BS solution $\underline{V(t, S_t)}$

$$V'_t + \frac{1}{2}\sigma^2 S^2 V''_{SS} + rSV'_S - rV = 0$$

$X_t = \ln S_t$

$$V'_S = \frac{\partial V}{\partial S} = \frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial S} = \frac{\partial V}{\partial x}\cdot\frac{1}{S}$$

$$V''_{SS} = \frac{\partial}{\partial S}\left(V'_S\right) = \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial x}\cdot\frac{1}{S}\right) =$$

$$= \underbrace{\frac{\partial}{\partial S}\frac{\partial V}{\partial x}}_{\underbrace{\frac{\partial}{\partial x}\cdot\frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial S}}_{= V''_{xx}\cdot\frac{1}{S}}}\cdot\frac{1}{S} - \frac{\partial V}{\partial x}\cdot\frac{1}{S^2}$$

$$\Rightarrow V''_{SS} = V''_{xx}\cdot\frac{1}{S^2} - V'_x\cdot\frac{1}{S^2}$$

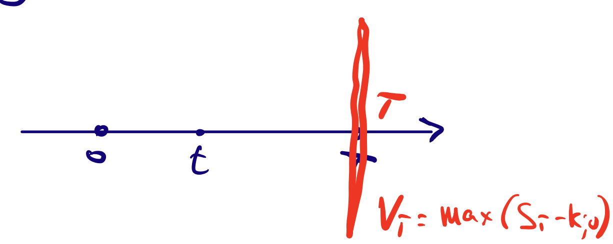
$$V'_S = V'_x\cdot\frac{1}{S} \qquad V''_{SS} = V''_{xx}\cdot\frac{1}{S^2} - V'_x\cdot\frac{1}{S^2}$$

$$V'_t + \frac{1}{2}\sigma^2 S^2 V''_{SS} + rSV'_S - rV = 0$$

$$\downarrow$$

$$V'_t + \frac{1}{2}\sigma^2\left(V''_{xx} - V'_x\right) + r\cdot V'_x - rV = 0$$

$$V'_t + \frac{1}{2}\sigma^2 V''_{xx} + \left(r - \frac{\sigma^2}{2}\right)V'_x - rV = 0$$



$$t = T - \tau \Rightarrow \tau = T - t$$

$$V'_t = \frac{\partial V}{\partial t} = \frac{\partial V}{\partial \tau}\cdot\frac{\partial \tau}{\partial t} = -V'_\tau$$

$$-V'_\tau + \frac{1}{2}\sigma^2 V''_{xx} + \left(r - \frac{\sigma^2}{2}\right)V'_x - rV = 0 \qquad V_{\tau=0} \checkmark$$

let $k = \frac{r}{\sigma^2/2}$

$$-\frac{1}{\frac{\sigma^2}{2}}\cdot V'_\tau + V''_{xx} + \left(\underbrace{\frac{r}{\frac{\sigma^2}{2}}}_{k} - 1\right)V'_x - \underbrace{\frac{r}{\frac{\sigma^2}{2}}}_{k} V = 0$$

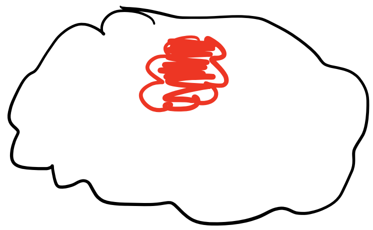
$$-\frac{1}{\sigma^2/2}\cdot V'_{\tilde\tau} + V''_{xx} + (k-1)V'_x - k\cdot V = 0$$

NOW $\quad \tau' = \tilde\tau\cdot\frac{\sigma^2}{2} \Rightarrow -\frac{1}{\sigma^2/2}\cdot V'_{\tilde\tau} = -\frac{1}{\sigma^2/2}\cdot\frac{\partial V}{\partial\tau'}\cdot\frac{\partial\tau'}{\partial\tilde\tau} =$

$$= -\frac{1}{\sigma^2/2}V'_{\tau'}\cdot\sigma^2/2 = -V'_{\tau'}$$

$$\tilde\tau = \tau'$$

$$-V'_{\tilde\tau} + V''_{xx} + (k-1)V'_x - k\cdot V = 0$$
$$V'_{\tilde\tau} = V''_{xx} + (k-1)V'_x - k\cdot V$$



$$V = e^{\alpha x+\beta\cdot\tau}\cdot u(x,\tau)$$

$$\Rightarrow V'_\tau = \beta e^{\cdots}\cdot u + e^{\cdots}u'_\tau$$
$$V'_x = \alpha e^{\cdots}u + e^{\cdots}\cdot u'_x = e^{\cdots}(\alpha u + u'_x)$$
$$V''_{xx} = \alpha e^{\cdots}(\alpha u + u'_x) + e^{\cdots}\cdot(\alpha u'_x + u''_{xx})$$

$$\beta\cancel{e^{\cdots}}u + \cancel{e}u'_\tau = \alpha\cancel{e^{\cdots}}(\alpha u + u'_x) + \cancel{e^{\cdots}}\cdot(\alpha u'_x + u''_{xx}) + (k-1)\cancel{e^{\cdots}}(\alpha u + u'_x) - k\cancel{e^{\cdots}}u$$
$$\beta u + u'_\tau = \alpha^2 u + \alpha u'_x + \alpha u'_x + u''_{xx} + (k-1)\alpha u + (k-1)u'_x - ku$$

$$u'_\tau = u''_{xx} + u'_x\cdot\underbrace{(2\alpha + k - 1)}_{=0} + u\cdot\underbrace{\left(-\beta + \alpha^2 + (k-1)\alpha - k\right)}_{=0}$$

$$\underline{\underline{\alpha = -\frac{k-1}{2}}}$$

$$\beta = \frac{(k-1)^2}{4} - \frac{(k-1)^2}{2} - k =$$
$$\beta = -\frac{(k-1)^2}{4} - k = -\frac{k^2-2k+1+4k}{4}$$
$$\beta = -\frac{k^2+2k+1}{4} = \underline{\underline{-\frac{(k+1)^2}{4}}}$$

