Time Series
Lecture 2

Peter Lukianchenko

21 January 2023

$$y_t = \underbrace{\alpha \cdot y_{t-1} + \beta \cdot y_{t-2}}_{AR} + \varepsilon_t + \underbrace{\gamma_1 \varepsilon_{t-1} + \gamma_2 \cdot \varepsilon_{t-2}}_{MA}$$

$\varepsilon_t \sim N(0; \sigma_t^2)$

ARIMA
ARFIMA
SARFIMA
SARFIMAX

ARMA models

GARCH

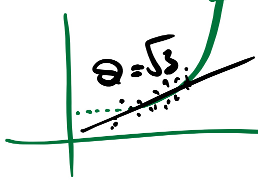
ARCH
GARCH
TGARCH
igarch
egarch
...

Autoregressive Moving-Average Models

$y_t = \theta^t \cdot y_0 + \dots$

$\{\varepsilon_t\}$ is a white noise

ARMA(1;0) = AR(1)



Unit Root Test

$H_0: \theta = 1$

$H_1: \theta < 1$

- ARMA(1, 1)

$y_{t-1} = \theta \cdot y_{t-2} + \varepsilon_{t-1}$

$y_t = \theta^2 y_{t-2} + \theta \varepsilon_{t-1} + \varepsilon_t$

$$\boxed{Y_t = \theta Y_{t-1} + \varepsilon_t + \cancel{\dots}}$$

$y_t = \{\theta^t \cdot y_0 + \varepsilon_t + \theta \varepsilon_{t-1} + \theta^2 \varepsilon_{t-2} + \dots + \theta^{t-1} \cdot \varepsilon_1$

$|\theta| < 1 \quad t \to +\infty \quad \theta^t \cdot y_0 \to 0$

$\boxed{AR(1) = MA(\infty)}$

- **ARMA(p, q)**

$$Y_t = \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + \cdots \theta_p Y_{t-p} + \varepsilon_t + \varphi_1 \varepsilon_{t-1} + \varphi_2 \varepsilon_{t-2} + \dots + \varphi_q \varepsilon_{t-q}$$

- More general ARMA(p, q)

AR(1) = ARMA(1;0) $\quad y_t = \theta_1 y_{t-1} + \varepsilon_t$

$$Y_t = c + \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + \cdots \theta_p Y_{t-p} + \varepsilon_t + \varphi_1 \varepsilon_{t-1} + \varphi_2 \varepsilon_{t-2} + \dots + \varphi_q \varepsilon_{t-q}$$

ARMA(1;1) $\quad y_t = \theta_1 \cdot y_{t-1} + \varepsilon_t + \varphi_1 \cdot \varepsilon_{t-1}$

ARMA(2;3) $\quad y_t = \theta_1 y_{t-1} + \theta_2 y_{t-2} + \varepsilon_t + \varphi_1 \varepsilon_{t-1} + \varphi_2 \cdot \varepsilon_{t-2} + \varphi_3 \cdot \varepsilon_{t-3}.$

Properties of ARMA(p,q) models

ARMA

White Noise

- **Stationarity**
  - $E[\varepsilon_t] = 0$ for all t
  - $\text{Var}(\varepsilon_t) = \sigma^2$ for all t
  - $Cov(\varepsilon_t, \varepsilon_{t+j}) = 0$ for all t and $j \neq 0$
- **Autocovariances**
  - $\gamma(0) = \sigma^2$
  - $\gamma(k) = 0$ for all $k \neq 0$
- **Autocorrelation**
  - $\rho(0) = 1$
  - $\rho(k) = 0$ for all $k \neq 0$
- **PACF**
  - $\alpha(0) = 1$
  - $\alpha(k) = 0$ for all $k > 0$

MA(1): $Y_t = \varepsilon_t + \varphi\varepsilon_{t-1}$

$E\varepsilon_t = 0$

$V\varepsilon_t = \sigma^2$

$Cov(\varepsilon_t, \varepsilon_j) = 0$

- **Stationarity**
  - $E[Y_t] = E[\varepsilon_t] + \varphi E[\varepsilon_{t-1}]$ for all t $\quad EY_t = 0 \ \forall t$
  - $\mathbf{Var(Y_t)} = Var(\varepsilon_t) + \varphi^2 Var(\varepsilon_{t-1}) + 2\varphi Cov(\varepsilon_t, \varepsilon_{t-1}) = \boldsymbol{(1+\varphi^2)\sigma^2}$ for all t
  - $\boldsymbol{Cov(Y_t, Y_{t+1}) = \varphi\sigma^2}$ for all t. $\boldsymbol{Cov(Y_t, Y_{t+k}) = 0}$, for all |k| > 1
- **Autocovariances**
  - $\boldsymbol{\gamma(0)} = Var(Y_t) = \boldsymbol{\sigma^2(1+\varphi^2)}, \boldsymbol{\gamma(1)} = Cov(Y_t, Y_{t+1}) = \boldsymbol{\varphi\sigma^2}$
  - $\boldsymbol{\gamma(k) = 0}$ for all $|k| > 1$
- **ACF** = autocorrelation fn

ACF

  - $\rho(0) = 1, \rho(1) = \frac{\varphi}{1+\varphi^2}$
  - $\boldsymbol{\rho(k) = 0}$ for all $|k| > 1$

$\rho(0) = \frac{\gamma(0)}{\gamma(0)} \quad \rho(1) = \frac{\gamma(1)}{\gamma(0)}, \ \rho(2) = \frac{\gamma(2)}{\gamma(0)}$

- **PACF**
  - complicated, but does not become 0 at some lag

$\boxed{\text{MA} \rightarrow \text{ACF}}$

MA(q): $Y_t = \varepsilon_t + \varphi_1 \varepsilon_{t-1} + \ldots + \varphi_q \varepsilon_{t-q}$

