Brownian Motion

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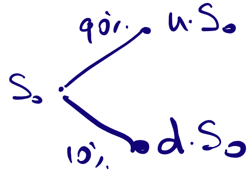
Risk Neutral Pricing — Binomial Tree

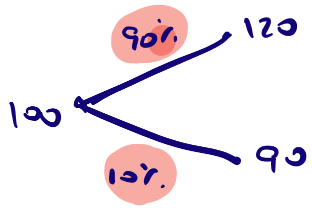
$\{\Omega, \mathbb{P}, \mathcal{F}\}$ real prob.

$\{\Omega, \mathbb{Q}, \mathcal{F}\}$ risk-neutral prob.

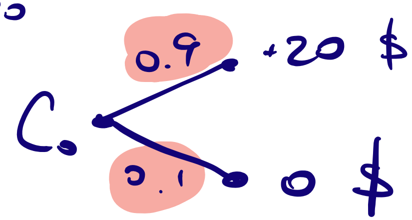


$\Rightarrow$ Call $= \max(S_T - K; 0)$

- $S_0 = 100$ $\quad u = 1.2$ $\quad d = 0.9$ $\quad K = 100$ $\quad r_f = 5\%$ $\Rightarrow$



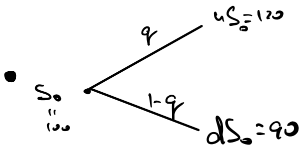
$\Rightarrow$



$\Rightarrow$

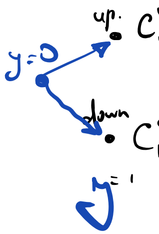
$y = 0$ $\quad y = 1$ $\qquad y = 0$ $\quad y = 1$

$$\Rightarrow C_0 = \frac{(0.9 \cdot 20 + 0.1 \cdot 0)}{1 + r_f} = \frac{18}{1 + r_f} = \frac{18}{1.05}$$



$B_0 \longrightarrow (1 + r_f) B_0$

$\Rightarrow$



$C_1^u = 20 = \alpha \cdot u \cdot S_0 + \beta \cdot (1 + r_f) B_0$

$C_1^d = 0 = \alpha d S_0 + \beta (1 + r_f) B_0$

$$\alpha \cdot u \cdot S_0 + \beta \cdot (1 + r_f) B_0 = 20$$
$$\alpha d S_0 + \beta (1 + r_f) B_0 = 0$$

$\Rightarrow$ $\alpha (u - d) S_0 = 20 \rightarrow \alpha = \frac{20}{(u-d) S_0} = \frac{1}{u-d} \cdot \frac{1}{5}$

$$\beta = \frac{-\alpha d S_0}{(1 + r_f) B_0} = \frac{\frac{1}{u-d} \cdot d \cdot \frac{100}{5}}{(1 + r_f) B_0} = \frac{20 d}{u - d} \cdot \frac{1}{(1 + r_f) B_0}$$

$P = \alpha \cdot S + \beta \cdot B$

$$P_0 = \frac{1}{u-d} \cdot 20 + \frac{20 d}{u - d} \cdot \frac{1}{(1 + r_f)} = 20 \cdot \left[\frac{1}{u-d} + \frac{d / 1 + r_f}{u - d} \right]$$

$$= \frac{20}{1 + r_f} \left[\frac{1 + r_f}{u - d} + \frac{d}{u - d} \right]$$

Ito’s Lemma

2.2. **Vasicek’s interest rate model.** We now present an application of the previous theory to the study of a quite famous model for the stochastic behaviour of the short rate, due to Vasicek. It is assumed that the instantaneous interest rate $(r_t)_{t\in\mathbb{R}_+}$ satisfies the dynamics

$$\mathrm{d}r_t = a(\rho - r_t)\mathrm{d}t + \sigma \mathrm{d}W_t, \tag{2.3}$$

where W is a standard Brownian motion, and a, ρ and σ are strictly positive real numbers. The above dynamics in (2.3) result in r being a so-called *mean-reverting* process: when far from level ρ, r tends to revert back toward that level with amplitude determined by a.

