1 Introduction

The goal of this document is to derive an ILC update law that inherently considers the impact of the next iteration, $j+1$ on the iteration after that, $j+2$. The process will be to similar to previous work in that we will start by forming a quadratic performance index, re-write that performance index in terms of only the design variable (sequence of control inputs), differentiate that performance index with respect to the design variable, set the result equal to the zero vector, then solve or rearrange to obtain an update law. This performance index will then include terms that depend on u_j, u_{j+1}, u_{j+2}, x_j, x_{j+1}, x_{j+2}, e_j, e_{j+1} an e_{j+2}.

2 Preliminaries

In this work iteration j is the one which was just completed, iteration $j+1$ is the iteration that is about to occur but has not yet begun. Also, assume the path is discretized into n_s steps, we have n_x state variables, and n_u control inputs.

2.1 Lifted System Model

Assume we are given the following a path-domain lifted system model of the form

$$\delta x_j = H_j \delta u_j + F_j \delta x_j^0 \tag{1}$$

where $\delta x_j \triangleq x_{j+1} - x_j$, $\delta u_j \triangleq u_{j+1} - u_j$ and $\delta x_j^0 \triangleq x_{j+1}^0 - x_j^0$. Here, x_{j+1}^0 and x_j^0 are the initial state vectors at iteration j and iteration $j+1$ respectively. Therefore, another expression for the lifted system model is

$$x_{j+1} = x_j + H_j \left(u_{j+1} - u_j\right) + F_j \left(x_{j+1}^0 - x_j^0\right) \tag{2}$$

2.2 Estimating x_{j+1} and e_{j+1}

Note that for our application the final state vector from iteration j is the initial state vector for iteration $j+1$. Therefore, we define E_F and E_I according to

$$E_F \triangleq \begin{bmatrix} \mathbb{0}^{n_x \times n_x(n_s-1)} & \mathbb{I}^{n_x \times n_x} \end{bmatrix} \tag{3}$$

$$E_I \triangleq \begin{bmatrix} \mathbb{I}^{n_x \times n_x} & \mathbb{0}^{n_x \times n_x(n_s-1)} \end{bmatrix} \tag{4}$$

so that

$$\hat{F}_j \triangleq F_j \left(E_F - E_I\right) \tag{5}$$

so that the expression for the lifted system model can be written as

$$\begin{aligned} x_{j+1} =& x_j + H_j \left(u_{j+1} - u_j\right) + \hat{F}_j x_j & (6)\\ =& x_j + \hat{F}_j x_j + H_j \left(u_{j+1} - u_j\right) & (7)\\ =& (\mathbb{I} + \hat{F}_j) x_j + H_j \left(u_{j+1} - u_j\right) & (8)\\ =& (\mathbb{I} + \hat{F}_j) x_j - H_j u_j + H_j u_{j+1} & (9)\\ =& x_j + \hat{F}_j x_j - H_j u_j + H_j u_{j+1} & (10) \end{aligned}$$

So this is our expression for x_{j+1} that will be used in the performance index later on. Next, lets look at the expressions for the error sequence, e_{j+1},

$$\begin{aligned} e_{j+1} &= r - x_{j+1} & (11)\\ &= r - (x_j + \hat{F}_j x_j - H_j u_j + H_j u_{j+1}) & (12)\\ &= r - x_j - \hat{F}_j x_j + H_j u_j - H_j u_{j+1} & (13)\\ &= e_j - \hat{F}_j x_j + H_j u_j - H_j u_{j+1} & (14) \end{aligned}$$

2.3 Estimating x_{j+1} and e_{j+1}

Next, lets find expressions for x_{j+2} and e_{j+2}. To estimate these quantities, I'm going to use the current lifted model. Note that this is inherently inaccurate, since that model is based upon linearizations around trajectories from iteration j. It would be nice if we could use the model based upon a linearization around the trajectories from iteration $j+1$ but since we haven't done that iteration yet, this is what we're stuck with. Note that this further motivates the introduction of the MPC-like structure later-on wherein once we get the trajectories from iteration $j+1$ we update our model. Anyways, the model I'm going to use looks like this

$$x_{j+2} = x_j + H_j \left(u_{j+2} - u_j \right) + F_j \left(x^0_{j+2} - x^0_j \right) \tag{15}$$

where that x^0_{j+2} is the final condition from iteration $j+1$. Therefore, noting that

