Tutoring Session 1

Probability Theory

1 Modeling with different distributions

Problem 1: Suppose that a system contains a certain type of component whose time, in years, to failure is given by T. The random variable T is modelled nicely by the exponential distribution with mean time to failure $\beta = 5$. If 5 of these components are installed in different systems, what is the probability that at least 2 are still functioning at the end of 8 years?

Problem 2: A multiple-choice exam has 200 questions. Each question has four possible answers, with one correct only. What is the probability that random guessing yields from 25 to 30 correct answers for the 80 of the 200 problems about which the student has no knowledge?

You may find it useful to use the *central limit theorem* to solve this exercise:

Let $X_i, i = 1, \ldots, n$ be a sequence of i.i.d. random variables with mean μ and variance σ^2. Then

$$\sqrt{n}\left(\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - \mu\right) \to \mathcal{N}\left(0, \sigma^2\right).$$

(For the mathematically inclined: the convergence is *in distribution.*)

2 Bayes

Problem 3: As an avid player of board games you have a nice collection of non-standard dice: You have a 3-sided, 5-sided, 7-sided, 11-sided and 20-sided die. The five dice are treasured in a beautiful purple velvet bag. Without looking, a friend of yours randomly chooses a die from the bag and rolls a 6. What is the probability that the 11-sided die was chosen? What is the probability that the 20-sided die was used for the role? Show your work!

Now your friend rolls (with the same die!) an 18. What is the probability now that the die is 11-sided? What is the probability that it is 20-sided? Show your work!

Problem 4: Suppose that 15 percent of the items produced in a certain plant are defective. If an item is defective, the probability is 0.9 that it will be moved to the waste. If an item is not defective, the probability is 0.2 that it is moved (accidentally) to the waste.

If an item is removed, what is the probability that it is defective?

Problem 5: A manufacturing firm employs three analytical plans for the design and development of a particular product. For cost reasons, all three are used at varying times. In fact, plans 1, 2, and 3 are used for 30%, 20%, and 50% of the products, respectively. The defect rate is different for the three procedures as follows:

$$p(D \mid P_1) = 0.01, \quad p(D \mid P_2) = 0.03, \quad p(D \mid P_3) = 0.02,$$

where $p(D \mid P_j)$ is the probability of a defective product, given plan j. If a random product was observed and found to be defective, which plan was most likely used and thus responsible?

3 Multivariate distributions

Problem 6: The fraction X of male runners and the fraction Y of female runners who compete in marathon races are described by the joint density function

$$f(x, y) = \begin{cases} 8xy & 0 \leq y \leq x \leq 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Find the covariance of X and Y.