# Brownian Motion

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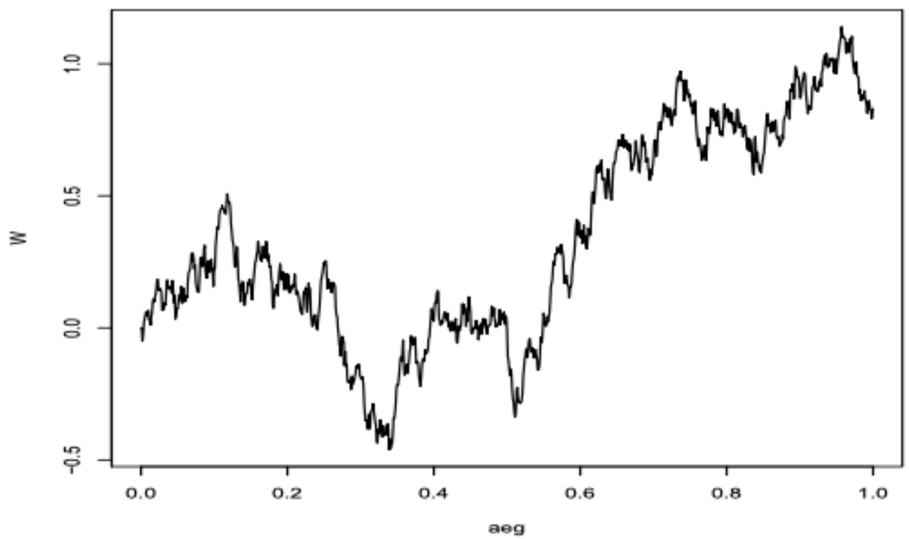
# Continuous time. Brownian Motion

- From the definition of BM it follows that also the increments of BM are normally distributed: by the stationarity of increments

$$W_t - W_s \overset{\mathrm{D}}{=} W_{t-s} - W_0 = W_{t-s} \sim N\left(0, C\sqrt{t-s}\right),$$

where $\overset{\mathrm{D}}{=}$ is to be read as "has same distribution as" .




A trajectory of standard Brownian motion

- Note that, in fact, the property (iv) can be deduced from properties (i)-(iii).

- BM is a mathematical model widely used in physics (diffusions), economics (price models) e.t.c.

# Stochastic integral

- Our aim here is to give a meaning to the integral $\int_0^t X_s \, dM_s$, where $X$ is an adopted process and $M$ is a martingale, e.g. Brownian motion. This new integral is rather different from the classical Stiltjes integral. In fact, we have already donea useful piece of work in this direction when considering discrete time martingales.
- For discrete time martingales we have defined the martingale transform as the process

$$Y_n = \sum_{i=1}^{n} C_i (X_i - X_{i-1}) =: (C \bullet X)_n$$

was interpreted as the total winnings of a player after the game $n$ (recall that $C_i$ is the stake of the player on game $i$ and $X_i - X_{i-1}$ is the net winnings per unit stake in game $i$.) Most importantly, we have shown (Theorem 2, slide 17) that if the process $C_i$ is predictable and $X$ is a martingale, then the process $Y$ is a martingale again. At the same time, the formula of the martingale transform above looks like a certain integral sum. Our next task is to define similar concept (stochastic integral) for continuous time martingales. Stochastic integral is an efficient tool to solve problems in various areas, including option pricing.

- We first show that it is not possible to integrate with respect to continuous time martingales in a traditional manner i.e. path-wise (as Riemann- Stiltjes integral).

## Definition 10

The $p-variation$ of a function $f$ over the interval $[a,b]$ is defined as

$$Var_p(f;a,b) = \lim_{||\pi||\to 0} sup \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p \,,$$

where $\pi_n$ is a partition of $[a,b]$ by cutting points $a = t_0 < \ t_1 < \cdots < t_n = b$, and $\|\pi_n\|$ is the length of the longest subinterval of $\pi_n$.

- It is easy to see that when f is continuous and the partition is fine enough, so that the increments $|f(t_i) - f(t_{i-1})|$ are small numbers (smaller than 1), then the higher is the order $p$ the less is the result. Thus for $p > q$ the $p$-variation is less than the $q$-variation.
- At the same time, while studying martingales, we have used the term ‘quadratic variation’: the quadratic variation of a martingale $M$ was a predictable process $A$ such that the difference $\mathrm{M}^2 - A$ is again martingale. For example, for a standard Brownian motion $W$ the process $W_t^2 - t$ is a martingale, and hence $t$ is the quadratic variation of standard Brownian motion.

