Define deviations in signals between iterations:

$$\delta x_j = x_{j+1} - x_j \tag{1}$$
$$\delta x_j^0 = x_{j+1}^0 - x_j^0 \tag{2}$$
$$\delta u_j = u_{j+1} - u_j \tag{3}$$
$$\delta e_j = e_{j+1} - e_j = r - x_{j+1} - e_j \tag{4}$$
$$\tag{5}$$

The expression for the lifted model is:

$$x_{j+1} = x_j + G_j \delta u_j + F_j \delta x_j^0 \tag{6}$$
$$= x_j + G_j \left(u_{j+1} - u_j\right) + F_j \left(x_{j+1}^0 - x_j^0\right) \tag{7}$$
$$= x_j + G_j u_{j+1} - G_j u_j + F_j x_{j+1}^0 - F_j x_j^0 \tag{8}$$
$$\tag{9}$$

Next, define two matrices that "pick off" the state vector from a lifted state vector at the first and last path step, E_I, and E_F. Because the last state vector from the previous iteration is the first state vector of the next iteration, we can then write the following relationships:

$$x_j^0 = E_I x_j \tag{10}$$
$$x_{j+1}^0 = E_F x_j \tag{11}$$

Using these to simplify the lifted model then gives

$$x_{j+1} = x_j + G_j u_{j+1} - G_j u_j + F_j E_F x_j - F_j E_I x_j \tag{12}$$
$$= \left(\mathbb{I} + F_j \left(E_F - E_I\right)\right) x_j + G_j u_{j+1} - G_j u_j \tag{13}$$
$$= \left(\mathbb{I} - F_j \Delta\right) x_j + G_j u_{j+1} - G_j u_j \tag{14}$$
$$\tag{15}$$

where $\Delta \triangleq E_I - E_F$.

Error at the next iteration based on the lifted model:

$$e_{j+1} = r - x_{j+1} = r - \left(\left(\mathbb{I} - F_j \Delta\right) x_j + G_j u_{j+1} - G_j u_j\right) \tag{16}$$
$$= r - \left(\mathbb{I} - F_j \Delta\right) x_j - G_j u_{j+1} + G_j u_j \tag{17}$$
$$\tag{18}$$

Write performance in terms of the control input sequence u_{j+1}

$$J_{j+1} = u_{j+1}^T Q_u u_{j+1} + \delta u_j^T Q_{\delta u} \delta u_j + e_{j+1}^T Q_e e_{j+1} + \delta e_j^T Q_{\delta e} \delta e_j + \delta x_j^T Q_{\delta x} \delta x_j - S_x x_{j+1} \tag{19}$$
$$= u_{j+1}^T Q_u u_{j+1} + \left(u_{j+1} - u_j\right)^T Q_{\delta u} \left(u_{j+1} - u_j\right) \tag{20}$$
$$+ \left(r - \left(\mathbb{I} - F_j \Delta\right) x_j - G_j u_{j+1} + G_j u_j\right)^T Q_e \left(r - \left(\mathbb{I} - F_j \Delta\right) x_j - G_j u_{j+1} + G_j u_j\right) + \left(r - \left(\mathbb{I} - F_j \Delta\right) x_j - G_j u_{j+1} + G_j u_j - e_j\right)^T Q_{\delta e} \left(r - \left(\mathbb{I} - F_j \Delta\right) x_j - G_j u_{j+1} + G_j u_j - e_j\right) \tag{21}$$
$$+ \left(\left(\mathbb{I} - F_j \Delta\right) x_j + G_j u_{j+1} - G_j u_j - x_j\right)^T Q_{\delta x} \left(\left(\mathbb{I} - F_j \Delta\right) x_j + G_j u_{j+1} - G_j u_j - x_j\right) - S_x \left(\left(\mathbb{I} - F_j \Delta\right) x_j + G_j u_{j+1} - G_j u_j\right) \tag{22}$$
$$\tag{23}$$

Noting that

$$r - \left(\mathbb{I} - F_j \Delta\right) x_j - G_j u_{j+1} + G_j u_j = e_j + F_j \Delta x_j - G_j u_{j+1} + G_j u_j \tag{24}$$
$$r - \left(\mathbb{I} - F_j \Delta\right) x_j - G_j u_{j+1} + G_j u_j - e_j = F_j \Delta x_j - G_j u_{j+1} + G_j u_j \tag{25}$$
$$\left(\mathbb{I} - F_j \Delta\right) x_j + G_j u_{j+1} - G_j u_j - x_j = -F_j \Delta x_j + G_j u_{j+1} - G_j u_j \tag{26}$$

