# Basic Linear Regression

by Grady Jensen

Reading Material:
"Pattern Recognition and Machine Learning" by Bishop [ch. 3.1, 3.2, 3.6]

Further extra reading:
"Machine Learning: A Probabilistic Perspective" by Murphy [ch. 7.2 - 7.3, 7.5.4,]

---

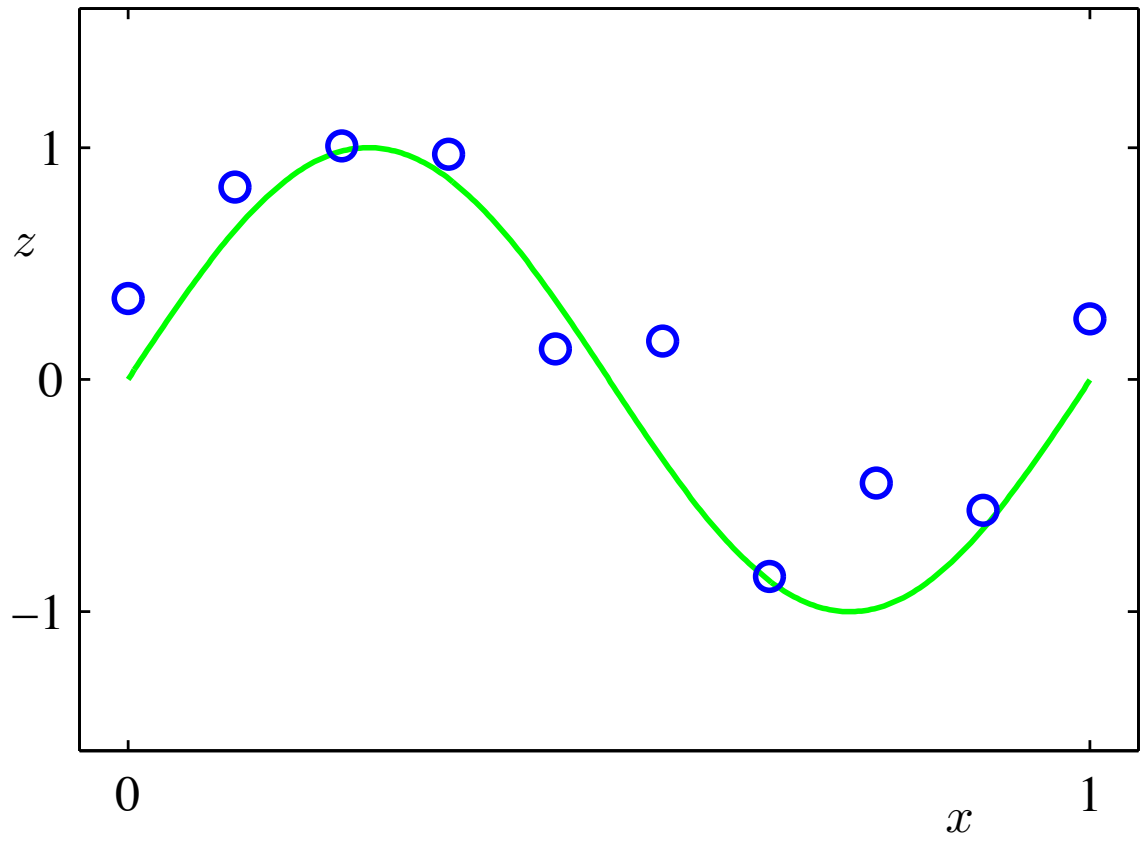
Note: these slides are adapted from slides originally by Christian Osendorfer
Most figures are from C. Bishop: "Pattern Recognition and Machine Learning"

# Notation

| Symbol | Meaning |
|---|---|
| $s$ | a scalar number is lowercase and not bold |
| $S$ | A vector is uppercase |
| $\mathbf{S}$ | a matrix is uppercase and bold |
| $y(X)$ | predicted value of inputs $X$ |
| $z$ | targets |
| $z_i$ | The target of the i'th example |
| $w_0$ | a bias term ( not to be confused with bias in general) |
| $\phi()$ | a basis function |
| $E()$ | an error function |
| $\widetilde{E}()$ | regularized error function |
| $\mathcal{D}$ | The training data |
| $\Phi^\dagger$ | Moore-Penrose pseudoinverse of $\Phi$ |

There is not a special symbol for vectors or matrices augmented by the bias term, $w_0$. Assume it is always included.

# A noisy real-valued function




$$\text{inputs: } X = (x_1, \ldots, x_N)^{\mathsf{T}} \tag{1}$$

$$\text{targets: } z = (z_1, \ldots, z_N)^{\mathsf{T}}, \quad z_i = y(x_i) + \epsilon = \sin(2\pi x_i) + \epsilon \tag{2}$$

Most figures are from C. Bishop: "Pattern Recognition and Machine Learning"

# Fitting a function to data.

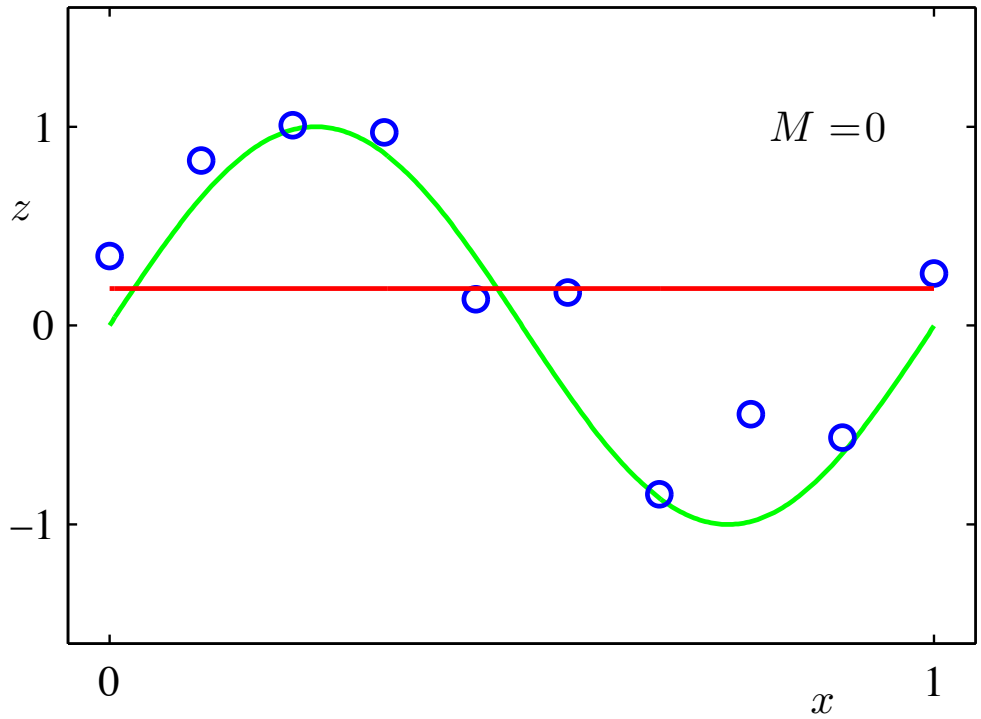
Common approach: Sum of squared Errors

Try to minimize:

$$E_{\mathcal{D}}(W) = \frac{1}{2}\sum^{N}[y(x_n, W) - z_n]^2 \tag{3}$$

We learn the coefficients, $W$, by minimizing some of the error function that gives us a measure for the "misfit" between our model and the data points we have.

