3^{-n} Decimal Problem

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1 Outline

1.1 Difinition of Problem

Consider a problem:

Definition

For $n \in \mathbb{N}$, define

$$A_n = \frac{1}{3^n 10^{3^k}}$$

and

$$A = \sum_{n=1}^{\infty} A_n$$

Definition

Given a real number $a = 0.a_1a_2a_3\ldots \in [0, 1)$ which does not end with repeating 9 and $n \in \mathbb{N}$, **the n-th term of a** is a_n.

Now given $n \in \mathbb{N}$, find the n-th to $(n + 9)$-th terms of A? Consider $n = 10^2, 10^8, 10^{16}$.

1.2 Outline of Solution

Since n is at most 10^{16}, it's sufficient to approximate A to be $\sum_{n=1}^{33} A_n$. Then we find k-th term of A by finding around k-th terms of A_n and sum them all. For $k \leq 16$, we find it in directly way, by finding its repetend. For $k > 16$, let $a = 16, b = k - 16$, then simulating 3^b divides the repetend of $1/3^a$. With some regularity, we can jump with the length 3^{a-2} when simulating. In conclusion, we get a solution with time complexity $O(\sqrt{n} \log n)$, which is good enough to halt in 1 minutes.

2 Properties of the repetend of $1/3^k$

Fact

1. If $a \in \mathbb{Q}$, then a has a repeating decimal representation.
2. If the repetend of $1/3^n = 0.a_1 \ldots a_k\overline{a_{k+1} \ldots a_{k+l}}$, then $k = 0$.
3. **Long division** algorithm

Definition

When $a \in [0, 1)$ is in the form $0.a_1a_2 \ldots a_k\overline{a_{k+1}a_{k+1} \ldots a_{k+l}}$ with least k and l, and $a_{k+1}, \ldots, a_{k+l}$ not all zero, $a_{k+1} \ldots a_{k+l}$ is called the **repetend** of a. The length of the repetend of a is l.

Property

The length of the repetend of $1/3$ is 1. For $n \geq 2$, the length of the repetend of $1/3^n$ is 3^{n-2}.

Proof. Induction on n. $1/3 = 0.\overline{3}$ and $1/9 = 0.\overline{1}$ are trivial. For $1/3^n$, we generate the repetend of $1/3^n$ by 3 dividing the repetend of $1/3^{n-1}$. Let the repetend of $1/3^{n-1} = 0.\overline{a_1 \ldots a_{3l}}$ and $1/3^{n-2} = 0.\overline{b_1 \ldots b_l}$. If $a_1 \ldots a_{3l} = 0 \pmod 3$, then $b_1 \ldots b_lb_1 \ldots b_lb_1 \ldots b_l = 0 \pmod 9$, thus $b_1 \ldots b_l = 0 \pmod 3$. Then the length of the repetend of $1/3^{n-1}$ can be $l < 3l$, a contradiction. If $a_0 \ldots a_{3l} = 1 \pmod 3$, then $a_1 \ldots a_{3l}a_1 \ldots a_{3l} = 10 \ldots 01 = 2 \pmod 3$. And $a_1 \ldots a_{3l}a_1 \ldots a_{3l}a_1 \ldots a_{3l} = 10 \ldots 010 \ldots 01 = 3 = 0 \pmod 3$. Hence the length of the repetend of $1/3^n$ is a factor of $9l$. On the other hand, it is similar for the case $a_1 \ldots a_{3l} = 2 \pmod 3$. Also, it is larger than $3l$, therefore, it is exactly $9l$. By induction, $9l = 3^{n-2}$.

Property

Let $n, m \in \mathbb{N} \cup \{0\}$, then

$$10^{n3^{m-2}} = 1 \pmod{3^m}$$

Proof. Consider $n = 1$. Induction on m. Clear for $m = 3, 9$. Given m, let $n = m - 2$. By induction, $10^{3^{n-3}} = 1 \pmod{3^{m-1}}$, then $10^{3^{n-2}} = 10^{3^{n-3}}10^{3^{n-3}}10^{3^{n-3}}$. Thus $10^{3^{n-2}} = 1 \pmod{3^m}$. And it's clear for $n = 0$ and $n > 1$.

3 Algorithm to solve the problem

We consider solving A_n only, then we can get solution by summing A_1 to A_{33}.

Algorithm

Find the repetend of 3^{-n}, brutal method.

