On Partitioning Reality

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Abstract: In Smith and Brogaard (2000, 2001) the notion of partition is introduced as a generalization of David Lewis's (1991) conception of a class as the mereological sum of its singletons. A partition is a mereological sum of *labeled cells*. Partitions thus conceived are involved in all human mapping, classifying, listing and theorizing activity. We here provide a more detailed formal characterization of partitions. We define notions of well-formedness and truth for partitions and we classify partitions along three axes: (a) degree of correspondence between partition cells and objects in reality; (b) degree of how well a partition represents the mereological structure of the domain it is projected onto; and (c) degree of completeness and exhaustiveness.

1. Introduction

1.1 Partitions as cognitive devices

Sorting, slicing, counting and parceling out, dividing and gathering into units and portions, listing, pigeonholing, cataloguing and checking off – all of these are activities performed by human beings in their everyday and scientific traffic with the world. Partitions are the cognitive devices designed and built by human beings to fulfill these various purposes. When making lists or classifying objects you are in every case, we shall suggest, employing a certain grid of cells and you are recognizing certain objects as being located in those cells. A partition, as we shall conceive it, is just such a grid of cells.

Some partitions are flat: they amount to nothing more than a mere list. Other partitions are hierarchical: they consist of cells and subcells in a hierarchical array. Some partitions are built in order to reflect independently existing divisions on the side of objects in the world. Other partitions – for example the partitions created by nightclub doormen or electoral redistricting commissions – are themselves such as to create the necessary divisions on the side of their objects. Quite different sorts of partitions, having cells of different resolutions and effecting slicings and unifyings of different types, can be applied simultaneously to the same domain of objects. The people in your building can be divided according to gender, zip code, social class, tax bracket, blood type, current location, golf handicap or (to bring this list to an end) according to Erdös number or blood cesium level.

As will be clear from what follows, the notion of partition that is hereby implied is only distantly related to the more familiar notion defined in terms of equivalence classes. The latter notion is indeed parasitic on the one presented here. This is because it carries with it the presupposition that the domain to which an equivalence relation is applied has already been divided up into units (the elements of the set with which we begin), and it is this very notion of division into units which our present theory is designed to illuminate.

1.2 Better than Sets

In Smith and Brogaard (2000, 2001) the notion of partition is introduced as a generalization of David Lewis's (1991) conception of classes as the mereological sums of their singletons. Partitions, as we conceive them, are similarly the mereological sums of their constituent *cells*. The cells within a partition may however manifest a range of properties which singletons lack. This is because, where a set is defined in terms of its members, a partition is a device that is, as it were, seeking out objects which might fall within its cells. The cells of a partition have identity conditions independently of whether there are objects located within them, identity conditions determined via systems of labels or via the positions of cells within a regular grid.

Our listing and sorting purposes may fail: just as, when we point our telescope in a certain direction we may fail to find what we are looking for, so, when we point our partition in a certain direction, it may be that no objects are located in its cells. There may, in this sense, be empty cells within a partition. This does not mean, however, that the theory of partitions recognizes some counterpart of the set theorist's empty set (an entity which is contained as a subset within every set). For the empty set is empty by necessity; a cell in a partition, in contrast, is at best empty *per accidens* (for example because of some failure of knowledge on our part: compare the treatment of 'partial sets' in Mislove *et al.* 1990).

Given its set-theoretical roots, our basic formal ontology of partitions will have two parts: (A) a theory of the relations between cells and the partitions in which they are housed, and (B) a theory of the relations between cells and objects in reality. The counterpart of (A) in a set-theoretic context would be the study of the relations among subsets of a single set; the counterpart of (B) would be the study of the relations between sets and their members. When developed in the set-theoretical framework, theories (A) and (B) are not independent. This is because the standard subset relation is itself defined in terms of the set-membership relation ('*x* is a subset of *y*' means: all the members of *x* are members of *y*). In the context of partition theory, we are able to keep the two theories strictly separate, and as we shall see, this makes partition theory in some respects much richer than its set-theoretic progenitor. On the other hand, however, partition theory reflects a rather modest, commonsensical attitude to ontology that places it at some distance from the high-rise projects of post-Cantorian set theory. Thus it has no counterpart of the distinction between two ways in which one set can be contained inside another (on the one hand as element, on the other hand as subset). We note, however, that it is precisely this latter feature of set theory which brings it about that its most intuitively appealing axiomatizations are inconsistent.

2. Partitions as System of Cells

2.1 Basic Conditions

Theory (A) studies properties partitions have in virtue of the relations between the cells from out of which they are built. All partitions involve cells arranged together in some sort of structure. The partition-theoretic feature of resolution or granularity applies in the first place to this structure of cells. The partition of the people in your building according to *number of days spent behind bars* is a more refined partition (has a larger number of cells) than the partition according to *number of years spent behind bars*, and this is so even if none of the people in your building has in fact spent any time at all in jail. Cells in partitions may be nested one inside another in the way in which, for example, the species *crow* is nested inside the species *bird* which in turn is nested inside the genus *vertebrate* in standard biological taxonomies.