Ito's Lemma

3. Itô-Doëblin Formula

3.1. **The formula.** Suppose that $X = (X_t)_{t \in \mathbb{R}_+}$ is an Itô process as in (1.11). If X is actually differentiable (i.e., if $\sigma \equiv 0$), the usual chain rule of calculus implies that $\mathrm{d}f(X_t) = f'(X_t)\mathrm{d}X_t$ holds for all $t \in \mathbb{R}_+$ and continuously differentiable functions f; in other words, $f(X_t) = f(X_0) + \int_0^t f'(X_u)\mathrm{d}X_u$. The Itô-Doeblin formula is a generalisation of this in the case where the "dW" part of an Itô process is not vanishing.

Theorem 3.1. *Suppose that $X = (X_t)_{t \in \mathbb{R}_+}$ is an Itô process, and let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function that is twice continuously differentiable. Then, $(f(X_t))_{t \in \mathbb{R}_+}$ is also an Itô process; in fact,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_u)\mathrm{d}X_u + \frac{1}{2}\int_0^t f''(X_u)\mathrm{d}[X, X]_u \tag{3.1}$$

holds for all $t \in \mathbb{R}_+$.

Ito's Lemma

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t$$

European option pricing PDE

$$u(t,x) := E\left[\mathrm{e}^{-r(T-t)} f(S_T) \mid S_t = x\right],$$

$$\frac{\partial}{\partial t}u(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}u(t,x) + rx\frac{\partial}{\partial x}u(t,x) - ru(t,x) = 0, \quad u(T,x) = f(x).$$

American option pricing PDE (HJB equation in form of variational inequality)

$$v(t,x) := \sup_{\tau\in\mathcal{T}_{t,T}} E\left[\mathrm{e}^{-r(\tau-t)} f(S_\tau) \mid S_t = x\right], \quad \mathcal{T}_{t,T} := \{\,\text{Stopping times } t \le \tau \le T\,\},$$

$$\frac{\partial}{\partial t}v(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}v(t,x) + rx\frac{\partial}{\partial x}v(t,x) - rv(t,x) \le 0, \quad v(t,x) \ge f(x), \quad v(T,x) = f(x),$$

with equality in the PDE on $\{\, x \mid v(t,x) > f(x) \,\}$.

Ito's Lemma

$$\frac{\partial}{\partial t}u(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}u(t,x) + rx\frac{\partial}{\partial x}u(t,x) - ru(t,x) = 0, \quad u(T,x) = (K-x)^+$$

- Change of variables: $y := \log(x/K)$, $\tau := \frac{1}{2}\sigma^2(T-t)$, $q := 2r/\sigma^2$ and $\widetilde{u}(\tau,y) := \frac{1}{K}\exp\left(\frac{1}{2}(q-1)y + \left(\frac{1}{4}(q-1)^2 + q\right)\tau\right)u(t,x)$, satisfying the heat equation

$$\frac{\partial}{\partial \tau}\widetilde{u}(\tau,y) = \frac{\partial^2}{\partial y^2}\widetilde{u}(\tau,y), \quad \widetilde{u}(0,y) = \left(e^{\frac{1}{2}(q-1)y} - e^{\frac{1}{2}(q+1)y}\right)_+ .$$

Ito’s Lemma

$$\frac{\partial}{\partial t}u(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}u(t,x) + rx\frac{\partial}{\partial x}u(t,x) - ru(t,x) = 0, \quad u(T,x) = (K-x)^+$$

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- Boundary conditions: Natural boundary condition for put option at $y \to \infty$ and from put-call-parity:

$$\widetilde{u}(\tau,y) = \exp\left(\frac{1}{2}(q-1)y + \frac{1}{4}(q-1)^2\tau\right) \text{ for } y \to -\infty, \quad \widetilde{u}(\tau,y) = 0 \text{ for } y \to \infty$$

