Basic of Stochastic Finance

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Martingales

Definition

A process X is called *martingale* (relative to $\{F_N\}$) if

1. X is adopted,

2. $E|Xn| < \infty$,

3. $E(X_n|\mathcal{F}_{n-1}) = X_{n-1}$ a. s. $(n \geq 1)$.

- ***Super****martingale* is defined similarly, except that (3) is replaced by

$$E(X_n|\mathcal{F}_{n-1}) \leq X_{n-1} \text{ a. s. } (n \geq 1)$$

- ***Sub****martingale* is defined with (3) replaced by

$$E(X_n|\mathcal{F}_{n-1}) \geq X_{n-1} \text{ a. s. } (n \geq 1)$$

- A supermartingale decreases on average, a submartingale increases on average in time.

Martingales: examples

One reason martingales are so powerful is that they model a situation where one gains progressively more information over time. Suppose that $\mathcal{U}$ is a set of objects, and $f : \mathcal{U} \to \mathbb{R}$. Let X be a random variable taking values in $\mathcal{U}$, and let $\{Y_i\}$ be another sequence of random variables. The associated *Doob martingale* is given by

$$X_i = \mathbb{E}[f(X) \mid Y_0, Y_1, \ldots, Y_i] \,.$$

In words, this is our "estimate" for the value of $f(X)$ given the information contained in $\{Y_0, \ldots, Y_i\}$. To see that this is always a martingale with respect to $\{Y_i\}$, observe that

$$\mathbb{E}[X_{i+1} \mid Y_0, \ldots, Y_i] = \mathbb{E}[\mathbb{E}[f(X) \mid Y_0, \ldots, Y_{i+1}] \mid Y_0, \ldots, Y_i] = \mathbb{E}[f(X) \mid Y_0, \ldots, Y_i] = X_i \,,$$

where we have used the tower rule of conditional expectations.

$X_n \sim \mathcal{F}$-adapted ; martingale

$$E[X_n \mid \mathcal{F}_{n-2}] = E[\underbrace{E[X_n \mid \mathcal{F}_{n-1}]}_{= X_{n-1}} \mid \mathcal{F}_{n-2}] = E[X_{n-1} \mid \mathcal{F}_{n-2}] = X_{n-2}$$

Martingales: examples

Balls in bins. Suppose we throw m balls into n bins one at a time. At step i, we place ball i in a uniformly random bin. Let $C_1, C_2, \ldots, C_m$ be the sequence of (random) choices, and let C denote the final configuration of the system, i.e. exactly which balls end up in which bins.

Now we can consider a functional like $f(C) = \#$ of empty bins. If $X_i = \mathbb{E}[f(C) \mid C_1, \ldots, C_i]$, then $\{X_i\}$ is a (Doob) martingale. It is straightforward to calculate that

$$\mathbb{E}[X_m] = \mathbb{E}[X_0] = \mathbb{E}[f(C)] = n \cdot \left(1 - \frac{1}{n}\right)^m .$$

Suppose we are interested the concentration of $X_m = f(C)$ around its mean value. Of course, we can write $X_m = Z_1 + \cdots Z_m$ where Z_i is the indicator of whether the ith been is empty after all the balls have been thrown. But note that the $\{Z_i\}$ variables are not independent—in particular, if I tell you that $Z_1 = 1$ (bin 1 is empty), it decreases slightly the likelihood that other bins are empty.