- **Stationarity**
  - automatically follows from stationarity of $\{\varepsilon_t\}$
- **Autocovariances**
  - $\boldsymbol{\gamma(0)} = Var(Y_t) = \boldsymbol{\sigma^2(1 + \varphi_1^2 + \ldots + \varphi_q^2)}$,
  - $\boldsymbol{\gamma(k) = \sigma^2(\varphi_k + \varphi_{k+1}\varphi_1 + \varphi_{k+2}\varphi_2 + \ldots + \varphi_q \varphi_{q-k})}$ for $k = 1, \ldots, q$
  - $\boldsymbol{\gamma(k) = 0}$ for $|k| > q$

ACF $\rho(0) = 1$

$\rho(1) \neq 0$

$\rho(\ldots) \neq 0$

$\rho(q) \neq 0$

$\rho(>q) = 0$

AR(1): $Y_t = \theta Y_{t-1} + \varepsilon_t$

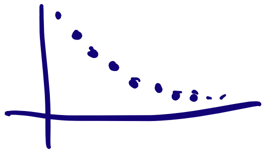
- Plug in the expression for $Y_{t-1}, Y_{t-2,}$ and so on:
  - $Y_t = \theta Y_{t-1} + \varepsilon_t$
  - $Y_{t-1} = \theta Y_{t-2} + \varepsilon_{t-1}$
  - $Y_t = \theta(\theta Y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t = \theta^2 Y_{t-2} + \theta\varepsilon_{t-1} + \varepsilon_t$

$$\boxed{\boldsymbol{Y_t = \theta^n Y_{t-n} + \sum_{j=0}^{n-1} \theta^j \varepsilon_{t-j}}}$$

- If $|\theta| \geq 1$, as $n \to \infty, \theta^n \to \infty$, and Y_t explodes.
- So we need **$|\theta| < 1$ for stationarity**.

AR(1): $Y_t = \theta Y_{t-1} + \varepsilon_t$! AR(1) $\longleftrightarrow$ PACF $\alpha(1) = \theta$

- **Stationarity:** stationary if $|\theta| < 1$. Then
  - $E[Y_t] = 0$ for all t
  - $\mathbf{Var}(\boldsymbol{Y_t}) = \theta^2 Var(Y_{t-1}) + Var(\varepsilon_t) = \frac{\sigma^2}{1-\theta^2}$ for all t
  - ! $\boldsymbol{Cov(Y_t, Y_{t-k}) = \theta^k \frac{\sigma^2}{1-\theta^2}}$ for all t, for all k
- **Autocovariances**
  - $\boldsymbol{\gamma(k) = \theta^k \frac{\sigma^2}{1-\theta^2}}$ for all k
- **ACF**
  - $\boldsymbol{\rho(k) = \theta^k}$ for all k



- **PACF**
  - $\alpha(1) = \theta$
  - $\alpha(k) = 0$ for all $|k| > 1$

$$\underbrace{Var\, Y_t}_{T} = \theta^2 \, Var\, Y_{t-1} + Var\, \varepsilon_t$$

$$T = \theta^2 \cdot T + \sigma^2$$

$$Var\, Y_t = T = \frac{\sigma^2}{1-\theta^2}$$

$$\frac{Cov(Y_t, Y_{t-k})}{Cov(Y_t, Y_{t-k-1})} = \theta$$

AR(1): $Y_t = \theta Y_{t-1} + \varepsilon_t$

- Can be derived in a different way: $(1 - \theta L)Y_t = \varepsilon_t$, so if $(1 - \theta L)$ has an inverse, Y_t can be written as

$$Y_t = (1 - \theta L)^{-1}\varepsilon_t = \sum_{j=0}^{\infty} \theta^j L^j \varepsilon_t$$

- So it is covariance-stationary, if $\sum_{j=0}^{\infty} |\theta^j| < \infty$, i.e., whenever $|\theta| < 1$.
- Now, $Cov(\varepsilon_t, Y_{t-1}) = \sum_{j=0}^{\infty} \theta^j Cov\left(\varepsilon_t, \varepsilon_{t-j}\right) = 0$, if $Cov\left(\varepsilon_t, \varepsilon_{t-j}\right) = 0$ for all $j > 0$. So, if $\{\varepsilon_t\}$ is a white noise, it holds.
- Also, $E[\varepsilon_t | Y_{t-1}] = E[\, \varepsilon_t | \varepsilon_{t-1}, \varepsilon_{t-2}, \dots]$, so if $\{\varepsilon_t\}$ is an MDS, the regression assumption is satisfied.

AR(1)

- Can be derived in a different way: $(1-\theta L)Y_t = \varepsilon_t$, so if $(1-\theta L)$ has an inverse, Y_t can be written as

$$Y_t = (1-\theta L)^{-1}\varepsilon_t = \sum_{j=0}^{\infty}\theta^j L^j \varepsilon_t$$

- So it is covariance-stationary and ergodic, if $\sum_{j=0}^{\infty}|\theta^j| < \infty$, i.e., whenever $|\theta| < 1$.
- Now, $\mathrm{Cov}(\varepsilon_t, Y_{t-1}) = \sum_{j=1}^{\infty}\theta^j \mathrm{Cov}(\varepsilon_t, \varepsilon_{t-j}) = 0$, if $\mathrm{Cov}(\varepsilon_t, \varepsilon_{t-j}) = 0$ for all $j > 0$. So, if $\{\varepsilon_t\}$ is a white noise, it holds.
- Also, $\mathrm{E}[\varepsilon_t | Y_{t-1}] = \mathrm{E}[\varepsilon_t | \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots]$, so if $\{\varepsilon_t\}$ is an MDS, the regression assumption is satisfied.