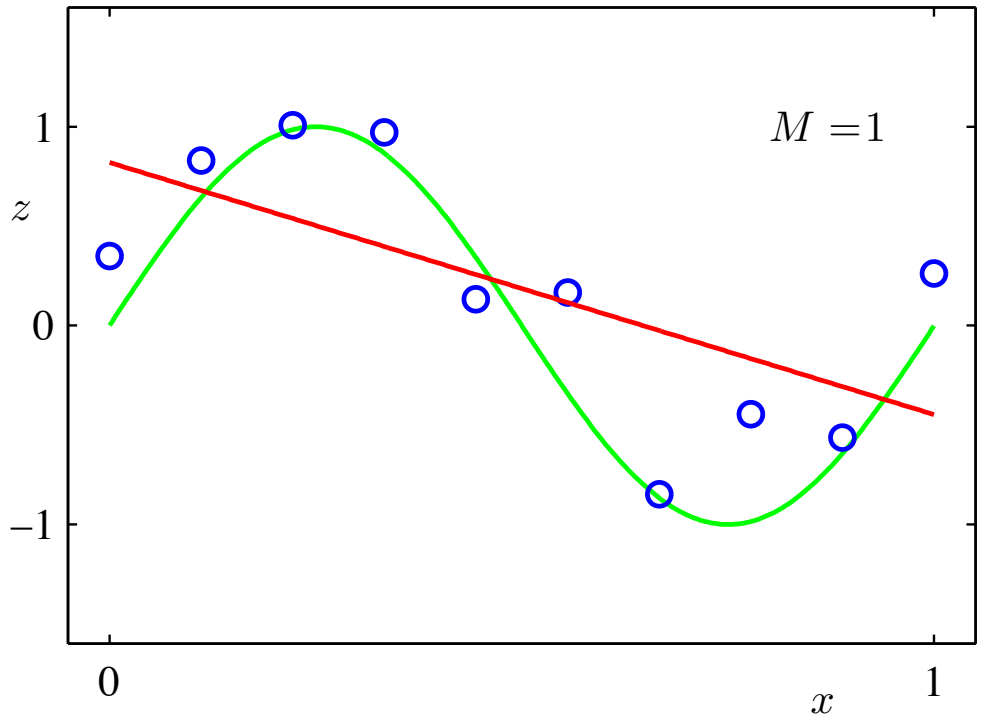
# Model: 0th order polynomial




$$y(x_n, w) = w_0$$

$$E_{\mathcal{D}}(w_0) = \frac{1}{2}\sum^{N}(w_0 - z_n)^2$$

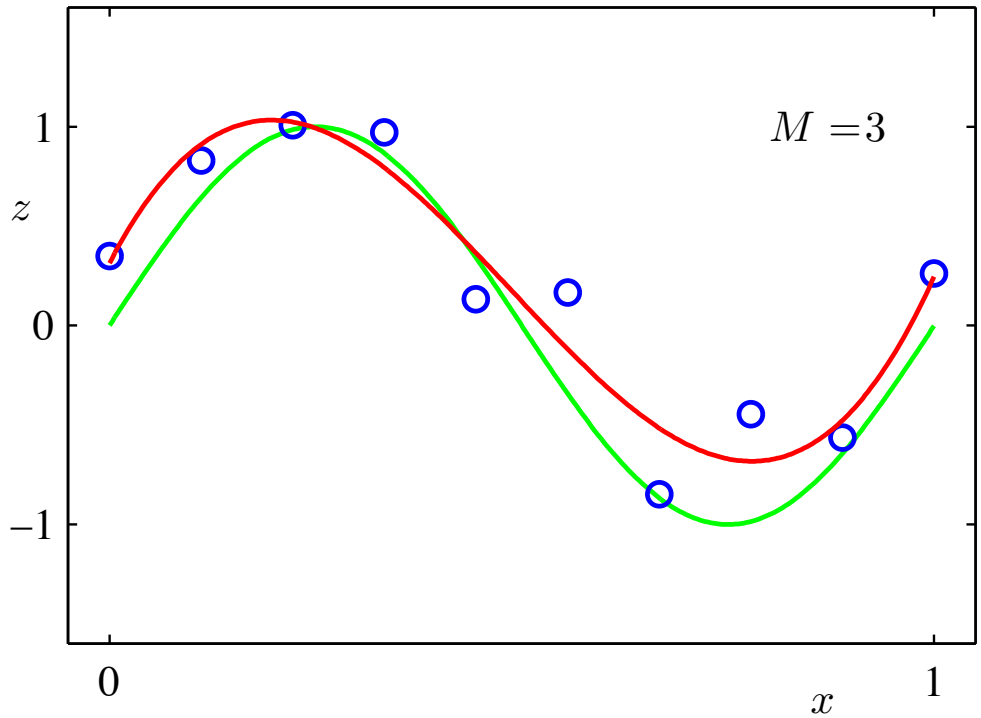
# Model: 1st order polynomial




$$y(x_n, W) = w_0 + w_1 x$$

$$E_{\mathcal{D}}(W) = \frac{1}{2} \sum^{N} (w_0 + w_1 x_n - z_n)^2$$

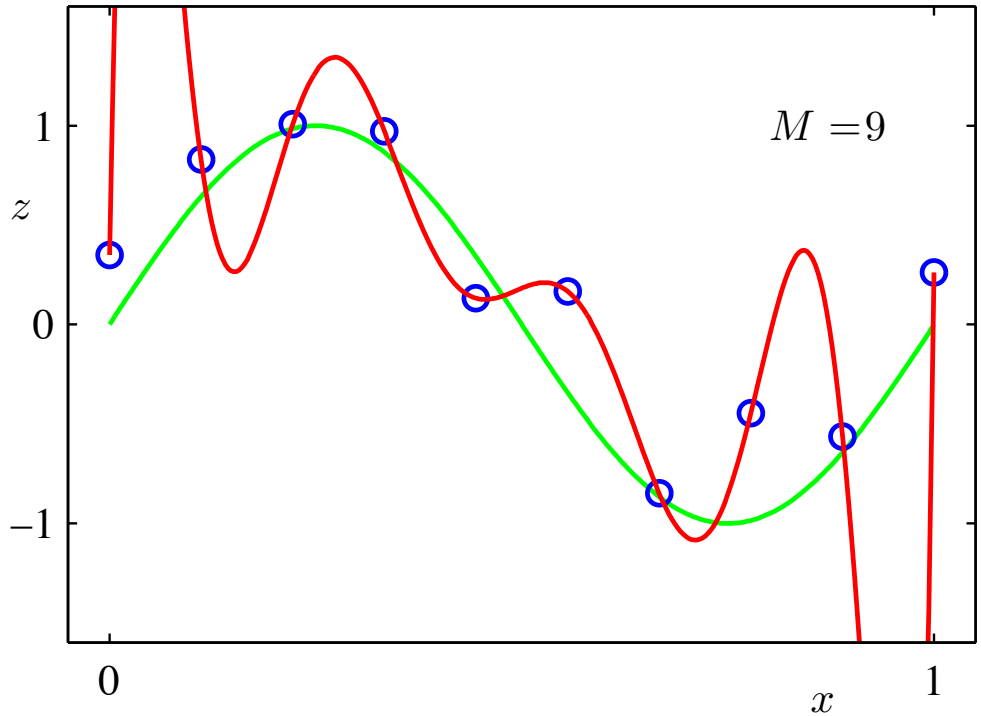
# Model: 3rd order polynomial




$$y(x_n, W) = w_0 + w_1 x + w_2 x^2 + w_3 x^3$$

$$E_{\mathcal{D}}(W) = \frac{1}{2} \sum^{N} \left[ y\left(x_n, W\right) - z_n \right]^2$$

# Model: 9th order polynomial




$$y(x_n, W) = \sum_{j=0}^{M} w_j x^j$$

$$E_{\mathcal{D}}(W) = \frac{1}{2} \sum^{N} \left[ y\left(x_n, W\right) - z_n \right]^2$$

# Problem Definition

We have input vectors $X$ and associated output values $z$. We want to describe the underlying functional relation between them.

What about the following simple model? (Look familiar?)

$$y(X, W) = w_0 + \sum_{j=1}^{M-1} w_j \phi_j(X) = W^{\mathsf{T}} \phi(X) \tag{4}$$

where

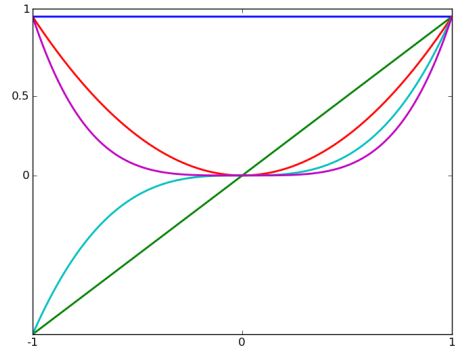
| | | | |
|---|---|---|---|
| $\phi$ | basis function | — | many choices, can be nonlinear |
| $w_0$ | bias | — | equivalent to defining $\phi_0 \equiv 1$ |

It is linear in $W$!