- The question arises whether such a coincidence of terminology is justified. The positive answer is provided by the following lemma. Consider a partition of the interval $[0, t]$.

$$\pi_n: 0 = t_0 < t_1 < \cdots < t_n = t$$

And let us denote

- $\Delta_i = t_i - t_{i-1}, i = 1, 2, \dots, n$
- $\Delta W_i = W_{t_i} - W_{t_{i-1}}$,
- $Q_n(t) = \sum_{i=1}^{n} \left(W_{t_i} - W_{t_{i-1}}\right)^2 = \sum_{i=1}^{n} \left(\Delta W_{t_i}\right)^2$

## Lemma 3

The following (mean-square) convergence takes place: $E[Q_n(t)-t]^2 \to 0, n \to \infty$.

**Proof.** The increments of Brownian motion $\Delta W_i$ are independent and $\Delta W_i \sim N(0, \sqrt{\Delta_i})$. Therefore

$$EQ_n\,(t) = \sum_{i=1}^{n} E(\Delta W_i)^2 = \sum_{i=1}^{n} \Delta_i = t$$

At whe same time wne variance

$$D\big(Q_n(t)\big) = \sum_{i=1}^{n} D\big((\Delta W_i)^2\big) = \sum_{i=1}^{n} E((\Delta W_i)^4 - \Delta_i^2)$$

It is well known that the 4-th order moment of a $N(0,1)$- distributed random variable is 3, thus $E(W_1^4) = 3$. Hence, by taking into account the stationarity of increments, we have

$$E(\Delta W_i)^4 = E\big(W_{t_i} - W_{t_{i-1}}\big)^4 = EW_{t_i - t_{i-1}}^4 = EW_{\Delta_i}^4 = E\big(\sqrt{\Delta_i} W_1\big)^4 = 3\Delta_i^2$$

from which

$$D\big(Q_n(t)\big) = 2\sum_{i=1}^{n} \Delta_i^2$$

Therefore, if $\|\pi_n\| = \max \Delta_i \to 0$, then

$$D\big(Q_n(t)\big) \leq 2\|\pi_n\| \cdot \sum_{i=1}^{n} \Delta_i = 2t\|\pi_n\| \to 0.$$

Since

$$D(Q_n(t) = E(Q_n(t) - t^2),$$

the proof is completed.

Now it is not difficult to show that *the variation (1-variation) of Brownian motion is unbounded*.

- **Corollary.** *The trajectories of Brownian motion have a.s. unbounded variation, i.e.* $Var_1\,(W;0,t) = \infty$ .
- **Proof:**

Obviously, the following inequalities are valid:

$$
\begin{aligned}
Q_n(t) \;&=\; \sum_{i=1}^{n} (\Delta W_i)^2 \\
&\le \max_i |\Delta W_i| \cdot \sum_{i=1}^{n} |\Delta W_i| \\
&\le \max_i |\Delta W_i| \cdot Var_1\,(W;0,t) \qquad (11)
\end{aligned}
$$

- We now let $\|\pi_n\| = \max \Delta_i \to 0$. Since the trajectories of Brownian motion are uniformly continuous in the interval $[0,t]$, we have the convergence $\max_i |\Delta W_i| \to 0$.
- Suppose now, in contrary, that $Var_1\,(W;0,t) < \infty$. Then the product (11) also tends to zero, while a smaller quantity $Q_n(t)$ converges to $t$ - a contradiction.
- Therefore, it must be that $Var_1\,(W;0,t) = \infty$.

# First example of stochastic integral

- In the next, we try to give meaning to the integral $\int_0^t W_s dW_s$. From the discussion above, it is clear that this integral can not exist in Riemann- Stiltjes sense, because of the unbounded variation of $W_t$. Thus for each separate trajectory the integral sums

$$S_n = \sum_{i=1}^{n} W_{t_i -1} \Delta W_i = \sum_{i=1}^{n} W_{t_i}(W_{t_i} - W_{t_{i-1}}) \qquad (12)$$

- do not converge as ordinary number sequence, when $n \to \infty$. However, by using $(W_{t_i} - W_{t_{i-1}})^2 = W_{t_i}^2 + W_{t_{i-1}}^2 - 2W_{t_i}W_{t_{i-1}}$, it is possible (after little algebra) to represent $S_n$ in the form

$$S_n = \frac{1}{2} W_t^2 - \frac{1}{2} Q_n(t)$$

- By Lemma 3, the mean-square convergence ($L^2$- convergence) $S_n \to \frac{1}{2} W_t^2 - \frac{1}{2} t$ takes place. The limit $\frac{1}{2} W_t^2 - \frac{1}{2} Q_n(t)$ is called *stochastic integral* and we write

$$\int_0^t W_s dW_s = \frac{1}{2} W_t^2 - \frac{1}{2} t. \qquad (13)$$

- We see that, in addition to Riemann-Stiltjes integral, an additional term — $t/2$ has appeared.