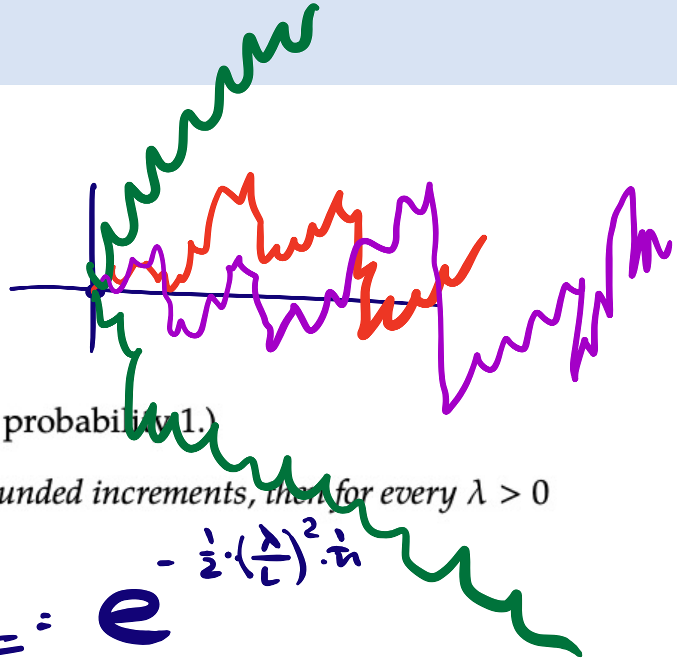
Martingales: examples

The Hoeffding-Azuma inequality

Say that a martingale $\{X_i\}$ has L-bounded increments if

$$|X_{i+1} - X_i| \leqslant L$$

for all $i \geqslant 0$. (The preceding inequality is meant to hold with probability 1.)



Theorem 2.1. *For every $L > 0$, if $\{X_i\}$ is a martingale with L-bounded increments, then for every $\lambda > 0$ and $n \geqslant 0$, we have*

$$\mathbb{P}[X_n \geqslant X_0 + \lambda] \leqslant e^{-\frac{\lambda^2}{2L^2 n}} = e^{-\frac{1}{2}\cdot\left(\frac{\lambda}{L}\right)^2 \cdot \frac{1}{n}}$$

$$\mathbb{P}[X_n \leqslant X_0 - \lambda] \leqslant e^{-\frac{\lambda^2}{2L^2 n}}$$

We will prove this in Section 3. It's useful to note the following special case of the theorem.

Corollary 2.2. *Suppose that $Z_1, Z_2, \ldots, Z_n$ are independent random variables taking values in the interval $[-L, L]$. Put $Z = Z_1 + \cdots + Z_n$ and $\mu = \mathbb{E}[Z]$. Then for every $\lambda > 0$, we have*

$$\mathbb{P}\big[Z \geqslant \mu + \lambda\big] \leqslant e^{-\lambda^2/(2L^2 n)}$$

$$\mathbb{P}\big[Z \geqslant \mu - \lambda\big] \leqslant e^{-\lambda^2/(2L^2 n)}$$

Continuous time. Brownian Motion

$\boxed{C=1}$

- By choosing the relationship between Δx andΔt such that$\frac{(\Delta x)^2}{\Delta t} = const =: C^2$ we get, on the limit, that $X_t \sim N(0, C\sqrt{t})$. The limiting process X_t, preserves some important features of SRW:
- (i) The increments of X; are independent i.e. for $0 \leq s \leq t \leq u \leq v$ the increments $X_t - X_s$ and $X_v - X_u$ are independent r.v. (the same is valid for any n time intervals).
- (ii) The increments of X_t are stationary i.e. the distribution of $X_{s+t} - X_s$, only depends on t (and not on s).

Definition 6

The random process $\{W_t, \ t \geq 0\}$ is called Brownian motion (Wiener process), if