AR(p)

Stationarity:

- AR(p) process is stationary, if $\Theta(L) = 1 - \theta_1 L - \ldots - \theta_p L^p$ can be inverted.
- Holds, if the roots of the (characteristic) polynomial $1 - \theta_1 x - \theta_2 x^2 \ldots - \theta_p x^p$ lie *outside* the unit circle.
- AR(1): $1 - \theta x = 0 \Rightarrow |x| = 1/|\theta| > 1$, if $|\theta| < 1$.
- Equivalent formulation: the process is stationary if the roots of the inverse characteristic polynomial $\lambda^p - \theta_1 \lambda^{p-1} - \ldots - \theta_{p-1}\lambda - \theta_p$ lie inside the unit circle
- AR(1): $\lambda - \theta = 0 \Rightarrow |\lambda| = |\theta| < 1$.
- **Necessary condition**: the coefficients of $\Theta(L)$ should add up to less than 1, i.e. $\sum_{j=1}^{p} \theta_p < 1$.
- **Sufficient condition**: the absolute values of coefficients of $\Theta(L)$ should add up to less than 1, i.e. $\sum_{j=1}^{p} |\theta_p| < 1$.

AR(p)

- **Stationarity**:
  - AR(p) process is stationary, if the roots of the (characteristic) polynomial $1-\theta_1 x-\theta_2 x^2 \ldots -\theta_p x^p$ lie *outside* the unit circle.
- **ACF**: can be computed recursively (***Yule-Walker equations***): for $k = 1, 2, \ldots$

$$\rho(k) = \theta_1 \rho(k-1) + \ldots + \theta_p \rho(k-p)$$

- **PACF**: First p $\alpha(k)$ are (in general) nonzero, and $\alpha(k) = 0$, for $|k| > p$.

ARMA(p,q)

- Can be written as

$$Y_t = \Psi(L)\,\varepsilon_t,$$

  if $\Theta(L)$ is invertible (i.e., has inverse), where $\Psi(L) = \Theta(L)^{-1}\Phi(L)$,

  $\Theta(L) = 1 - \theta_1 L - \ldots - \theta_p L^p$ and $\Phi(L) = 1 + \varphi_1 L - \ldots + \varphi_q L^q$.

- ARMA(p,q) process is **stationary**, if and only if the lag polynomial corresponding to the AR part is invertible.

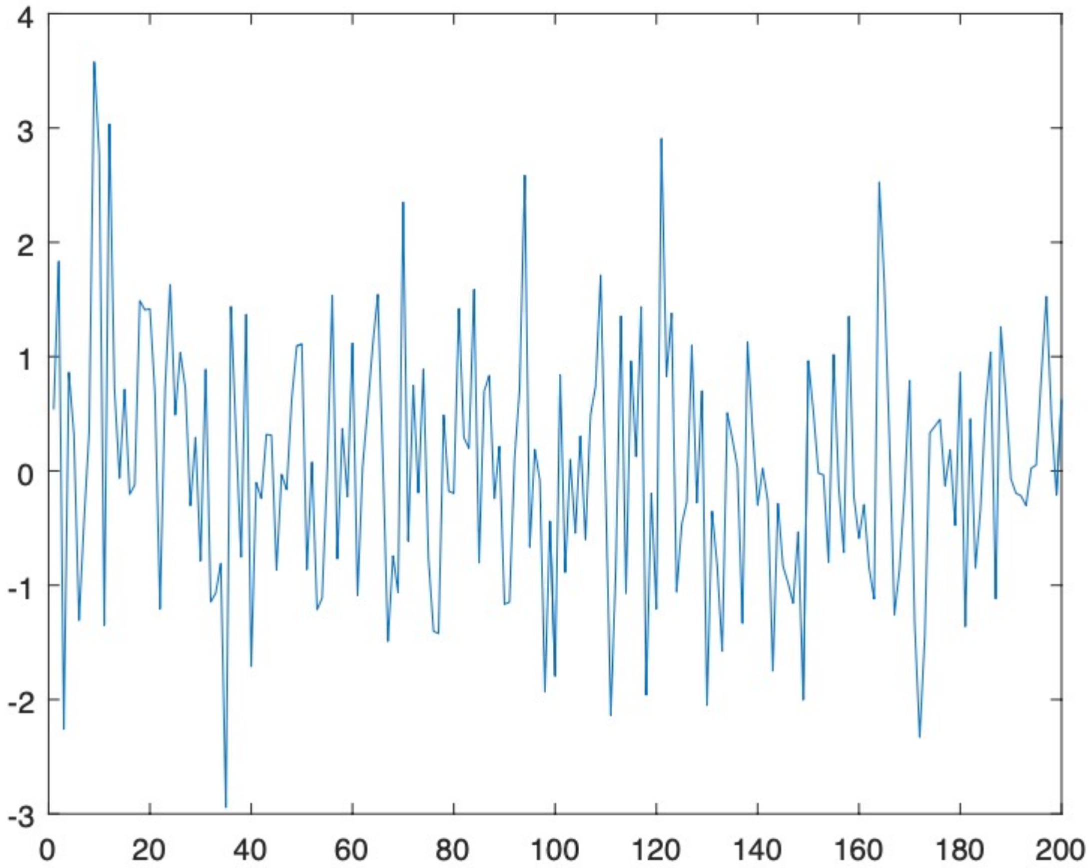
- Stationary ARMA(p,q) can be written as MA(∞).

- ACF and PACF: combination of ACFs and PACFs for AR(p) and MA(q) (none is zero after a certain lag, but decays exponentially fast)

