Small Exercises 1

Probability Theory

These small exercises are meant to prepare the inverted classroom lecture. Keep your answers short: two or three sentences, sometimes even less, should suffice. The purpose is to shape or question your intuition, not your mathematical rigour.

Your solutions are submitted *online* in a form found under the link at the bottom of the page. The submission deadline is *before* the respective inverted classroom!

The homework exercise sheet is due later.

Problem 1: Explain the concept of a random variable in 140 characters or less.

Random variables are proxies to probability spaces, easing further computation by mapping an outcome to only its relevant properties. (133 characters.)

Problem 2: What is the highest value a probability mass function can take?

1, as it directly corresponds to a probability, which is limited between 0 and 1.

Problem 3: What is the highest value a probability density function can take?

∞.

In other words: There is no such value. As an example, take a Gaussian distribution with $\mu = 0$ and $\sigma^2 > 0$. Now the pdf at $x = 0$,

$$f\left(0 \mid \mu = 0, \sigma^2\right) = \frac{1}{\sqrt{2\pi}\sigma},$$

grows beyond any boundary for $\sigma \to 0$.

Problem 4: Is the probability density function of a continuous distribution always continuous?

No, only the cumulative distribution function (cdf) is.

Strictly speaking, any continuous distribution with bounded support works, e.g., the uniform distribution is discontinuous at 0 and 1. A more insightful example is a mixture: The random variable X

is a fair random draw from the outcomes of $Y \sim \mathrm{U}(-2, 2)$ and $Z \sim \mathrm{U}(-1, 1)$. It has pdf

$$f(x) = \begin{cases} \frac{1}{8} & x \in [-2, -1) \cup [1, 2) \\ \frac{3}{8} & x \in [-1, 1) \\ 0 & \text{else,} \end{cases}$$

which is obviously discontinuous. At the same time, the cdf, the integral over the domain, is everywhere continuous.

Problem 5: Which plots have positive, which have negative Pearson correlation? Which plot shows the highest correlation?

This is a trick question. All scatter plots were designed such that they have zero correlation. This clearly shows that correlation only measures linear relationships. All data sets are highly dependent.

Problem 6: Let A and B be independent events. Does that imply that they are also conditionally independent given a third event C?

No. Here is a minimal counterexample: You conduct two coin tosses, A is the event that the first one shows up heads, B the same for the second toss. These events are independent. Conditioned on the third event C "both flips showed the same result", A and B are no longer independent.

Problem 7: Let X and Y be independent random variables with

$$\mathbb{E}[X] = \mathbb{E}[Y],$$
$$\mathrm{Var}[X] = \mathrm{Var}[Y].$$

Are X and Y identically distributed?

In general no. Here are three very different distributions with mean 0 and variance 1:

- the standard normal distribution $\mathcal{N}(0, 1)$,
- the uniform distribution on $\left[-\sqrt{3}, \sqrt{3}\right]$,
- the discrete (!) Bernoulli distribution on the slighlty modified support $\{-1, 1\}$.

The statement holds only for Gaussian distributions (because they are fully determined by their first two moments).