(i) $W(0) = 0$,

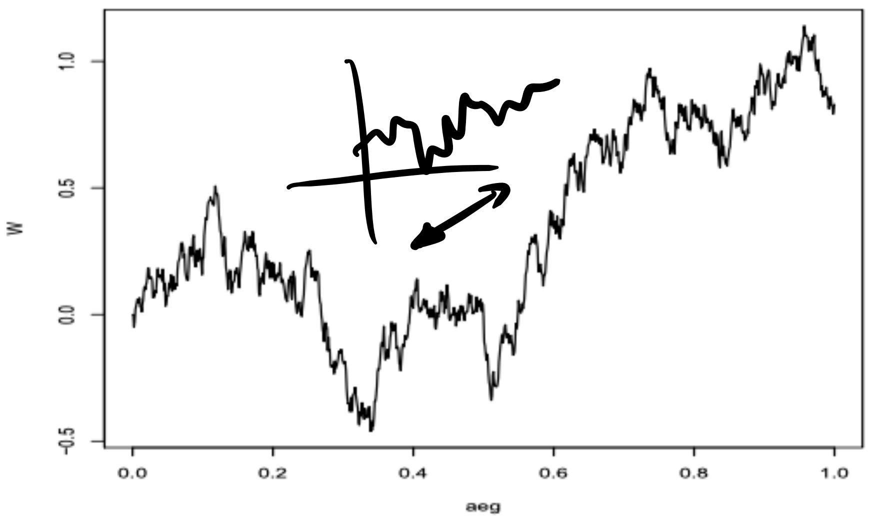
(ii) for all $t > 0$ the r.v. $W_t \sim N(0, C\sqrt{t})$, where $C > 0$ is a constant.

Continuous time. Brownian Motion

- From the definition of BM it follows that also the increments of BM are normally distributed: by the stationarity of increments

$$W_t - W_s \overset{\mathrm{D}}{=} W_{t-s} - W_0 = W_{t-s} \sim N\left(0, C\sqrt{t-s}\right),$$

where $\overset{\mathrm{D}}{=}$ is to be read as "has same distribution as" .



A trajectory of standard Brownian motion

- Note that, in fact, the property (iv) can be deduced from properties (i)-(iii).

- BM is a mathematical model widely used in physics (diffusions), economics (price models) e.t.c.

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$W_t \sim$ Wiener's Proc.

$E[W_t] = E[W_t - 0] = E[\underbrace{W_t - W_0}_{\sim N(0;\, t-0)}] = 0$

$E[W_t^2] = Var(W_t) + \underbrace{E^2(W_t)}_{=0} = Var(W_t - W_0) = t - 0 = t$

$t > s$ $\quad E[W_t \cdot W_s] = Cov(W_t, W_s) + \underbrace{EW_t}_{=0} \cdot \underbrace{EW_s}_{=0} =$

$= Cov(W_t, W_s) = \cancel{Cov(W_t - W_0, W_s - W_0)}$

$Cov((W_t - W_s) + W_s, W_s) = Cov(W_t - W_s, W_s) + Cov(W_s, W_s)$

$Cov(W_t - W_s, W_s - W_0) = 0$ $\quad Var\, W_s = Var(W_s - W_0) = s$

$t > s$ $\quad E[W_t \cdot W_s] = s$

$E[W_t \cdot W_s] = \min(t, s)$

$corr(W_t, W_s) = \dfrac{\min(t,s)}{\sqrt{t \cdot s}} = \left\{ \text{if } s < t \right\} = \sqrt{\dfrac{s}{t}}.$

Finite-dimensional distributions of SBM

- The joint distribution of $\left(W_{t_1}, W_{t_2}, \dots, W_{t_n}\right)$ where $0 < t_1 < t_2 < \cdots < t_n$ can easily be calculated.
- For each t_i, the density of W_{t_i} is

$$f_{W_{t_i}}(x) = \frac{1}{\sqrt{2\pi t_i}} e^{-\frac{x^2}{2t_i}}$$

Provided that $C = 1$.

Since the equalities $\begin{cases} W_{t_1} = x_1 \\ W_{t_2} = x_2 \\ \dots \\ W_{t_n} = x_n \end{cases}$ are equivalent to the equalities $\begin{cases} W_{t_1} = x_1 \\ W_{t_2} = x_2 \\ \dots \\ W_{t_n} = x_n \end{cases}$

And since the increments ate independent, we have

- $f_{W_{t_1},\dots,W_{t_n}}(x_1, \dots, x_n) = f_{W_{t_1}, W_{t_2}-W_{t_1}, \dots, W_{t_n}-W_{t_{n-1}}}(x_1, x_2 - x_1, \dots, x_n - x_{n-1}) = f_{W_{t_1}}(x_1) \cdot f_{W_{t_2}-W_{t_1}}(x_2 - x_1) \cdot \dots \cdot f_{W_{t_n}-W_{t_{n-1}}}(x_n - x_{n-1}) = \frac{1}{\sqrt{2\pi t_1}} e^{-\frac{x_1^2}{2t_1}} \frac{1}{\sqrt{2\pi(t_2-t_1)}} e^{-\frac{(x_2-x_1)^2}{2(t_2-t_1)}} \cdot \dots \cdot \frac{1}{\sqrt{2\pi(t_n-t_{n-1})}} e^{-\frac{(x_n-x_{n-1})^2}{2(t_n-t_{n-1})}}$
- The formula obtained can be used for many purposes.

