2 3  0.17
1 7  0.19
0 2  0.26
1 2  0.36
1 3  0.29
2 7  0.34
6 2  0.40
3 6  0.52
6 0  0.58
6 4  0.93
```

MST edge (black)



non-MST edge (gray)

```
% java MST tinyEWG.txt
0-7 0.16
1-7 0.19
0-2 0.26
2-3 0.17
5-7 0.28
4-5 0.35
6-2 0.40
1.81
```

Minimum spanning tree API

Q. How to represent the MST?

	public class MST	
	MST(EdgeWeightedGraph G)	*constructor*
Iterable<Edge>	edges()	*edges in MST*
double	weight()	*weight of MST*

```
public static void main(String[] args)
{
   In in = new In(args[0]);
   EdgeWeightedGraph G = new EdgeWeightedGraph(in);
   MST mst = new MST(G);
   for (Edge e : mst.edges())
      StdOut.println(e);
   StdOut.printf("%.2f\n", mst.weight());
}
```

```
% java MST tinyEWG.txt
0-7 0.16
1-7 0.19
0-2 0.26
2-3 0.17
5-7 0.28
4-5 0.35
6-2 0.40
1.81
```

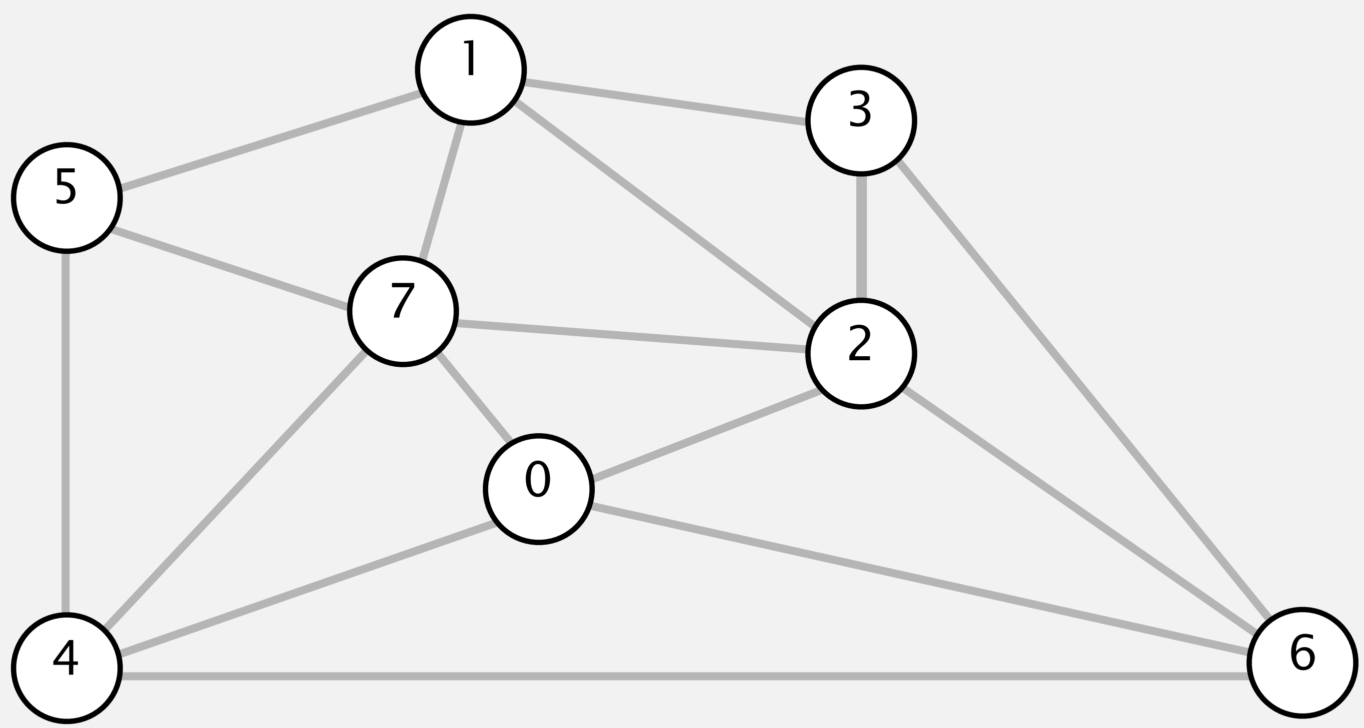
Minimum Spanning Trees

- introduction
- edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.



an edge-weighted graph

graph edges sorted by weight

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

Kruskal's algorithm demo

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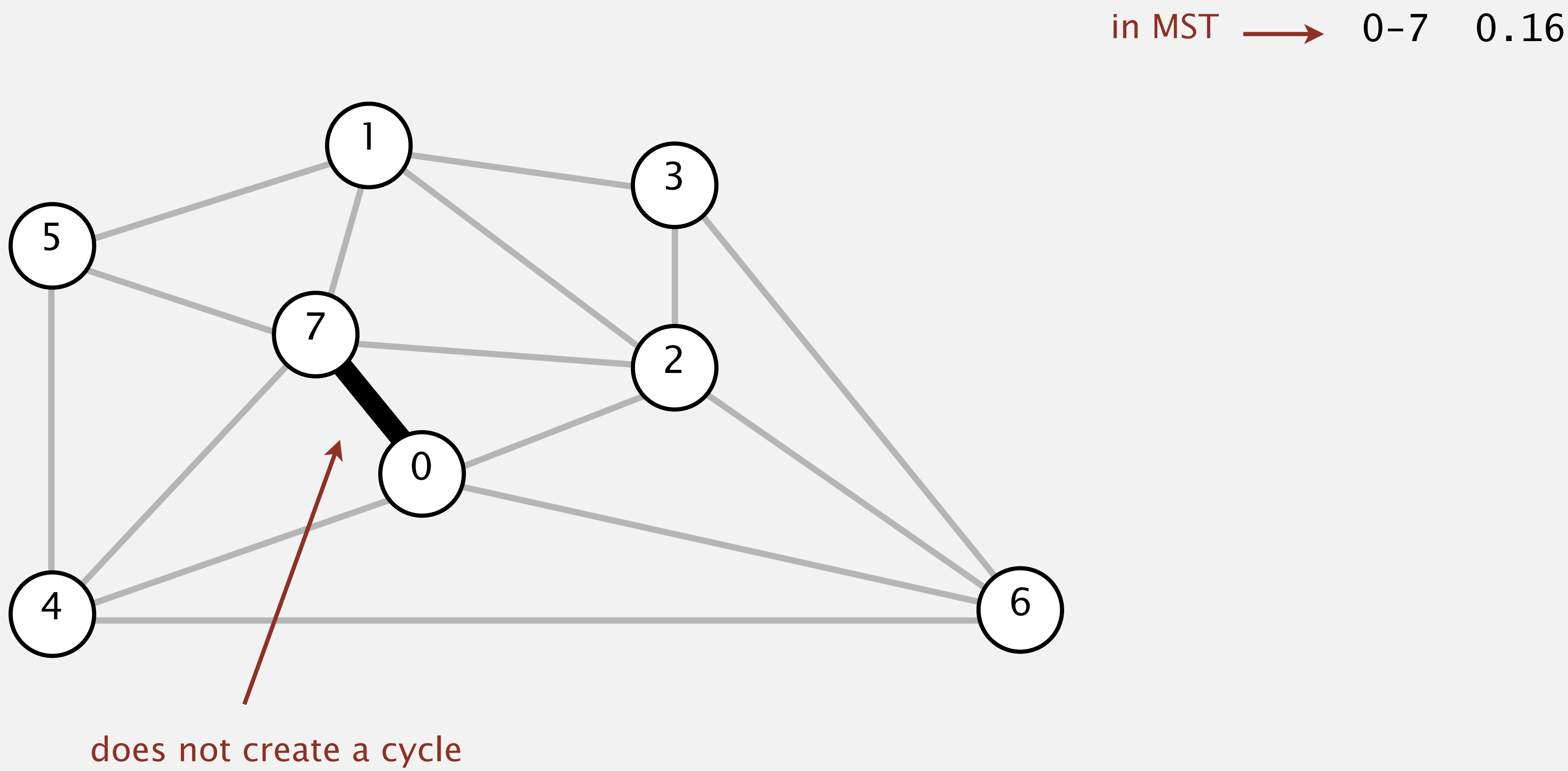

Kruskal's algorithm demo

Consider edges in ascending order of weight.

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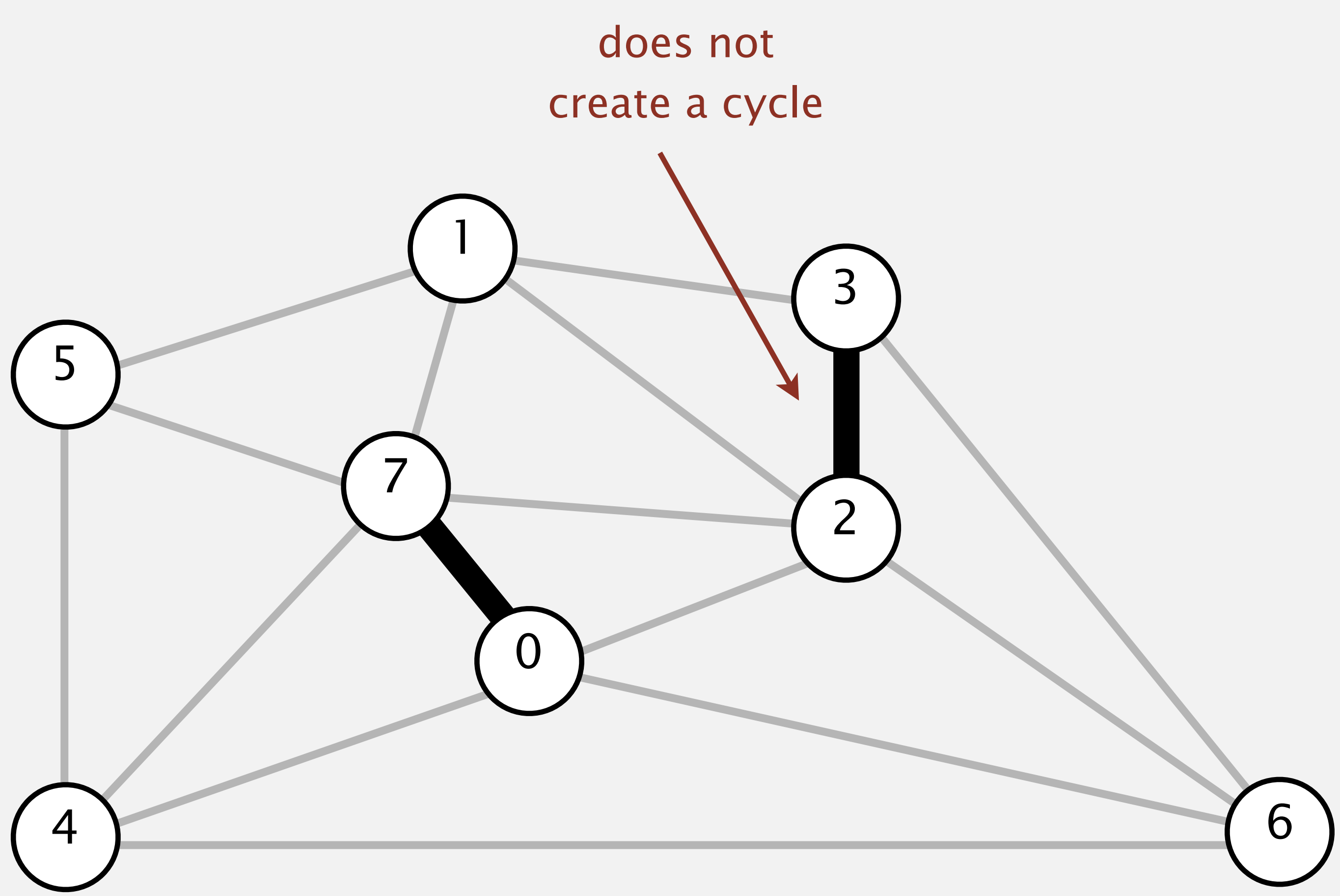


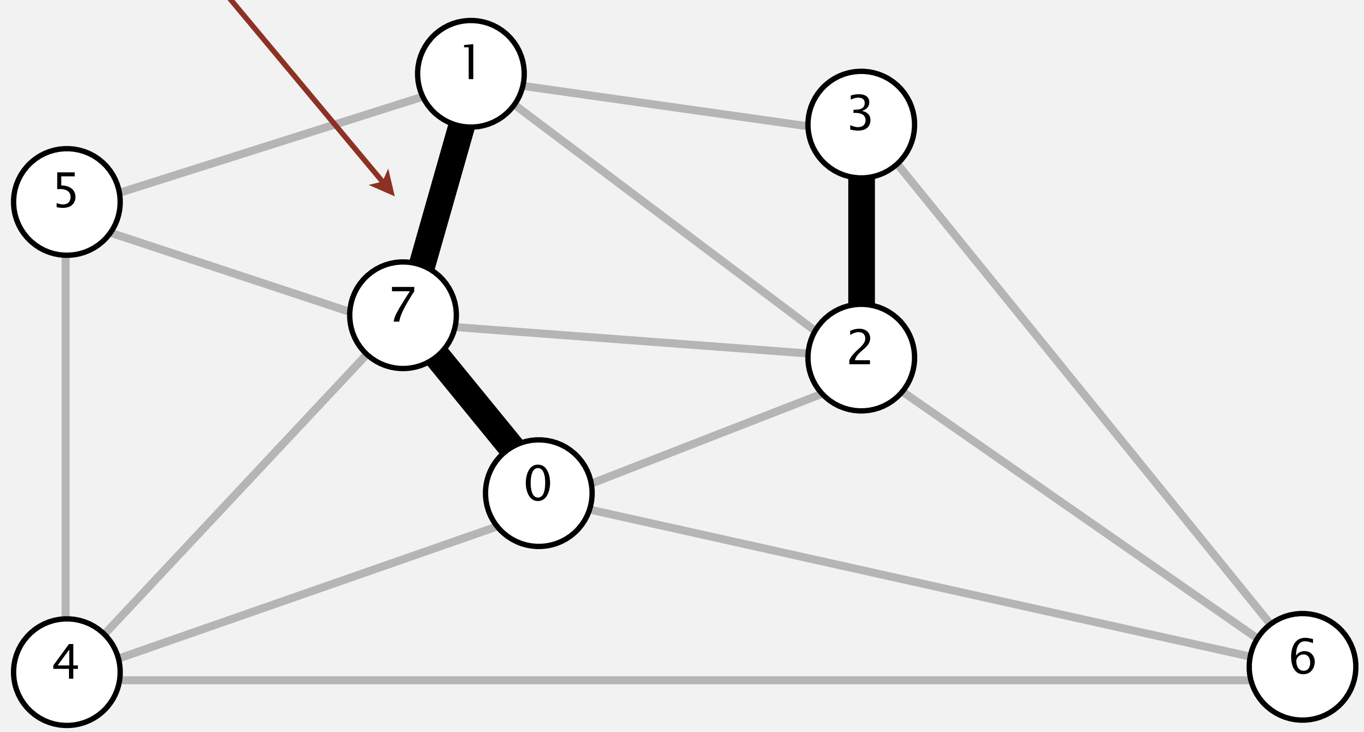
	0-7	0.16
in MST →	2-3	0.17

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

does not create a cycle



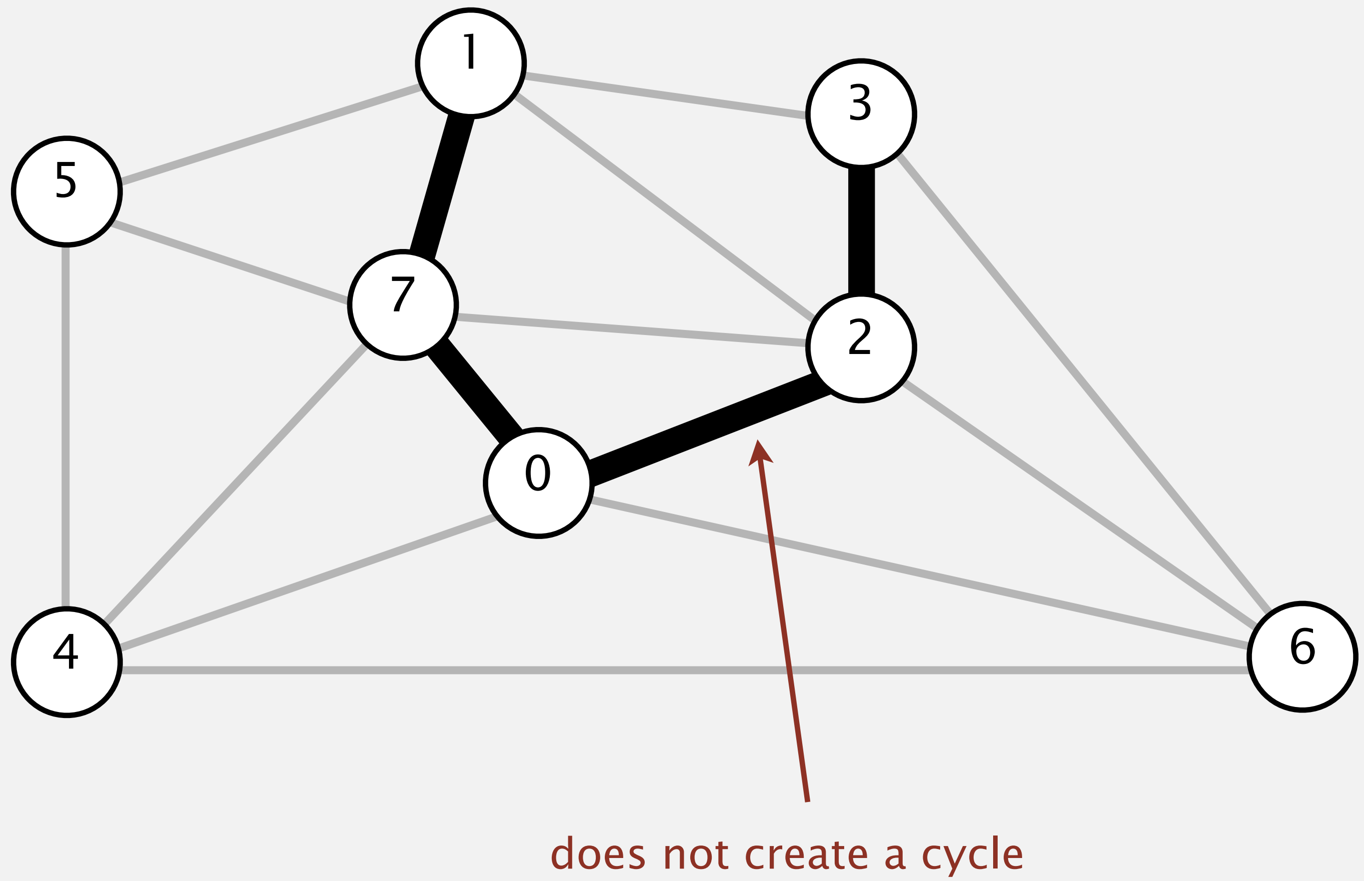
0-7	0.16
2-3	0.17
1-7	0.19

in MST → 1-7 0.19

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.



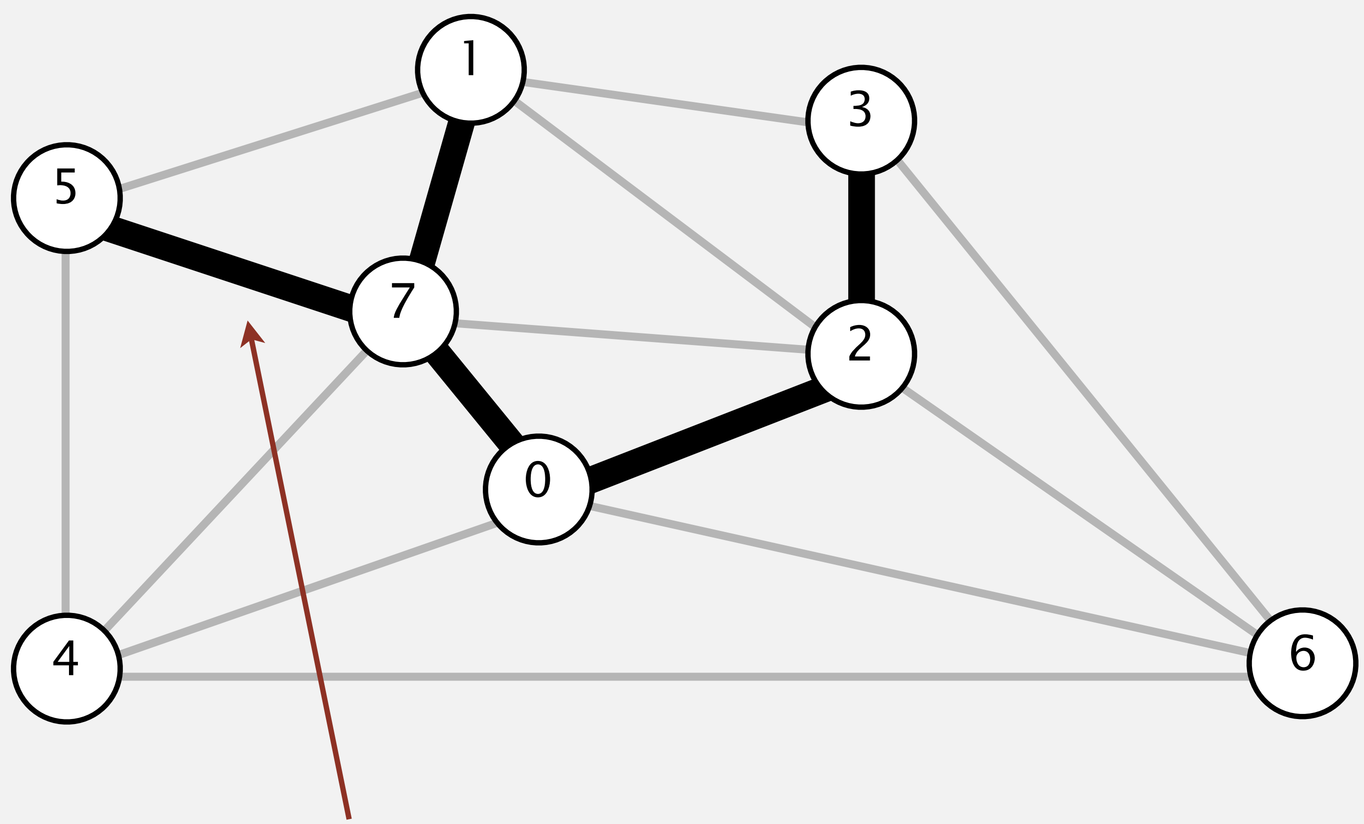
in MST →

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.



does not create a cycle

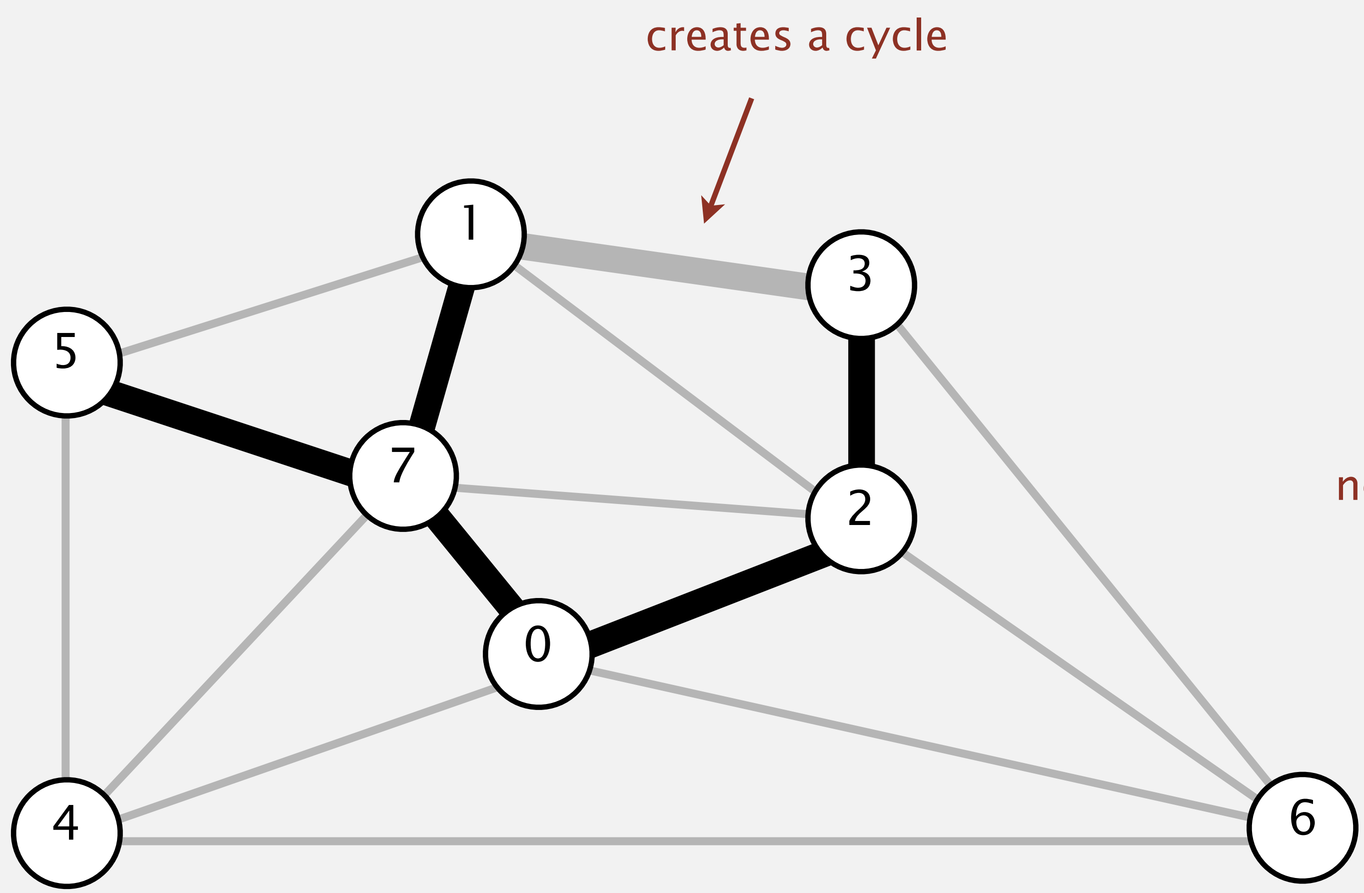
0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28

in MST → 5-7 0.28

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.



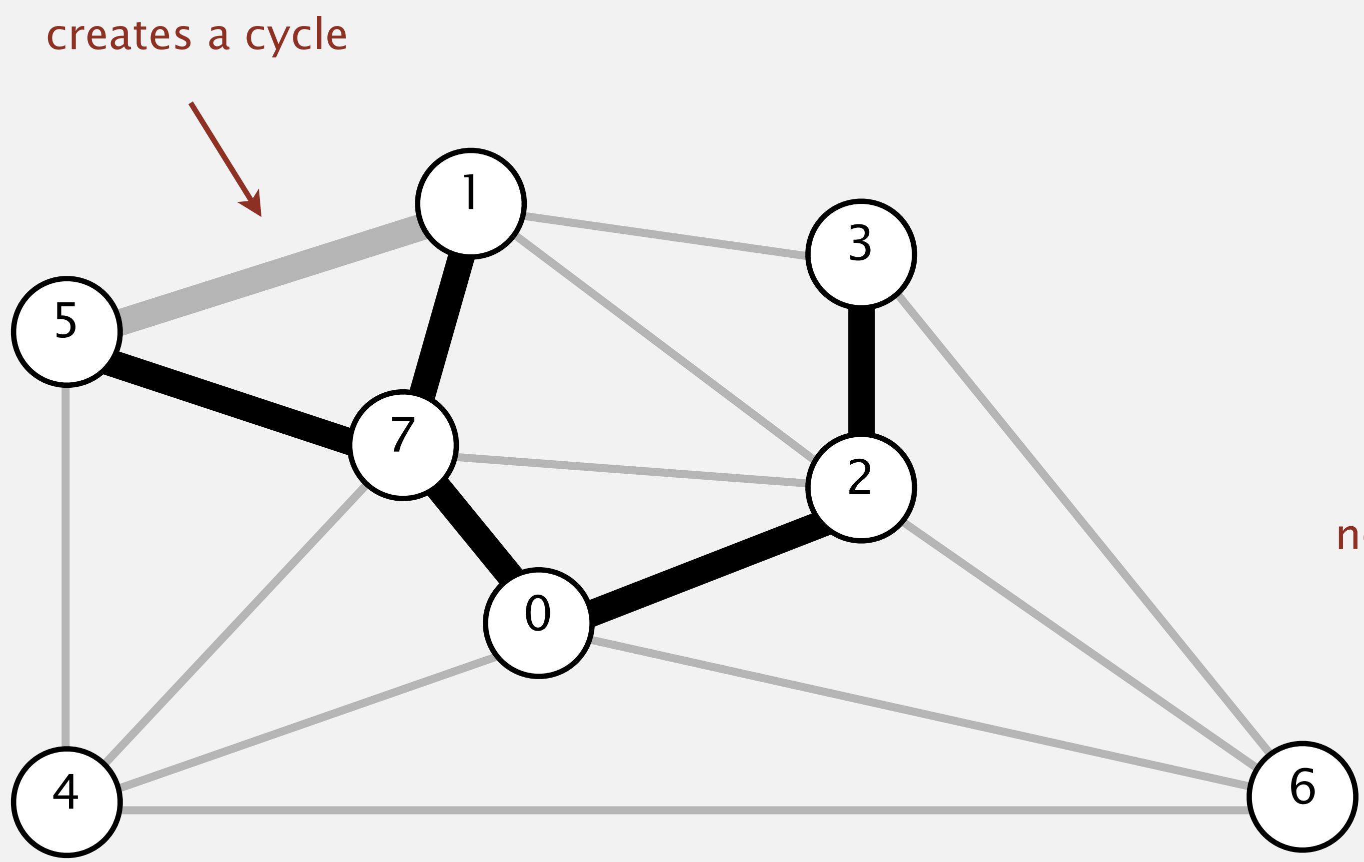
0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29

not in MST → 1-3 0.29

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.



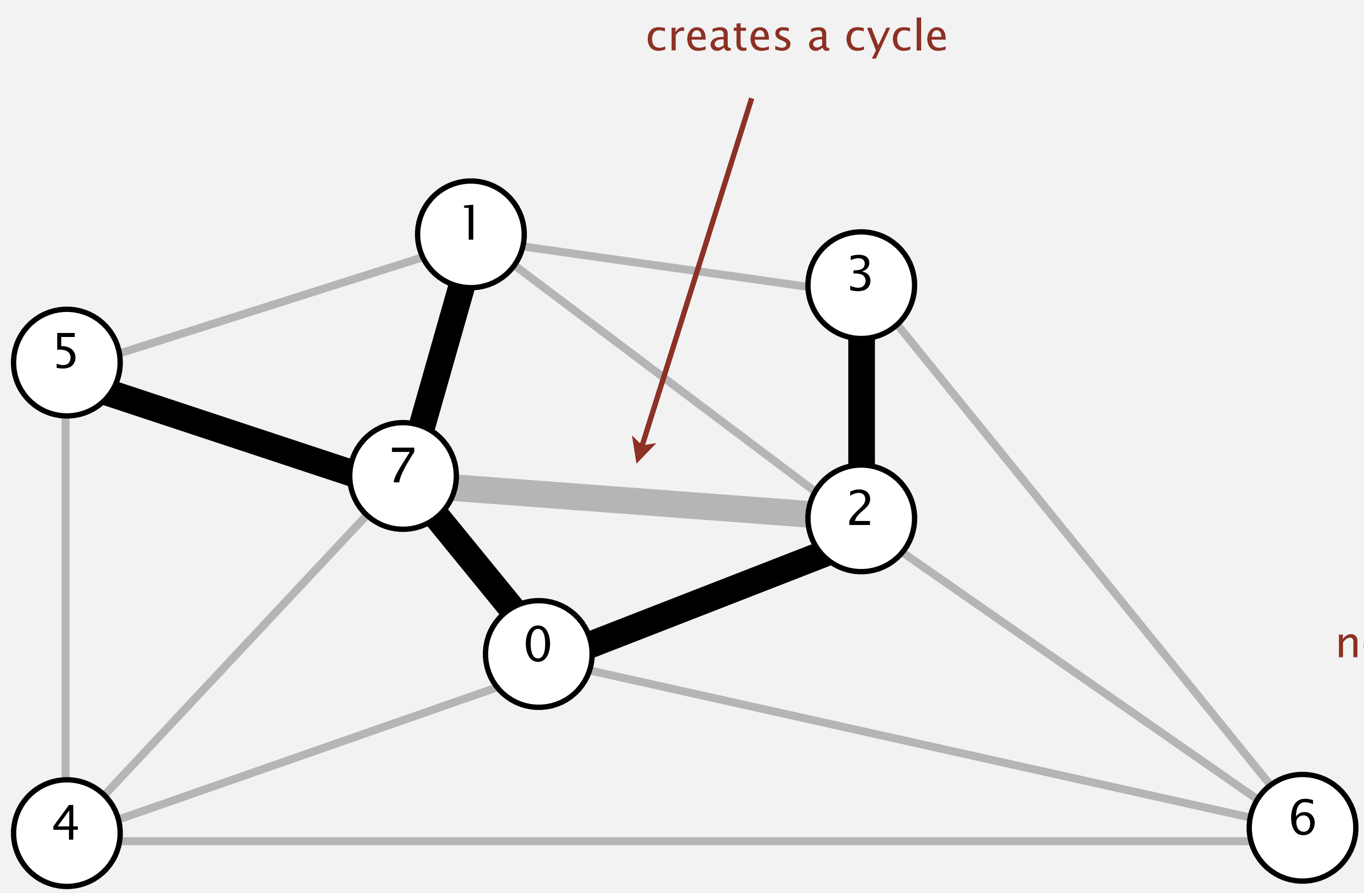
0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32

not in MST

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.



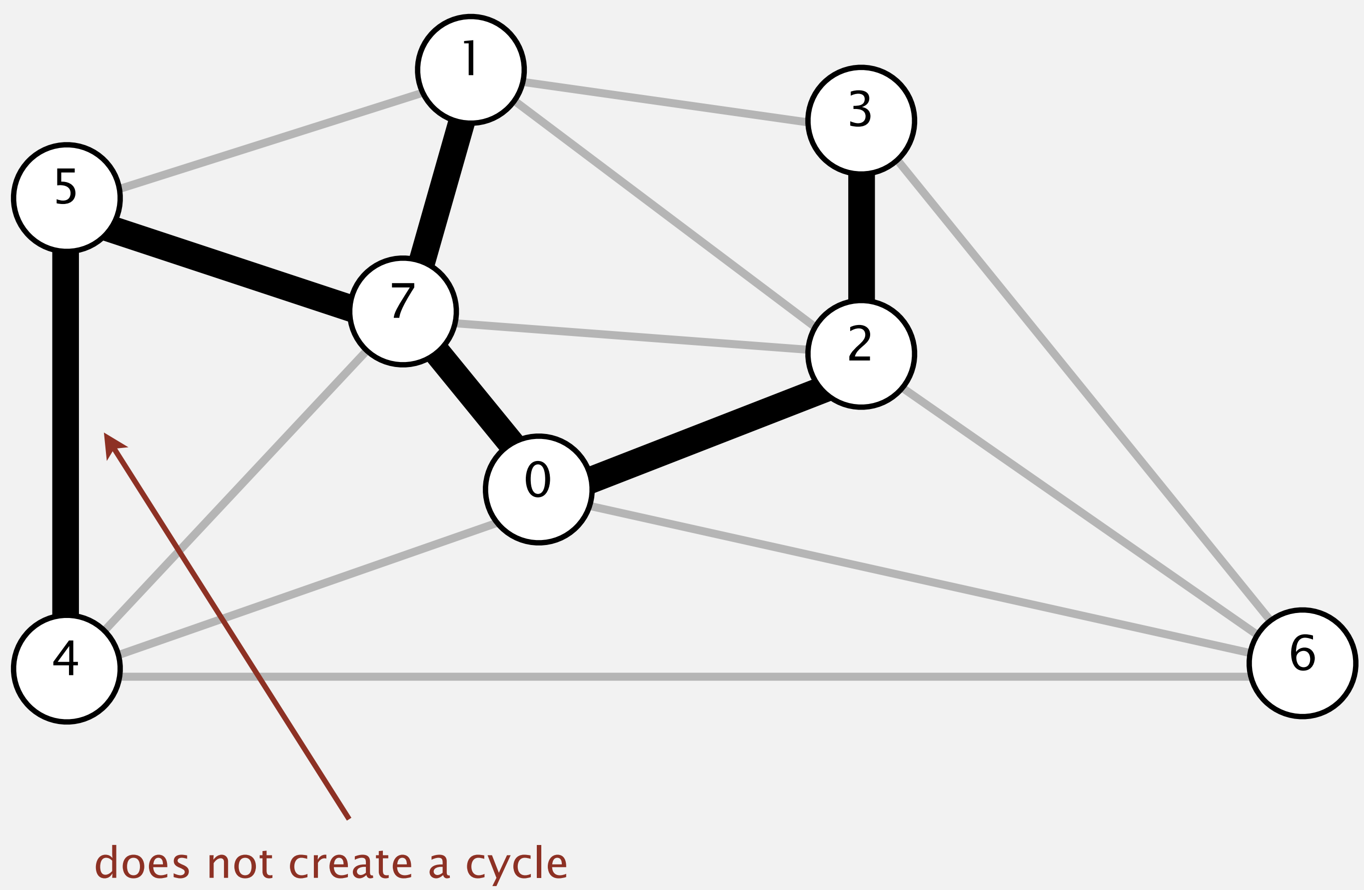
0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34

not in MST

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.



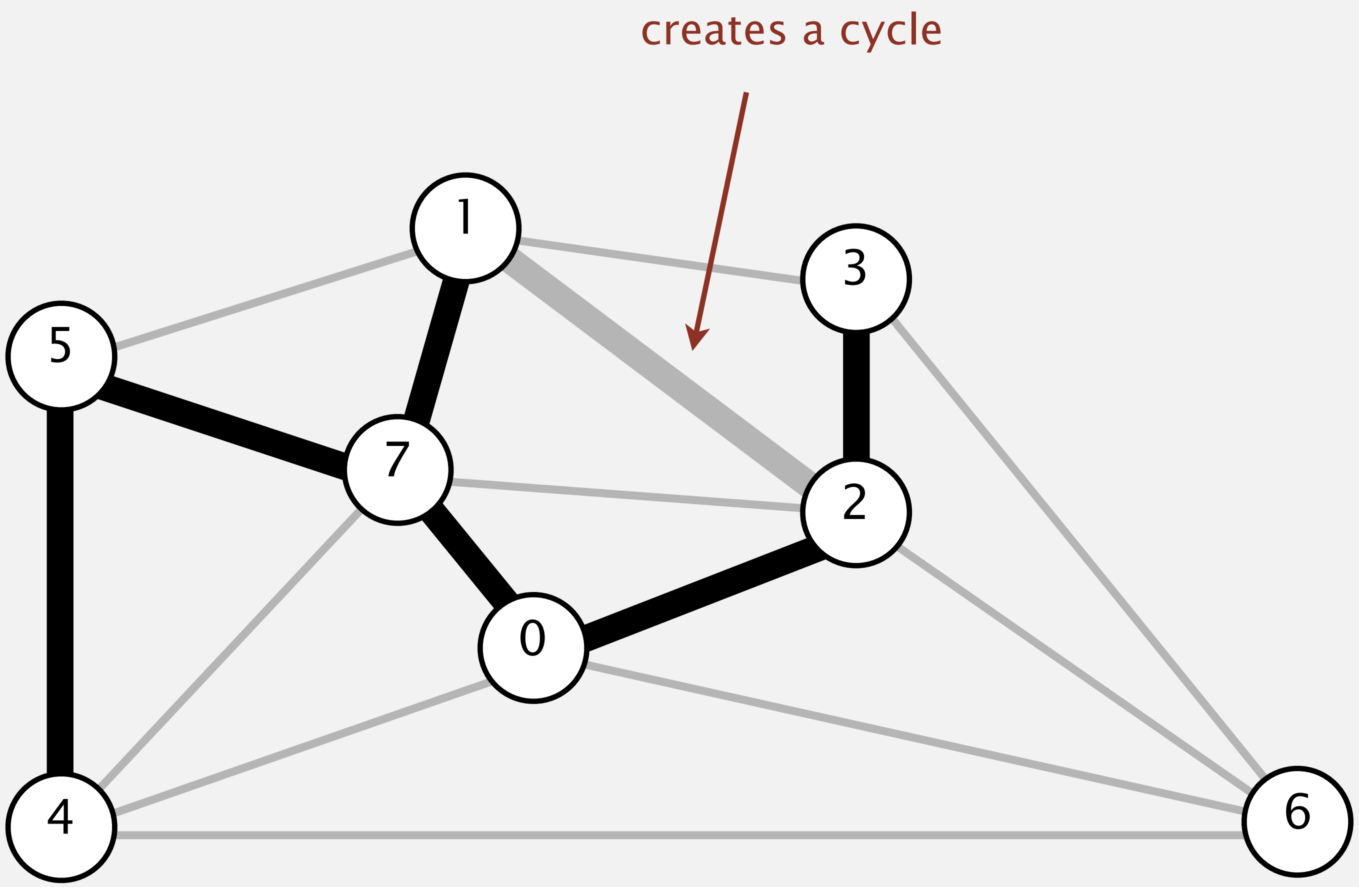
in MST →

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.



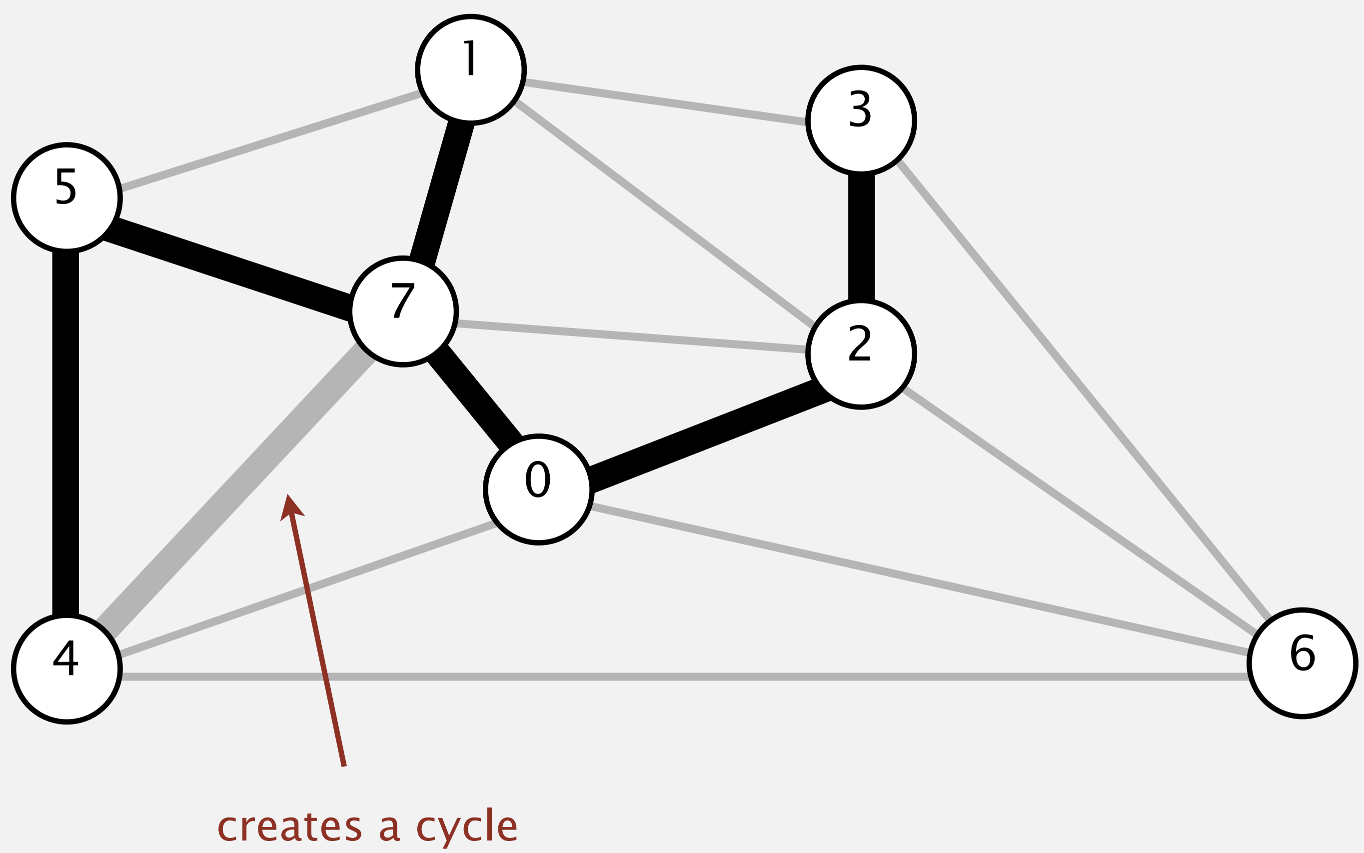
not in MST

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.



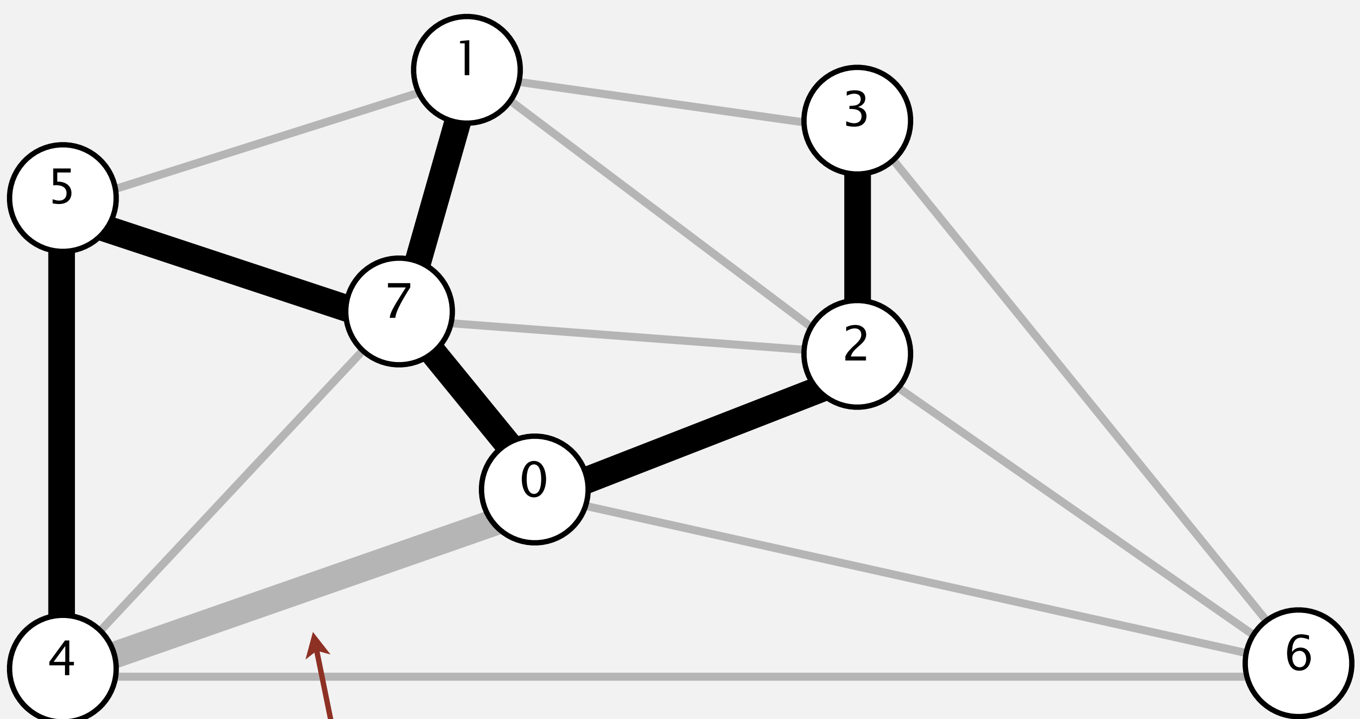
0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37

not in MST → 4-7 0.37

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.



creates a cycle

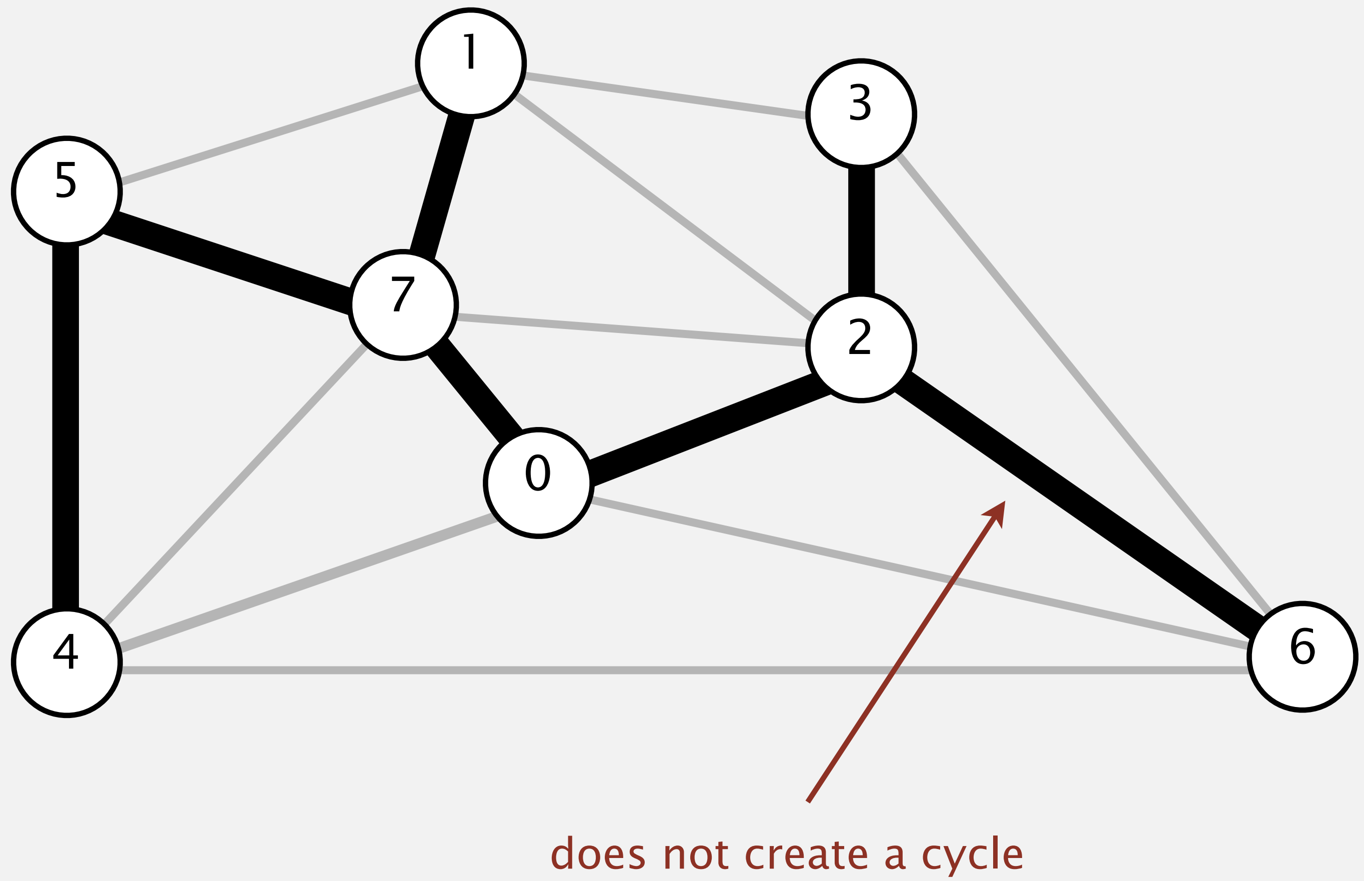
not in MST

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.



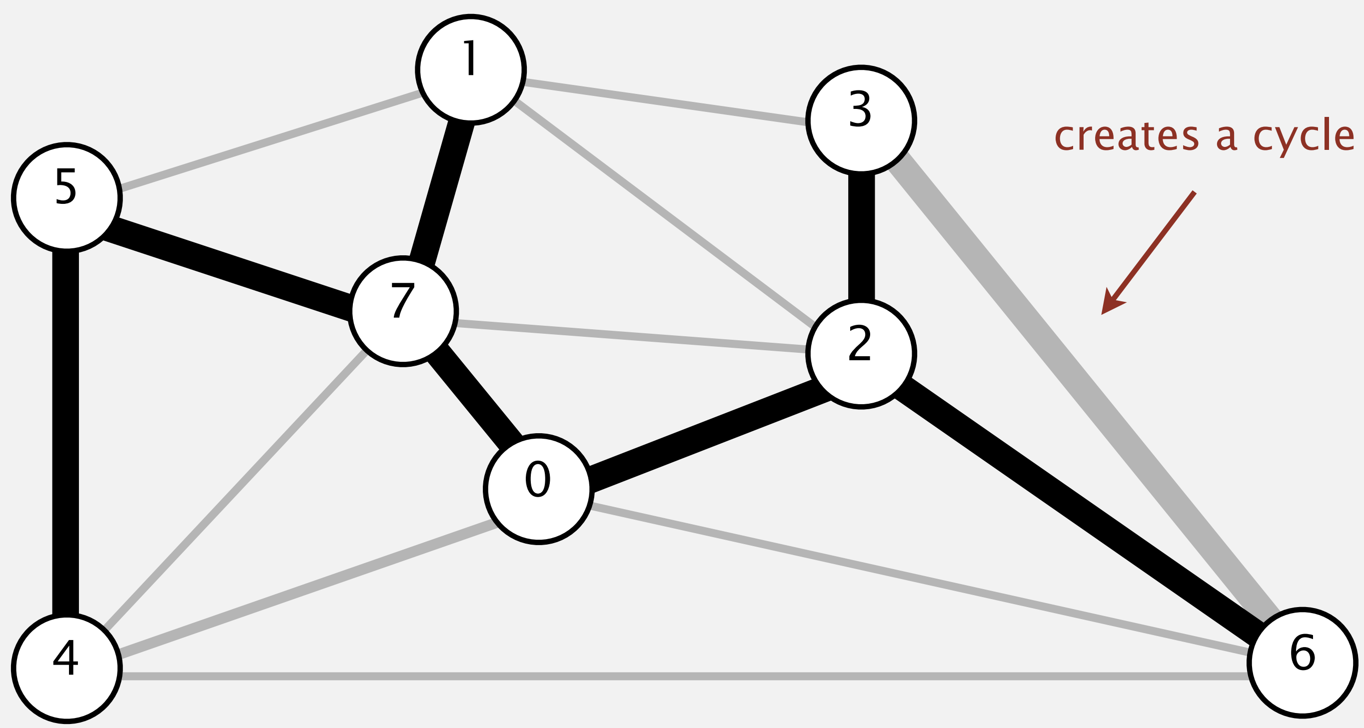
0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40

in MST

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

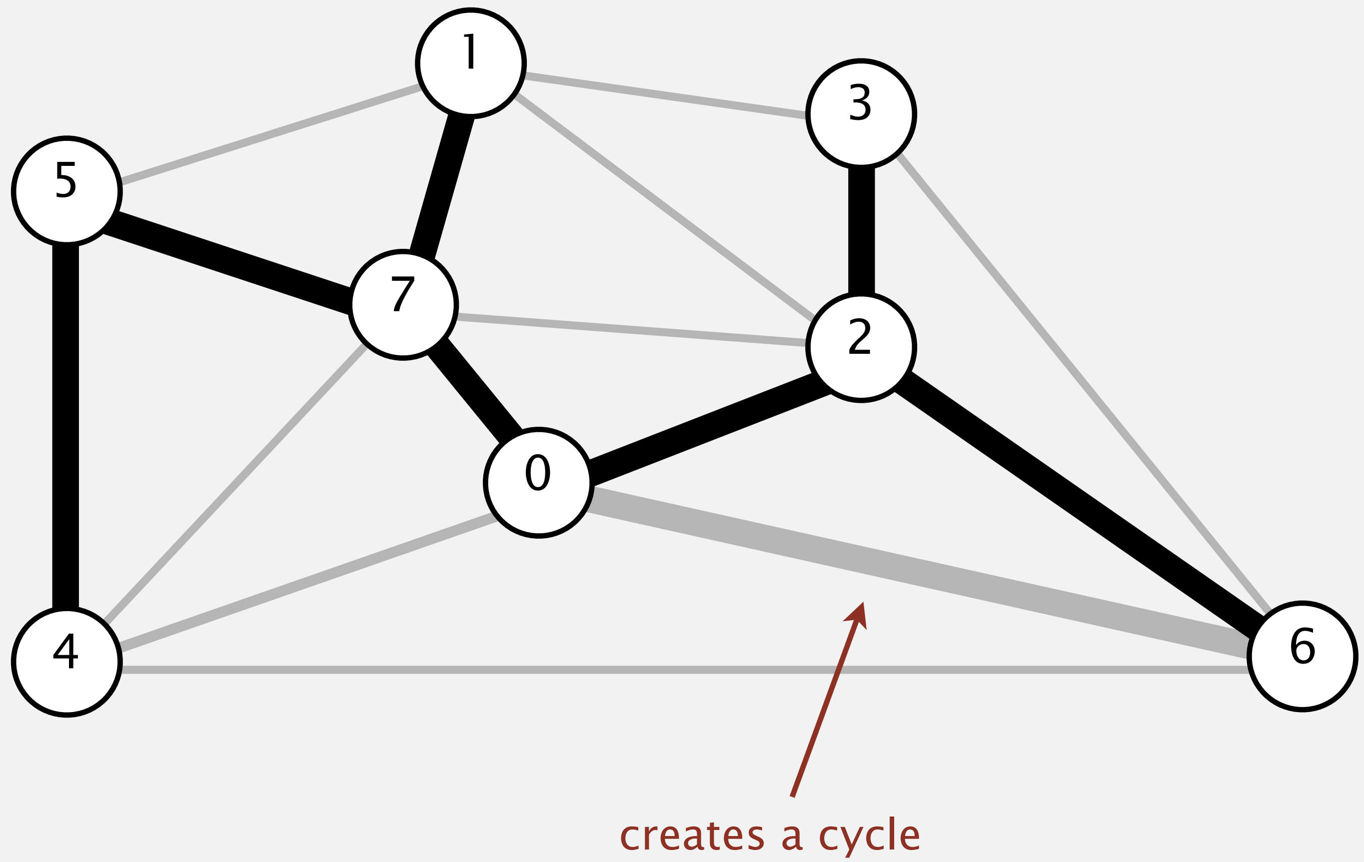


0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
not in MST → 3-6	0.52

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.



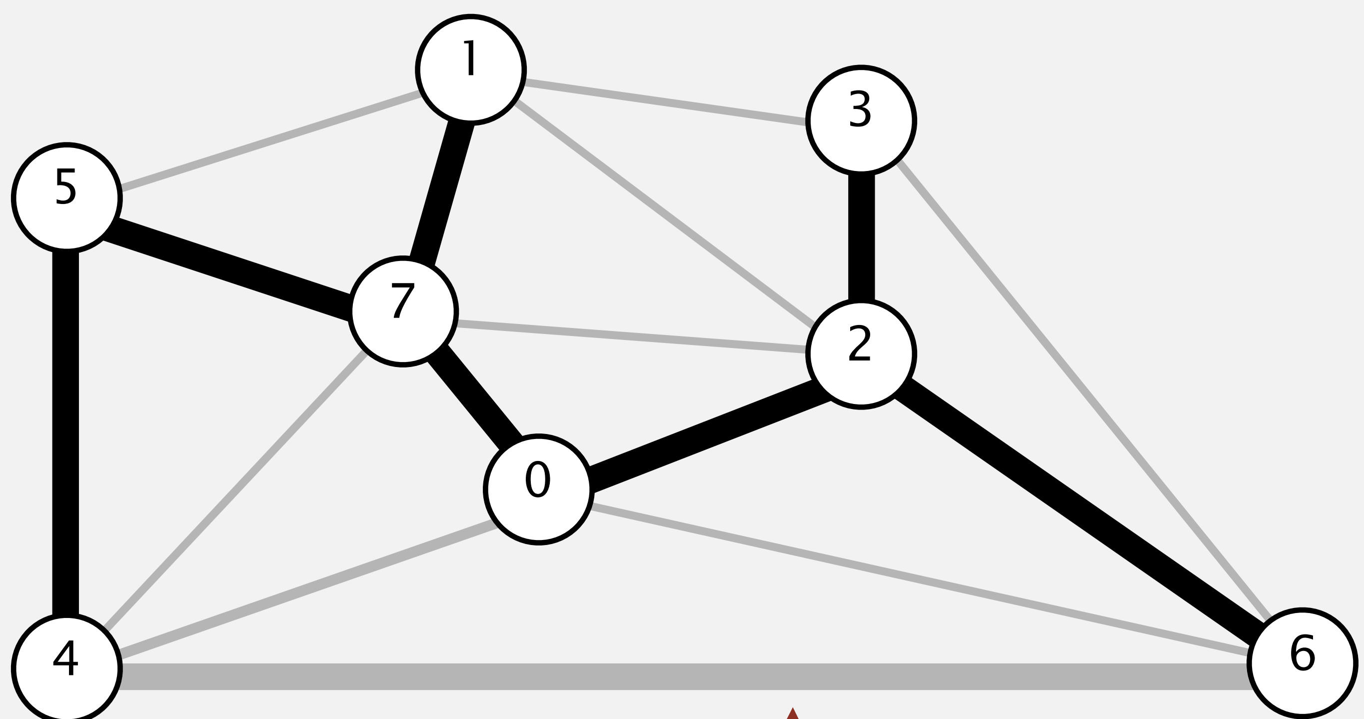
0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58

not in MST →

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.



creates a cycle

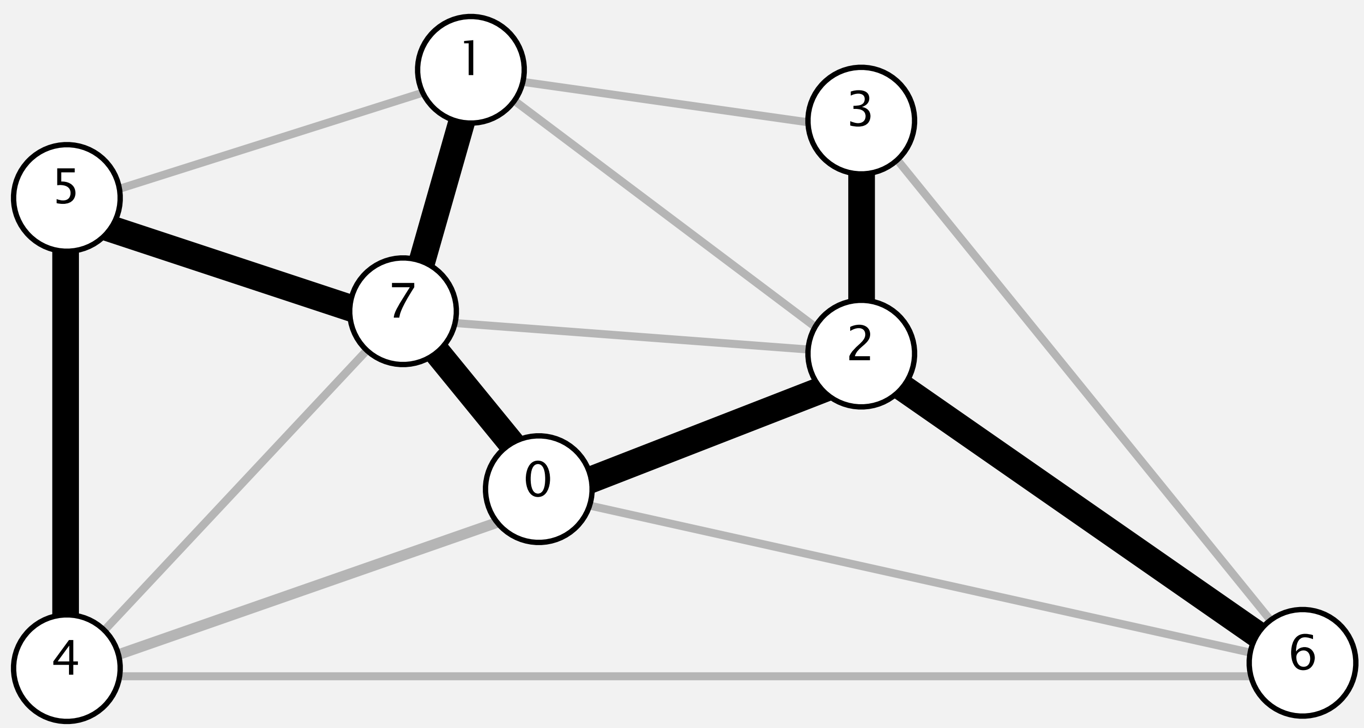
0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

not in MST

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.



a minimum spanning tree

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

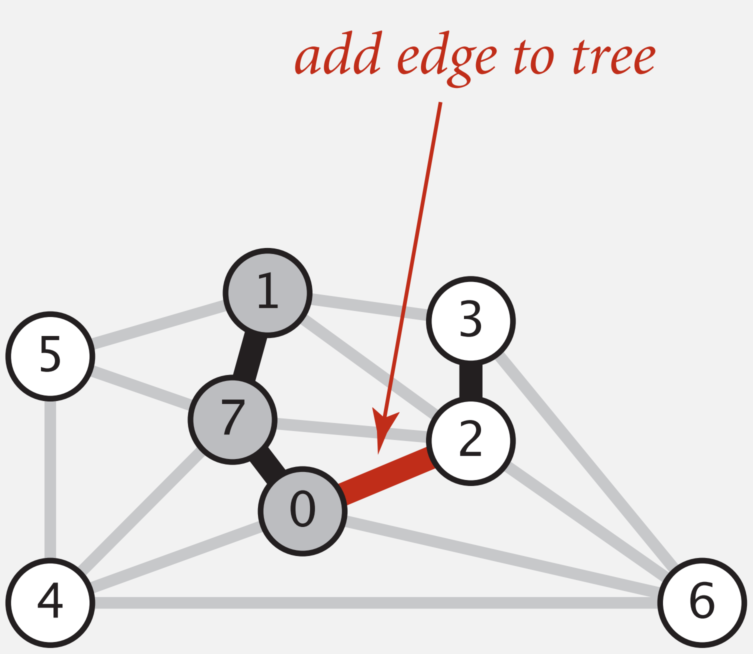
Kruskal's algorithm: visualization

Kruskal's algorithm: correctness proof

Proposition. [Kruskal 1956] Kruskal's algorithm computes the MST.

Pf. Kruskal's algorithm is a special case of the greedy MST algorithm.

- Suppose Kruskal's algorithm colors the edge $e = v–w$ black.
- Cut = set of vertices connected to v in tree T.
- No crossing edge is black (by the algorithm).
- No crossing edge has lower weight. Why?

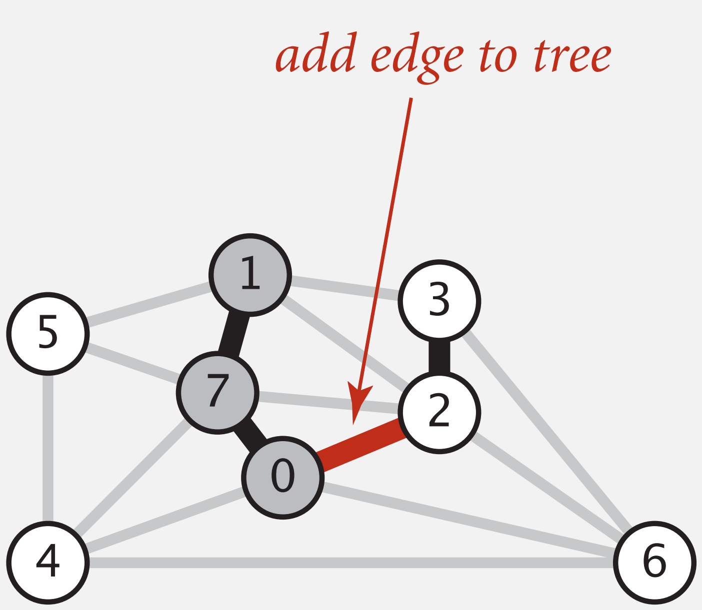


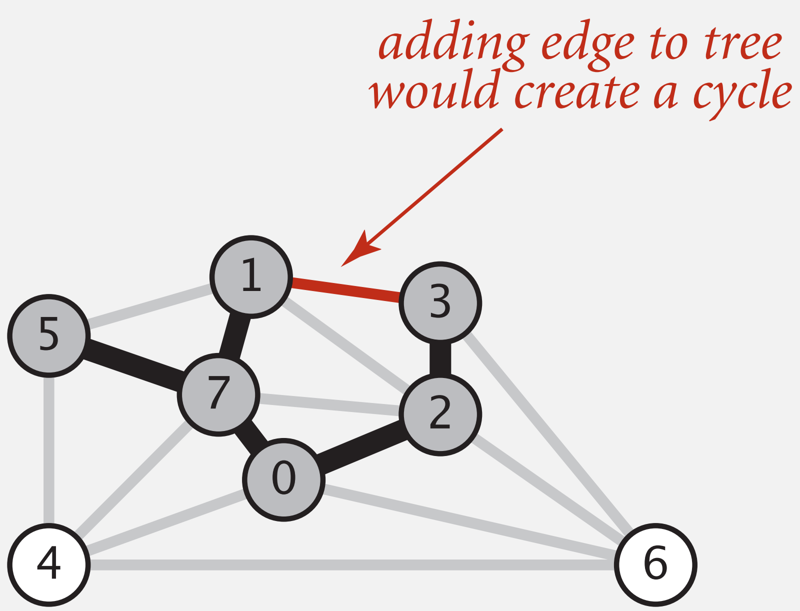
Kruskal's algorithm: implementation challenge

Challenge. Would adding edge $v–w$ to tree T create a cycle? If not, add it.

How difficult?

- $E + V$
- V ← run DFS from v, check if w is reachable (T has at most V – 1 edges)
- $\log V$
- $\log^* V$ ← use the union-find data structure !
- 1



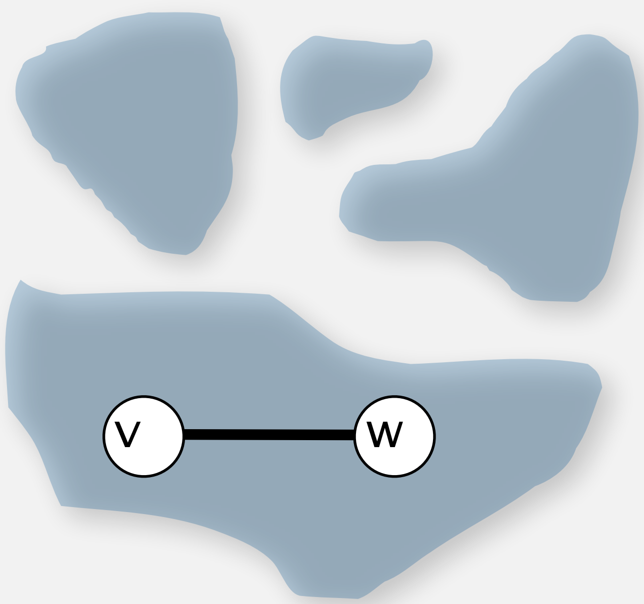


Kruskal's algorithm: implementation challenge

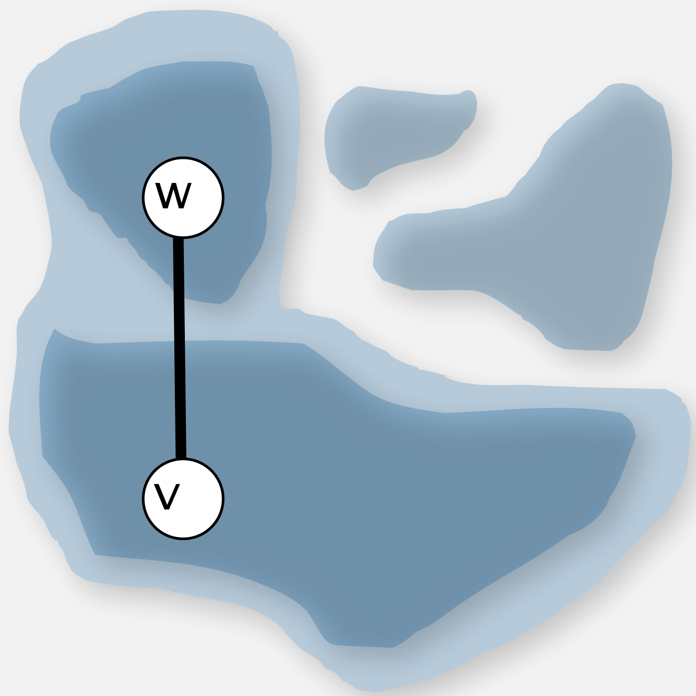
Challenge. Would adding edge $v–w$ to tree T create a cycle? If not, add it.

Efficient solution. Use the union-find data structure.

- Maintain a set for each connected component in T.
- If v and w are in same set, then adding $v–w$ would create a cycle.
- To add $v–w$ to T, merge sets containing v and w.



Case 1: adding v–w creates a cycle



Case 2: add v–w to T and merge sets containing v and w

Kruskal's algorithm: Java implementation