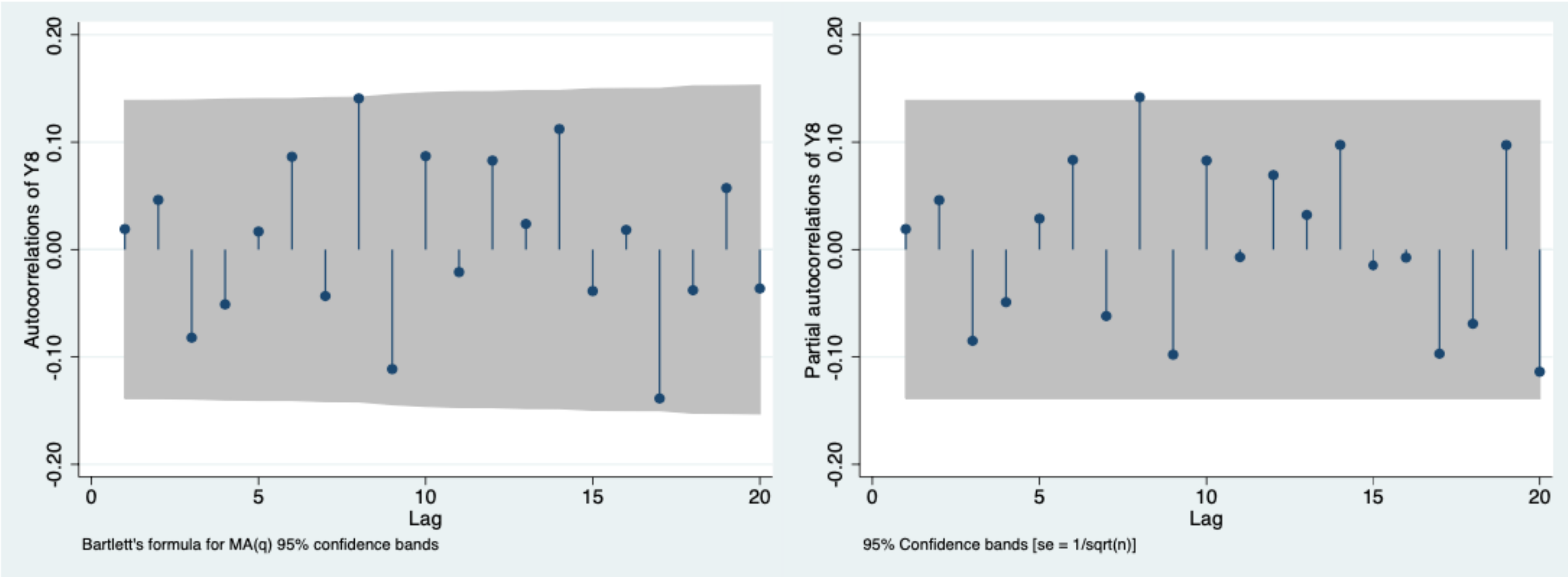
ARMA(p,q)

Process	ACF	PACF
WN	$\rho(k) = 0$	$\alpha(k) = 0$
AR(1)	$\rho(k) = \theta^k$	$\alpha(1) = \theta, \alpha(k) = 0$ for $k > 1$
AR(p)	Exponentially decays to 0, may oscillate	First p are non-zero; $\alpha(k) = 0$, for $k > p$
MA(1)	$\rho(1) = \varphi$, $\rho(k) = 0$ for $k > 1$	Exp. decays to 0, may oscillate; $sign(\alpha(1)) = sign(\varphi)$
MA(q)	First q $\rho(k)$ are non-zero, $\rho(k) = 0$, for $k > q$	Exp. decays to 0, may oscillate
ARMA(1,1)	$sign(\rho(1)) = sign(\theta + \varphi)$; exp. decays (oscillating if $\theta < 0$)	$\alpha(1) = \rho(1)$; exp. decays (oscillating if $\theta > 0$)
ARMA(p,q)	Starts exp. decaying (may oscillate) at lag q	Starts exp. decaying (may oscillate) at lag p

What is this process?

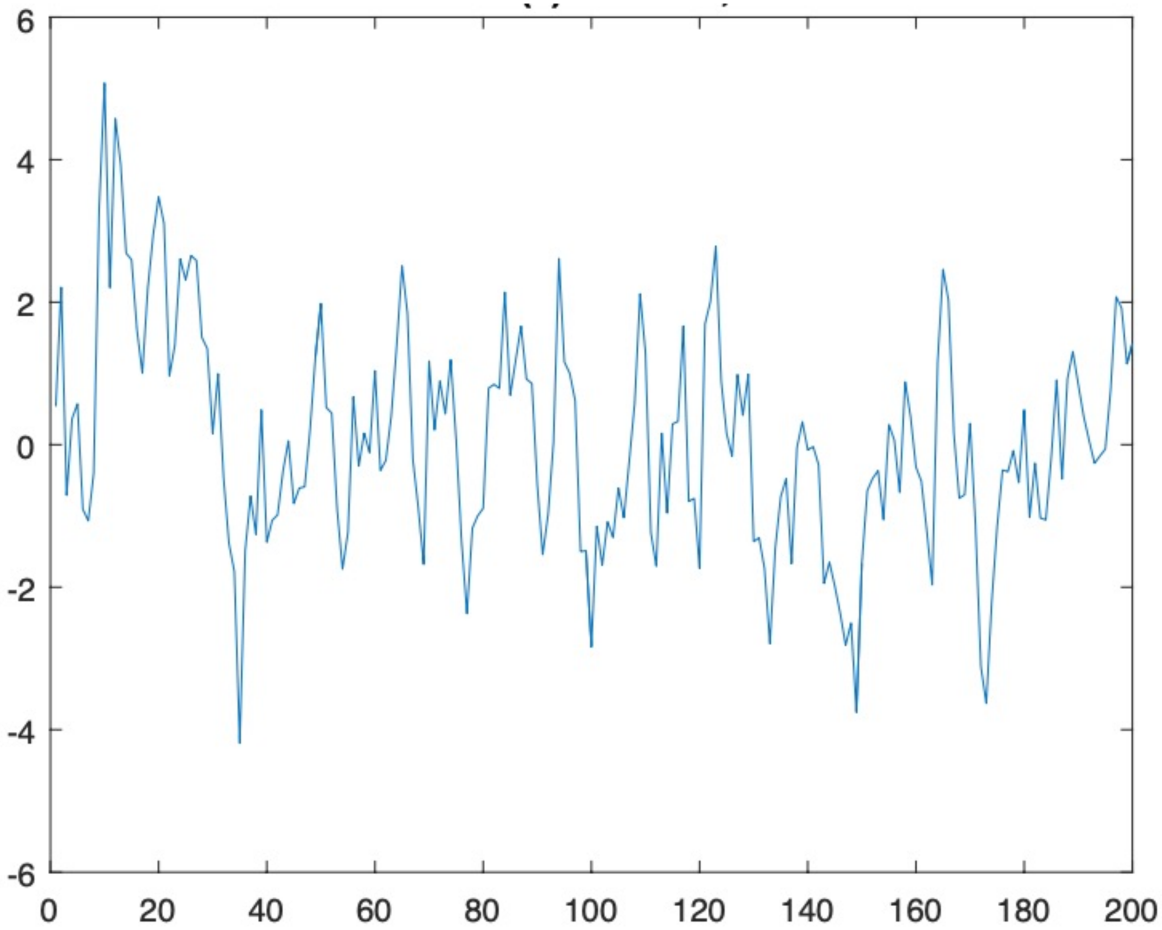


What is this process?



