A Unified Theory of Granularity, Vagueness, and Approximation

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Abstract: We propose a view of vagueness as a semantic property of names and predicates. All entities are crisp, on this semantic view, but there are, for each vague name, multiple portions of reality that are equally good candidates for being its referent, and, for each vague predicate, multiple classes of objects that are equally good candidates for being its extension. We provide a new formulation of these ideas in terms of a theory of granular partitions. We show that this theory provides a general framework within which we can understand the relation between vague terms and concepts and the corresponding crisp portions of reality. We also sketch how it might be possible to formulate within this framework a theory of vagueness which dispenses with the notion of truth-value gaps and other artifacts of more familiar approaches. Central to our approach is the idea that judgments about reality involve in every case (1) a separation of reality into foreground and background of attention and (2) the feature of granularity. On this basis we attempt to show that even vague judgments made in naturally occurring contexts are not marked by truth-value indeterminacy. We distinguish, in addition to crisp granular partitions, also vague partitions, and reference partitions, and we explain the role of the latter in the context of judgments that involve vagueness. We conclude by showing how reference partitions provide an effective means by which judging subjects are able to temper the vagueness of their judgments by means of approximations.

1. Introduction

Consider the proper name 'Mount Everest'. This refers to some mereological whole, a certain giant formation of rock. A mereological whole is the sum of its parts, and Mount Everest certainly contains its *summit* as part. But it is not so clear which parts along the foothills of Mount Everest are parts of the mountain and which belong to its neighbors. Thus it is not clear which mereological sum of parts of reality actually constitutes Mount Everest. One option is to hold that there are multiple candidates, no one of which can claim exclusive rights to serve as the referent of this name. Each of these many candidates has the summit, with its height of 29,028 feet, as part. These candidates differ, however, regarding which parts along the foothills are included as parts of Mount Everest and which are not (see the right part of Figure 1).

Consider, analogously, the predicate ‘is a bald male’. Bill Clinton certainly does not belong to the extension of this predicate, and Yul Brunner certainly does. But how about Bruce Willis? It would seem that there are some candidates for the extension of this predicate in which Bruce Willis is included, and certain others in which he is not.

Varzi (2001) refers to the above as a *de dicto* view of vagueness. It treats vagueness not as a property of objects but rather as a semantic property of names and predicates. There are, for each vague name, multiple portions of reality that are equally good candidates for being its referent, and, for each vague predicate, multiple classes of objects that are equally good candidates for being its extension. There are some, for example Tye (1990), who are happy to include in their ontology vague objects and regions and thus defend a *de re* view of vagueness. In a quantitative formalism this might result in what Fisher (1996) calls fuzzy objects and regions. The important point is that on this *de re* view one needs to extend one’s ontology in such a way as to include new, special sorts of regions and objects in addition to the crisp objects and regions one has already recognized. This not only brings added ontological commitments but implies also that one needs to investigate the question whether vague location (of vague objects in vague regions) is or is not the same relation as the more familiar crisp location of old.

Given the *de dicto* point of view there is no need to extend our ontology in this way. We need, rather, to reconceptualize the relationships between terms and concepts on the one hand, and crisp objects and locations out there in the world on the other. Such relationships are not one-one, but rather one-many, and we can think of their targets, tentatively, as multiple *products of demarcation*. Note that this reconceptualization is not intended as an account of what is involved cognitively when we use vague terms or predicates. Normal subjects in normal (which means: non-philosophical) contexts are not aware of the existence of such multiple targets. Rather, the simultaneous demarcation of a multiplicity of crisp referents or extensions takes place as it were behind the scenes. What we offer here is a proposal for dealing theoretically with the ontology of that particular type of links a cognitive subject to some correlated reality when vague terms or predicates are used. While it is not our primary purpose here to throw light on what the cognitive subject thinks is going on when using such terms or predicates, the fact that many of the matters with which we deal fall beneath the threshold of her concern is itself something which the theory of granular partitions is able to illuminate.

The *de dicto* view of vagueness goes hand in hand with the doctrine of supervaluationism, which is based on a redefinition of the notion of truth to accommodate the multiplicity of candidate precisifications which the *de dicto* view sees as being associated with vague names or predicates. The basic idea is that, when determining the truth of an assertion containing a vague name or predicate, it is necessary to take into account all its candidate referents or extensions. In order to evaluate such an assertion semantically, we must effectively run through these candidates in succession and determine, for each particular choice, whether it makes the assertion true or false. An assertion such as ‘Yul Brunner was bald’ is *supertrue* because it is true for all such choices. An assertion such as ‘Bill Clinton is bald’ is *superfalse* because it is false for all such choices.

Figure 1: Left: a partition, with cells *Everest*, *Lhotse* and *The Himalayas*. Right: A part of the Himalayas seen from space, with Mount Lhotse (left) and Mount Everest (right).

The problems arise in regard to sentences which are *indeterminate*, in the sense that they come out true for some choices and false for others. The core of these problems is captured in the so-called Sorites paradox (Hyde 1996). Consider Bill Clinton. He is certainly not bald, and losing one hair will not make him bald. This seems to hold quite generally: if Clinton is not bald and he loses one hair, then he is still not bald. Following this chain of reasoning if we start from a non-bald Clinton, then Clinton will still not be bald even if he has only 10 hairs left on his head. This is because, intuitively, losing one hair does not cause the transition to baldness. A similar chain of reasoning can be constructed in the case of Mount Everest. The summit is part of the mountain. If x is a part of a mountain, then every molecule that is connected to x is also part of the mountain. Following this chain of reasoning, we end up concluding that Berlin is part of Mount Everest. In this paper we will provide a framework for understanding how such chains of reasoning are broken in normal contexts of assertion.

We shall concentrate our attentions in what follows on *the case of singular reference*, i.e., reference via names and definite descriptions to concrete portions of reality such as mountains and deserts, leaving for another occasion the task of extending the account to the case of vague predication. We shall concentrate primarily on spatial examples. As will become clear, however, it is one advantage of the framework here defended, that it can be generalized automatically beyond the spatial case.

2. Judgments, Supervaluation, and Context

2.1 Judgments and supervaluation

As already pointed out above, supervaluation is based on a redefinition of the notion of truth to accommodate a multiplicity of possible referents. It draws on the recognition that a sentence can often be assigned a determinate truth-value independently of how the referents of its constituent singular terms are more precisely specified, or in other words, independently of how we might restrict such reference to just one (or just some few) of the many portions of reality which are candidate precisifications. A sentence is called *supertrue*, on this account, if and only if it is true (and *superfalse* if and only if it is false) for all such precisifications. If, on the other hand, it is true under some ways of precisifying and false under others, then it is said to fall down a supervaluational truth-value gap. Its truth-value is indeterminate.

The technique of supervaluation evolved as part of standard model-theoretic semantic. Thus it has been studied primarily as it applies to formulae of artificial languages conceived in context-free fashion. As Smith and Brogaard (2001) point out, however, the degree and type of vagueness by which the singular terms of natural language are affected varies in significant ways according to the contexts in which such terms are used. They therefore argue that, if the supervaluationistic method is to be extended to natural language, then it will be necessary to *contextualize* the theory by conceiving semantic evaluations as being applied not to *sentences* but to the *judgments* which such sentences express. It is, after all, through judgments – sentences as used assertively in specific contexts – that terms are projected onto reality by the subjects who use them.

> It then transpires that the very same sentence may be used in different contexts to express distinct judgments even where the singular terms involved refer to what is intuitively the same parcel of reality. The supervaluations of the given judgments will then look very different, even though the sentences in question are, as syntactic objects, one and the same (Smith and Brogaard 2001).

This context-dependence of vagueness has important consequences. For while it is easy to concoct examples of sentences neither supertrue nor superfalse when such sentences are treated out of context – much of the philosophical literature on vagueness is devoted to the discussion of examples of this sort – it is much less easy to find examples of such sentences when we confine ourselves to assertions which would naturally arise in the specific types of contexts which human beings actually inhabit. This is for reasons of pragmatics: such contexts have features which make it difficult if not impossible for judgments to occur within them which are marked by indeterminacy.

2.2 Context dependence

To get an idea of what we have in mind, consider the sentence:

[A] This cavity is part of Mount Everest,

uttered by someone pointing to a small cave near the summit of the mountain. Certainly, if we conceive matters entirely in abstraction from all contexts, then there are some precisified referents of ‘Mount Everest’ which would make this sentence come out true, and others which would make it come out false. The sentence comes out true, for example, if we could allow precisifications of Mount Everest to be defined spatially, for example by means of a rule (R_1) to the effect that it is a sufficient condition for *x* to be part of Mount Everest that *x* occupies a spatial location which lies within the convex hull of the mountain as depicted on relief maps. The sentence comes out false, on the other hand, for all precisifications which conform to another, no less attractive rule (R_2), to the effect that if *x* is a part of a mountain above a certain minimal size, then *x* is made of rock.

Different sorts of rules for determining allowable precisifications will now be in operation in different sorts of contexts. Imagine, for example, that the sentence [A] is uttered by a speleologist on commencing the exploration of the cave. For her the cave is certainly a part of Mount Everest; she uses rule R_1 as a matter of course. Moreover, the fact that the cave is filled with air is in this context critical: if it would be filled with rock, it would not be a cave. When she uses [A] to express a judgment in her specific speleological context, then the resultant judgment is reasonably evaluated as true for all possible precisifications consistent with this context; and hence, as supertrue.

Consider, on the other hand a geologist analyzing probes collected by drilling holes in the rock. For him, rule R_2 is in operation: portions of Mount Everest are constituted out of rock in every case. Here we see in play the factor of pragmatics: the geologist would not *use* [A], or anything like [A], to make a judgment. Even the negation of [A], i.e., 'This cavity is *not* part of Mount Everest' is not judgeable in his geological context.

Some sentences have the feature that they are judgeable only in certain exceptional contexts. Consider for example the sentence:

[A′] This hole is part of my jacket.

In most everyday contexts [A′] is simply not judgeable. And if it is judgeable (for example because the hole is a design feature of the jacket), then it comes out supertrue.

Consider the following example:

[B] This glass is empty,

and contrast the behavior of this sentence in two distinct contexts. In the first, C_1, it is used to express a judgment by a drunkard in a seedy bar just after taking the last sip of beer from his glass. In the second, C_2, its negation is used to express a judgment by a hygiene inspector inspecting the same glass just a few seconds later. We have here two distinct judgments, which we can abbreviate loosely as: J_1 = (B, C_1) and J_2 = (not-B, C_2). J_1 is supertrue, since the glass contains, on all precisifications, nothing left to drink. And J_2 is supertrue also: the hygiene inspector sees all the bacteria inside the glass and on no precisification consistent with what she sees would the sentence [B] be evaluated as true.

Judgments, to repeat, are always made in contexts. Hence, to evaluate a judgment as to its truth (supertruth) or falsehood (superfalsehood) is to evaluate that judgment *in its context*. A judgment is supertrue if and only if it is true under all contextually appropriate ways of putting members of the pertinent 'many' into the extensions of the corresponding terms; and analogously for superfalsehood. If a sentence is not judgeable in a given context, then in that context it does not even reach the point where it can serve as a proper object of supervaluation.

Can a sentence be judgeable in a context and yet still be indeterminate as to its truth-value? It is this question with which we shall deal in what follows. The notion of ‘context’ is of course itself notoriously problematic. The primary advantage of our proposal here will lie in the fact that the framework we advance enables us to rephrase our question in a way which does not rely on the use of this problematic notion.