$$\begin{aligned}
x^0_j &= E_I x_j && (16)\\
x^0_{j+2} &= E_F x_{j+1} && (17)\\
&= E_F \left(x_j + \hat{F}_j x_j - H_j u_j + H_j u_{j+1} \right) && (18)\\
&= E_F x_j + E_F \hat{F}_j x_j - E_F H_j u_j + E_F H_j u_{j+1} && (19)
\end{aligned}$$

makes the lifted model

$$\begin{aligned}
x_{j+2} &= x_j + H_j \left(u_{j+2} - u_j \right) + F_j \left(E_F x_j + E_F \hat{F}_j x_j - E_F H_j u_j + E_F H_j u_{j+1} - E_I x_j \right) && (20)\\
&= x_j + H_j u_{j+2} - H_j u_j + F_j E_F x_j + F_j E_F \hat{F}_j x_j - F_j E_F H_j u_j + F_j E_F H_j u_{j+1} - F_j E_I x_j && (21)\\
&= x_j + \left(F_j E_F + F_j E_F \hat{F}_j - F_j E_I \right) x_j - \left(F_j E_F + \mathbb{I} \right) H_j u_j + F_j E_F H_j u_{j+1} + H_j u_{j+2} && (22)
\end{aligned}$$

which makes the expression for e_{j+2}.

$$\begin{aligned}
e_{j+2} &= r - x_{j+2} && (23)\\
&= r - \left(x_j + \left(F_j E_F + F_j E_F \hat{F}_j - F_j E_I \right) x_j - \left(F_j E_F + \mathbb{I} \right) H_j u_j + F_j E_F H_j u_{j+1} + H_j u_{j+2} \right) && (24)\\
&= r - x_j - \left(F_j E_F + F_j E_F \hat{F}_j - F_j E_I \right) x_j + \left(F_j E_F + \mathbb{I} \right) H_j u_j - F_j E_F H_j u_{j+1} - H_j u_{j+2} && (25)\\
&= e_j - \left(F_j E_F + F_j E_F \hat{F}_j - F_j E_I \right) x_j + \left(F_j E_F + \mathbb{I} \right) H_j u_j - F_j E_F H_j u_{j+1} - H_j u_{j+2} && (26)\\
& && (27)
\end{aligned}$$

3 Deriving Optimal Learning Filters

The plan here is basically the same as previous work, write a performance index, differentiate it with respect to the control sequence, set the result equal to zero and solve or rearrange it to get an update law. The slight difference is that we want to form things such that the control sequence includes the control sequence for both iterations $j+1$ and $j+2$.

3.1 Forming the Performance Index

So, I'm going to shoot for a super-generic performance index that penalizes the values of and iteration-to-iteration deviations in the control input, state, and error. We also want to include a linear term on the states which can be used to incentive maximizing a given state. So then I'll choose:

$$\begin{aligned} J = & u_{j+2}^T Q_u u_{j+2} + u_{j+1}^T Q_u u_{j+1} + \left(u_{j+2} - u_{j+1}\right)^T Q_{\delta u} \left(u_{j+2} - u_{j+1}\right) + \left(u_{j+1} - u_j\right)^T Q_{\delta u} \left(u_{j+1} - u_j\right) && (28) \\ & + e_{j+2}^T Q_e e_{j+2} + e_{j+1}^T Q_e e_{j+1} + \left(e_{j+2} - e_{j+1}\right)^T Q_{\delta e} \left(e_{j+2} - e_{j+1}\right) + \left(e_{j+1} - e_j\right)^T Q_{\delta e} \left(e_{j+1} - e_j\right) && (29) \\ & + x_{j+2}^T Q_x x_{j+2} + x_{j+1}^T Q_x x_{j+1} + \left(x_{j+2} - x_{j+1}\right)^T Q_{\delta x} \left(x_{j+2} - x_{j+1}\right) + \left(x_{j+1} - x_j\right)^T Q_{\delta x} \left(x_{j+1} - x_j\right) && (30) \\ & + S_x x_{j+1} + S_x x_{j+2} && (31) \end{aligned}$$