```
public class KruskalMST
{
   private Queue<Edge> mst = new Queue<Edge>();

   public KruskalMST(EdgeWeightedGraph G)
   {
      MinPQ<Edge> pq = new MinPQ<Edge>(G.edges());

      UF uf = new UF(G.V());
      while (!pq.isEmpty() && mst.size() < G.V()-1)
      {
         Edge e = pq.delMin();
         int v = e.either(), w = e.other(v);
         if (!uf.connected(v, w))
         {
            uf.union(v, w);
            mst.enqueue(e);
         }
      }
   }

   public Iterable<Edge> edges()
   {  return mst;  }
}
```

build priority queue (or sort)

greedily add edges to MST

edge v–w does not create cycle

merge sets

add edge to MST

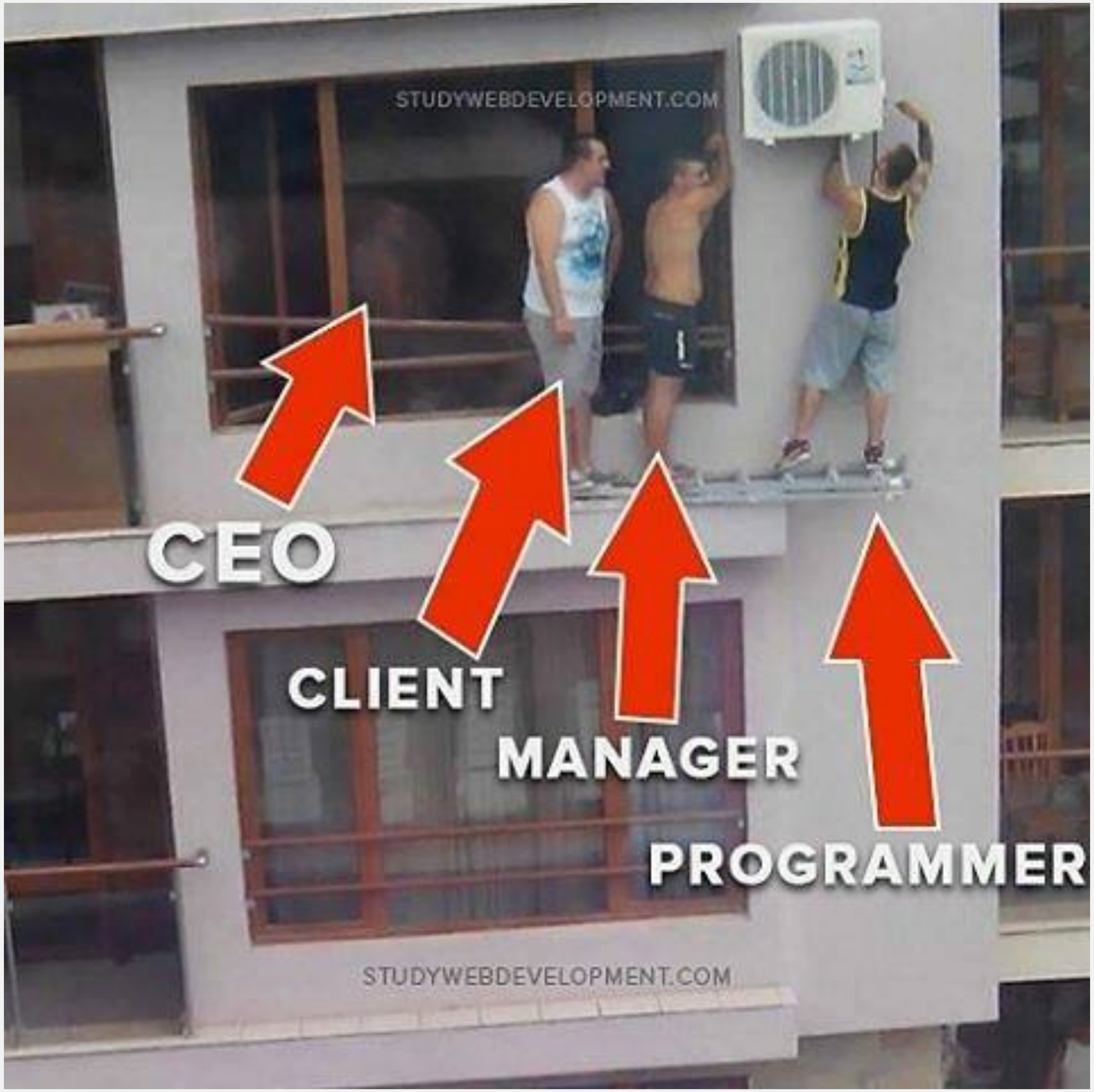
Kruskal's algorithm: running time

Proposition. Kruskal's algorithm computes MST in time proportional to $E \log E$ (in the worst case).

Pf.

operation	frequency	time per op
build pq	1	E
delete-min	E	$\log E$
union	V	$\log^* V$ †
connected	E	$\log^* V$ †

† amortized bound using weighted quick union with path compression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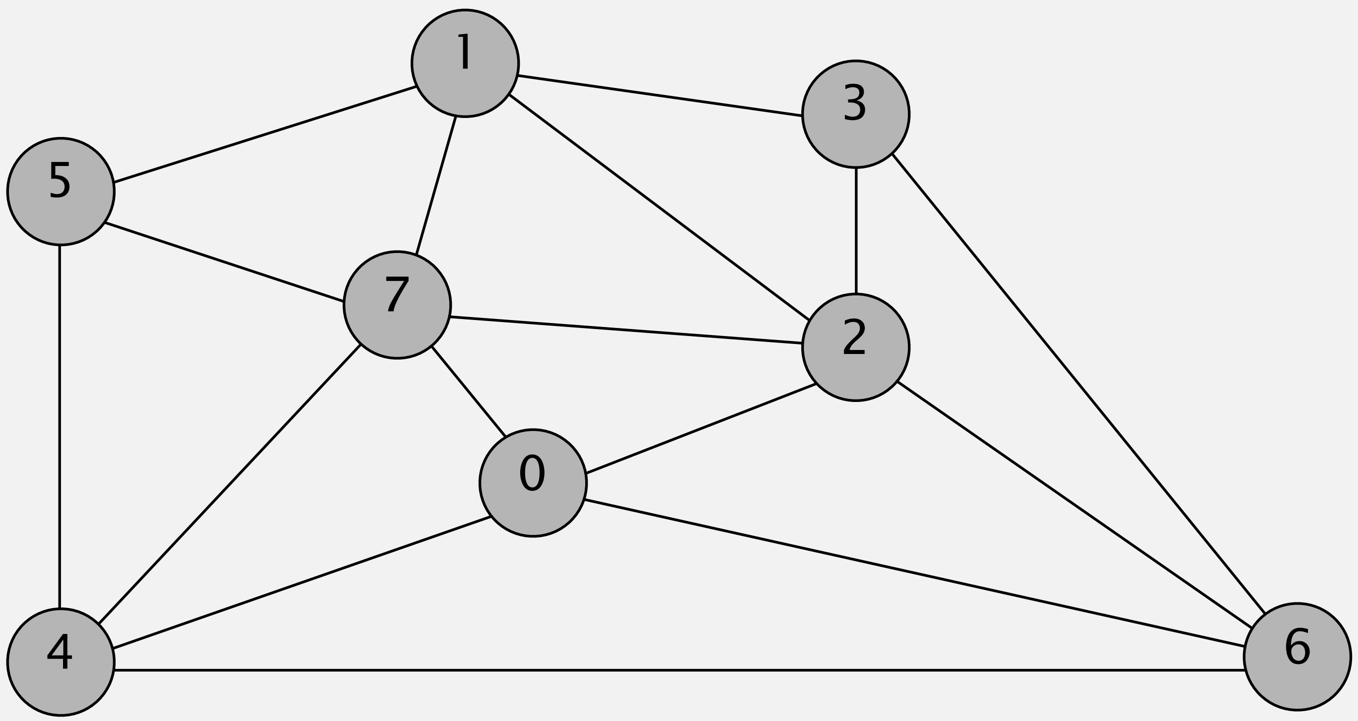
Minimum Spanning Trees

- *introduction*
- *edge-weighted graph API*
- *Kruskal's algorithm*
- *Prim's algorithm*

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



an edge-weighted graph

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
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•

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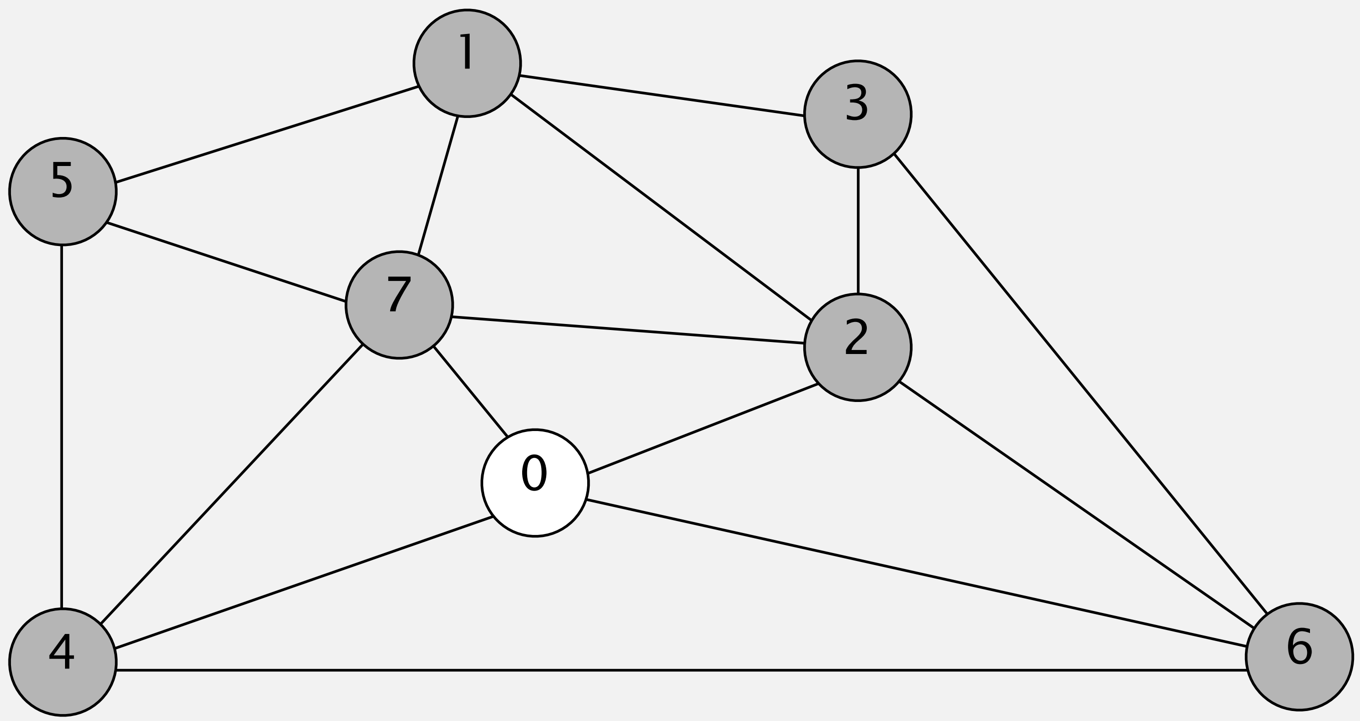

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
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- Repeat until $V - 1$ edges.

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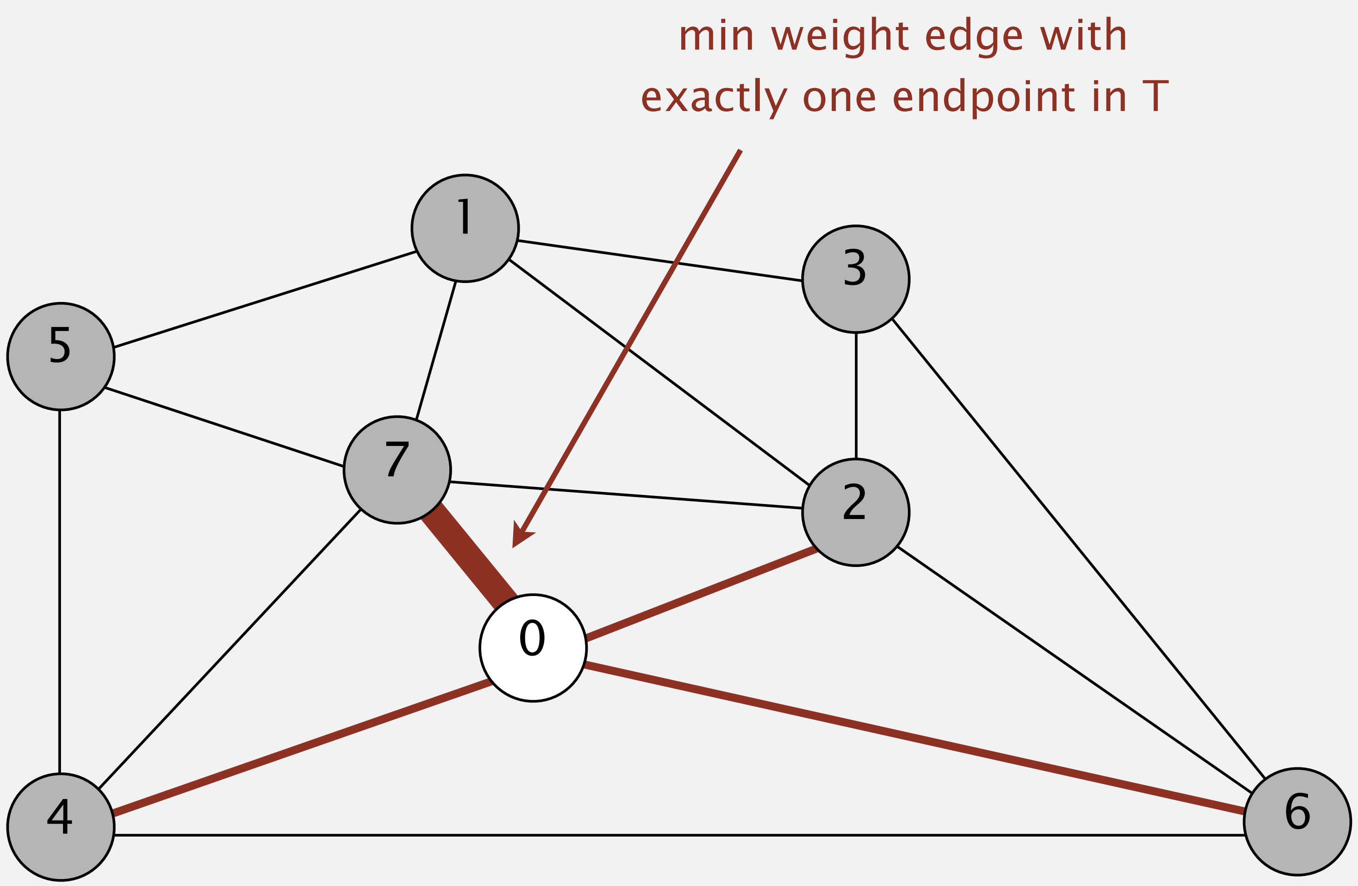

edges with exactly one endpoint in T (sorted by weight)

in MST →

0-7	0.16
0-2	0.26
0-4	0.38
6-0	0.58

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

-

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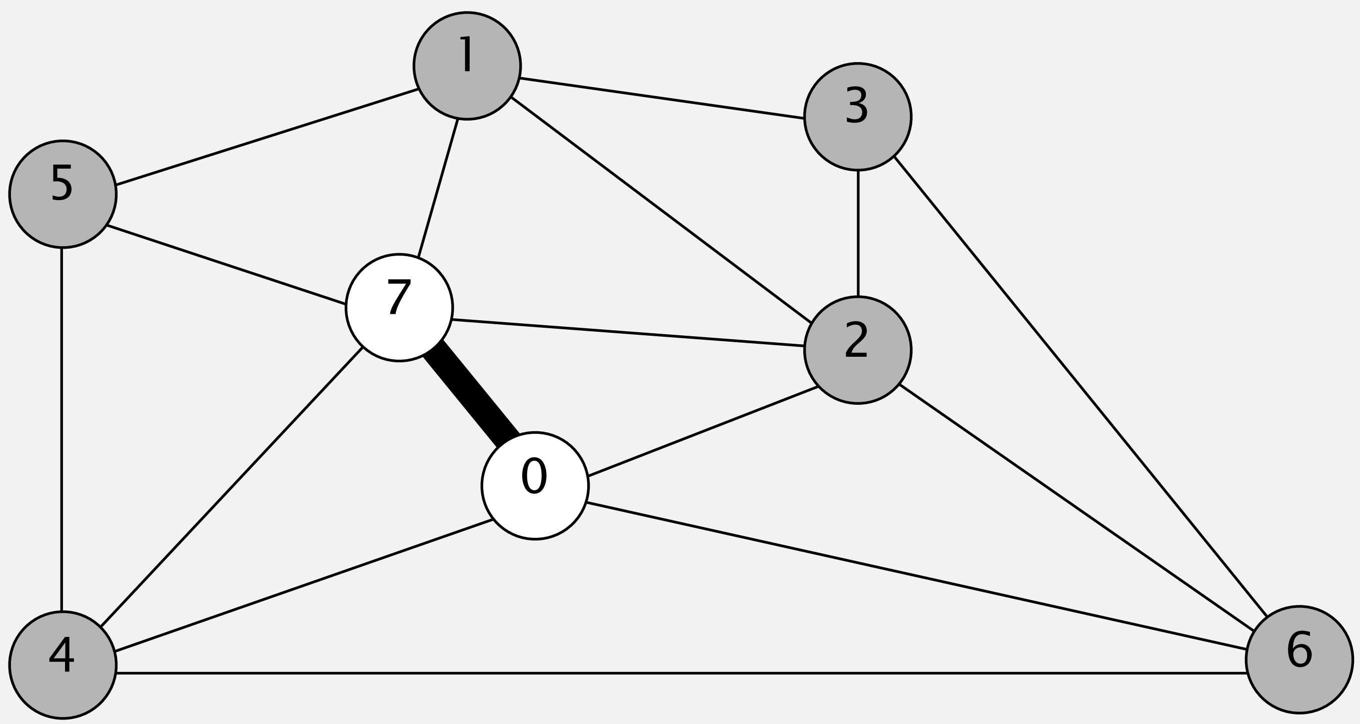

MST edges

0-7

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
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6-0	0.58
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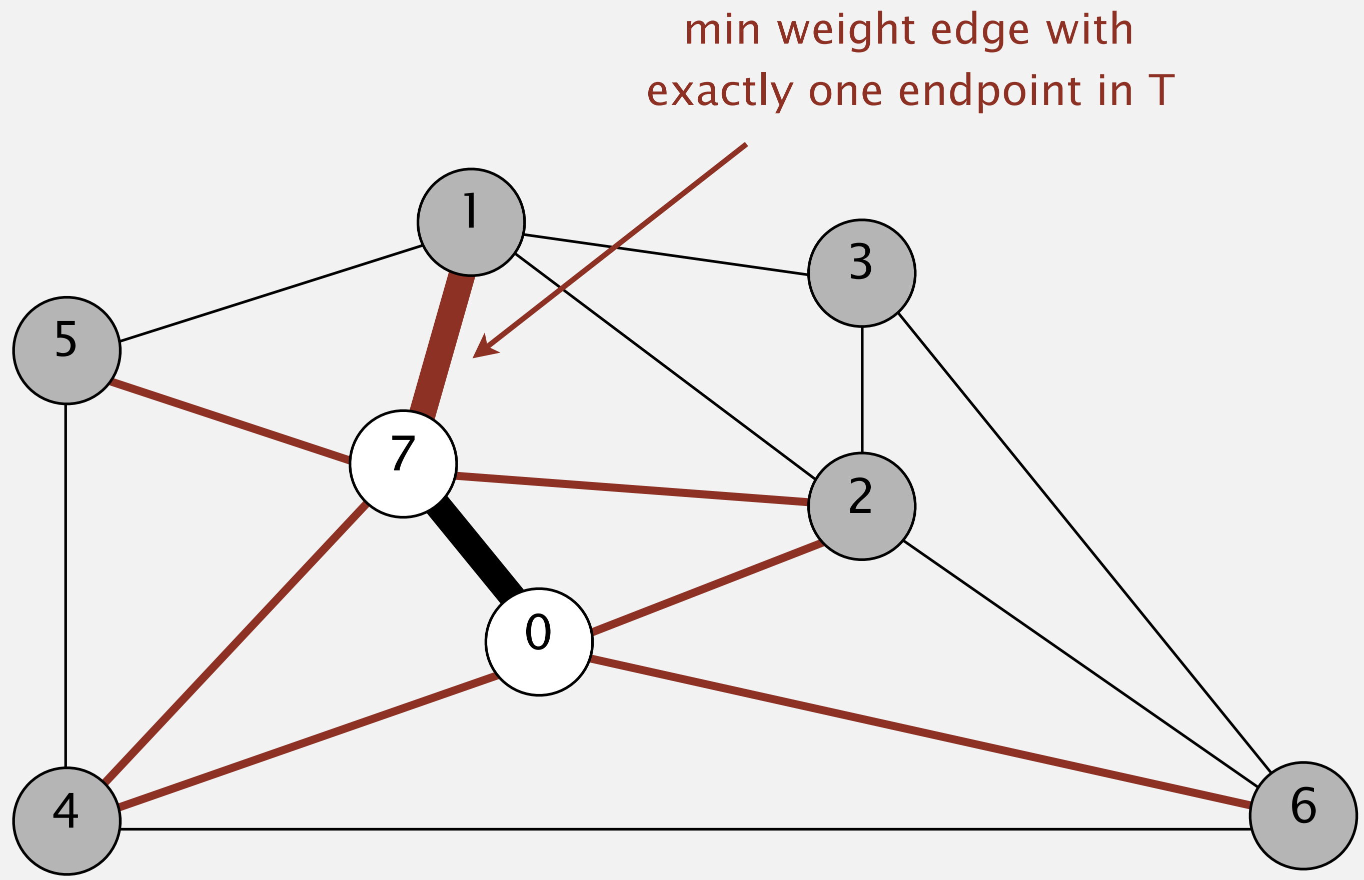

edges with exactly one endpoint in T (sorted by weight)

in MST →

1-7	0.19
0-2	0.26
5-7	0.28
2-7	0.34
4-7	0.37
0-4	0.38
6-0	0.58

MST edges

0-7

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

•

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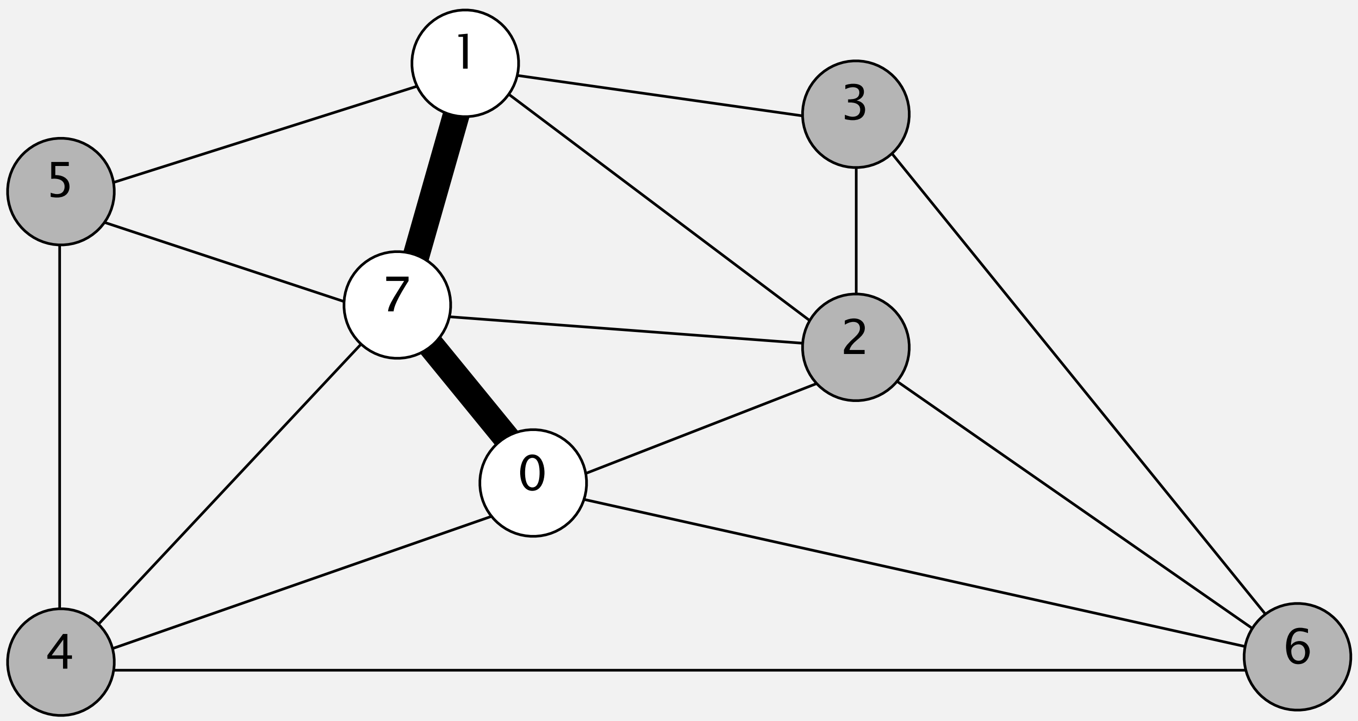


MST edges

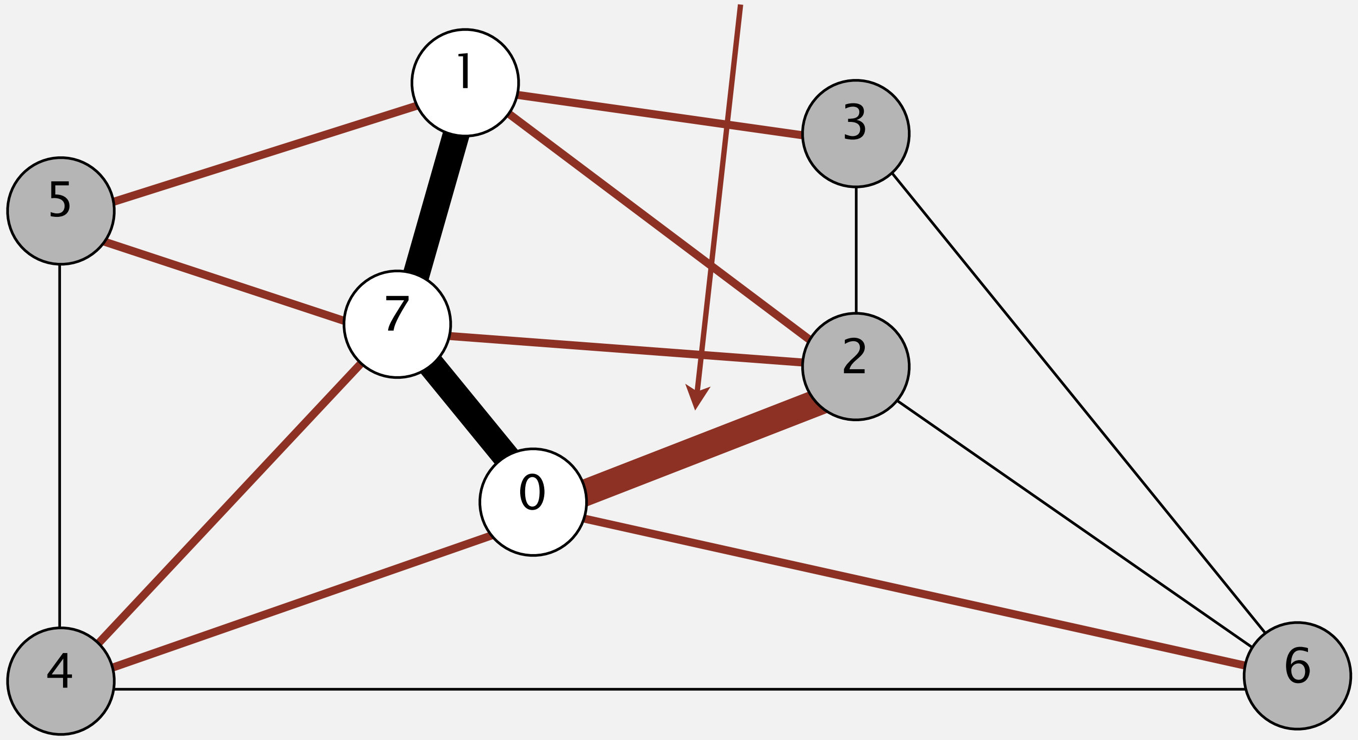
0-7 1-7

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

min weight edge with
exactly one endpoint in T



edges with exactly
one endpoint in T
(sorted by weight)

in MST

0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
1-2	0.36
4-7	0.37
0-4	0.38
6-0	0.58

MST edges

0-7 1-7

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.
-

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
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1-3	0.29
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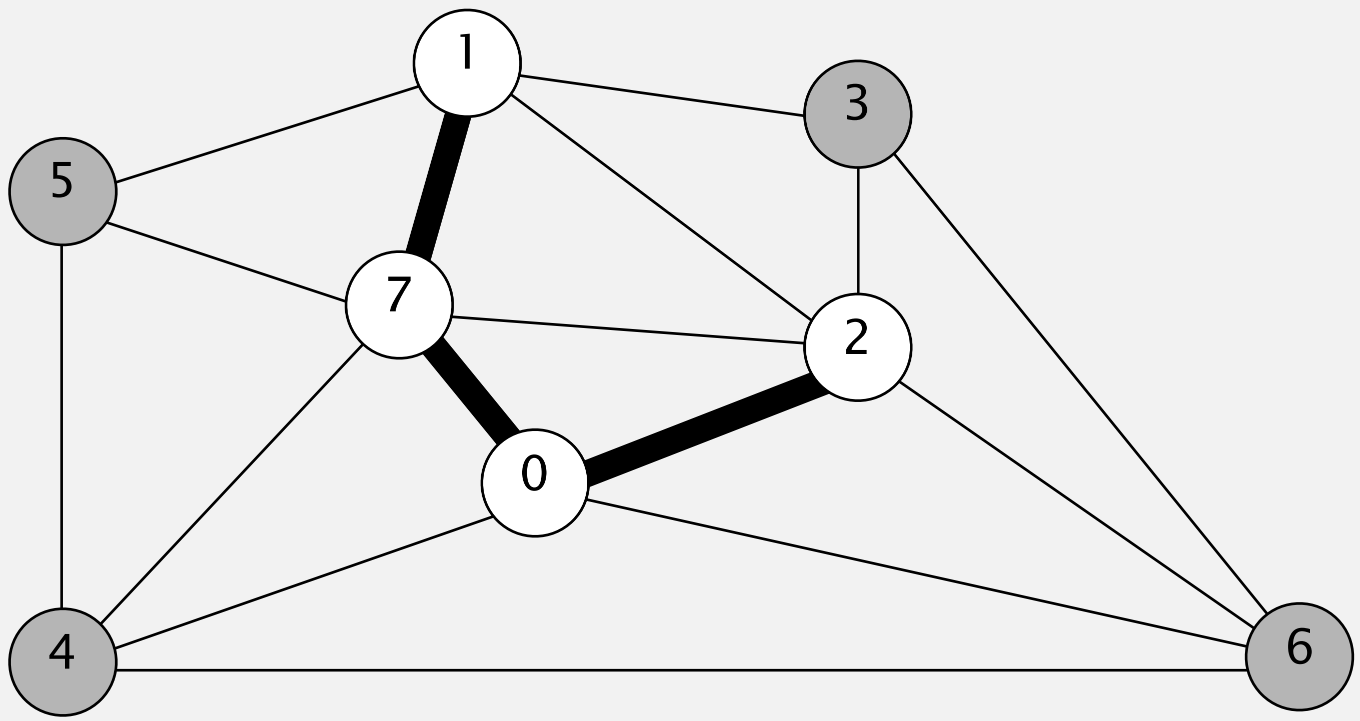
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



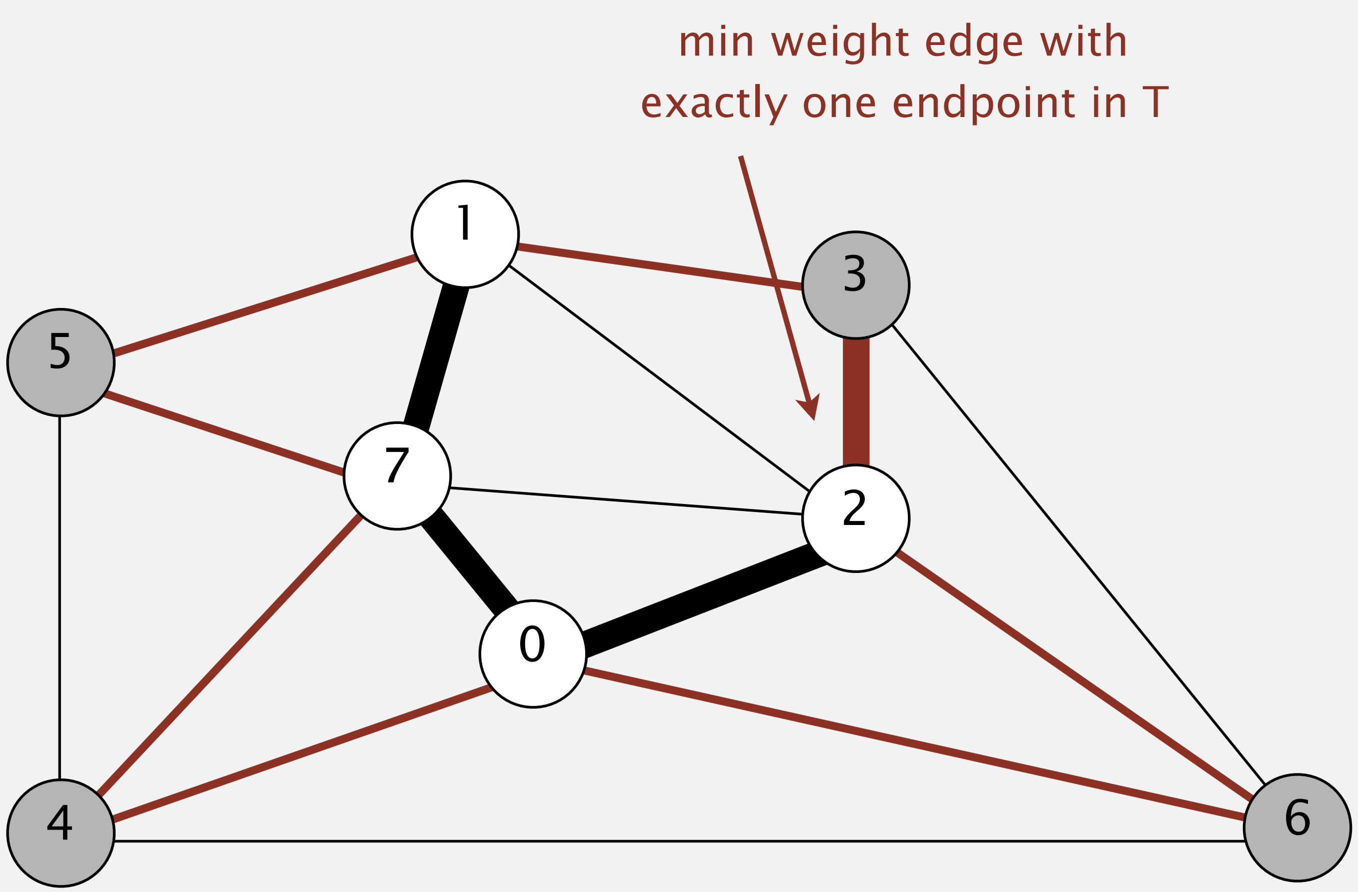
MST edges

0-7 1-7 0-2

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



edges with exactly one endpoint in T (sorted by weight)

in MST → 2-3 0.17

2-3	0.17
5-7	0.28
1-3	0.29
1-5	0.32
4-7	0.37
0-4	0.38
6-2	0.40
6-0	0.58

MST edges

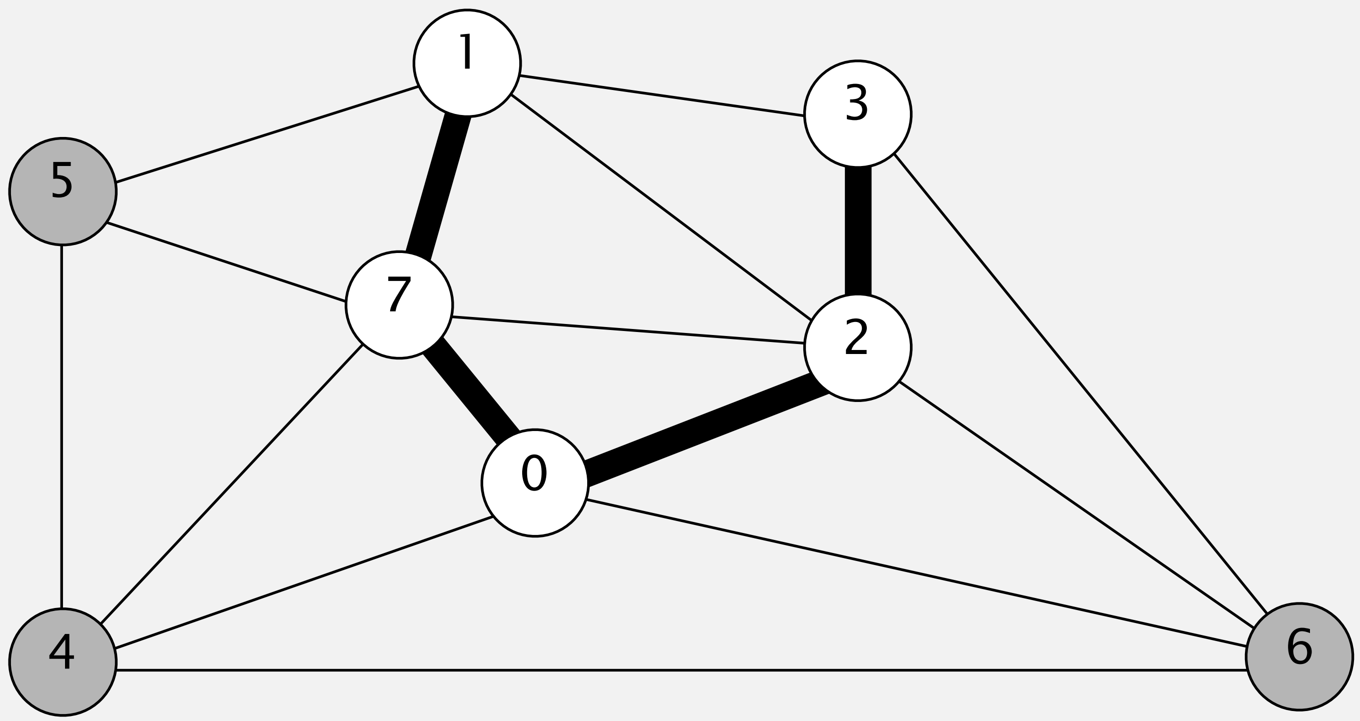
0-7 1-7 0-2

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

•

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



MST edges

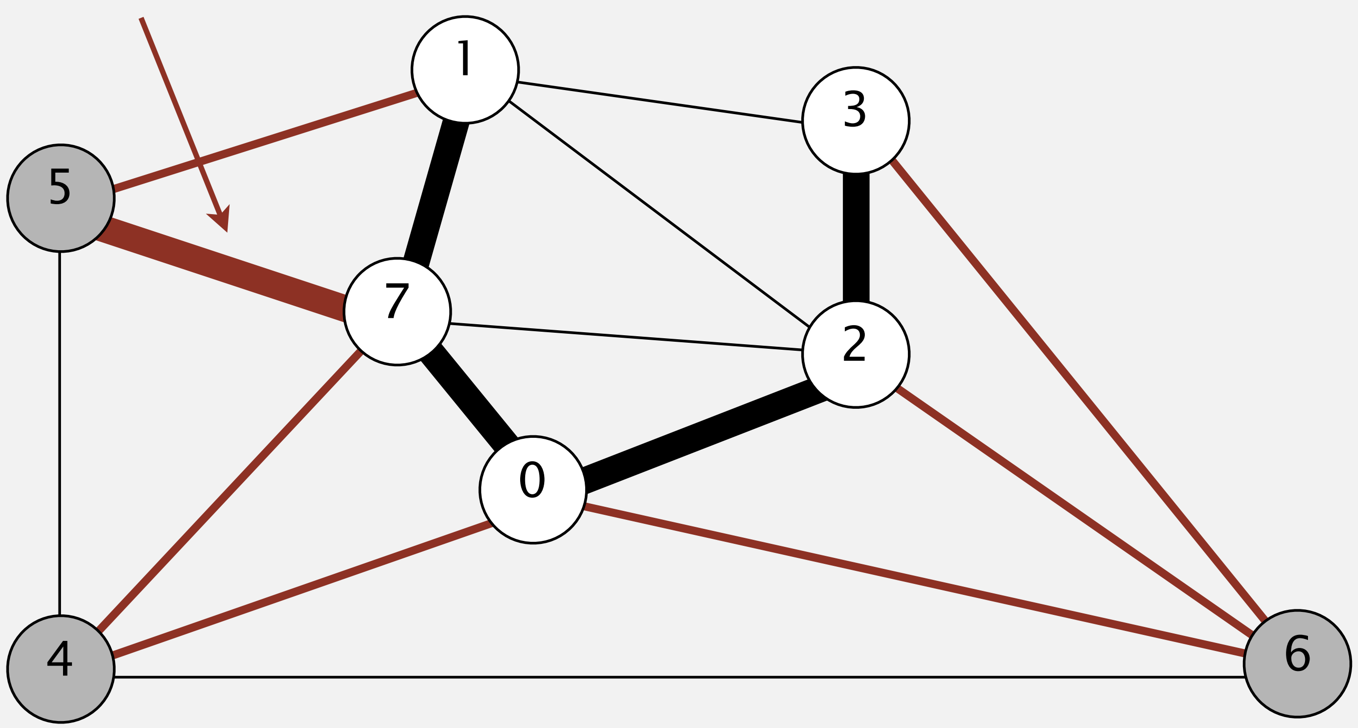
0-7 1-7 0-2 2-3

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

•

min weight edge with
exactly one endpoint in T



edges with exactly
one endpoint in T
(sorted by weight)

in MST →	5-7	0.28
	1-5	0.32
	4-7	0.37
	0-4	0.38
	6-2	0.40
	3-6	0.52
	6-0	0.58

MST edges

0-7 1-7 0-2 2-3

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

•

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
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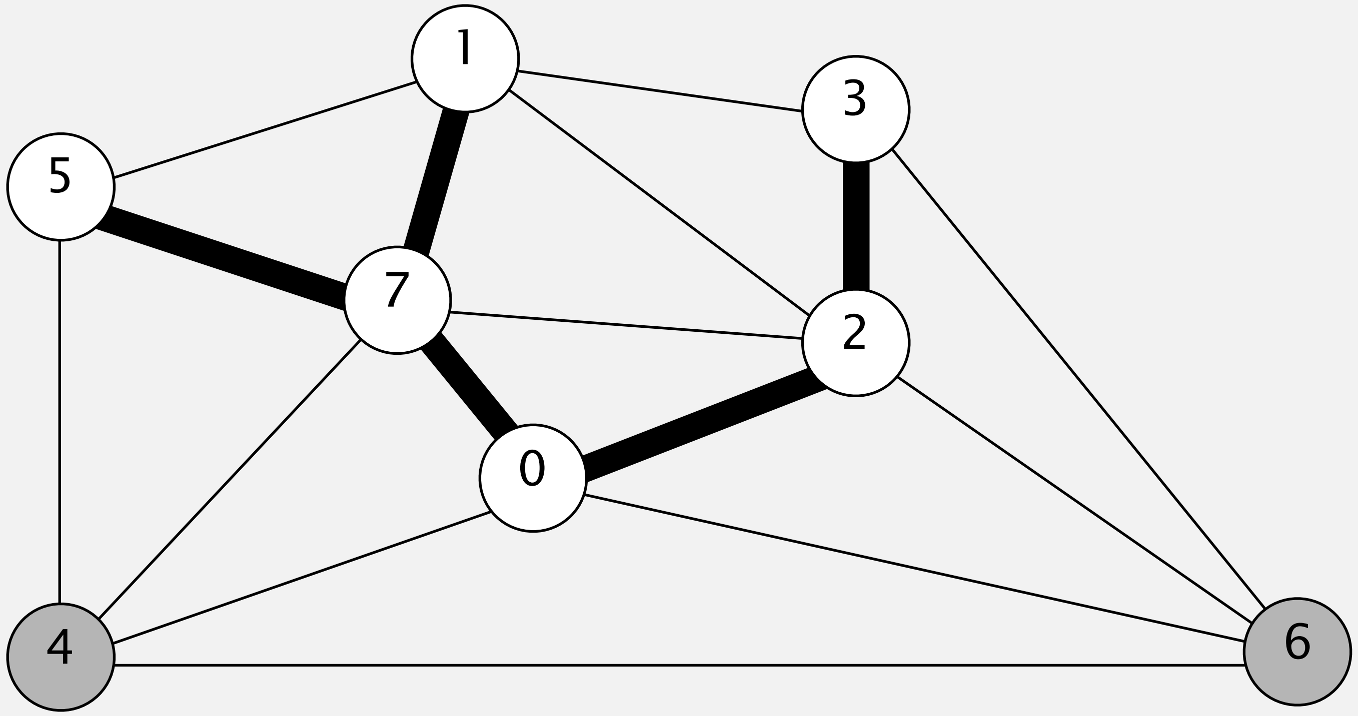
6-0	0.58
6-4	0.93



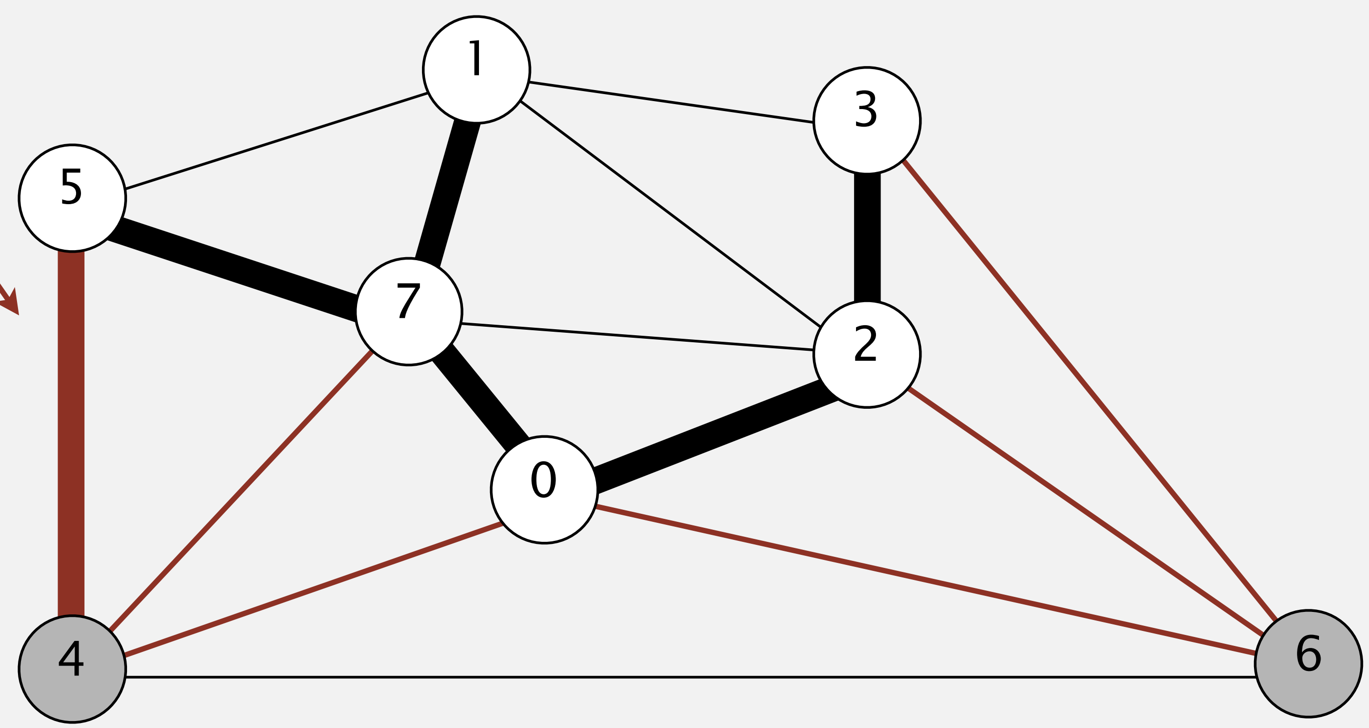
MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

min weight edge with exactly one endpoint in T



edges with exactly one endpoint in T (sorted by weight)

in MST →	4-5	0.35
	4-7	0.37
	0-4	0.38
	6-2	0.40
	3-6	0.52
	6-0	0.58

MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

•

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
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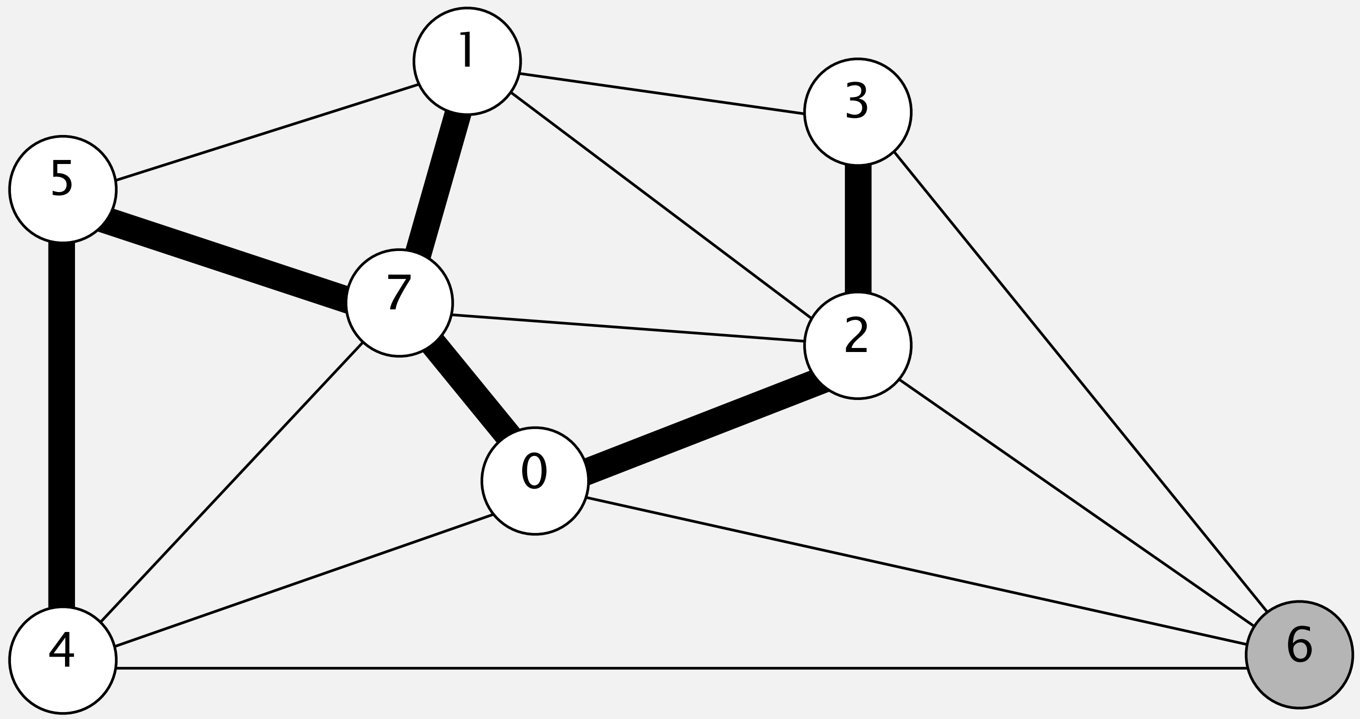
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



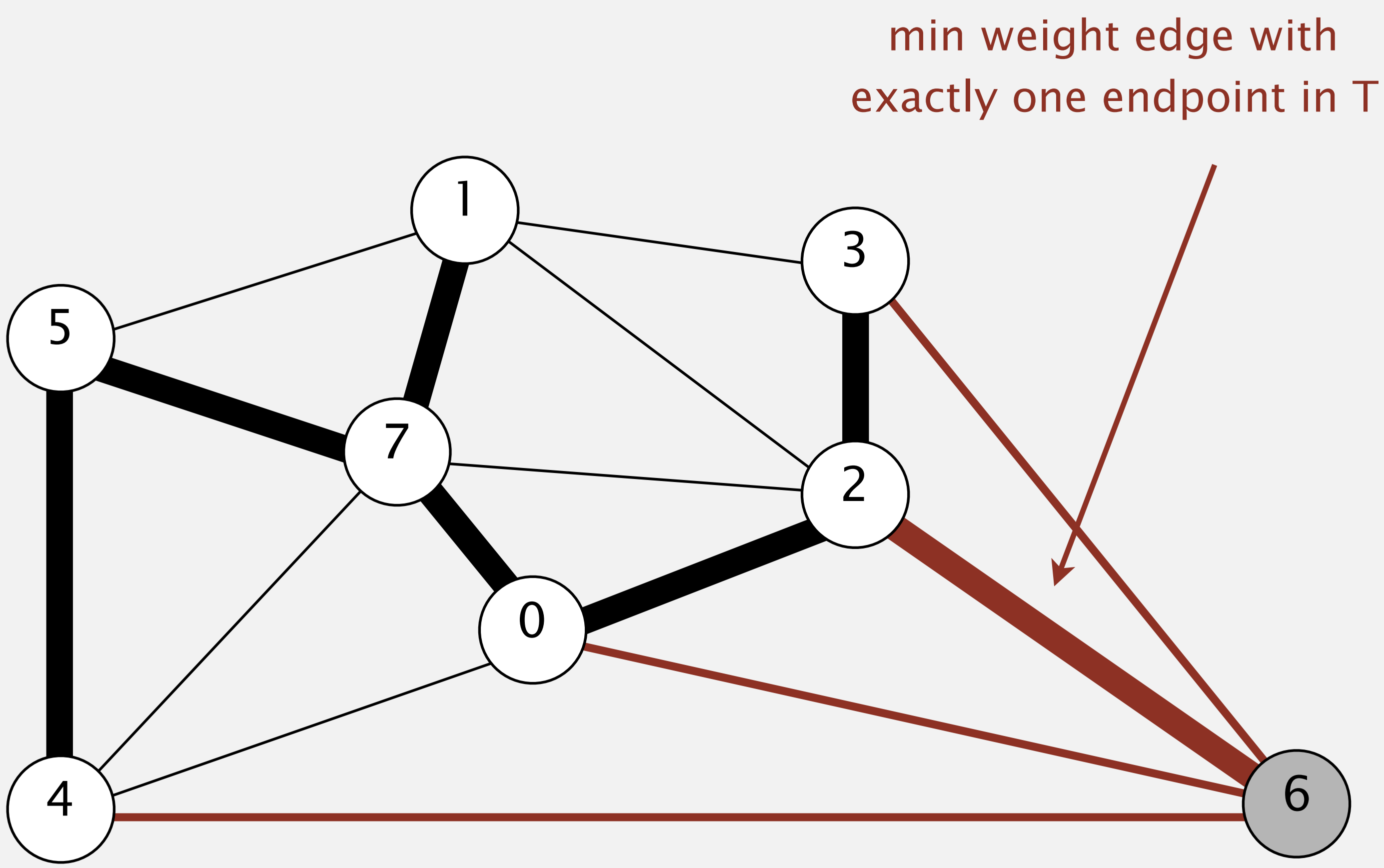
MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



edges with exactly one endpoint in T (sorted by weight)

in MST → 6-2 0.40

6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

•

0-7	0.16
2-3	0.17
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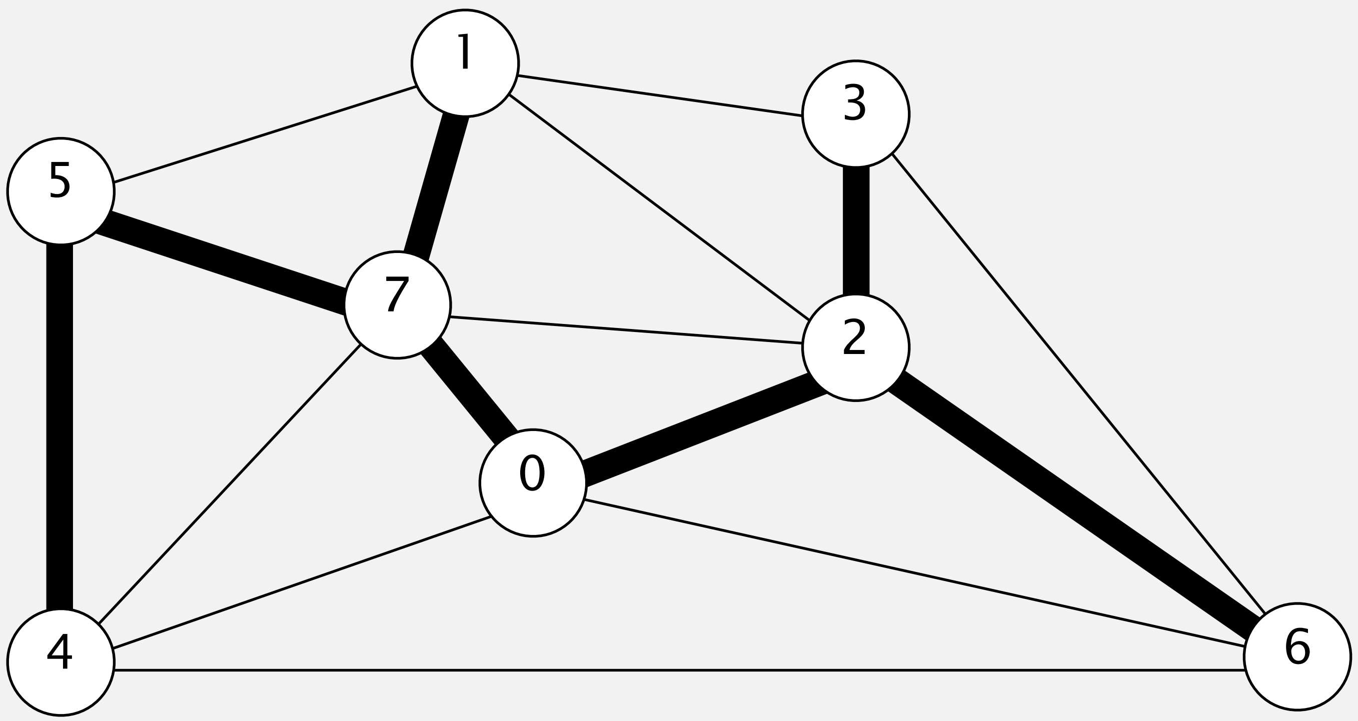
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



MST edges

0-7 1-7 0-2 2-3 5-7 4-5 6-2

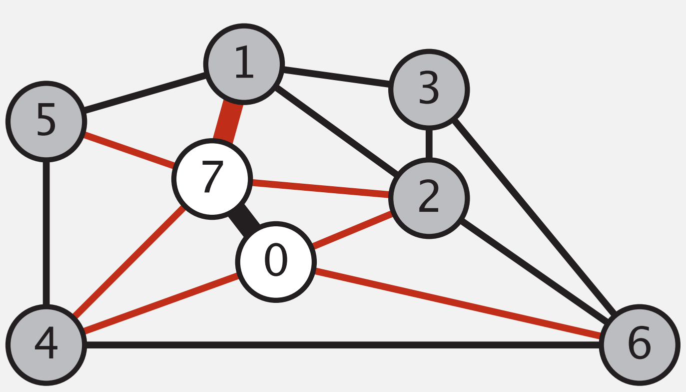
Prim's algorithm: implementation challenge

Challenge. Find the min weight edge with exactly one endpoint in T.

How difficult?

- E ← try all edges
- $E \log E$
- $\log E$ ← use a priority queue!
- $\log^* E$
- 1

1-7 is min weight edge with exactly one endpoint in T



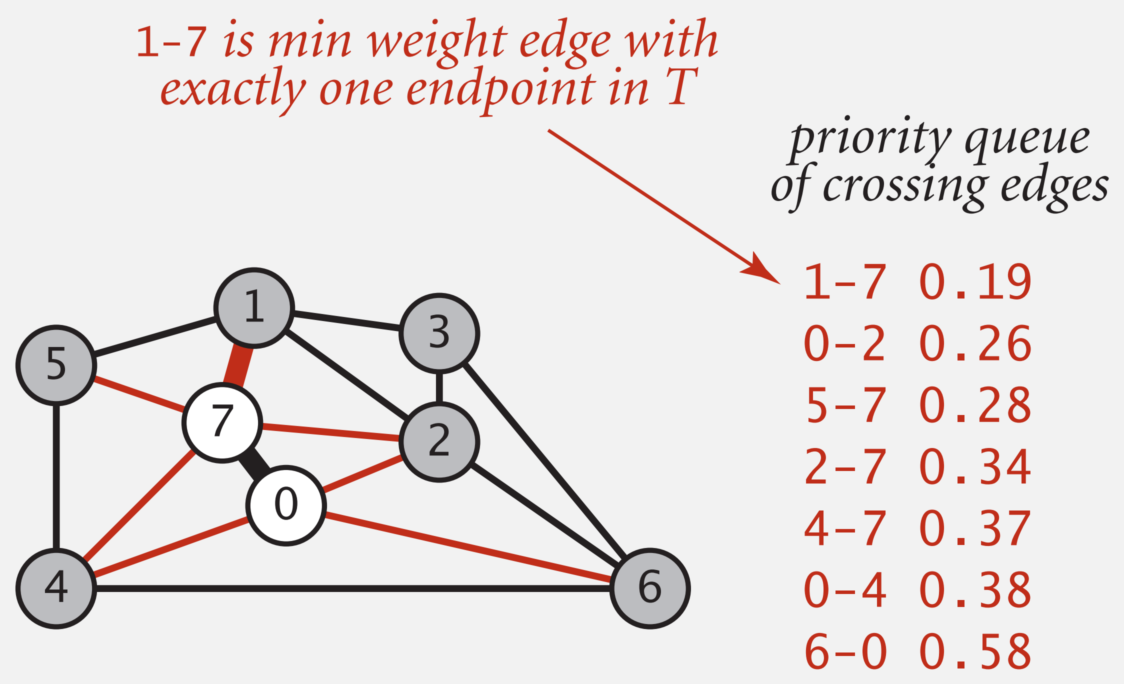