We say that one cell, z_1, is a subcell of another, z_2, if the first is contained in the latter and we write $z_1 \subseteq z_2$ in order to designate this relationship. The subcell relation $\subseteq$ is a reflexive, transitive, and antisymmetric relation. The nestedness of cells inside a partition yields chains of cells satisfying $z_1 \supseteq z_2 \supseteq \ldots \supseteq z_n$. We shall demand, as the first of several master conditions which we shall impose on all partitions, that:

MA1: The chains $z_1 \supseteq z_2 \supseteq \ldots \supseteq z_n$ in a partition are always finite.

We shall call the cells at the ends of such chains *minimal*. MA1 leaves open the issue as to whether partitions themselves are finite: thus it does not rule out the possibility that a given cell within a partition might have infinitely many immediate subcells.

Following Smith (1991) we can define the partition-theoretic sum and product of cells within partitions as follows. The partition-theoretic sum $z = z_1 \cup z_2$ of two cells in a partition as a $\subseteq$-minimal cell satisfying a constraint to the effect that $z_1 \subseteq z$ and $z_2 \subseteq z$. Notice that this partition-theoretic sum is distinct from the mereological sum of cells. (Relative to the standard geopolitical partition of the surface of the globe, the partition-theoretic sum of Malta and Cyprus is: the British Commonwealth.) The partition-theoretic product, $z = z_1 \cap z_2$, of two cells is defined only if z_1 and z_2 are not mereologically disjoint. If it is defined, then it yields a largest subcell shared in common by z_1 and z_2.

Every partition A ('partition' is '*Aufteilung*' in German) has a maximal cell. We define the notion of a maximal cell of a partition as follows:

$$M(z_1, A) \equiv Z(z_1, A) \text{ and } \forall z: Z(z, A) \rightarrow z \subseteq z_1,$$

where '$Z(z, A)$' means that z is a cell in the partition A. (We shall henceforth omit the condition $Z(z, A)$ in cases where it is clear that we are talking about cells within some fixed partition A. In addition, initial universal quantifiers will be taken as understood.) We now define the root of a partition as the unique maximal cell:

$$r(A) = z \equiv M(z, A) \text{ and } \forall z_1: M(z_1, A) \rightarrow z = z_1$$

and demand that

MA2: $\exists z: r(A) = z$

Every partition has a root cell, which is such that all the cells in the partition are included in it as subcells.

We shall demand that every pair of distinct cells in a partition stand to each other within the partition either in the subcell relation or in the relation of mereological disjointness. Thus if two cells overlap, then only because one is a subcell of the other:

MA3: $\exists z: (z = z_1 \cap z_2) \rightarrow z_1 \subseteq z_2 \text{ or } z_1 \supset z_2.$

2.2 Trees

Philosophers since Aristotle have recognized that nested partitions can be represented as branching structures which mathematicians nowadays called trees. We here take the possibility of such representation as a well-formedness constraint on what will be allowed to count as a partition within the terms of our theory.

MA4: Partitions can be represented as trees.

Trees are special sorts of graphs (they are directed graphs without cycles). Here we are interested more specifically in rooted trees, which is to say: trees with a single topmost node to which all other vertices are connected, either directly or indirectly, via edges. In a rooted tree, every pair of vertices is connected by one and only one chain (or sequence of edges). The directedness of an edge should be conceived as proceeding down the tree from top to bottom (from ancestors to descendants).

At the formal level the duality of trees and partitions can be proved by showing inductively that every tree can be transformed into a structure, analogous to a Venn diagram, in which partition cells are represented as topologically simple and regular non-intersecting regions of the plane. Conversely every array of non-intersecting, possibly nested rings in the plain can be transformed into a tree in such a way that each ring is represented by a node in the tree, and each directed link in the tree represents an *immediately contains* relation between the corresponding pair of nested rings. We can for illustrative purposes think of each partition as such a planar map.

2.3 Partition-theoretic sum and maximal cells

When we consider a partition as a planar map, then minimal cells correspond to smallest regions within the diagram. It is important to recognize that it is not in general the case that every mereological sum of such regions has a corresponding cell in the partition. There is no cell in our standard biological partition of the animal kingdom consisting of rabbits together with jellyfish, and there is no cell, in our standard geopolitical partition of the surface of the globe consisting of Hong Kong together with Algeria. It is for this reason that the mereological sum of cells and the partition-theoretic sum of cells defined above are distinct operations. Since, by MA2, partitions always have a maximal cell, the partition-theoretic sum is always defined. However the results produced by this operation are not necessarily identical to those produced by the operation of taking standard mereological sums. We can however assert that the mereological sum is in every case a part of the partition-theoretic sum. That is, we can write: $\forall z_1,z_2: z_1 + z_2 \leq z_1 \cup z_2$, where + abbreviates the mereological sum of cells and $\leq$ abbreviates the mereological part-of relation. (Notice that we cannot use the subcell relation, here, since $z_1 + z_2$ is not necessarily a cell of the partition with which we are dealing.) $\subseteq$ is the result of restricting $\leq$ to the natural units within a mereological whole. Set theory is what results when this restriction is abandoned.

