1 Introduction

This document is a derivation of the general form of learning filters for receding horizon iterative learning control. Suppose that we are given a lifted system model of the form

$$
\begin{aligned}
x_{j+1} &= x_j + G_j(u_{j+1} - u_j) + F_j x_j^0 && (1)\\
&= x_j + G_j(u_{j+1} - u_j) + \hat{F}_j x_j && (2)\\
&= x_j + \hat{F}_j x_j + G_j(u_{j+1} - u_j) && (3)\\
&= \left(\mathbb{I} + \hat{F}_j\right) x_j + G_j(u_{j+1} - u_j) && (4)
\end{aligned}
$$

Then our predictions of the state sequence at $j+2$, $j+3$, and so on are

$$
\begin{aligned}
x_{j+2} &= x_j + G_j(u_{j+2} - u_j) + \hat{F}_j x_{j+1} && (5)\\
&= x_j + G_j(u_{j+2} - u_j) + \hat{F}_j \left(x_j + G_j(u_{j+1} - u_j) + \hat{F}_j x_j\right) && (6)\\
&= x_j + \hat{F}_j x_j + G_j(u_{j+2} - u_j) + \hat{F}_j \left(G_j(u_{j+1} - u_j) + \hat{F}_j x_j\right) && (7)\\
&= x_j + \hat{F}_j x_j + \hat{F}_j^2 x_j + G_j(u_{j+2} - u_j) + \hat{F}_j G_j(u_{j+1} - u_j) && (8)\\
&= \left(\mathbb{I} + \hat{F}_j + \hat{F}_j^2\right) x_j + +G_j(u_{j+2} - u_j) + \hat{F}_j G_j(u_{j+1} - u_j) && (9)
\end{aligned}
$$

$$
\begin{aligned}
x_{j+3} &= x_j + G_j(u_{j+3} - u_j) + \hat{F}_j x_{j+2} && (10)\\
&= x_j + G_j(u_{j+3} - u_j) + \hat{F}_j \left(x_j + \hat{F}_j x_j + \hat{F}_j^2 x_j + G_j(u_{j+2} - u_j) + \hat{F}_j G_j(u_{j+1} - u_j)\right) && (11)\\
&= x_j + \hat{F}_j x_j + \hat{F}_j^2 x_j + \hat{F}_j^3 x_j + G_j(u_{j+3} - u_j) + \hat{F}_j G_j(u_{j+2} - u_j) + \hat{F}_j^2 G_j(u_{j+1} - u_j) && (12)\\
&= \left(\mathbb{I} + \hat{F}_j + \hat{F}_j^2 + \hat{F}_j^3\right) x_j + G_j(u_{j+3} - u_j) + \hat{F}_j G_j(u_{j+2} - u_j) + \hat{F}_j^2 G_j(u_{j+1} - u_j) && (13)\\
& && (14)
\end{aligned}
$$

So the general form for x_{j+N} where $N \in \mathbb{N}^+$ is

$$
\begin{aligned}
x_{j+N} &= \left(\sum_{k=0}^{N} \hat{F}_j^k\right) x_j + \left(\sum_{k=0}^{N-1} \hat{F}_j^{N-1-k} G_j\left(u_{j+k+1} - u_j\right)\right) && (15)\\
&= \left(\mathbb{I} + \sum_{k=1}^{N} \hat{F}_j^k\right) x_j + \left(\sum_{k=0}^{N-1} \hat{F}_j^{N-1-k} G_j\left(u_{j+k+1} - u_j\right)\right) && (16)\\
& && (17)
\end{aligned}
$$

Therefore, if we form the uber-lyfted vectors

$$
\mathbf{x}_{j+1} \triangleq \begin{bmatrix} x_{j+1} \\ x_{j+2} \\ \vdots \\ x_{j+N-1} \\ x_{j+N} \end{bmatrix}, \quad \mathbf{u}_{j+1} \triangleq \begin{bmatrix} u_{j+1} \\ u_{j+2} \\ \vdots \\ u_{j+N-1} \\ u_{j+N} \end{bmatrix} \tag{18}
$$

then we can write an expression for $\mathbf{x}_{j+1}$ in terms of $\mathbf{u}_{j+1}$

