1 Introduction

This document is a derivation of the general form of learning filters for receding horizon iterative learning control. Suppose that we are given a lifted system model of the form

$$
\begin{aligned}
x_{j+1} &= x_j + G_j(u_{j+1} - u_j) + F_j\left(x_{j+1}^0 - x_j^0\right) && (1)\\
&= x_j + G_j(u_{j+1} - u_j) + F_j\left(x_j^F - x_j^0\right) && (2)\\
&= x_j + G_j(u_{j+1} - u_j) + F_j\left(E_F - E_I\right)x_j && (3)\\
&= x_j + G_j(u_{j+1} - u_j) + F_jE_Fx_j - F_jE_Ix_j && (4)\\
&= x_j + G_j(u_{j+1} - u_j) + \hat{F}_jx_j && (5)\\
&= x_j + \hat{F}_jx_j + G_j(u_{j+1} - u_j) && (6)\\
&= \left(\mathbb{I} + \hat{F}_j\right)x_j + G_j(u_{j+1} - u_j) && (7)
\end{aligned}
$$

Then our predictions of the state sequence at $j+2$, $j+3$, and so on are

$$
\begin{aligned}
x_{j+2} &= x_j + G_j(u_{j+2} - u_j) + F_j\left(x_{j+2}^0 - x_j^0\right) && (8)\\
&= x_j + G_j(u_{j+2} - u_j) + F_j\left(E_Fx_{j+1} - E_Ix_j\right) && (9)\\
&= x_j + G_j(u_{j+2} - u_j) + F_jE_Fx_{j+1} - F_jE_Ix_j && (10)\\
&= x_j + G_j(u_{j+2} - u_j) - F_jE_Ix_j + F_jE_Fx_{j+1} && (11)\\
&= x_j + G_j(u_{j+2} - u_j) - F_jE_Ix_j + F_jE_F\left(x_j + G_j(u_{j+1} - u_j) + F_jE_Fx_j - F_jE_Ix_j\right) && (12)\\
&= x_j + G_j(u_{j+2} - u_j) - F_jE_Ix_j + F_jE_Fx_j + F_jE_FG_j(u_{j+1} - u_j) + F_jE_FF_jE_Fx_j - F_jE_FF_jE_Ix_j && (13)\\
&= x_j + G_j(u_{j+2} - u_j) + F_jE_FG_j(u_{j+1} - u_j) - F_jE_Ix_j + F_jE_Fx_j + F_jE_FF_jE_Fx_j - F_jE_FF_jE_Ix_j && (14)\\
&= x_j + G_j(u_{j+2} - u_j) + F_jE_FG_j(u_{j+1} - u_j) + F_j\left(E_F - E_I\right)x_j + F_jE_FF_jE_Fx_j - F_jE_FF_jE_Ix_j && (15)\\
&= x_j + G_j(u_{j+2} - u_j) + F_jE_FG_j(u_{j+1} - u_j) + F_j\left(E_F - E_I\right)x_j + F_jE_FF_j\left(E_F - E_I\right)x_j && (16)
\end{aligned}
$$