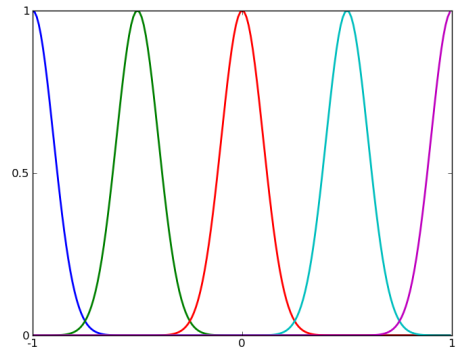
$$E_{\mathcal{D}}(W) = \frac{1}{2} \sum^{N} \left[ W^{\mathsf{T}} \phi(x_n) - z_n \right]^2 \tag{5}$$

# Typical Basis Functions

Polynomials $\phi_j(x) = x^j$

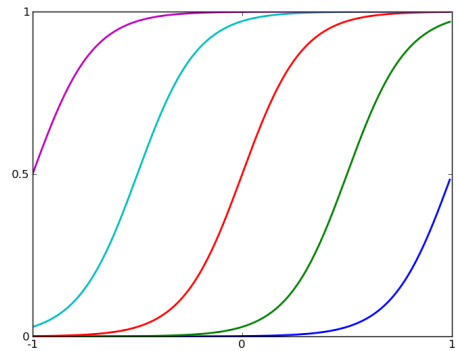



Gaussian $\phi_j(x) = \mathrm{e}^{\frac{-(x-\mu_j)^2}{2s^2}}$




Logistic Sigmoid $\phi_j(x) = \sigma(\frac{x-\mu_j}{s})$,

where $\sigma(a) = \frac{1}{1+\mathrm{e}^{-a}}$



# Optimal Solution

To compute the optimal solution, we need to minimise the error function $E_{\mathcal{D}}$ with respect to $W$:

$$E_{\mathcal{D}}(W) = \frac{1}{2}\sum^{N}\left(W^T\phi(x_n) - z_n\right)^2 = \frac{1}{2}(\mathbf{\Phi}W - z)^T(\mathbf{\Phi}W - z) \quad (6)$$

with

$$\mathbf{\Phi} = \begin{pmatrix} \phi_0(X_1) & \phi_1(X_1) & \dots & \phi_{M-1}(X_1) \\ \phi_0(X_2) & \phi_1(X_2) & & \vdots \\ \vdots & \vdots & \ddots & \\ \phi_0(X_N) & \phi_1(X_N) & \dots & \phi_{M-1}(X_N) \end{pmatrix}$$

being the design matrix of $\phi$.

# Optimal Solution

Therefore we can compute the gradient error function w.r.t. $W$ by taking the derivative of $E_{\mathcal{D}}(W)$.

$$\nabla_W E_{\mathcal{D}}(W) = \frac{\partial}{\partial W} \frac{1}{2}(\mathbf{\Phi} W - Z)^T(\mathbf{\Phi} W - Z) \tag{7}$$

$$= -\mathbf{\Phi}^T(\mathbf{\Phi} W - Z) \tag{8}$$

$$= 0 \tag{9}$$

(The steps from (7) to (8) are not trivial, but Eqs. (71) and (81) of the matrix cook book can help you.) Setting the gradient to zero we get normal equations of (ordinary) least squares problem:

$$W_{\text{optimal}} = \underbrace{(\mathbf{\Phi}^{\mathsf{T}}\mathbf{\Phi})^{-1}\mathbf{\Phi}^{\mathsf{T}}}_{=\mathbf{\Phi}^{\dagger}} z. \tag{10}$$

$\mathbf{\Phi}^{\dagger}$ is called Moore-Penrose pseudo-inverse of $\mathbf{\Phi}$ (because for an *invertible* square matrix, $\mathbf{\Phi}^{\dagger} = \mathbf{\Phi}^{-1}$).

Second derivative: $\mathbf{\Phi}^T\mathbf{\Phi}$ is (semi) positive definite.

# Computational aspect

Computing the solution $W_{\text{Optimal}}$ using the normal equations is not such a great idea. Why?

- If $\mathbf{\Phi}$ is not full rank, $(\mathbf{\Phi}^{\mathsf{T}}\mathbf{\Phi})^{-1}$ does not exist (why? how can rank deficiency happen?)
- Even if $\mathbf{\Phi}$ is full rank, it can be *ill-conditioned* (???), (i.e., $\kappa(\mathbf{\Phi})$ is large), $\mathbf{\Phi}^{\mathsf{T}}\mathbf{\Phi}$ will be even worse ($\kappa(\mathbf{\Phi}^{\mathsf{T}}\mathbf{\Phi}) = \kappa(\mathbf{\Phi})^2$).

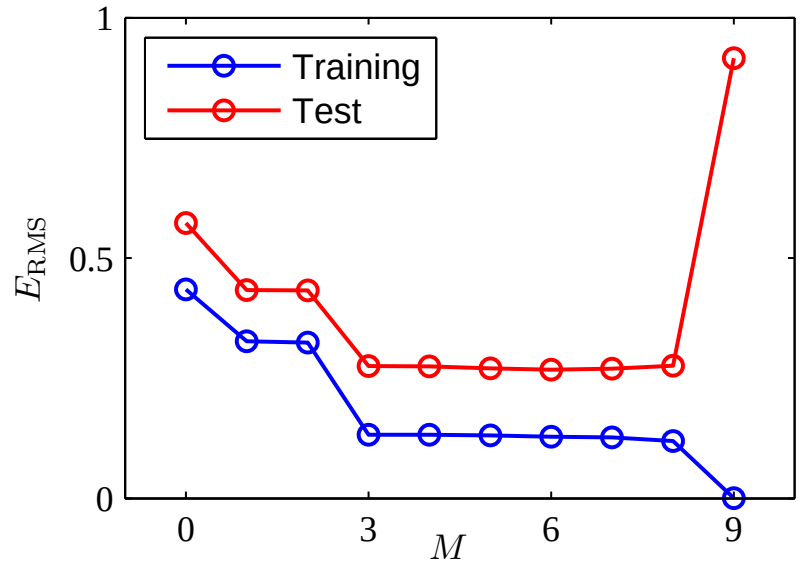
Applied numerical computing :)

# Fighting Overfitting (Split up data)

Intuitively, the use of high level polynomials causes the equation to fit itself to the noise in the data.

- Use training and test sets of the data
- Use Root Mean Square (RMS) to normalize for differences in set sizes.

$$E_{\mathrm{RMS}} = \sqrt{\frac{2E(W^*)}{N}}$$

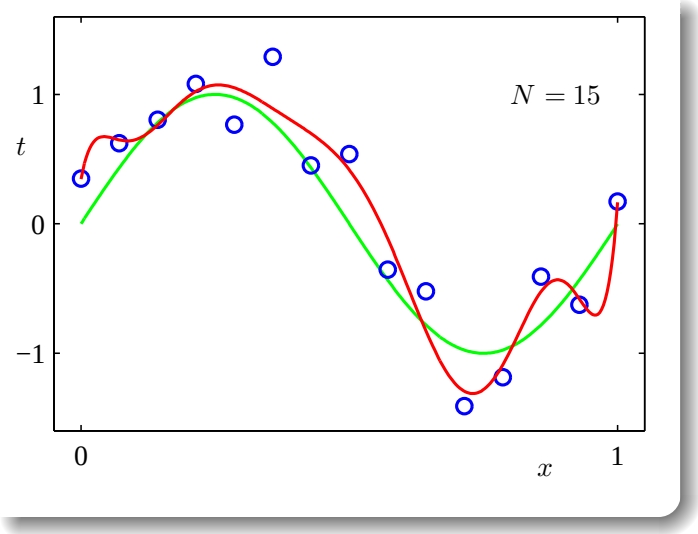


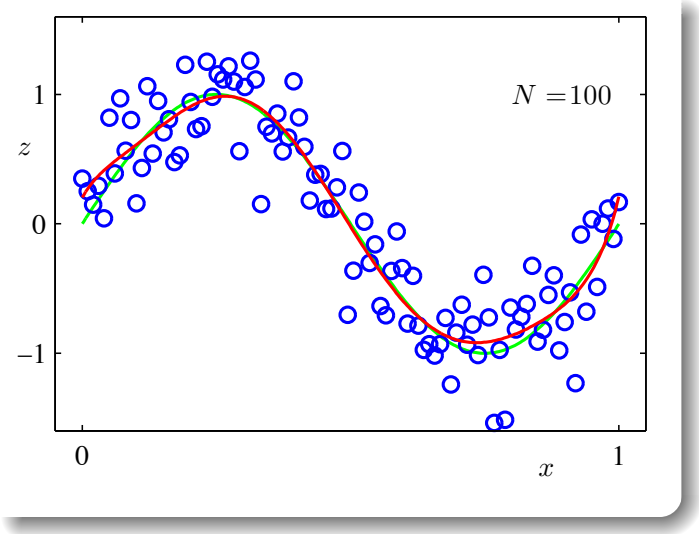

$W^*$ is the value of $W$ that minimizes the error function $E$

# Fighting Overfitting

## Heuristic:

# of data points $>$ 5 or 10 times number of parms.


Sadly, the number of parameters isn't always a good measure of complexity. A much better approach is to choose complexity based on problem to be solved.

