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A Logical Theory of Dependence[1]

§1 [Introduction]

In the course of the logical analysis of *Gestalt* carried out by Mr. Oppenheim and myself[2] we often dealt with the notion of dependence[3] and therefore we felt the need to attach a precise sense to that notion.

As far as I know the general notion of dependence has not been analysed by modern logicists before. Therefore I hope that the attempt to carry out this analysis will be of some interest for scientists independently of its connection with the *Gestalt* problem.

A popular example of the use of 'dependence' is furnished by the statement that a commercial price at a given time depends upon demand and supply at that time. The analysis of such an instance and similar ones leads to the following statements concerning the logical form of the propositions involved:

(1) Anything said to depend upon something else is – or at least can be described as – a *function*.

(2) What something is said to depend upon is a *class* generally consisting *of* several *functions*. In special cases this class may have only one element.

(3) All the functions involved in the same statement of dependence must have the *same argument*,[4] i.e. it must be possible to use the same letter, say 'z', as the argument for all the functions occurring in one formula.

Consequently a statement of dependence may be symbolized as follows:

$$R(f,\varphi)_x$$

where 'f' means the function said to depend, 'φ' the class of functions upon which f depends and 'x' is the variable for the common argument.

§2 [Dependence]

2.1 I am not quite sure whether there is a unique consistent meaning common to all cases in which the word 'dependence' and its derivatives are used in daily life and science. Therefore I suggest we distinguish between several kinds of dependence. One of them and perhaps the most important one may be described by the following statement:

(E) If, for some argument x_1, every function belonging to φ, i.e. every function upon which f depends, takes the same values as for the argument x_2, then f itself must take *equal* values for x_1 and x_2 as well.

Thus, if, in our example, at the time t_1 the demand for the article a is equal to the demand at the time t_2 and if the same holds for the supply, then the price of a at the time t_2 will be equal to it at the time t_1. This condition is necessary and sufficient for the price to depend upon demand and supply.[5]

It is, to be sure, easy to find other examples which fulfil the condition (E) and which nevertheless would not be considered as cases of dependence in everyday life. We need indeed only choose either a class φ such that at least one of its elements takes *different* values for every two different arguments or one can put down a constant instead of f, i.e. a function taking the *same* value for all arguments. However, in my opinion, from such trivial cases, well known to logicians, no serious objection can be derived against my suggestion of describing one sort of dependence by the statement (E).

2.2 A formal definition on the basis of statement (E) runs as follows: Let φ be a class of functions all of them with x as argument and let f be a single function of the same argument. Then I define:

D1 $\quad \mathrm{Equidep}\,(f,\varphi)_x =_{df} (x_1)(x_2)[(g)(g\varepsilon\varphi \supset g(x_1) = g(x_2)) \supset f(x_1) = f(x_2)].$

For certain purposes it will be convenient to introduce a special symbol for the expression 'f takes equal values in x_1 and x_2'. Therefore we write:

D2 $\quad Eq(f, x_1, x_2) =_{df} f(x_1) = f(x_2),$

and consequently

'$\overset{\rightarrow\subset}{Eq}(x_1, x_2)$' will be short for '$\hat{f}\{Eq(f, x_1, x_2)\}$'.

Thereby the definition D1 may be transformed and our first theorem reads:

T1 $\quad \text{Equidep}\,(f,\varphi)_x \equiv (x_1)(x_2)(\varphi \subset \overset{\rightarrow\subset}{Eq}(x_1, x_2) \supset Eq(f, x_1, x_2)).$

A further convenient abbreviation is expressed by the definition

D3 $\quad E(\varphi) =_{df} \hat{f}\,\{\text{Equidep}\,(f, \varphi)\}.$

The following theorems are obvious consequences of our definitions:[6]

T2 $\quad f \varepsilon E([f]),$
T3 $\quad \varphi \subset \psi \supset E\,(\varphi) \subset E\,(\psi)$
T4 $\quad \varphi \subset E\,(\varphi).$

The class of constants which we already mentioned can be defined as follows:

D4 $\quad \text{const} =_{df} \hat{f}\,\{(x_1)(x_2) f(x_1) = f(x_2)\}$

Then we have

T5 $\quad \text{const} = E(\Lambda)$

where 'Λ' designates the null-class of functions.
From T3 and T5 can be inferred:

T6 $\quad \text{const} \subset E\,(\varphi).$

Now I will define a notion which I have mentioned before also, namely that of a function the values of which are different for different arguments. For this notion we use the symbol 'mon' because monotone functions in mathematics are special cases of it.