$E[W_s \mid \mathcal{F}_s] = W_s$

$E[W_s] = 0$

② $W_t \sim$ martingale $t > s$

$E[W_t \mid \mathcal{F}_s] = E[W_t - W_s + W_s \mid \mathcal{F}_s] = E[W_t - W_s \mid \mathcal{F}_s] + E[W_s \mid \mathcal{F}_s] = W_s$, where $E[W_t - W_s \mid \mathcal{F}_s] = 0$, $E[W_s \mid \mathcal{F}_s] = W_s$

③ Brownian Bridge $s < t$

$\{W_t^2 - t\}_{t>0}$ - martingale - ?

$E[W_t^2 - t \mid \mathcal{F}_s] = E(W_t^2 \mid \mathcal{F}_s) - E(t \mid \mathcal{F}_s) = E(W_t^2 \mid \mathcal{F}_s) - t = W_s^2 - s$, where $E(t \mid \mathcal{F}_s) = t$

$E[W_t^2 \mid \mathcal{F}_s] = E[((W_t - W_s) + W_s)^2 \mid \mathcal{F}_s] =$

$= \underbrace{E[(W_t - W_s)^2 \mid \mathcal{F}_s]}_{= t-s} + 2\underbrace{E[(W_t - W_s) W_s \mid \mathcal{F}_s]}_{} + \underbrace{E[W_s^2 \mid \mathcal{F}_s]}_{= W_s^2}$

$E[(W_t - W_s) W_s \mid \mathcal{F}_s] = 2 \cdot [E[W_t \cdot W_s \mid \mathcal{F}_s] - E[W_s^2 \mid \mathcal{F}_s]]$, where $E[W_t \cdot W_s \mid \mathcal{F}_s] = W_s \cdot E[W_t \mid \mathcal{F}_s] = W_s^2$, $E[W_s^2 \mid \mathcal{F}_s] = W_s^2$

$= t - s + W_s^2 = W_s^2 - s + t$

Conditional Distribution

- Let's use the formula above to solve one particular problem. Suppose we know that at time t BM has taken value $W_t = B$. Let s be an earlier time, $s < t$.
- What is the conditional distribution of W_s given the event $W_t = B$? It is known that the conditional density is the ratio of joint density and the density of the condition, we can calculate

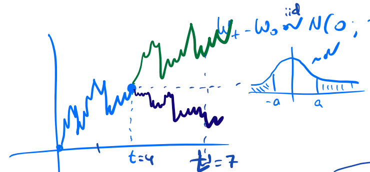
$$f_{W_s|W_t}(x|B) = \frac{f_{W_s,W_t}(x,B)}{f_{W_t(B)}} = \frac{\frac{1}{\sqrt{2\pi s}} e^{\frac{x^2}{2s}} \cdot \frac{1}{\sqrt{2\pi(t-s)}} e^{\frac{(B-x)^2}{2(t-s)}}}{\frac{1}{\sqrt{2\pi t}} e^{\frac{B^2}{2t}}} = \ldots . = \frac{1}{\sqrt{2\pi \frac{t}{s}(t-s)}} e^{\frac{t(x-B\frac{s}{t})^2}{2s(t-s)}}$$

- Hence, the conditional distribution of W_s is normal distribution with mean and variance $\frac{s(t-s)}{t}$.