Start by multiplying everything out

$$\begin{aligned} J = & u_{j+2}^T Q_u u_{j+2} + u_{j+1}^T Q_u u_{j+1} + u_{j+2}^T Q_{\delta u} u_{j+2} - 2u_{j+2}^T Q_{\delta u} u_{j+1} + u_{j+1}^T Q_{\delta u} u_{j+1} + u_{j+1}^T Q_{\delta u} u_{j+1} - 2u_{j+1}^T Q_{\delta u} u_j + u_j^T Q_{\delta u} u_j && (32) \\ & + e_{j+2}^T Q_e e_{j+2} + e_{j+1}^T Q_e e_{j+1} + e_{j+2}^T Q_{\delta e} e_{j+2} - 2e_{j+2}^T Q_{\delta e} e_{j+1} + e_{j+1}^T Q_{\delta e} e_{j+1} + e_{j+1}^T Q_{\delta e} e_{j+1} - 2e_{j+1}^T Q_{\delta e} e_j + e_j^T Q_{\delta e} e_j && (33) \\ & + x_{j+2}^T Q_x x_{j+2} + x_{j+1}^T Q_x x_{j+1} + x_{j+2}^T Q_{\delta x} x_{j+2} - 2x_{j+2}^T Q_{\delta x} x_{j+1} + x_{j+1}^T Q_{\delta x} x_{j+1} + x_{j+1}^T Q_{\delta x} x_{j+1} - 2x_{j+1}^T Q_{\delta x} x_j + x_j^T Q_{\delta x} x_j && (34) \\ & + S_x x_{j+1} + S_x x_{j+2} && (35) \end{aligned}$$

Gather like terms

$$\begin{aligned} J = & u_{j+2}^T (Q_u + Q_{\delta u}) u_{j+2} + u_{j+1}^T (Q_u + 2Q_{\delta u}) u_{j+1} - 2u_{j+2}^T Q_{\delta u} u_{j+1} - 2u_{j+1}^T Q_{\delta u} u_j + u_j^T Q_{\delta u} u_j && (36) \\ & + e_{j+2}^T (Q_e + Q_{\delta e}) e_{j+2} + e_{j+1}^T (Q_e + 2Q_{\delta e}) e_{j+1} - 2e_{j+2}^T Q_{\delta e} e_{j+1} - 2e_{j+1}^T Q_{\delta e} e_j + e_j^T Q_{\delta e} e_j && (37) \\ & + x_{j+2}^T (Q_x + Q_{\delta x}) x_{j+2} + x_{j+1}^T (Q_x + 2Q_{\delta x}) x_{j+1} - 2x_{j+2}^T Q_{\delta x} x_{j+1} - 2x_{j+1}^T Q_{\delta x} x_j + x_j^T Q_{\delta x} x_j && (38) \\ & + S_x x_{j+1} + S_x x_{j+2} && (39) \end{aligned}$$

So now the goal is to write expressions for the state and error trajectories over the next *two* iterations in terms of the control input over the next two iterations. To do this, define some super-lifted vectors and some block matrices

$$\mathbf{u}_{k+1} \triangleq \begin{bmatrix} u_{j+1} \\ u_{j+2} \end{bmatrix} \quad \mathbf{x}_{k+1} \triangleq \begin{bmatrix} x_{j+1} \\ x_{j+2} \end{bmatrix} \quad \mathbf{e}_{k+1} \triangleq \begin{bmatrix} e_{j+1} \\ e_{j+2} \end{bmatrix} \quad \mathbf{r} \triangleq \begin{bmatrix} r \\ r \end{bmatrix} \tag{40}$$

$$\mathbf{Q}_u \triangleq \begin{bmatrix} Q_u + 2Q_{\delta u} & -Q_{\delta u} \\ -Q_{\delta u} & Q_u + Q_{\delta u} \end{bmatrix} \quad \mathbf{Q}_e \triangleq \begin{bmatrix} Q_e + 2Q_{\delta e} & -Q_{\delta e} \\ -Q_{\delta e} & Q_e + Q_{\delta e} \end{bmatrix} \quad \mathbf{Q}_x \triangleq \begin{bmatrix} Q_x + 2Q_{\delta x} & -Q_{\delta x} \\ -Q_{\delta x} & Q_x + Q_{\delta x} \end{bmatrix} \quad \mathbf{S}_x \triangleq \begin{bmatrix} S_x & S_x \end{bmatrix} \tag{41}$$

$$\mathbf{q}_{\delta u} \triangleq \begin{bmatrix} -2Q_{\delta u} & \mathbb{0} \end{bmatrix} \quad \mathbf{q}_{\delta e} \triangleq \begin{bmatrix} -2Q_{\delta e} & \mathbb{0} \end{bmatrix} \quad \mathbf{q}_{\delta x} \triangleq \begin{bmatrix} -2Q_{\delta x} & \mathbb{0} \end{bmatrix} \tag{42}$$

so one iteration in the k-domain here would include two iterations in the j domain. Now write J in terms of these lifted-super vectors. We can now use these to re-write the performance index