3. Granular Partitions

3.1 Foreground, Background and Granularity

Our fundamental idea is that every use of language to make a judgment about reality brings about a certain *granular partition*. Already every act of singular reference and every act of perception effects a partition of reality into a *foreground* domain, within which the object of reference is located, and a *background* domain, which comprehends all the entities beyond. When one moves ones attention from *this* to *that* (for example from this chair to that table) then one brings about an *ontological regrouping* of foreground and background: objects in one’s environment that previously served as foreground are now in the background, while objects previously in the background are now advanced to the front.

Sometimes there occurs not regrouping but what we might call *ontological zooming*. The hygiene inspector first sees the glass, which serves as foreground object of her attention; then she focuses more carefully on the tiny particles of soap and beer clinging to the walls of the glass. She sees the world first through a coarser and then through a finer grid.

To produce an ontological theory of such partitioning, of ontological zooming and regrouping, will be somewhat tricky. This is because the results of partitioning are granular in every case, and this means that they cannot be understood along any simple mereological lines. For it is not as if one connected, compact (hole-free) portion of reality would be foregrounded or set into relief in relation to its surroundings in such a way that the latter – the background of our cognitive act – could itself be identified simply as the mereological complement of what is foregrounded. For if an object – say Leeds, or the ice cream in your hand – are included in the foreground domain, this does not at all imply that all the *parts* of this object are also included therein. For to say that partitions are granular is to say that they do not recognize parts beneath a certain size. The separate roads and buildings in Leeds are not foregrounded by the partition you create when you use the term ‘Leeds’, for example, when planning your trip to England next month; the separate molecules of the ice cream are not foregrounded by the partition you create when you look down to the ice cream in your hand prior to eating. This means that the ontology of foreground and background structure is ontologically more complex than has hitherto been supposed. Simple mereology will not suffice (Bittner 1997).

The complexity of the foreground/background structure has consequences also for the issue of vagueness. For it means that each partitioning of a portion of reality into foreground and background is compatible with a range of possible views as to the ultimate constituents of the objects in the foreground. The granularity involved in our partitioning activity effectively allows us to trace over the lower-level constituents of those objects which are set into relief. It is this very granularity which is thus in fact responsible for the vagueness of our terms and concepts, for it allows us to ignore questionable parts and thus also to ignore questions as to the precise boundaries of the objects with which we have to deal.

The theory of granular partitions is advanced in our earlier papers (Smith and Brogaard to appear), (Bittner and Smith 2001), (Smith and Bittner 2001) as a solution to the problem of how to deal with granularity in a mereological framework. Granular partitions are defined as systems of cells conceived as projecting onto reality in something like the way in which a bank of flashlights projects onto reality when it carves out cones of light in the darkness. Consider, for example the simple partition of the Himalayas that is depicted in the left part of Figure 1 above. This partition contains cells labeled ‘Everest’ and ‘Lhotse’, together with one maximal cell labeled ‘the Himalayas’. These cells project onto different parts of that portion of reality that is depicted in the right part of Figure 1. They carve *mountains* out of a certain formation of rock. They do not do this physically, but rather by establishing fiat boundaries in reality, represented by the black lines in the right part of the figure. (Smith 1995), (Smith 2001), (Bittner and Smith 2001). Fiat boundaries are in a way like the boundaries of a light-cone that is projected during daylight. The fiat boundaries are there, but we cannot see them. Thus we have to use indirect means (for example maps and compasses and complex calculations) in order to discover where they lie. In some cases we may have good grounds for believing that we have crossed them. For example a sudden increase in slope may tell us that we have crossed the boundary of Mount Everest. In some cases fiat boundaries have become associated with suitable bona fide props, for example with systems of pegs or fences in reality. Surveying is about establishing relations between fiat boundaries and real, physical landmarks of these sorts. (Moffitt and Bouchard 1987), (Bittner 1999).

The problematic nature of the cases which concern us here, however, lies in the fact that the fiat boundaries with which we have to deal are not in any determinate place, but exist rather as multiple systems of boundaries projected onto reality through cognitive acts of a range of different sorts. Vagueness is, on the *de dicto* view, entirely a matter of the fiat realm. Everything which exists in the bona fide physical world – the world as it is before we come along with our partitions and our fiat borders– is crisp. (We leave aside the problems which arise for this thesis at very small scales.)

3.2 Judgments, Partitions and Contexts

Judgments and partitions are closely related. Consider the judgments J_1 = (B, C_1) and J_2 = (not-B, C_2) referred to above. Corresponding to J_1 and J_2 are two partitions, Pt_1 and Pt_2. Both contain cells labeled ‘glass’ and ‘beer’, similar to the cells in the partition in the left part of Figure 1. But Pt_2 has in addition cells labeled ‘bacteria’, ‘mold’, ‘chlorine’, and so forth. Moreover Pt_1 and Pt_2 do not differ only in their complement of cells; they differ also in the way the cells they share in common are projected onto reality. The cell labeled ‘beer’ in the drunkard’s partition projects (tries to project) onto drinkable amounts of beer. The corresponding cell in the partition of the hygiene inspector projects even onto amounts of beer that are visible only under a microscope. Reflecting on such examples reveals a way in which partitions, by means of their cell structure, can stand proxy for contexts in a theory of judgment designed to take account of the context-dependence of vagueness. The number and arrangement of cells within a partition and the ways in which these cells project onto reality – which means above all the granularity at which they are targeted upon objects in reality – serve as formally tractable surrogates for those features of contexts which are relevant to the understanding of vagueness as a semantic (*de dicto*) phenomenon.

Let us return to our partition of the Himalayas. There are, we can now say, multiple equally good ways of projecting the cell ‘Mount Everest’ onto the corresponding formation of rock. Each is slightly different as regards the location of the mountain boundaries which are projected among the pertinent foothills. Each projection targets just one possible candidate precisification. Each has, in other words, an ontological correlate that is entirely *crisp.* Vagueness arises only because there is not one such admissible projection, but rather very many.

4. A Theory of Granular Partitions: A Brief Outline

4.1 Partitions as System of Cells

The theory of granular partitions has two parts: (A) a theory of the relations between cells and the partitions in which they are housed, and (B) a theory of the relations between cells and objects in reality.

Theory (A) studies the properties granular partitions have in virtue of the relations between and the operations performed upon the cells from out of which they are built. All such partitions involve cells arranged together in some grid-like structure. This structure is intrinsic to the partition itself; that is to say, it is what it is independently of the objects onto which it might be projected. As we shall see this part of the theory applies equally well to crisp as to vague partitions.

The cells in a partition may be arranged in a simple side-by-side fashion, for example in our partition of *the Beatles* into *John*, *Paul*, *George* and *Ringo*. Cells may also be nested one inside another in the way in which, for example, the species *crow* is nested inside the species *bird* which is nested in turn inside the genus *vertebrate* in standard biological taxonomies. It is the possibility of this nesting which more than anything else distinguishes granular partitions as here understood from partitions in the more familiar mathematical sense (partitions generated by equivalence relations).

We define the cell structure, A, of a partition, Pt, as a system of cells, z_0, z_1, …, . We write Z(z, A_{Pt}) as an abbreviation for 'z is a cell in the cell-structure A of the partition Pt'. We say that z_1 is a subcell of z_2 if the two cells are in the same cell structure and the first is contained in the latter, and we write $z_1 \subseteq_A z_2$ in order to designate this relationship. In the remainder we omit subscripts wherever the context is clear. We then impose four axioms (or 'master conditions') on all partitions, as follows:

> MA1: The subcell relation $\subseteq$ is reflexive, transitive, and antisymmetric.
> MA2: The cell structure of a partition is always such that chains of nested cells are of finite length.
> MA3: If two cells overlap, then one is a subcell of the other.
> MA4: Each partition contains a unique maximal cell.

These conditions together ensure that each partition can be represented as a tree (a directed graph with a root and no cycles).

4.2 Partitions in their Projective Relation to Reality

Theory (B) arises in reflection of the fact that partitions are more than just systems of cells. They are constructed in such a way as to project upon reality. Intuitively, this projection corresponds to the way proper names project onto or refer to the objects they denote and to the way our acts of perception are related to their objects. (Projection is close to what philosophers call 'intentionality'.) When projection is successful, then we say that the object targeted by a cell is located in that cell. We then write 'P(z, o)' as an abbreviation for: cell z is projected onto object o, and 'L(o, z)' as an abbreviation for: object o is located at cell z. Intuitively, being located in a cell is like being illuminated by a spotlight. That location is not simply the converse of projection follows from the fact that a cell may project without there being anything onto which it is projected (as a spotlight can cast its beam without striking any object). Because location is what results when projection succeeds, location presupposes projection. An object is never located in a cell in a partition unless as a result of the fact that this cell has been projected upon that object. This is the first of our master conditions for theory (B):

MB1 $\quad L(o, z) \rightarrow P(z, o)$.

Partitions are cognitive artifacts. Objects can come to be located in the cells of our partitions only if we have constructed cells of the appropriate sort and targeted them in the right direction. We then say that the partition in question is *transparent to* the corresponding portion of reality. We can formulate this condition of transparency as follows:

MB2 $\quad P(z, o) \rightarrow L(o, z)$.

In what follows we shall assume conditions MB1 and MB2 as master conditions governing all partitions. Thus in the restricted context of this paper MB1 and MB2 collapse to $L(o, z) \leftrightarrow P(z, o)$. MB2 serves to guarantee that objects are actually located at the cells that project onto them. In a more general theory of granular partitions, MB2 will be weakened to allow misprojection, for example where an object is wrongly named or wrongly classified.

In order to ensure that projection and location satisfy the intuitions underlying our spotlight analogy, we demand further that projection and location are functional relations, i.e., that every cell projects onto just one object and every object is located in just one cell:

MB3 $\quad P(z, o_1)$ and $P(z, o_2) \rightarrow o_1 = o_2$
MB4 $\quad L(o, z_1)$ and $L(o, z_2) \rightarrow z_1 = z_2$

For partitions satisfying MB3, each cell is projected onto one single object. (One rather than two; there is no overcrowding.) For partitions satisfying MB4 objects are in every case located at single cells. Notice that this excludes the sort of redundancy which would be involved where a single partition would contain distinct cells (for example labeled ‘Mount Everest’ and ‘Chomlungma’) both projecting onto what is (modulo the factor of vagueness) the same formation of rock. Notice also that ‘object’ here is used in a very wide sense, to include also scattered wholes. Thus a partition of the animal kingdom might involve a cell labeled *cat* which projects onto that object which is the mereological sum of all live cats.

We will assume that partitions are *complete* in the sense that every cell projects onto at least one object, i.e., that there are no empty cells (no cells projecting outwards into the void):

MB5 $\quad Z(z, A) \rightarrow \exists o\colon L(o, z)$

Consequently, projection is a total function.

Location, however, is typically a partial function. This is because human beings are not omnipotent in their partitioning power. Thus for any given partition directed towards some domain of concrete reality there will always be objects which its referential spotlights do not reach. Even where we have a partition whose domain is just one single object, we can assume that there will be parts of this object – atoms, or sub-atomic particles – onto which no cell is projected. (This, again, is what is meant when we say that partitions are ‘granular’.) In the context of this paper we will assume that the constraints MB1–5 are always satisfied, i.e., projection and location are always functional, and there are no empty cells.

4.3 Recognizing and Preserving Mereological Structure

Partitions reflect the basic part-whole structure of reality in virtue of the fact that the cells in a partition are themselves such as to stand in the relation of part to whole. This means that, given the master conditions expressed within the framework of theory (A) above, partitions have at least the potential to reflect the mereological structure of the domain onto which they are projected. And in felicitous cases this potential is realized. We write ‘p(z)’ to designate the object located in the cell z. By MB5, p(z) is always defined. We say that the cells z_1 and z_2 *reflect the mereological relationship between the objects onto which they are projected* if and only if the following holds:

$$\text{DR1} \qquad RS(z_1, z_2) \equiv z_1 \subset z_2 \rightarrow p(z_1) < p(z_2)$$

If z_1 is a proper subcell of z_2 in a given partition, then the object onto which z_1 projects is a proper part of the object onto which z_2 projects. A partition reflects the mereological structure of the domain it is projected onto if and only if each pair of cells satisfies DR1, a condition we impose on all partitions, as follows:

$$\text{MB6:} \qquad Z(z_1, A) \text{ and } Z(z_2, A) \rightarrow RS(z_1, z_2)$$

It follows from MB6 that everything onto which a cell in a partition is projected is a part of that onto which the root-cell is projected.