White noise

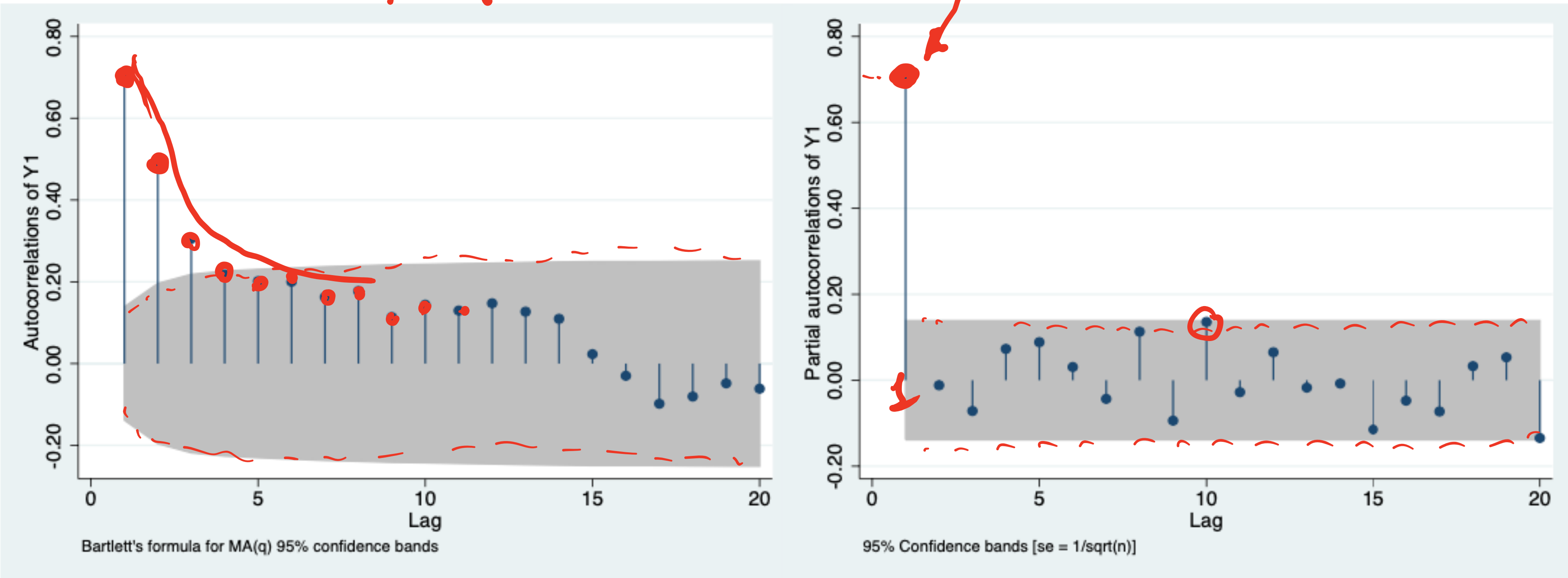
What is this process?



$y_t = \theta y_{t-1} + \varepsilon_t$

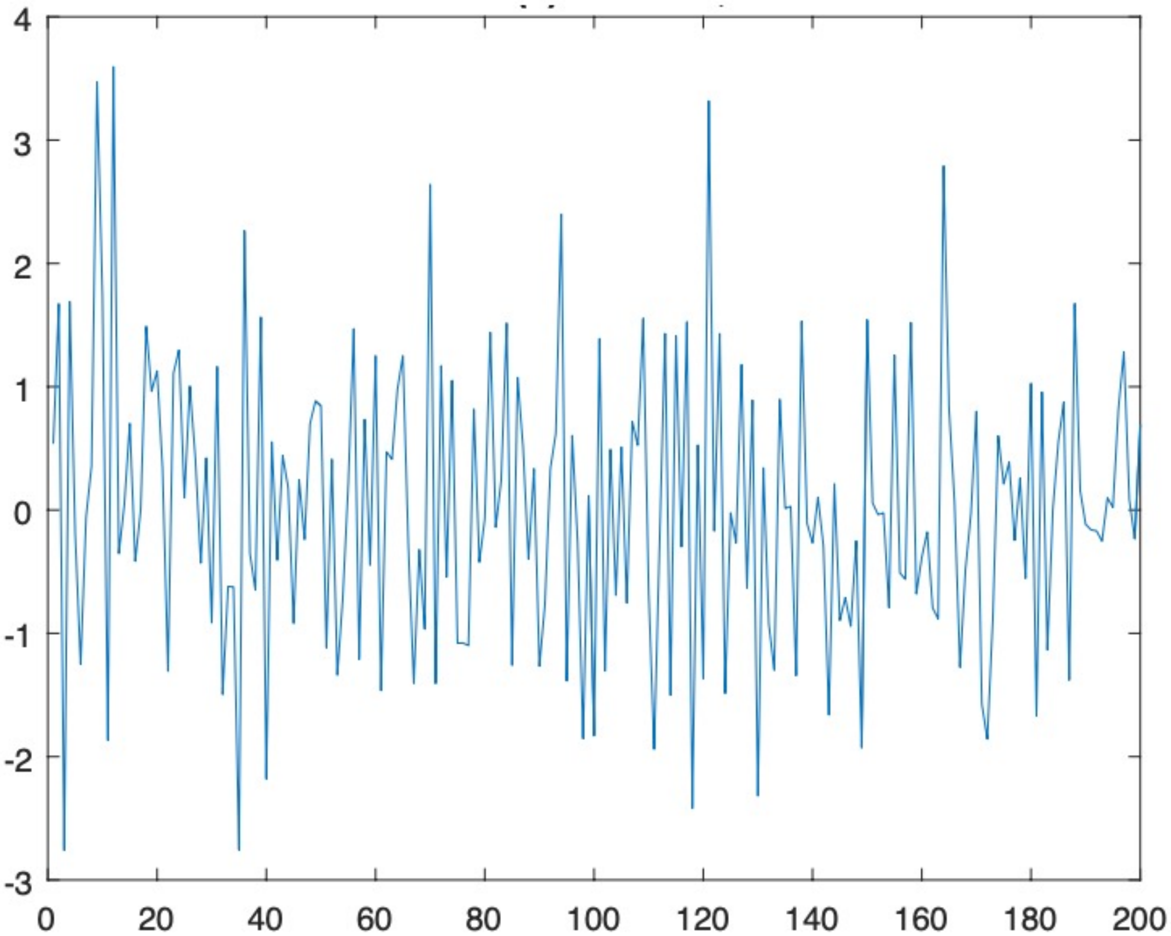
What is this process?