D5 $\quad \text{mon} =_{df} \hat{f}\{(x_1)\,(x_2)\,(Eq\,(f, x_1, x_2) \supset x_1 = x_2\}.$

It will be easily seen that if the class φ has among its elements at least one 'mon', every function must stay in Equidep to φ. Hence:

T7 $\exists!\text{mon} \cap \varphi \supset E(\varphi) = V$

where 'V' symbolizes the universal class of functions.

The two following theorems may also be easily verified:

T8 $\psi \subset E(\varphi) \supset E(\varphi\psi) \subset E(\varphi)$,

T9 $E(E(\varphi)) = E(\varphi)$.

The last theorem shows that the class $E(\varphi)$ is closed with respect to the operation $E(\varphi)$.

2.3 Our next step will be the definition of some one-place predicates for classes of functions in term of 'Equidep'. I begin with a notion which can be considered as one – out of several possible – formalizations of *independence*.

D6 $\text{inequidep}(\varphi) =_{df} (f)(f \varepsilon \varphi \supset \sim\text{Equidep}(f, \varphi - [f]))$

i.e. φ is inequidep, when and only when none of its elements stands in Equidep to its complementary class, the term 'complementary class' being an expression for the symbol '$\varphi - [f]$'. Therefrom the following theorem can be derived.

T10 $\text{inequidep}(\varphi) \equiv (\psi)((\psi \subset \varphi . E(\psi) = E(\varphi)) \supset \psi = \varphi)$.

Secondly I suggest a formalization for the well known notion of *interdependence*:

D7 $\text{interequidep}(\varphi) =_{df} (f)(f \varepsilon \varphi \supset \text{Equidep}(f, \varphi - [f]))$,

i.e. φ is interequidep if every one of its elements stands in Equidep to its complementary class. The following notion is stronger than the preceding one:

D8 $\text{interequidepend}(\varphi) =_{df} (\psi)((\psi \neq \Lambda . \psi \subset \varphi) \supset \varphi \subset E(\psi))$,

i.e. φ is interequidepend, when and only when every element of φ stands in Equidep to any existent sub-class of φ. Substituting '$[f]$' for 'ψ' in D8 we get after an easy transformation:

T11 $\text{interequidepend}(\varphi) \equiv (f)(f \,\varepsilon\, \varphi \supset \varphi \subset E([f]))$ and
T12 $\text{interequidepend}(\varphi) \equiv (f)(g)(f, g \,\varepsilon\, \varphi \supset \text{Equidep}(f, [g]))$,

i.e. φ is interequidepend when and only when each of its elements stands in Equidep to every class the only element of which belongs to φ. Another obvious consequence of D7 is:

T13 $\text{interequidepend}(\varphi) \supset \text{interequidep}(\varphi)$.[7]

§3 [Vardep and Equidep]

3.1 The notion 'Equidep' is being based on the statement (E), i.e. on the assumption that *equality* of the values of f is implied by the *equality* of the corresponding values of the other functions involved (that was the very reason for the choice of the symbol). Now it seems to be equally evident that a function which is said to depend upon other functions must *vary* with them. In order to explain this new notion let us consider a method often employed by scientists in testing the dependence of one phenomenon on other phenomena. Suppose we have a certain phenomenon *a* and want to test its dependence upon a group of phenomena: *b, c, d.* Then we often proceed in the following way: first we keep *b* and c constant and let *d* alone vary; then, if *a* varies also, we infer that *d* is one of the phenomena upon which *a* is depending. Suppose we do the same thing with *c* and find that *a* does not vary when *c* alone among the group *b, c, d* has been made to vary. In that case we would say that *a* does *not* depend upon *c*, etc.