$$
\begin{aligned}
x_{j+3} &= x_j + G_j(u_{j+3} - u_j) + F_j\left(x_{j+3}^0 - x_j^0\right) && (17)\\
&= x_j + G_j(u_{j+3} - u_j) + F_j\left(E_Fx_{j+2} - E_Ix_j\right) && (18)\\
&= x_j + G_j(u_{j+3} - u_j) + F_jE_Fx_{j+2} - F_jE_Ix_j && (19)\\
&= x_j + G_j(u_{j+3} - u_j) - F_jE_Ix_j + F_jE_Fx_{j+2} && (20)\\
&= x_j + G_j(u_{j+3} - u_j) - F_jE_Ix_j + F_jE_F\left(x_j + G_j(u_{j+2} - u_j) + F_jE_FG_j(u_{j+1} - u_j) - F_jE_Ix_j + F_jE_Fx_j + F_jE_FF_jE_Fx_j - F_jE_FF_jE_Ix_j\right) && (21)\\
&= x_j + G_j(u_{j+3} - u_j) - F_jE_Ix_j + F_jE_Fx_j + F_jE_FG_j(u_{j+2} - u_j) + F_jE_FF_jE_FG_j(u_{j+1} - u_j) - F_jE_FF_jE_Ix_j + F_jE_FF_jE_Fx_j + F_jE_FF_jE_FF_jE_Fx_j - F_jE_FF_jE_FF_jE_Ix_j && (22)\\
&= x_j + G_j(u_{j+3} - u_j) + F_jE_FG_j(u_{j+2} - u_j) + F_jE_FF_jE_FG_j(u_{j+1} - u_j) - F_jE_Ix_j + F_jE_Fx_j - F_jE_FF_jE_Ix_j + F_jE_FF_jE_Fx_j + F_jE_FF_jE_FF_jE_Fx_j - F_jE_FF_jE_FF_jE_Ix_j && (23)\\
&= x_j + G_j(u_{j+3} - u_j) + F_jE_FG_j(u_{j+2} - u_j) + F_jE_FF_jE_FG_j(u_{j+1} - u_j) + F_j\left(E_F - E_I\right)x_j - F_jE_FF_jE_Ix_j + F_jE_FF_jE_Fx_j + F_jE_FF_jE_FF_jE_Fx_j - F_jE_FF_jE_FF_jE_Ix_j && (24)\\
&= x_j + G_j(u_{j+3} - u_j) + F_jE_FG_j(u_{j+2} - u_j) + F_jE_FF_jE_FG_j(u_{j+1} - u_j) + F_j\left(E_F - E_I\right)x_j + F_jE_FF_j\left(F_F - E_I\right)x_j + F_jE_FF_jE_FF_j\left(E_F - E_I\right)x_j && (25)
\end{aligned}
$$

$$
\begin{aligned}
x_{j+1} &= x_j && + G_j(u_{j+1} - u_j) && + F_j\left(E_F - E_I\right)x_j && (26)\\
x_{j+2} &= x_j && + G_j(u_{j+2} - u_j) + F_jE_FG_j(u_{j+1} - u_j) && + F_j\left(E_F - E_I\right)x_j + F_jE_FF_j\left(E_F - E_I\right)x_j && (27)\\
x_{j+3} &= x_j + G_j(u_{j+3} - u_j) && + F_jE_FG_j(u_{j+2} - u_j) + F_jE_FF_jE_FG_j(u_{j+1} - u_j) && + F_j\left(E_F - E_I\right)x_j + F_jE_FF_j\left(F_F - E_I\right)x_j + F_jE_FF_jE_FF_j\left(E_F - E_I\right)x_j && (28)
\end{aligned}
$$

Therefore, if we form the uber-lyfted vectors

$$\mathbf{x}_{j+1} \triangleq \begin{bmatrix} x_{j+1} \\ x_{j+2} \\ \vdots \\ x_{j+N-1} \\ x_{j+N} \end{bmatrix}, \quad \mathbf{u}_{j+1} \triangleq \begin{bmatrix} u_{j+1} \\ u_{j+2} \\ \vdots \\ u_{j+N-1} \\ u_{j+N} \end{bmatrix} \tag{29}$$

then we can write an expression for $\mathbf{x}_{j+1}$ in terms of $\mathbf{u}_{j+1}$

$$\mathbf{x}_{j+1} = \mathbf{I}_x x_j + \begin{bmatrix} \mathbb{I} \\ \mathbb{I} + F_j E_F \\ \mathbb{I} + F_j E_F + F_j E_F F_j E_F \\ \vdots \\ \mathbb{I} + \sum_{k=1}^{N_i-1} \prod_{m=1}^{k} F_j E_F \\ \mathbb{I} + \sum_{k=1}^{N_i} \prod_{m=1}^{k} F_j E_F \end{bmatrix} F_j \left(E_F - E_I\right) x_j + \begin{bmatrix} G_j & \mathbb{0} & \mathbb{0} & \dots & \mathbb{0} \\ F_j E_F G_j & G_j & \mathbb{0} & \dots & \mathbb{0} \\ F_j E_F F_j E_F G_j & F_j E_F G_j & G_j & \dots & \mathbb{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left(\prod_{m=1}^{N_i-1} F_j E_F\right) G_j & \left(\prod_{m=1}^{N_i-2} F_j E_F\right) G_j & \dots & G_j & \mathbb{0} \\ \left(\prod_{m=1}^{N_i} F_j E_F\right) G_j & \left(\prod_{m=1}^{N_i-1} F_j E_F\right) G_j & \dots & F_j E_F G_j & G_j \end{bmatrix} \left(\mathbf{u}_{j+1} - \mathbf{I}_u u_j\right) \tag{30}$$