We demand further that granular partitions satisfy a constraint to the effect that if objects recognized by a given partition stand to each other in a relation of part to whole, then the cells in which these objects are located stand to each other in the subcell relation. We first of all define what it is for an object to be recognized by a partition:

$$\text{DR2} \qquad R(o, A) \equiv \exists z\text{: } Z(z, A) \text{ and } L(o, z),$$

and we write ‘l(o)’ to designate the cell in which an object recognized by a given partition is located. We then set:

$$\text{DR3} \qquad RS^{+}(o_1, o_2) \equiv o_1 < o_2 \rightarrow l(o_1) \subset l(o_2)$$

We can now formulate the condition:

$$\text{MB6}^{+}\text{:} \qquad R(A, o_1) \text{ and } R(A, o_2) \rightarrow RS^{+}(o_1, o_2),$$

which asserts that all partitions are *mereologically monotone.*

4.4 The domain of a partition

Each partition has a certain *domain*, which we can define as that portion of reality upon which its maximal cell is projected. This is a certain mereological sum: it is, as it were, the total mass of stuff upon which the partition sets to work: thus it is stuff as it exists independently of any of the divisions or demarcations effected by the partition itself through its constituent cells. Since the scope of partition theory is so general, the domain of a partition may comprehend not only concrete particulars and their constituents (atoms, molecules, limbs, organs), but also groups or populations of individuals (for example biological species and genera, battalions and regiments, archipelagos and diasporas) and their constituent members. In some cases, for example when drawing gridded maps, we project the cells of our partitions deliberately onto regions of space. (A more general theory than the one advanced here might allow that even partitions of this last sort may have cells which are empty. For example they may fail to project onto any *actual* region of space, as in the case of a map of Middle Earth.)

We can define the domain of a partition, D(Pt), simply as *the object onto which its root cell is projected*. By functionality of projection and location there can be only one such object. That every partition has a non-empty domain follows from MB5. We now can define a granular partition as a triple Pt = (A, P, L), where A is a system of cells such that MA1–4 hold and P and L are projection and location relations such that MB1–6^+ hold for the relationship between the cell structure A and the portion of reality onto which it projects.

5 Judgments

A judgment is a pair J = (S, Pt) where S is a sentence and Pt is a granular partition (which stands proxy for the context in which the judgment is made). It will take us too far afield to provide a general partition-theoretic account of truth for judgments here (to include, for example, compound judgments and judgments expressed by a sentence involving non-referring singular terms). It will suffice for our purposes to provide brief treatments of one or two simple examples, which have been chosen for illustrative purposes.

5.1 Judgments about mereological relationships

Consider the left part of Figure 1, with its three partition cells labeled 'The Himalayas', 'Everest', and 'Lhotse'. These labels are the partition-theoretic counterparts of (*inter alia*) the names we use in judgments. Consider the sentence 'Everest is part of the Himalayas' uttered in the context represented by the partition Pt. This judgment is of the form 'a stands in R to b' where 'a' is replaced by 'Everest' and 'b' is replaced by 'the Himalayas' and 'R' is replaced by the binary predicate 'part-of'.

Given a judgment J = (S, Pt), the relationship between S and Pt is provided by a *labeling function* which assigns the names of the objects referred to in S to cells of Pt = (A, P, L). We say that λ is a *labeling* relating the partition Pt to the sentence S if and only if the following holds:

(1) λ maps the sentence S as a whole onto the root cell of the partition Pt;

(2) λ maps proper names appearing in S to cells in A in such a way that each cell gets uniquely labeled and each name has a unique corresponding cell;

(3) the co-domain of λ exhausts the cell-structure of Pt.

Condition (1) ensures that the judgment as a whole has a well-defined scope, namely the domain of Pt. In the specific case of $J_1 = (S_1, Pt)$, where S is the sentence 'Mount Everest is part of the Himalayas' and Pt is the partition shown in the left part of Figure 1. The sentence S_1 as a whole is mapped by λ onto the root cell of the given partition. Condition (2) ensures, in conjunction with the assumption (MB5) that there are no empty cells, that each cell is uniquely labeled by a name contained in S. The association of partition cells to the names occurring in the corresponding judgment corresponds to our discussion of ontological regrouping above. The judgment J_1 brings into the foreground Mount Everest, the Himalayas, and the part-of relation which holds between them; it forces everything else, including Mount Lhotse, Leeds, Bill Clinton, the ice cream in your hand, into the background of our attentions. Condition (3) ensures that the corresponding partition contains the cells 'Everest' and 'The Himalayas' but not a cell labeled 'Lhotse.' In this sense the labeling function always maps onto partitions that are *minimal* with respect to the sentence used in making the corresponding judgment.

Imagine a partition similar to the one represented in Figure 1, but without the cell 'Lhotse'. Here we can establish a labeling function between S_1 and Pt, but we need to acknowledge that the root cell has a special status in the following sense: the labeling λ maps both the sentence S_1 as a whole and also the name 'the Himalayas' onto the root cell of the given partition. Consequently the inverse of λ is not a function. This does not however violate condition (2), since the latter demands only the unique correspondence between *names* and *cells*. Formally we demand: (i) λ is a total function on proper names in S; and (ii) the inverse of the restriction of λ to proper names in S, $(\lambda|_S)^{-1}$, is a total function in $(A - \{r(A)\})$. Note that the root cell is often not targeted explicitly by any name at all.

We now say that a judgment of the form 'a is part of b' is *true in the context represented by* Pt if and only if

- (i) Pt represents a portion of reality in such a way that MA1–4 and MB1–6^+ hold;
- (ii) there is a labeling function λ with the properties specified above; and
- (iii) the cell labeled 'a' is a subcell of the cell labeled 'b' in the partition Pt.

5.2 Judgments about spatial relationships

Consider the judgment $J_2 = (S_2, Pt_2)$, with S_2 = 'Lhotse lies to the west of Everest' uttered while discussing the location of mountains in the Himalayas. This we shall interpret as a judgment of the form 'a is F' where 'F' is identified with the predicate 'west of Everest'. We can then interpret J_2 as a judgment to the effect that 'Lhotse' is a part of the mereological whole formed by the sum of all things that lie to the west of Mount Everest.

Let Pt_2 be the partition shown in

Figure 2 which consists of three nested cells. The labeling function λ maps S_2 onto the root-cell, which projects onto the Himalayas. This reflects the fact that the judgment was uttered in the context of a discussion of the relative locations of the mountains of the Himalayas. The predicate 'west of Everest' is mapped onto the middle cell which is projected onto the mereological sum of all the parts of the Himalayas that are to the west of Mount Everest. Finally, the name ‘Lhotse’ is mapped onto the cell in the center, which projects onto Mount Lhotse.

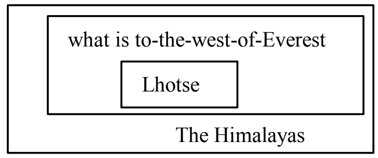

Figure 2: A partition corresponding to the judgment ‘Lhotse is to the west of Everest’.

We can now regard the judgment ‘Lhotse is to the west of Everest’ as being of the more general form ‘a is F.’ We define a labeling, for judgments of this form, as follows. λ is a *labeling* relating the partition Pt to the sentence S = ‘a is F’ if and only if the following holds:

(1) λ maps the sentence S as a whole onto the root cell of the partition Pt;
(2) λ maps the subject term of S onto a unique cell z_i in A;
(3) the co-domain of λ exhausts the cell-structure of Pt.

A judgment of the form ‘a is F’ is then *true in the context represented by* Pt if and only if

(i) Pt is a partition of reality for which MA1–4 and MB1–6^+ hold;
(ii) there is a labeling function λ with the properties specified above; and
(iii) the cell labeled ‘a’ is a subcell of the cell labeled ‘F’ in the partition Pt.

5.3 The perspective of the semantic theorists and of the judging subject

When discussing the truth of judgments in partition-theoretic terms, we must take two distinct perspectives into account: the perspective of the *semantic theorist* and the perspective of the *judging subject*. It is critical to carefully separate these two different views.

The perspective of the semantic theorist considers the truth of a judgment in relatively abstract terms as a correspondence between language and reality. (We are attempting, in all of the above, to be consistent with the standard notion of truth as correspondence.) The judgment J = (S, Pt) is true if and only if there is labeling function of the appropriate sort linking S to the cells of the partition Pt, and a projection function linking these cells in turn to corresponding portions of reality. This is of course very preliminary, and the range of examples treated is meager in the extreme, but it will provide a sufficient basis for what follows nonetheless.

The judging subject succeeds in making a true judgment because he is able to effect a separation of reality into foreground and background and to bring to bear a perspective on reality that has a certain appropriate granularity. Only as a result of these selective features of his attention is he able to establish a relation to reality of the sort that is required to make a true judgment.

6. Vague Granular Partitions

6.1 Vagueness of projection

The core of the theory of granular partitions is presented in (Smith and Brogaard, to appear). Our paper (Bittner and Smith 2001) gives a formal account of the concepts of cell and projection. The present paper provides a formal account of the phenomenon of vagueness in partition-theoretic terms.

When projection is vague, then (to pursue our earlier spotlight analogy) not only can you not see the fiat boundaries carved out by the projections, you can know only roughly where they lie. What this 'roughly' ('vaguely') means is explained from the *de dicto* point of view as follows. It is as if there were many overlapping portions of reality that are equally good candidates for falling within the light-cone of your flashlight. Thus there are many alternative ways in which fiat boundaries for Mount Everest might be carved out among its foothills. The judging subject knows roughly where they lie – above all he knows that they must include the summit – but he cannot see or measure them directly. And this is not merely an epistemological problem: thus it is not merely that we do not know the facts about where the boundary of the mountain lies. There are no facts that specify where this boundary is located.

Partition theory enables us to understand how, through their use of terms and concepts, judging subjects effect corresponding demarcations on the side of objects in reality. What we as partition theorists need to do now is to show how the use of terms and concepts can effect not only crisp demarcations of reality – as in the case of postal districts and census tracts – but also vague demarcations, as in the case of mountains and deserts and unregulated wetlands. The extension of the theory of granular partitions is modeled on the supervaluationist understanding of vagueness, but it follows the contextualized version of supervaluation suggested in (Smith and Brogaard 2001). Where, in the crisp case, each partition is characterized by a *single* projection relation and a *single* location relation, in the vague case we need to give up the constraint that each partition is associated with a single projection/location relation. Theory (A) is unaffected by this change, but we will need to provide modified axioms for theory (B) in such a way that crispness is included as just one special case.

6.2 Vague partitions

A *vague* granular partition $Pt^v = (A, P^v, L^v)$ is a triple such that A is a system of cells for which MA1-4 hold and P^v and L^v are *classes* of projection and location relations, with properties which will be discussed below. Consider Figure 3, which depicts a vague partition $Pt^V = (A, P^v, L^v)$ of the Himalayas. This has a cell structure A, as shown in the left part of Figure 3, which is in fact identical to the corresponding part of Figure 1. In the right part of the figure, in contrast, there is a multiplicity of possible candidate projections for the cells in A, indicated by boundary regions depicted via cloudy ovoids. The boundaries of the actual candidates onto which the cells ‘Lhotse’ and ‘Everest’ are projected under the various P_i in P^v are included somewhere within the clouds of regions depicted in the figure.