Evidently the meaning of the term "dependence" that we have just described is not identical with the one defined previously although many logical relations hold between the two. Let the symbol of the new concept be

'$\text{Vardep}\,(f, \varphi)_x$'.

3.2 The definition is as follows:

D9 $\text{Vardep}(f, \varphi) =_{df} (x_1)(x_2)\,[E!(\imath g)(g \,\varepsilon\, \varphi . \dot{-} \text{Eq}(g, x_1, x_2)) \supset \dot{-} Eq(f, x_1, x_2)]$

i.e. the relation Vardep holds between the function f and the class of functions φ with respect to the argument variable x, when and only when for every pair of arguments x_1 and x_2 for which one and only one element of φ takes different values, f takes different values as well. In analogy to D3 we put down:

D10 $\mathrm{V}(\varphi) =_{df} \hat{f}\{\mathrm{Vardep}\,(f,\varphi)\}$

The following theorems need no explanation:

T14 $f \varepsilon \mathrm{V}([f])$
T15 $\mathrm{V}(\Lambda) = \mathrm{V}$
T16 $\varphi \subset \mathrm{const} \supset \mathrm{V}(\varphi) = \mathrm{V}$
T17 $\mathrm{mon} \subset \mathrm{V}(\varphi)$.

If only two functions are being considered, Vardep is in a certain sense the converse of Equidep. Which is expressed by:

T18 $\mathrm{Vardep}\,(f, [g]) \equiv \mathrm{Equidep}\,(g, [f])$.

3.3 Now we proceed to the definition of a one-place predicate which corresponds to 'inequidep':

D11 $\mathrm{invardep}\,(\varphi) =_{df} (f)\,(f \varepsilon \varphi \supset \sim \mathrm{Vardep}\,(f, \varphi - [f]))$,

and of another one corresponding to 'interequidep':

D12 $\mathrm{intervardep}\,(\varphi) =_{df} (f)\,(f \varepsilon \varphi \supset \mathrm{Vardep}\,(f, \varphi - [f]))$.[8]

The following theorems state some relations holding between the various notions:

T19 $\mathrm{invardep}\,(\varphi) \supset \sim \mathrm{interequidep}\,(\varphi)$
T20 $\mathrm{interequidep}\,(\varphi) \supset (f)\,\mathrm{Vardep}\,(f, \varphi)$
T21 $\mathrm{invardep}\,(\varphi) \supset Nc\text{'}\varphi > 1$
T22 $Nc\text{'}\varphi > 1 \supset (\mathrm{interequidep}\,(\varphi) \equiv \mathrm{intervardep}\,(\varphi))$:

This shows that there is no difference between interequidep and intervardep except for the trivial case when φ has only one element.

Under the same restrictive condition, invardep is implied by inequidep:

T23 $Nc'\varphi > 1 \supset (\text{inequidep}(\varphi) \supset \text{invardep}(\varphi))$.

If φ consists of exactly two elements inequidep is equivalent to invardep:

T24 $Nc'\varphi = 2 \supset (\text{inequidep}(\varphi) \equiv \text{invardep}(\varphi))$.

Considering the theorems about Equidep, Vardep and their derivatives, one gets the impression that some kind of duality may hold for these notions. It appears the more likely from the following theorem which obviously corresponds to T19:

T25 $Nc'\varphi > 1 \supset (\text{inequidep}(\varphi) \supset \sim\text{intervardep}(\varphi))$.

§4 [Equivardep]

We saw that in some cases Equidep is a good approximation for the meaning of 'dependence' in current language and that in other cases Vardep can be used for the same purpose. Hence it seems natural to suppose that the conjunction of the two will be a still better approximation. Although I cannot decide yet whether it is the case or not I will state the definition of such a concept and mention some of its properties:

D13 $\text{Equivardep} \equiv_{df} \text{Equidep} \cap \text{Vardep}$
T26 $\text{Equivardep}(f, [g]) \equiv \text{Equivardep}(g, [f])$
T27 $\text{Equivardep}(f, [g]) \equiv \text{interequidep}([g, f])$
T28 $(\text{Equivardep}(f, \varphi). \varphi - [g] \subset \text{const}) \supset Eq(g, x_1, x_2) \equiv Eq(f, x_1, x_2)$.