Here, $\mathbf{I} \triangleq \begin{bmatrix} \mathbb{I} & \dots & \mathbb{I} \end{bmatrix}^T$If we define $\mathbf{F}_j$ to be the first matrix, and $\mathbf{G}_j$ to be the second matrix, then our system model as lifted in the iteration domain is

$$\mathbf{x}_{j+1} = \left(\mathbf{I}_x + \mathbf{F}_j\right) x_j + \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}_u u_j\right) \tag{31}$$

Next, form the uber-lyfted vector $\mathbf{r} \triangleq \mathbf{I}r$, then the uber-lyfted error sequence $\mathbf{e}_{j+1}$ is

$$\begin{aligned} \mathbf{e}_{j+1} &= \mathbf{I}_r r - \mathbf{x}_{j+1} && (32) \\ &= \mathbf{I}_r r - \left(\mathbf{I}_x + \mathbf{F}_j\right) x_j - \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}_u u_j\right) && (33) \\ &= \mathbf{I}_r r - \mathbf{I}_x x_j - \mathbf{F}_j x_j - \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}_u u_j\right) && (34) \\ &= \mathbf{I}_e e_j - \mathbf{F}_j x_j - \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I} u_j\right) && (35) \end{aligned}$$

Note that $\mathbf{I}_e = \mathbf{I}_x = \mathbf{I}_r$. Now, we want to form a performance index that includes terms for 7 things

- a penalty on the size of the control input for each future iteration
- a penalty on the deviation in the control input for each future iteration
- a penalty on the size of the state for each future iteration
- a penalty on the deviation in the state for each future iteration
- a penalty on the size of the error for each future iteration
- a penalty on the deviation in the error for each future iteration
- an economic incentive on the state for each future iteration

We can write this as

$$\begin{aligned} \mathbf{J}_{j+N} = \sum_{k=1}^{N} (& u_{j+k}^T Q_u u_{j+k} + \left(u_{j+k} - u_{j+k-1}\right)^T Q_{\delta u} \left(u_{j+k} - u_{j+k-1}\right) && (36) \\ & + x_{j+k}^T Q_x x_{j+k} + \left(x_{j+k} - x_{j+k-1}\right)^T Q_{\delta x} \left(x_{j+k} - x_{j+k-1}\right) && (37) \\ & + e_{j+k}^T Q_e e_{j+k} + \left(e_{j+k} - e_{j+k-1}\right)^T Q_{\delta e} \left(e_{j+k} - e_{j+k-1}\right) && (38) \\ & + S_x x_{j+k}) && (39) \\ & && (40) \end{aligned}$$

$$\mathbf{J}_{j+N} = \sum_{k=1}^{N}(u_{j+k}^T Q_u u_{j+k} + x_{j+k}^T Q_x x_{j+k} + e_{j+k}^T Q_e e_{j+k} + S_x x_{j+k}) + \tag{41}$$

$$\sum_{k=1}^{N}((u_{j+k} - u_{j+k-1})^T Q_{\delta u} (u_{j+k} - u_{j+k-1}) \tag{42}$$

$$+ (x_{j+k} - x_{j+k-1})^T Q_{\delta x} (x_{j+k} - x_{j+k-1}) \tag{43}$$

$$+ (e_{j+k} - e_{j+k-1})^T Q_{\delta e} (e_{j+k} - e_{j+k-1})) \tag{44}$$

$$\mathbf{J}_{j+N} = \sum_{k=1}^{N}(u_{j+k}^T Q_u u_{j+k} + x_{j+k}^T Q_x x_{j+k} + e_{j+k}^T Q_e e_{j+k} + S_x x_{j+k}) + \tag{45}$$

