more problems

problem 1: prove that $p(A|C) = \sum_B p(A|B)p(B|C)$ if A is conditionally independent of C, given B: $A \perp\!\!\!\perp C \mid B$ that is: $p(AC|B) = p(A|B)p(C|B)$

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$$p(AC|B) = p(A|B)p(C|B)$$

$$p(AC|B) = p(A|B)\frac{p(B|C)p(C)}{p(B)} \qquad \text{(Bayes)}$$

$$p(AC|B)p(B) = p(A|B)p(B|C)p(C)$$

$$p(ABC) = p(A|B)p(B|C)p(C) \qquad \Big|\ \sum_B ()$$

(left side: summing out B)

$$p(AC) = p(C)\sum_B p(A|B)p(B|C) \qquad \Big|\ \frac{1}{p(C)}$$

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problem 2: *Prove*

$$p(x, y|z) = p(x|z)p(y|x, z)$$

and also

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$$p(x,y|z) = p(x|z)p(y|x,z)$$ "Conditional Total Probability"

$and\ also$

$$p(x|y,z) = \frac{p(y|x,z)p(x|z)}{p(y|z)}$$ "Conditional Bayes"

$$p(x,y|z) = \frac{p(x,y,z)}{p(z)} = \frac{p(y|x,z)p(x,z)}{p(z)} = p(y|x,z)p(x|z)$$

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problem 3: *Consider three variable distributions which admit the factorisation*

$$p(a, b, c) = p(a|b)p(b|c)p(c)$$

where all variables are binary. How many parameters are needed to specify distributions of this form?

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$2 + 2 + 1 = 5$. This is compared to the 7 parameters in a general three binary-variable distribution.

$$p(abc) = p(c)\, p(ab|c) = \overset{1}{p(c)}\ \overset{2}{p(b|c)}\ \overset{4}{p(a|bc)}$$ (chain rule (holds generally))

$$= \underset{1}{p(c)}\ \underset{2}{p(b|c)}\ \underset{2}{p(a|c)}$$ (using conditional independence $a \perp\!\!\!\perp b \mid c$ which implies $p(a|bc) = p(a|b)$)

problem 4: *The weather in London can be summarised as: if it rains one day there's a 70% chance it will rain the following day; if it's sunny one day there's a 40% chance it will be sunny the following day.*

1. *Assuming that the prior probability it rained yesterday is 0.5, what is the probability that it was raining yesterday given that it's sunny today?*

2. *If the weather follows the same pattern as above, day after day, what is the probability that it will rain on any day (based on an effectively infinite number of days of observing the weather)?*

3. *Use the result from part 2 above as a new prior probability of rain yesterday and recompute the probability that it was raining yesterday given that it's sunny today.*

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1. The transition matrix $p(w_{t+1}|w_t)$ for $w \in \{\mathsf{rain}, \mathsf{run}\}$ is

$$\begin{pmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{pmatrix}$$

We just use then Bayes' rule to compute the probability.

$$p(w_t = \mathsf{rain}|w_{t+1} = \mathsf{sun}) = \frac{p(w_{t+1} = \mathsf{sun}|w_t = \mathsf{rain})p(w_t = \mathsf{rain})}{p(w_{t+1} = \mathsf{sun}|w_t = \mathsf{rain})p(w_t = \mathsf{rain}) + p(w_{t+1} = \mathsf{sun}|w_t = \mathsf{sun})p(w_t = \mathsf{sun})} = 3/7$$

where we used $p(\mathsf{rain}) = p(\mathsf{sun}) = 0.5$

2. In this case, the stationary distribution is $p(rain) = 2/3, p(sun) = 1/3$, which can be found either by explicitly solving for the stationary distribution, or just multiplying the transition matrix

3. Using $p(rain) = 2/3, p(sun) = 1/3$ in the previous calculation, we arrive at $p(w_t = \mathsf{rain}|w_{t+1} = \mathsf{sun}) = 3/5$.