```
1-7 0.19
0-2 0.26
5-7 0.28
2-7 0.34
4-7 0.37
0-4 0.38
6-0 0.58
```

Prim's algorithm: lazy implementation

Challenge. Find the min weight edge with exactly one endpoint in T.

Lazy solution. Maintain a PQ of edges with (at least) one endpoint in T.

- Key = edge; priority = weight of edge.
- Delete-min to determine next edge $e = v–w$ to add to T.
- Disregard if both endpoints v and w are marked (both in T).
- Otherwise, let w be the unmarked vertex (not in T):
  - add to PQ any edge incident to w (assuming other endpoint not in T)
  - add e to T and mark w



Prim's algorithm: lazy implementation

```
public class LazyPrimMST
{
   private boolean[] marked;   // MST vertices
   private Queue<Edge> mst;    // MST edges
   private MinPQ<Edge> pq;     // PQ of edges

    public LazyPrimMST(WeightedGraph G)
    {
       pq = new MinPQ<Edge>();
       mst = new Queue<Edge>();
       marked = new boolean[G.V()];
       visit(G, 0);

       while (!pq.isEmpty() && mst.size() < G.V() - 1)
       {
          Edge e = pq.delMin();
          int v = e.either(), w = e.other(v);
          if (marked[v] && marked[w]) continue;
          mst.enqueue(e);
          if (!marked[v]) visit(G, v);
          if (!marked[w]) visit(G, w);
       }
   }
}
```

assume G is connected

repeatedly delete the
min weight edge e = v–w from PQ

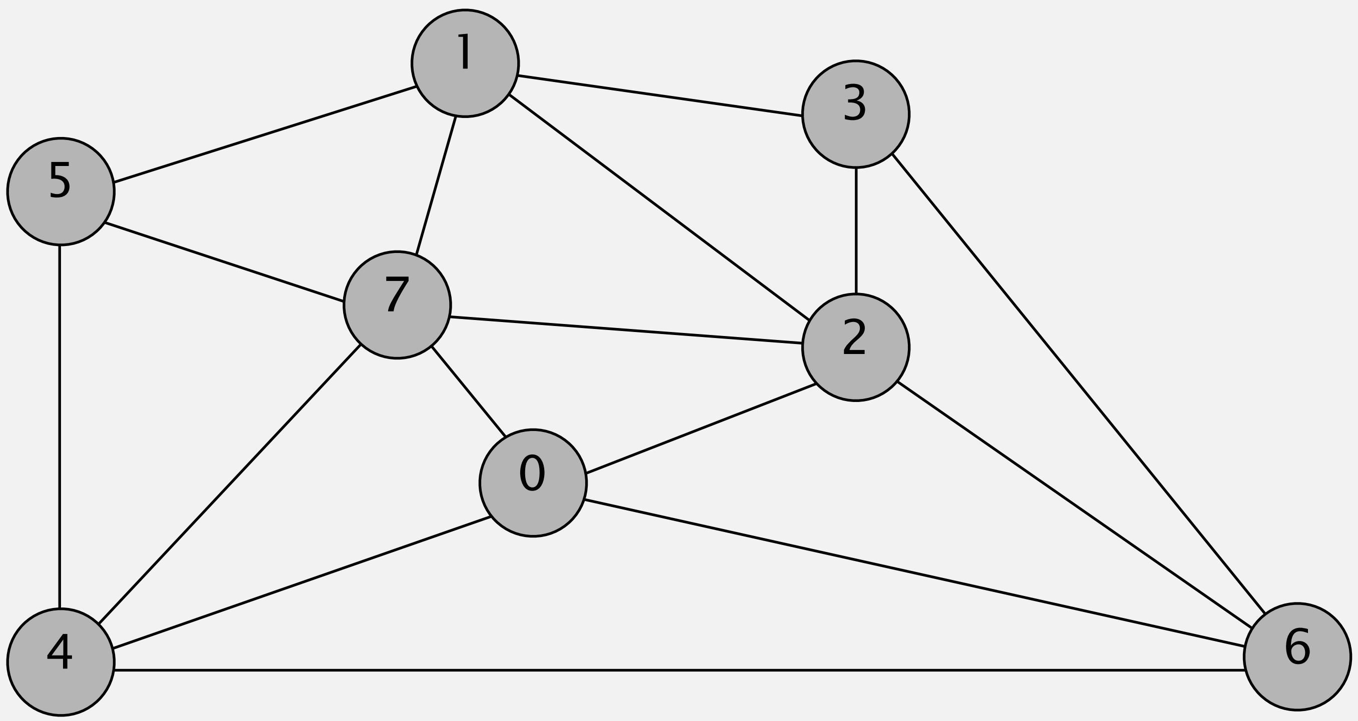
ignore if both endpoints in T

add edge e to tree

add v or w to tree

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



an edge-weighted graph

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2-3	0.17
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5-7	0.28
1-3	0.29
1-5	0.32
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4-7	0.37
0-4	0.38
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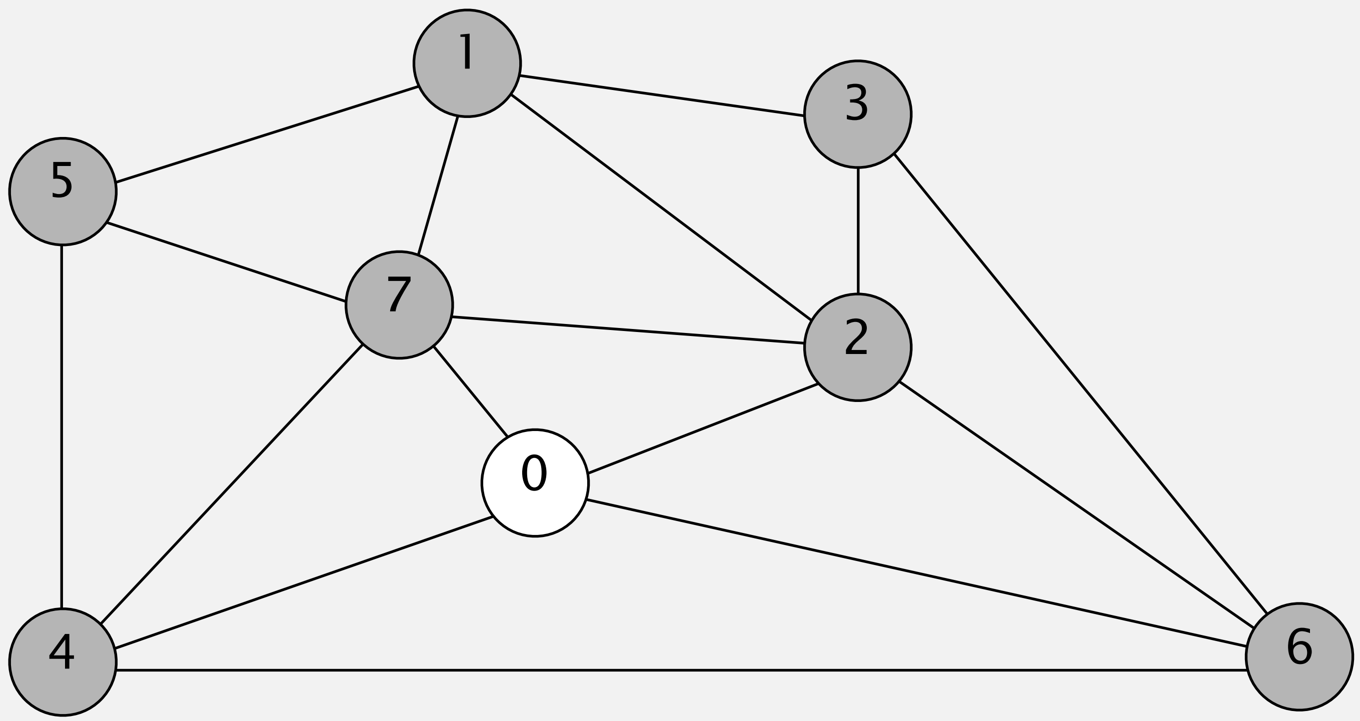
3-6	0.52
6-0	0.58
6-4	0.93

Prim's algorithm: lazy implementation demo

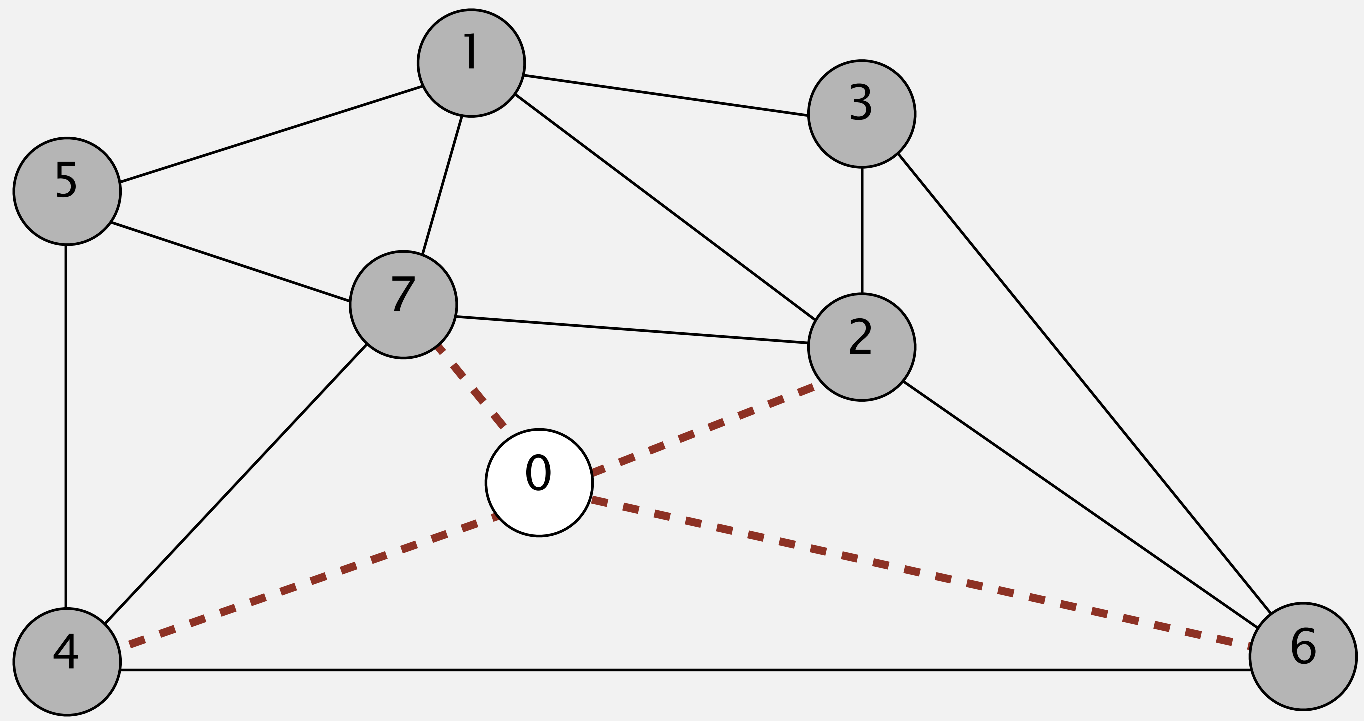
- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

add to PQ all edges incident to 0



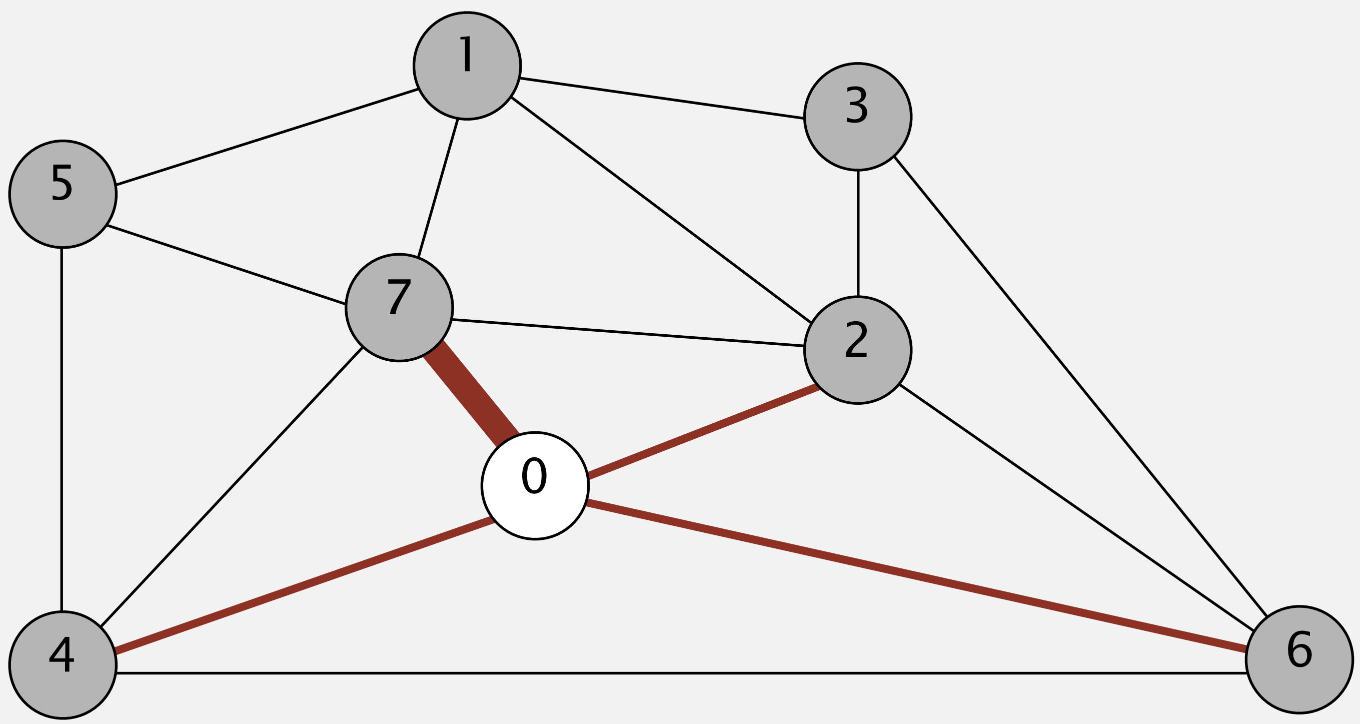
edges on PQ
(sorted by weight)

*	0-7	0.16
*	0-2	0.26
*	0-4	0.38
*	6-0	0.58

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

delete 0-7 and add to MST

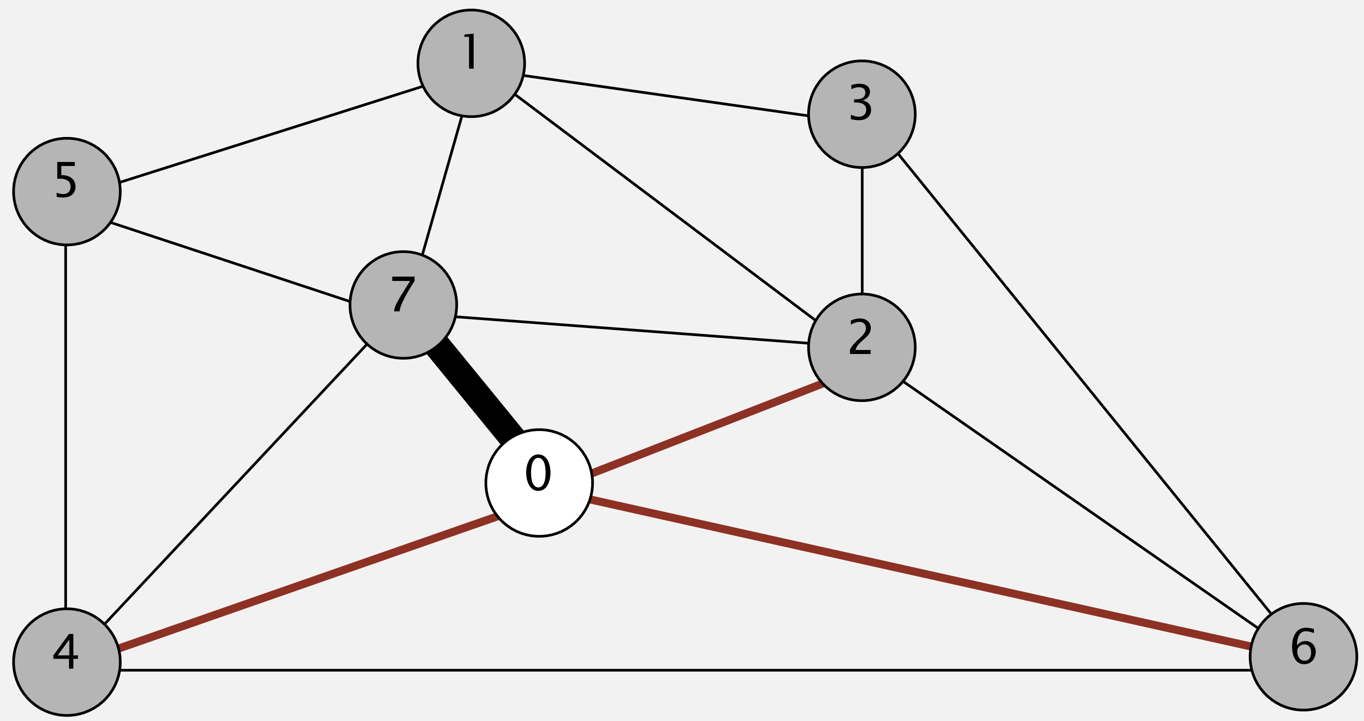


edges on PQ
(sorted by weight)

0-7	0.16
0-2	0.26
0-4	0.38
6-0	0.58

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



edges on PQ
(sorted by weight)

0-2	0.26
0-4	0.38
6-0	0.58

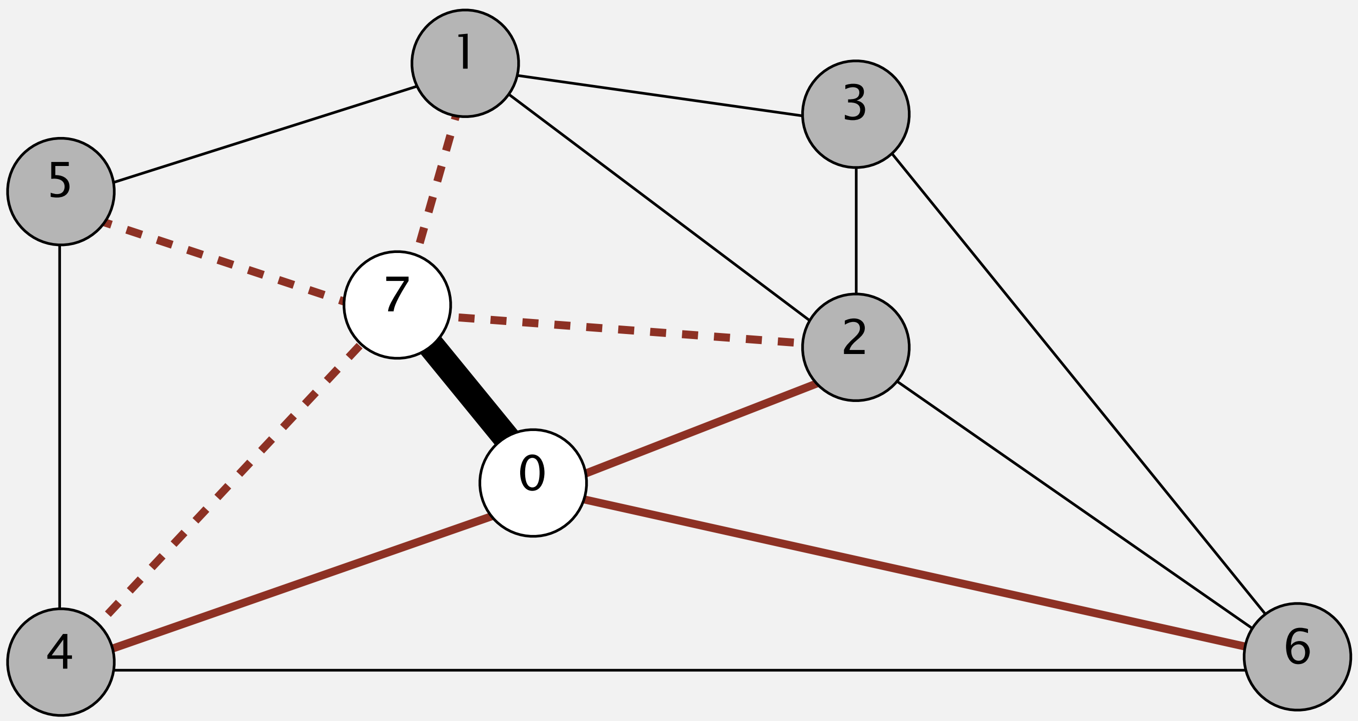
MST edges

0-7

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

add to PQ all edges incident to 7



edges on PQ
(sorted by weight)

*	1-7	0.19
	0-2	0.26
*	5-7	0.28
*	2-7	0.34
*	4-7	0.37
	0-4	0.38
	6-0	0.58

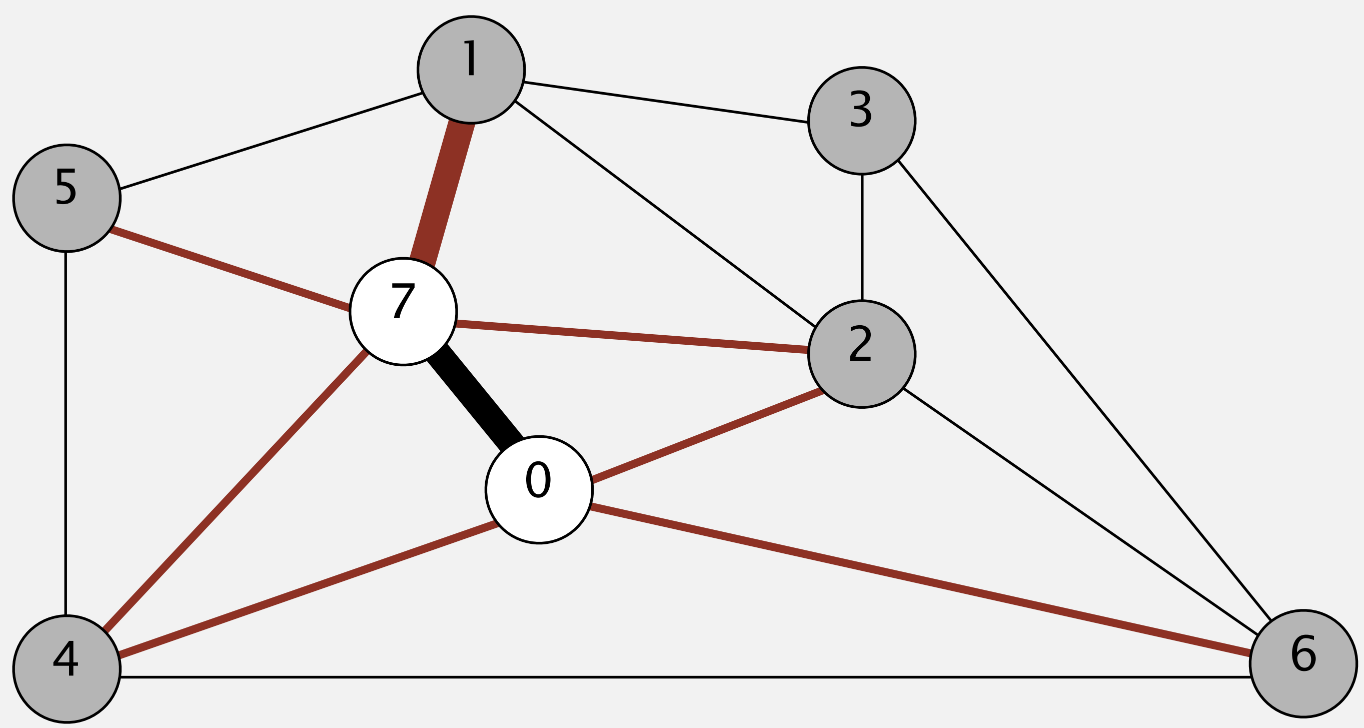
MST edges

0-7

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

delete 1-7 and add to MST



edges on PQ
(sorted by weight)

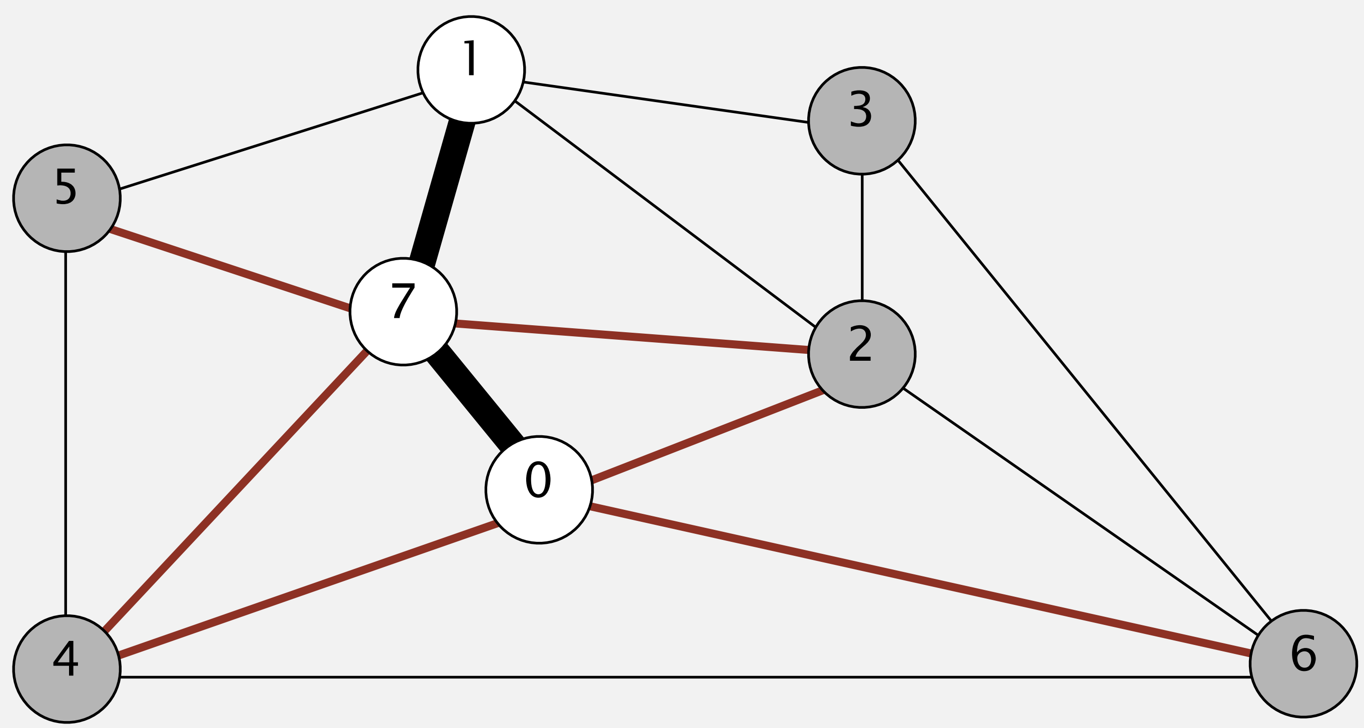
1-7	0.19
0-2	0.26
5-7	0.28
2-7	0.34
4-7	0.37
0-4	0.38
6-0	0.58

MST edges

0-7

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



edges on PQ
(sorted by weight)

0-2	0.26
5-7	0.28
2-7	0.34
4-7	0.37
0-4	0.38
6-0	0.58

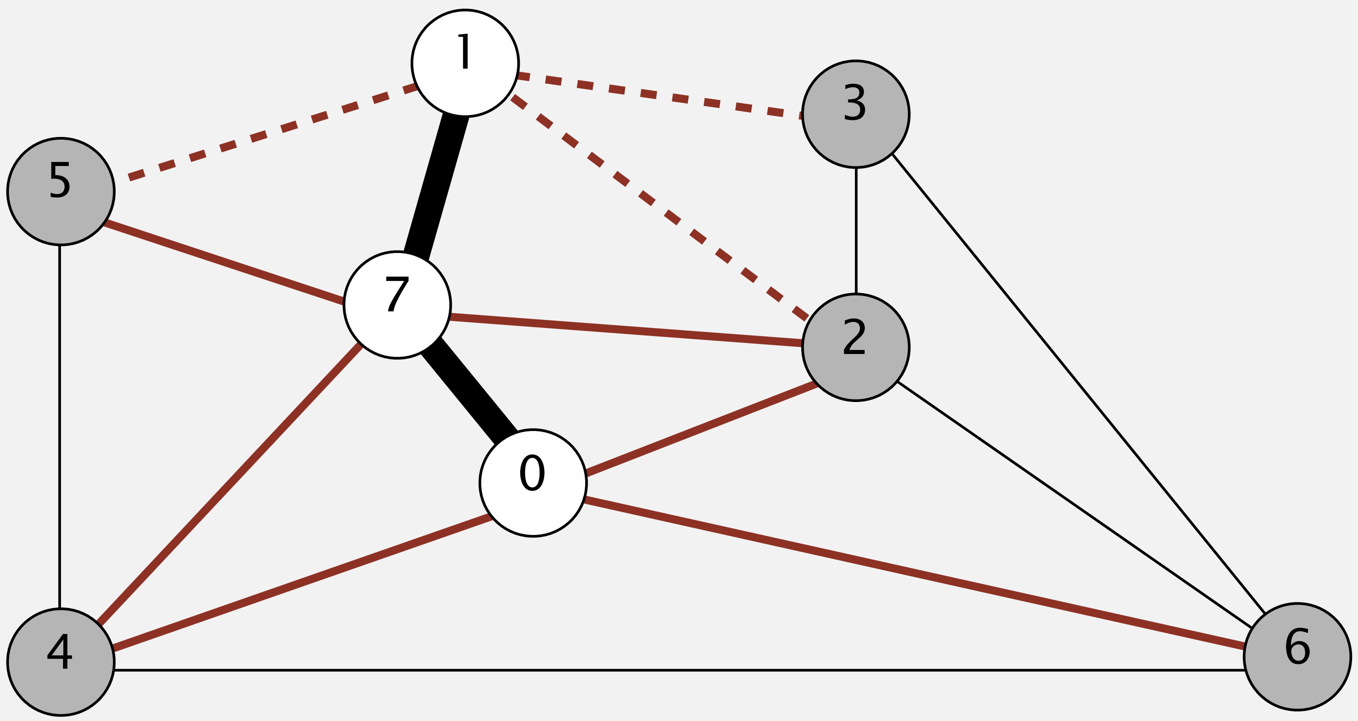
MST edges

0-7 1-7

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

add to PQ all edges incident to 1



edges on PQ
(sorted by weight)

	0-2	0.26
	5-7	0.28
*	1-3	0.29
*	1-5	0.32
	2-7	0.34
*	1-2	0.36
	4-7	0.37
	0-4	0.38
	6-0	0.58

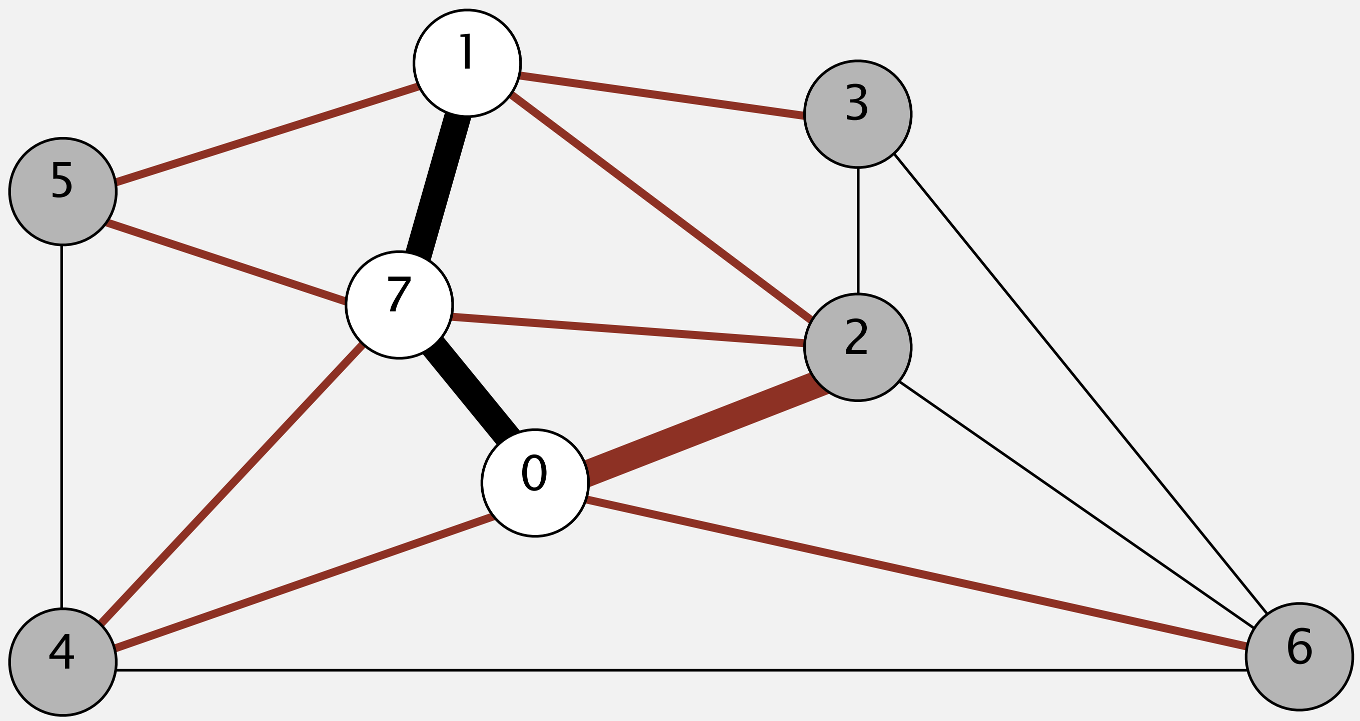
MST edges

0-7 1-7

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

delete edge 0-2 and add to MST



edges on PQ
(sorted by weight)

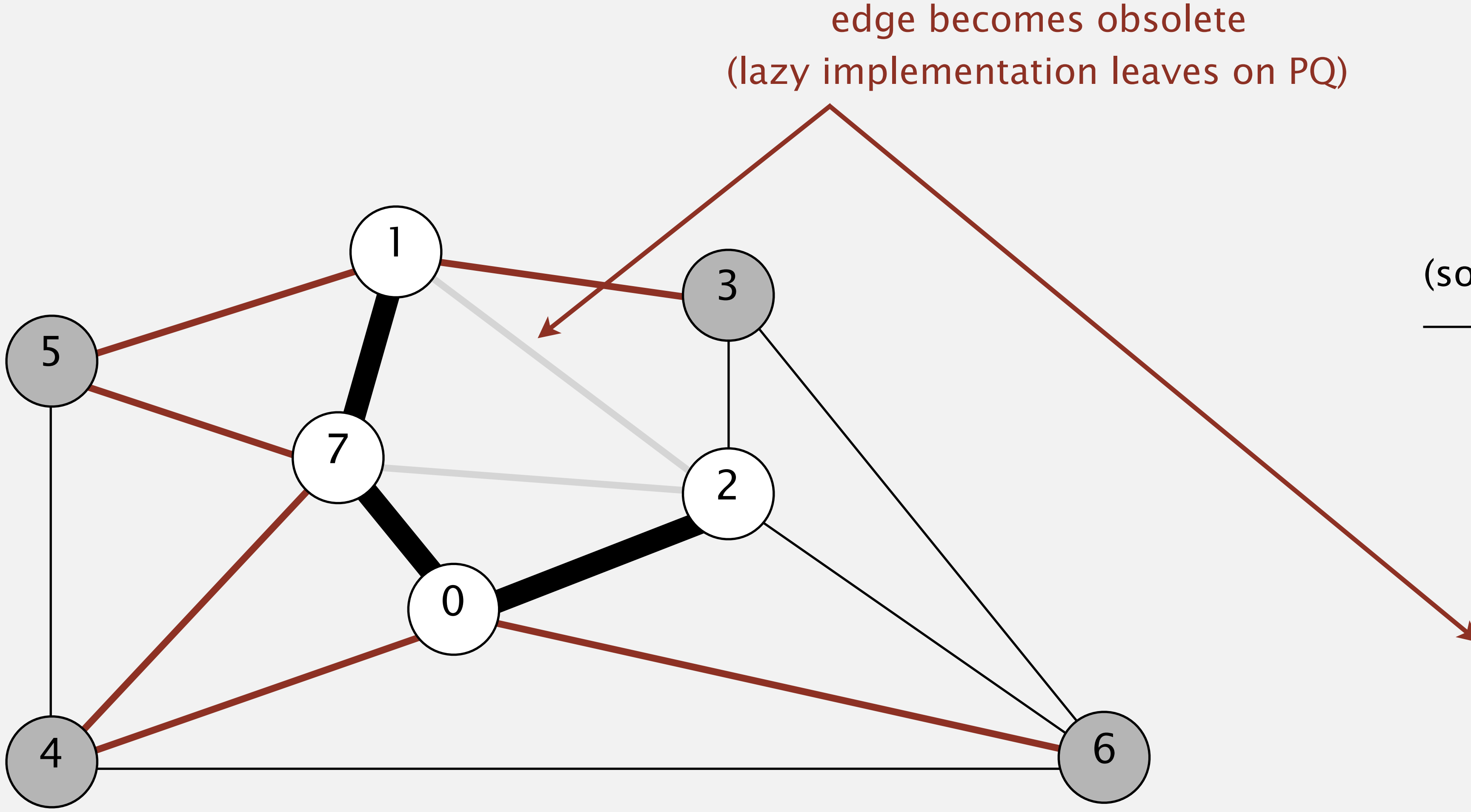
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
1-2	0.36
4-7	0.37
0-4	0.38
6-0	0.58

MST edges

0-7   1-7

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



edges on PQ
(sorted by weight)

5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
1-2	0.36
4-7	0.37
0-4	0.38
6-0	0.58

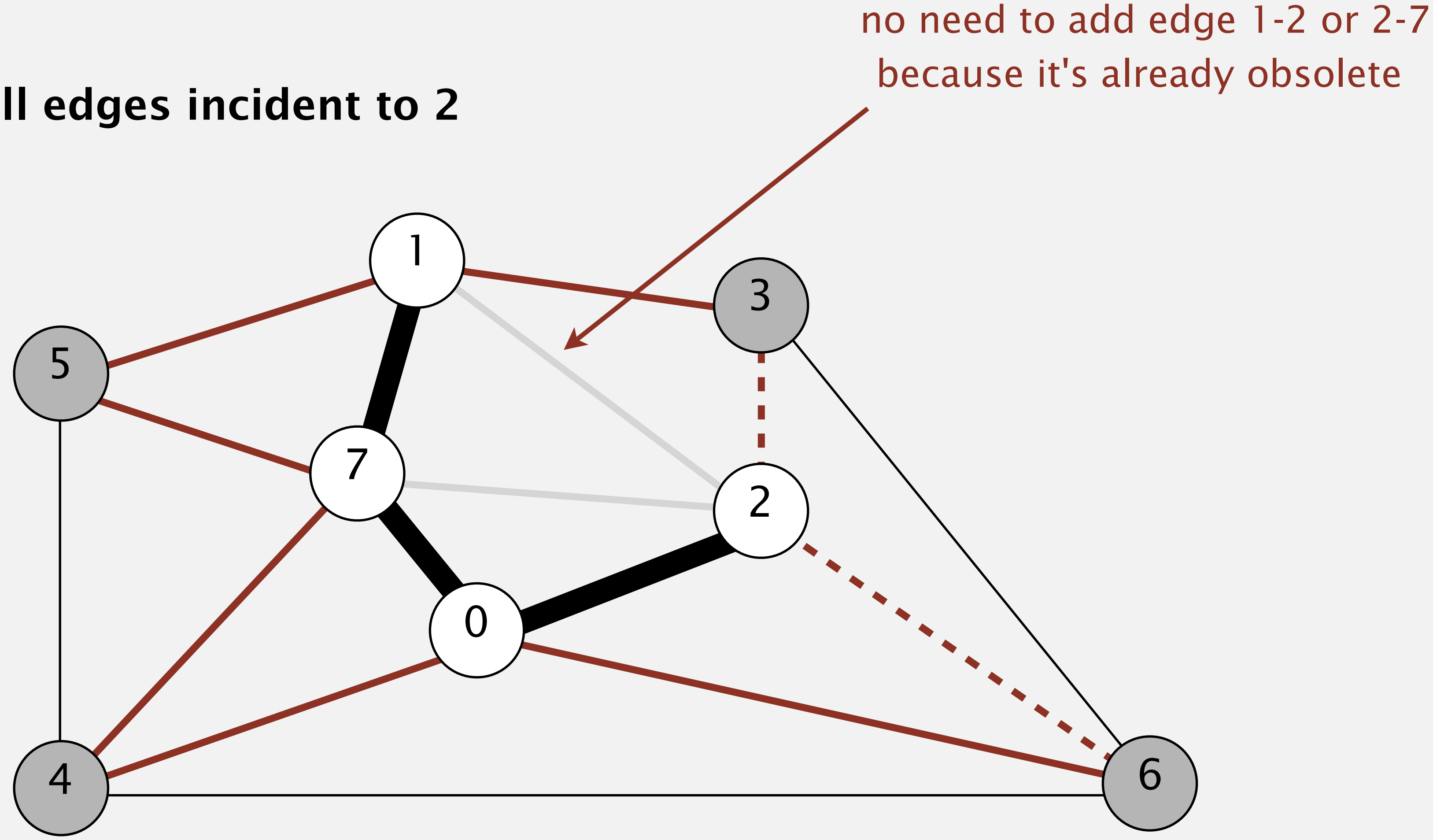
MST edges

0-7 1-7 0-2

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

add to PQ all edges incident to 2



edges on PQ
(sorted by weight)

*	2-3	0.17
	5-7	0.28
	1-3	0.29
	1-5	0.32
	2-7	0.34
	1-2	0.36
	4-7	0.37
	0-4	0.38
*	6-2	0.40
	6-0	0.58

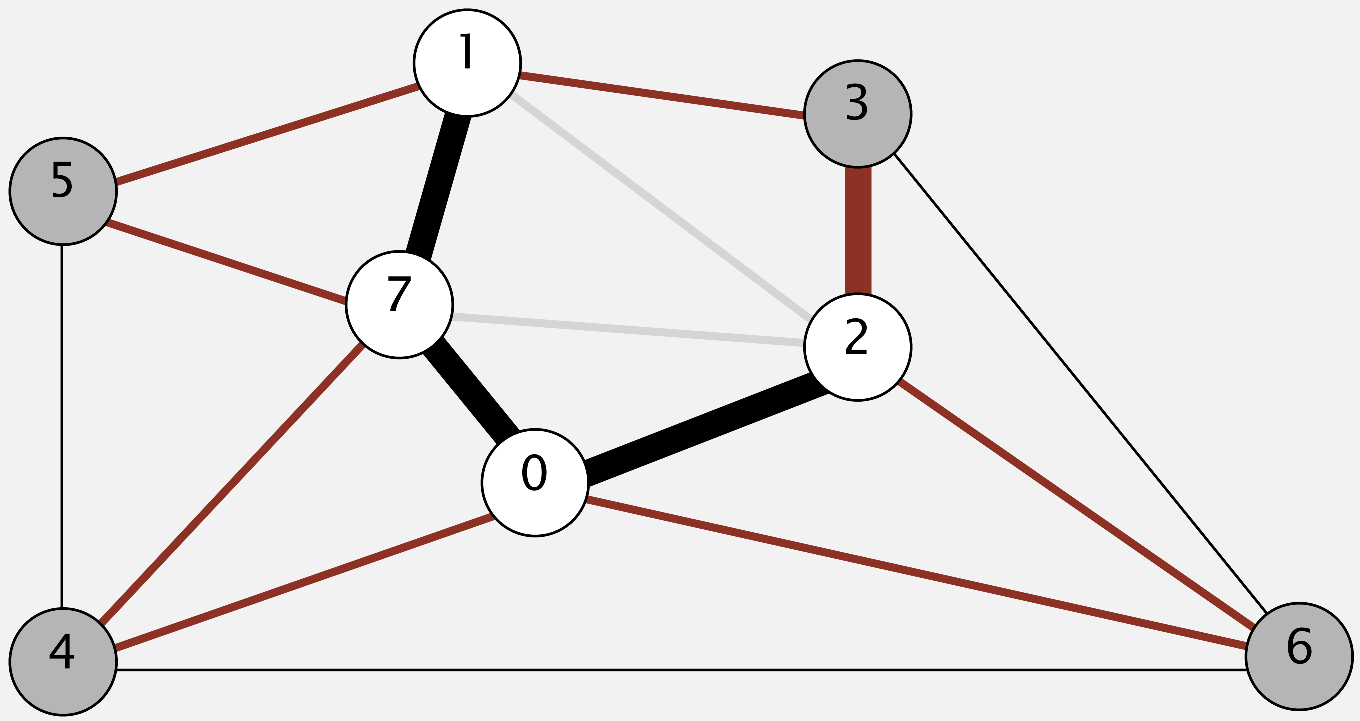
MST edges

0-7 1-7 0-2

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