$$\sum_{k=2}^{N}((u_{j+k} - u_{j+k-1})^T Q_{\delta u} (u_{j+k} - u_{j+k-1}) \tag{46}$$

$$+ (x_{j+k} - x_{j+k-1})^T Q_{\delta x} (x_{j+k} - x_{j+k-1}) \tag{47}$$

$$+ (e_{j+k} - e_{j+k-1})^T Q_{\delta e} (e_{j+k} - e_{j+k-1})) \tag{48}$$

$$+ (u_{j+1} - u_j)^T Q_{\delta u}(u_{j+1} - u_j)(x_{j+1} - x_j)^T Q_{\delta x}(x_{j+1} - x_j)(e_{j+1} - e_j)^T Q_{\delta e}(e_{j+1} - e_j) \tag{49}$$

which has an equivalent block form

$$\begin{aligned}\mathbf{J}_{j+N} =& \mathbf{u}_{j+1}^T \mathbf{Q_u} \mathbf{u}_{j+1} + \mathbf{u}_{j+1}^T \mathbf{D}_u^T \mathbf{Q_{\delta u}} \mathbf{D}_u \mathbf{u}_{j+1} + (u_{j+1} - u_j)^T Q_{\delta u} (u_{j+1} - u_j) \\ &+ \mathbf{x}_{j+1}^T \mathbf{Q_x} \mathbf{x}_{j+1} + \mathbf{x}_{j+1}^T \mathbf{D}_x^T \mathbf{Q_{\delta x}} \mathbf{D}_x \mathbf{x}_{j+1} + (x_{j+1} - x_j)^T Q_{\delta x} (x_{j+1} - x_j) \\ &+ \mathbf{e}_{j+1}^T \mathbf{Q_e} \mathbf{e}_{j+1} + \mathbf{e}_{j+1}^T \mathbf{D}_e^T \mathbf{Q_{\delta e}} \mathbf{D}_e \mathbf{e}_{j+1} + (e_{j+1} - e_j)^T Q_{\delta e} (e_{j+1} - e_j) \\ &+ \mathbf{S_x} \mathbf{x}_{j+1}\end{aligned} \tag{50}$$

$$\begin{aligned}\mathbf{J}_{j+N} =& \mathbf{u}_{j+1}^T \mathbf{Q_u} \mathbf{u}_{j+1} + \mathbf{u}_{j+1}^T \mathbf{D}_u^T \mathbf{Q_{\delta u}} \mathbf{D}_u \mathbf{u}_{j+1} + (\mathbf{E}_u \mathbf{u}_{j+1} - u_j)^T Q_{\delta u} (\mathbf{E}_u \mathbf{u}_{j+1} - u_j) \\ &+ \mathbf{x}_{j+1}^T \mathbf{Q_x} \mathbf{x}_{j+1} + \mathbf{x}_{j+1}^T \mathbf{D}_x^T \mathbf{Q_{\delta x}} \mathbf{D}_x \mathbf{x}_{j+1} + (\mathbf{E}_x \mathbf{x}_{j+1} - x_j)^T Q_{\delta x} (\mathbf{E}_x \mathbf{x}_{j+1} - x_j) \\ &+ \mathbf{e}_{j+1}^T \mathbf{Q_e} \mathbf{e}_{j+1} + \mathbf{e}_{j+1}^T \mathbf{D}_e^T \mathbf{Q_{\delta e}} \mathbf{D}_e \mathbf{e}_{j+1} + (\mathbf{E}_e \mathbf{e}_{j+1} - e_j)^T Q_{\delta e} (\mathbf{E}_e \mathbf{e}_{j+1} - e_j) \\ &+ \mathbf{S_x} \mathbf{x}_{j+1}\end{aligned} \tag{51}$$

$$\mathbf{J}_{j+N} = \mathbf{u}_{j+1}^T \left(\mathbf{Q_u} + \mathbf{D}_u^T \mathbf{Q_{\delta u}} \mathbf{D}_u + \mathbf{E}_u^T Q_{\delta u} \mathbf{E}_u\right) \mathbf{u}_{j+1} - 2u_j^T Q_u \mathbf{E}_u \mathbf{u}_{j+1} + u_j^T Q_u u_j \tag{52}$$