$$J = \mathbf{u}_{k+1}^T \mathbf{Q}_u \mathbf{u}_{k+1} + u_j^T \mathbf{q}_{\delta u} \mathbf{u}_{k+1} + u_j^T Q_{\delta u} u_j + \mathbf{e}_{k+1}^T \mathbf{Q}_e \mathbf{e}_{k+1} + e_j^T \mathbf{q}_{\delta e} \mathbf{e}_{k+1} + e_j^T Q_{\delta e} e_j + \mathbf{x}_{k+1}^T \mathbf{Q}_x \mathbf{x}_{k+1} + x_j^T \mathbf{q}_{\delta x} \mathbf{x}_{k+1} + x_j^T Q_{\delta x} x_j + \mathbf{S}_x \mathbf{x}_{k+1} \tag{43}$$

3.2 Differentiating the Performance Index

So the goal will be to differentiate this with respect to $\mathbf{u}_{k+1}$ but to do that, first we need to write $\mathbf{x}_{k+1}$ and $\mathbf{e}_{k+1}$ in terms of $\mathbf{u}_{k+1}$

$$\mathbf{x}_{k+1} = \begin{bmatrix} x_{j+1} \\ x_{j+2} \end{bmatrix} \tag{44}$$

$$= \begin{bmatrix} x_j + \hat{F}_j x_j - H_j u_j + H_j u_{j+1} \\ x_j + \left(F_j E_F + F_j E_F \hat{F}_j - F_j E_I\right) x_j - \left(F_j E_F + \mathbb{1}\right) H_j u_j + F_j E_F H_j u_{j+1} + H_j u_{j+2} \end{bmatrix} \tag{45}$$

$$= \begin{bmatrix} \mathbb{1} \\ \mathbb{1} \end{bmatrix} x_j + \begin{bmatrix} \hat{F}_j \\ F_j E_F + F_j E_F \hat{F}_j - F_j E_I \end{bmatrix} x_j - \begin{bmatrix} \mathbb{1} \\ F_j E_F + \mathbb{1} \end{bmatrix} H_j u_j + \begin{bmatrix} \mathbb{1} & \mathbb{0} \\ F_j E_F & \mathbb{1} \end{bmatrix} H_j \mathbf{u}_{k+1} \tag{46}$$

$$= \mathbf{I} x_j + \mathbf{F}_j x_j + \mathbf{G}_j u_j + \mathbf{H}_j \mathbf{u}_{k+1} \tag{47}$$

$$\mathbf{e}_{k+1} = \begin{bmatrix} e_{j+1} \\ e_{j+2} \end{bmatrix} \tag{48}$$

$$= \begin{bmatrix} e_j - \hat{F}_j x_j + H_j u_j - H_j u_{j+1} \\ e_j - \left(F_j E_F + F_j E_F \hat{F}_j - F_j E_I\right) x_j + \left(F_j E_F + \mathbb{1}\right) H_j u_j - F_j E_F H_j u_{j+1} - H_j u_{j+2} \end{bmatrix} \tag{49}$$

$$= \begin{bmatrix} \mathbb{1} \\ \mathbb{1} \end{bmatrix} e_j - \begin{bmatrix} \hat{F}_j \\ F_j E_F + F_j E_F \hat{F}_j - F_j E_I \end{bmatrix} x_j + \begin{bmatrix} \mathbb{1} \\ F_j E_F + \mathbb{1} \end{bmatrix} H_j u_j - \begin{bmatrix} \mathbb{1} & \mathbb{0} \\ F_j E_F & \mathbb{1} \end{bmatrix} H_j \mathbf{u}_{k+1} \tag{50}$$

$$= \mathbf{I} e_j - \mathbf{F}_j x_j - \mathbf{G}_j u_j - \mathbf{H}_j \mathbf{u}_{k+1} \tag{51}$$

with $\mathbf{M}_j$, $\mathbf{F}_j$, $\mathbf{H}_j$, and $\mathbf{I}$ defined appropriately

These expressions mean that

$$\frac{d\mathbf{x}_{k+1}}{d\mathbf{u}_{k+1}} = \mathbf{H}_j \quad \text{and} \quad \frac{d\mathbf{e}_{k+1}}{d\mathbf{u}_{k+1}} = -\mathbf{H}_j \tag{52}$$

