Machine Learning Worksheet 01

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Problem 1

From the problem statements we know that:

$p(Terrorist) = 0.01$, $p(Scanned \mid Terrorist) = 0.95$
$p(NotTerrorist) = 0.99$, $p(Scanned \mid NotTerrorist) = 0.05$

And we can further calculate the probability of a random person being scanned is:

$p(Scanned) = p(S \mid T) \cdot p(T) + p(S \mid NT) \cdot p(NT) = 0.059$

After that we can get the solution by applying Bayes rule

$$p(Terrorist \mid Scanned) = \frac{p(S \mid T)p(T)}{p(S)} = \frac{0.95 \cdot 0.01}{0.059} \approx 16.1\%$$

The result is about 16.1%

Problem 2

The chances of there being 0, 1 or 2 red balls are tied to the result of coin toss which is a binomial distribution as follows

$p(R = 0) = \binom{2}{0} \cdot 0.5 \cdot 0.5 = 0.25$
$p(R = 1) = \binom{2}{1} \cdot 0.5 \cdot 0.5 = 0.5$
$p(R = 2) = \binom{2}{2} \cdot 0.5 \cdot 0.5 = 0.25$

Then we need to know the probability of 3 red balls are being drawn from the box which is

$p(ThreeRed) = 0 \cdot 0 \cdot 0 \cdot p(R = 0) + 0.5 \cdot 0.5 \cdot 0.5 \cdot p(R = 1) + 1 \cdot 1 \cdot 1 \cdot p(R = 2) = 0.3125$

Finally, we can once again use Bayes rule to get the result

$$p(R = 2 \mid ThreeRed) = \frac{p(ThreeRed \mid R = 2)p(R = 2)}{p(ThreeRed)} = \frac{1 \cdot 0.25}{0.3125} = 0.8$$

The solution is 0.8

Problem 3

This problem is just getting the expected value of two random variables.

$$E[Tail] = \sum_{x=1}^{\infty} x \cdot (\frac{1}{2})^x \cdot \frac{1}{2} = \frac{1}{2} \cdot (\frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + \cdots)$$

Assume $y = \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + \cdots$

$$y - \frac{1}{2}y = \frac{1}{2}y = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{1}{2} \cdot 2 = 1$$

So then we get $y = 2$ and thus

$$E[Tail] = \frac{1}{2} \cdot y = 1$$

Similarly, the expected value of head is

$$E[Head] = \sum_{x=0}^{\infty} 1 \cdot (\frac{1}{2})^x \cdot \frac{1}{2} = \frac{1}{2} \cdot (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...) = \frac{1}{2} \cdot 1 \cdot 2 = 1$$

Problem 4

$$mean = E[X] = \int_a^b \frac{1}{b-a} x dx = \frac{1}{b-a}(\frac{1}{2}b^2 - \frac{1}{2}a^2) = \frac{(b-a)(b+a)}{2(b-a)} = \frac{a+b}{2}$$

$$variance = E[X^2] - E[X]^2 = \int_a^b \frac{1}{b-a} x^2 dx - \frac{(a+b)^2}{4} = \frac{1}{b-a} \cdot \frac{b^3 - a^3}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{(b-a)(a^2 + ab + b^2)}{3(b-a)} - \frac{(a+b)^2}{4} = \frac{a^2 - 2ab + b^2}{12} = \frac{(a-b)^2}{12}$$

Problem 5

$$E_Y[E_{X|Y}[X]] = E_Y[\int p(x \mid y) x dx] = \int \int x p(x \mid y) dx p(y) dy = \int \int x p(x \mid y) p(y) dx dy$$

$$\int \int x p(y \mid x) p(x) dx dy = \int x p(x) dx \int p(y \mid x) dy = E[X] \cdot 1 = E[X]$$

$$E_Y[Var_{x|Y}[X]] + Var[E_{X|Y}[X]] = E_Y[E_{X|Y}[X^2] - E_{X|Y}[X]^2] + E_Y[E_{X|Y}[X]^2] - E_Y[E_{X|Y}[X]]^2$$

$$= E_Y[E_{X|Y}[X^2]] - E_Y[E_{X|Y}[X]]^2 = E[X^2] - E[X]^2 = Var[X]$$

Problem 6

We create a new random variable Y which is the sum of X_1 through X_n.
Since X_1 to X_n are i.i.d. random variables, Y has the mean of $nE[X_1]$ and the variance of $nVar[X_1]$.
Then let us look at the equation

$$p\big(\mid \frac{1}{n}\sum_{i=1}^{n} X_i - E[X_i] \mid > \epsilon\big) = p\big(\mid \sum_{i=1}^{n} X_i - E[X_i] \mid > n\epsilon\big) = p\big(\mid Y - E[Y] \mid > n\epsilon\big)$$

according to Chebyshevs inequality

$$p\big(\mid Y - E[Y] \mid > n\epsilon\big) \leq \frac{Var(Y)}{n^2\epsilon^2}$$

we already know that the variance of X is finite, so the variance of Y will also be finite. As n approaches ∞, the term on the right will approach 0. So we have

$$p\big(\mid Y - E[Y] \mid > n\epsilon\big) \leq 0 \quad as\ n \to \infty$$