# Stochastic integral of simple process

- Let $T = [0, \infty)$ and let $\{\mathcal{F}_t, t \in T\}$ be a filtration. Let $M$ be a continuous square integrable martingale w.r.t. the filtration $\{\mathcal{F}_t, t \in T\}$

**Definition 12**

The process $\eta$ is called a *simple process* if there exists a finite number of time instances $0 = t_0 < t_1 < \cdots < t_n = \infty$ and random variables $\xi_i$, $i = 1, 2, \dots, n$ with finite variances such that $\xi_i$ is $\mathcal{F}_{t_{i-1}}$-measurable for each $i$ and

$$\eta_t = \sum_{i=1}^{n} I_{[t_{i-1}, t_i]}(t)\xi_i.$$

# Stochastic integral of simple process wrt to a martingale

**Definition 13**

The *stochastic integral of a simple process η with respect to a martingale M* is defined as the process

$$Int_t = \int_0^t \eta_s dM_s := \sum_{i=1}^{n_t} \xi_i (M_{t_i} - M_{t_{i-1}}) + \xi_{n_t+1} \left( M_t - M_{t_{n_t}} \right)$$

$$\equiv \sum_{i=1}^{n_t} \xi_i \left( M_{t \wedge t_i} - M_{t \wedge t_{i-1}} \right),$$

where $n$ is an integer such that $t_{n_t} \leq t \leq t_{n_t+1}$

- Notice that the last formula has the structure of a martingale transform. Therefore, by Theorem 2, the process $Int_t$, is a martingale as well.
- It is easy to see that by choosing $\eta \equiv 1$, we obtain a useful formula

$$\int_0^t dM_s := M_t - M_0$$

as in the case of classical R-S integral.

# Stochastic integral of continuous process

Let $M$ be a martingale and let $Z$ be an adopted and continuous process such that

$$E(\int_0^t Z_s^2 d\langle M\rangle_s < \infty \ \forall\, t > 0$$

## Definition 14

The *stochastic integral of a process $Z$ with respect to a martingale $M$* is defined as the limit (in $\mathcal{L}^2$)

$$Int_t = \int_0^t Z_s dM_s := \lim_{\|\pi_n\|\to 0} \sum_{i=1}^{n} Z_{t_{i-1}} \left(M_{t_i} - M_{t_{i-1}}\right)$$

where $\pi_n$, is a partition of $[0, t]$ into $n$ subintervals:

$$0 = t_0 < t_1 < \cdots < t_n = t.$$

## Theorem 7

Stochastic integral has following properties:

(i) The process $(Int_t)_{t\in[0,\infty)}$square integrable martingale.

(ii) The quadratic variation of $Int$ is the process $\langle Int\rangle_t = \int_0^t Z_s dM_s$

# Itô formula and its applications

Consider an stochastic process $X_t$, which can be decomposed as

$$X_t = X_0 + V_t + M_t \tag{14}$$

Where

- $M_t$ is a martingale with continuous trajectories that have non-zero quadratic variation, $Var_2 > 0$ (such trajectories are called 'rough', e.g. Brownian motion),
- $V_t$ is an adopted process with continuously differentiable (or 'smooth') trajectoies that have finite variation, $Var_1 < \infty$ , and zero quadratic variation, $Var_2 = 0$ .
- Such processes $X_t$ are called *semimartingales*.

We are interested in an expression for the increment $f(X_t) — f(X_0)$ for some functions $f$.

In classical analysis, it is known that if $X_t$ is a "common" process (e.g. with continuously differentiable trajectories) then the Newton-Leibnizi formula gives

$$f(X_t) - f(X_0) = \int_0^t f'(X_s)dX_s$$

- However, for more general processes (14) the answer is different, as shows the Itô formula below.

# Itô formula

- Let $f$ be twice continuously differentiable function. Then

$$f(X_t) - f(X_0) = \int_0^t f'(X_s)dV_s + \int_0^t f'(X_s)dM_s + \frac{1}{2}\int_0^t f''(X_s)d\langle M\rangle_s \quad (15)$$

- The second integral at the right-hand side of Itô formula is a stochastic integral, whereas the other two are traditional (Riemann-Stiltjes) integrals. Hence, the stochastic integral can be expressed in terms of usual integrals.