# Controlling overfitting with regularization

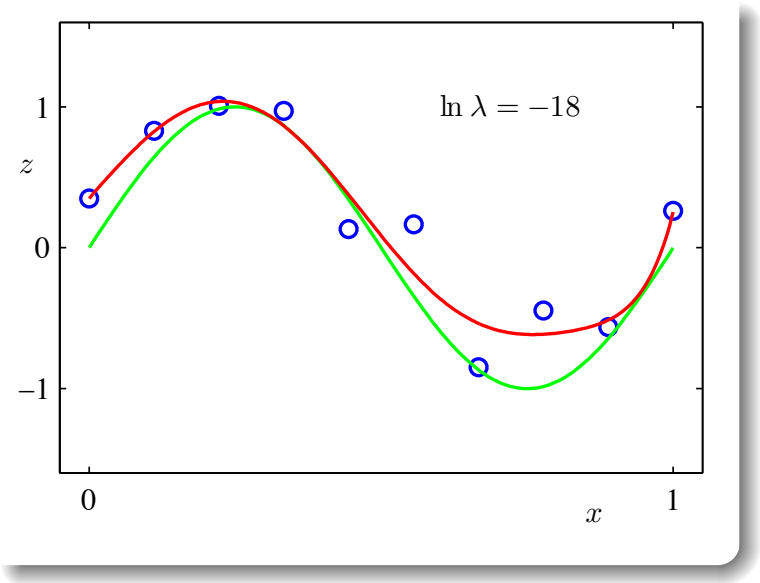
MLE often suffers from overfitting $\rightsquigarrow$ use regularisation:

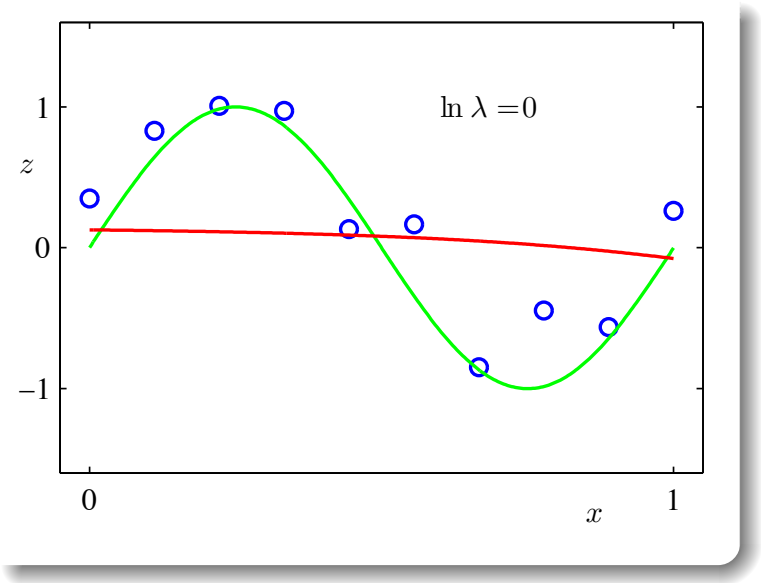
$$\widetilde{E_{\mathcal{D}}}(W) = \frac{1}{2}\sum^{N}\left[W^{\mathsf{T}}\phi(x_n) - z_n\right]^2 + \frac{\lambda}{2}\|W\|_2 \qquad (11)$$

Where,

$$\|W\| = W^T W = w_0^2 + w_1^2 + w_2^2 + \cdots + w_m^2$$

$\lambda =$ weight of regularization term




Penalty term discourages coefficients from reaching large values...

Techniques of regularization, like sum of squares, are called shrinkage methods in statistics because they reduce the value of coefficients.

# Regularization

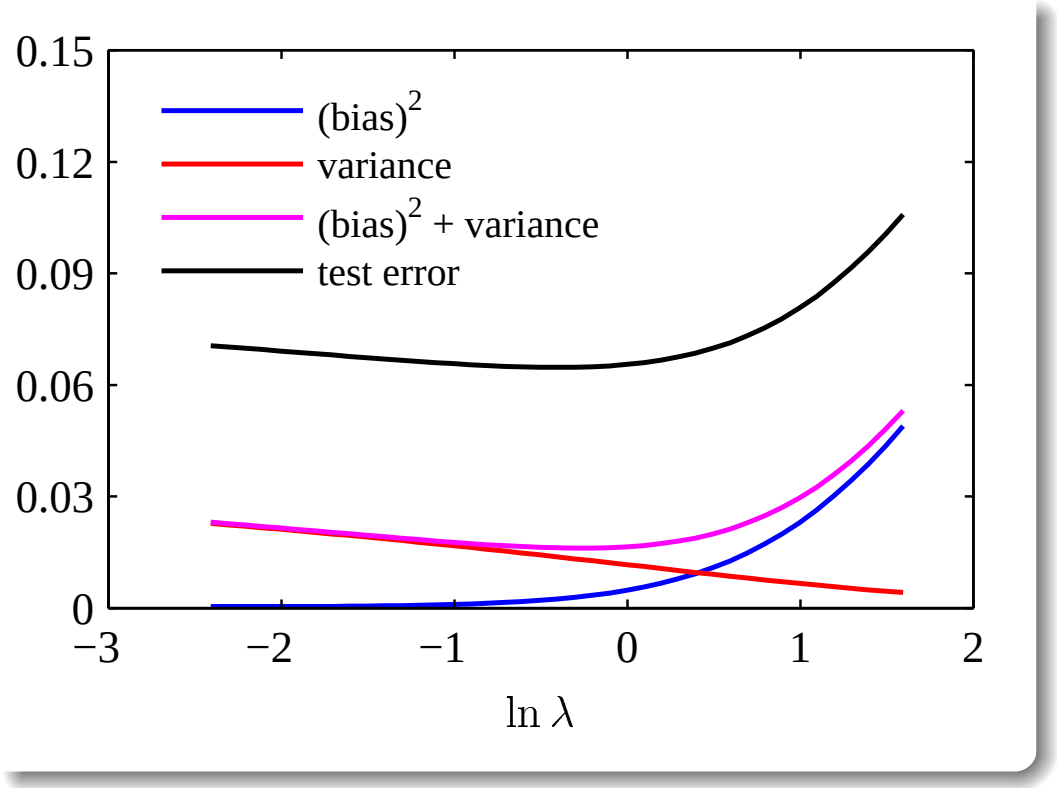
$$\widetilde{E_{\mathcal{D}}}(W) = \frac{1}{2}\sum^{N}\left[W^{\mathsf{T}}\phi(x_n) - z_n\right]^2 + \frac{\lambda}{2}\|W\|_q \qquad (12)$$

$\Rightarrow$ this is like a Lagrange term specifying an additional constraint:

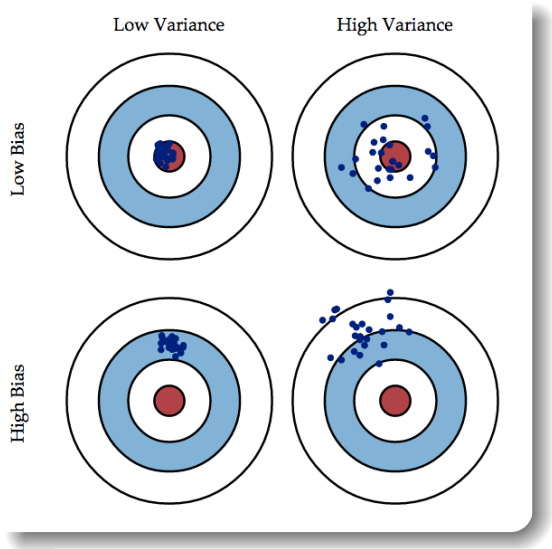
$$\sum^{M}\|w_j\|_q \leq \eta$$

$\hookrightarrow$ most often, use quadratic regulariser ($\ell 2$, *ridge* regression): $q = 2$, i.e.