delete 2–3 and add to MST



edges on PQ
(sorted by weight)

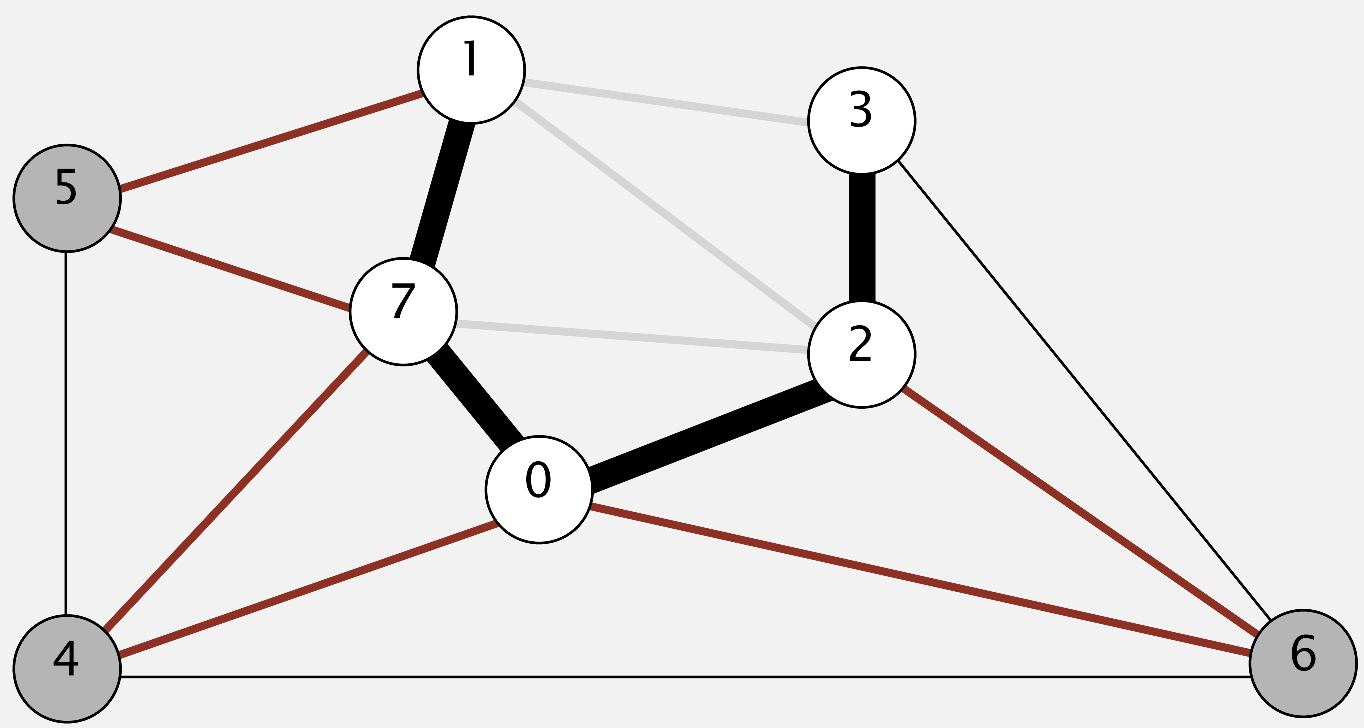
*	2-3	0.17
	5-7	0.28
	1-3	0.29
	1-5	0.32
	2-7	0.34
	1-2	0.36
	4-7	0.37
	0-4	0.38
*	6-2	0.40
	6-0	0.58

MST edges

0-7 1-7 0-2

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



edges on PQ
(sorted by weight)

5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
6-0	0.58

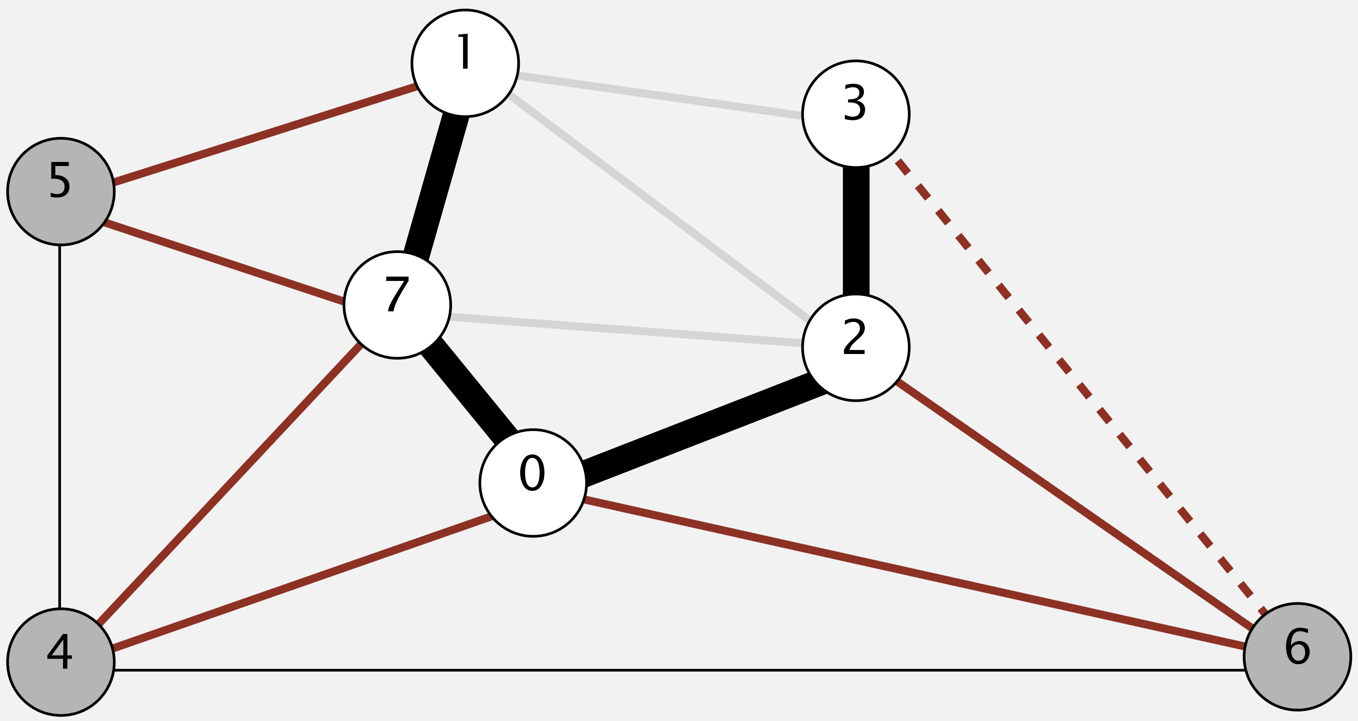
MST edges

0-7 1-7 0-2 2-3

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

add to PQ all edges incident to 3



edges on PQ
(sorted by weight)

	5-7	0.28
	1-3	0.29
	1-5	0.32
	2-7	0.34
	1-2	0.36
	4-7	0.37
	0-4	0.38
	6-2	0.40
*	3-6	0.52
	6-0	0.58

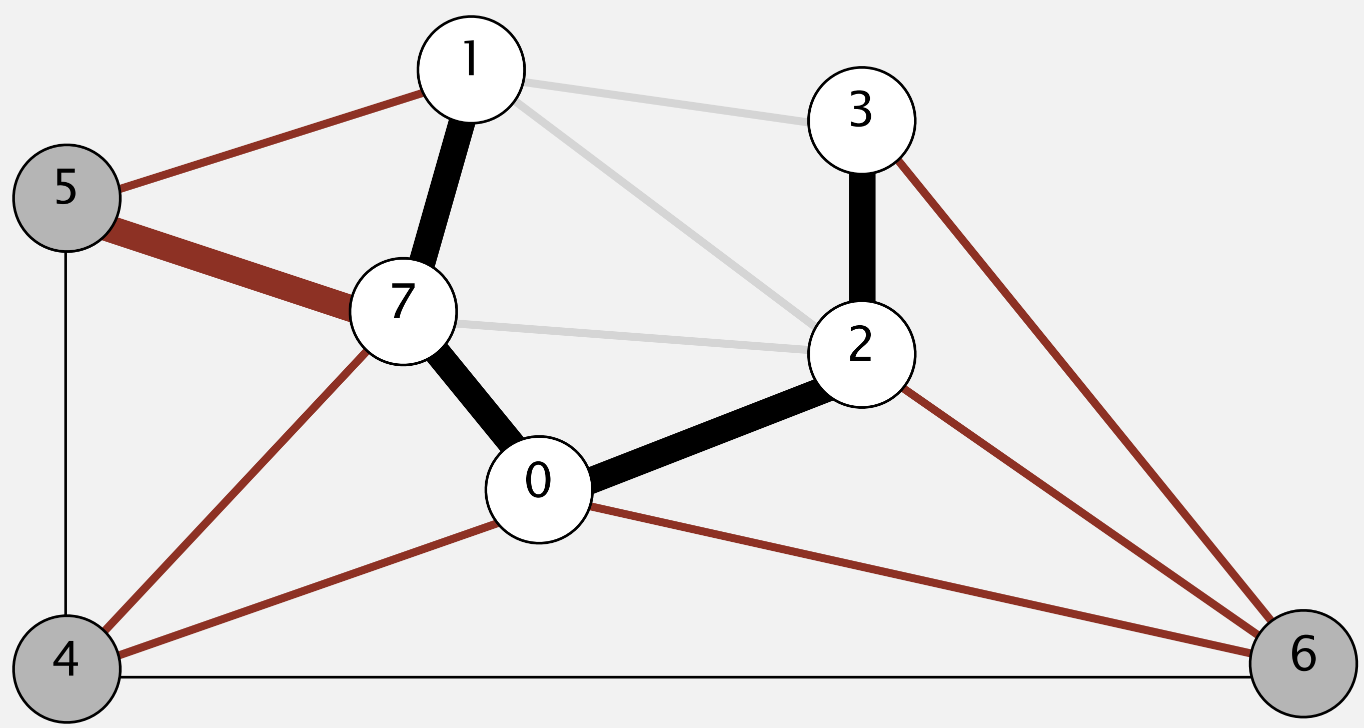
MST edges

0-7 1-7 0-2 2-3

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

delete 5–7 and add to MST



edges on PQ
(sorted by weight)

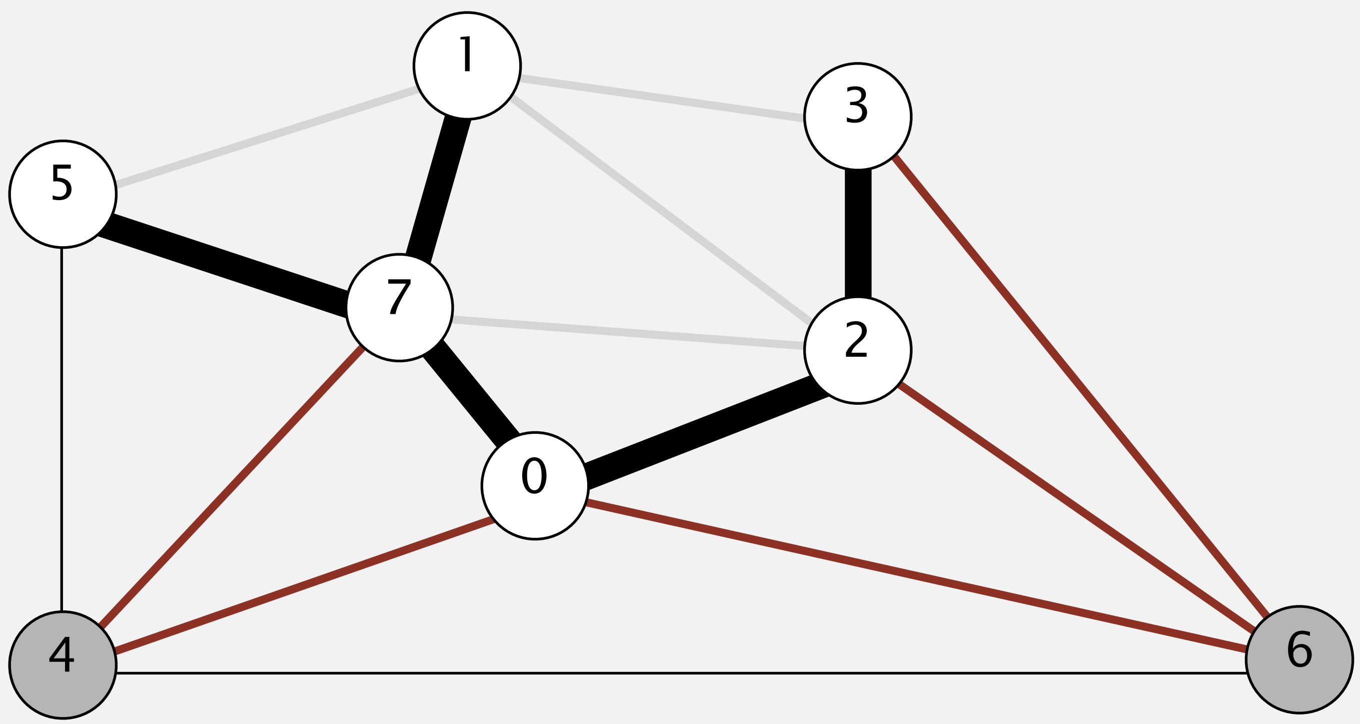
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58

MST edges

0–7 1–7 0–2 2–3

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



edges on PQ
(sorted by weight)

1-3	0.29
1-5	0.32
2-7	0.34
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58

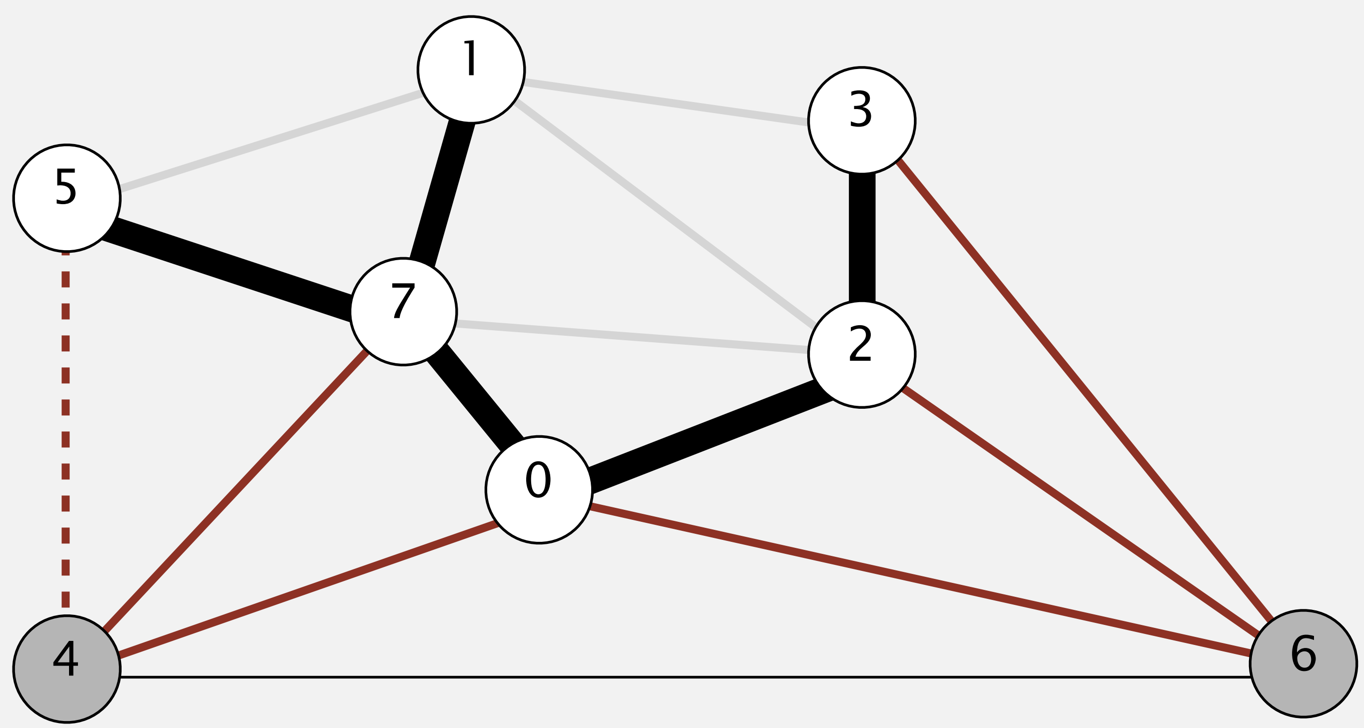
MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

add to PQ all edges incident to 5



edges on PQ
(sorted by weight)

	1-3	0.29
	1-5	0.32
	2-7	0.34
*	4-5	0.35
	1-2	0.36
	4-7	0.37
	0-4	0.38
	6-2	0.40
	3-6	0.52
	6-0	0.58

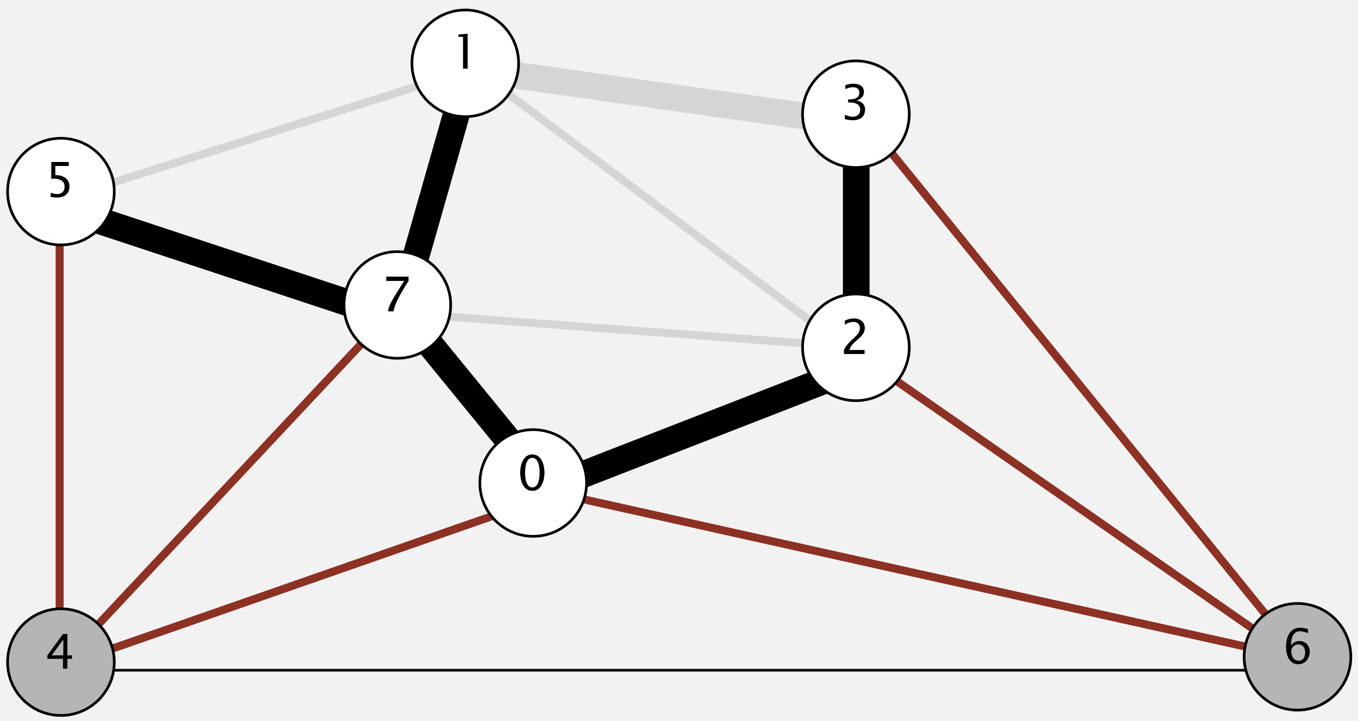
MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

delete 1–3 and discard obsolete edge



edges on PQ
(sorted by weight)

1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58

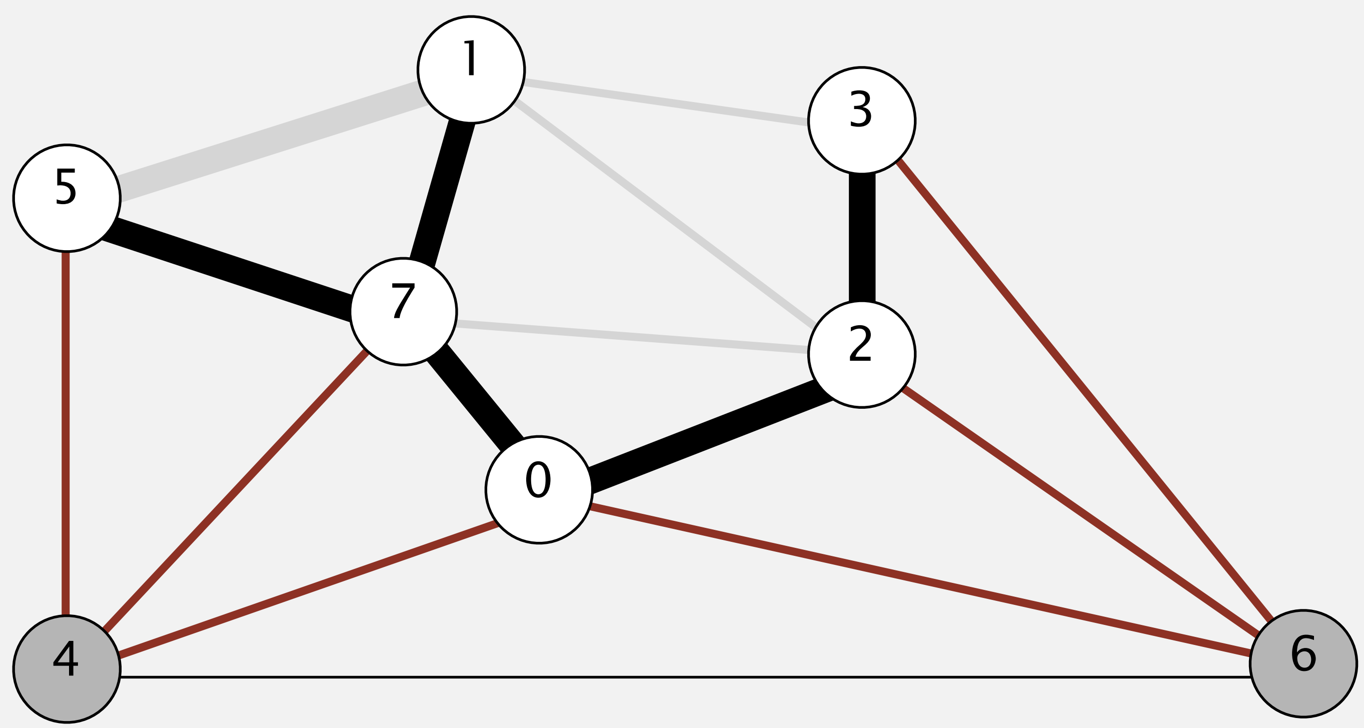
MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

delete 1–5 and discard obsolete edge



edges on PQ
(sorted by weight)

1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58

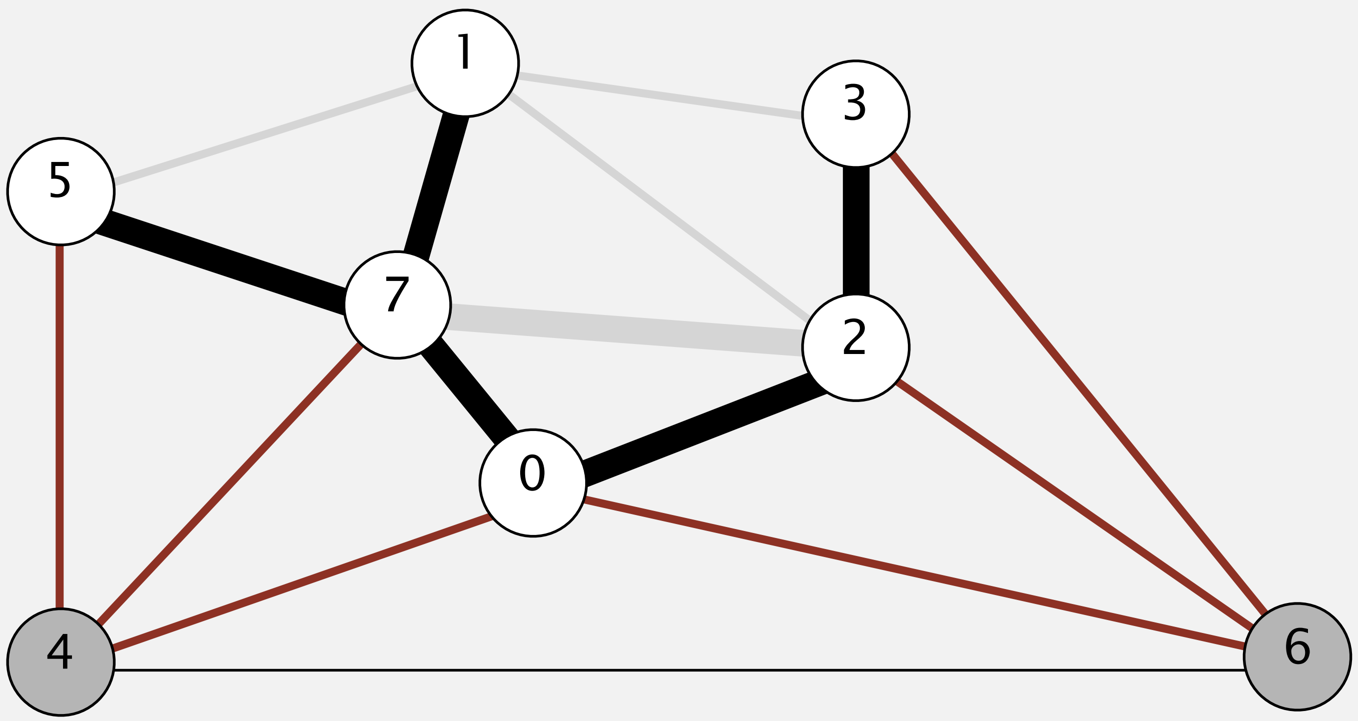
MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

delete 2-7 and discard obsolete edge



edges on PQ
(sorted by weight)

2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58

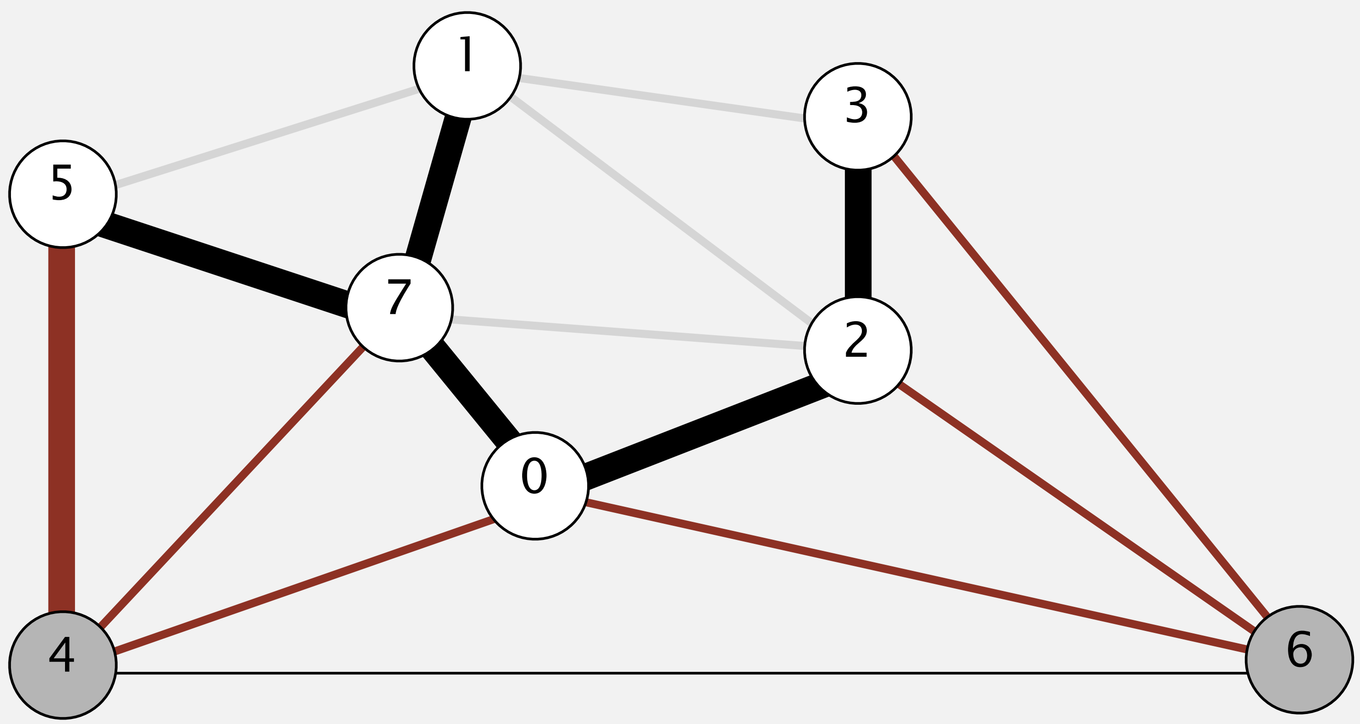
MST edges

0-7   1-7   0-2   2-3   5-7

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

delete 4-5 and add to MST



edges on PQ
(sorted by weight)

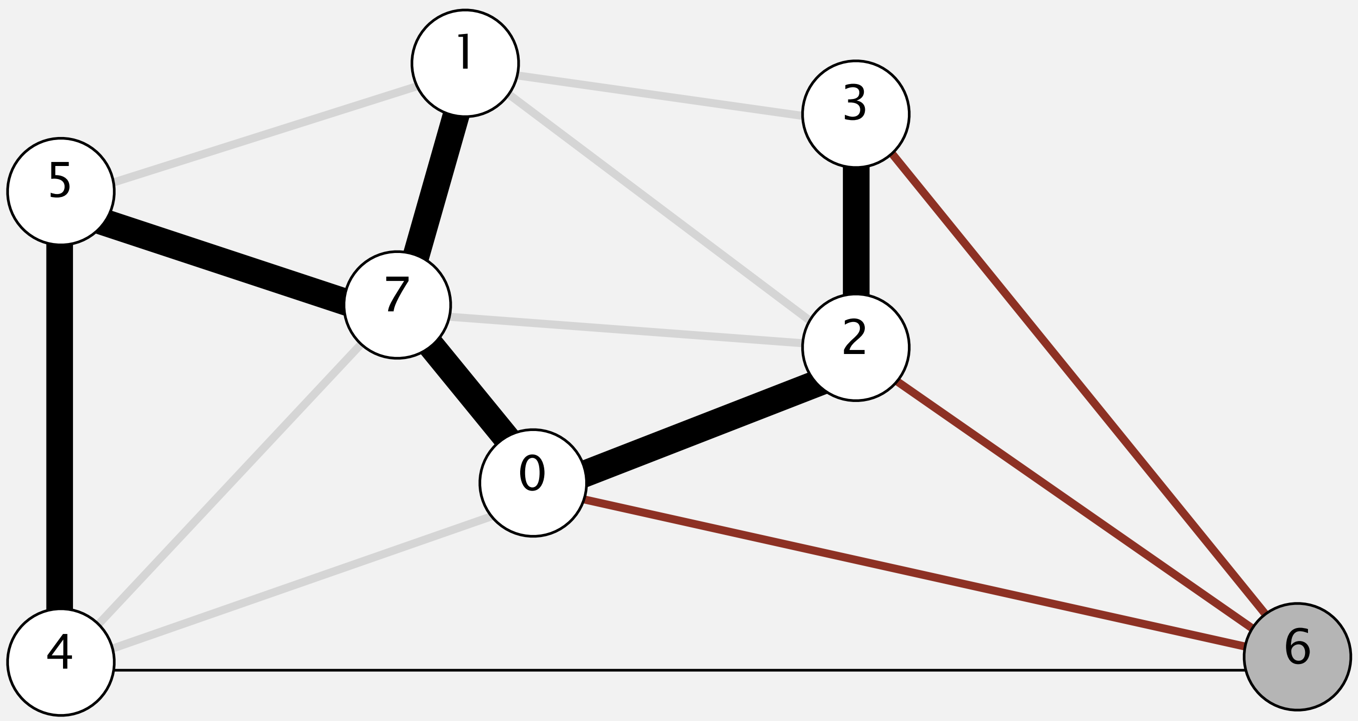
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58

MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



edges on PQ
(sorted by weight)

1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58

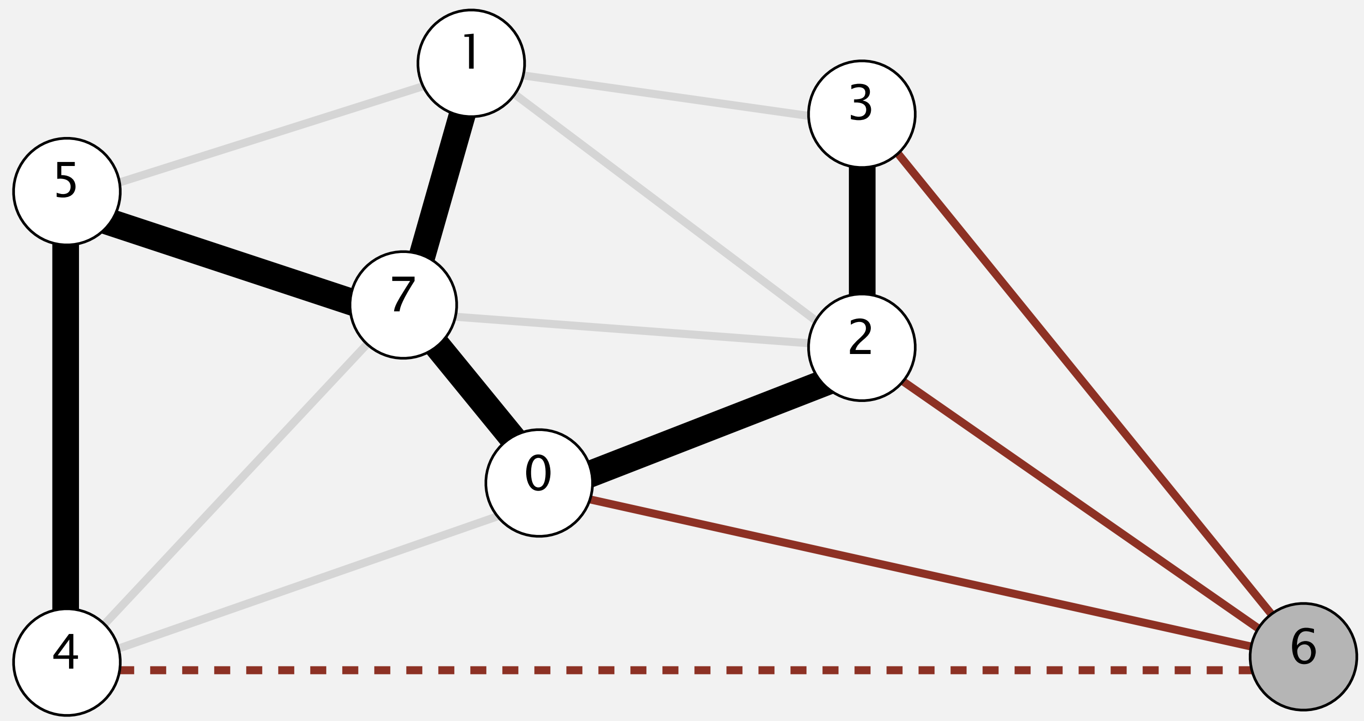
MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

add to PQ all edges incident to 4



edges on PQ
(sorted by weight)

1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
* 6-4	0.93

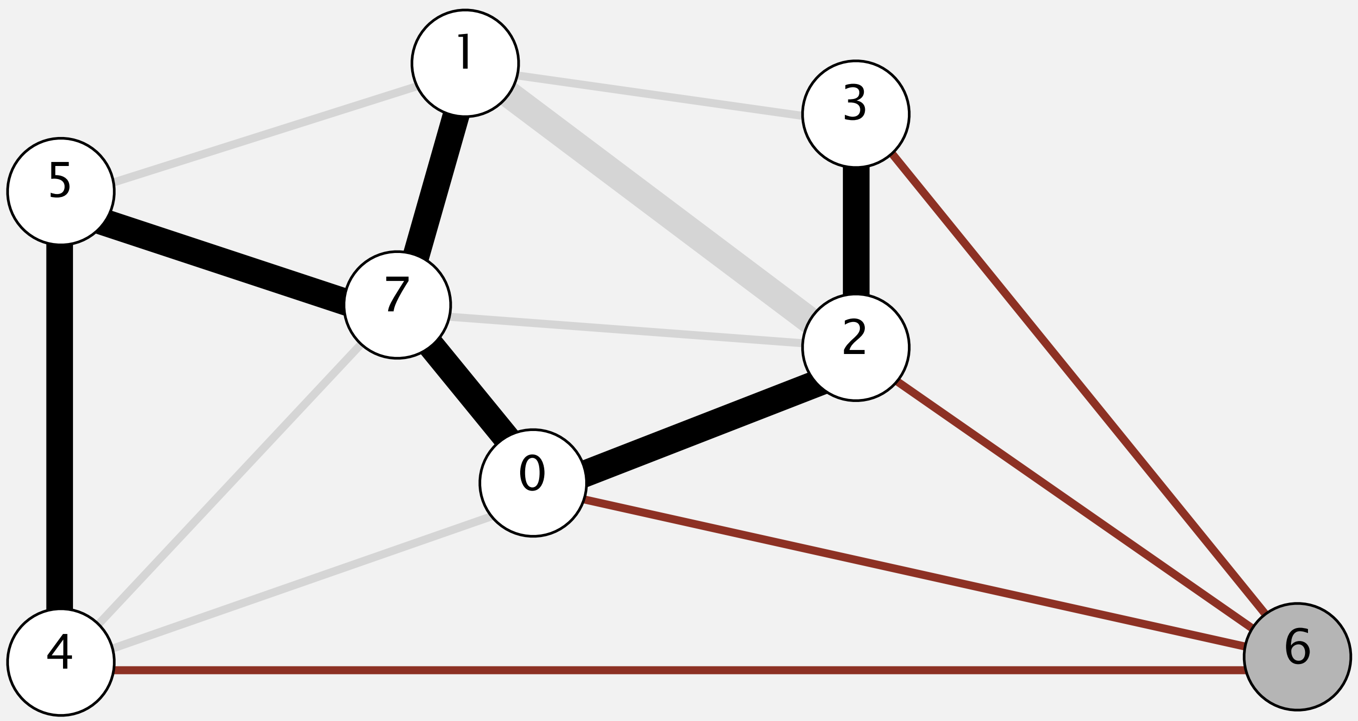
MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

delete 1–2 and discard obsolete edge



edges on PQ
(sorted by weight)

1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

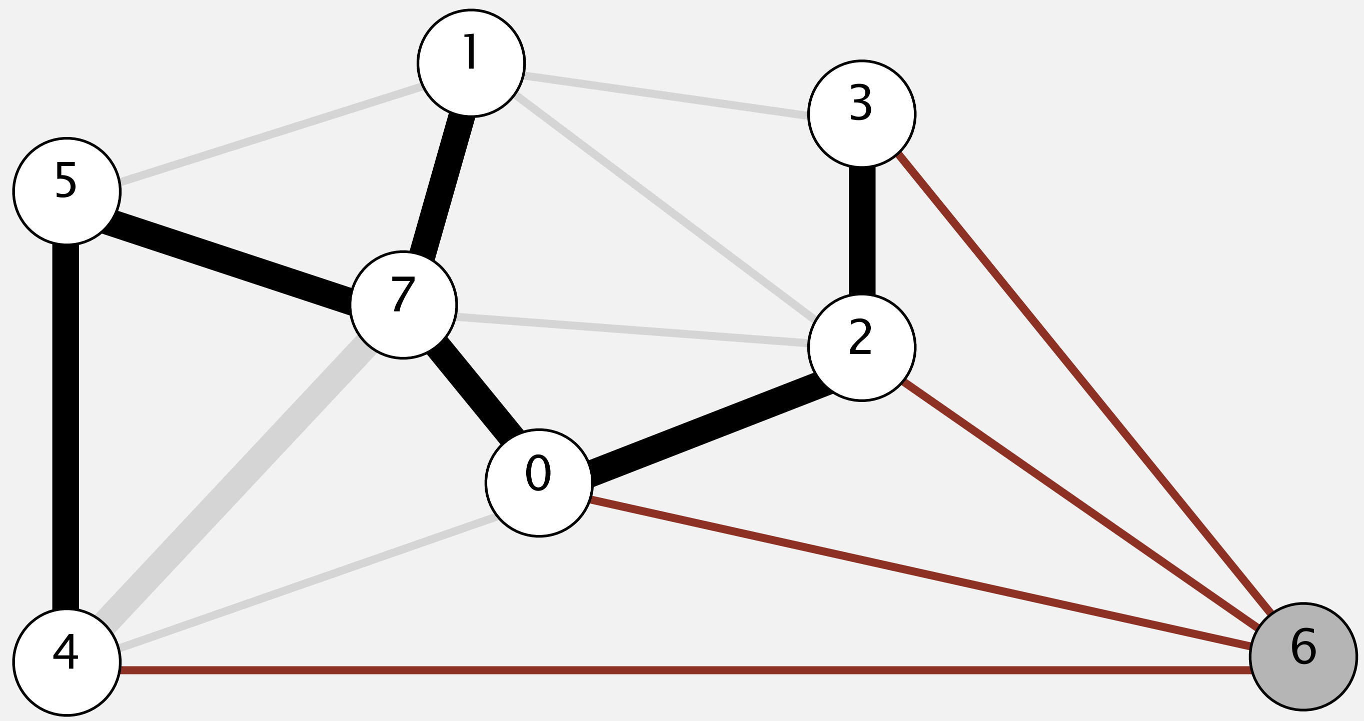
MST edges

0-7   1-7   0-2   2-3   5-7   4-5

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

delete 4-7 and discard obsolete edge



edges on PQ
(sorted by weight)

4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

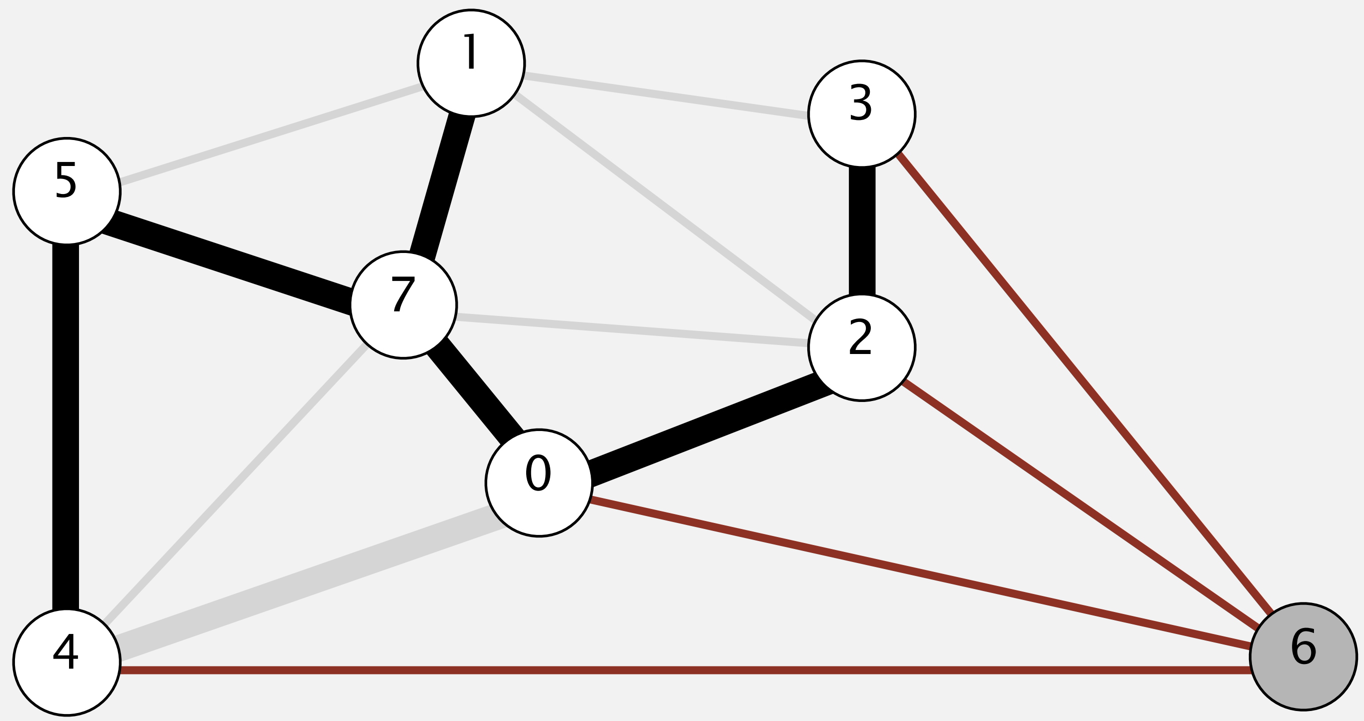
MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

delete 0–4 and discard obsolete edge



edges on PQ
(sorted by weight)

0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

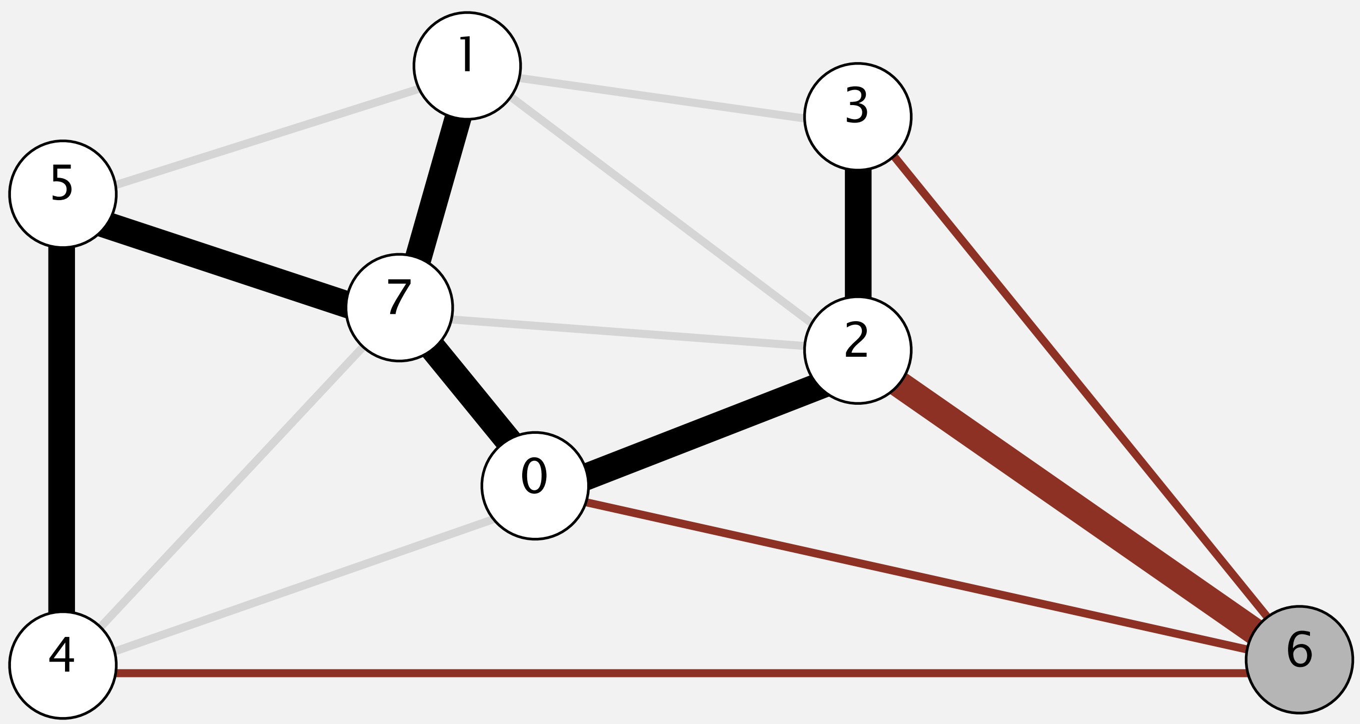
MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

delete 6–2 and add to MST



edges on PQ
(sorted by weight)

6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

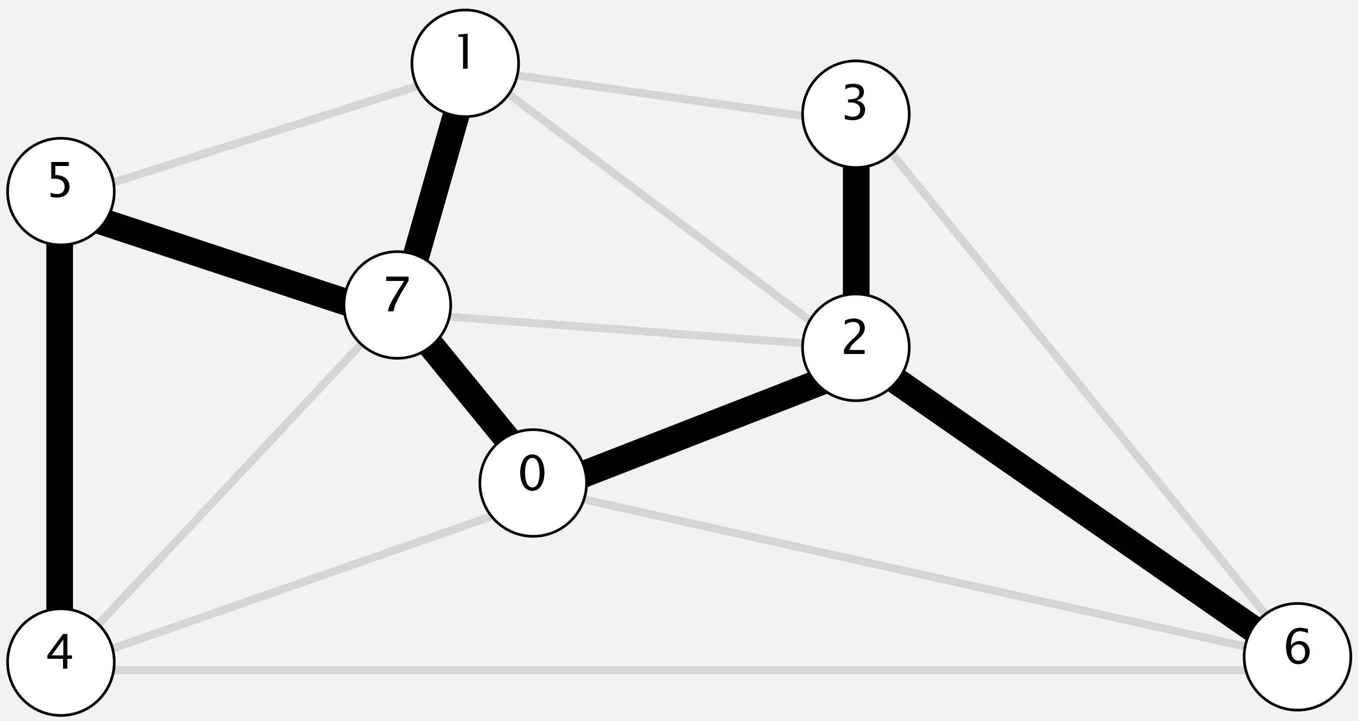
MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.

delete 6-2 and add to MST



edges on PQ
(sorted by weight)

3-6	0.52
6-0	0.58
6-4	0.93

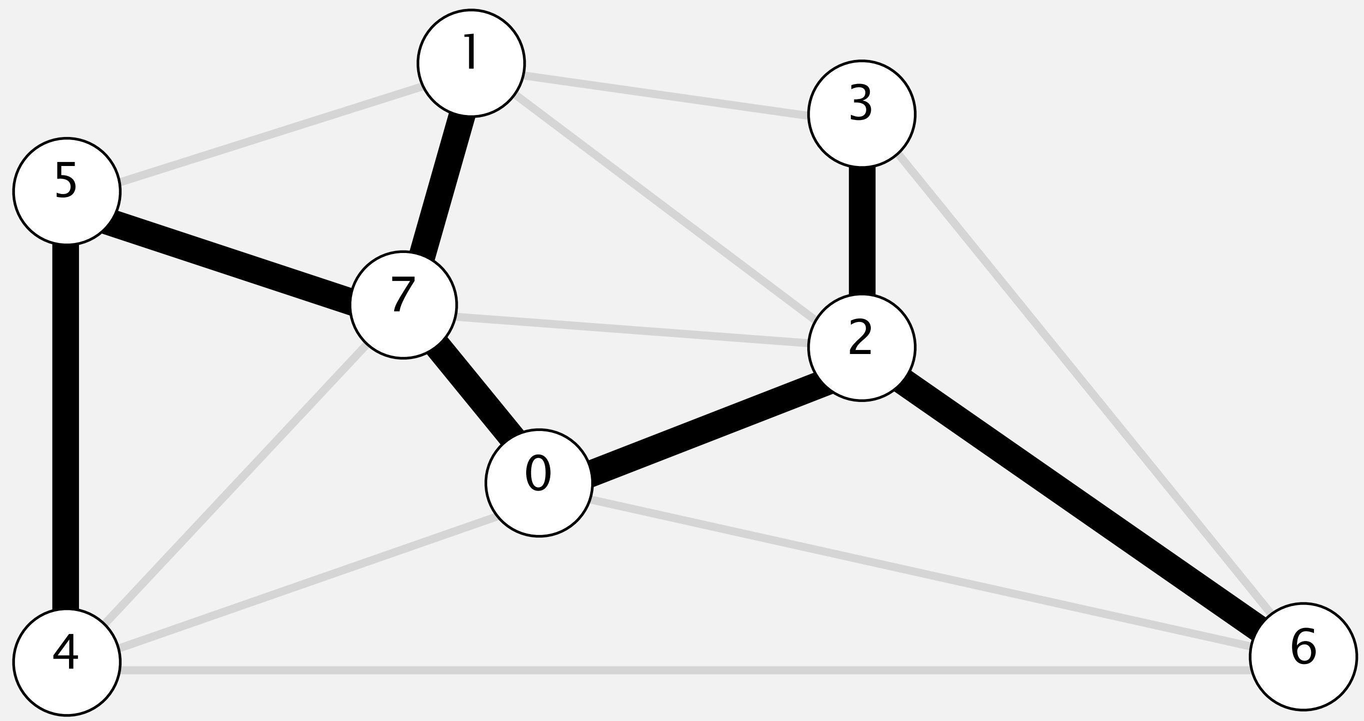
MST edges

0-7 1-7 0-2 2-3 5-7 4-5 6-2

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

stop since V-1 edges



edges on PQ
(sorted by weight)

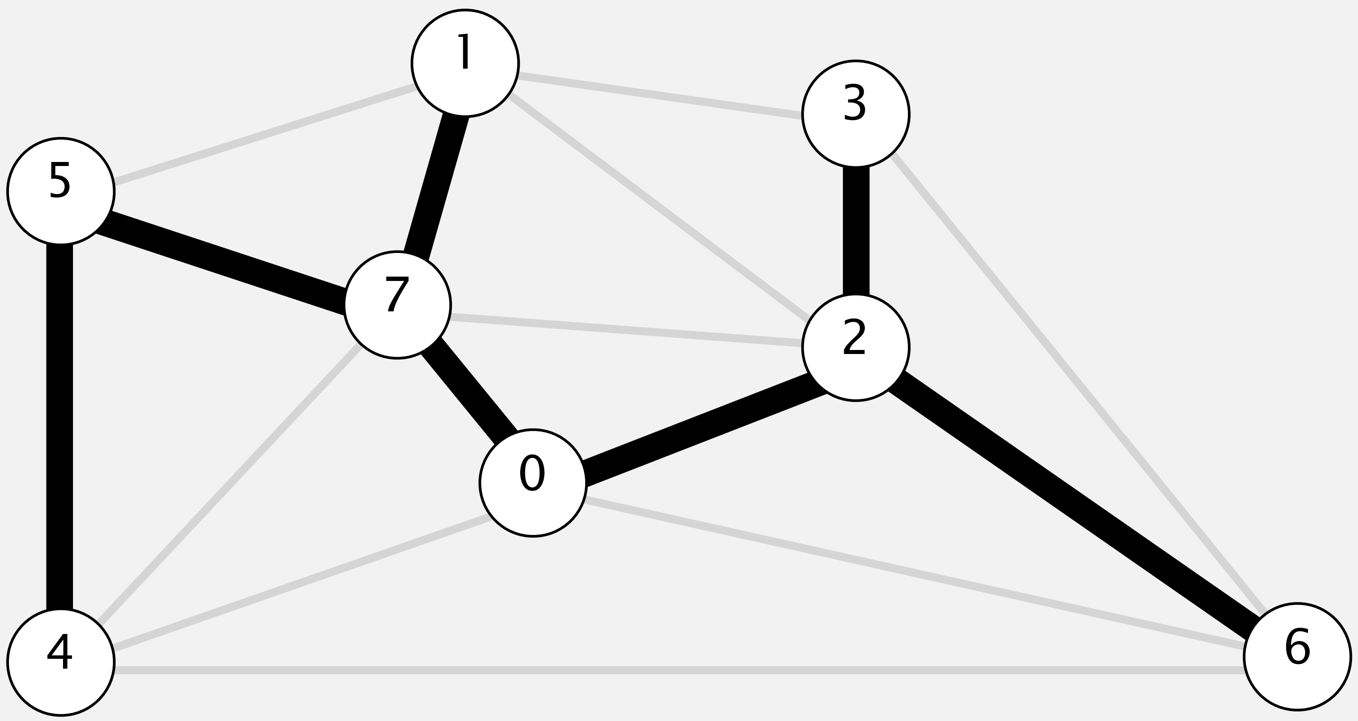
3-6	0.52
6-0	0.58
6-4	0.93

MST edges

0-7 1-7 0-2 2-3 5-7 4-5 6-2

Prim's algorithm: lazy implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



MST edges

0-7 1-7 0-2 2-3 5-7 4-5 6-2

Prim's algorithm: lazy implementation