Ito's Lemma

$$\frac{\partial}{\partial t}v(t,x)+\frac{1}{2}\sigma^2x^2\frac{\partial^2}{\partial x^2}v(t,x)+rx\frac{\partial}{\partial x}v(t,x)-rv(t,x) \leq 0, \quad v(t,x) \geq (K-x)^+, \quad v(T,x) = (K-x)^+$$

The same transformation as above gives:

$$\left(\frac{\partial}{\partial \tau}\widetilde{v}(\tau,y) - \frac{\partial^2}{\partial y^2}\widetilde{v}(\tau,y)\right)(\widetilde{v}(\tau,y) - g(\tau,y)) = 0, \quad \frac{\partial}{\partial \tau}\widetilde{v}(\tau,y) - \frac{\partial^2}{\partial y^2}\widetilde{v}(\tau,y) \geq 0,$$

$$\widetilde{v}(\tau,y) \geq g(\tau,y), \quad \widetilde{v}(0,y) = g(0,y),$$

$$\widetilde{v}(\tau,y) = g(\tau,y) \text{ for } y \to -\infty, \quad \widetilde{v}(\tau,y) = 0 \text{ for } y \to \infty,$$

where

$$g(y,\tau) := \exp\left(\frac{1}{4}(q+1)^2\tau\right)\left(e^{\frac{1}{2}(q-1)y} - e^{\frac{1}{2}(q+1)y}\right)^+$$

Ito's Lemma

- Forward difference quotient: $f'(x) = \dfrac{f(x+h) - f(x)}{h} + \mathcal{O}(h)$
- Backward difference quotient: $f'(x) = \dfrac{f(x) - f(x-h)}{h} + \mathcal{O}(h)$
- Central difference quotient: $f'(x) = \dfrac{f(x+h) - f(x-h)}{h} + \mathcal{O}(h^2)$
- Central difference quotient: $f''(x) = \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2} + \mathcal{O}(h^2)$

Replace derivatives in time and space by finite difference quotients based on grids.

Ito's Lemma

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Replace derivatives in time and space by finite difference quotients based on grids.

- Notation: $t_i := i\Delta t$, $i = 0, \ldots, N$, $\Delta t := T/N$. $x_j := a + j\Delta x$, $j = 0, \ldots, M$, $\Delta x := (b-a)/M$.
- Solving heat equation with appropriate boundary conditions on $[a, b]$, setting $u_{i,j} := u(t_i, x_j)$ – and similarly its FD approximation $\bar{u}_{i,j}$.

Ito's Lemma

Based on the approximations:

$$\frac{\partial}{\partial t}u(t_i,x_j)=\frac{u_{i+1,j}-u_{i,j}}{\Delta t}+\mathcal{O}(\Delta t),\qquad \frac{\partial^2}{\partial x^2}u(t_i,x_j)=\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\Delta x^2}+\mathcal{O}(\Delta x^2).$$

Explicit FD scheme for the heat equation

With $\lambda := \frac{\Delta t}{\Delta x^2}$ set $\overline{u}_{i+1,j} := \overline{u}_{i,j} + \lambda(\overline{u}_{i,j+1} - 2\overline{u}_{i,j} + \overline{u}_{i,j-1})$, $i = 0,\dots,N-1$, $j = 1,\dots,M-1$.