$$
\mathbf{x}_{j+1} = \left(\mathbf{I} + \begin{bmatrix} \mathbb{0} \\ \hat{F}_j \\ \vdots \\ \hat{F}_j + \ldots + \hat{F}_j^{N-2} + \hat{F}_j^{N-1} \\ \hat{F}_j + \ldots + \hat{F}_j^{N-1} + \hat{F}_j^{N} \end{bmatrix}\right) x_j + \begin{bmatrix} G_j & \mathbb{0} & \mathbb{0} & \ldots & \mathbb{0} \\ \hat{F}_j G_j & G_j & \mathbb{0} & \ldots & \mathbb{0} \\ \hat{F}_j^2 G_j & F_j G_j & G_j & \ldots & \mathbb{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{F}_j^{N-2} 2G_j & F_j^{N-3} G_j & \ldots & G_j & \mathbb{0} \\ \hat{F}_j^{N-1} 2G_j & F_j^{N-2} G_j & \ldots & \hat{F}_j G_j & G_j \end{bmatrix} \left(\mathbf{u}_{j+1} - \mathbf{I} u_j\right) \tag{19}
$$

Here, $\mathbf{I} \triangleq \begin{bmatrix} \mathbb{I} & \ldots & \mathbb{I} \end{bmatrix}^T$ If we define $\mathbf{F}_j$ to be the first matrix, and $\mathbf{G}_j$ to be the second matrix, then our system model as lifted in the iteration domain is

$$
\mathbf{x}_{j+1} = \left(\mathbf{I} + \mathbf{F}_j\right) x_j + \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I} u_j\right) \tag{20}
$$

Next, form the uber-lyfted vector $\mathbf{r} \triangleq \mathbf{I}r$, then the uber-lyfted error sequence $\mathbf{e}_{j+1}$ is

$$\begin{aligned}
\mathbf{e}_{j+1} &= \mathbf{r} - \mathbf{x}_{j+1} && (21)\\
&= \mathbf{r} - (\mathbf{I} + \mathbf{F}_j)\, x_j - \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}u_j\right) && (22)\\
&= \mathbf{I}e_j - \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}u_j\right) && (23)\\
& && (24)
\end{aligned}$$

Now, we want to form a performance index that includes terms for 7 things

- a penalty on the size of the control input for each future iteration
- a penalty on the deviation in the control input for each future iteration
- a penalty on the size of the state for each future iteration
- a penalty on the deviation in the state for each future iteration
- a penalty on the size of the error for each future iteration
- a penalty on the deviation in the error for each future iteration
- an economic incentive on the state for each future iteration

We can write this as

$$\begin{aligned}
\mathbf{J}_{j+N} = \sum_{k=1}^{N} &(u_{j+k}^T Q_u u_{j+k} + \left(u_{j+k+1} - u_{j+k}\right)^T Q_{\delta u} \left(u_{j+k+1} - u_{j+k}\right) \\
&+ x_{j+k}^T Q_x x_{j+k} + \left(x_{j+k+1} - x_{j+k}\right)^T Q_{\delta x} \left(x_{j+k+1} - x_{j+k}\right) \\
&+ e_{j+k}^T Q_e e_{j+k} + \left(e_{j+k+1} - e_{j+k}\right)^T Q_{\delta e} \left(e_{j+k+1} - e_{j+k}\right) \\
&+ S_x x_{j+k})
\end{aligned} \tag{25}$$

which has an equivalent block form

$$\begin{aligned}
\mathbf{J}_{j+N} =& \mathbf{u}_{j+1}^T \mathbf{Q}_{\mathbf{u}} \mathbf{u}_{j+1} + \mathbf{u}_{j+1}^T \mathbf{D}_u^T \mathbf{Q}_{\delta\mathbf{u}} \mathbf{D}_u \mathbf{u}_{j+1} \\
&+ \mathbf{x}_{j+1}^T \mathbf{Q}_{\mathbf{x}} \mathbf{x}_{j+1} + \mathbf{x}_{j+1}^T \mathbf{D}_x^T \mathbf{Q}_{\delta\mathbf{x}} \mathbf{D}_x \mathbf{x}_{j+1} \\
&+ \mathbf{e}_{j+1}^T \mathbf{Q}_{\mathbf{e}} \mathbf{e}_{j+1} + \mathbf{e}_{j+1}^T \mathbf{D}_e^T \mathbf{Q}_{\delta\mathbf{e}} \mathbf{D}_e \mathbf{e}_{j+1} \\
&+ \mathbf{S}_{\mathbf{x}} \mathbf{x}_{j+1}
\end{aligned} \tag{26}$$