Let $V := \emptyset, d := 1, A = ()$
while $d \notin V$ **do**
  $V := V \cup \{d\}$
  $d := 10d$
  Let $a \in \mathbb{N} \cup \{0\}$ be maximum such that $a3^n \leq d$
  $A := (A, a)$
  $d := d - a3^n$
end while
return A

The above algorithm is allowed for $n \leq 17$, at the limit of time and space.

4 Derivation for $n > 17$

For $n > 17$, let $m = 17, l = n - 17$. By long division algorithm, to get the k-th term of $1/3^n$, we can simulate $\frac{3^{-m}}{3^l}$, where we replace 3^{-m} with $0.\overline{a_1 \ldots a_{3^{m-2}}}$, the repetend of 3^{-m}. To get the k-th term of 3^{-n}, it's equivalent to first compute the remainder, $a_1 \ldots a_{k-1} \mod 3^l$, where we define $a_i = a_j \Leftrightarrow i = j \pmod{3^{m-2}}$. Next, we split $a_1 \ldots a_{k-1}$ into N or $N+1$ pieces, where N is maximum satisfying $N3^{m-2} \leq k-1$, say $a_1 \ldots a_{N3^{m-2}} a_{N3^{m-2}+1} \ldots a_{k-1}$. Although $N \approx 10^{16}$, we have fast algorithm to compute $a_1 \ldots a_{N3^{m-2}} \mod 3^l$, by

$$
\begin{aligned}
& a_1 \ldots a_{N3^{m-2}} \pmod{3^l} \\
=& (a_1 \ldots a_{3^{m-2}})(10^{(N-1)3^{m-2}} + 10^{(N-2)3^{M-2}} + \ldots + 10^0) \pmod{3^l} \\
=& (a_1 \ldots a_{3^{m-2}} \mod 3^l) \sum_{i=0}^{N-1} (10^{i3^{m-2}} \mod 3^l) \\
=& (a_1 \ldots a_{3^{m-2}} \mod 3^l) N
\end{aligned}
\tag{1}
$$

Hence we have the following method.

Algorithm

Given $k, n \in \mathbb{N}$, $n > 17$, find k-th term of 3^{-n}

$m := 17$, $l := n - 17$, $a_1 \ldots a_{3^{m-2}} :=$ the repetend of 3^{-m}
$N :=$ the maximum satisfying $N3^{m-2} \leq k - 1$
$a := (a_1 \ldots a_{3^{m-2}} \mod 3^l)N \mod 3^l$
return $aa_1a_2 \ldots a_{k-1-N3^{m-2}} \mod 3^l$

Thus we have a method to get k-th term of A_n. Each step of the above algorithm is $O(\sqrt{k})$. Thus totally $O(\sqrt{k}\log k)$ to find all A_n, $1 \le n \le 33$.

5 Carry of the summation of A_n

The problem ask $(n-1)$-th term to $(n+9)$-th terms, but we can't only fetch those term, we also need to take $(n+10)$-th, $(n+11)$-th or more into consideration, since it may carry to $(n+9)$-th term of A.

Property

For $n = 10^{16}$, we need to consider at most $(n+11)$-th term.

Proof. Let the carry from 3^{-34}-th term be c. Then the carry from $(3^{-34}-1)$-th term is at most $\lfloor (c + 9 \times 33)/10 \rfloor$, the carry from $(3^{-34}-2)$-th term is at most $\lfloor (\lfloor (c+9\times 33)/10 \rfloor + 9\times 33)/10 \rfloor$. Define a sequence $a_1 = c$, $a_n = \lfloor (a_{n-1}+297)/10 \rfloor$. Since a_n is bounded above by sequence $b_1 = c$, $b_n = (b_{n-1} + 297)/10$. And solve it that $b_n = b_1 \times 10^{-(n-1)} + 29.7 \sum_{i=0}^{n-2} 10^i = c10^{1-n} + 29.7/90(10^n - 10)$. Now take $n = 3^{34} - 3^{33} + 9$, which is large enough that a_n is at most 2 digits. Thus if we consider $(n+11)$-th term, it will not affect $(n+9)$-th term.

6 Conclude

In conclusion, we have the following method to solve the problem.

Algorithm

Given $k \le 10^{16}$, find the $(k-1)$-th term to $(k+9)$-th term of A.

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for $n = 1$ **to** 33 **do**
  if $k \le 3^{17}$ **then**
    apply algorithm of section 3, find $(k-1)$-th to $(k+11)$-th term of A_n
  else
    apply algorithm of section 4, find $(k-1)$-th to $(k+11)$0th term of A_n.
  end if
end for
add those terms
return the $(k-1)$-th to $(k+9)$-th term