- *Example 1.* Let $W_t$ be standard Brownian motion and $X_t = W_t$ (i.e. the decomposition (14) contains only the martingale part while $X_0 = V_t = 0$. Using Itô formula, show that $\int_0^t W_s dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}t$(that, in fact, we know already from before).

- Hint: Take $f(x) = x^2$.

# The proof of Itô formula

- We present only the basic idea of the proof. For simplicity, let $V_t = 0$ and let $M_t = W_t$ − a standard Brownian motion. Since the quadratic variation is $\langle W \rangle_s = s$, we need to show that

$$f(X_t) - f(X_0) = \int_0^t f'(X_s)dW_s + \frac{1}{2}\int_0^t f''(X_s)ds \quad (16)$$

- Let us consider a partition of the interval $[0, t]$ by points

$0 = t_0 < t_1 < \cdots < t_n = t$. Then

$$f(X_t) - f(X_0) \equiv \sum_{i=1}^{n} [f(X_{t_i}) - f(X_{t_{i-1}})].$$

# The proof of Itô formula

- In each subinterval we apply usual Taylor's formula:

$$f(X_t) - f(X_0) = \sum_{i=1}^{n} f'(X_{t_{i-1}})(W_{t_i} - W_{t_{i-1}}) + \frac{1}{2}\sum_{i=1}^{n} f''(X_{\xi_i})(W_{t_i} - W_{t_{i-1}})^2$$

where $\xi_i \in [t_i, t_{i-1}]$ .

- When the partition gets finer, so that the maximum length of subintervals tends to zero, the first sum converges (in mean square) to stochastic integral $\int_0^t f'(X_s)dW_s$. At the same time, in the second term the square of the increment of the Brownian motion $\left(W_{t_i} - W_{t_{i-1}}\right)^2$ can be approximated by its mean value $\Delta_i = t_i - t_{i-1}$ (related calculations were made in the proof of the *lemma 6* where it came out that the variance $D\left(\left(W_{t_i} - W_{t_{i-1}}\right)^2\right) = 2\Delta_i^2 \to 0$).
- Therefore, the second term can be approximated by the expression $\frac{1}{2}\sum_{i=1}^{n} f''(X_{\xi_i})\Delta_i$, which converges to usual integral $\frac{1}{2}\int_0^t f''(X_s)ds$.

# Example 2

- A widely used model in financial mathematics to describe the behaviour of a stock price $S_t$ is the following:

$$dS_t = S_t(\mu dt + \sigma dW_t) \qquad (17)$$

where

- $\mu$ shows relative change of the price per time unit (a constant in this model),
- $dW_t$ is the random part of the price change, that within a short time interval $\Delta t$ behaves like an increment of a Brownian motion,
- $\sigma$ shows the importance of the random component in the development of the price (volatility parameter).

The relationship above is, in fact, a short notation of the following equation:

$$S_t - S_0 = \int_0^t S_s \mu ds + \int_0^t S_s \sigma dW_s \qquad (18)$$

- The equation (17) is called stochastic differential equation (SDB) . Our aim is to solve this SDE for $S_t$. It turns out that the easiest way to do that is first to find $\ln(\frac{S_t}{S_0})$
- Hint: apply Itô formula for $f(x) = Inx$.

# Itô formula in differential form

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle M\rangle_t \qquad (19)$$

- The formula (19) comes immediately from the Itô formula (15), being simply its shorter (and more convenient) notation. Here the expression $d\langle M\rangle_t$, is the usual differential of the function $\langle M\rangle_t$, however $dX_t$ is a symbol of (nn *stochastic differential*), whose precise meaning is given by the notion of stochastic integral $\int_0^t f'(X_s)dX_s$.
- *Example 3*. Solve the problem in previous *Example 2* (slide 34), using the differential form of Itô formula.

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# A generalization of Itô formula

- Let the function $f$ also depend on time, $f = f(X_t, t)$, where the process $X_t$ is still of the form (18). Assuming that the function $f(x, t)$ is smooth enough (sufficiently times differentiable), the following formula is valid:

$$df(X_t) = \frac{\partial f}{\partial t}(X_t, t)dt + \frac{\partial f}{\partial t}(X_t, t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(X_t, t)d\langle M\rangle_t \qquad (20)$$

- *Example 4*. Assume that a stock price $S_t$ behaves in accordance with the following model:

$$dS_t = S_t(\mu dt + \sigma dW_t)$$

where $r$ is the risk free interest rate and $\sigma$ is the price volatility.