3. Partitions in their Projective Relation to Reality

3.1 Projection

Partitions are more than just systems of cells. They are constructed to serve as pictures or maps of reality, and in this they are analogous to propositions (*Elementarsätze*) as described by Wittgenstein in the *Tractatus*. A proposition, for Wittgenstein, is built out of simple signs (names) arranged in a certain order. Each name stands in a projective relation to a corresponding object in the world: it cannot fail to strike its target. If a proposition is true, then its simple signs stand to each other within the proposition as the corresponding objects stand to each other in the world. It is in this sense that a true atomic proposition is a picture or map of a state of affairs in reality. That a proposition is a complex of names arranged in a certain order is in our present context equivalent to the thesis that a partition is a complex of cells arranged in a certain order.

A proposition is a complex of names 'in its projective relation to the world.' (*Tractatus*, 3.12). And so for us here: a partition is a complex of cells in its projective relation to the world. The projection may be effected either directly by the user of the partition – for example in looking through the cells of the grid and recording what objects are detected on the other side – or indirectly, with the help of proper names or other referring devices such as systems of coordinates or taxonomic labels.

In some cases the cells of a partition project but there are no objects for them to project onto. (Consider the partition cataloguing Aztec gods.) Here, however, we are interested primarily in partitions which do not project out into thin air in this way. We write 'P(z, o)' as an abbreviation for: cell z is projected onto object o. In what follows we shall assume that a unique projection is defined for each partition. In a more general theory we can allow projections to vary with time while the partition remains fixed (for example when we use a territorial grid of cells to map the presence of birds of given species in given areas from one moment to the next).

3.2 Location

That not all partitions are true follows from the fact that the relation P(z, o) is marked by a certain taint of subjectivity and by a certain possibility of failure. L(o, z), like the relation of set-membership, is free of these taints. It is determinately true, of every object o, and of every cell z, that o is either located, or not located, in z. If we pursue the image of a partition as a net or grid that is projected over reality, then the idea is that, when once the net has been thrown (projected) over a certain domain, then the objects in that domain will fall where they may – they will sort themselves out intrinsically into the corresponding cells. In what follows we make the simplifying assumption that objects are exactly located at their cells in the way in which Wyoming is exactly located at the cell 'Wyoming' in the partition of the US into States or in the way in which your brother Norse is exactly located at the cell 'Norse' in the partition (list) of your family members. In a more general theory we shall liberalize the location relation in such a way as to allow also for partial or rough location (Bittner and Stell 1998, Casati and Varzi 1995).

When projection succeeds, then location is what results. Projection and location thus correspond to the two directions of fit – from mind to world and from world to mind – between an assertion and the corresponding truthmaking portion of reality. (Searle 1983, Smith 1999) The relation 'L(o, z)' abbreviates: object o is located at cell z. Location presupposes projection: an object is never located in a cell unless through the projection relation associated with the relevant partition. Thus:

MB1 $\quad L(z, o) \rightarrow P(o, z)$

In the case where no errors have been made in the construction of a partition, L(o, z) holds if and only if P(z, o). This is because, in such a case, if a partition projects a given cell onto a given object, then that object is indeed located in the corresponding cell. Very many partitions – from automobile component catalogues to our maps of states and nations – have this quality without further ado, and it is such partitions upon which we shall concentrate in what follows. Such partitions are *transparent to* the corresponding portion of reality. In this case projection and location are converse relations with respect to the partition in question. Formally we write:

DTr: $\mathrm{Tr}(A) \equiv \forall z \forall o: Z(z, A) \rightarrow (P(z, o) \rightarrow L(o, z))$

We now impose a further master condition to the effect that:

MB2: All partitions are transparent in the sense of DTr.

3.3 Transparency

The master condition MB2 ensures that objects are actually located at the cells that project onto them. Notice however that a transparent partition, according to our definition, may still have empty cells. A list of the members of your local football team does not lose its quality of transparency because one of the players has, unbeknownst to the compiler of the list, died. A component catalogue does not lose its quality of transparency because given components are, for longer or shorter periods, out of stock. Empty cells may be needed in the context of scientific partitions, too, in order to leave room for what, on the side of the objects, may be discovered in the future. (The Periodic Table leaves empty cells for chemical elements of types which have yet to be detected.)