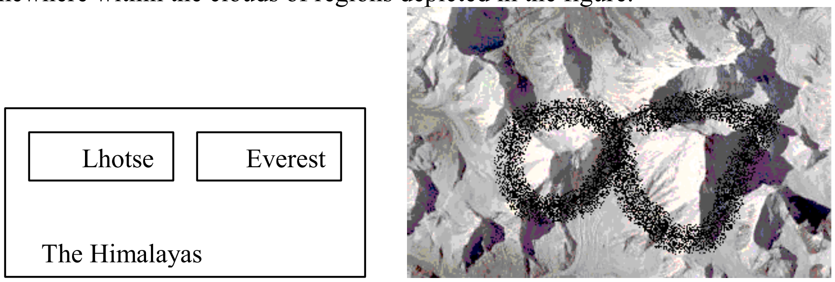

Figure 3: A vague partition of the Himalayas

The projection and location relations in these classes form pairs (P_i, L_j), which are such that each P_i has a corresponding unique L_j and vice versa, satisfying the following conditions (where the notation ‘∃!’ abbreviates: ‘there exists one and only one’):

MB1^V $\forall j$: $L_j(o, z) \rightarrow \exists! i\ P_i(z, o)$
MB2^V $\forall i$: $P_i(z, o) \rightarrow \exists! j$: $L_j(o, z)$

In the context of this paper MB1^V and MB2^V can be simplified as: $\forall i \exists! j$: $P_i(z,o) \leftrightarrow L_j(o,z)$.

We also demand that all P_i and all L_j are functional in the sense discussed in the crisp case:

MB3^V $P_i(z, o_1)$ and $P_i(z, o_2) \rightarrow o_1 = o_2$
MB4^V $L_j(o, z_1)$ and $L_j(o, z_2) \rightarrow z_1 = z_2$

We demand further that cells project onto some object (are non-empty) under every projection:

MB5^V $Z(z, A) \rightarrow \forall j\ \exists o$: $L_j(o, z)$

The modified versions of the axioms enforcing the preservation of mereological structure and mereological monotony for the pairs (P_j, L_i) satisfying MB1^V and MB2^V then read:

MB6^V $Z(z_1, A)$ and $Z(z_2, A) \rightarrow \forall i\ RS_i(z_1, z_2)$

MB6^{+V} $R_i(A, o_1)$ and $R_i(A, o_2) \rightarrow RS_i^+(o_1, o_2)$,

with definitions

DR1^V $RS_i(z_1, z_2) \equiv z_1 \subset z_2 \rightarrow p_i(z_1) < p_i(z_2)$

$DR2^V \quad R_i(o, A) \equiv \exists z: Z(z, A) \text{ and } L_i(o, z)$

$DR3^V \quad RS_i^+(o_1, o_2) \equiv o_1 < o_2 \rightarrow l_i(o_1) \subset l_i(o_2)$

We call all partitions $Pt_{ij} = (A, P_i, L_j)$ with pairs (P_i, L_j) satisfying MB1–MB6$^+$ *crispings* of the vague partition Pt^V. The domain of a vague partition is the mereological sum of the domains of all crispings. From MB5^V it follows that the domain of each crisping is non-empty, i.e., $\forall i, \exists o: o = D(Pt_i)$.

Consider a partition with cells labeled with vague proper names. Intuitively, each pair of projection and location relations (P_i, L_j) then recognizes exactly one candidate precisified referent for each such cell. The precise candidates carved out by each (P_i, L_j) are all slightly different. But each is perfectly crisp and thus it has all of the properties of crisp partitions discussed in the previous sections. This means that, even under conditions of vagueness, the principal properties of partitions are preserved. Note that the vague partition Pt^V has just one single system of cells but many projection and location relations. The one system of cells projects in multiple ways onto reality. Each of the projections and each of the corresponding location relations behaves as it would in a standard, crisp partition.

We can now consider two pairs of projection and location relations, (P_i, L_j) and (P_m, L_n), both satisfying MB1^V–6^{+V}. We then have two distinct crisp partitions $Pt_{ij} = (A, P_i, L_j)$ and $Pt_{mn} = (A, P_m, L_n)$. The cell structure is identical in both cases; both have the same minimal cells, 'Mount Everest' and 'Mount Lhotse', contained in the same maximal cell 'the Himalayas'. In both cases these cells project onto neighbouring formations of rock, which are disjoint in the sense that $\neg\exists x: x = P_i(\text{'Everest'}) * P_i(\text{'Lhotse'})$ and $\neg\exists x: x = P_m(\text{'Everest'}) * P_m(\text{'Lhotse'})$, where '*' signifies mereological intersection. (That such disjointness should hold for each projection relation is a *penumbral condition* in the sense of Fine (1975).) Again, it is important to recognize that the presence of vagueness does not mean that any of our conditions governing partitions are violated. Vagueness *de dicto* is captured at the partition level via *multiple ways* of projecting crisply. Each of these ways of projecting crisply must satisfy the conditions on partitions set forth above.

6.3 Judgments and vagueness

We can now define the notions of supertruth, superfalsehood, and indeterminacy for judgments, $J = (S, Pt^V)$ with respect to a vague partition $Pt^V = (A, P^V, L^V)$. We assume that the cell structure A satisfies MA1–4 and that all of its crisp $Pt_{ij} = (A, P_i, L_j)$ are such that MB1^V–6^{+V} hold. A judgment J is then *supertrue* with respect to a vague granular partition Pt^v if and only if it is true with respect to *all* of the crisp partitions $Pt_{ij} = (A, P_i, L_j)$. A judgment J is *superfalse* with respect to Pt^v if and only if it is true with respect to *none* of the crisp partitions $Pt_{ij} = (A, P_i, L_j)$. It is *indeterminate* otherwise.

As in the crisp case we need to take into account both the perspective of the semantic theorist and the perspective of the judging subject. The former is reflected in our use of a contextualized supervaluationary semantics, which captures those features of the matters in hand which fall beneath the threshold of awareness of the judging subject. As to the latter we note first of all that, as in the crisp case, important aspects of judging are the separation of reality into foreground and background of attention and the fact that judgments about reality are made at a certain level of granularity.

7. Unity and vagueness

7.1 Unity conditions

When making a judgment to the effect that a is part of b, you apply a *unity condition* which provides you with the means to determine which parts of reality are to form a certain whole. We shall see that the study of the vagueness of judgments of the form J^V = (‘a is part of b’, Pt^V) is closely related to questions of the vagueness of unity conditions.

When recognizing wholes as sums of parts, judging subjects draw upon unity conditions that specify what sums of parts they are concerned with. In the case of Mount Everest, the pertinent unity condition might be formulated, in first approximation, along the following lines:

U1 (1) The summit is part of Mount Everest. (2) If x is a part of Mount Everest and y is connected to x then y is a part of Mount Everest.

We can assume for present purposes that clause (1) is unproblematic. Not so for clause (2), however, for this makes the unity condition incapable of determining which outlying portions of reality are parts of the mountain and which are not. It is because of this that paradoxes of the Sorites type can arise. U1 has the structure of an inductive definition. It specifies a start condition and a condition on how to add parts to Mount Everest, but it does not specify where to *stop* adding parts. This means that if we take (1) and (2) in U1 as true premises, then it is logically sound to infer that portions of reality are parts of Mount Everest that clearly fall outside it.

We cannot simply dismiss U1. Clause (2) captures the *continuous structure* of the formation of rock to which the concept *mountain* applies, that is, it captures the fact that mountains are never scattered wholes; they are always such that we can form chains of connected parts a_1, a_2, a_3, … But what determines the outer limits of such chains? Where does the mountain stop? As will by now be clear, there is no generally applicable and context-independent stop condition that can be inferred from a general concept such as *mountain*.

Consider now the relationship between the unity condition U1 and a judgment of the form J^V = (‘a is part of Everest’, Pt^V). The two are closely related in the following sense: U1 governs the way Pt^V projects onto reality in the sense that the cell ‘Everest’ must project onto a topologically connected whole (clause (2) of U1) which contains the summit (clause (1) of U1). On the other hand judgment J^V in the context represented by Pt^V also places limits on the range of admissible precisifications in the sense that it projects boundaries onto reality which serve to break the unlimited chains in the needed fashion. The problem is that these limits, i.e., the projected boundaries, are themselves subject to vagueness, and it is this which threatens the possibility of truth-value indeterminacy. Our task will be to show how this possibility is prevented from becoming actual in natural contexts, and thus to show that even judgments expressed by sentences involving vague terms have determinate truth-values.

To this end, we need to discuss the range of relevant kinds of contexts. Two cases in particular are of importance, distinguished by the *kinds of boundaries* that can provide stop conditions of the needed sort:

1. Contexts in which our use of the corresponding term brings a single crisp boundary into existence.
2. Contexts in which our use of the corresponding term brings a vague boundary (i.e., a multiplicity of crisp boundary candidates) into existence.

7.2 The single (crisp) boundary case

Contexts of the first type are illustrated by those cases where judging subjects themselves have the *authority* (the partitioning power) to bring a crisp boundary into existence. Suppose that you have been delegated by some government agency to establish the boundaries of Mount Everest for purposes of regulating the activities of climbers. Your partition – we can imagine that it is set forth in some document *D* – would then come very close to being fully crisp, i.e. only one single projection relation would be involved, and the boundary of Mount Everest would in relevant contexts coincide with the boundary imposed by you. This has the consequence that, in the given contexts, the incomplete unity condition that comes with the underlying general concept is completed contextually, as follows:

U2 (1) The summit is part of Mount Everest. (2) x is part of Mount Everest if and only if: (i) there is some y which is part of Mount Everest and x is connected to y, and (ii) x is part of the projection of the cell labeled 'Everest' in the partition determined by the document *D*.

U2 has the advantage of blocking the admission of unlimited chains of connected parts. Moreover U2 still enforces the continuity of parts of the mountain in the spirit of U1. Using U2 the truth-value of a judgment of the form J = ('a is part of Mount Everest', Pt) is fixed in a determinate manner for each a, and truth-value indeterminacy cannot arise.

7.3 The multiple (vague) boundary case

Contexts where judging subjects have the authority and the need to bring a precise boundary into existence are, it must be admitted, very rare. Fortunately however there is in most contexts no need for the high degree of precision which such contexts represent. In most contexts, that is to say, we get along with a created boundary that is *just precise enough*. This means that it is precise to the degree to which it matters where it lies, and therefore also just precise enough to enable the judging subject to make a determinate judgment. In most cases, therefore, it will manifest a certain degree of vagueness, and the actual degree of vagueness (or the degree of precision) will depend on context. Where vagueness is involved indeterminate cases threaten to arise. To this end we must show, following (Smith and Brogaard 2001), that in naturally occurring contexts where boundaries are *just precise enough*, sentences which would have indeterminate truth-values are unjudgeable.

In instructing your staff to set up the tables in your restaurant each evening, you establish where the line between smoking and non-smoking zones is to be drawn by using a sentence like:

[C] The boundary of the smoking zone goes here,

while pointing with your finger in such a way to bisect the restaurant floor. You thereby also indicate on which tables the ashtrays are to be placed. You specify vaguely where the boundary lies. This means that, with your vague gesture ,you bring a whole multitude of equally good boundary-candidates into existence.

Our concept of a smoking zone is, after all, one of a whole with boundaries which are not precisely defined by sharp lines, fences, or walls. This is reflected by a unity condition along the lines of U3:

U3 x is part of the smoking zone if and only if x is part of one of a multitude of equally good smoking-zone-candidates that were brought into existence by your initial specification of the boundary location .