The last theorem is of practical import in the field of science. Indeed it implies that if f stands in Equivardep to φ and one succeeds in keeping constant all the elements of φ but g, then a strict correlation holds between f and g.

§5 [Related Notions]

The two fundamental notions we defined, namely Equidep and Vardep are in many cases either too weak or too strong. But as I will show next, one can define certain other notions which are either stronger or weaker than Equidep and Vardep, and some of them will prove to be their limit cases. Yet I cannot develop these concepts in detail here and shall have to confine myself to a summary enumeration.

5.1 Let us start with two stronger notions: In the first place it is often convenient to submit the class φ to a *minimum* condition in the following way:

$$\text{D14}\quad \text{Equidepmin}\,(f, \varphi) =_{df} \text{Equidep}\,(f, \varphi).\,(\psi)\,(\psi \subset \varphi.\,\text{Equidep}\,(f, \psi) : \supset \psi = \varphi),$$

and similarly for Vardepmin.

Secondly, if the arguments and the values of the functions involved can be represented as points of two topological spaces, Equidep and Vardep can be reinforced by establishing that *Eq* holds not only for single points but for whole *environments*. I omit the formal definition of this concept.

5.2 As an example of a weaker notion I will mention one which I believe to be very useful but which has not yet been worked out in detail. It results from the introduction of the notion of *probability* in this context. So we are led to consider different degrees of dependence. The following description gives an idea of what is meant:

This sort of dependence holds between a function f and a class φ with respect to a certain probability function $p(d)$, when and only when the probability for the value of $f(x_2)$ to be found within the interval d in the environment of $f(x_1)$ is $p(d)$, provided that every element of φ has equal values for x_1 and x_2.

In a corresponding manner probability may also be combined with Vardep.

§6 [Logical and Causal Dependence]

A last and very important problem is concerned with the distinction between *logical* and *causal* dependence. This distinction is a semantical one. It might be formalized by means of Carnap's notions of *L*- and *F*-truth.[9] I want to suggest the following formulation: we may speak of logical dependence if the definiens of D1 or D9 is an *L*-true sentence, and of causal dependence if it is an *F*-true sentence. However, the topic needs further investigation.

§7 [Concluding Remark]

The definitions which I have proposed here are nothing but attempts to solve the problem of dependence. Most of these concepts might not be applicable yet to the practical course of science. However I firmly believe and hope that further developments of these investigations will finally prove to be fairly useful for all sorts of scientists.

Notes

[1] Paper sent in for the fifth International Congress for the Unity of Science (Cambridge, Mass. 1939), [and here reprinted for the first time. Section headings have been added.]

[2] Cf. K. Grelling and P. Oppenheim, "Logical Analysis of Gestalt as Functional Whole", this volume. The theory developed here has been worked out thanks to the cooperation of Mr. Oppenheim and the writer. No idea expressed here should be attributed to one of us in particular. Besides I am indebted to Mr. C. G. Hempel for some very useful suggestions.

[3] Instead of 'dependence' one may, *mutatis mutandis*, just as well use 'determination' which seems to be in a certain sense the converse of dependence.

[4] In order to avoid unnecessary complications, I am dealing with functions of one argument only. The generalization of the theory for functions of more arguments would make no fundamental difficulty.

[5] For readers acquainted with theoretical economics I want to point out that, of

course, values of the functions, demand and supply, cannot be represented by single numbers. These values are themselves functions. But that does not invaliditate the aforesaid statement.

[6] I omit the symbol for the argument when all the functions involved have the same unique argument.

[7] It is interesting to find that our theory of the Equidep relation shows a certain correspondence with the theory of *consequence*, i.e. of the well known relation holding between a proposition and a class of propositions. I can only point out this correspondence here. I will devote a special paper to it later on.

[8] The notions 'interequidep' and 'intervardep' both designated as 'interdependence' as well as 'interequidepend' and 'intervardepend' designated as 'strict interdependence' are of special importance in the above mentioned communication by Grelling and Oppenheim.

[9] Cf. Carnap, *Foundations of Logic and Mathematics*, International Encyclopedia of Unified Science, Vol. 1, No. 3, section 7.