MB2 tells us only that, if a cell in a partition projects upon some object, then that object is indeed located in the corresponding cell. It does not tell us what happens in case a cell fails to project onto anything at all. DTr is correspondingly only a first step along the way towards a definition of *truth* for partitions. Truth (as Wittgenstein very nearly recognized) involves three notions: transparency, completeness (all cells are occupied), and structural mapping or isomorphism.

Following Smith and Brogaard (2000) we define the notion of *recognition*, of an object by a cell, as follows. An object o is *recognized* by a partition if and only if it is located in some cell in the partition:

DR1: $R(A, o) \equiv \exists z: Z(z, A) \text{ and } L(o, z)$

The relation defined in DR1 is the partition-theoretic analogue of the standard set-membership relation.

3.4 Reflecting mereological structure

That upon which a partition is projected is a certain domain of objects in reality. We shall conceive the domain of a partition as a mereological sum. It is, as it were, the total mass of stuff upon which the partition sets to work: thus it is stuff as it is prior to any of the divisions or demarcations effected by the partition itself. The domains of partitions will comprehend not only individual objects and their constituents (atoms, molecules, limbs, organs), but also groups or populations of individuals (for example biological species and genera, battalions and divisions, archipelagos and diasporas) and their constituent members. Spatial partitions, for example categorical coverages (Frank *at al.* 1997), are one important family of partitions in our more general sense. The domains of spatial partitions are the mereological sums of spatial regions.

Partitions – think again of Venn diagrams – reflect the basic part-whole structure of reality through the fact that the cells in a partition are themselves such as to stand in the relation of part to whole. This means that, given the master conditions expressed within the framework of theory (A) above, partitions have at least the potential to reflect the mereological structure of the domain onto which they are projected. And in felicitous cases this potential is realized.

We say that the cells z_1 and z_2 *reflect* the mereological relationship between the objects onto which they are projected if and only if the following holds:

DR2: $RS(z_1, z_2) \equiv \forall o_1, o_2: (R(z_1, o_1) \text{ and } R(z_2, o_2)) \rightarrow (z_1 \subseteq z_2 \rightarrow o_1 \leq o_2)$.

This means that if z_1 is a subcell of z_2 then any object recognized by z_1 is a part of any object recognized by z_2. A partition reflects the mereological structure of the domain it is projected onto if and only if each pair of cells recognizes in this way the mereological structure on the side of their objects:

DR3: $RS(A) \equiv \forall z_1, z_2: (Z(z_1, A) \text{ and } Z(z_2, A)) \rightarrow RS(z_1, z_2)$

Mereological structure here includes the structure of identity (identity is the limit case of the relation a part bears to its whole which is arrived at when we allow the part to approximate without limit to its whole).

We then impose a new master condition:

MB3 All partitions are *structure reflecting* in the sense of DR3.

What this means is that all partitions are such that if one cell is a subcell of another, then any object recognized by the first cell is a part of any object recognized by the second. All of David Lewis's classes are structure reflecting in this sense, at least in those cases where we are dealing with classes built up out of members which themselves stand in mereological relations.

3.5 Tracing over mereological structure

MB3 is still very weak. It does little more than ensure that partitions which satisfy it do not misrepresent the mereological relationships between their objects. But partitions might still be blind to (trace over) such relationships, since $\subseteq$ and $\leq$ both represent *partial* rather than total orderings. There may be objects (or cells) that do not stand in the relations $\leq$ (or $\subseteq$) to each other at all.

Objects that do not stand to each other in the part-whole relation are either disjoint or they are such as to overlap mereologically. The absence of a part-whole relation between two cells z_1 and z_2 within a given partition means: 'the partition does not know (or does not care) how z_1 and z_2 are related.' We note that the minimal cells in a partition do not stand in the relation $\subseteq$ to each other. From this we are entitled to infer nothing at all about the mereological relations among the corresponding objects. Consider, for example, a partition that contains cells that recognize John and his arm, i.e., L(John, z_1) and L(John's arm, z_2).

Partitions may trace over mereological relationships between the objects they recognize, but MB3 is strong enough to ensure that, if a partition tells us something about the mereological relationships on the side of the objects which it recognizes, then what it tells us is true.

3.6 Domains of a partition

We are now able to specify what we mean by 'domain of a partition.' Roughly, the domain of a partition is the mereological sum of all those objects onto which it successfully projects. Firstly we define what it means to be a *part* of the domain of a partition in the standard mereological sense:

DD1 $\quad D(x, A) \equiv \exists y\, (x \leq y \text{ and } \forall z \forall u\text{: } (Z(z, A) \text{ and } L(u, z)) \rightarrow u \leq y)$

This can be read as follows: object x is a part of a domain of the partition A if and only if there is some y of which x is a part and it holds of y that everything that is located at some cell of A is a part of y.

