Time Series
Lecture 4

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Diebold-Mariano test

- Compares two sequences of forecasts: $\{\hat{Y}_{1t}\}$ and $\{\hat{Y}_{2t}\}$
- Forecasts are the primitives, not models
- Look at the loss differential:

  $d_{12t} = L(e_{1t}) - L(e_{2t}) = (Y_t - \hat{Y}_{1t})^2 - (Y_t - \hat{Y}_{2t})^2$
- Assumption DM: $\{d_{12t}\}$ is covariance-stationary
- Two forecasts are equally good if $\mathrm{E}[d_{12t}] = 0$. That's H_0.
- Form the test statistic:

$$t = \frac{\frac{1}{T}\sum_{t=1}^{T} d_{12t}}{\sqrt{\hat{\sigma}_d / T}},$$

  where $\sigma_d = \sum_{j=-\infty}^{+\infty} \gamma_d(j)$
- $t \to^d \mathcal{N}(0,1)$
- If $t < -z_\alpha$, $\{\hat{Y}_{1t}\}$ is preferable; if $t > z_\alpha$, $\{\hat{Y}_{2t}\}$ is preferable.

MSE

MAE

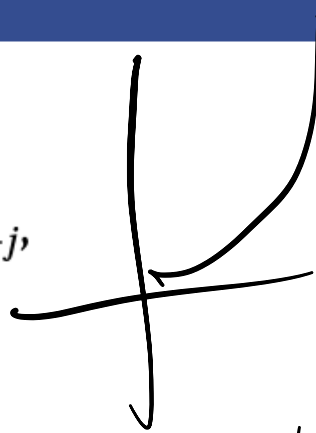
(MAPE)

Type of Non-Stationary TimeSeries

- Time trend
- Unit root
- Structural break in levels
- Structural break in variance

Trend-Stationary TimeSeries

$$Y_t = \mu + \delta t + \Psi(L)\,\varepsilon_t = \mu + \delta t + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j},$$



where $\sum_{j=1}^{\infty} |\psi_j| < \infty$

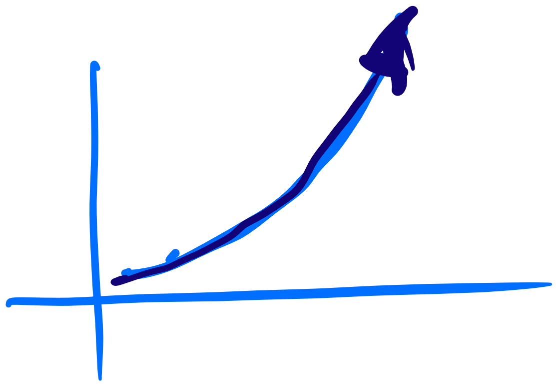
- $Y_t - \delta t$ is stationary
- Forecasts:
  - $\hat{Y}_{t+h|t} = \mu + \delta(t+h) + \psi_h \varepsilon_t + \psi_{h+1} \varepsilon_{t-1} + \ldots$
  - Forecast error: $e_{t+h|t} = \varepsilon_{t+h} + \psi_1 \varepsilon_{t+h-1} + \ldots + \psi_{h-1} \varepsilon_{t+1}$
  - Variance of the forecast error: $\mathrm{Var}(e_{t+h|t}) = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2 < \infty$
- Impulse response to a shock: $\frac{\partial Y_{t+h}}{\partial \varepsilon_t} = \psi_h \to 0$, as $h \to \infty$

Trend-Stationary TimeSeries

- Estimate the trend + arma
- If there is no trend, $\hat{\delta} \rightarrow^{p} 0$
- If there is a trend, but just estimate arma, you'll get something close to a unit root (model is misspecified)
- Trends might be logarithmic or quadratic

$-\bar{\delta} \cdot \ln t$

$-\bar{\delta} \cdot t^2$



Difference stationary TS

$$Y_t = \mu + Y_{t-1} + \Psi(L)\,\varepsilon_t = \mu + Y_{t-1} + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j},$$

where $\sum_{j=1}^{\infty} |\psi_j| < \infty$

- $(1-L)\,Y_t = Y_t - Y_{t-1}$ is stationary
- Forecasts (for simplicity, let $\Psi(L) = I$):
  - $\hat{Y}_{t+h|t} = \mu h + Y_t$
  - Forecast error: $e_{t+h|t} = \sum_{j=1}^{h} \varepsilon_{t+j}$
  - Variance of the forecast error: $\mathrm{Var}(e_{t+h|t}) = \sigma^2 h \to \infty$, as $h \to \infty$
- Impulse response to a shock: $\frac{\partial Y_{t+h}}{\partial \varepsilon_t} = 1$ $\forall h$