ACF AR(1) PACF



AR(1) with $\theta_1 = 0.7$

What is this process?

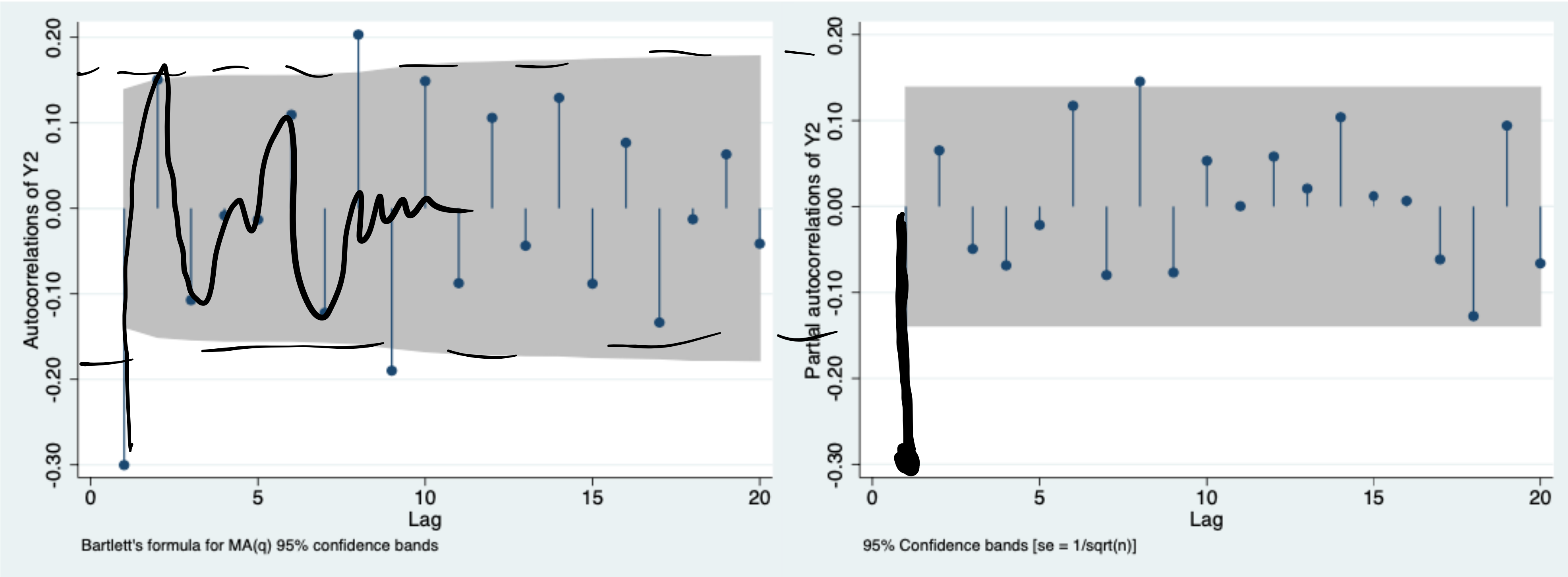


$y_t = \alpha \cdot y_{t-1} + u_t, \; u_t \sim N(0; \sigma^2)$

$y_t = 0 \quad y_1 = u_1; y_2 = \alpha u_1 + u_2$