```
public class LazyPrimMST
{
   private boolean[] marked;    // MST vertices
   private Queue<Edge> mst;     // MST edges
   private MinPQ<Edge> pq;      // PQ of edges

    public LazyPrimMST(WeightedGraph G)
    {
        pq = new MinPQ<Edge>();
        mst = new Queue<Edge>();
        marked = new boolean[G.V()];
        visit(G, 0);

        while (!pq.isEmpty() && mst.size() < G.V() - 1)
        {
           Edge e = pq.delMin();
           int v = e.either(), w = e.other(v);
           if (marked[v] && marked[w]) continue;
           mst.enqueue(e);
           if (!marked[v]) visit(G, v);
           if (!marked[w]) visit(G, w);
        }
   }
}
```

assume G is connected

repeatedly delete the
min weight edge e = v–w from PQ

ignore if both endpoints in T

add edge e to tree

add v or w to tree

Prim's algorithm: lazy implementation

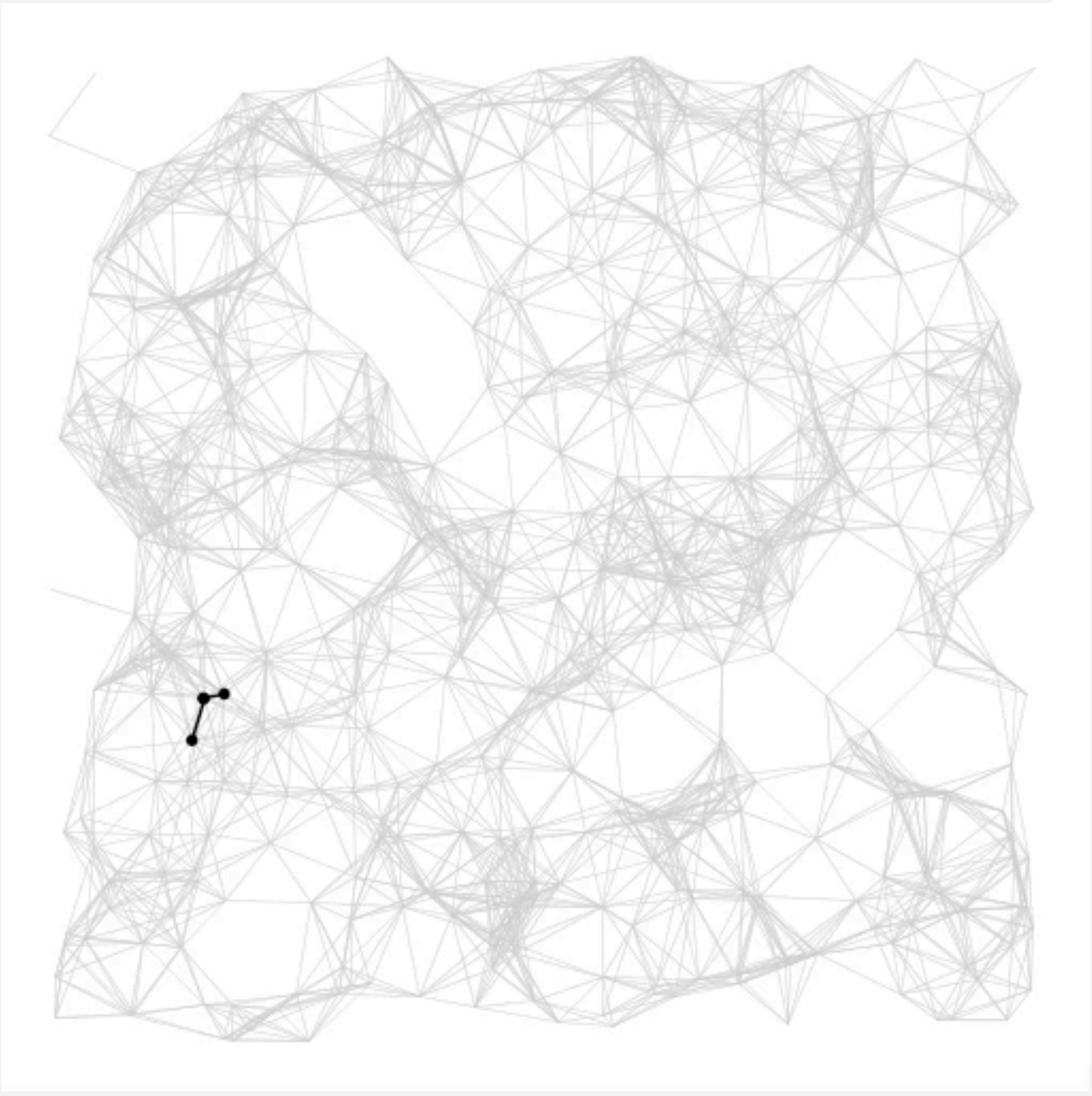
```
private void visit(WeightedGraph G, int v)
{
   marked[v] = true;
   for (Edge e : G.adj(v))
      if (!marked[e.other(v)])
         pq.insert(e);
}