$$\mathbf{J}_{j+N} = \mathbf{u}_{j+1}^T \left(\mathbf{Q}_{\mathbf{u}} + \mathbf{D}_u^T \mathbf{Q}_{\delta\mathbf{u}} \mathbf{D}_u\right) \mathbf{u}_{j+1} + \mathbf{x}_{j+1}^T \left(\mathbf{Q}_{\mathbf{x}} + \mathbf{D}_x^T \mathbf{Q}_{\delta\mathbf{x}} \mathbf{D}_x\right) \mathbf{x}_{j+1} + \mathbf{e}_{j+1}^T \left(\mathbf{Q}_{\mathbf{e}} + \mathbf{D}_e^T \mathbf{Q}_{\delta\mathbf{e}} \mathbf{D}_e\right) \mathbf{e}_{j+1} + \mathbf{S}_{\mathbf{x}} \mathbf{x}_{j+1} \tag{27}$$

$$\mathbf{J}_{j+N} = \mathbf{u}_{j+1}^T \hat{\mathbf{Q}}_{\mathbf{u}} \mathbf{u}_{j+1} + \mathbf{x}_{j+1}^T \hat{\mathbf{Q}}_{\mathbf{x}} \mathbf{x}_{j+1} + \mathbf{e}_{j+1}^T \hat{\mathbf{Q}}_{\mathbf{e}} \mathbf{e}_{j+1} + \mathbf{S}_{\mathbf{x}} \mathbf{x}_{j+1} \tag{28}$$

where $\hat{\mathbf{Q}}_{\mathbf{u}}$, $\hat{\mathbf{Q}}_{\mathbf{x}}$, and $\hat{\mathbf{Q}}_{\mathbf{e}}$ are defined appropriately

Now look at the gradient of each term with respect to $\mathbf{u}_{j+1}$,

$$\frac{d}{d\mathbf{u}_{j+1}} \mathbf{u}_{j+1}^T \hat{\mathbf{Q}}_{\mathbf{u}} \mathbf{u}_{j+1} = 2\mathbf{u}_{j+1}^T \hat{\mathbf{Q}}_{\mathbf{u}} \tag{29}$$

$$\begin{aligned}
\frac{d}{d\mathbf{u}_{j+1}} \mathbf{x}_{j+1}^T \hat{\mathbf{Q}}_{\mathbf{x}} \mathbf{x}_{j+1} &= \frac{d}{d\mathbf{u}_{j+1}} \left((\mathbf{I} + \mathbf{F}_j)\, x_j + \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}u_j\right)\right)^T \hat{\mathbf{Q}}_{\mathbf{x}} \left((\mathbf{I} + \mathbf{F}_j)\, x_j + \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}u_j\right)\right) && (30)\\
&= 2\left((\mathbf{I} + \mathbf{F}_j)\, x_j + \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}u_j\right)\right)^T \hat{\mathbf{Q}}_{\mathbf{x}} \mathbf{G}_j && (31)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{d\mathbf{u}_{j+1}} \mathbf{e}_{j+1}^T \hat{\mathbf{Q}}_{\mathbf{e}} \mathbf{e}_{j+1} &= \frac{d}{d\mathbf{u}_{j+1}} \left(\mathbf{I}e_j - \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}u_j\right)\right)^T \hat{\mathbf{Q}}_{\mathbf{e}} \left(\mathbf{I}e_j - \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}u_j\right)\right) && (32)\\
&= -2\left(\mathbf{I}e_j - \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}u_j\right)\right)^T \hat{\mathbf{Q}}_{\mathbf{e}} \mathbf{G}_j && (33)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{d\mathbf{u}_{j+1}} \mathbf{S}_x \mathbf{x}_{j+1} &= \frac{d}{d\mathbf{u}_{j+1}} \mathbf{S}_x \left((\mathbf{I} + \mathbf{F}_j)\, x_j + \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}u_j\right)\right) && (34)\\
&= \mathbf{S}_x \mathbf{G}_j && (35)
\end{aligned}$$

So then the gradient of the performance index is

$$\frac{d}{d\mathbf{J}_{j+1}} = 2\mathbf{u}_{j+1}^T\hat{\mathbf{Q}}_{\mathbf{u}} + 2\left(\left(\mathbf{I}+\mathbf{F}_j\right)x_j + \mathbf{G}_j\left(\mathbf{u}_{j+1} - \mathbf{I}u_j\right)\right)^T\hat{\mathbf{Q}}_{\mathbf{x}}\mathbf{G}_j - 2\left(\mathbf{I}e_j - \mathbf{G}_j\left(\mathbf{u}_{j+1} - \mathbf{I}u_j\right)\right)^T\hat{\mathbf{Q}}_{\mathbf{e}}\mathbf{G}_j + \mathbf{S}_x\mathbf{G}_j \tag{36}$$