Difference Stationary TS

- Work with $Z_t = (1 - L)Y_t = Y_t - Y_{t-1}$, which is stationary
- Need to determine if there is a unit root
- Look at ACF (but might confuse with just large $\theta < 1$)
- Do statistical testing

Dickey Fuller Test

- Model:

$$Y_t = \theta Y_{t-1} + \varepsilon_t$$

- True process: $Y_t = Y_{t-1} + \varepsilon_t$
- The null: $H_0 : \theta = 1$ vs $H_1 : |\theta| < 1$
- Estimate by OLS, form the test statistic:

$$t_n = \frac{\hat{\theta} - 1}{s.e.(\hat{\theta})}$$

- What's the distribution?
- Test with significance level α: Reject H_0 if $t_n < DF_n^\alpha$

Augmented Dickey Fuller Test

- Model:

$$Y_t = c + \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + \varepsilon_t$$

- True process has a unit root: $\theta_1 + \theta_2 = 1$
- Write the equation:

$$\begin{aligned}
Y_t - Y_{t-1} &= c + \theta_1 Y_{t-1} - Y_{t-1} + \theta_2 Y_{t-1} + \theta_2 Y_{t-2} - \theta_2 Y_{t-1} + \varepsilon_t \\
\Delta Y_t &= c + (\theta_1 + \theta_2 - 1) Y_{t-1} - \theta_2 \Delta Y_{t-1} + \varepsilon_t \\
\Delta Y_t &= c + \theta^* Y_{t-1} + \theta_2^* \Delta Y_{t-1} + \varepsilon_t
\end{aligned}$$