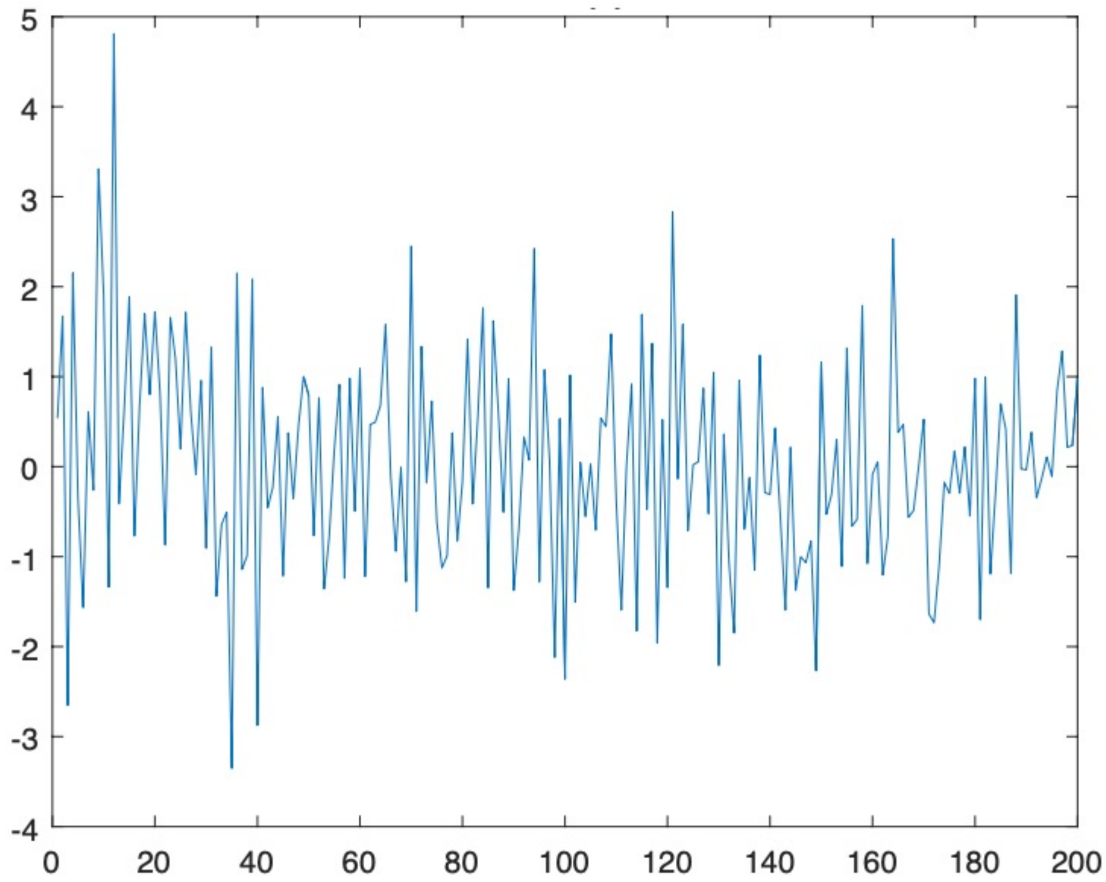
$y_3 = \alpha^2 u_1 + \alpha u_2 + u_3$

What is this process?



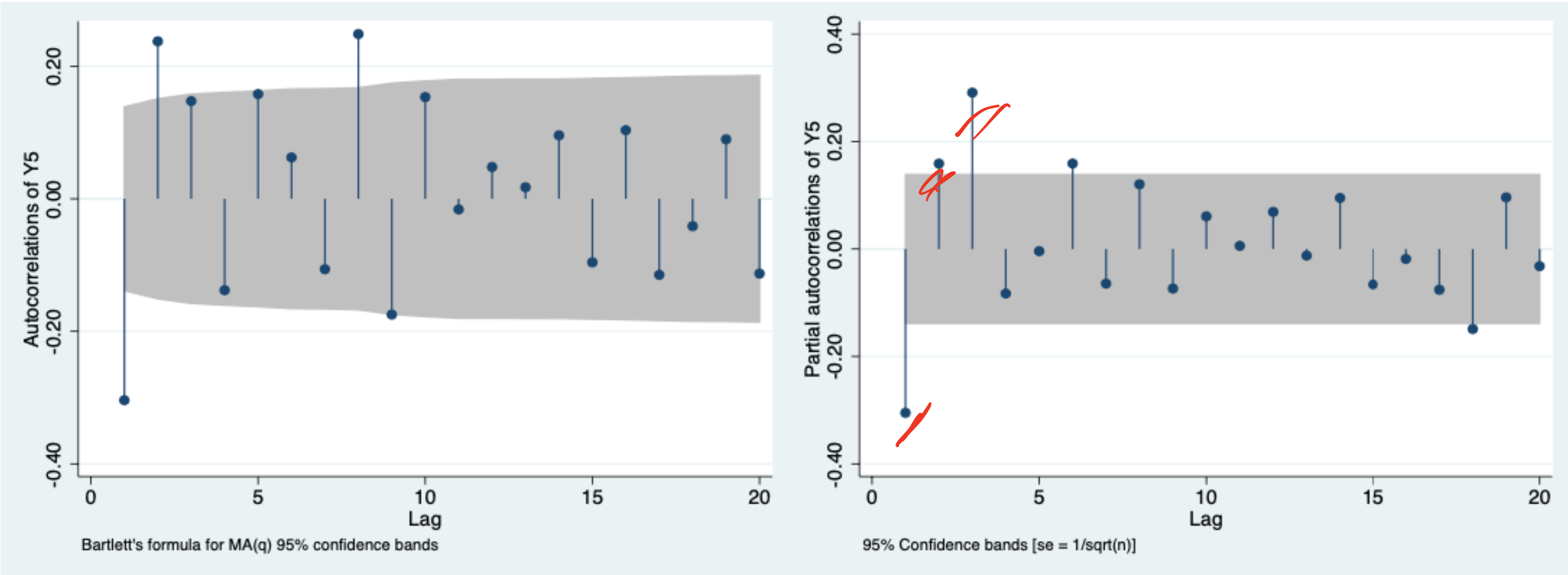
AR(1) with $\theta_1 = -0.3$

What is this process?