$\boxed{u'_\tau = u''_{xx}}$ - Heat Equation

Green's Kernel

$$u(x,\tau) = \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{\sqrt{\tau}} \cdot \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4\tau}} \cdot u(x,\tau=0)\,dy$$

$$V(x,\tau=0) = \max(S_t - k;0) = \max(e^x - k;0)$$

$= e^x - k$ if $S_t > k$

if $S_t \leq k$ $=0$

$$u(x,\tau=0) = \frac{V(x,\tau=0)}{e^{\cdots}}$$

$$u(x,0) = \left(e^{\frac{k+1}{2}x} - e^{\frac{k-1}{2}x}\right)^+ = \max(\ldots;0)$$

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\tau}} \cdot \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4\tau}} \cdot \underbrace{u(x,\tau=0)}_{u_0}\,dy$$

$y = x + z\cdot\sqrt{2\tau}$ $\quad dy = \sqrt{2\tau}\,dz$

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\tau}} \int_{-\infty}^{+\infty} e^{-\frac{z^2}{2}} \cdot u_0 \cdot \sqrt{2\tau}\,dz$$

$$u_0 = e^{\frac{k+1}{2}(x+z\sqrt{2\tau})} - e^{\frac{k-1}{2}(x+z\sqrt{2\tau})} \geq 0 \Rightarrow$$

$$\Rightarrow z \geq -\frac{x}{\sqrt{2\tau}}$$

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\frac{x}{\sqrt{2\tau}}}^{+\infty} e^{-\frac{z^2}{2}} \left(e^{\frac{k+1}{2}(x+z\sqrt{2\tau})} - e^{\frac{k-1}{2}(x+z\sqrt{2\tau})}\right) dz =$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\frac{x}{\sqrt{2\tau}}}^{+\infty} e^{\frac{k+1}{2}x + \frac{k+1}{2}z\cdot\sqrt{2\tau} - \frac{z^2}{2}}\,dz - \int_{-\frac{x}{\sqrt{2\tau}}}^{+\infty} e^{\frac{k-1}{2}x + \frac{k-1}{2}z\cdot\sqrt{2\tau} - \frac{z^2}{2}}\,dz\right]$$

$$\frac{k+1}{2}x + \frac{k+1}{2}z\sqrt{2\tau} - \frac{z^2}{2} = -\frac{1}{2}\left[z^2 - 2z\sqrt{2\tau}\,\frac{(k+1)}{2}\right] + \frac{k+1}{2}x =$$

$$= -\frac{1}{2}\left[z^2 - 2z\sqrt{2\tau}\cdot\frac{k+1}{2} + \frac{\tau}{2}(k+1)^2\right] + \frac{k+1}{2}x + \frac{\tau}{4}(k+1)^2$$

$$= -\frac{1}{2}\left[z - \sqrt{\tau/2}\cdot(k+1)\right]^2 + \frac{k+1}{2}x + \frac{\tau}{4}(k+1)^2$$

$$\int_{-\frac{x}{\sqrt{2\tau}}}^{+\infty} e^{\frac{k+1}{2}x + \frac{k+1}{2}z\sqrt{2\tau} - \frac{z^2}{2}}\, dz = e^{\frac{k+1}{2}x + \frac{\tau}{4}(k+1)^2} \cdot \int_{-\frac{x}{\sqrt{2\tau}}}^{+\infty} e^{-\frac{\left[z - \sqrt{\frac{\tau}{2}}(k+1)\right]^2}{2}}\, dz =$$

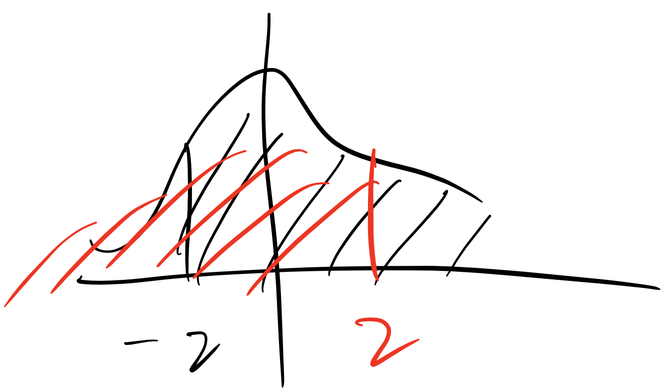
let $m = z - \sqrt{\frac{\tau}{2}}(k+1) \Rightarrow dm = dz$

$$= e^{\frac{k+1}{2}x + \frac{\tau}{4}(k+1)^2} \cdot \int_{-\left(\frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k+1)\right)}^{+\infty} e^{-\frac{m^2}{2}}\, dm$$

$$"P\left(M > -\left(\frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k+1)\right)\right) = P\left(M < +\left(\frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k+1)\right)\right)$$

$$" \; N(d_1)$$

$$d_1 = +\left(\frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k+1)\right)$$



$$V_t = S_t N(d_1) - K e^{-r(T-t)} N(d_2)$$

$$d_1 = \frac{\ln\frac{S_t}{K} + \left(r - \frac{\sigma^2}{2}\right)\sqrt{T-t}}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \left(r - \frac{\sigma^2}{2}\right)\sqrt{T-t}$$