U3 like U2 has the advantage of blocking the unlimited chains of connected parts. The question then arises whether a judgment of the form J = (‘This table is part of the smoking zone’, Pt^V) can be such as to have an indeterminate truth-value. Inspection reveals that the apparent vagueness of the boundary-specification does not affect the determinacy of those judgments which judging subjects such as the restaurant staff or customers might actually make. Whether an ashtray is or is not placed on a table is, after all, a completely determinate matter. To capture the pragmatic constrains on judgeability in the given context, U3 needs to be revised in such a way that it *does not admit arbitrary parts* but only parts of certain size:

U4 x is part of the smoking zone if and only if: (1) x is greater than or equal to one table in size; and (2) x is part of one of a multitude of equally good smoking-zone-candidates that were brought into existence by the initial specification of the boundary location.

In addition, U4 gains a twin, which determines the analogous condition for parthood in relation to the non-smoking zone which is its (partition-theoretic) complement:

U4′ x is part of the non-smoking zone if and only if: (1) x is greater than or equal to one table in size; and (2) x is part of one of a multitude of equally good non-smoking-zone-candidates that were brought into existence by the initial specification of the boundary location.

The vagueness of your specification of the location of the boundary of the smoking area does not affect the determinacy of the truth-value of the judgments made in the resultant context. U4 ensures that judgments of the form of J are either supertrue or superfalse. It also ensures that a judgment of the form 'This nicotine molecule is part of the smoking zone' cannot be uttered in the given context, since the unity condition U4 does not admit molecules as parts of smoking (or non-smoking) zones. A judgment of this form reflects an illegitimate mixing of granularities. If judgments of the given form are to be judgeable, then more precise specifications of the relevant boundaries would needed to be made by those involved, and this would mean creating a new context.

8. Degrees of Vagueness and Crispness

We can see that the achievement of an *appropriate degree* of vagueness or crispness within given naturally occurring contexts is critical for avoiding truth-value indeterminacy. In this section we discuss a range of examples which further strengthen this point.

Imagine two neighboring countries, one with the death penalty and one without. Even if the border between the two countries is fiat in nature (no wall, no fence), still, if you murder somebody on one side of the border you will be liable to die, and if you commit your crime on the other side of the border you will be liable to go to jail. Here it does not seem that indeterminacy can arise. This will hold even if you commit the crime while your body spans the border of the two countries (a one-dimensional fiat spatial entity, whose location can nowadays be determined with considerable accuracy). This is because, since this is the sort of case where your exact location relative to the boundary *matters to the proceedings of the courts*, these courts will themselves have developed mechanisms to remove indeterminacy by fiat from their judgments in light of the fact that the same person cannot both be hanged, and not hanged, for the same crime.

Imagine that you are wandering across the desert somewhere in the borderlands between Libya and Egypt pointing towards a grain of sand on the ground, and that you pronounce the sentence:

[D] This grain of sand belongs to Egypt.

No corresponding judgment will have been made, according to the view we are here defending. This is the case not because the specification of the boundary between Libya and Egypt is vague. Rather, it is because speaker and audience would not take the given sentence seriously as expressing a judgment, because again, it reflects an illegitimate mixing of granularities.

If, on the other hand, the need to determine the ownership of every grain of sand were to arise (for example because sand has become more valuable than gold), then means would be devised – and new sorts of contexts created – which would allow the corresponding judgments to be made and their truth-to be determined, at least in principle, unequivocally. For so long as this is not the case, however, there is no way to determine the truth-value of a judgment like [D]. Consequently, too, any attempt to make a judgment of this kind in our present contexts must fail on pragmatic grounds.

Imagine that you are with a party of climbers somewhere in the foothills of Mountain Everest and that one of your number, pointing to some imaginary line on the ground, uses the sentence:

[E] This is the boundary of Mount Everest

in order to make a judgment. We argue that in the given context (a context in which it is obvious to all parties that there is no law or treaty which establishes where, in or around its foothills, the boundary of the mountain lies) someone using [E] would not succeed in making a judgment. Rather, he would be seen as making some sort of joke. This is because a judgment J = (D, Pt) of this form would invoke a *crisp* partition Pt = (A, P, L), and it is pragmatically impossible to invoke crisp partitions in contexts where both speaker and audience know that vague partitions are the best that can be achieved. Corresponding attempts to make judgments will not be taken seriously.

It is, though, possible to conceive of contexts in which it is necessary to refer to the boundary of Mount Everest no matter how vague it might be. Suppose you make a judgment of the form:

[F] We will cross the boundary of Mount Everest within the next hour.

The admissible candidate boundaries for Mount Everest are hereby delimited as falling within a certain range, projected out onto the path ahead and determined as a function of travel time (all under the assumption that the judgment in question is true).

In this case you, as judging subject, do not care where precisely the border is crossed because you are aware that you yourself are in a sense creating this border. The judgment concerns the *approximate* location of a boundary which has no legal or other formal status beyond that which is intended by you in the given context. The way in which the location of the boundary is specified is then once again just precise enough: *it is such that it can be crossed within the next hour*. It is then easy to see how your judgment might be either supertrue or superfalse. It is supertrue if, after a few minutes, you embark on a steep rise, which continues uninterrupted until you reach the summit. It is superfalse if you discover (or could discover), two hours after making your judgment, that you were over-optimistic: a new, wide valley suddenly appears between you and the mountain.

The crucial question is: under what conditions might the given judgment be indeterminate in truth-value? Bear in mind that there is here no crisply pre-established boundary; it is you the judger who determines – roughly – where the boundary lies. Can you determine that the boundary will be located in such a way as to dissect the family of admissible precisifications associated with the judgment you express by [F] into two disjoint sub-families, the first crossable within the hour, the second not? We think not. There is here only *just enough precision*. The necessary degree of precision to give rise to indeterminacy is not available.

9. Boundaries limiting vagueness

We argued that sentences containing vague names need to be considered as vehicles for judgment and thus that they must be analyzed semantically *in the contexts in which they are actually used*. Our overarching project is to show that, when considering judgments in their contexts, indeterminacy of truth-value is at least a much rarer phenomenon than is commonly supposed. Another large family of contexts must now be considered, they are contexts which involve the specification of constraints that delimit the range of admissible candidates. These are contexts which allow judgers to impose boundaries onto reality that resolve or at least limit the vagueness of their acts of reference. Thus in this section we focus on the judging subject and on his role in delimiting the degree of vagueness of his judgment by imposing fiat boundaries onto reality.

9.1 Vagueness and approximation

How do judging subjects impose boundaries vaguely? Consider judgment [F]: ‘We will cross the boundary of Mount Everest within the next hour’. This judgment specifies a *range* of admissible candidates by using the phrase ‘cross … within the next hour’. The judging subject thereby delimits the range of admissible candidates. Consider the left part of Figure 4. Boundaries delimiting admissible candidates are imposed by specifying a time interval that translates to travel distance along a path – time serves here as *frame of reference*. The boundaries are defined by the current location of the judging subjects (marked: ‘now’) and their location after the specified time has passed (marked: ‘in one hour’). Boundaries of admissible candidates of reference cross the path between these two boundaries. In general we call the first boundary the *exterior boundary* and the second the *interior boundary*. Exterior and interior boundaries are imposed onto reality by judging subjects in a process we call *approximation.*

In the process of approximation the judging subject projects a granular partition onto reality. This granular partition serves as *frame of reference* in terms of which the judging subject is able to both specify and constrain the range of admissible candidates of vague reference. In being projected onto reality this granular partition imposes fiat boundaries that limit the vagueness of reference in the sense discussed above in the context of judgment [F]. In the examples shown in Figure 4 the cell-structure of the partition serving as frame of reference (the *reference partition*) consists of three cells that are labeled ‘exterior’ and ‘core’ (projecting on the path left of ‘in one hour’ in the left part), and ‘where-the-boundary-candidates-are’. The cell *exterior* then projects onto the path west of ‘now’ in the left part of the figure, the cell *core* projects onto the path east of 'in one hour' and the cell *where-the-boundary-candidates-are* projects onto the region enclosed by the two boundaries.

Consider the sentences: 'We will cross the boundary of Mount Everest in the next ten seconds' and 'We will cross the boundary of Mount Everest in the next 10 years'. Both sentences are certainly not judgeable in most naturally occurring contexts. In the first case this is because the specified range of admissible candidates is much too fine, in the second case because it is much to coarse. We will discuss the relationships between degree of vagueness (the higher the degree of vagueness the larger the range within which admissible candidates occur) and the specification of constraints on the range of admissible candidates in more detail below. For now it is sufficient that the degree of vagueness and the specification of constraints on the range of admissible candidates need to be of compatible scale in the sense sketched above. In the remainder we consider constraints on the range of admissible candidates that are compatible with the degree of vagueness in force in a given context unless explicitly stated otherwise.

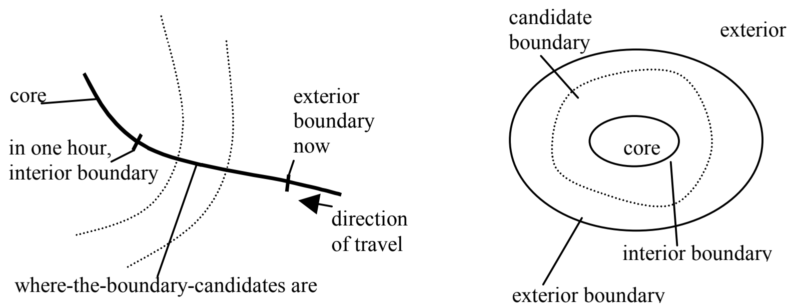

Figure 4: Boundaries that limit vagueness

9.2 Higher-order vagueness

In a slightly more complex case, the boundaries that are imposed to delimit the vagueness of the reference of a judging subject are of the sort illustrated by the sentence:

> [G] We will cross the boundary of Mount Everest not earlier than 60 minutes and not later than 90 minutes from now

Here there is created a zone within which all the various admissible candidate boundaries must lie.

This phenomenon is extensively discussed in the literature, e.g. (Cohn and Gotts 1996), (Clementini and Felice 1996), (Roy and Stell to appear), but it raises the problem of higher-order vagueness. For when considered in a context-free manner, the exterior and interior boundaries are themselves subject to vagueness. We hypothesize, again, that when contexts are taken into account, and when we restrict our attentions to naturally occurring contexts, then this higher-order vagueness is, not indeed eliminated, but at least constrained in such a way that truth-value indeterminacy of judgments cannot arise.

In order to support this hypothesis we need once more to consider the range of possible cases in which a judging subject establishes an object-boundary from scratch by specifying constraints on the possible location of the boundaries of admissible candidates. There are two fundamentally different ways in which this might occur:

(1) Existing bona fide or fiat boundaries are re-used, as for example in the case where someone judges ‘Ohio is north of the Ohio river’.

(2) New fiat boundaries are imposed, as for example in the case where someone judges ‘We will cross the boundary of Mount Everest within the next hour’.

We shall discuss each of these in turn.

9.3 Re-using existing boundaries

There is one crisp granular partition with which we are all familiar. It has exactly 50 cells, which project onto the 50 constituent states of the United States of America. A fragment of this partition is presented in the left and right parts of Figure 5. In the foreground of the figure we see in addition an area of bad weather, represented by a dark dotted region that is subject to vagueness *de dicto* in the sense discussed above. Wherever the boundaries of this object might be located, they certainly lie skew to the boundaries of the relevant states. But the figure also indicates (with the help of suitable labeling) that there are parts of the area of bad weather that are also parts of Wyoming, others which are parts of Montana, others which are parts of Utah, and yet others which are parts of Idaho.