public Iterable<Edge> mst()
{  return mst;  }
```

add v to T

for each edge e = v–w, add to PQ if w not already in T

Prim's algorithm: visualization



Lazy Prim's algorithm: running time

Proposition. Lazy Prim's algorithm computes the MST in time proportional to $E \log E$ and extra space proportional to E (in the worst case).

Pf.

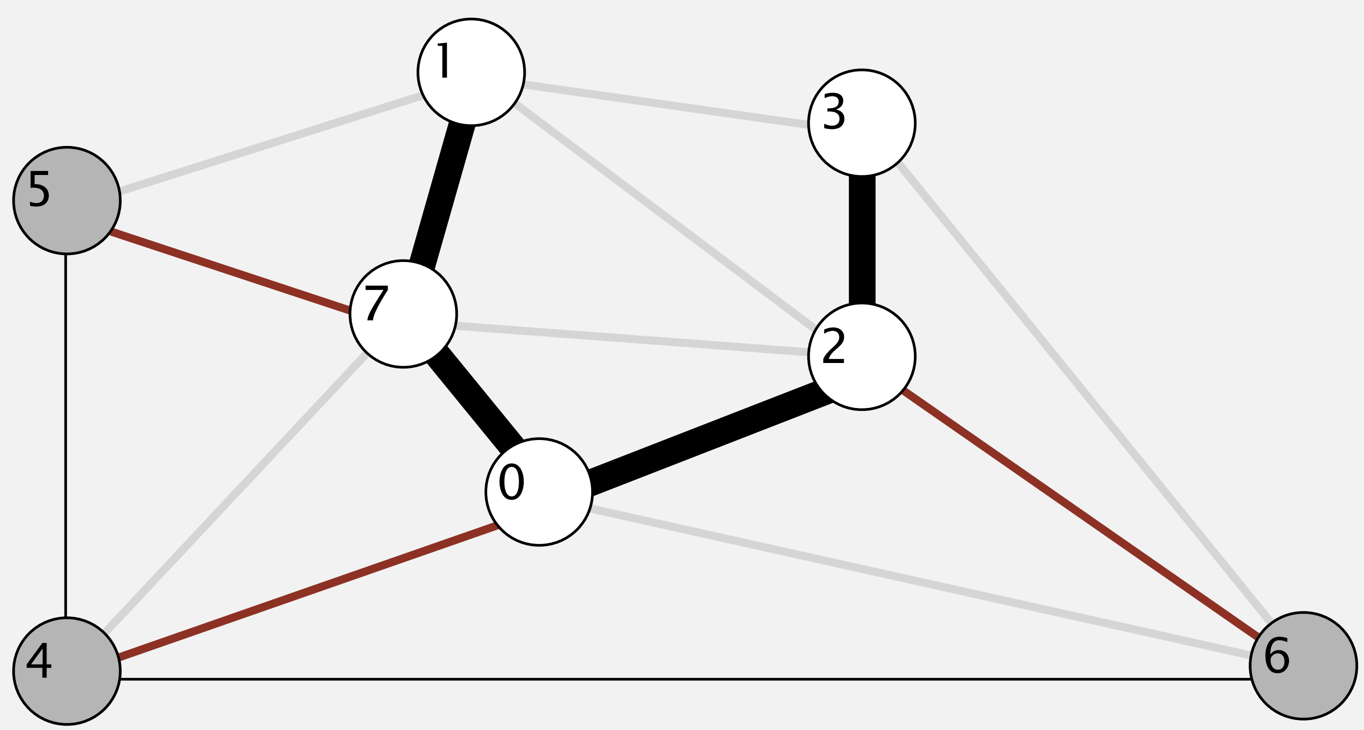
operation	frequency	binary heap
delete min	E	$\log E$
insert	E	$\log E$

Prim's algorithm: eager implementation

Challenge. Find min weight edge with exactly one endpoint in T.

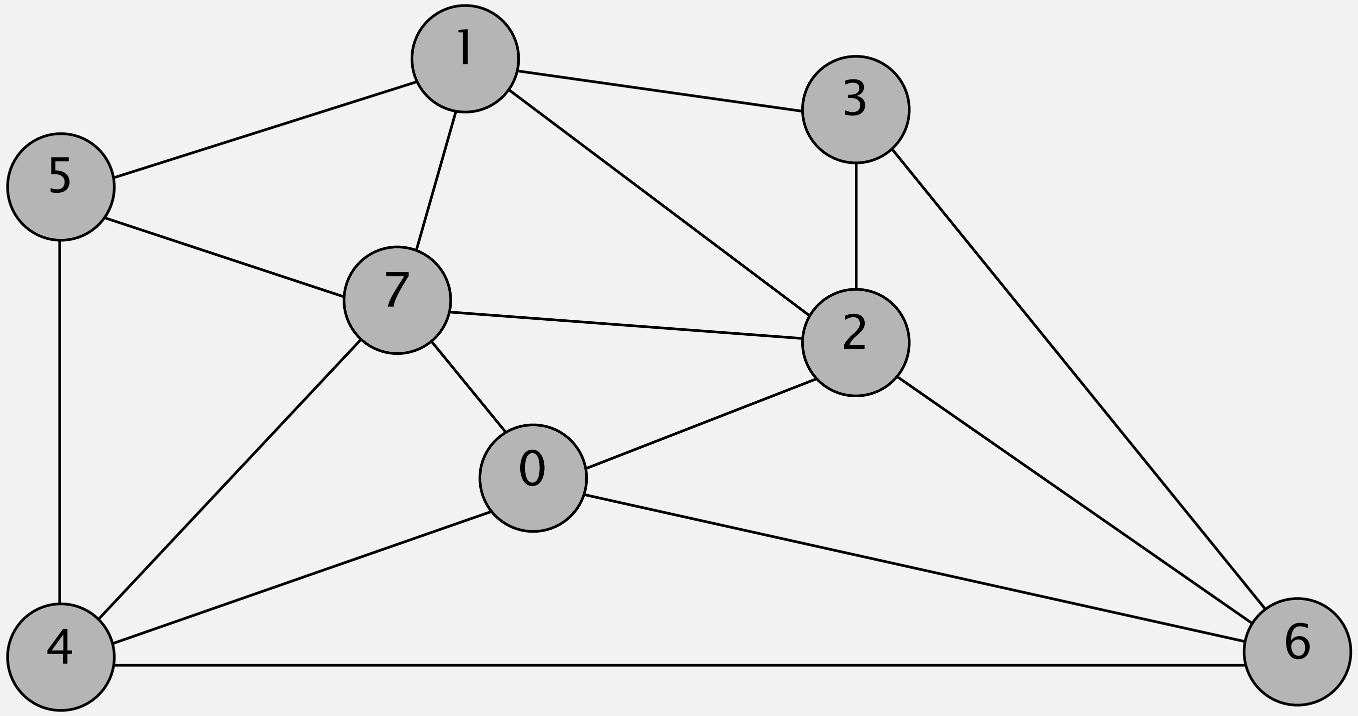
Observation. For each vertex v, need only shortest edge connecting v to T.

- MST includes at most one edge connecting v to T. Why?



Prim's algorithm: eager implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



an edge-weighted graph

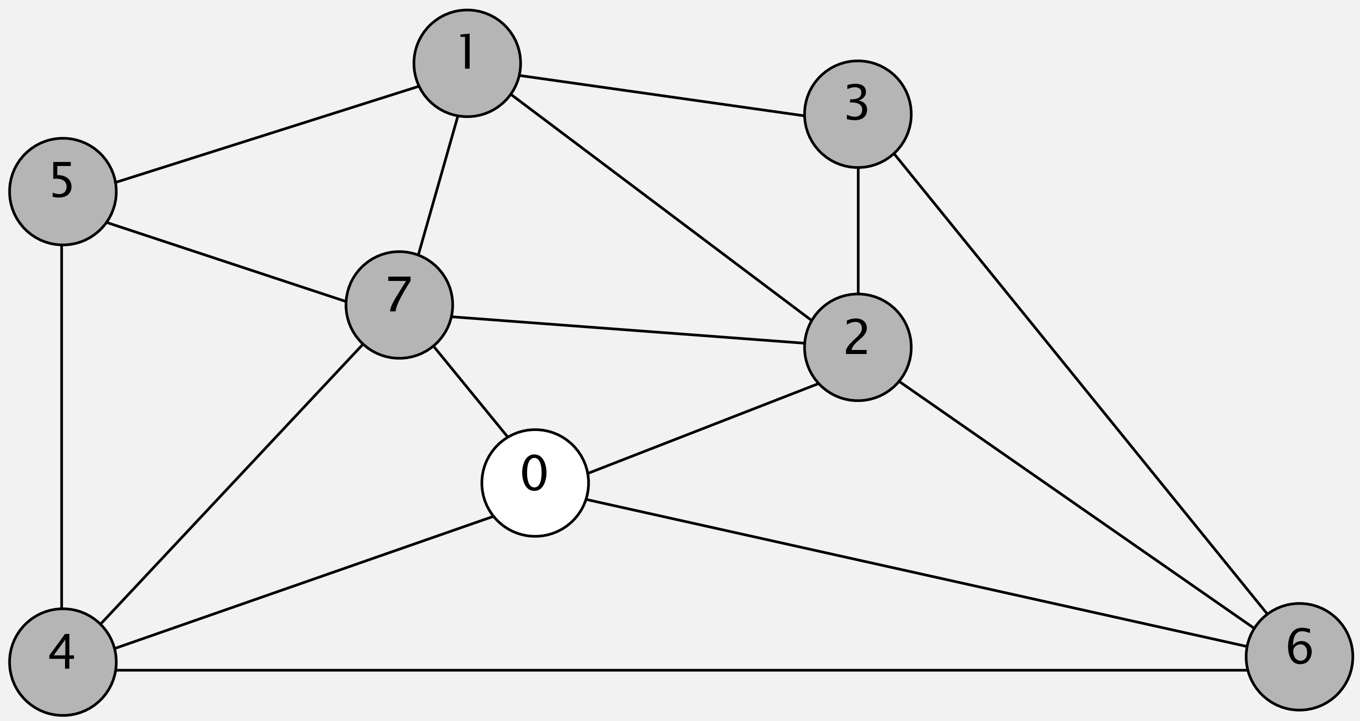
0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

Prim's algorithm: eager implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

•

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.39
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

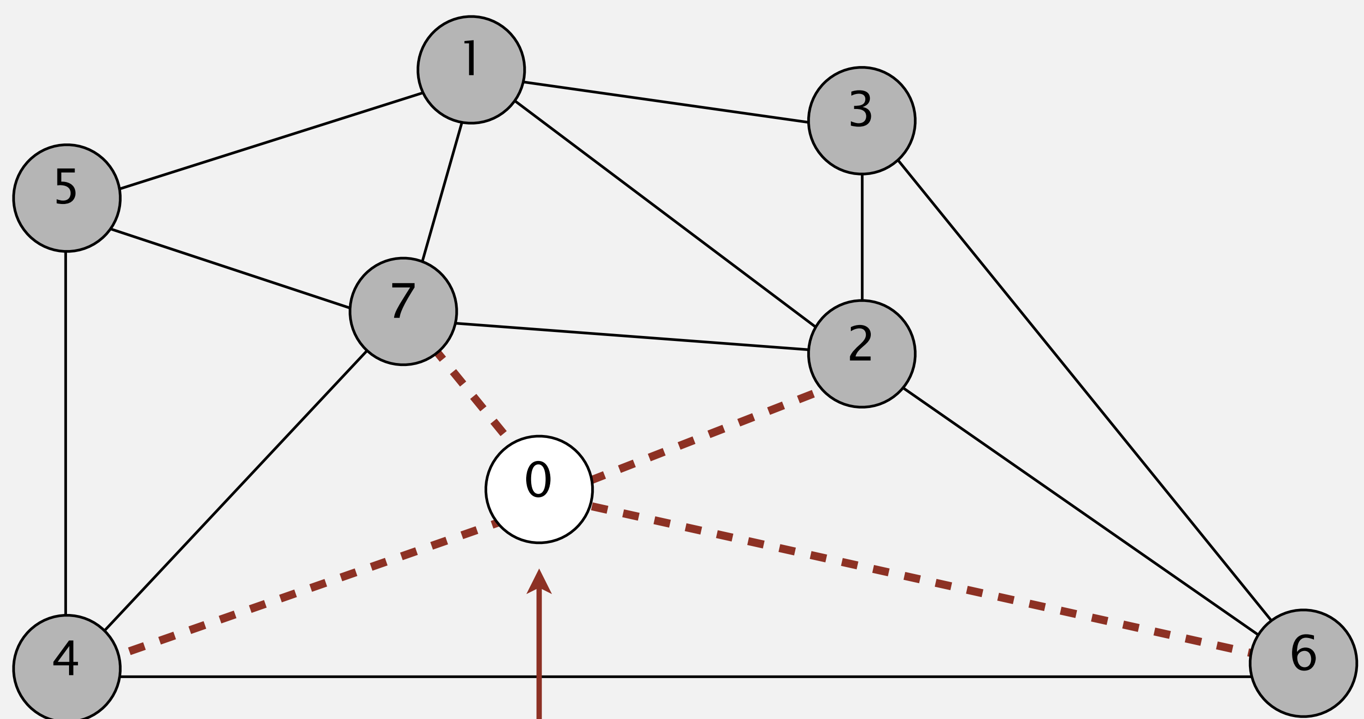


v	edgeTo[]	distTo[]
0	-	-

Prim's algorithm: eager implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.39
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



add vertices 7, 2, 4, and 6 to PQ

v	edgeTo[]	distTo[]
0	–	–
7	0–7	0.16
2	0–2	0.26
4	0–4	0.38
6	6–0	0.58

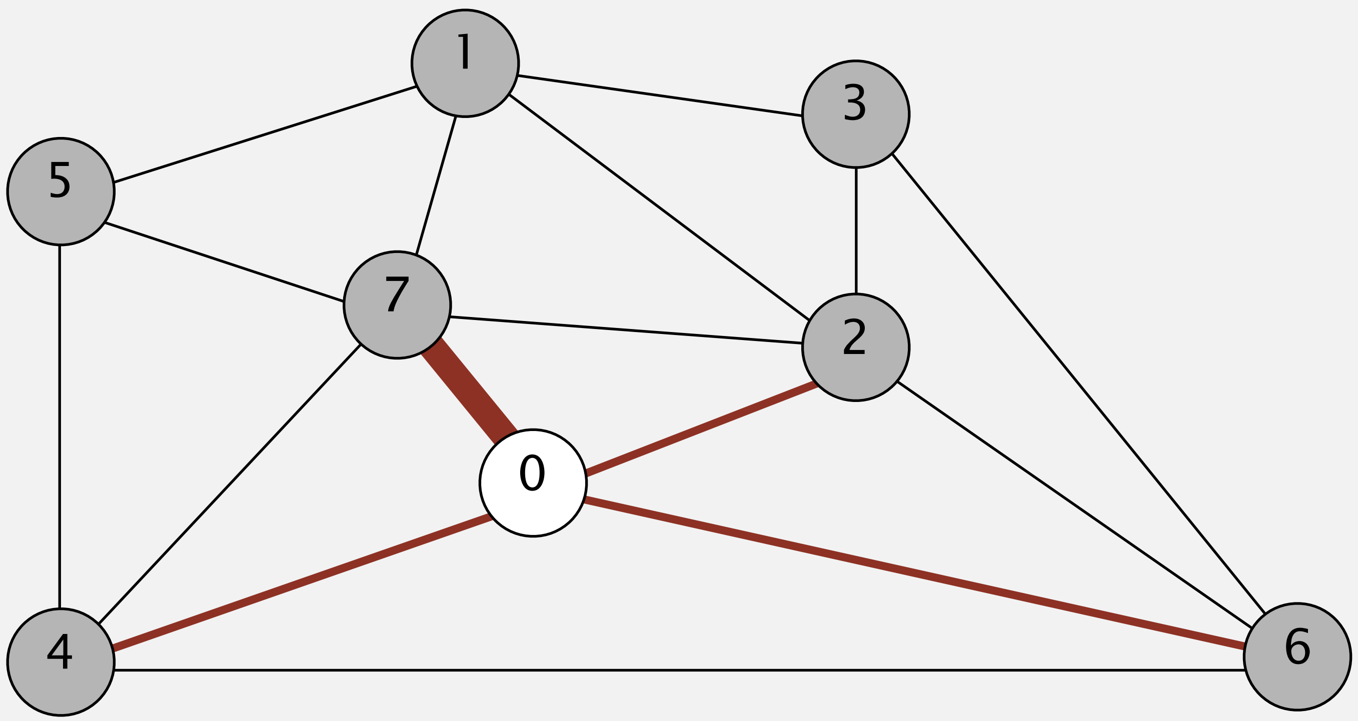
vertices on PQ
(sorted by weight)

Prim's algorithm: eager implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

•

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



v	edgeTo[]	distTo[]
0	–	–
7	0-7	0.16
2	0-2	0.26
4	0-4	0.38
6	6-0	0.58

vertices on PQ
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2-3	0.17
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2-7	0.34
4-5	0.35
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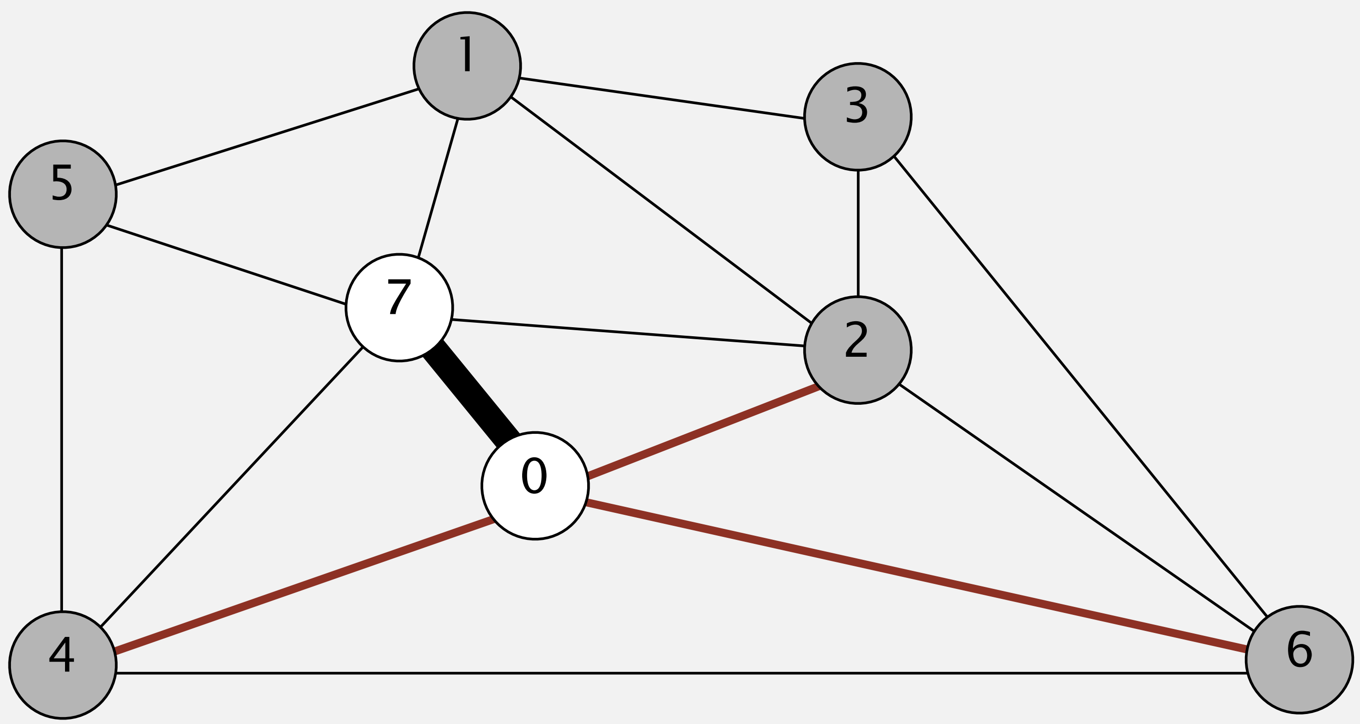
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



v	edgeTo[]	distTo[]
0	–	–
7	0–7	0.16
2	0–2	0.26
4	0–4	0.38
6	6–0	0.58

MST edges

0–7

Prim's algorithm: eager implementation demo

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- Add to T the min weight edge with exactly one endpoint in T.
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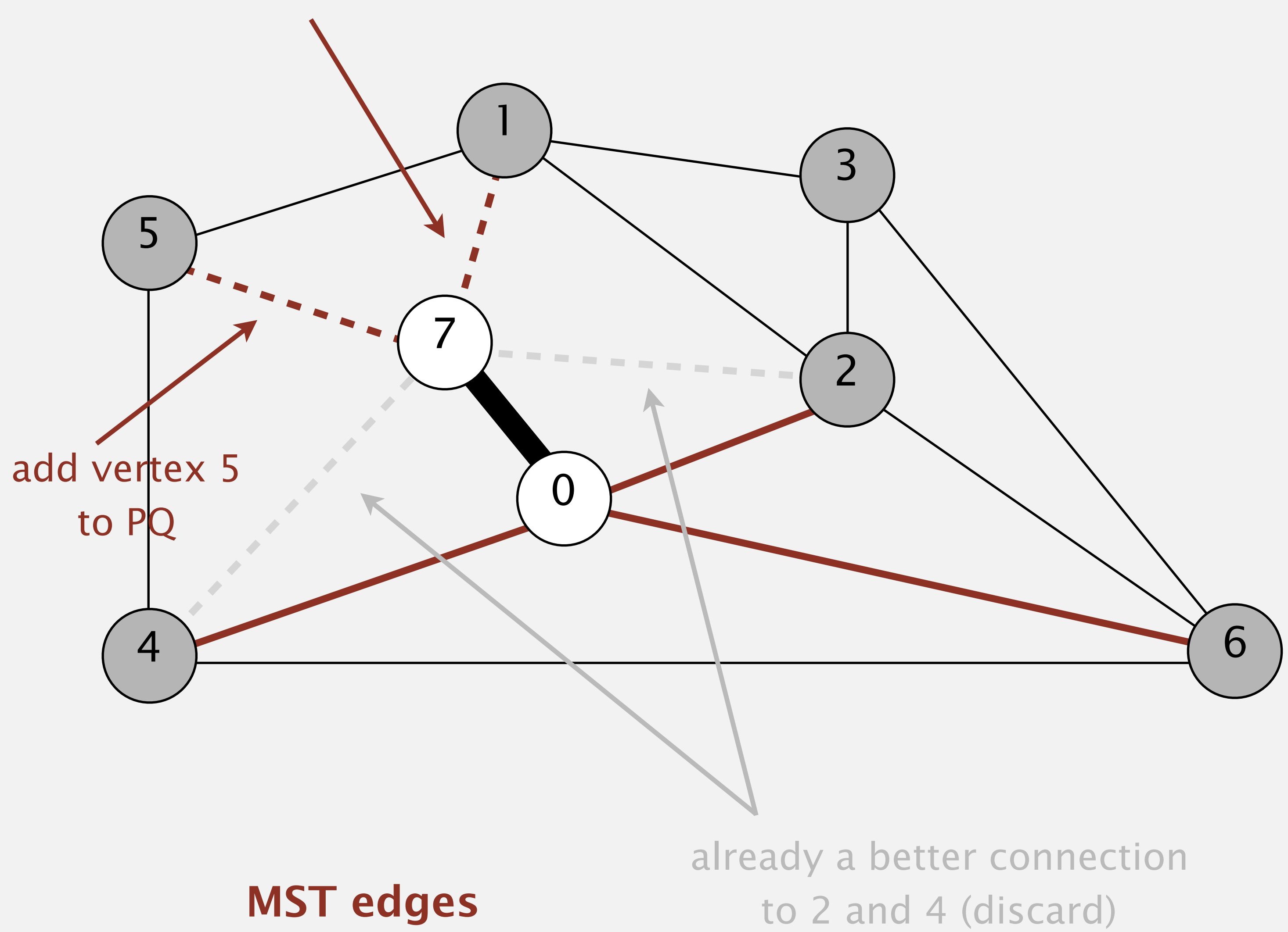
0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.39
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



MST edges

0-7

v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
5	5-7	0.28
4	0-4	0.38
6	6-0	0.58

vertices on PQ (sorted by weight)

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- Start with vertex 0 and greedily grow tree T.
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0-7	0.16
2-3	0.17
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0-2	0.26
5-7	0.28
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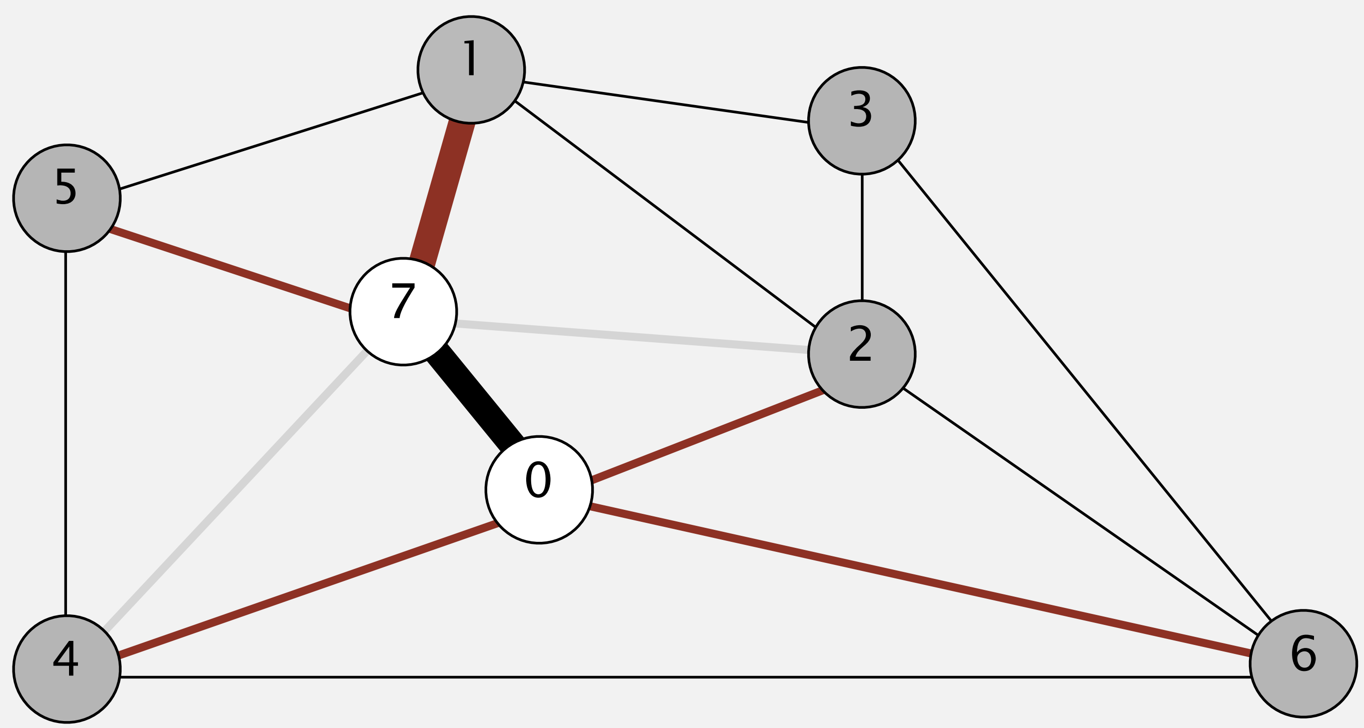

v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
5	5-7	0.28
4	0-4	0.38
6	6-0	0.58

vertices on PQ
(sorted by weight)

MST edges

0-7 1-7

Prim's algorithm: eager implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

•

0-7	0.16
2-3	0.17
1-7	0.19
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5-7	0.28
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1-2	0.36
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0-4	0.38
6-2	0.40
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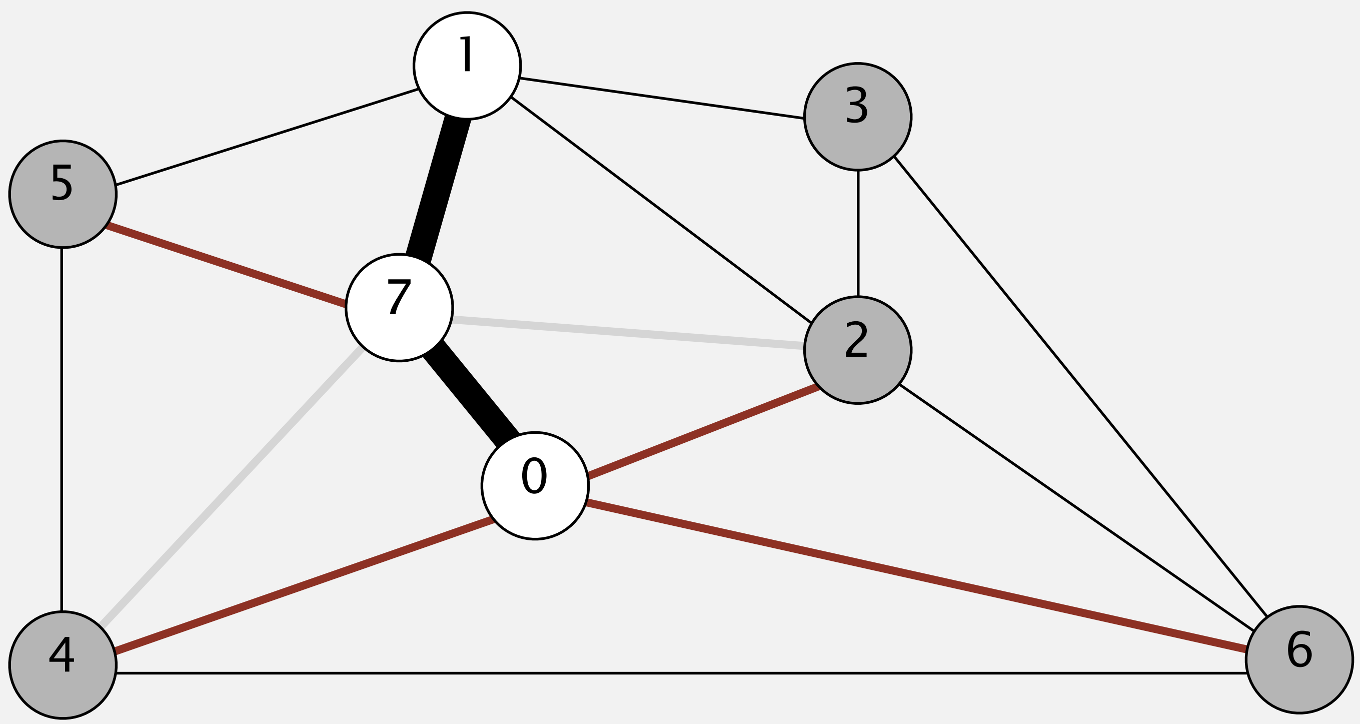
6-0	0.58
6-4	0.93



MST edges

0-7 1-7

v	edgeTo[]	distTo[]
0	–	–
7	0–7	0.16
→ 1	1–7	0.19
2	0–2	0.26
5	5–7	0.28
4	0–4	0.38
6	6–0	0.58

vertices on PQ
(sorted by weight)

Prim's algorithm: eager implementation demo

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0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
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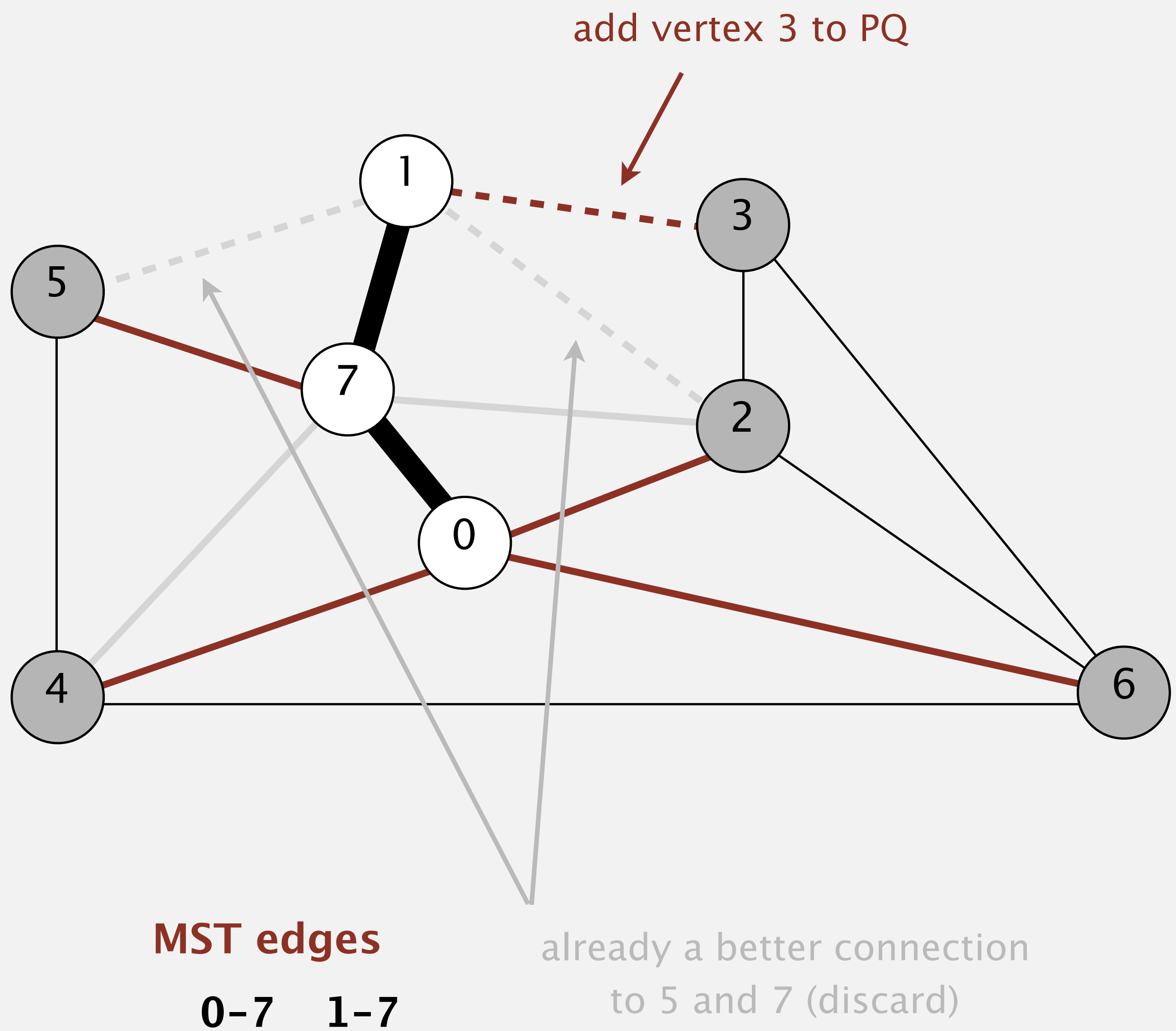
6-0	0.58
6-4	0.93



MST edges

0-7 1-7

v	edgeTo[]	distTo[]
0	–	–
7	0–7	0.16
1	1–7	0.19
2	0–2	0.26
5	5–7	0.28
3	1–3	0.29
4	0–4	0.38
6	6–0	0.58

Prim's algorithm: eager implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
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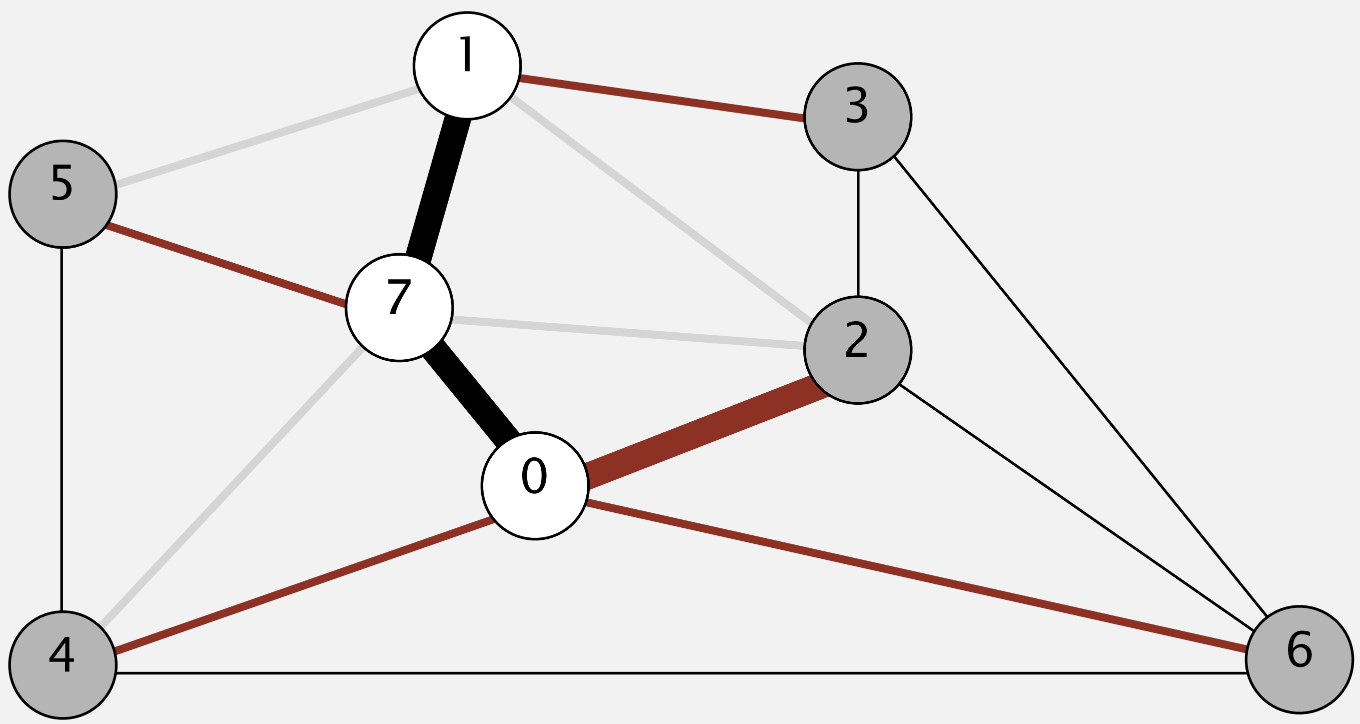
-

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
5	5-7	0.28
3	1-3	0.29
4	0-4	0.38
6	6-0	0.58

MST edges

0-7 1-7

Prim's algorithm: eager implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
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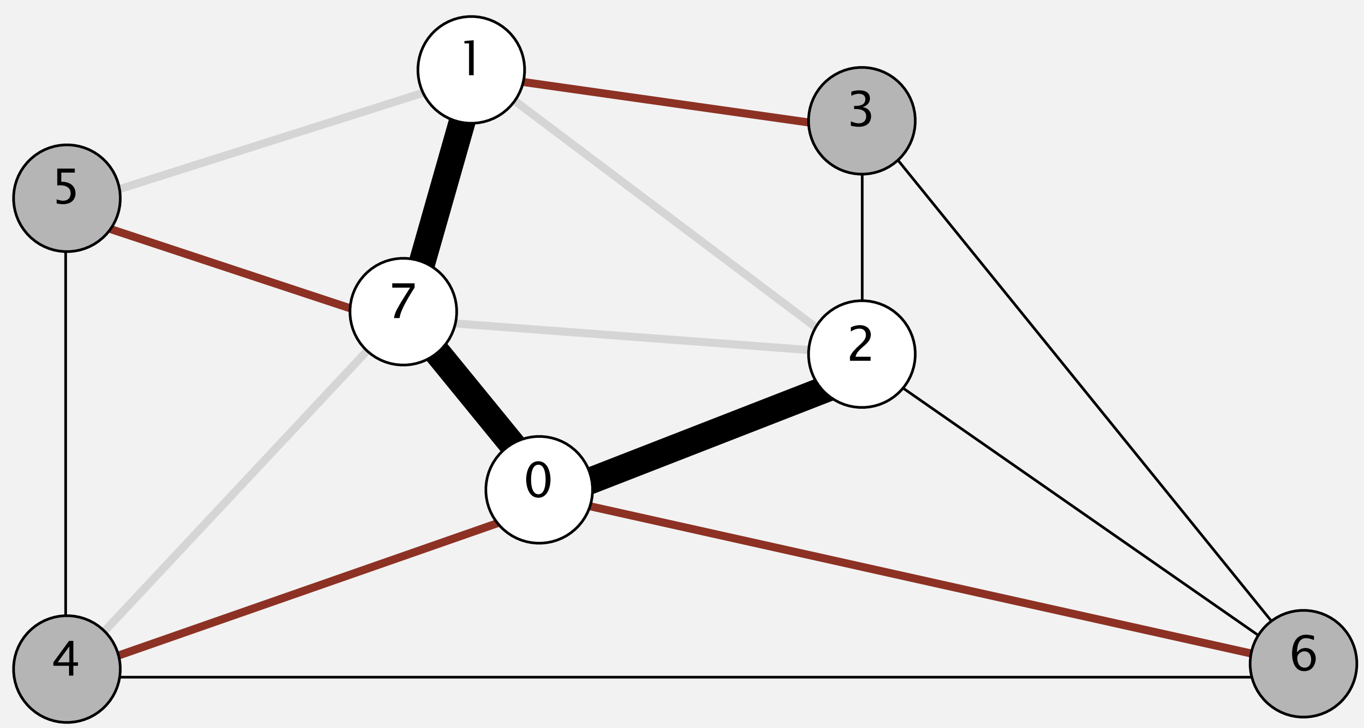
6-0	0.58
6-4	0.93



v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