$$+ \mathbf{x}_{j+1}^T \left(\mathbf{Q_x} + \mathbf{D}_x^T \mathbf{Q_{\delta x}} \mathbf{D}_x + \mathbf{E}_x^T Q_{\delta x} \mathbf{E}_x\right) \mathbf{x}_{j+1} - 2x_j^T Q_x \mathbf{E}_x \mathbf{x}_{j+1} + x_j^T Q_x x_j \tag{53}$$

$$+ \mathbf{e}_{j+1}^T \left(\mathbf{Q_e} + \mathbf{D}_e^T \mathbf{Q_{\delta e}} \mathbf{D}_e + \mathbf{E}_e^T Q_{\delta e} \mathbf{E}_e\right) \mathbf{e}_{j+1} - 2u_j^T Q_e \mathbf{E}_e \mathbf{e}_{j+1} + e_j^T Q_u e_j \tag{54}$$

$$+ \mathbf{S_x} \mathbf{x}_{j+1} \tag{55}$$

Since we're going to differentiate, I'm going to drop the terms that don't depend on $\mathbf{u}_{j+1}$

$$\mathbf{J}_{j+N} = \mathbf{u}_{j+1}^T \left(\mathbf{Q_u} + \mathbf{D}_u^T \mathbf{Q_{\delta u}} \mathbf{D}_u + \mathbf{E}_u^T Q_{\delta u} \mathbf{E}_u\right) \mathbf{u}_{j+1} - 2u_j^T Q_u \mathbf{E}_u \mathbf{u}_{j+1} \tag{56}$$

$$+ \mathbf{x}_{j+1}^T \left(\mathbf{Q_x} + \mathbf{D}_x^T \mathbf{Q_{\delta x}} \mathbf{D}_x + \mathbf{E}_x^T Q_{\delta x} \mathbf{E}_x\right) \mathbf{x}_{j+1} - 2x_j^T Q_x \mathbf{E}_x \mathbf{x}_{j+1} \tag{57}$$

$$+ \mathbf{e}_{j+1}^T \left(\mathbf{Q_e} + \mathbf{D}_e^T \mathbf{Q_{\delta e}} \mathbf{D}_e + \mathbf{E}_e^T Q_{\delta e} \mathbf{E}_e\right) \mathbf{e}_{j+1} - 2u_j^T Q_e \mathbf{E}_e \mathbf{e}_{j+1} \tag{58}$$

$$+ \mathbf{S_x} \mathbf{x}_{j+1} \tag{59}$$

$$\mathbf{J}_{j+N} = \mathbf{u}_{j+1}^T \hat{\mathbf{Q}}_{\mathbf{u}} \mathbf{u}_{j+1} - 2u_j^T Q_u \mathbf{E}_u \mathbf{u}_{j+1} + \mathbf{x}_{j+1}^T \hat{\mathbf{Q}}_{\mathbf{x}} \mathbf{x}_{j+1} - 2x_j^T Q_x \mathbf{E}_x \mathbf{x}_{j+1} + \mathbf{e}_{j+1}^T \hat{\mathbf{Q}}_{\mathbf{e}} \mathbf{e}_{j+1} - 2u_j^T Q_e \mathbf{E}_e \mathbf{e}_{j+1} + \mathbf{S}_{\mathbf{x}} \mathbf{x}_{j+1} \tag{60}$$

where $\hat{\mathbf{Q}}_{\mathbf{u}}$, $\hat{\mathbf{Q}}_{\mathbf{x}}$, and $\hat{\mathbf{Q}}_{\mathbf{e}}$ are defined appropriately