In the sorts of contexts (represented by more or less coarse-grained partitions) which we humans normally inhabit, it is impossible to refer to any crisp boundary when making judgments about the location of a bad weather region of the sort described. However it is possible to describe its (current) location relative to the underlying US-state partition. We, the judging subjects, then deliberately employ this partition as our frame of reference and we describe the *relationships* that hold between all admissible referents of the vague term ‘area of bad weather’ and the cells of this partition. In terms of spatial relations this means in the given case that all admissible candidates partially overlap the states of Wyoming, Montana, Utah, and Idaho and that they do not overlap any other state. Consequently, if a judging subject can specify for every partition cell a unique relation that holds for all admissible candidate referents of a vague term, this is a *determinate way to effect vague reference*. A meteorologist may use a finer approximation, which means that she will employ a finer-grained partition as frame of reference in order to make a more specific judgment about the current location of the bad weather region. Thus she might use cells labeled Eastern Idaho, Southern Montana, Western Wyoming, and Northern Utah, and so on. The latter yield a fiat boundary of the sort depicted in the right part of Figure 5.

Notice that all these boundaries existed already before the judgments which use them as frame of reference were made. They are only re-used in order to formulate constraints on the possible location of admissible candidates of the correspondingly vague referring term. Judging subjects re-use existing boundaries in this way in order to make determinate judgments about approximate locations. They do so because this is a convenient and determinate way to make vague reference, and it has even greater utility when the frame of reference is a commonly accepted one, as in the present case. It is important to see, again, that frames of reference are chosen in natural (normal) contexts in such a way that there is no truth-value indeterminacy in judgments effecting vague reference.

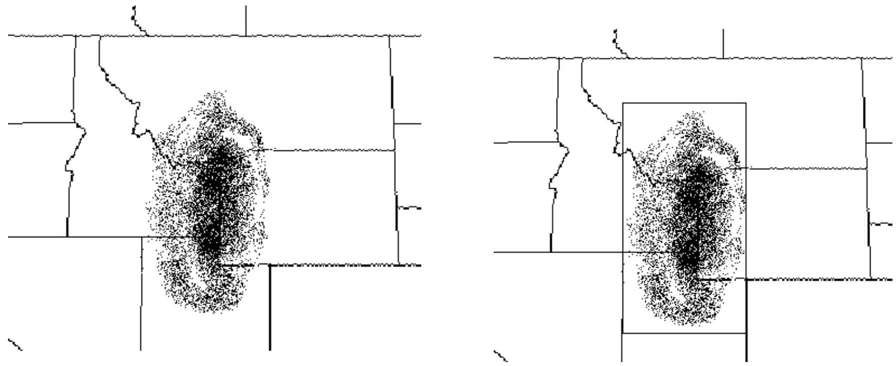

Figure 5: States of the United States with a bad weather system

Consider now the issue of higher-order vagueness, i.e., the question whether or not the boundaries re-used in order to delimit vague reference are subject to vagueness themselves. If the boundaries re-used by a judging subject are of the bona-fide sort – if, that is, they are boundaries in physical reality – then they are crisp by definition, at least at those levels of granularity pertinent to everyday human behavior.

A first examination of the example above shows that the boundaries in a frame of reference like that determined by the states of the United States, too, are crisp. One can see easily that there are many frames of reference that impose crisp boundaries in the same sense and that such boundaries are re-used in judgments by judging subjects in the way discussed above.

There are, however cases where the boundaries that are re-used are not crisp (i.e. where the boundaries in question are a multiplicity of crisp boundaries). Consider the judgment

[H] The path taken by the whales follows the Gulf Stream.

Obviously both ‘the path taken by the whales’ and ‘the Gulf Stream’ are vague terms, and no less obviously the vague reference of the latter constrains – if only vaguely – the range of admissible candidate referents of the former. But are there contexts in which [H] expresses a judgment whose truth-value is indeterminate? Again, we think not. Precisely because the judging subject avoids imposing crisp boundaries it is impossible to construct cases of truth-value indeterminacy, since once again: the requisite degree of precision is here not available for use on the level of semantic evaluation in the way required.

9.4 Imposing fiat boundaries

If the judging subject has the freedom to impose fiat boundaries in order to delimit the vagueness of reference of a certain term, then these delimiting boundaries may or may not be imposed upon reality crisply. Consider the approximation-based approach to representing vagueness underlying our treatment of sentence [F] ('We will cross the boundary of Mount Everest within the next hour'). In this case the judging subject has the authority to impose a (new, transient) boundary that limits the extent of the vagueness of the term 'Everest'. Consider the left part of Figure 4, depicting the boundary imposed by a subject who makes a judgment by using the sentence [F]. Since [F] is judged in a specific spato-temporal location, there is no vagueness in reference to the imposed fiat exterior boundary. The spatio-temporal location of the judging subject now is perfectly crisp.

In most normal cases however the end-boundary of the time-interval will not be taken to to impose a crisp boundary. Rather, in most contexts and for most judging subjects the intended meaning of [F] is better captured by the judgment:

> [I] We will cross the boundary of Mount Everest in the next hour *or so*.

This judgment is obviously subject to higher-order vagueness in the sense that there is a multitude of admissible candidate referents for 'in the next hour or so'. Yet the judgment has a determinate truth-value nonetheless. This is because, as we saw, the vagueness of the approximation does not allow us to construct cases that give raise to truth-value indeterminacy.

The higher-order vagueness presented in cases such as [I] is restricted in the sense that the degree of vagueness of the interval boundaries does not exceed the degree of vagueness of the boundaries of the underlying objects. For example, [I] cannot be used by a judging subject in order to project onto reality in such a way that among the admissible boundary-candidates there will be included some which can be crossed only after several hours of travel. This means that reference-frame-boundaries delimiting vagueness must be (much) crisper than the boundaries whose vagueness they delimit, i.e., the degree of vagueness of the reference-frame delimiting boundaries ('one hour or so') must be (significantly) less than the degree of vagueness of the object boundaries (in the case the boundaries of the mountain) to which reference is made. From this it follows that if we assume that in naturally occurring contexts the delimiting of vagueness of object boundaries which is effected by using crisp boundaries does not cause truth-value indeterminacy, then in those same contexts higher-order vagueness cannot cause truth-value indeterminacy either.

10. Judgments, vagueness, and approximation

Given that judging subjects are (in certain contexts) able to impose boundaries onto reality (to re-use boundaries already existing in reality) in order to delimit the vagueness of their acts of reference, then we need to define in a formal manner what this means from the perspective of the semantic theorist who employs a supervaluationist framework. In order to do so we first define the notion of *approximating judgments* of which a judgment like [F] ‘We will cross the boundary of Mount Everest within the next hour’, is a specific instance. Second, we discuss the relationship between the underlying supervaluationist semantic of vague names like ‘Everest’ and the vagueness-limiting boundaries imposed by approximating judgments like [F]. Thirdly, we define truth-conditions for approximating judgments.

10.1 Approximating judgments

Approximating judgments are judgments like [F] ‘We will cross the boundary of Mount Everest within the next hour’, [G] ‘We will cross the boundary of Mount Everest not earlier than 60 minutes and not later than 90 minutes from now’, [H] ‘The path taken by the whales follows the Gulf Stream’, [I]‘We will cross the boundary of Mount Everest in the next hour *or so*’.

Approximate judgments are a special class of judgments that contain vague names which contain in addition a crisp reference to boundaries that delimit this vagueness. Consider the approximating judgments [F] and [G]. These judgments are vague because they contain the vague name ‘Everest’. They are also approximating since they contain the reference to boundaries delimiting the vagueness of the name ‘Everest’ by referring to boundaries that delimit admissible candidates of reference for ‘Everest’: ‘[having a boundary that will be crossed] not earlier than 60 minutes and not later than 90 minutes from now’ in [G], or ‘within the next hour’ (i.e., between now and 60 minutes from now) in [F]. In this paper we consider approximating judgments which contain a single vague name and some reference to boundaries delimiting the vagueness of this name. More complex cases are possible – for example cases where the reference frame itself involves a certain degree of vagueness – but consideration of these is omitted here since their treatment follows the same basic pattern.

From the perspective of the partition theorist an approximating judgment J^A, if uttered successfully, imposes *two* partitions onto reality: a vague partition Pt^V representing the supervalutationist semantics of the vague name N and a *reference partition* Pt^R which delimits the vagueness of reference of N.

We say that an *approximating judgment* is a triple $J^A{}_1 = (S, Pt^V, Pt^R)$, consisting of a sentence, S, together with two granular partitions, Pt^V and Pt^R. The context of an approximating judgment is represented by the two partitions taken together.

Consider, for example, the approximating judgment $J^A{}_1 = ([F], Pt^V, Pt^R)$, with the sentence [F] containing the vague name ‘Everest’ and the constraint on admissible boundaries of admissible candidates of ‘Everest’ being expressed by ‘[having a boundary that will be crossed] in the next hour’. The vague name ‘Everest’ imposes a vague partition Pt^V with a corresponding cell labeled ‘Everest’ projecting onto the multiplicity of admissible candidate referents of the name ‘Everest’, as indicated in Figure 3.

The judgment [F] (through its ‘in the next hour’ part) also imposes a *reference partition* Pt^R onto reality as depicted in the left part of Figure 4. Intuitively, this reference partition projects onto reality in a way that constrains admissible candidate referents for ‘Everest’, i.e., it constrains admissible projections of the cell ‘Everest’ in Pt^V. The latter must be such that they will be crossed by the judging subject between now and in one hour.

Another example of an approximating judgment is J^A_2 = ([K], Pt^V, Pt^R) with the sentence:

> [K] The area of bad weather extends over parts of Wyoming, parts of Montana, parts of Utah, and parts of Idaho.

The corresponding vague partition Pt^V contains a cell labeled ‘the area of bad weather’ projecting onto a multiplicity of admissible candidates. The judgment J^A_2 reuses the partition depicted in the right part of Figure 5 as reference partition P^R. The latter constrains admissible projections of the cell labeled ‘the area of bad weather’ in Pt^V in such a way that each candidate of reference that is targeted by a projection P_i of Pt^V must extend over parts of reality targeted by the cells labeled ‘Wyoming’, ‘Utah’, ‘Montana’, and ‘Idaho’ in Pt^R. Moreover Pt^R *implicitly* constrains admissible projections of the cell ‘the area of bad weather’ in such a way that *no* candidate of reference targeted by a projection P_i of Pt^V can extend over parts of reality targeted by cells of Pt^R with labels not mentioned in [K].

We now continue by giving a formal partition-theoretic definition of reference partitions of the sort described.

10.2 Partition theory and approximation

The idea behind the theory of approximation is that a (crisp) granular partition can be used as a *frame of reference* (a generalized coordinate frame (Bittner 1997)), which allows us to describe the *approximate location* of objects. The partition-theoretic notion of approximation is closely related to the notion of approximation defined using rough sets (Pawlak 1982). Consider a vague name such as ‘Mount Everest’ and the corresponding multiplicity of admissible candidates formed by crisp portions of reality in the domain of the Himalayas. Consider another crisp partition structuring this same domain but without recognizing any of the candidates referred to by the name ‘Everest’ directly. This might be a partition working with the boundaries of India, Tibet and Nepal and their constituent provinces, or a partition formed by a raster of cells aligned to lines of latitude and longitude (as in the right part of Figure 6). Such a reference partition has the power to recognize all these admissible candidates *indirectly*, i.e., without explicitly projecting onto them.

To see how this works, we introduce the three concepts of full overlap *(fo)*, partial overlap *(po)*, and non-overlap *(no)*, concepts which we shall use to generalize the notions of projection and location, as follows. Let o be an object that is not directly recognized by a given partition and let x be an object that is located at the cell z of our partition (the cell z projects onto x). x is, in the cases mentioned, a region of space on the surface of the Earth. The constants *fo*, *po*, *no* will now be used to measure the degree of coverage of the object x by the object o.