lag

OLS

$\theta_1, \theta_2 : \mathcal{E}$

$H_0 : \theta^* = 0$

$H_1 : \theta^* \neq 0$

$|\theta_1 + \theta_2| < 1$

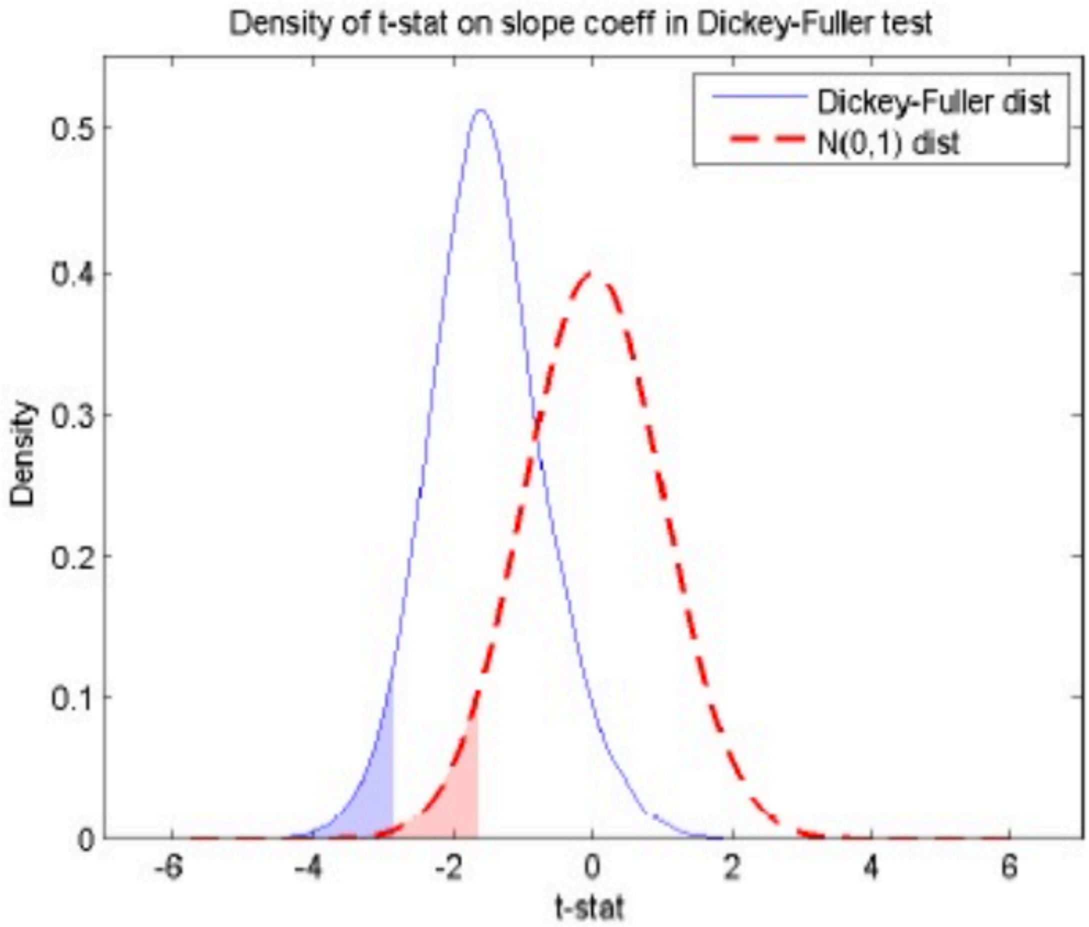
unit root

$H_0 : \theta^* = 0$

$H_1 : \theta^* < 0$ no unit root

- Estimate by OLS, form the test statistic $t_n = \frac{\hat{\theta}^*}{s.e.(\hat{\theta}^*)}$
- The same distribution as before
- Test with significance level α: Reject H_0 if $t_n < DF_n^\alpha$

DF Distribution



Augmented DF Test

- Model:

$$Y_t = c + \theta_1 Y_{t-1} + \dots + \theta_p Y_{t-p} + \varepsilon_t$$

- The process has a unit root: $\theta_1 + \dots + \theta_p = 1$
-

$$\Delta Y_t = c + \theta^* Y_{t-1} + \theta_2^* \Delta Y_{t-1} + \dots + \theta_p^* \Delta Y_{t-p+1} + \varepsilon_t$$

- Estimate by OLS, form the test statistic:

$$t_n = \frac{\hat{\theta}^*}{s.e.(\hat{\theta}^*)}$$

- The same distribution as before
- Test with significance level α: Reject H_0 if $t_n < DF_n^\alpha$

Augmented DF Test

- Distribution is different in 4 different cases:

Case number	True Model	Estimated Model
1	$Y_t = Y_{t-1} + \varepsilon_t$	$Y_t = \theta Y_{t-1} + \varepsilon_t$
2	$Y_t = Y_{t-1} + \varepsilon_t$	$Y_t = c + \theta Y_{t-1} + \varepsilon_t$
3	$Y_t = c + Y_{t-1} + \varepsilon_t$	$Y_t = c + \theta Y_{t-1} + \varepsilon_t$
4	$Y_t = c + Y_{t-1} + \varepsilon_t$	$Y_t = c + \theta Y_{t-1} + \delta t + \varepsilon_t$

Philips Perron Test

- Model:

$$Y_t = c + \theta_1 Y_{t-1} + \varepsilon_t$$

- True process has a unit root: $Y_t = Y_{t-1} + \varepsilon_t$
- But allow $\{\varepsilon_t\}$ to be serially correlated
- Two alternative test statistics (ρ and τ)
- Same distribution as DF
- Applicable to cases 1,2,4

Kwiatowski, Philips, Schmidt, Shin Test

$\mu_t = \mu_{t-1} = \mu_{t-2}$

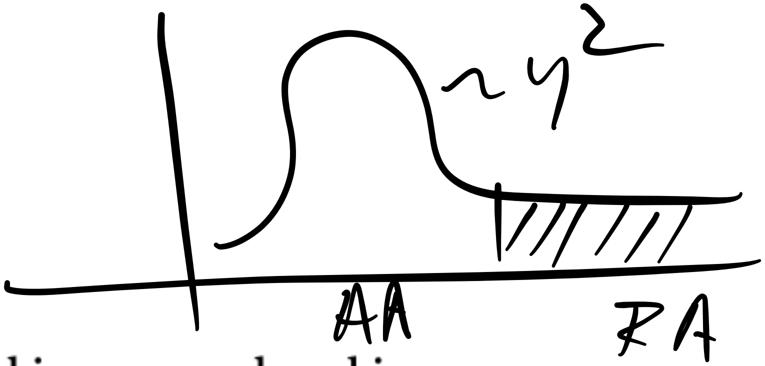
$Y_{t+1} = \ldots + \mu_{t+1} + \varepsilon_{t+1}$

- Model:

$$Y_t = trend + \mu_t + \varepsilon_t, \text{ where } \mu_t = \mu_{t-1} + u_t,$$

ε_t is $I(0)$, possibly heteroskedastic

- $H_0 : \sigma_u^2 = 0$ (i.e. $\mu_t = const$)
- The test is against one-sided alternative



$\delta \cdot t$

- The distribution depends on which trend is assumed and is non-standard

~~δt~~

- Reject the null at 5% level if *KPSS* is larger than 95% quantile of its distribution

Determining d in ARIMA (p,d,q)

- Test whether there is a unit root in $\{Y_t\}$
- If reject, set $d = 0$
- If fail to reject, consider $Z_t = \Delta Y_t$ and test whether there is a unit root in $\{Z_t\}$
- If reject, set $d = 1$
- If fail to reject, consider $W_t = \Delta Z_t = \Delta^2 Z_t$ and test whether there is a unit root in $\{W_t\}$
- ...