So differentiating J gives:

$$\frac{dJ}{d\mathbf{u}_{k+1}} = 2\mathbf{u}_{k+1}^T \mathbf{Q}_u + u_j^T \mathbf{q}_{\delta u} - 2\mathbf{e}_{k+1}^T \mathbf{Q}_e \mathbf{H}_j - e_j^T \mathbf{q}_{\delta e} \mathbf{H}_j + 2\mathbf{x}_{k+1}^T \mathbf{Q}_x \mathbf{H}_j + x_j^T \mathbf{q}_{\delta x} \mathbf{H}_j + \mathbf{S}_x \mathbf{H}_j \tag{53}$$

3.3 Rearranging to Form an Update Law

Transposing everything

$${\frac{dJ}{d\mathbf{u}_{k+1}}}^T = 2\mathbf{Q}_u \mathbf{u}_{k+1} + \mathbf{q}_{\delta u}^T u_j - 2\mathbf{H}_j^T \mathbf{Q}_e \mathbf{e}_{k+1} - \mathbf{H}_j^T \mathbf{q}_{\delta e}^T e_j + 2\mathbf{H}_j^T \mathbf{Q}_x \mathbf{x}_{k+1} + \mathbf{H}_j^T \mathbf{q}_{\delta x}^T x_j + \mathbf{H}_j^T \mathbf{S}_x^T \tag{54}$$

Now, sub in expressions from above for $\mathbf{x}_{k+1}$ and $\mathbf{e}_{k+1}$.

$${\frac{dJ}{d\mathbf{u}_{k+1}}}^T = 2\mathbf{Q}_u \mathbf{u}_{k+1} + \mathbf{q}_{\delta u}^T u_j - 2\mathbf{H}_j^T \mathbf{Q}_e \left(\mathbf{I} e_j - \mathbf{F}_j x_j - \mathbf{G}_j u_j - \mathbf{H}_j \mathbf{u}_{k+1}\right) - \mathbf{H}_j^T \mathbf{q}_{\delta e}^T e_j + 2\mathbf{H}_j^T \mathbf{Q}_x \left(\mathbf{I} x_j + \mathbf{F}_j x_j + \mathbf{G}_j u_j + \mathbf{H}_j \mathbf{u}_{k+1}\right) + \mathbf{H}_j^T \mathbf{q}_{\delta x}^T x_j + \mathbf{H}_j^T \mathbf{S}_x^T \tag{55}$$

Multiply everything out

$${\frac{dJ}{d\mathbf{u}_{k+1}}}^T = 2\mathbf{Q}_u \mathbf{u}_{k+1} + \mathbf{q}_{\delta u}^T u_j \tag{56}$$

$$- 2\mathbf{H}_j^T \mathbf{Q}_e \mathbf{I} e_j + 2\mathbf{H}_j^T \mathbf{Q}_e \mathbf{F}_j x_j + 2\mathbf{H}_j^T \mathbf{Q}_e \mathbf{G}_j u_j + 2\mathbf{H}_j^T \mathbf{Q}_e \mathbf{H}_j \mathbf{u}_{k+1} - \mathbf{H}_j^T \mathbf{q}_{\delta e}^T e_j \tag{57}$$

$$+ 2\mathbf{H}_j^T \mathbf{Q}_x \mathbf{I} x_j + 2\mathbf{H}_j^T \mathbf{Q}_x \mathbf{F}_j x_j + 2\mathbf{H}_j^T \mathbf{Q}_x \mathbf{G}_j u_j + 2\mathbf{H}_j^T \mathbf{Q}_x \mathbf{H}_j \mathbf{u}_{k+1} + \mathbf{H}_j^T \mathbf{q}_{\delta x}^T x_j + \mathbf{H}_j^T \mathbf{S}_x^T \tag{58}$$

