3^{-n} Decimal Problem

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1 Outline

1.1 Difinition of Problem

Consider a problem:

Definition

For $n \in \mathbb{N}$, define
$$A_n = \frac{1}{3^n 10^{3^k}}$$
and
$$A = \sum_{n=1}^{\infty} A_n$$

Definition

Given a real number $a = 0.a_1a_2a_3\ldots \in [0,1)$ which does not end with repeating 9 and $n \in \mathbb{N}$, **the n-th term of a** is a_n.

Now given $n \in \mathbb{N}$, find the n-th to $(n+9)$-th terms of A? Consider $n = 10^2, 10^8, 10^{16}$.

1.2 Outline of Solution

Since n is at most 10^{16}, it's sufficient to approximate A to be $\sum_{n=1}^{33} A_n$. Then we find k-th term of A by finding around k-th terms of A_n and sum them all. For $k \leq 16$, we find it in directly way, by finding its repetend. For $k > 16$, let $a = 16, b = k - 16$, then simulating 3^b divides the repetend of $1/3^a$. With some regularity, we can jump with the length 3^{a-2} when simulating. In conclusion, we get a solution with time complexity $O(\sqrt{n})$, which is good enough to halt in 1 minutes.

2 Properties of the repetend of $1/3^k$

Fact

1. If $a \in \mathbb{Q}$, then a has a repeating decimal representation.
2. If the repetend of $1/3^n = 0.a_1 \ldots a_k\overline{a_{k+1} \ldots a_{k+l}}$, then $k = 0$.
3. **Long division** algorithm

Definition

When $a \in [0, 1)$ is in the form $0.a_1a_2 \ldots a_k\overline{a_{k+1}a_{k+1} \ldots a_{k+l}}$ with least k and l, and $a_{k+1}, \ldots, a_{k+l}$ not all zero, $a_{k+1} \ldots a_{k+l}$ is called the **repetend** of a. The length of the repetend of a is l.

Property

The length of the repetend of $1/3$ is 1. For $n \geq 2$, the length of the repetend of $1/3^n$ is 3^{n-2}.

Proof. Induction on n. $1/3 = 0.\overline{3}$ and $1/9 = 0.\overline{1}$ are trivial. For $1/3^n$, we generate the repetend of $1/3^n$ by 3 dividing the repetend of $1/3^{n-1}$. Let the repetend of $1/3^{n-1} = 0.\overline{a_1 \ldots a_{3l}}$ and $1/3^{n-2} = 0.\overline{b_1 \ldots b_l}$. If $a_1 \ldots a_{3l} = 0 \pmod 3$, then $b_1 \ldots b_lb_1 \ldots b_lb_1 \ldots b_l = 0 \pmod 9$, thus $b_1 \ldots b_l = 0 \pmod 3$. Then the length of the repetend of $1/3^{n-1}$ can be $l < 3l$, a contradiction. If $a_0 \ldots a_{3l} = 1 \pmod 3$, then $a_1 \ldots a_{3l}a_1 \ldots a_{3l} = 10 \ldots 01 = 2 \pmod 3$. And $a_1 \ldots a_{3l}a_1 \ldots a_{3l}a_1 \ldots a_{3l} = 10 \ldots 010 \ldots 01 = 3 = 0 \pmod 3$. Hence the length of the repetend of $1/3^n$ is a factor of $9l$. On the other hand, it is similar for the case $a_1 \ldots a_{3l} = 2 \pmod 3$. Also, it is larger than $3l$, therefore, it is exactly $9l$. By induction, $9l = 3^{n-2}$.

Property

Let $n, m \in \mathbb{N}$, $n \geq \max\{2, m\}$, then

$$10^{3^{n-2}} = 1 \pmod{3^m}$$

Proof. Induction on m. Clear for $m = 3, 9$. Given m, let $n = m-2$. By induction, $10^{3^{n-3}} = 1 \pmod{3^{m-1}}$, then $10^{3^{n-2}} = 10^{3^{n-3}}10^{3^{n-3}}10^{3^{n-3}}$. Thus $10^{3^{n-2}} = 1 \pmod{3^m}$. And it's clear for $n > \max\{2, m\}$.

3 Algorithm to solve the problem

Algorithm

Find the repetend of 3^{-n}, brutal method.

Let $V := \emptyset, d := 1, A = ()$
while $d \notin V$ **do**
  $V := V \cup \{d\}$
  $d := 10d$
  Let $a \in \mathbb{N} \cup \{0\}$ be maximum such that $a3^n \leq d$
  $A := (A, a)$
  $d := d - a3^n$
end while
return A

Algorithm

Find the repetend of 3^{-n}, with the fact that the length of the repetend of 3^{-n} is 3^{n-2}.

Let $d := 1, A = ()$
for $i := 1$ **to** 3^{n-2} **do**
  $d := 10d$
  Let $a \in \mathbb{N} \cup \{0\}$ be maximum such that $a3^n \leq d$
  $A := (A, a)$
  $d := d - a3^n$
end for
return A

The above 2 algorithms are allowable for $n \leq 17$, at the limit of time and space.

4 Derivation for $n > 17$

Consider $n > 17$, we can't store the whole repetend, but we still can find specified terms. If we want to find k-th term. It's equivalent to find the remainder of 10^{k-1} % 3^n. then do one step of long division.

Let $m = 17, l = n - 17$. Factor 10^{k-1} into

$$10^{m-2} \ldots 10^{m-2} 10^{k-1-N(m-2)},$$

where 10^{m-2} repeat N times and N is the maximum that $N(m-2) \leq k-1$. Let $a_1 \ldots a_{3^{m-2}}$ be the repetend of 3^{m-2}. Notice that $1/3^n = 1/3^m 1/3^l = 0.\overline{a_1 \ldots a_{3^{m-2}}}/3^l$.

Thus

$$\begin{aligned} & 10^{k-1} \pmod{3^n} \\ =& 10^{m-2} \dots 10m - 210^{k-1-N(m-2)} \pmod{3^n} \end{aligned} \tag{1}$$

Definition