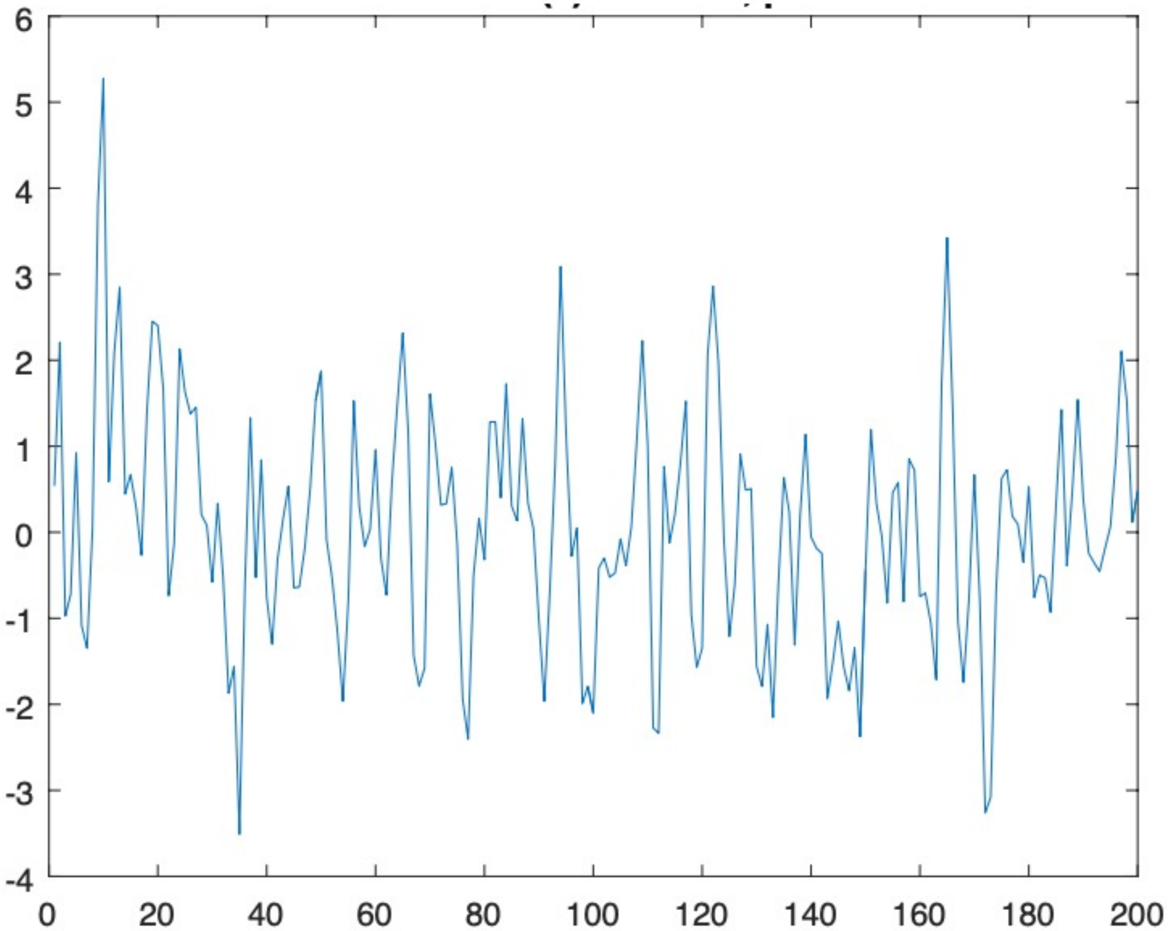
What is this process?

MA AR



$$\text{AR(3)}: \; Y_t = -0.3Y_{t-1} + 0.2Y_{t-2} + 0.3Y_{t-3} + \varepsilon_t$$

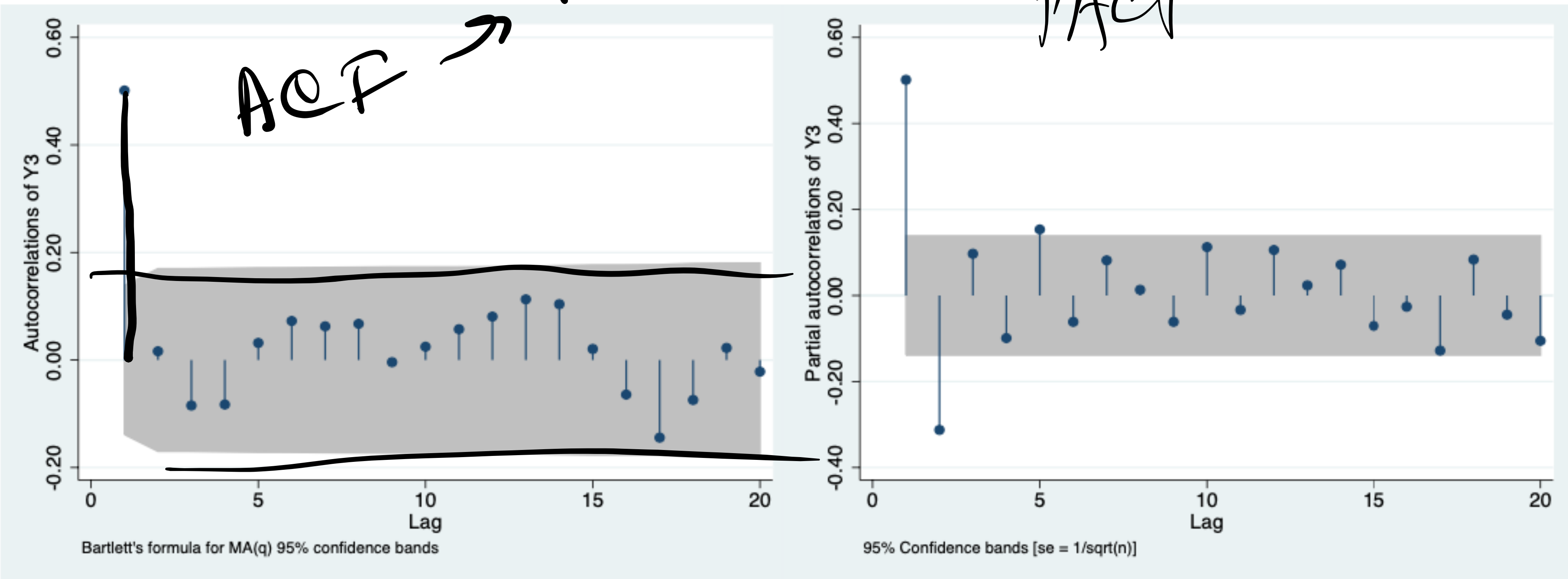
What is this process?



What is this process?

ACF → MA(?)

PACF



MA(1) with $\varphi_1 = 0.7$