We can simplify this expression for J_{j+1}:

$$
\begin{aligned}
J_{j+1} = & u_{j+1}^T Q_u u_{j+1} + (u_{j+1} - u_j)^T Q_{\delta u} (u_{j+1} - u_j) && (27) \\
& + (e_j + F_j \Delta x_j - G_j u_{j+1} + G_j u_j)^T Q_e (e_j + F_j \Delta x_j - G_j u_{j+1} + G_j u_j) + (F_j \Delta x_j - G_j u_{j+1} + G_j u_j)^T Q_{\delta e} (F_j \Delta x_j - G_j u_{j+1} + G_j u_j) && (28) \\
& + (-F_j \Delta x_j + G_j u_{j+1} - G_j u_j)^T Q_{\delta x} (-F_j \Delta x_j + G_j u_{j+1} - G_j u_j) - S_x ((\mathbb{I} - F_j \Delta) x_j + G_j u_{j+1} - G_j u_j) && (29) \\
& && (30)
\end{aligned}
$$

Now, differentiate this expression with respect to the next control input sequence, u_{j+1} note that all Q's are symmetric:

$$
\begin{aligned}
\frac{dJ_{j+1}}{du_{j+1}} = & 2u_{j+1}^T Q_u + 2(u_{j+1} - u_j)^T Q_{\delta u} && (31) \\
& - 2(e_j + F_j \Delta x_j - G_j u_{j+1} + G_j u_j)^T Q_e G_j - 2(F_j \Delta x_j - G_j u_{j+1} + G_j u_j)^T Q_{\delta e} G_j && (32) \\
& + 2(-F_j \Delta x_j + G_j u_{j+1} - G_j u_j)^T Q_{\delta x} G_j - S_x G_j && (33)
\end{aligned}
$$

Gather all the u_{j+1} and u_j terms:

$$
\begin{aligned}
\frac{dJ_{j+1}}{du_{j+1}} = & 2u_{j+1}^T \left(Q_u + Q_{\delta u} + G_j^T (Q_e + Q_{\delta_e} + Q_{\delta x}) G_j\right) - 2u_j^T \left(Q_{\delta u} + G_j^T (Q_e + Q_{\delta_e} + Q_{\delta x}) G_j\right) && (34) \\
& - 2(e_j + F_j \Delta x_j)^T Q_e G_j - 2(F_j \Delta x_j)^T Q_{\delta e} G_j && (35) \\
& - 2(F_j \Delta x_j)^T Q_{\delta x} G_j - S_x G_j && (36)
\end{aligned}
$$

Gather the x_j terms

$$
\frac{dJ_{j+1}}{du_{j+1}} = 2u_{j+1}^T \left(Q_u + Q_{\delta u} + G_j^T (Q_e + Q_{\delta_e} + Q_{\delta x}) G_j\right) - 2u_j^T \left(Q_{\delta u} + G_j^T (Q_e + Q_{\delta_e} + Q_{\delta x}) G_j\right) - 2e_j^T Q_e G_j - 2x_j^T \Delta^T F_j^T (Q_e + Q_{\delta e} + Q_{\delta x}) G_j - S_x G_j \tag{37}
$$

Set this equal to zero, divide through by 2 and re-arrange so that the u_{j+1} term is on the left side and the others are on the right:

$$
u_{j+1}^T \left(Q_u + Q_{\delta u} + G_j^T (Q_e + Q_{\delta_e} + Q_{\delta x}) G_j\right) = u_j^T \left(Q_{\delta u} + G_j^T (Q_e + Q_{\delta_e} + Q_{\delta x}) G_j\right) + e_j^T Q_e G_j + x_j^T \Delta^T F_j^T (Q_e + Q_{\delta e} + Q_{\delta x}) G_j + \frac{1}{2} S_x G_j \tag{38}
$$

Transpose both sides, note that all Q's are symmetric and in general for any matrix A and a symmetric B, $A^T BA$ is symmetric:

$$
\left(Q_u + Q_{\delta u} + G_j^T (Q_e + Q_{\delta_e} + Q_{\delta x}) G_j\right) u_{j+1} = \left(Q_{\delta u} + G_j^T (Q_e + Q_{\delta_e} + Q_{\delta x}) G_j\right) u_j + G_j^T Q_e e_j + G_j^T (Q_e + Q_{\delta e} + Q_{\delta x}) F_j \Delta x_j + \frac{1}{2} G_j^T S_x^T \tag{39}
$$

So then the update law is:

$$
u_{j+1} = L_u u_j + L_e e_j + L_x x_j + L_c \tag{40}
$$

Where

$$
\begin{aligned}
L_0 & \triangleq \left(Q_u + Q_{\delta u} + G_j^T (Q_e + Q_{\delta_e} + Q_{\delta x}) G_j\right)^{-1} && (41) \\
L_u & \triangleq L_0 \left(Q_{\delta u} + G_j^T (Q_e + Q_{\delta_e} + Q_{\delta x}) G_j\right) && (42) \\
L_e & \triangleq L_0 G_j^T Q_e && (43) \\
L_x & \triangleq L_0 G_j^T (Q_e + Q_{\delta e} + Q_{\delta x}) F_j \Delta && (44) \\
L_c & \triangleq \frac{1}{2} L_0 G_j^T S_x^T && (45)
\end{aligned}
$$

A couple thoughts/sanity checks:

- if Q_u is a matrix of zeros then L_u reduces to the identity matrix, which is what I had before (first page)
- the L_x term now includes the final/initial condition deviation