5	5-7	0.28
3	1-3	0.29
4	0-4	0.38
6	6-0	0.58

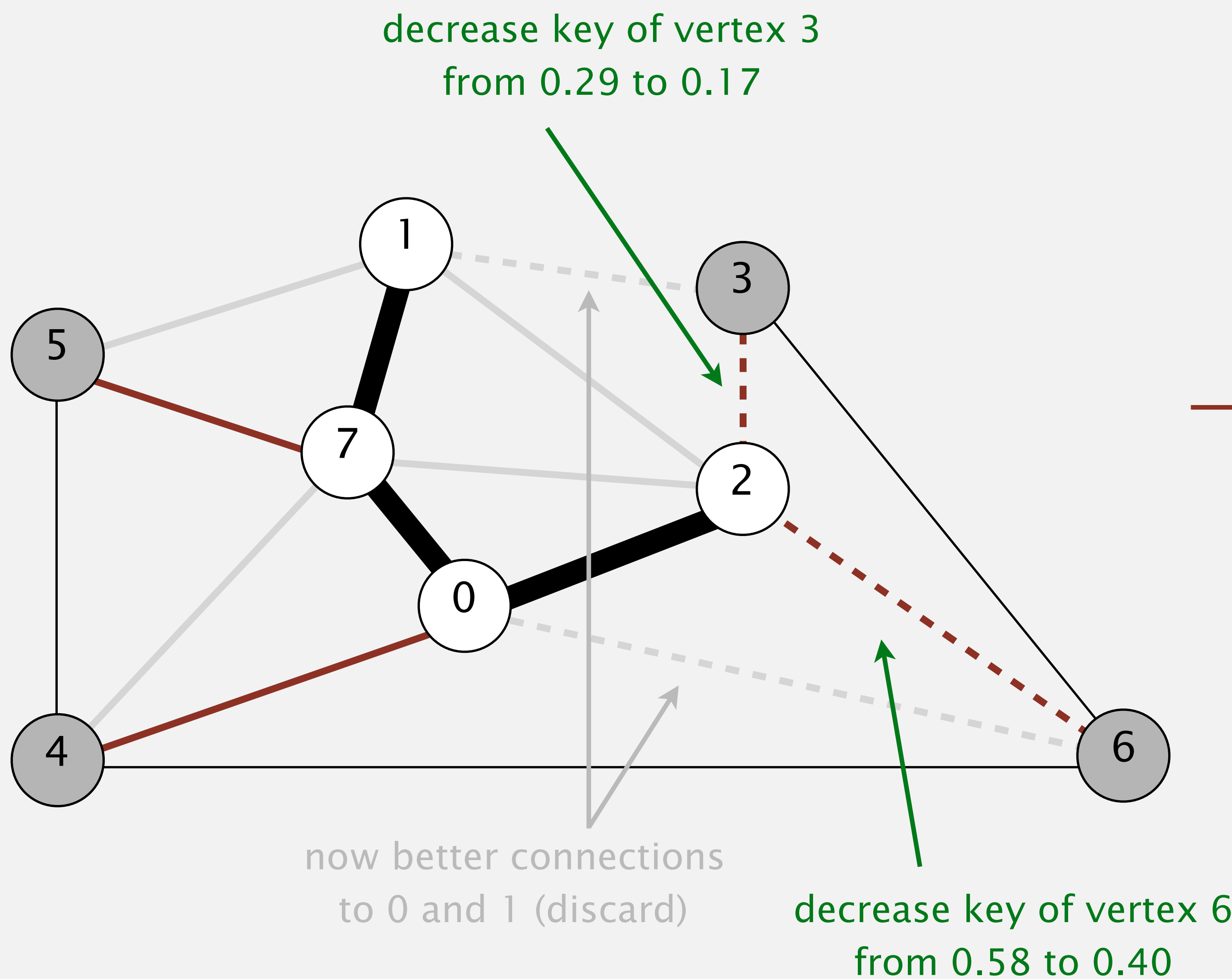
MST edges

0-7   1-7   0-2

Prim's algorithm: eager implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
3	~~1-3~~ 2-3	~~0.29~~ 0.17
5	5-7	0.28
4	0-4	0.38
6	~~6-0~~ 6-2	~~0.58~~ 0.40

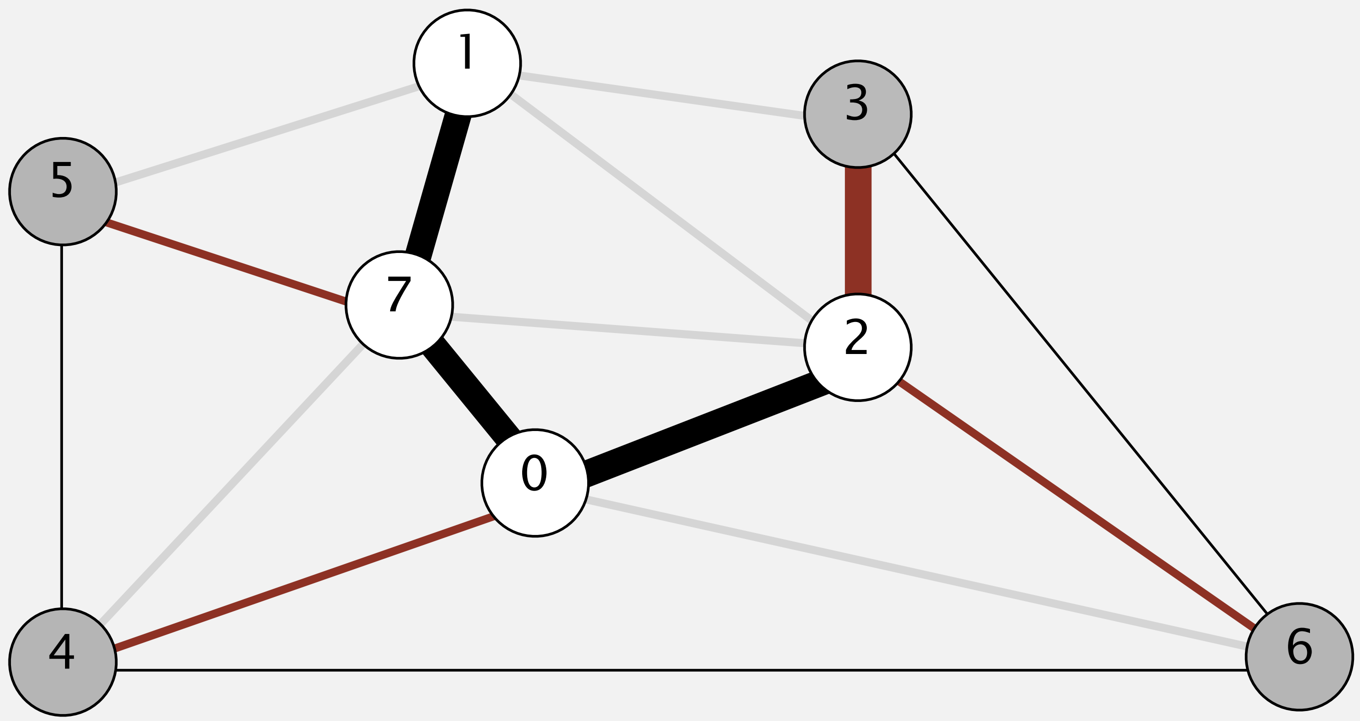
MST edges

0-7 1-7 0-2

Prim's algorithm: eager implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.
-

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
3	2-3	0.17
5	5-7	0.28
4	0-4	0.38
6	6-2	0.40

MST edges

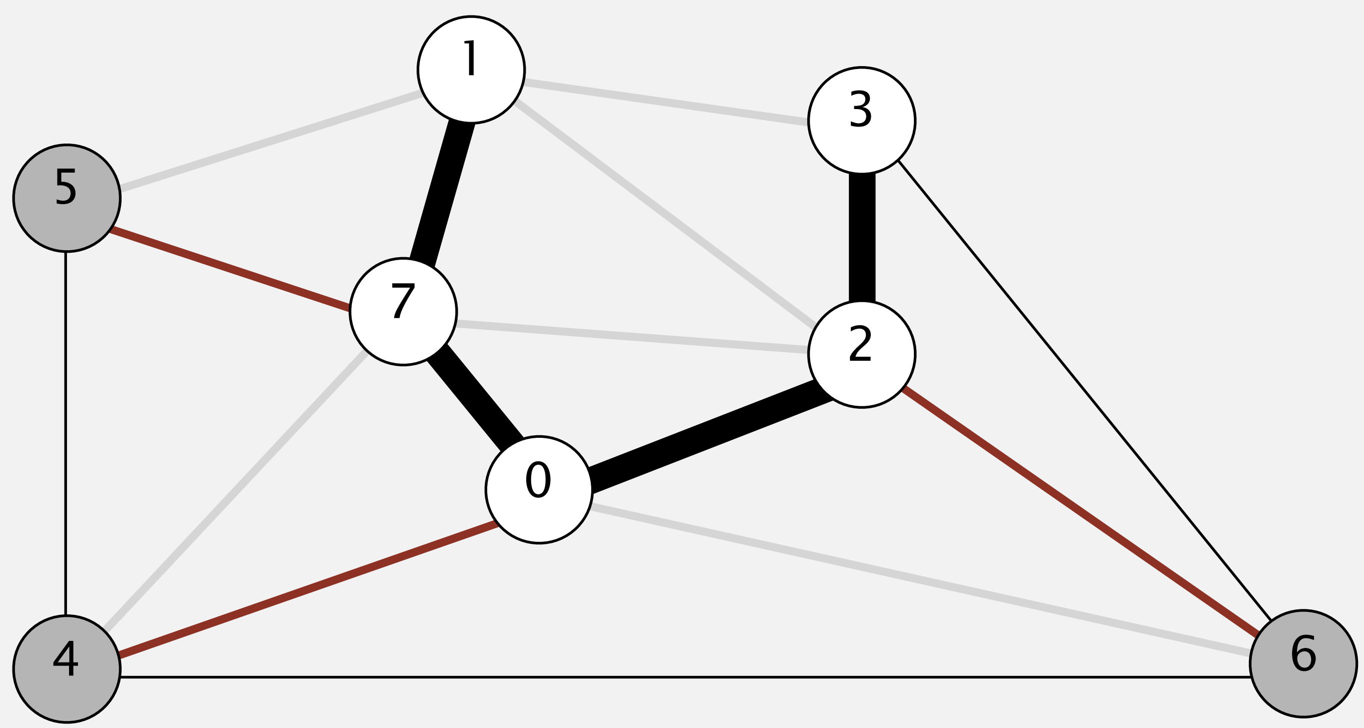
0-7 1-7 0-2 2-3

Prim's algorithm: eager implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

•

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
3	2-3	0.17
5	5-7	0.28
4	0-4	0.38
6	6-2	0.40

MST edges

0-7 1-7 0-2 2-3

Prim's algorithm: eager implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

•

0-7	0.16
2-3	0.17
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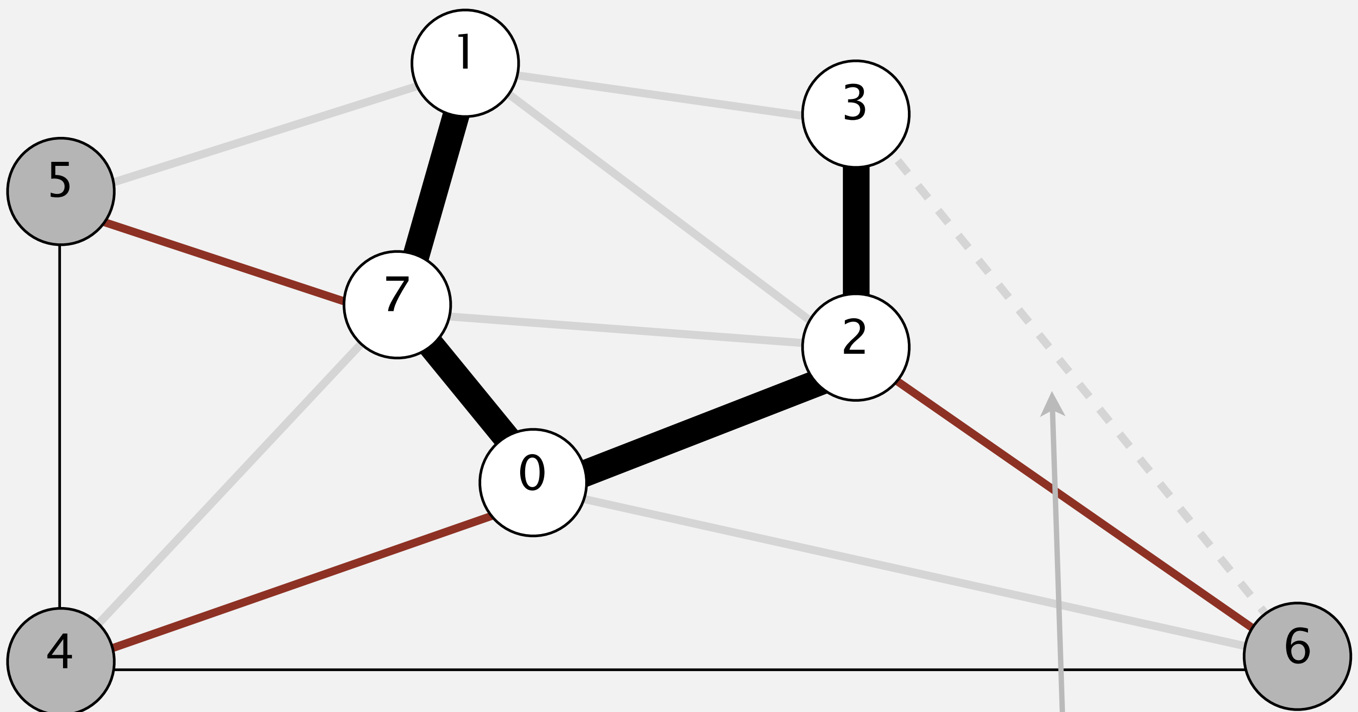

already a better connection
to 6 (discard)

v	edgeTo[]	distTo[]
0	–	–
7	0–7	0.16
1	1–7	0.19
2	0–2	0.26
3	2–3	0.17
5	5–7	0.28
4	0–4	0.38
6	6–2	0.40

MST edges

0-7 1-7 0-2 2-3

Prim's algorithm: eager implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

•

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2-3	0.17
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4-5	0.35
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4-7	0.37
0-4	0.38
6-2	0.40
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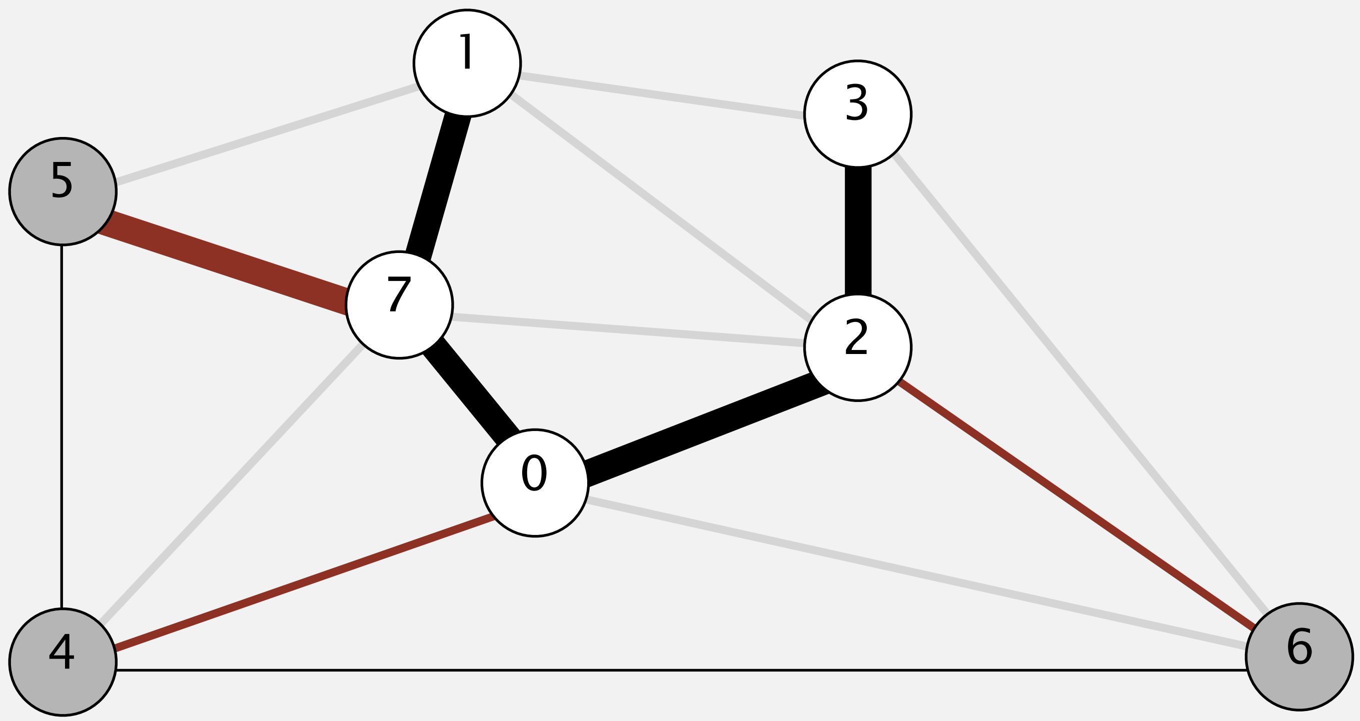
6-0	0.58
6-4	0.93



v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
3	2-3	0.17
5	5-7	0.28
4	0-4	0.38
6	6-2	0.40

MST edges

0-7 1-7 0-2 2-3

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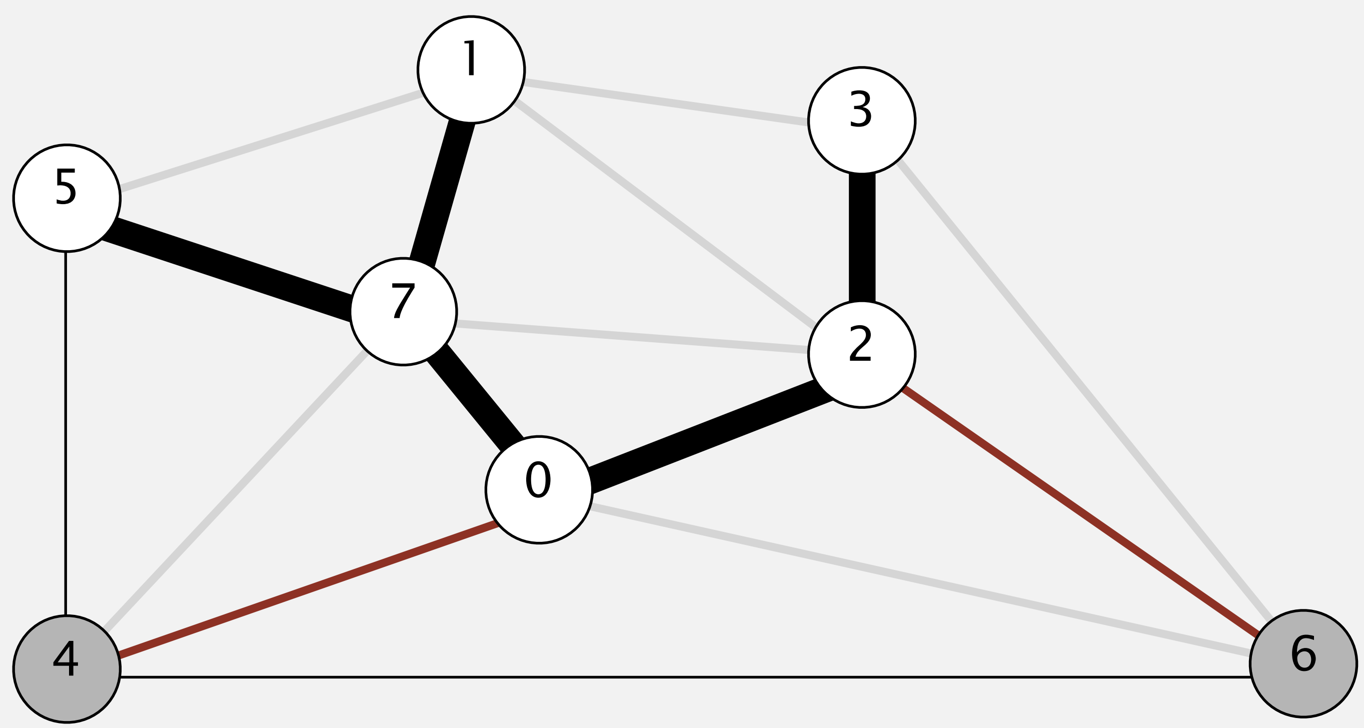
6-4	0.93



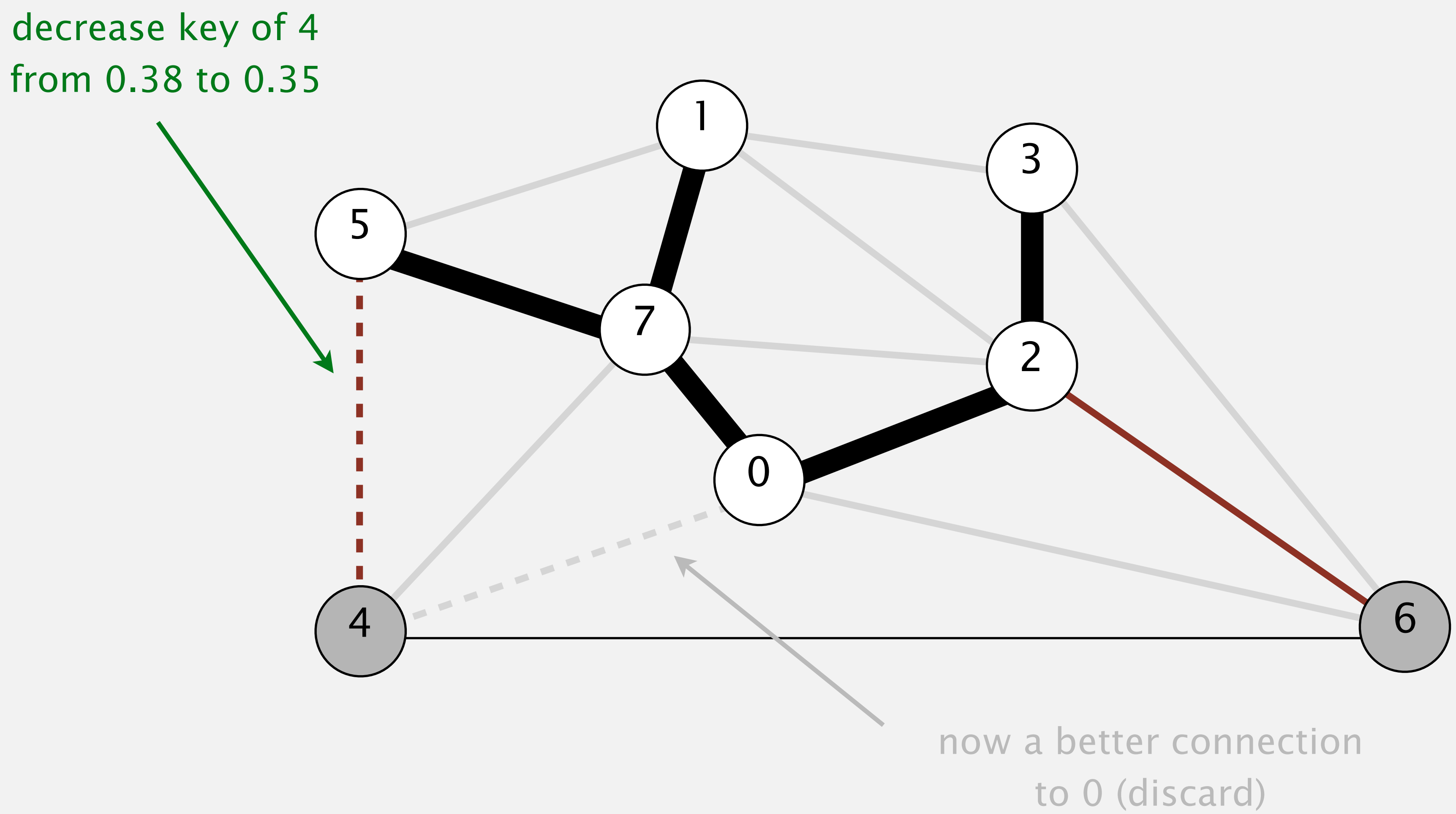
v	edgeTo[]	distTo[]
0	–	–
7	0–7	0.16
1	1–7	0.19
2	0–2	0.26
3	2–3	0.17
5	5–7	0.28
4	0–4	0.38
6	6–2	0.40

MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm: eager implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
3	2-3	0.17
5	5-7	0.28
4	~~0-4~~ 4-5	~~0.38~~ 0.35
6	6-2	0.40

MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm: eager implementation demo

- Start with vertex 0 and greedily grow tree T.
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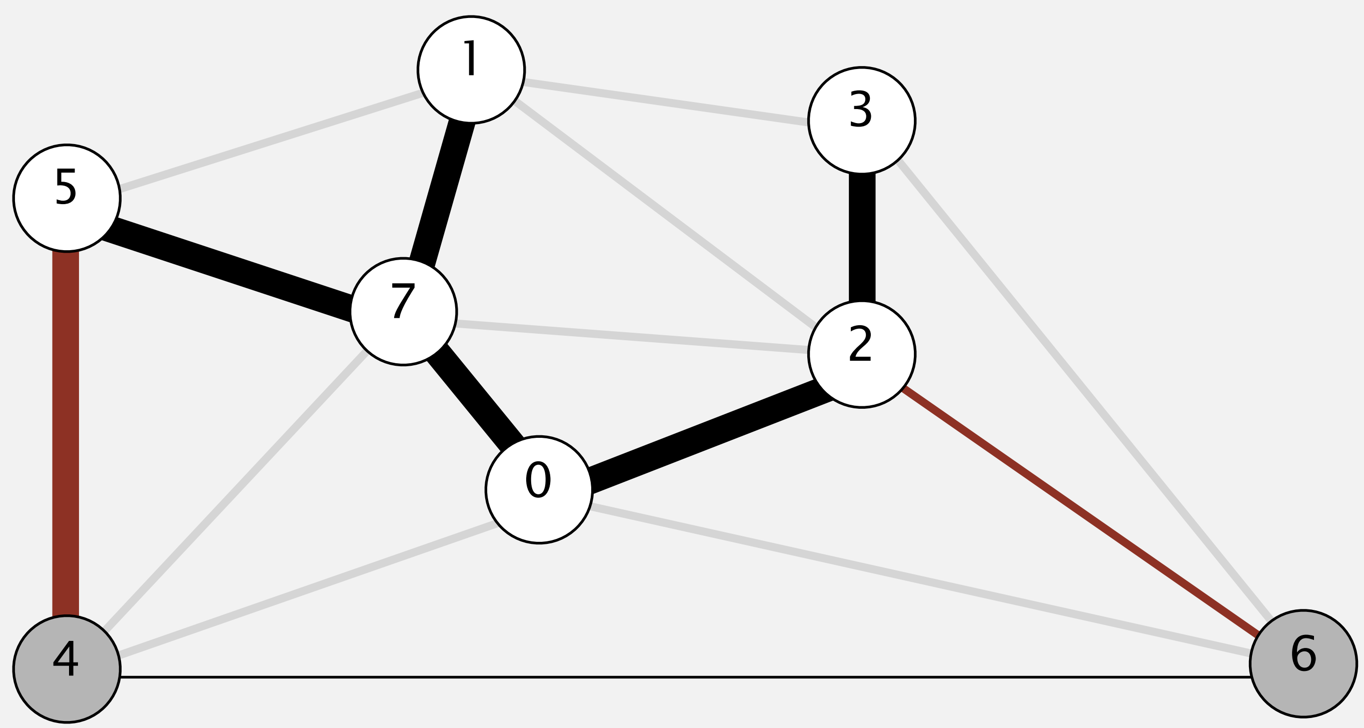
•



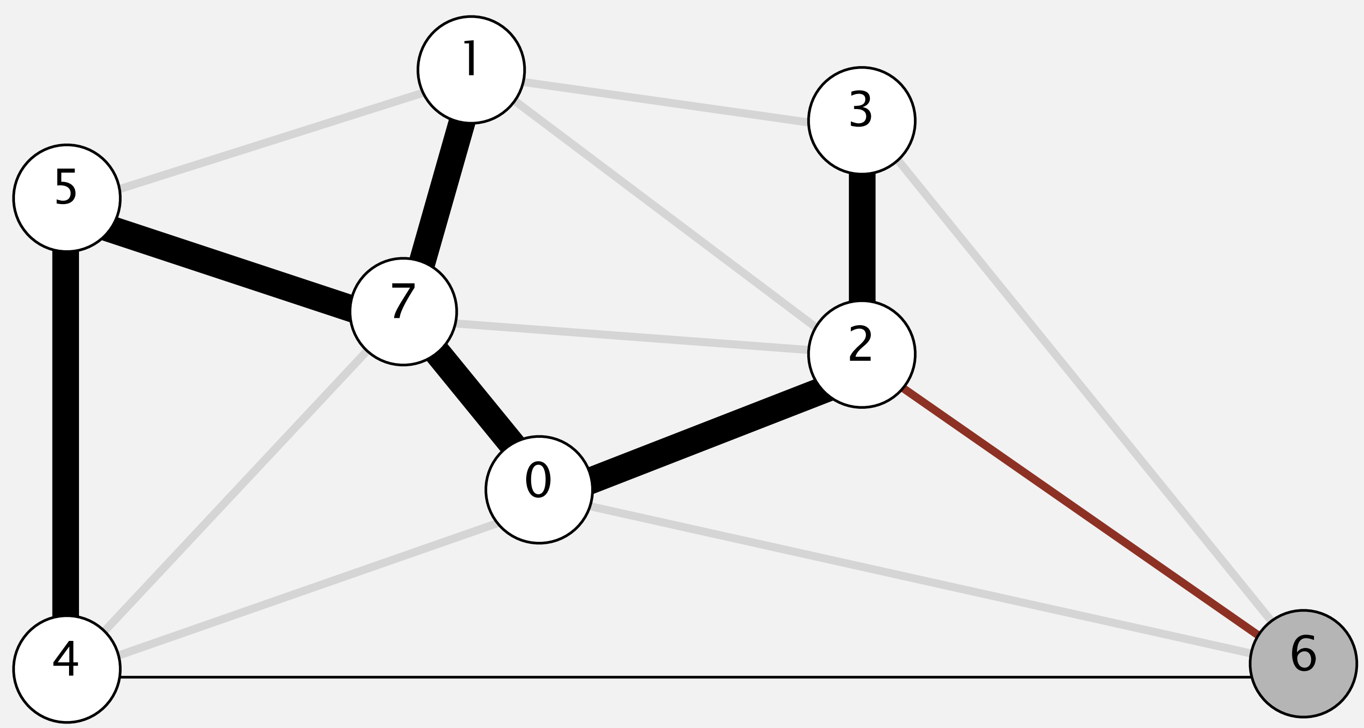
v	edgeTo[]	distTo[]
0	-	-
7	0–7	0.16
1	1–7	0.19
2	0–2	0.26
3	2–3	0.17
5	5–7	0.28
4	4–5	0.35
6	6–2	0.40

MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm: eager implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



v	edgeTo[]	distTo[]
0	-	-
7	0–7	0.16
1	1–7	0.19
2	0–2	0.26
3	2–3	0.17
5	5–7	0.28
4	4–5	0.35
6	6–2	0.40

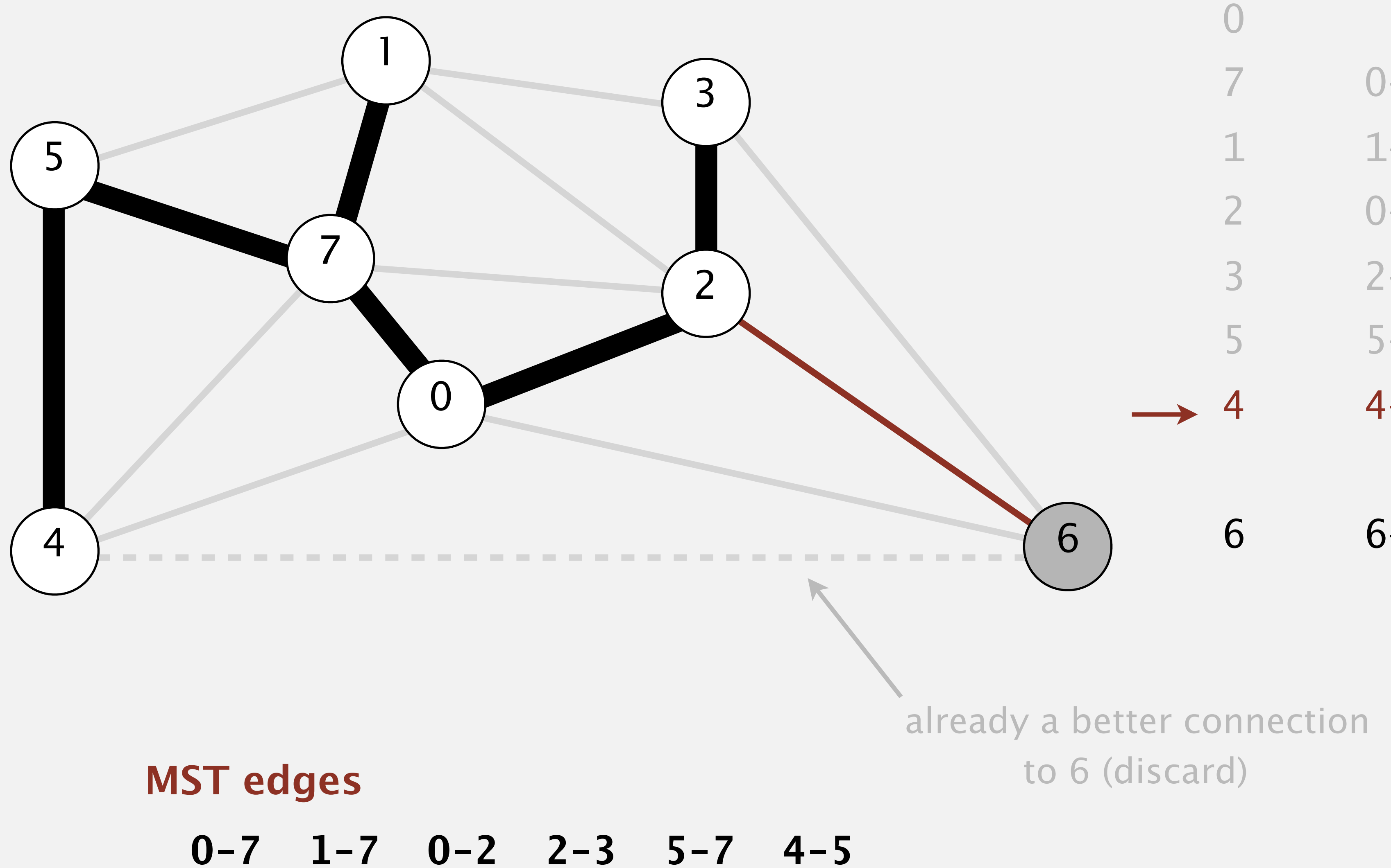
MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm: eager implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

•

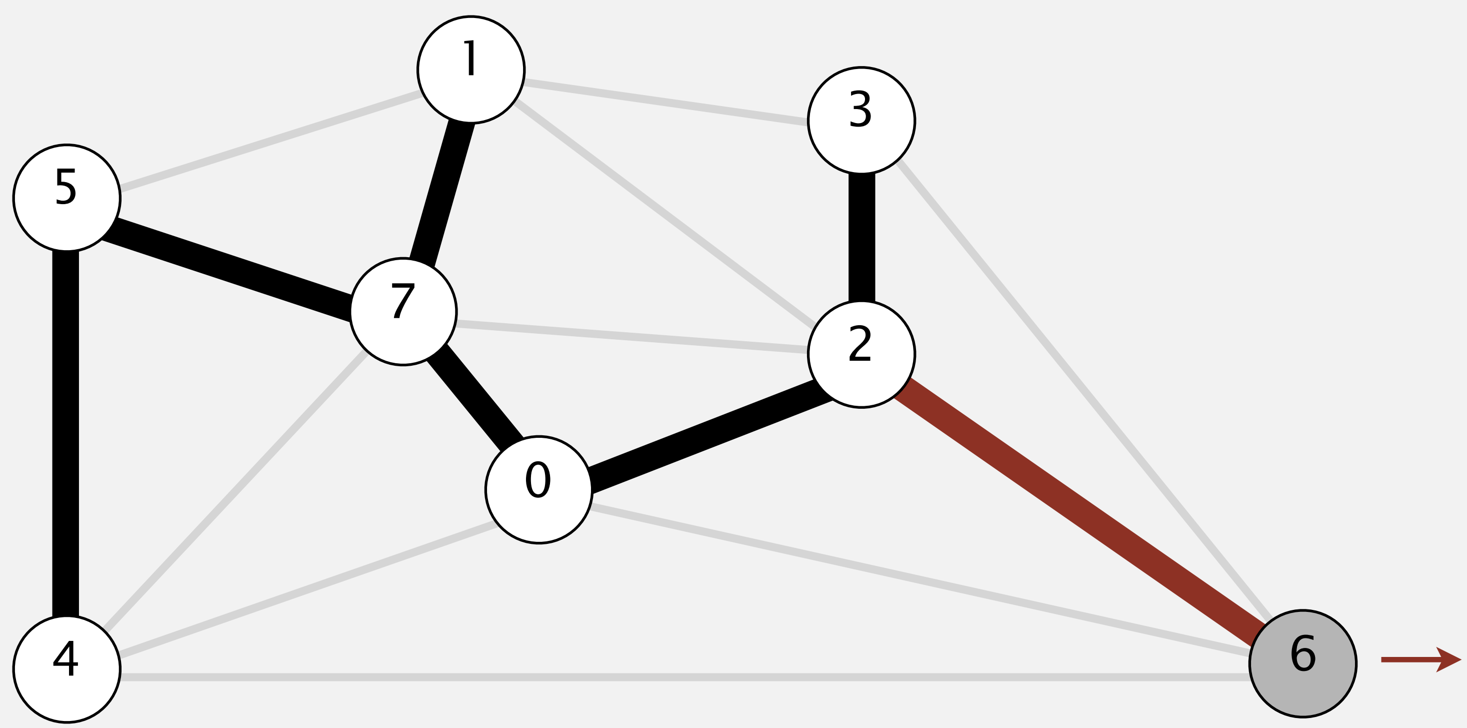


v	edgeTo[]	distTo[]
0	–	–
7	0–7	0.16
1	1–7	0.19
2	0–2	0.26
3	2–3	0.17
5	5–7	0.28
4	4–5	0.35
6	6–2	0.40

Prim's algorithm: eager implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

•



v	edgeTo[]	distTo[]
0	–	–
7	0–7	0.16
1	1–7	0.19
2	0–2	0.26
3	2–3	0.17
5	5–7	0.28
4	4–5	0.35
6	6–2	0.40

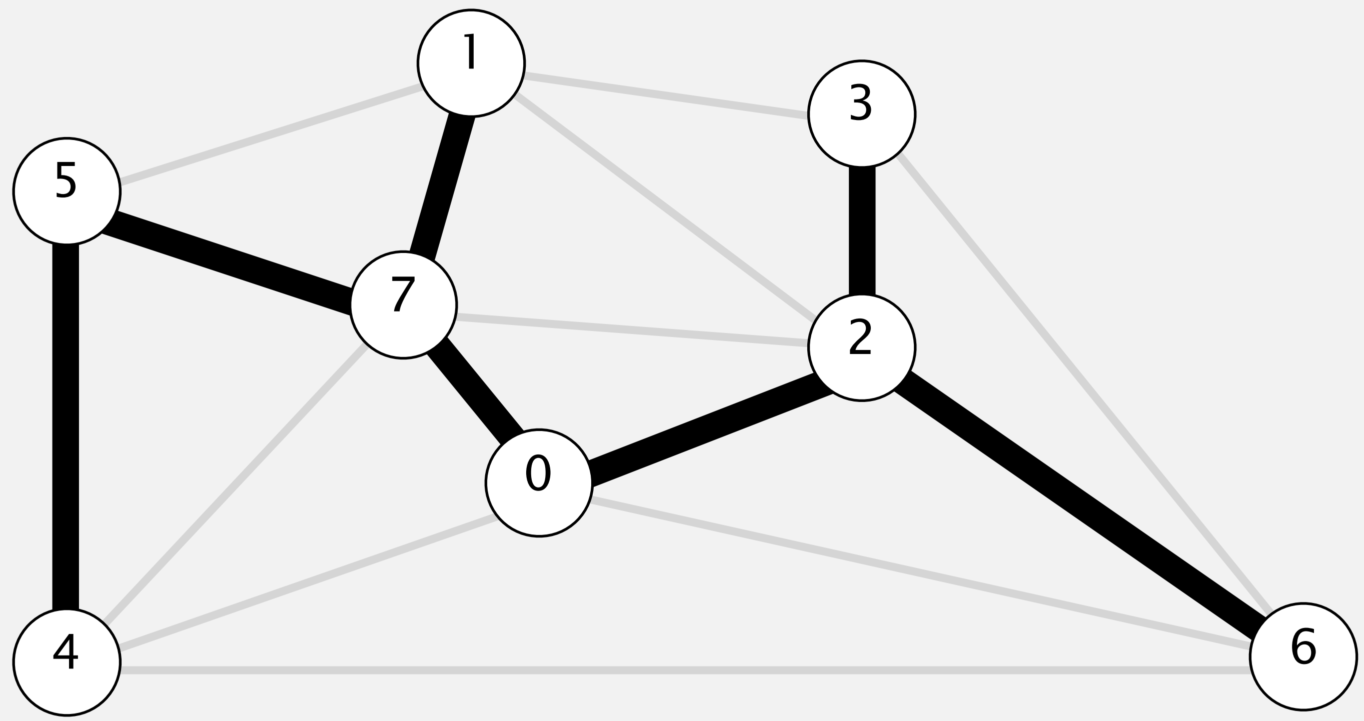
MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm: eager implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

•



v	edgeTo[]	distTo[]
0	-	-
7	0–7	0.16
1	1–7	0.19
2	0–2	0.26
3	2–3	0.17
5	5–7	0.28
4	4–5	0.35
6	6–2	0.40

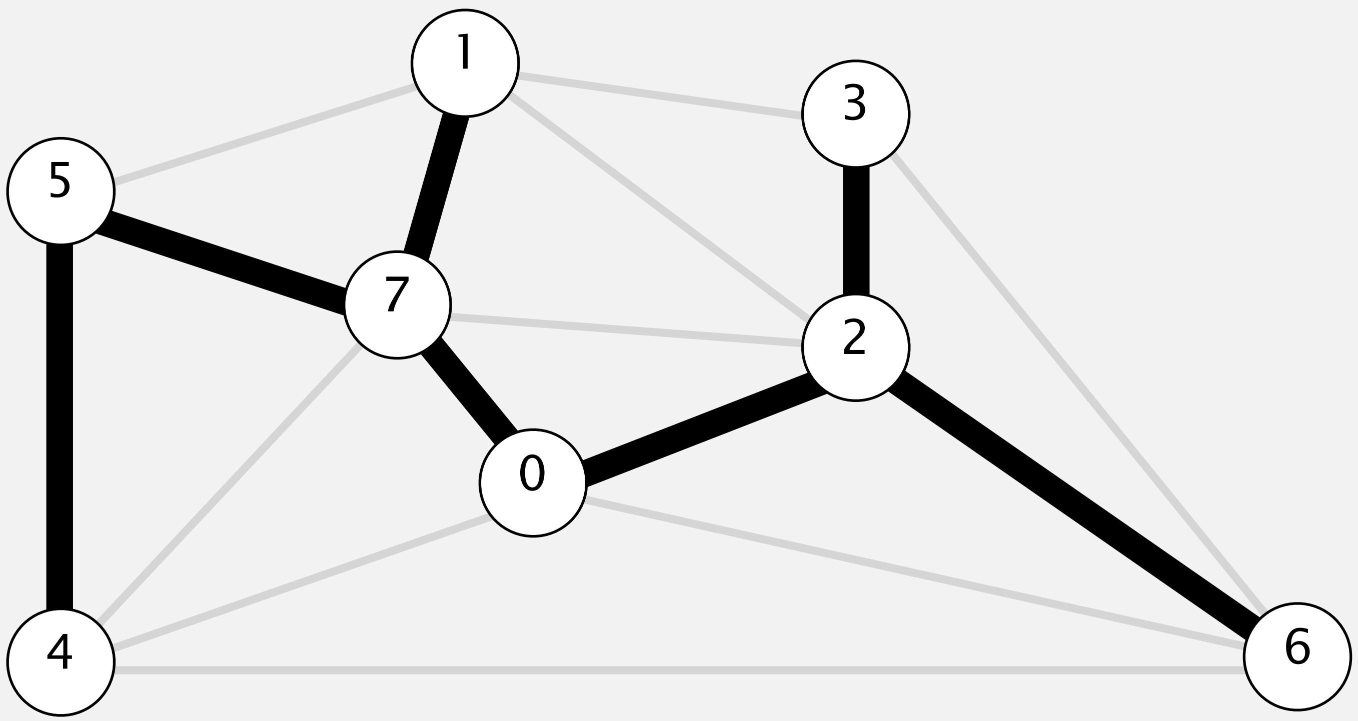
MST edges

0-7 1-7 0-2 2-3 5-7 4-5 6-2

Prim's algorithm: eager implementation demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

•



v	edgeTo[]	distTo[]
0	-	-
7	0–7	0.16
1	1–7	0.19
2	0–2	0.26
3	2–3	0.17
5	5–7	0.28
4	4–5	0.35
6	6–2	0.40

MST edges

0-7 1-7 0-2 2-3 5-7 4-5 6-2

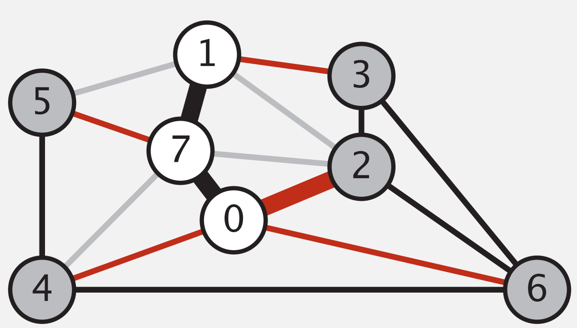
Prim's algorithm: eager implementation

Challenge. Find min weight edge with exactly one endpoint in T.

pq has at most one entry per vertex

Eager solution. Maintain a PQ of vertices connected by an edge to T, where priority of vertex v = weight of shortest edge connecting v to T.

- Delete min vertex v and add its associated edge $e = v–w$ to T.
- Update PQ by considering all edges $e = v–x$ incident to v
  - ignore if x is already in T
  - add x to PQ if not already on it
  - decrease priority of x if $v–x$ becomes shortest edge connecting x to T



0		
1	1-7	0.19
2	0-2	0.26
3	1-3	0.29
4	0-4	0.38
5	5-7	0.28
6	6-0	0.58
7	0-7	0.16

red: on PQ

black: on MST

Prim's algorithm: eager implementation