We call the relation $L^R(o, z, \omega)$ the *rough location* of the object o with respect to the cell z and the relation $P^R(z, o, \omega)$ the *rough projection* of the cell z onto o. In both relations the value ω characterizes the degree of overlap of the object targeted by the cell z with the actual object o, i.e., it takes the value *fo*, *po*, or *no*. Consider the left part of Figure 5. There the relation *po* holds between all admissible candidate referents (BWA_i) of 'the area of bad weather' and Montana, i.e., $\forall i$: $L^R(BWA_i, \text{Montana}, po)$. The relation *no* holds between all BWA_i and Oregon, i.e., $\forall i$: $L^R(BWA_i, \text{Oregon}, no)$.

We can characterize the relationships between exact and rough location and exact and rough projection as follows:

$$L^R(o, z, fo) \equiv \forall x: L(x, z) \rightarrow x \leq o$$
$$P^R(z, o, fo) \equiv \forall x: P(z, x) \rightarrow x \leq o$$
$$L^R(o, z, po) \equiv \forall x: L(x, z) \rightarrow \exists y: y < x \text{ and } y \leq o$$
$$P^R(z, o, po) \equiv \forall x: P(z, x) \rightarrow \exists y: y < x \text{ and } y \leq o$$
$$L^R(o, z, no) \equiv \forall x: L(x, z) \rightarrow \neg\exists y: y \leq x \text{ and } y \leq o$$
$$P^R(z, o, no) \equiv \forall x: P(z, z) \rightarrow \neg\exists y: y \leq x \text{ and } y \leq o$$

The notion of rough location gives rise to an equivalence relation in the domain of objects with respect to a given reference partition Pt, as follows:

$$o_1 \approx_R o_2 \equiv \forall z: L^R(z, o_1, \omega) \leftrightarrow L^R(z, o_2, \omega).$$

This can be interpreted as meaning that two objects are equivalent with respect to the granular partition Pt^R if and only if they have an identical rough location with respect to all cells of this partition. $\approx_R$ can thus be interpreted as meaning indiscernibility with respect to the frame of reference provided by the partition Pt^R.

Consider the left part of Figure 6, which shows the approximation, by means of a rectangular frame with two cells, of one transparent ellipsoid region and one gray colored region consisting of two separate parts. With respect to the underlying crisp partition indicated by the two raster cells, both regions are equivalent: $L^R(z_1, o_1, po)$, $L^R(z_1, o_2, po)$, $L^R(z_2, o_1, po)$, $L^R(z_2, o_2, po)$. In order to take into account the topological properties of the approximated objects one can either use more sophisticated methods of approximation (Bittner and Stell 2000), or one can take into account the topological structure of the approximated objects (Randell, Cui et al. 1992). In either cases one needs as underlying theory a mereotopology (Varzi 1994) rather than pure mereology. Both alternatives go beyond the scope of this paper.

The equivalence relation $\approx_R$ induces equivalence classes of indiscernible objects relative to Pt^R. We write $[o_1]$ in order to denote the class of objects that are $\approx_R$-equivalent to o_1. Below we will show that approximating judgments like ([F], Pt^V, Pt^R) project onto reality in such a way that all admissible candidate referents (e.g., $p^V{}_i$('Everest')) are equivalent with respect to some $\approx_R$ (e.g., the partition Pt^R imposed by '[such that its border can be crossed] in the next hour'). This means that all admissible candidates of reference have identical relations to the portions of reality onto which the cells of the reference partition project. In this way all admissible candidates can be recognized in determinate fashion, albeit indirectly.

We define a *reference partition* as a fourtuple, $Pt^R = (A, P^R, L^R, \Omega)$ where A is a cell structure, P^R and L^R are a rough projection and location relations, and Ω is the set of values (*fo*, *po*, *no*), indicating the degrees of overlap distinguished. The equivalence relation $\approx_R$ is given indirectly by L^R and Ω. We demand that the following counterparts of $MB1^R$–4^R hold for reference partitions:

$MB1^R$ $\quad L^R(o, z, \omega) \to P^R(z, o, \omega)$
$MB2^R$ $\quad P^R(z, o, \omega) \to L^R(o, z, \omega)$
$MB3^R$ $\quad \forall o{:}\ (L^R(o, z_1, \omega) \text{ and } L^R(o, z_2, \omega)) \to z_1 = z_2$
$MB4^R$ $\quad \forall z{:}\ (P^R(z, o_1, \omega) \text{ and } P^R(z, o_2, \omega)) \to o_1 \approx_{Pt} o_2$

$MB3^R$ tells us that if two cells are such that all objects cover the targets of these cells in the same way then these cells must be identical. $MB4^R$ tells us that if two objects have the same relations (*fo*, *po*, *no*) to all cells of a granular partition Pt then the two objects are indiscernible with respect to this partition.

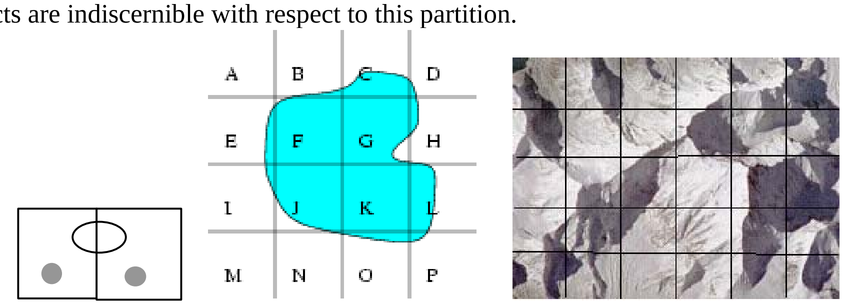

Figure 6: Rough approximation

Every reference partition Pt^R has a *crisp skeleton* $Pt^S = (A, P^S, L^S)$. Where Pt^S is a crisp granular partition with the following properties: (1) A is identical to the cell structure in Pt^R; (2) P^S is the restriction of P^R to triples of the form (z, o, *fo*) for objects o that are recognized directly by single cells z in A; (3) L^S is the restriction of L^R to triples of the form (o, z, *fo*) for objects o that are recognized directly by single cells z in A; (4) Pt^S satisfies MA1–4, MB1–6^+. It is the crisp skeleton of the reference partition which actually establishes the frame of reference for the approximation, i.e., projects onto the boundaries that delimit vagueness.

Consider the judgment ‘The area of bad weather extends over parts of Wyoming, parts of Montana, parts of Utah, and parts of Idaho’ and the corresponding formal structure $J^A = ([K], Pt^V, Pt^R)$. The crisp skeleton of the reference partition Pt^R is the partition Pt^S which recognizes the federal states of the US (Figure 5) and establishes the frame of reference for the approximation. Consider Figure 4. The crisp skeletons of the reference partitions contain the cells ‘core’, ‘exterior’, and ‘where-the-boundary-candidates-are’ (this cell structure is shared in both cases with the reference partition Pt^R). The *crisp* projection (Pt^S) of these cells establishes the (interior and exterior) boundaries which limit the vagueness as discussed above. In both cases the reference partition (Pt^R), on the other hand, indirectly recognizes the admissible candidate referents. Consequently, the correspondence between the cell structure A in reference partitions, Pt^R, and reality is completely determined by the crisp projection and location relations of the underlying crisp skeleton Pt^S. The reference partition is built upon the crisp skeleton, i.e., the frame of reference, provided by Pt^S.

10.3 Truth for approximating judgments

Approximate judgments J^A are defined by a sentence S containing a vague name N and by a frame of reference delimiting the boundary-candidates of the referent of N. This results in a structure $J^A = (S, Pt^V, Pt^R)$ where Pt^V represents the vagueness of the vague name N and Pt^R represents the frame of reference. In order to be *true* the partitions Pt^V and Pt^R must stand in a particular, well-defined relationship to each other.

Truth conditions are necessary and sufficient conditions that need to be satisfied for a judgment to be true. The definition of truth conditions for approximating judgments is based on the truth conditions for granular partitions and vague partitions as defined above. If an approximating judgment has been made (and this means: made in such a way that its truth can at least in principle be evaluated by speaker and audience), then this means that appropriate partitions Pt^V and Pt^R have been successfully invoked by the judging subjects involved. These can now be used on the level of semantic evaluation to define truth conditions for the corresponding judgment.

Consider again the approximating judgment $J^A = ([K], Pt^V, Pt^R)$ with [K] = 'The area of bad weather extends over parts of Wyoming, …'. Let $Pt^{\mathbf{V}}$ be a vague partition $Pt^V = (A, P^V, L^V)$, representing the vague reference of the name 'the area of bad weather,' and let Pt^R be the reference partition, $Pt^R = (A', P^R, L^R, \Omega)$, depicted in the right part of Figure 5. Both partitions have roughly the same domain, but the cell structures A and A′ are completely distinct. We say that Pt^R *approximates* the vague reference of the name 'the area of bad weather' if and only if all admissible candidate referents, i.e., all portions of reality carved out by the multiplicity of projections of the cell labeled 'the area of bad weather' in Pt^V are equivalent with respect to $\approx_R$.

More generally we can consider approximating judgments such as [K], [G], [F] as instances of the general form:

$$J^A = (S = \text{'N stands in relation } R_1 \text{ to } N_1, \ldots, R_n \text{ to } N_n\text{'}, Pt^V, Pt^R).$$

Consider judgment [K]: here 'N' translates to 'the area of bad weather', R_1 translates to *po* (extends over parts of), N_1 translates to Wyoming, and so on. In the case of judgment [F] 'N' translates to 'Everest', N_1 translates to the path the judging subject takes 'in the next hour', and R_1 translates to *po*.

In general we say that N is a vague name, $N_1, \ldots, N_n$ are crisp names, and the R_i range over the relationships *fo*, *po*, *no*. The approximate judgment of the form J^A has a reference partition Pt^R, which *approximates* the multiplicity of admissible referents of the vague name N in S in accordance with the vague partition Pt^V, if and only if the following holds:

I. the labeling λ^R maps the names $N_1, \dots, N_n$ onto cells $z_1, \dots, z_n$ in Pt^R, i.e., $(\lambda^R(N_i)) = z_i$
   the labeling λ^V maps the name N onto the cell z in Pt^V, i.e., $N = (\lambda^{V\,-1})(z)$

II. the relations $R_i(p^V(z), p^R(z_i))$ hold.

The approximating judgment $J^A = (S, Pt^V, Pt^R)$ is then *true* if and only if

(a) Pt^R approximates the candidate referents onto which the cell $\lambda^V(N)$ vaguely projects;
(b) Pt^V represents its domain in such a way that MA1–4 and $MB1^V$–6^{V+} hold;
(c) Pt^R represents its domain in such a way that MA1–4 and $MB1^R$–6^{R+} hold.

From this it follows that in a true approximating judgment $J^A = (S, Pt^V, Pt^R)$ all admissible candidate referents of the vague name N in S are equivalent with respect to $\approx_R$.

Consequently, the judgment $J^A = ([F], Pt^V, Pt^R)$ is true if and only if all boundary candidates are crossed within the specified interval. However the underlying notion of approximation allows the judging subject to choose the interval generously enough to make sure that the boundaries of all admissible candidates are crossed. Notice, however, that the judging subject does not have the freedom to make the interior boundary arbitrarily small and the exterior boundary arbitrarily large as discussed in the section on the relationship between vagueness and approximation above. Consider the judgment $J^A = ([G], Pt^V, Pt^R)$. One can easily verify that this judgment comes out true in terms of the truth conditions specified.