- Show that then the discounted price process $e^{-rt}S_t$ is a martingale.
- Hint: use $f(x, t) = e^{-rt}x$.

# Example 5

- Find $d(e^{-rt}V(S_t, t)$, where $V(s, t)$ is known twice continuously differentiable function and the price process $S_t$ is driven by the equation $dS_t = S_t(\mu dt + \sigma dW_t)$.
- Hint: Take $f(s, t) = e^{-rt}V(s, t)$.

# Itô 's Lemma

$$G \equiv G(x,t)$$

- Useful for understanding finance literature
- Continuous time extension of a Taylor series

$$dG = \frac{\partial G}{\partial t} dt + \frac{\partial G}{\partial x} dx + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} dt^2 + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} dx^2 + \frac{\partial^2 G}{\partial t \partial x} dxdt$$

- Plug in $dx$ and note that $(dt)^y=0$ if $y>1$

$$dG = \left( \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} bdz$$

# Ito's Lemma

*Exercise 1*. If $W$ is a standard Brownian motion, show that the process $X$ defined via $X_t = x + \mu t + \sigma W_t$ for $t \in \mathbb{R}_+$ is a $\mathrm{BM}(x, \mu, \sigma^2)$. Furthermore, if $X$ is a $\mathrm{BM}(x, \mu, \sigma^2)$, show that there exists a standard Brownian motion $W$ such that $X_t = x + \mu t + \sigma W_t$ holds for $t \in \mathbb{R}_+$.

# Samuelson, Merton, Black and Scholes model

- Observe that $L = \log(S)$ is a $\mathrm{BM}(\log(S_0), \gamma, \sigma^2)$. It follows that we can write $L_t = \log(S_0) + \gamma t + \sigma W_t$ for $t \in \mathbb{R}_+$, where $W$ is a standard Brownian motion. In other words, upon exponentiating, we have

$$S_t = S_0 \exp(\gamma t + \sigma \mathrm{W}_t, t \in \mathbb{R}_+ \quad (21)$$

- This is the famous model of Samuelson, Merton, Black and Scholes. In fact, one usually sees the later model given in the form of dynamics, i.e.,

$$\frac{dS_t}{S_t} = \alpha dt + \sigma dW_t, \text{for } t \in \mathbb{R}_+. \quad (22)$$

- Above, a is the *rate of return* of the risky asset and a its volatility. Observe that (22) is a *stochastic differential equation* that describes the infinitesimal relative change in the value of the risky asset.

# Vasicek's Interest Rate Model

- We now present an application of the previous theory to the study of a quite famous model for the stochastic behaviour of the short rate, due to Vasicek.
- It is assumed that the instantaneous interest rate $(r_t)_{t \in \mathbb{R}_+}$, satisfies the dynamics

$$(25)\ dr_t = a(\rho - r_t)dt + \sigma dW_t,$$

- where $W$ is a standard Brownian motion, and $a, \rho$ and $\sigma$ are strictly positive real numbers.
- The above dynamics in (25) result in $r$ being a so-called *mean-reverting process*: when far from level $\rho$, $r$ tends to revert back toward that level with amplitude determined by $a$.

# Itô – Doëblin formula

- Suppose that $X = (X_t)_{t\in\mathbb{R}_+}$, is an Itô process as in (1.11). If $X$ is actually differentiable (i.e., if $\sigma \equiv 0$), the usual chain rule of calculus implies that

$$df(X_t) = f'(X_t)dX_t$$

- holds for all $t \in \mathbb{R}_+$ and continuously differentiable functions $f$; in other words, $f(X_t) = f(Xo) +$
- $Ss\ f'(Xu)dXy$. The Itô-Doeblin formula is a generalization of this in the case where the "$dW$" part
- of an Itô process is not vanishing.

**Theorem 1.8**

Suppose that $X = (X_t)_{t\in\mathbb{R}_+}$, is an Itô process, and let $f: \mathbb{R} \to \mathbb{R}$ be a function that is twice continuously differentiable. Then, $\big(f(X_t)\big)_{t\in\mathbb{R}_+}$, is also an Itô process; in fact,

$$f(X_t) = f(X_0) + \int_0^t f'(X_u)dX_u + \frac{1}{2}\int_0^t f''(X_u)d[X,X]_u$$

holds for all $t \in \mathbb{R}_+$.

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2$$

Thank you for your attention!

See next week!