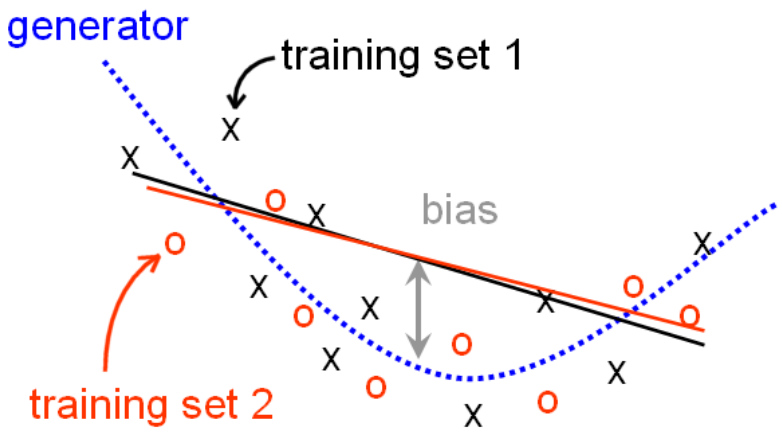
$$\sum^{M}\|w_j\|_2 = W^{\mathsf{T}}W$$

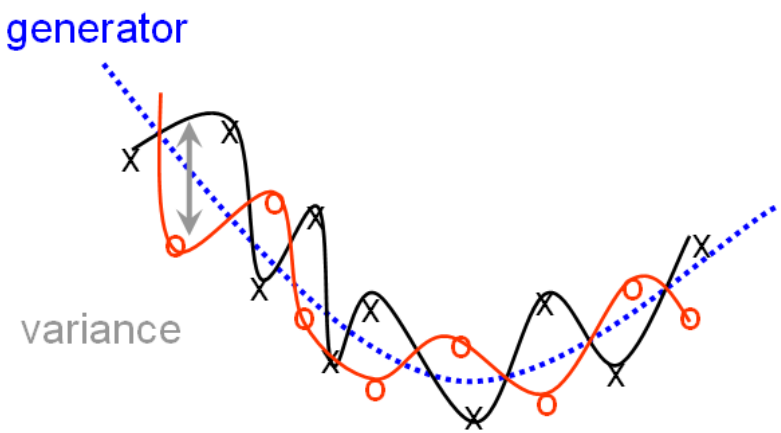


# Bias and Variance




$\longrightarrow$ Regularization $\longleftarrow$
mediates between these extremes.



# Bayesian Linear Regression

by Grady Jensen

Reading Material:
"Pattern Recognition and Machine Learning" by Bishop [ch. 3.3 - 3.3.2, 3.4 - 3.5]

Further extra reading:
"Machine Learning: A Probabilistic Perspective" by Murphy [ch. 7.4 - 7.6]

---

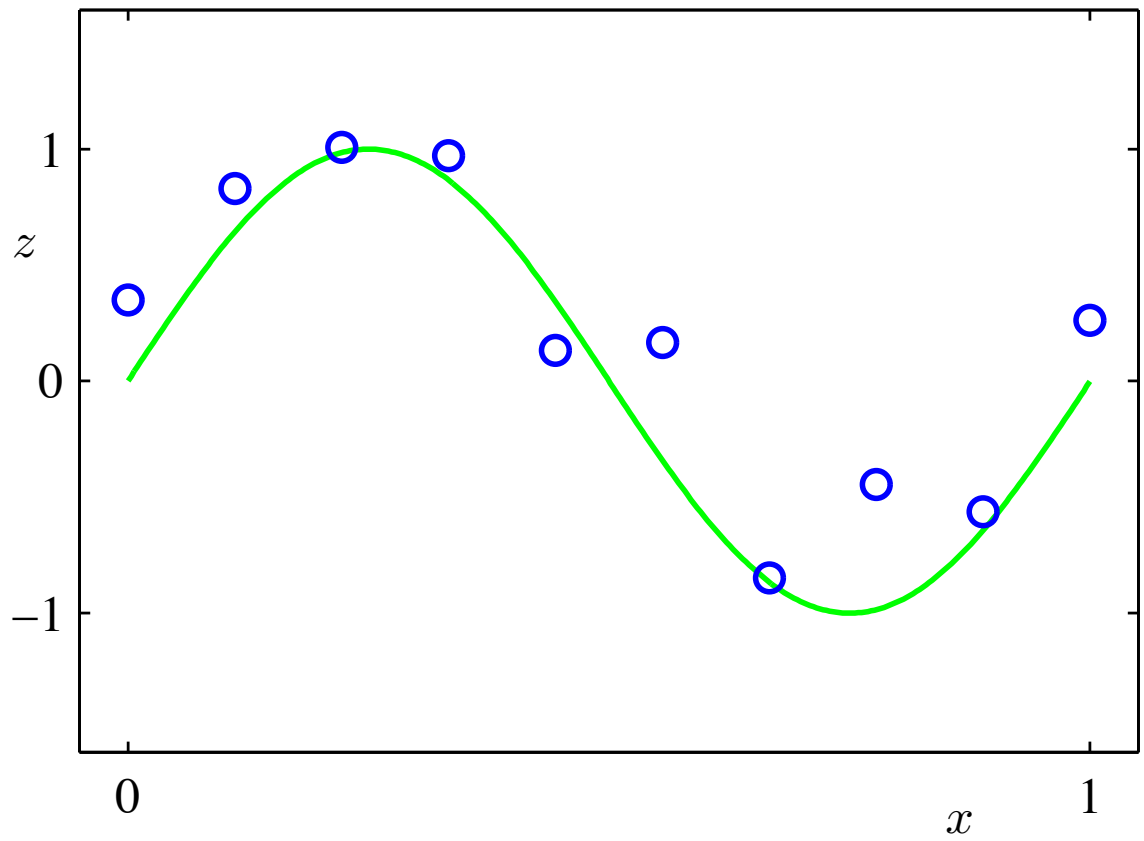
Note: these slides are adapted from slides originally by Christian Osendorfer
Most figures are from C. Bishop: "Pattern Recognition and Machine Learning"

# Notation

| Symbol | Meaning |
|---|---|
| $s$ | a scalar number is lowercase and not bold |
| $S$ | A vector is uppercase |
| $\mathbf{S}$ | a matrix is uppercase and bold |
| $y(X)$ | predicted value of inputs $X$ |
| $z$ | targets |
| $z_i$ | The target of the i'th example |
| $w_0$ | a bias term ( not to be confused with bias in general) |
| $\phi()$ | a basis function |
| $E()$ | an error function |
| $\widetilde{E}()$ | regularized error function |
| $\mathcal{D}$ | The training data |

There is not a special symbol for vectors or matrices augmented by the bias term, $w_0$. Assume it is always included.

# A noisy real-valued function




$$\text{inputs: } X = (x_1, \ldots, x_N)^{\mathsf{T}} \tag{1}$$

$$\text{targets: } z = (z_1, \ldots, z_N)^{\mathsf{T}}, \quad z_i = y(x_i) + \epsilon = \sin(2\pi x_i) + \epsilon \tag{2}$$

# Problem Definition

We have input vectors $X$ and associated output values $z$. We want to describe the underlying functional relation between them.

What about the following simple model? (Look familiar?)

$$y(X, W) = w_0 + \sum_{j=1}^{M-1} w_j \phi_j(X) = W^{\mathsf{T}} \phi(X) \tag{3}$$

where

| | | | |
|---|---|---|---|
| $\phi$ | basis function | — | many choices, can be nonlinear |
| $w_0$ | bias | — | equivalent to defining $\phi_0 \equiv 1$ |

$$E_{\mathcal{D}}(W) = \frac{1}{2} \sum^{N} \left[ W^{\mathsf{T}} \phi(x_n) - z_n \right]^2 \tag{4}$$

# Bayesian Regression

Remember from eq (2) at the start of the lecture,

$$z(x) = y(x, W) + \underbrace{\epsilon}_{\text{noise}}$$

Let's assume the values of the data have gaussian noise with a mean equal to $y(x, W) = w_0 + w_1 x + w_2 x^2 + \cdots + w_m x^M = \sum_{j=0}^{M} w_j x^j$ then,

$$p(z \mid x, W, \beta) = \mathcal{N}(z \mid y(x, W), \beta^{-1}) \approx y(x, W) \tag{5}$$

---

Remember: Without any loss of generality, $y(x, W) = W^T \phi(X)$

# Bayesian Likelihood

Assume the points are drawn i.i.d. and we get a likelihood function of

$$p(Z \mid X, W, \beta) = \prod_{n=1}^{N} \mathcal{N}(z_n \mid y(x_n, W), \beta^{-1}) \tag{6}$$

Or formulated in a more explicit way:

$$\ln p(Z \mid X, W, \beta) = \frac{-\beta}{2} \sum_{n=1}^{N} [y(x_n, W) - z_n]^2 + \frac{N}{2} \ln \beta - \frac{N}{2} \ln 2\pi \tag{7}$$

Why do we use $\ln$ ?

# Bayesian Maximum Likelihood

$$W_{\mathrm{ML}} = \frac{\partial}{\partial W} \ln p(Z \mid X, W, \beta) = 0 \quad (8)$$

$$\Leftrightarrow \frac{\partial}{\partial W} \frac{-\beta}{2} \sum_{n=1}^{N} [y(x_n, W) - z_n]^2 = 0 \quad (9)$$

$$\Leftrightarrow \frac{\partial}{\partial W} \frac{-1}{2} \sum_{n=1}^{N} [y(x_n, W) - z_n]^2 = 0$$

Scaling likelihood by a constant only changes height, not location with respect to $w$

$$\Leftrightarrow \underbrace{\frac{\partial}{\partial W} \frac{1}{2} \sum_{n=1}^{N} [y(x_n, W) - z_n]^2}_{= E_{\mathcal{D}}(W)} = 0$$

Minimize instead of maximize

see equation 4 from the Problem definition slide

What does this mean?