We now the define the *minimal* domain of a partition:

DD2: $\quad MD(A) \equiv$ the mereological intersection of all x satisfying D(x, A)

We can also define what it means to be a part of the domain of a single cell:

DD3: $\quad D(x, z) \equiv \exists y\, (x \leq y \text{ and } \forall u\text{: } L(u, z) \rightarrow u \leq y)$.

Something is part of the domain of a cell if and only if there is something in which it is included as part which itself includes everything that is located at this cell. The minimal domain of a cell is then defined in the obvious way as follows:

DD4: $\quad D(z) \equiv$ the mereological intersection of all x satisfying D(x, z)

3.7 Truth and granularity

We can define what it is for a partition to be *true of* a given domain as follows:

DT: $T(A, x) \equiv Tr(A)$ and $RS(A)$ and $D(x, A)$

Note that DT is still rather easily satisfied. The making of lists is normally an unproblematic exercise. If there is a maximal object (the whole universe), then DT is satisfied by the Spinoza partition – which consists of one cell labeled ‘everything’ – in relation to this object. DT is satisfied by a partition with three cells: a root cell, labeled ‘animals’ and two sub-cells labeled ‘dogs’ and ‘cats’. The given partition is true of the mereological sum of all animals, but its truth falls far short of a certain sort of desirable completeness.

The cells of a partition carry with them the feature of granularity. Because they function like singletons in set theory, they recognize only single whole units, the counterparts of set-theoretic elements or members. If a partition recognizes not only wholes but also one or more parts of such wholes, then this is because there are additional cells in the partition which do this recognizing job. Consider, for example, a partition that recognizes human beings, i.e., it has cells that project onto John, Mary, and so forth. This partition does not recognize parts of human beings – such as John’s arm or Mary’s shoulder – unless we add extra cells for this purpose. If a partition recognizes wholes and their parts, then it is not necessarily the case that it also reflects the mereological relationships between the two. It is not hard to imagine a partition with cells recognizing John, Mary, and that arm over there, where the cell recognizing the arm is not a subcell of the cell recognizing John. (Let the arm be Kosovo, let John and Mary be Serbia and Albania, respectively.)

In relation to this granularity of partitions, we can once more call in the aid of Wittgenstein:

> In the proposition there must be exactly as many things distinguishable as there are in the state of affairs which it represents. They must both possess the same logical (mathematical) multiplicity (4.04).

Wittgenstein himself takes care of this issue of granularity by insisting that the world is made up of discrete simples (an assumption adopted also in Galton 1999), and by insisting further that all partitions (propositions) picture complexes of such simples. This is a simplifying assumption, which the theory of partitions will enable us to avoid. The theory of partitions inherits from mereology the feature that it is consistent with both an axiom to the effect that atoms exist and with the negation of this axiom. The theory thus enables us to remain neutral as to the existence of any ultimate simples in reality from out of which other objects would be constructed via summation.

Consider some standard biological partition of the animal kingdom. Definition DD1 implies that besides the species dog and your dog Fido, also Fido’s DNA-molecules, proteins, and atoms are parts of the domain of this partition. Often we need a more restricted notion of parthood that takes the granularity imposed by the partition into account. We define the *restricted* parthood relation:

DG1: $x \sqsubseteq_A y \equiv x \leq y \text{ and } \exists z_1, z_2\text{: } R(z_1, x) \text{ and } R(z_2, y) \text{ and } z_1 \subseteq z_2$

to mean that x is a part of y in A if and only if x is a part of y, and x is recognized by a subcell of a cell recognizing y

Consider the usual common-sense (i.e., non-scientific) partition of the animal kingdom. These partitions contain cells recognizing dogs and mammals but no cells recognizing DNA molecules. Relative to this common-sense partition, DNA molecules are not parts of the animal kingdom in the sense defined by DG1. Of course, DNA molecules are parts of the animal kingdom in the unrestricted mereological sense.

4. Varieties of True Partitions

In this section we discuss some of the more important conditions which true partitions are standardly called upon to satisfy, in addition to the master conditions specified above. We classify such partitions along three essentially orthogonal axes: (a) degree of correspondence to objects; (b) degree of structural fit; and (c) degree of completeness.

4.1 Functionality constraints

Projection is functional

Partitions which possess the maximum degree of correspondence to objects must first of all be such as to rule out ambiguity on the side of their cell labels (or on the side of whatever it is in virtue of which projection is effected). This means that they must be such that their associated projection is a *functional* relation:

$$\text{CFP:} \quad P(z_1, o_1) \text{ and } P(z_2, o_2) \rightarrow (o_1 = o_2 \rightarrow z_1 = z_2)$$

For partitions satisfying CFP, cells are projected onto single objects (one rather than two).