Gather like terms

$$
\begin{aligned}
\frac{dJ}{d\mathbf{u}_{k+1}}^{T} =& 2\mathbf{Q}_u\mathbf{u}_{k+1} + 2\mathbf{H}_j^T\mathbf{Q}_e\mathbf{H}_j\mathbf{u}_{k+1} + 2\mathbf{H}_j^T\mathbf{Q}_x\mathbf{H}_j\mathbf{u}_{k+1} && (59)\\
& + \mathbf{q}_{\delta u}^T u_j + 2\mathbf{H}_j^T\mathbf{Q}_e\mathbf{G}_j u_j + 2\mathbf{H}_j^T\mathbf{Q}_x\mathbf{G}_j u_j && (60)\\
& - 2\mathbf{H}_j^T\mathbf{Q}_e\mathbf{I}e_j - \mathbf{H}_j^T\mathbf{q}_{\delta e}^T e_j && (61)\\
& + 2\mathbf{H}_j^T\mathbf{Q}_e\mathbf{F}_j x_j + 2\mathbf{H}_j^T\mathbf{Q}_x\mathbf{I}x_j + 2\mathbf{H}_j^T\mathbf{Q}_x\mathbf{F}_j x_j + \mathbf{H}_j^T\mathbf{q}_{\delta x}^T x_j && (62)\\
& + \mathbf{H}_j^T\mathbf{S}_x^T && (63)\\
=& \left(2\mathbf{Q}_u + 2\mathbf{H}_j^T\mathbf{Q}_e\mathbf{H}_j + 2\mathbf{H}_j^T\mathbf{Q}_x\mathbf{H}_j\right)\mathbf{u}_{k+1} && (64)\\
& + \left(\mathbf{q}_{\delta u}^T + 2\mathbf{H}_j^T\mathbf{Q}_e\mathbf{G}_j + 2\mathbf{H}_j^T\mathbf{Q}_x\mathbf{G}_j\right) u_j && (65)\\
& - \left(2\mathbf{H}_j^T\mathbf{Q}_e\mathbf{I} + \mathbf{H}_j^T\mathbf{q}_{\delta e}^T\right) e_j && (66)\\
& + \left(2\mathbf{H}_j^T\mathbf{Q}_e\mathbf{F}_j + 2\mathbf{H}_j^T\mathbf{Q}_x\mathbf{I} + 2\mathbf{H}_j^T\mathbf{Q}_x\mathbf{F}_j + \mathbf{H}_j^T\mathbf{q}_{\delta x}^T\right) x_j && (67)\\
& + \mathbf{H}_j^T\mathbf{S}_x^T && (68)
\end{aligned}
$$

Set it equal to zero vector and divide by 2

$$
\begin{aligned}
\vec{0} =& \left(\mathbf{Q}_u + \mathbf{H}_j^T(\mathbf{Q}_e + \mathbf{Q}_x)\mathbf{H}_j\right)\mathbf{u}_{k+1} && (69)\\
& + \left(\frac{1}{2}\mathbf{q}_{\delta u}^T + \mathbf{H}_j^T(\mathbf{Q}_e + \mathbf{Q}_x)\mathbf{G}_j\right) u_j && (70)\\
& - \mathbf{H}_j^T\left(\mathbf{Q}_e\mathbf{I} + \frac{1}{2}\mathbf{q}_{\delta e}^T\right) e_j && (71)\\
& + \mathbf{H}_j^T\left(\mathbf{Q}_e\mathbf{F}_j + \mathbf{Q}_x\mathbf{I} + \mathbf{Q}_x\mathbf{F}_j + \frac{1}{2}\mathbf{q}_{\delta x}^T\right) x_j && (72)\\
& + \frac{1}{2}\mathbf{H}_j^T\mathbf{S}_x^T && (73)
\end{aligned}
$$

So solve this for $\mathbf{u}_{k+1}$

$$
\begin{aligned}
\mathbf{u}_{k+1} &= \mathbf{L}_u u_j + \mathbf{L}_e e_j + \mathbf{L}_x x_j + \mathbf{L}_c && (74)\\
&\text{where} && (75)\\
\mathbf{L}_0 &\triangleq \left(\mathbf{Q}_u + \mathbf{H}_j^T(\mathbf{Q}_e + \mathbf{Q}_x)\mathbf{H}_j\right)^{-1} && (76)\\
\mathbf{L}_u &\triangleq -\mathbf{L}_0\left(\frac{1}{2}\mathbf{q}_{\delta u}^T + \mathbf{H}_j^T(\mathbf{Q}_e + \mathbf{Q}_x)\mathbf{G}_j\right) && (77)\\
\mathbf{L}_e &\triangleq \mathbf{L}_0\mathbf{H}_j^T\left(\mathbf{Q}_e\mathbf{I} + \frac{1}{2}\mathbf{q}_{\delta e}^T\right) && (78)\\
\mathbf{L}_x &\triangleq -\mathbf{L}_0\mathbf{H}_j^T\left(\mathbf{Q}_e\mathbf{F}_j + \mathbf{Q}_x\mathbf{I} + \mathbf{Q}_x\mathbf{F}_j + \frac{1}{2}\mathbf{q}_{\delta x}^T\right) && (79)\\
\mathbf{L}_c &\triangleq -\mathbf{L}_0\frac{1}{2}\mathbf{H}_j^T\mathbf{S}_x^T && (80)
\end{aligned}
$$