Now look at the gradient of each term with respect to $\mathbf{u}_{j+1}$,

$$\frac{d}{d\mathbf{u}_{j+1}} \left(\mathbf{u}_{j+1}^T \hat{\mathbf{Q}}_{\mathbf{u}} \mathbf{u}_{j+1} - 2u_j^T Q_u \mathbf{E}_u \mathbf{u}_{j+1} \right) = 2\mathbf{u}_{j+1}^T \hat{\mathbf{Q}}_{\mathbf{u}} - 2u_j^T Q_u \mathbf{E}_u \tag{61}$$

$$\frac{d}{d\mathbf{u}_{j+1}} \left(\mathbf{x}_{j+1}^T \hat{\mathbf{Q}}_{\mathbf{x}} \mathbf{x}_{j+1} - 2x_j^T Q_x \mathbf{E}_x \mathbf{x}_{j+1} \right) = \frac{d}{d\mathbf{u}_{j+1}} \left(\left(\left(\mathbf{I}_x + \mathbf{F}_j \right) x_j + \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}_u u_j \right) \right)^T \hat{\mathbf{Q}}_{\mathbf{x}} \left(\left(\mathbf{I}_x + \mathbf{F}_j \right) x_j + \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}_u u_j \right) \right) - 2x_j^T Q_x \mathbf{E}_x \mathbf{x}_{j+1} \right) \tag{62}$$

$$= 2 \left(\left(\mathbf{I}_x + \mathbf{F}_j \right) x_j + \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}_u u_j \right) \right)^T \hat{\mathbf{Q}}_{\mathbf{x}} \mathbf{G}_j - 2x_j^T Q_x \mathbf{E}_x \mathbf{G}_j \tag{63}$$

$$\frac{d}{d\mathbf{u}_{j+1}} \left(\mathbf{e}_{j+1}^T \hat{\mathbf{Q}}_{\mathbf{e}} \mathbf{e}_{j+1} - 2u_j^T Q_e \mathbf{E}_e \mathbf{e}_{j+1} \right) = \frac{d}{d\mathbf{u}_{j+1}} \left(\left(\mathbf{I}_e e_j - \mathbf{F}_j x_j - \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}_u u_j \right) \right)^T \hat{\mathbf{Q}}_{\mathbf{e}} \left(\mathbf{I} e_j - \mathbf{F}_j x_j - \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}_u u_j \right) \right) - 2u_j^T Q_e \mathbf{E}_e \mathbf{e}_{j+1} \right) \tag{64}$$

$$= -2 \left(\mathbf{I}_e e_j - \mathbf{F}_j x_j - \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}_u u_j \right) \right)^T \hat{\mathbf{Q}}_{\mathbf{e}} \mathbf{G}_j + 2e_j^T Q_e \mathbf{E}_e \mathbf{G}_j \tag{65}$$

$$\frac{d}{d\mathbf{u}_{j+1}} \mathbf{S}_x \mathbf{x}_{j+1} = \frac{d}{d\mathbf{u}_{j+1}} \mathbf{S}_x \left(\left(\mathbf{I}_x + \mathbf{F}_j \right) x_j + \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}_u u_j \right) \right) \tag{66}$$

$$= \mathbf{S}_x \mathbf{G}_j \tag{67}$$

So then the gradient of the performance index is

$$\frac{d}{d\mathbf{J}_{j+1}} = 2\mathbf{u}_{j+1}^T \hat{\mathbf{Q}}_{\mathbf{u}} - 2u_j^T Q_u \mathbf{E}_u \tag{68}$$

$$+ 2 \left(\left(\mathbf{I}_x + \mathbf{F}_j \right) x_j + \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}_u u_j \right) \right)^T \hat{\mathbf{Q}}_{\mathbf{x}} \mathbf{G}_j - 2x_j^T Q_x \mathbf{E}_x \mathbf{G}_j \tag{69}$$

$$- 2 \left(\mathbf{I}_e e_j - \mathbf{F}_j x_j - \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}_u u_j \right) \right)^T \hat{\mathbf{Q}}_{\mathbf{e}} \mathbf{G}_j + 2e_j^T Q_e \mathbf{E}_e \mathbf{G}_j \tag{70}$$

$$+ \mathbf{S}_x \mathbf{G}_j \tag{71}$$

Setting that equal to the zero vector

$$\vec{0}^T = \mathbf{u}_{j+1}^T \hat{\mathbf{Q}}_{\mathbf{u}} - u_j^T Q_u \mathbf{E}_u \tag{72}$$