Location is functional

Consider a partition labeled 'heavenly bodies' and having just three minimal cells labeled 'The Morning Star' 'The Evening Star', and 'Venus', respectively. As we know, all three cells project onto the same object. Yet even so, it is still perfectly consistent with our definitions that this partition is true – that its distinct cells truly, though unknowingly, recognize the same object; for these cells are minimal, and thus neutral about the possible mereological relations obtaining on the side of that onto which they project. It is not unusual that we give different names (or coordinates, or class-labels) to things in cases where we do not know that they are actually the same. A good partition, though, should clearly be one in which such errors are avoided.

Partitions manifesting the highest degree of correspondence to objects must, in other words, be ones in which location is a *functional* relation:

$$\text{CFL:} \quad L(o_1, z_1) \text{ and } L(o_2, z_2) \rightarrow (z_1 = z_2 \rightarrow o_1 = o_2)$$

In partitions that satisfy CFL, location is a function, i.e., objects are located at single cells (one rather than two). The location function is partial: no partition is omniscient.

4.2 Structural constraints

We required of true partitions that they reflect the mereological structure of the domain they recognize. Remember that such reflection is to be understood in such a way that it leaves room for the possibility that a partition is merely neutral about (traces over) some aspects of the mereological structure of its target domain. Taking this into account, we can order partitions according to the degree to which they actually represent the mereological structure on the side of the objects onto which they are projected. At the one extreme we have: (1) partitions that completely reflect the mereological relations holding between the objects they recognize. At the other extreme are (2): partitions that completely trace over the mereological structure of the objects they recognize. Between these two extremes we have partitions that reflect some but not all of the mereological structure of the objects they recognize.

Under heading (1) are those true partitions which satisfy the weak converse of MB3, which means that if o_1 is part of o_2, and if both o_1 and o_2 are recognized by the partition, then the cell at which o_1 is located is a subcell of the cell at which o_2 is located. Formally we can express this as follows:

$$\text{CMM} \qquad o_1 \leq o_2 \text{ and } L(o_1, z_1) \text{ and } L(o_2, z_2) \rightarrow z_1 \subseteq z_2$$

We call partitions satisfying CMM *mereologically monotonic*. An example of a mereologically monotonic partition is the perfect catalogue discussed below.

Some simple list partitions – which is to say partitions consisting exclusively of minimal cells – satisfy CMM. This is so for example of the partition recognizing the subdivision of the United States into its constituent states. The latter satisfies CMM because there is (at this level of granularity) no parthood structure on the side of its objects.

Under heading (2) are those simple list partitions which do not satisfy CMM but which do satisfy: $z_1 \subseteq z_2 \rightarrow (z_1 = z_2 \text{ or } z_2 = r(A))$. An example is a partition of the animal kingdom represented as a flat list with minimal cells labeled *arachnid*, *chordate*, *dog*, *horse*, *mammal*, *vertebrate*, etc., in the way in which these terms might occur in the index to a zoology text.

4.3 Completeness constraints

Completeness

We have allowed partitions to contain empty cells, i.e., cells that do not project onto any object. We now consider partitions which satisfy the constraint that every cell recognizes some object:

CC: $Z(z, A) \rightarrow \exists o: R(z, o)$

We say that partitions that satisfy CC *project completely*. Notice that this condition is independent of the functional or relational character of projection and location. Of particular interest, however, are partitions that project completely and in such a way that projection is a total function. These are partitions which satisfy both CFP and CC.

Exhaustiveness

We have accepted that partitions may have objects in their target domain which are yet not located at any cell. Such partitions are often not very satisfying: governments want *all* their subjects to be located in some cell of their partition of taxable individuals. They want their partitions to satisfy a completeness constraint to the effect that every object in the domain is indeed recognized. In this case we say location is complete. Alternatively we say that the partition *exhausts* its domain. Unfortunately we cannot use

(*) $D(o, A) \rightarrow \exists z: Z(z, A) \text{ and } R(z, o)$

to capture the desired constraint, since the tax authorities do not (as of this writing) want to tax the separate molecules of their subjects.

We believe that it will be necessary to promote several restricted forms of exhaustiveness, each one of which will approximate in different ways to the condition of unrestricted exhaustiveness expressed in (*), which is unrealizable in a non-atomistic world.

To see how one such exhaustiveness condition might look, in first (schematic) approximation, let us introduce a sortal predicate φ that singles out the kinds of objects our taxation partition is supposed to recognize (for example, human beings rather than parts of human beings). We now demand that the taxation partition recognize all of those objects in its domain which satisfy φ:

CE $D(o, A) \text{ and } \varphi(o) \rightarrow \exists z: Z(z, A) \text{ and } R(z, o)$.

CE in effect asserts the completeness of one partition *relative to* another, the φ-totalizer partition, which consists exclusively of minimal cells in which all and only the objects satisfying φ are located. Note that the idea underlying CE is closely related to the idea of granularity. An alternative means of formulating an exhaustiveness condition like CE, is via a restriction on object size.