Maxima of Brownian Motion

- If $a > 0$, then $P\left\{\max_{0\le s\le t} W_s \ge a\right\} = P\{T_a \le t\} = 2\left[1 - \Phi\left(\frac{|a|}{\sqrt{t}}\right)\right]$
- If $a < 0$, then $P\left\{\max_{0\le s\le t} W_s \ge a\right\} = 1$

$$P(\max X_i > \ldots) = 1 - \underbrace{P(\max X_i < \ldots)}_{= \left(P(X_i < \ldots)\right)^n}$$



$W_t - W_0 \overset{iid}{\sim} N(0; t-0)$

$P(Z>a) = P(Z<-a)$

$EY = 0$

$\operatorname{Var} Y = \operatorname{Var}(W_7 - W_4) + 64 \operatorname{Var}(W_4 - W_0)$, with $7-4=3$, $4-0=4$

$= 3 + 4 \cdot 64 = \underline{\underline{259}}$.

$P(W_7 > 9 \cdot W_4) = P(W_7 - 9W_4 > 0) = P(\underbrace{(W_7 - W_4) - 8(W_4 - W_0)}_{Y} > 0)$

$= P\left(\frac{Y}{\sqrt{259}} > 0\right) = P(Z \sim N(0;1) > 0) = \frac{1}{2}$

Brownian Motion between Two Boundaries

Let $A > 0, B > 0$. Let us find the probability that, starting from 0, BM reaches level A before $-B$. Recall that in the case of SRW the answer to the same question is $\frac{B}{A+B}$. As the same answer remains true for any time and space steps sizes, we have

$$P\{W_t reaches\ A\ before\ -B\} = P\{T_A < T_B\} = \frac{B}{A+B}$$

Exercise

Let W_t be a standard Brownian motion. Assume that W_1= 2. Find the probability that $\mathrm{W}_5 < 0$.

- Without any given condition we would have $P\{W_5 < 0\} \; = \; 3$, since $\mathrm{W}_t \sim N(0, \sqrt{t})$.
- However, under the condition W_1= 2the increment $W_5 - W_1 \sim N\left(0, \sqrt{4}\right) -$ this distribution only depends on the length of the time interval and not on its location. Therefore we have

$P\{W_5 < 0 | W_1 = 2\} = P\{W_5 - W_1 < -2 | W_1 = 2\} \; =$

$= \; P\{W_5 - W_1 < -2\} \; = \; P\{W_4 < \; -2\} \; =$

$= \; P\{N(0, \sqrt{4}) < -2\} \; = \Phi\left(\frac{-2}{2}\right) \; = \Phi(-1) \; = 0.16$

Continuous time. Filtration

- Many important concepts and results that are known for martingales with discrete time can be transferred to continuous time without any major change.
- Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let t be a real-valued parameter interpreted as time. Most often, t takes values from the half-line R^+ or finite interval $(0, T)$.

Definition 7

A family of σ-algebras $\{\mathcal{F}_t, t \geq 0\}$ is called a *filtration*, if

1) all its members $\mathcal{F}_t$ are sub- σ -algebras of $\mathcal{F}$ and,
2) for $s < t$ one has $\mathcal{F}_s \subseteq \mathcal{F}_t$.

As in the case of dicsrete time, we are mainly interested in the natural filtration $\{\mathcal{F}_t^X, t \geq 0\}$, generated by a random process X. As before, $\mathcal{F}_t^X$ contains the information induced by the random process X within the time interval $[0, t]$. It means that an event A $\in \mathcal{F}_t^X$ if and only if one can decide whether A occurred or not on the basis of the trajectory $\{X_s, 0 \leq s \leq t\}$ that the process X generates by time t.

Definition 8

If $\{Y_t, t \geq 0\}$ is a random process such that for each t the random variable Y is $\mathcal{F}_t$-measurable, then it is said that the process Y is *adopted to filtration* $\{\mathcal{F}_t, t \geq 0\}$.

Examples

1. The random process $Z_t = \int_0^t X_s ds$ is adopted to the filtration $\{\mathcal{F}_t^X, t \geq 0\}$, since knowing the path of X within time interval $[0, t]$ is sufficient to determine Z_t.