It is important to stress, again, that Pt^V and Pt^R are distinct partitions with different cell structures, projections and so on. Notice that choosing a raster that is as fine as possible for the approximating partition will not necessarily yield a better approximation, since the cells forming the boundary approximation must partially overlap the union of the boundary locations of the vague partition. However in most cases it will be not be too hard to find a partition that absorbs vagueness in the sense discussed here. Obviously, good approximations are harder to find than very coarse approximations.

11 Judging subject and approximation

When analyzing approximating judgments the partition theorist needs to go beyond purely semantic considerations. He needs to take the perspective of the judging subject into account. In this section we discuss the relationship between judging subjects and the reference partitions they utilize. We start by discussing properties of reference partitions and their use as frames of reference. Secondly, when uttering an approximating judgment judging subjects often have the choice between alternative reference partitions, some coarser, some finer, and this represents a new variant of the ontological zooming mentioned above. The choice of a particular reference partition determines its usability as a frame of reference and the *precision* of the resulting approximation. We will show below that judging subjects often utter approximate judgments that approximate as *precisely as necessary* rather than as precisely as possible. Thirdly, we shall see that judging subjects have more effective means to determine the truth of vague judgments than would be involved in passing through a potentially infinite number of candidate referents in a supervaluationist fashion.

11.1 Reference partitions

The types of granular partitions that are used as frames of reference, i.e., the crisp skeleton of the reference partition, characteristically have the following properties: (1) they are relatively stable, i.e., they do not change over time (we can also demand that they are specifiable in some easily communicable way); (2) they are mereologically monotonic; (3) they do not contain empty cells; (4) the mereological sum of the relevant minimal cells is identical to the root cell; and (5) the mereological sum of the targets of all partition cells is identical to the domain of the partition. Reference partitions are often spatial or temporal in nature.

The first characteristic property of a reference partition is that it is relatively static with respect to the objects to which it relates. This means that (a) the cell structure is fixed and that (b) the objects onto which it projects do not change (they are, for example, spatial regions tied to the surface of the Earth). Consider the examples in Figure 5. The granular partition projecting onto the United States has existed for more than one hundred years without significant changes, where each area of bad weather changes continuously throughout the course of its (brief) existence. In fact Figure 5 itself needs to be considered as a snapshot. (Smith and Brogaard, to appear) Due to the relative stability of the reference partition in a case such as this, it provides useful information to say that the area of bad weather was located in parts of Montana, Idaho, Wyoming, and Utah at such and such a time. Every child learns this reference partition at school, and it is used for all sorts of purposes thereafter (Stevens and Coupe 1978).

Secondly, the crisp skeleton of reference partitions must be mereologically monotonic ($MB6^+$) in order to assure that the mereological structure of the underlying domain is preserved.

Thirdly, reference partitions do not have empty cells, i.e., every cell projects onto some object. Fourth, reference partitions do not contain 'empty space' (in contrast, say, to the periodic table, which contains cells kept in reserve to project onto elements yet to be discovered. This means that the mereological sum of all minimal cells sums up the root cell. Formally we write:

$$\text{CF} \qquad \exists z, z_1,\ldots,z_n\text{: Max}(z)\ z = +_{\text{Min}(z_i)}\ z_i\ ,$$

where the predicates Max and Min hold of the root and minimal cells, respectively, of our reference partition and + is the operation of taking mereological sums. (See (Bittner and Smith 2001) for details.)

Fifth, reference partitions are such that the mereological sum of all objects located at minimal cells is identical to that onto which the root cell projects. Formally we write:

$$\text{CE} \qquad \exists z, z_1, \ldots, z_n\text{: Max}(z) \text{ and } p(z) = +_{\text{Min}(z_i)}\, p(z_i)\ ,$$

where p(z) returns the object onto which z is projected.

Conditions (3)-(5) ensure that crisp reference partitions are full, exhaustive, and complete in the sense of (Smith and Bittner 2001). In this class there fall all spatial partitions with minimal cells which sum together to exhaust a certain space. Important groups of reference partitions are partitions imposed by quantities of all kinds (Johansson 1989, chapter 4); temporal partitions like calendars (Bittner to appear), and spatial partitions like political subdivisions or the subdivision of the Earth into cells bounded by lines or latitude and longitude.

11.2 Precision of approximation

When analyzing approximating judgments it is often not sufficient to determine only truth or falsehood but one has to evaluate also the precision involved. We start by taking the perspective of the semantic theorist and classify portions of reality in terms their relationships to admissible candidate referents of a vague name. Afterwards we compare these partitions with reference partitions imposed by judging subjects.

Consider the vague projection of the cell 'Everest' as depicted in Figure 3. From the perspective of the semantic theorist, we need to take into account the corresponding unity condition, which would look like this:

U5 (1) The summit is part of Mount Everest; (2) if x is part of Mount Everest and y is connected to x then y is part of Mount Everest if and only if y is part of the projection P_i of the cell 'Everest'.

For each cell z in a vague partition Pt^V we can now classify corresponding portions of reality into three classes: core parts with respect to z, boundary parts with respect to z, and exterior parts with respect to z.

We say that x is a *core part* of the object onto which the cell *z* projects if and only if, under *all* projections P in P^V, x is a part of all admissible candidate referents of the vague name corresponding to z. In symbols:

$$\text{core}_V(x, z) \equiv \forall p \in P^V\text{: } x \leq p(z)$$

Consider, for example, the cell labeled 'Everest' in Pt^V. There are portions of reality that are parts of Mount Everest under all projections in Pt_V, for example, the summit. Thus core_V(summit, 'Everest').

x is a *boundary part* of the object onto which the cell *z* projects if and only if there are some projections in P^V under which x satisfies the associated unity condition and there are other projections in P^V under which x fails to satisfy the associated unity condition:

$$\text{boundary}_V(x, z) \equiv \exists p \in P^V: x \leq p(z) \text{ and } \exists p \in P^V: \neg(x \leq p(z))$$

For example, there are portions of reality that are parts of Mount Everest under some projections but not under others.

x is an *exterior part* of the object onto which the cell *z* projects if and only if x is not reached by any projection of the cell *z*.

$$\text{exterior}_V(x, z) \equiv \forall p \in P^V: \neg(x \leq p(z))$$

For example, there are parts of reality – such as Berlin – that are not reached by any of the projections of the cell 'Everest' in any natural context.

Consequently, from the semantic perspective every vague name creates a partition of reality into core, boundary, and exterior parts. We define the *semantic partitioning* of reality with respect to a vague name N to be a partitioning of this sort generated in reflection of the vagueness of N.

In order to evaluate the precision of a true approximating judgment, we now need to consider the *deviation* of the location of two pairs of boundaries: (a) the interior and exterior boundaries imposed by the judging subject as a frame of reference for her approximation; and (b) the boundaries imposed by the semantic partitioning.

Consider, again, the approximating judgment $J^A = ([F], Pt^V, Pt^R)$. The reference partition Pt^R (shown in the left part of Figure 4) is a special instance of the class of reference partitions that impose two fiat boundaries onto reality: an interior boundary of the approximation and an exterior boundary of the approximation. As discussed above, this often results in a partition structure similar to the one depicted in the right part of Figure 4. The (geometric) projection of this partition onto the path the judging subject takes on her journey towards the summit of Mount Everest results in the reference partition shown in the left part of Figure 4.

In more complex reference partitions, like the political subdivision of the US used in judgment [K] (Figure 5), the interior boundary of the approximation coincides with the boundary of the mereological sum of targets of partition cells for which the relation *fo* holds (e.g., the boundary of the cell K in the middle of Figure 6). The exterior boundary of the approximation coincides with the boundary of the mereological sum of targets of partition cells for which the relation *po* holds (e.g., the outer boundary of the mereological sum of the cells [A,…, P] minus the cells A and M in the middle of the figure). This corresponds to the boundaries of the mereological sums of the lower and upper approximations in the sense of rough set theory (Pawlak 1982).

We say that an approximation is *precise* for the name N if and only if (1) the interior boundary of the approximation coincides with the boundary separating the core from the surrounding parts of the semantic partitioning; and (2) the exterior boundary of the approximation coincides with the boundary separating exterior parts from what lies within. A true approximating judgment is otherwise imprecise.

Due to the nature of vagueness most approximations will be imprecise. Moreover, as discussed above in our section on degrees of vagueness and crispness, judging subjects will utter approximating judgments which are *as precise as necessary* in the given context. This means that the limits imposed on vagueness will be such that, for the class of objects in the foreground of the judger's attentions, only judgments with determinate truth-values can be made, for example judgments about the tables or ashtrays relative to the smoking zone as discussed in our example [C].

11.3 Determining the truth of judgments

The partition theorist holds that judging subjects use approximations in order to formulate judgments which truth-value can be determined more easily than going through all the (potentially infinite) candidates of vague reference in a supervaluationist fashion. To see this, consider the vague reference of the name 'Everest'. Within a certain range there are arbitrarily many, indeed infinitely many, admissible candidates of reference. When making a judgment about Mount Everest a finite being like a human judging subject cannot go through all these infinite candidate referents in order to determine the truth-value of the judgment in a supervaluationist fashion. Consequently more effective cognitive means need to be at work.

In order to see how approximations fit into the picture, consider the definition of the truth of an approximating judgment: An approximating judgment $J^A = (S, Pt^V, Pt^R)$ with the vague name N is S is true if and only if the partitions Pt^V and Pt^R project in a structure preserving manner onto reality (MA1–4, $MB1^V$–6^V and $MB1^R$–4^R hold) and all admissible candidates of reference of N (projected onto reality by Pt^V) are equivalent with respect to the approximation dictated by Pt^R.

An important point is that the above definition does not specify that the approximation is precise. On the other hand, the conceptual structure underlying vague names such as 'Everest' or 'area of bad weather' is such that it provides a (vague) unity condition which ensures that admissible candidate referents are *not* arbitrarily scattered mereological wholes. These two aspects give the judging subject the power to place vagueness-delimiting boundaries in such a way that *she does not need to go through all admissible candidates* in order to determine the truth of an approximating judgment. All the judging subject has to check is whether or not the judgment is true regarding the limiting boundaries, and she has the freedom to choose a degree of precision *most convenient* for her purposes. Thus she can place the boundaries in 'safe' places, i.e., in places that are such that the truth or falsehood of the judgment can be determined easily for these boundaries and without indeterminacy.

Consider the approximating judgment [G]: 'We will cross the boundary of Mount Everest in the next hour'. [G] can be uttered seriously only in contexts where the judging subject has means to determine that the boundary to Mount Everest has not yet been crossed. After *one hour* the judging subject checks again and will come to the result that the judgment was either true or false. During this hour the judging subject might have checked occasionally whether or not the boundary has been crossed already, but she certainly does not go through infinitely many candidates of reference while heading uphill.

12. Conclusions

In this paper we proposed an application of the theory of granular partitions to the phenomenon of vagueness, a phenomenon which is itself seen as a semantic property of names and predicates. We defended a supervaluationistic theory of the underlying semantics, but we argued that it is insufficient to consider the vagueness of names and predicates in a context-free fashion. Rather vague names and predicates must be evaluated as they appear within judgments actually made in natural contexts. We then argued that judgments add context to sentences in a way that helps to resolve the dilemma posed by vagueness. Note that this does not mean that vagueness is somehow eliminated. Vague names and predicates are still as vague as they always were. Rather, we showed that the framework of granular partitions can provide the framework for understanding how, in real-world contexts, judgments with indeterminate truth-values are systematically avoided. We also showed that the use of frames of reference in making approximating judgments can be formulated very naturally in partition-theoretic terms, and that the framework of granular partitions then helps us to understand the relationships between vagueness and approximation.

Acknowledgements

This work was supported in part by DARPA under the Command Post of the Future program and by the National Science Foundation under the Research on Learning and Education program and under Research Grant BCS-9975557: “Geographic Categories: An Ontological Investigation”).

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