The tax office probably does not care too much about empty cells in its partition, nor is it bothered too much by the idea of charging you twice. The main issue is to catch everything above a certain resolution at least once. This is the intuition behind CE. If you are a law-abiding citizen, you will accept CE, but you will insist that the partition not locate you in two separate cells, i.e., that you do not get charged twice. This means that you want the tax partition to satisfy CE and CFP. There might be a pedantic clerk in the tax office who does not rest until he has made sure that all empty cells have been removed. Partitions that will satisfy you, the government, and the clerk in the tax office must satisfy CC, CE, and CFP. From this it follows that these partitions also satisfy CFL: projection and location are total functions and one is the inverse of the other.

5. The Set-Theoretical Partition

We already mentioned that sets and partitions are closely related. We now take a closer look at the relationships between the partitions as defined above and classes in Lewis's sense. We consider the following issues, several of which have been addressed already in passing in the above. What exactly are the set-theoretic counterparts of cells, partitions, partition-theoretic intersection and union, projection, location and so on? Which of our axioms are satisfied also by sets? Which axioms are not satisfied by sets? Can we apply to sets our classification of partitions in terms of the satisfaction of completeness, exhaustiveness, and correspondence constraints?

5.1 Constraints satisfied by sets

Sets can be divided into three families: pure sets, built up exclusively out of the empty set with no *Urelemente*; classes in David Lewis's sense, built up exclusively out of *Urelemente*; and mixed cases. We restrict our attention here to Lewisian classes, imposing a further restriction to the effect that the *Urelemente* are themselves such as to be capable of standing in mereological relations to each other (thus we restrict ourselves, effectively, to sets of objects in the world of space and time). We follow Lewis in identifying classes as the mereological sums of their singletons. The relations P and L can then both be identified with the class-membership relation in the sense that x is a member of a certain class if and only if it is recognized by a certain cell (singleton) within a mereological sum which is the class. The partition-theoretic subcell relation $\subseteq$ then corresponds to the usual subset relation and the objects of non-minimal cells are identified as the mereological sums of the objects of the minimal cells from out of which they are constructed. Classes so conceived satisfy the following conditions:

- MA1 (the minimal chain condition), which is the analogue of the set-theoretic *Begründungsaxiom*.
- MA2 (root cell): this follows trivially given Lewis's conception: all singletons are parts of the corresponding class taken as a whole.
- MB1 (only what is projected is located): this follows trivially from the fact that the relations of location and projection are both identified with the class-theoretic membership relation.
- MB2 (transparency): as for MB1.
- MB3 (structure reflection): this follows trivially from the fact that the objects of non-minimal cells are the mereological sums of the objects of the underlying minimal cells.
- CFL (functionality of location): as for MB3.

5.2 Constraints not satisfied by sets

There are classes which have a structure which precludes them from being considered as partitions in the sense defended here. Consider the class {{a, b}, {a, c}}. Since we have {a} $\subseteq$ {a, b} and {a} $\subseteq$ {a, c}, any corresponding partition violates both MA3 and MA4. Of course there are many classes that do satisfy MA3 and MA4, e.g., the set {a, b, c}. Since classes trivially satisfy our remaining master conditions, it follows that the latter are properly included in the family of partitions as here conceived.

5.3 Completeness, exhaustiveness, and structural correspondence

Classes that satisfy MA3 and MA4 can be evaluated in terms of completeness, exhaustiveness and structural correspondence in the same way in which we evaluate partitions in general.

- CFP (functionality of projection) fails: let a be the mereological sum of b and c, and consider a partition with minimal cells {a}, {b}, {c} and one additional cell {b, c}. Then set z_1 = {a} and z_2 = {b, c}.
- CMM (monotonicity for location) fails: consider the set {John's arm, John}
- CC (every cell is occupied) fails for sets in general but it is satisfied by classes in David Lewis's sense, since these are sets in whose constitution the empty set plays no role.
- CE (exhaustiveness): the usual set-theoretic comprehension axiom is precisely a device which (in its unrestricted formulation) seeks to guarantee exhaustiveness in the sense of CE.

In sum, partition theory is a generalization of set theory which results when one effects a careful separation of the set-membership relation and the cell/subcell structure within a set. Equivalently, partition theory is what results when the set-membership relation is split apart into the two components of projection and location.

6. Two Examples

Partitions are, we repeat, natural cognitive devices. We assume that most of them are true in the sense that they are transparent and structure reflecting (they satisfy our master conditions MA1-4 and MB1-3). If we imagine the system of cells of a partition ranged over against a system of objects, with objects located in all the cells of the partition (under a certain relation of projection), then in the best case we have a partition that is mereologically monotone (CMM), projects completely (CC) and exhaustively (CE), and establishes a functional relationship to the domain it maps (CFP, CFL). We find examples of such perfection above all in the abstract, fiat domains of databases and spatial subdivisions. When we deal with objects in the world of flesh and blood reality, in contrast, there is no exact fit between partitions and the corresponding objects.