$$+ \left(\left(\mathbf{I}_x + \mathbf{F}_j \right) x_j + \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}_u u_j \right) \right)^T \hat{\mathbf{Q}}_{\mathbf{x}} \mathbf{G}_j - x_j^T Q_x \mathbf{E}_x \mathbf{G}_j \tag{73}$$

$$- \left(\mathbf{I}_e e_j - \mathbf{F}_j x_j - \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}_u u_j \right) \right)^T \hat{\mathbf{Q}}_{\mathbf{e}} \mathbf{G}_j + e_j^T Q_e \mathbf{E}_e \mathbf{G}_j \tag{74}$$

$$+ \frac{1}{2} \mathbf{S}_x \mathbf{G}_j \tag{75}$$

Transposing the left and right hand sides

$$\vec{0} = \hat{\mathbf{Q}}_{\mathbf{u}} \mathbf{u}_{j+1} - \mathbf{E}_u^T Q_u u_j \tag{76}$$

$$+ \mathbf{G}_j^T \hat{\mathbf{Q}}_{\mathbf{x}} \left(\left(\mathbf{I}_x + \mathbf{F}_j \right) x_j + \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}_u u_j \right) \right) - \mathbf{G}_j^T \mathbf{E}_x^T Q_x x_j \tag{77}$$

$$- \mathbf{G}_j^T \hat{\mathbf{Q}}_{\mathbf{e}} \left(\mathbf{I}_e e_j - \mathbf{F}_j x_j - \mathbf{G}_j \left(\mathbf{u}_{j+1} - \mathbf{I}_u u_j \right) \right) + \mathbf{G}_j^T \mathbf{E}_e^T Q_e e_j \tag{78}$$

$$+ \frac{1}{2} \mathbf{G}_j^T \mathbf{S}_x^T \tag{79}$$

Now multiply all the terms out

$$\vec{0} = \hat{\mathbf{Q}}_{\mathbf{u}}\mathbf{u}_{j+1} - \mathbf{E}_u^T Q_u u_j \tag{80}$$
$$+ \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{x}}\mathbf{I}_x x_j + \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{x}}\mathbf{F}_j x_j + \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{x}}\mathbf{G}_j\mathbf{u}_{j+1} - \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{x}}\mathbf{G}_j\mathbf{I}_u u_j - \mathbf{G}_j^T\mathbf{E}_x^T Q_x x_j \tag{81}$$
$$- \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{e}}\mathbf{I}_e e_j + \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{e}}\mathbf{F}_j x_j + \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{e}}\mathbf{G}_j\mathbf{u}_{j+1} - \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{e}}\mathbf{G}_j\mathbf{I}_u u_j + \mathbf{G}_j^T\mathbf{E}_e^T Q_e e_j \tag{82}$$
$$+ \frac{1}{2}\mathbf{G}_j^T\mathbf{S}_x^T \tag{83}$$

Now gather $\mathbf{u}_{j+1}$, $\mathbf{u}_j$, e_j and x_j terms.

$$\vec{0} = \hat{\mathbf{Q}}_{\mathbf{u}}\mathbf{u}_{j+1} + \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{x}}\mathbf{G}_j\mathbf{u}_{j+1} + \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{e}}\mathbf{G}_j\mathbf{u}_{j+1} \tag{84}$$
$$- \mathbf{E}_u^T Q_u u_j - \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{x}}\mathbf{G}_j\mathbf{I}_u u_j - \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{e}}\mathbf{G}_j\mathbf{I}_u u_j \tag{85}$$
$$- \mathbf{G}_j^T\mathbf{E}_x^T Q_x x_j + \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{x}}\mathbf{I}_x x_j + \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{x}}\mathbf{F}_j x_j + \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{e}}\mathbf{F}_j x_j \tag{86}$$
$$- \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{e}}\mathbf{I}_e e_j + \mathbf{G}_j^T\mathbf{E}_e^T Q_e e_j \tag{87}$$
$$+ \frac{1}{2}\mathbf{G}_j^T\mathbf{S}_x^T \tag{88}$$

