Machine Learning Worksheet 4

Multivariate Normal

1 Basic Gaussian

Problem 1: We decomposed the covariance matrix $\mathbf{\Sigma}$ into $\boldsymbol{U}\mathbf{\Lambda}\boldsymbol{U}^T$ with the orthonormal matrix $\boldsymbol{U}$ and the diagonal matrix of eigenvalues $\mathbf{\Lambda}$. Show that:

$$\boldsymbol{U}\mathbf{\Lambda}^{-1}\boldsymbol{U}^T = \sum_{i=1}^{d} \frac{1}{\lambda_i}\boldsymbol{u}_i\boldsymbol{u}_i^T$$

Problem 2: Let $\boldsymbol{L}$ be an invertible n by n matrix over $\mathbb{R}$. Show that the linear transformation $Y = \boldsymbol{L}X$ of a Gaussian is again a Gaussian. Use the moment parametrisation and the change of variable theorem to get there.

2 Conditioned Gaussian

You got a dataset with pairs $\boldsymbol{x}$ and $\boldsymbol{y}$. You computed the covariance of the concatenated datapoints $\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}$. The covariance matrix looks like this: $\begin{pmatrix} \mathbf{\Sigma}_{X,X} & \mathbf{\Sigma}_{X,Y} \\ \mathbf{\Sigma}_{Y,X} & \mathbf{\Sigma}_{Y,Y} \end{pmatrix}$. For simplicity we assume that $\boldsymbol{\mu}_X$ and $\boldsymbol{\mu}_Y$ are $\mathbf{0}$.

Problem 3: The underlying process of the probability distribution $p(\boldsymbol{y} \mid \boldsymbol{x})$ can be expressed as $Y = \boldsymbol{T}X + Z$ where $Z \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_{Y|X})$. What is the value of the matrix $\boldsymbol{T}$?

Problem 4: What is the covariance of Z?