6.1 The Dewey Catalogue System

Examples of ideal partitions are provided by the various classifying systems developed for purposes of library cataloguing. The Dewey Catalogue System, for example, rests on a partition of the domain of books which corresponds to a tree with a topmost node (labelled *books*), under which are ranged categories like

200 Religion

300 Social Sciences

400 Language

500 Natural Sciences and Mathematics

and sub-categories like:

390 Customs, Etiquette, Folklore, Costume, Wedding Planning

450 Italian, Romanian, Rhaeto-Romanic

590 Zoological Sciences

630 Agriculture, Gardening, Pets

With the aid of additional digits, for example representing author and year of publication, exactly one reference-number is hereby generated for every published book and this number is taken over, at least ideally, by all libraries which use the Dewey System.

The classes and sub-classes under which each book falls can then, conversely, be inferred from the reference-number with which the book has been assigned. From the fact that a given book has the number 595.789, we can infer, by looking up the classification, that it deals with butterflies, and thus also with lepidoptera (595.78), and also with insects (595.7) and with invertebrate animals (595).

In what sense, now, does the Dewey partition satisfy the conditions CMM, CC, CE, CFP and CFL of complete correspondence and mereological isomorphism? We said that each book has from the very beginning exactly one reference number in the Dewey partition. This means that there is no room for vagueness or imprecision in the way in which the partition applies to the books thus referenced. Each object fits exactly into its assigned cell within the partition because each object has in effect been constructed by the system itself to serve this purpose. For it is properly speaking not concrete physical books which are classified by such a catalogue, but rather the abstract data-units which represent them in the system. Books and entries in the catalogue thus stand in perfect one-to-one correspondence because *to be a book* means: *to have an entry in the catalogue.* Consequently projection and location are inverse functions (CFP, CFL) at the level of minimal cells. This exact correspondence also holds for non-minimal cells (with labels such as 300 and 390). The latter project onto certain whole classes of data-items corresponding to collections of books on the same subject. Since every book has an entry in the catalogue, it follows that the catalogue (partition) exhausts the domain of books onto which it is projected, i.e., CE holds for all minimal cells where the sortal predicate φ is defined in such a way that it returns *true* only for books. Moreover, there are no empty entries (cells) in the catalogue, i.e., CC holds. Consider a small village library whose books are organized using the Dewey system. This library does not have a physical copy of Husserl's *Logical Investigations*. But still there is no empty cell in the catalogue; rather there is no corresponding cell (i.e., no data entry) at all. Even if a book is stolen from the library then this, too, does not mean that there is an empty cell in the library partition. For, again, the corresponding data items refers not to some physical book, but rather to an abstract book counterpart; the cell in this case might at worst be addressed by an additional label to the effect that *this book has been stolen*.

Of course a catalogue system may have imperfections of its own. There may be books in Turkhmeni of which the catalogue knows nothing. This, however, just draws attention to another feature of partitions: that they have a certain – always limited – domain of objects, and when we talk of fit or correspondence then always as a relation between the partition and this associated, limited domain. As Wittgenstein nearly has it (5.156): 'A partition may well be an incomplete picture of a certain situation, but it is always a complete picture of *something.*'

6.2 Biological partitions

Biological partitions project onto the physical reality constituted by animals, plants and bacteria. They form sophisticated and complex tree structures of various types. We note, though, that even the standard partitions used by biologists will fall short of being exhaustive, since new species are being discovered every day. We note also that biological partitions may contain empty cells, corresponding to the fact that species may become extinct. It may further be the case that more than one name is used for the same species, i.e., objects may be located at more than one cell. (Note that in spite of these shortcomings, all standard biological partitions are nonetheless already true.)

Biologists strive to purify their partitions in such a way that the mentioned shortcomings are progressively eliminated. They strive also to ensure that their partitions recognize the mereological structure of their target domain as exactly as possible. Consequently mereological monotony, i.e., the satisfaction of CMM, is an important aspect of biological partitions.

7. Conclusion

There are alternative strategies for constructing biological partitions, just as there are alternatives to the Dewey System as a framework for classification in the world of books. Indeed many complex domains are such that we possess a plurality of equally powerful systems for dividing up the domain into cells and subcells. Such systems may employ quite different hierarchical organisations, yet still be such that they satisfy the conditions of partition-theoretic transparency to equal degrees. There are maps of different scales.

There are those who hold that the existence of competing biological classifications must imply that all such classifications are false, or that the world of biological phenomena is somehow 'socially constructed'. Partition theory makes clear, however, how there can be equally transparent skew partitions of one and the same domain – and this suggests that the theory may have a role to play also in the resolution of common philosophical confusions.

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