# Bayesian ML continued...

Now let's take the derivative with respect to $\beta$ to get the maximum likelihood value for $\beta$:

$$\frac{\partial}{\partial \beta} \ln(Z \mid X, W, \beta) = 0 \tag{10}$$

$$\Leftrightarrow \quad \frac{\partial}{\partial \beta} \frac{-\beta}{2} \sum_{n=1}^{N} [y(x_n, W) - z_n]^2 + \frac{N}{2} \ln \beta - \frac{N}{2} \ln 2\pi = 0 \tag{11}$$

$$\Leftrightarrow \quad \frac{-1}{2} \sum_{n=1}^{N} [y(x_n, W) - z_n]^2 + \frac{N}{\beta} = 0 \tag{12}$$

$$\Leftrightarrow \quad \frac{1}{\beta_{\mathrm{ML}}} = \frac{1}{N} \sum_{n=1}^{N} [y(x_n, W_{\mathrm{ML}}) - z_n]^2 \tag{13}$$

# Predictive

Plugging in the $W_{ML}$ and $\beta_{ML}$ into our likelihood we get a predictive distribution that allows us to make predictions for new values of $x$.

$$p(z \mid x, W_{\mathrm{ML}}, \beta_{\mathrm{ML}}) = \mathcal{N}(z \mid y(x, W_{\mathrm{ML}}), \beta_{\mathrm{ML}}^{-1}) \tag{14}$$

Up until now we have been using maximum likelihood to solve our problems. An issue with ML approaches is that they can overfit the data (when there isn't a lot of data) because it chooses the best parameters to fit the training data. In practice, this means that any new points, that were not trained on, will classify horribly.

# Maximum A Posteriori (a.k.a. MAP)

Recap: For the coin flip experiment, we introduced prior information to prevent overfitting. By analogy:

| | train data | likelihood | prior | posterior |
|---|---|---|---|---|
| coin: | $\mathcal{D} = X$ | $p(\mathcal{D} \mid \theta)$ | $p(\theta \mid a, b)$ | $p(\theta \mid \mathcal{D})$ |
| regr.: | $\mathcal{D} = \{X, Z\}$ | $p(Z \mid X, W, \beta)$ | $p(W \mid \cdot)$ | $p(W \mid X, Z, \cdot)$ |

Prior: How to find a good one?

A Gaussian prior encourages small parameter values; this discourages the curve of the function from varying wildly.

- recall (from Eq. (6)) that our likelihood function $p(Z \mid X, W, \beta)$ is a Gaussian,
- treat precision $\beta = 1/\sigma^2$ as a known parameter (for now),
- know that the conjugate prior for a Gaussian with known variance is also a Gaussian.

# Posterior Distribution

We can actually find a general closed expression for the posterior! Yay, conjugate priors!

Posterior parameter distribution

$$p(W \mid Z) = \mathcal{N}(W \mid M_N, \boldsymbol{S}_N) \tag{15}$$

with

$$\text{Mean} = M_N = \boldsymbol{S}_N \left( \boldsymbol{S}_0^{-1} M_0 + \beta \boldsymbol{\Phi}^{\mathsf{T}} Z \right) \tag{16}$$

$$\text{Covariance} = \boldsymbol{S}_N^{-1} = \boldsymbol{S}_0^{-1} + \beta \boldsymbol{\Phi}^{\mathsf{T}} \boldsymbol{\Phi} \tag{17}$$

$$S_0^{-1} = \text{prior's covariance}$$

Properties of the posterior:

- Since we again have a Gaussian, the MAP solution (i.e the mode) equals the mean: $W_{\text{MAP}} = M_N$.
- In the limit of an infinitely broad prior, $\boldsymbol{S}_0^{-1} \to 0$, $W_{\text{MAP}} \to W_{\text{ML}}$
- For $N = 0$, i.e. no data points, we get the prior back.

---

$\Phi$ is the design matrix, i.e. a matrix of basis functions $\phi_m(X)$, for each $m$ in $M$.

# Prior distribution example

Let's introduce an isotropic gaussian distribution with zero mean prior over $w$.

$$p(W \mid \alpha) = \mathcal{N}(W \mid 0, \alpha^{-1}\mathbf{I}) = (\frac{\alpha}{2\pi})^{\frac{(M+1)}{2}} e^{(-\frac{\alpha}{2} W^T W)} \tag{18}$$

Where, $\alpha =$ precision of the distribution

$M + 1 =$ # of elements in the vector $W$ for an Mth order polynomial .

This means that the posterior will have the same form as equation 15, with the following differences:

$$M_n = \beta S_n \Phi^T Z$$

$$S_N^{-1} = \alpha I + \beta \Phi^T \Phi$$

# MAP Solution

$$\text{Since } p(W \mid X, Z, \alpha, \beta) \propto p(Z \mid X, W, \beta)\, p(W \mid \alpha), \tag{19}$$

we have a new error function to which we can apply negative log and minimize.

$$E_{MAP} = -\ln p(W \mid X, Z, \alpha, \beta)$$

$$E_{MAP} = \frac{\beta}{2}\sum_{n=1}^{N}[y(x_n, W) - z_n]^2 - \frac{N}{2}\ln\beta + \frac{N}{2}\ln 2\pi - \left(\frac{\alpha}{2\pi}\right)^{\frac{M+1}{2}} + \frac{\alpha}{2}W^T W$$

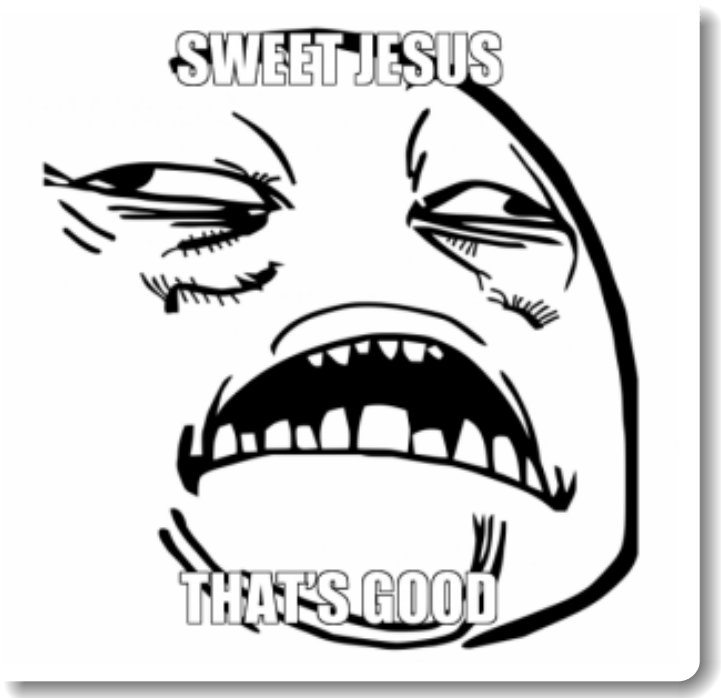
$$\frac{\partial}{\partial W}\frac{\beta}{2}\sum_{n=1}^{N}[y(x_n, W) - z_n]^2 - \frac{N}{2}\ln\beta + \frac{N}{2}\ln 2\pi - \left(\frac{\alpha}{2\pi}\right)^{\frac{M+1}{2}} + \frac{\alpha}{2}W^T W = 0$$
$$\Leftrightarrow \frac{\partial}{\partial W}\frac{\beta}{2}\sum_{n=1}^{N}[y(x_n, W) - z_n]^2 + \frac{\alpha}{2}W^T W = 0$$

$$\Leftrightarrow \frac{\partial}{\partial W}\underbrace{\frac{1}{2}\sum^{N}\left[W^{\mathsf{T}}\phi(x_n) - z_n\right]^2 + \frac{\lambda}{2}\|W\|_2}_{\Rightarrow \widetilde{E_{\mathcal{D}}}(W)} = 0, \text{ where } \lambda = \frac{\alpha}{\beta}$$

Well, that certainly looks familiar!