```
public class PrimMST
{
   private Edge[] edgeTo;          // shortest edge from tree vertex
   private double[] distTo;        // distTo[w] = edgeTo[w].weight()
   private boolean[] marked;       // true if v on tree
   private IndexMinPQ<Double> pq;  // eligible crossing edges

   public PrimMST(EdgeWeightedGraph G)
   {
      edgeTo = new Edge[G.V()];
      distTo = new double[G.V()];
      marked = new boolean[G.V()];
      for (int v = 0; v < G.V(); v++)
         distTo[v] = Double.POSITIVE_INFINITY;
      pq = new IndexMinPQ<Double>(G.V());

      distTo[0] = 0.0;
      pq.insert(0, 0.0);              // Initialize pq with 0, weight 0.
      while (!pq.isEmpty())
         visit(G, pq.delMin());       // Add closest vertex to tree.
   }

   private void visit(EdgeWeightedGraph G, int v)
   {  // Add v to tree; update data structures.
      marked[v] = true;
      for (Edge e : G.adj(v))
      {
         int w = e.other(v);
         if (marked[w]) continue;     // v-w is ineligible.
         if (e.weight() < distTo[w])
         {  // Edge e is new best connection from tree to w.
            edgeTo[w] = e;
            distTo[w] = e.weight();
            if (pq.contains(w)) pq.change(w, distTo[w]);
            else                pq.insert(w, distTo[w]);
         }
      }
   }

   public Iterable<Edge> edges()    // See Exercise 4.3.21.
   public double weight()           // See Exercise 4.3.31.
}
```

Eager Version's Prim's algorithm

The eager version of Prim's algorithm uses extra space proportional to V and time proportional to$E \log V$ (in the worst case) to compute the MST of a connected edge- weighted graph with E edges and V vertices.

lazy Prim	E	$E \log E$
eager Prim	V	$E \log V$
Kruskal	E	$E \log E$