$$\vec{0} = \left(\hat{\mathbf{Q}}_{\mathbf{u}} + \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{x}}\mathbf{G}_j + \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{e}}\mathbf{G}_j\right)\mathbf{u}_{j+1} \tag{89}$$
$$- \left(\mathbf{E}_u^T Q_u + \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{x}}\mathbf{G}_j\mathbf{I}_u + \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{e}}\mathbf{G}_j\mathbf{I}_u\right) u_j \tag{90}$$
$$- \left(\mathbf{G}_j^T\mathbf{E}_x^T Q_x - \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{x}}\mathbf{I}_x - \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{x}}\mathbf{F}_j - \mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{e}}\mathbf{F}_j\right) x_j \tag{91}$$
$$- \left(\mathbf{G}_j^T\hat{\mathbf{Q}}_{\mathbf{e}}\mathbf{I}_e - \mathbf{G}_j^T\mathbf{E}_e^T Q_e\right) e_j \tag{92}$$
$$+ \frac{1}{2}\mathbf{G}_j^T\mathbf{S}_x^T \tag{93}$$

$$\vec{0} = \left(\hat{\mathbf{Q}}_{\mathbf{u}} + \mathbf{G}_j^T\left(\hat{\mathbf{Q}}_{\mathbf{x}} + \hat{\mathbf{Q}}_{\mathbf{e}}\right)\mathbf{G}_j\right)\mathbf{u}_{j+1} \tag{94}$$
$$- \left(\mathbf{E}_u^T Q_u + \mathbf{G}_j^T\left(\hat{\mathbf{Q}}_{\mathbf{x}} + \hat{\mathbf{Q}}_{\mathbf{e}}\right)\mathbf{G}_j\mathbf{I}_u\right) u_j \tag{95}$$
$$- \mathbf{G}_j^T\left(\mathbf{E}_x^T Q_x - \hat{\mathbf{Q}}_{\mathbf{x}}\mathbf{I}_x - \hat{\mathbf{Q}}_{\mathbf{x}}\mathbf{F}_j - \hat{\mathbf{Q}}_{\mathbf{e}}\mathbf{F}_j\right) x_j \tag{96}$$
$$- \mathbf{G}_j^T\left(\hat{\mathbf{Q}}_{\mathbf{e}}\mathbf{I}_e - \mathbf{E}_e^T Q_e\right) e_j \tag{97}$$
$$+ \frac{1}{2}\mathbf{G}_j^T\mathbf{S}_x^T \tag{98}$$

Solving this for $\mathbf{u}_{j+1}$ gives the optimal learning filters and the update law

$$\mathbf{u}_{j+1} = L_u u_j + L_e e_j + L_x x_j + L_c \tag{99}$$

$$L_0 \triangleq \left(\hat{\mathbf{Q}}_{\mathbf{u}} + \mathbf{G}_j^T \left(\hat{\mathbf{Q}}_{\mathbf{x}} + \hat{\mathbf{Q}}_{\mathbf{e}}\right) \mathbf{G}_j\right)^{-1} \tag{100}$$

$$L_u \triangleq L_0 \left(\mathbf{E}_u^T Q_u + \mathbf{G}_j^T \left(\hat{\mathbf{Q}}_{\mathbf{x}} + \hat{\mathbf{Q}}_{\mathbf{e}}\right) \mathbf{G}_j \mathbf{I}_u\right) \tag{101}$$

$$L_x \triangleq L_0 \mathbf{G}_j^T \left(\mathbf{E}_x^T Q_x - \hat{\mathbf{Q}}_{\mathbf{x}} \mathbf{I}_x - \hat{\mathbf{Q}}_{\mathbf{x}} \mathbf{F}_j - \hat{\mathbf{Q}}_{\mathbf{e}} \mathbf{F}_j\right) \tag{102}$$

$$L_e \triangleq L_0 \mathbf{G}_j^T \left(\hat{\mathbf{Q}}_{\mathbf{e}} \mathbf{I}_e - \mathbf{E}_e^T Q_e\right) \tag{103}$$

$$L_c \triangleq -\frac{1}{2} L_0 \mathbf{G}_j^T \mathbf{S}_x^T \tag{104}$$