# Examples of Closed Form Posteriors

| Likelihood | Prior | Posterior Name |
|---|---|---|
| Gaussian | Uniform | Least Squares |
| Gaussian | Gaussian | Ridge |
| Gaussian | Laplace | Lasso |
| Laplace | Uniform | Robust Regression |
| Student | Uniform | Robust Regression |



# Online Linear Regression

by Grady Jensen

Reading Material:
"Pattern Recognition and Machine Learning" by Bishop[ch. 3.1.3, 3.1.4, 3.3.2]

Further extra reading:
"Machine Learning: A Probabilistic Perspective" by Murphy [ch. 7.6.2 - 7.6.3]

---

Note: these slides are adapted from slides originally by Christian Osendorfer
Most figures are from C. Bishop: "Pattern Recognition and Machine Learning"

# Notation

| Symbol | Meaning |
|---|---|
| $s$ | a scalar number is lowercase and not bold |
| $S$ | A vector is uppercase |
| $\mathbf{S}$ | a matrix is uppercase and bold |
| $y(X)$ | predicted value of inputs $X$ |
| $z$ | targets |
| $z_i$ | The target of the i'th example |
| $w_0$ | a bias term ( not to be confused with bias in general) |
| $\phi()$ | a basis function |
| $E()$ | an error function |
| $\widetilde{E}()$ | regularized error function |
| $\mathcal{D}$ | The training data |
| $t$ | a typesetting error that should be $z$ |

There is not a special symbol for vectors or matrices augmented by the bias term, $w_0$. Assume it is always included.

# Data Processing


The algorithm for learning maximum likelihood estimates for $W$ assumes that all data points are available at once (*offline* learning, *batch* learning).

What if large data sets are involved so that batch processing of all points at once is infeasable? What if data points arrive over time (sequentially), and possibly should be discarded as soon as possible?

# Online Learning: The least-mean-squares (LMS) algorithm

The basic idea is to update $W$ after each newly arriving data point by applying the following technique which is also called stochastic gradient descent.

$$W_{(\tau+1)} = W_\tau - \eta \nabla E_n \tag{1}$$

$E_n$ represents the error function of the nth data point and we assume that the overall error is given by $E = \sum_n E_n$ and $\tau$ is the iteration number.

The parameter $\eta$ is called the learning rate and needs to be chosen carefully in order to achieve convergence of the algorithm.

For the least squares algorithm, its update would look like this:

$$W_{(\tau+1)} \quad = \quad W_\tau - \eta \left( z_n - {W_\tau}^T \, \phi(X_n) \right) \phi(X_n)^T \tag{2}$$

# Recursive Least Squares

Determine the *optimal* learning rate for linear regression.

Defining $\boldsymbol{R}_\tau = {\boldsymbol{\Phi}_\tau}^T \boldsymbol{\Phi}_\tau$, we get a recursive relationship over time:

$$W_{(\tau+1)} = W_\tau + {\boldsymbol{R}_{(\tau+1)}}^{-1} X_{(\tau+1)} \left( z_{(\tau+1)} - {X_{(\tau+1)}}^T W_\tau \right)$$

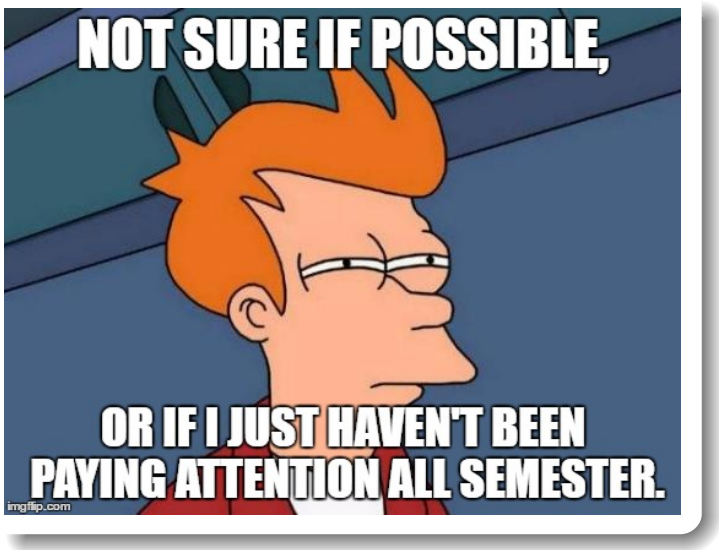
$${\boldsymbol{R}_{(\tau+1)}}^{-1} = {\boldsymbol{R}_\tau}^{-1} - \frac{{\boldsymbol{R}_\tau}^{-1} X_{(\tau+1)} {X_{(\tau+1)}}^T {\boldsymbol{R}_\tau}^{-1}}{\left(1 + {X_{(\tau+1)}}^T {\boldsymbol{R}_\tau}^{-1} X_{(\tau+1)}\right)}$$

- Initial value for ${\boldsymbol{R}^{(0)}}^{-1}$ is usually a diagonal matrix with large entries on its diagonal.
- No matrix inversions are necessary!
- Versions with weighting/forgetting factors are also possible.

# Bayesian Online

What if we don't think of this as an optimization problem, but rather take a Bayesian viewpoint?

Is it possible for us to frame the problem in such a way that takes advantage of Baye's rule to make sequential updates to our approximation to the hidden distribution we are trying to model?



# MAP Solution

Remember, if we introduce a gaussian prior:

$$p(W) = \mathcal{N}(W \mid M_0, \boldsymbol{S}_0) \qquad [M_0\text{: mean, } \boldsymbol{S}_0\text{: covariance matrix}] \quad (3)$$

Often we don't know much about the prior distribution anyway. For a suitably designed model with independent parameters $W$, the following prior is usually reasonable (i.e. an isotropic gaussian):

$$p(W \mid \alpha) = \mathcal{N}(W \mid M_0 = 0, \boldsymbol{S}_0 = \alpha^{-1}\boldsymbol{I}) \quad (4)$$

This results in the posterior:

$$p(W \mid Z, \alpha, \beta) \propto p(Z \mid W, \beta) p(W \mid \alpha)$$

# Posterior Distribution for a Simple Prior

If we look at our simplified case as shown in equation 4 from the previous slide:

$$p(W \mid \alpha) = \mathcal{N}(W \mid M_0 = 0, \boldsymbol{S}_0 = \alpha^{-1}\boldsymbol{I})$$

the posterior parameters simplify to:

$$M_N = \beta \boldsymbol{S}_N \boldsymbol{\Phi}^{\mathsf{T}} Z \tag{5}$$

$$\boldsymbol{S}_N^{-1} = \alpha \boldsymbol{I} + \beta \boldsymbol{\Phi}^{\mathsf{T}} \boldsymbol{\Phi} \tag{6}$$

# Predictive Distribution

Usually, we want to know output $z$ for new values of $X$—the model parameters $W$ are just a means to achieve this. To predict $z$, evaluate

$$\underbrace{p(z \mid X, Z, \alpha, \beta)}_{\text{new posterior}} = \int_W \underbrace{p(z \mid X, W, \beta)}_{\text{likelihood}} \underbrace{p(W \mid Z, \alpha, \beta)}_{\text{old posterior}} \, \mathrm{d}W \tag{7}$$

(coin flip analogy: $p(x \mid \mathcal{D}, a, b) = \int_0^1 p(x \mid \theta) p(\theta \mid \mathcal{D}, a, b) \, \mathrm{d}\theta$)

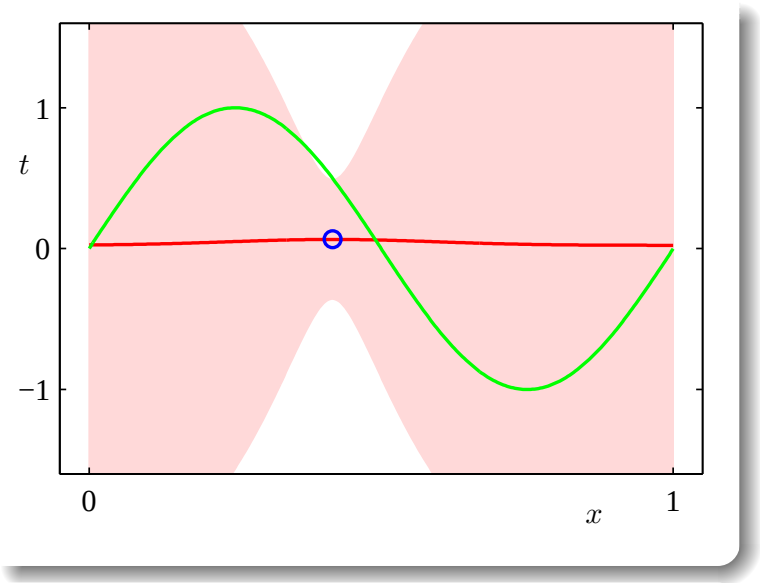
Predictive distribution

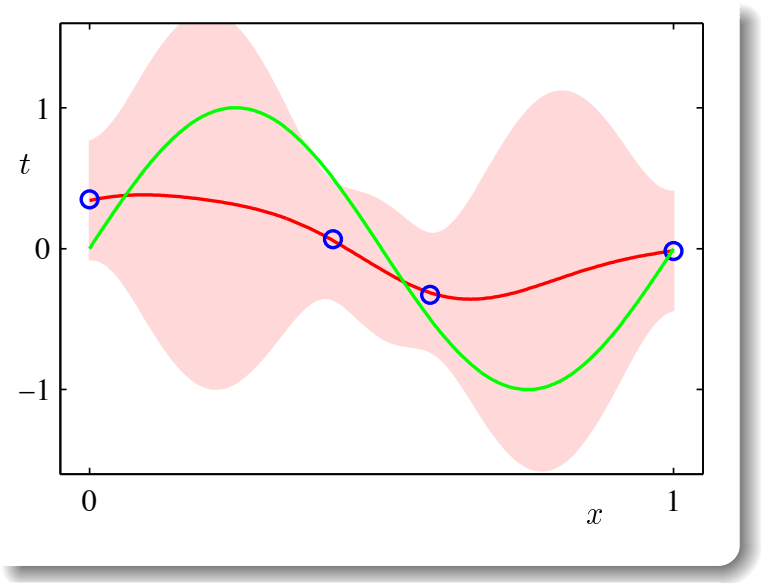
$$p(z \mid X, Z, \alpha, \beta) = \mathcal{N}(z \mid M_N^\mathsf{T} \phi(X), \sigma_N^2(X)) \tag{8}$$