Setting this equal to zero vector

$$\vec{0}^T = \mathbf{u}_{j+1}^T\hat{\mathbf{Q}}_{\mathbf{u}} + \left(\left(\mathbf{I}+\mathbf{F}_j\right)x_j + \mathbf{G}_j\left(\mathbf{u}_{j+1} - \mathbf{I}u_j\right)\right)^T\hat{\mathbf{Q}}_{\mathbf{x}}\mathbf{G}_j - \left(\mathbf{I}e_j - \mathbf{G}_j\left(\mathbf{u}_{j+1} - \mathbf{I}u_j\right)\right)^T\hat{\mathbf{Q}}_{\mathbf{e}}\mathbf{G}_j + \frac{1}{2}\mathbf{S}_x\mathbf{G}_j \tag{37}$$

$$\vec{0} = \hat{\mathbf{Q}}_{\mathbf{u}}\mathbf{u}_{j+1} + \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{x}}\left(\left(\mathbf{I}+\mathbf{F}_j\right)x_j + \mathbf{G}_j\left(\mathbf{u}_{j+1} - \mathbf{I}u_j\right)\right) - \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{e}}\left(\mathbf{I}e_j - \mathbf{G}_j\left(\mathbf{u}_{j+1} - \mathbf{I}u_j\right)\right) + \frac{1}{2}\mathbf{G}_j^T\mathbf{S}_x^T \tag{38}$$

Now gather $\mathbf{u}_{j+1}$, $\mathbf{u}_j$, e_j and x_j terms.

$$\vec{0} = \left(\hat{\mathbf{Q}}_{\mathbf{u}} + \mathbf{G}_j^T\left(\hat{\mathbf{Q}}_{\mathbf{x}} + \hat{\mathbf{Q}}_{\mathbf{e}}\right)\mathbf{G}_j\right)\mathbf{u}_{j+1} + \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{x}}\left(\left(\mathbf{I}+\mathbf{F}_j\right)x_j - \mathbf{G}_j\mathbf{I}u_j\right) - \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{e}}\left(\mathbf{I}e_j + \mathbf{G}_j\mathbf{I}u_j\right) + \frac{1}{2}\mathbf{G}_j^T\mathbf{S}_x^T \tag{39}$$

$$= \left(\hat{\mathbf{Q}}_{\mathbf{u}} + \mathbf{G}_j^T\left(\hat{\mathbf{Q}}_{\mathbf{x}} + \hat{\mathbf{Q}}_{\mathbf{e}}\right)\mathbf{G}_j\right)\mathbf{u}_{j+1} - \mathbf{G}_j^T\left(\hat{\mathbf{Q}}_{\mathbf{x}} + \hat{\mathbf{Q}}_{\mathbf{e}}\right)\mathbf{G}_j\mathbf{I}u_j + \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{x}}\left(\mathbf{I}+\mathbf{F}_j\right)x_j - \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{e}}\mathbf{I}e_j + \frac{1}{2}\mathbf{G}_j^T\mathbf{S}_x^T \tag{40}$$

Solving this for $\mathbf{u}_{j+1}$ gives the optimal learning filters and the update law

$$\mathbf{u}_{j+1} = L_u u_j + L_e e_j + L_x x_j + L_c \tag{41}$$

$$L_0 \triangleq \left(\hat{\mathbf{Q}}_{\mathbf{u}} + \mathbf{G}_j^T\left(\hat{\mathbf{Q}}_{\mathbf{x}} + \hat{\mathbf{Q}}_{\mathbf{e}}\right)\mathbf{G}_j\right)^{-1} \tag{42}$$

$$L_u \triangleq L_0\mathbf{G}_j^T\left(\hat{\mathbf{Q}}_{\mathbf{x}} + \hat{\mathbf{Q}}_{\mathbf{e}}\right)\mathbf{G}_j\mathbf{I} \tag{43}$$

$$L_x \triangleq -L_0\mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{x}}\left(\mathbf{I}+\mathbf{F}_j\right) \tag{44}$$

$$L_e \triangleq L_0\mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{e}}\mathbf{I} \tag{45}$$

$$L_c \triangleq -\frac{1}{2}L_0\mathbf{G}_j^T\mathbf{S}_x^T \tag{46}$$