problem 5: . *Sally is new to the area and listens to some friends discussing about another female friend. Sally knows that they are talking about either Alice or Bella but doesn't know which. From previous conversations Sally knows some independent pieces of information: She's 90% sure that Alice has a white car, but doesn't know if Bella's car is white or black. Similarly, she's 90% sure that Bella likes sushi, but doesn't know if Alice likes sushi. Sally hears from the conversation that the person being discussed hates sushi and drives a white car. What is the probability that the friends are talking about Alice?*

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Assuming that $p(\mathsf{Alice}) = p(\mathsf{Bella}) = 0.5$, $\mathrm{dom}(f) = \{\mathsf{Alice}, \mathsf{Bella}\}$, and $p(sushi, car|f) = p(sushi|f)p(car|f)$, then

$$p(\mathsf{Alice}|\mathsf{dislike\ sushi}, \mathsf{white\ car}) = \frac{p(\mathsf{dislike\ sushi}, \mathsf{white\ car}|\mathsf{Alice})p(\mathsf{Alice})}{p(\mathsf{dislike\ sushi}, \mathsf{white\ car}|\mathsf{Alice})p(\mathsf{Alice}) + p(\mathsf{dislike\ sushi}, \mathsf{white\ car}|\mathsf{Bella})p(\mathsf{Bella})}$$

$$= \frac{0.5 \times 0.9 \times 0.5}{0.5 \times 0.9 \times 0.5 + 0.1 \times 0.5 \times 0.5} = 0.9$$

Example: Probabilistic Reasoning (Source: Barber p34)

Sally comes home to find that the burglar alarm is sounding ($A = 1$). Has she been burgled ($B = 1$), or was the alarm triggered by an earthquake ($E = 1$)? She turns the car radio on for news of earthquakes and finds that the radio broadcasts an earthquake alert ($R = 1$).

Using Bayes' rule, we can write, without loss of generality,

$$p(B, E, A, R) = p(A|B, E, R)p(B, E, R) \tag{3.1.16}$$

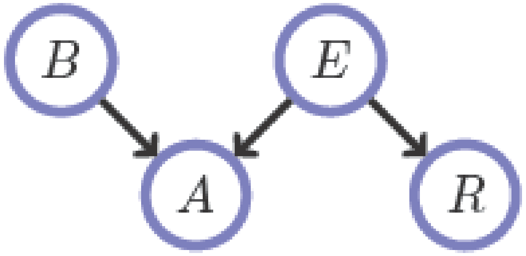
We can repeat this for $p(B, E, R)$, and continue

$$p(B, E, A, R) = p(A|B, E, R)p(R|B, E)p(E|B)p(B) \tag{3.1.17}$$

However, the alarm is surely not directly influenced by any report on the Radio – that is, $p(A|B, E, R) = p(A|B, E)$. Similarly, we can make other conditional independence assumptions such that

$$p(B, E, A, R) = p(A|B, E)p(R|E)p(E)p(B) \tag{3.1.18}$$

as depicted in fig(3.1b).



Specifying conditional probability tables

Alarm = 1	Burglar	Earthquake
0.9999	1	1
0.99	1	0
0.99	0	1
0.0001	0	0

Radio = 1	Earthquake
1	1
0	0

The remaining tables are $p(B = 1) = 0.01$ and $p(E = 1) = 0.000001$. The tables and graphical structure fully specify the distribution. Now consider what happens as we observe evidence.

Initial Evidence: The Alarm is sounding

$$p(B = 1|A = 1) = \frac{\sum_{E,R} p(B = 1, E, A = 1, R)}{\sum_{B,E,R} p(B, E, A = 1, R)} \tag{3.1.19}$$

$$= \frac{\sum_{E,R} p(A = 1|B = 1, E)p(B = 1)p(E)p(R|E)}{\sum_{B,E,R} p(A = 1|B, E)p(B)p(E)p(R|E)} \approx 0.99 \tag{3.1.20}$$

Additional Evidence: The Radio broadcasts an Earthquake warning: A similar calculation gives $p(B = 1|A = 1, R = 1) \approx 0.01$. Thus, initially, because the Alarm sounds, Sally thinks that she's been burgled. However, this probability drops dramatically when she hears that there has been an Earthquake. That is, the Earthquake 'explains away' to an extent the fact that the Alarm is ringing. See `demoBurglar.m`.