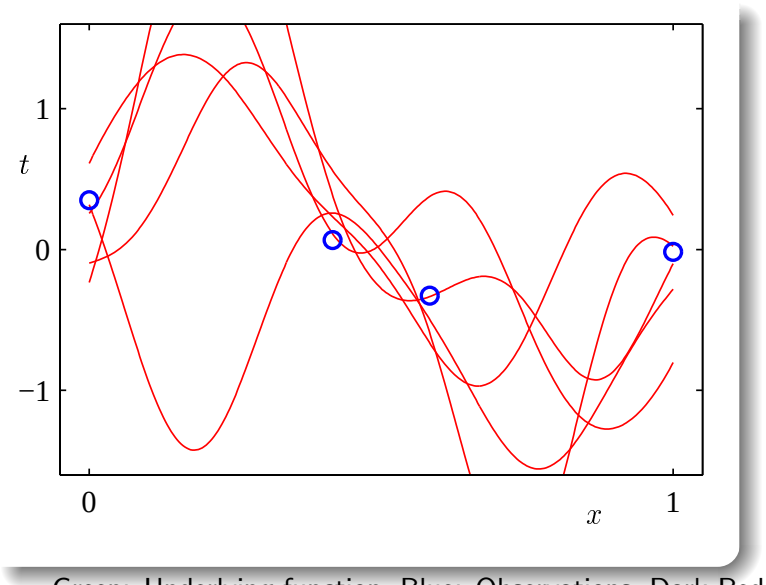
with variance

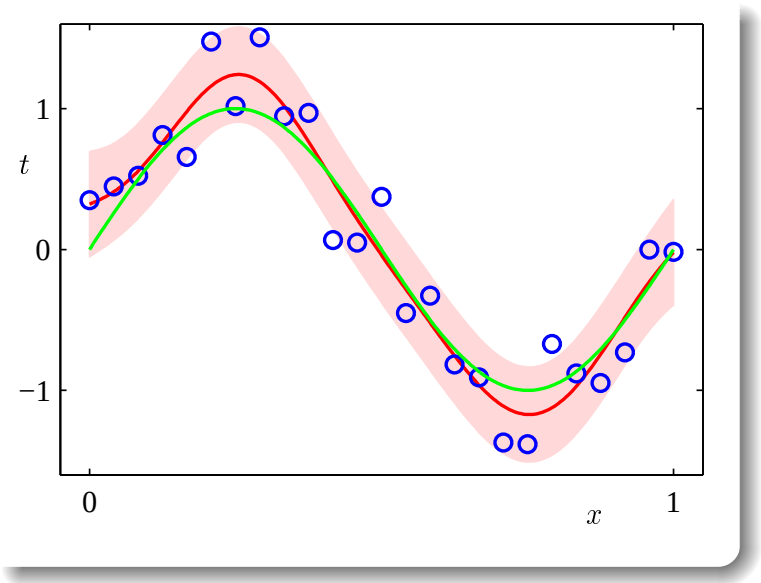
$$\sigma_N^2(X) = \frac{1}{\beta} + \phi(X)^\mathsf{T} \boldsymbol{S}_N \phi(X). \tag{9}$$

# Example of Posterior Predictive




Green: Underlying function, Blue: Observations, Dark-Red: Mode

# A simple example

Bayesian regression for the target values

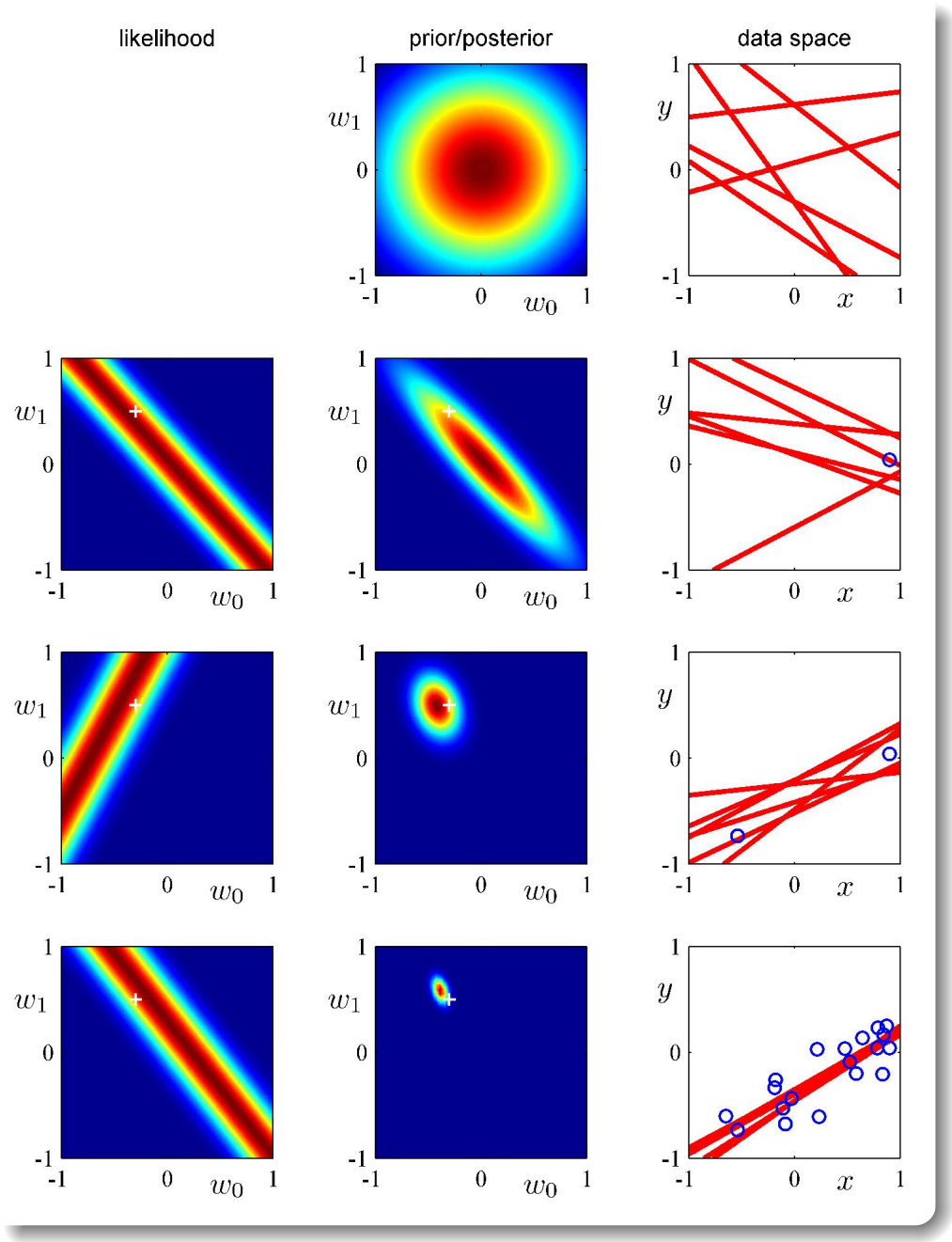
$$z_n = -0.3 + 0.5x_n + \epsilon$$

where $\epsilon$ is a Gaussian noise term ($\sigma = 0.2$).

To model this, we set $\phi(x) = \begin{bmatrix} 1 \\ x \end{bmatrix}$ and thus

$$y(x, W) = w_0 + w_1 x$$

Sequential Estimation: The demo shows how the posterior's breadth gets smaller as more and more points $t$ are taken into account, and how its mode converges to the optimum (=correct) values of the weights (white cross).


likelihood
prior/posterior
data space
$w_0$
$w_1$
$x$
$y$