2. The process $M_t = \max\limits_{0 \leq s \leq t} W_s$ is adopted to the filtration $\{\mathcal{F}_t^W, t \geq 0\}$.

$W_t - W_s \sim N(0, t-s)$

3. The process $Z_t = W_{t+1}^2 - W_t^2$ is not adopted to the filtration $\{\mathcal{F}_t^W, t \geq 0\}$.

$J_n = \sum_{i=1}^{n} (Y_i - Y_{i-1})^2$ $\qquad Y_i = W_{\frac{it}{n}}$

$0 \quad \frac{t}{n} \quad \frac{2t}{n} \quad \frac{it}{n} \quad t$

$\sum_{i=1}^{n} (Y_i - Y_{i-1})^2$

$* \; E[J_n] = E\left[\sum_{i=1}^{n} (Y_i - Y_{i-1})^2\right] = \sum_{i=1}^{n} E\left[(Y_i - Y_{i-1})^2\right] = \sum_{i=1}^{n} \underbrace{Var(Y_i - Y_{i-1})}_{= Var\left(W_{\frac{it}{n}} - W_{\frac{(i-1)t}{n}}\right)} + \underbrace{E^2[Y_i - Y_{i-1}]}_{=0} =$

$= \sum_{i=1}^{n} \frac{t}{n} = n \cdot \frac{t}{n} = t.$

$* \; Var[J_n] = Var\left[\sum_{i=1}^{n} (Y_i - Y_{i-1})^2\right] = \sum_{i=1}^{n} Var\left((Y_i - Y_{i-1})^2\right) = \sum_{i=1}^{n} Var\left[\left(\frac{Y_i - Y_{i-1}}{\sqrt{t/n}}\right)^2 \cdot \frac{t}{n}\right] =$

$\left\{ \begin{array}{l} X \sim N(0;1) \\ X^2 \sim \chi^2 \\ Var(\chi_1^2) = 2 \end{array} \right\}$

$Y_i - Y_{i-1} \sim N\left(0, \frac{t}{n}\right)$ $\quad = \frac{t^2}{n^2} \sum_{i=1}^{n} Var\left(\left(\frac{Y_i - Y_{i-1}}{\sqrt{t/n}}\right)^2\right)$

$\frac{Y_i - Y_{i-1}}{\sqrt{t/n}} \sim N(0;1)$ $\quad = \frac{t^2}{n^2} \cdot \sum_{i=1}^{n} 2 = \frac{2t^2}{n}$

$E[J_n] = t$

$Var(J_n) = \frac{2t^2}{n} \xrightarrow[n \to +\infty]{} 0 \Rightarrow J_n \xrightarrow[n \to \infty]{} t$

$\cancel{\int_0^t W_u \, dW_u = \frac{W_t^2 - W_s^2}{2}}$

$dt \qquad (dW_t)^2 \to dt$

$dt \cdot dt = 0$ $dt \cdot dW_t = 0$ $dW_t \cdot dW_t = dt$

$(dW_t)^3 = dW_t^2 \, dW_t = dt \cdot dW_t = 0$ Itô's Lemma

$(dW_t)^4 = 0$

Continuous time. Martingales

$f(t, W_t)$ $df = f_t' \cdot dt + f_w' \cdot dW + \frac{1}{2} f_{ww}'' (dW_t)^2$

Definition 9

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space endowed with a filtration $\{\mathcal{F}_t, t \geq 0\}$. A random process $\{M_t, t \geq 0\}$ is called a martingale, if

1. M is adopted to the filtration $\{\mathcal{F}_t, t \geq 0\}$,
2. $E|M_t| < \infty, \forall\, t$
3. For any $s \leq t$ we have $E(M_t|\mathcal{F}_s) = M_s$ a.s.

- If the equality in (3) is replaced by the inequality $\leq$ $(or\ \geq)$, then we speak about a supermartingale (respectively submartingale).

Remark: Similarly, a martingale can be defined on a finite time interval $[0, T]$. In the following mostly the filtration $\{\mathcal{F}_t^W, t \geq 0\}$, generated by a standard Brownian motion W, is used.