- Boundary and initial conditions for the European put option:

$$\overline{u}_{0,j}=\left(e^{\frac{1}{2}(q-1)x_j}-e^{\frac{1}{2}(q+1)x_j}\right)^+,\quad \overline{u}_{i+1,0}=\exp\left(\frac{1}{2}(q-1)a+\frac{1}{4}(q-1)^2t_{i+1}\right),\quad \overline{u}_{i+1,M}=0$$

- Up to the boundary conditions,

$$\overline{u}_{i+1,:}=A(\lambda)\overline{u}_{i,:},\quad A(\lambda):=\begin{pmatrix}1-2\lambda & \lambda & 0 & \cdots & 0\\ \lambda & 1-2\lambda & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \lambda\\ 0 & \cdots & 0 & \lambda & 1-2\lambda\end{pmatrix}.$$

Example (Instability of the explicit FD scheme)

Consider the heat equation with $u(0,x) = \sin(\pi x)$, $x \in [0,1]$, $u(t,0) \equiv u(t,1) \equiv 1$. Then $u(t,x) = \sin(\pi x)e^{-\pi^2 t}$. Compute $u(0.5, 0.2)$:

- Explicit solution: $u(0.5, 0.2) = 0.004227$.
- FD with $\Delta x = 0.1$, $\Delta t = 0.0005$: $u(0.5, 0.2) \approx \overline{u}_{1000,2} = 0.00435$.
- FD with $\Delta x = 0.1$, $\Delta t = 0.01$: $u(0.5, 0.2) \approx \overline{u}_{50,2} = -1.5 \times 10^8$.

- The explicit FD scheme is prone to instability, i.e., explosive error propagation.
- $x \mapsto Ax$ is stable iff the spectral radius is bounded by 1. For $A(\lambda)$ this can be proved to be the case when $\lambda \leq 1/2$.

Theorem

The explicit FD scheme converges is stable and converges when $\Delta t \leq \frac{1}{2}\Delta x^2$ (plus technical conditions). In this case, the error behaves like $O(\Delta t) + O(\Delta x^2)$.

Ito's Lemma

Based on the approximations:

$$\frac{\partial}{\partial t}u(t_i, x_j) = \frac{u_{i,j} - u_{i-1,j}}{\Delta t} + \mathcal{O}(\Delta t), \qquad \frac{\partial^2}{\partial x^2}u(t_i, x_j) = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta x^2}$$

Implicit FD scheme for the heat equation

Define $\overline{u}_{i,:}$ as solution of the system $\overline{u}_{i-1,j} = \overline{u}_{i,j} + \frac{\Delta t}{\Delta x^2}(-\overline{u}_{i,j+1} + 2\overline{u}_{i,j} - \overline{u}_{i,j-1})$, $i = 1, \ldots, N$, $j = 1, \ldots, M-1$.

- Up to boundary conditions:

$$A\overline{u}_{i,:} = \overline{u}_{i-1,:}, \quad A := \begin{pmatrix} 1+2\lambda & -\lambda & 0 & \cdots & 0 \\ -\lambda & 1+2\lambda & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -\lambda \\ 0 & \cdots & 0 & -\lambda & 1+2\lambda \end{pmatrix}.$$

Theorem

The implicit FD scheme is unconditionally stable and converges with $\mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2)$.

Ito's Lemma

- Assume that the Cauchy problem is well-posed.
- Notation: u ...solution of the PDE, u^i ... u at time t_i discretized on the x-grid, $\overline{u}^i$ ...FD approximation, given by $B_1\overline{u}^{i+1} = B_0\overline{u}^i + f^i$.
- Consistency: $\left\|B_1 u^{i+1} - (B_0 u^i + f^i)\right\| \to 0$ as $\Delta t = T/N, \Delta x = (b-a)/M \to 0$, $i = 0, \ldots, N$.
- Stability: there is a constant C s.t. $\left\|(B_1^{-1}B_0)^N\right\| \le C$ uniformly in N. (Note: B_1, B_0 depend on N via $\Delta t, \Delta x$.)
- Convergence: Consider $i(N)$ s.t. $t_i \to t$ as $N \to \infty$. Then $\left\|\overline{u}^{i(N)} - u^{i(N)}\right\| \to 0$ as $N \to \infty$.

Theorem (Lax–Richtmyer; Lax equivalence principle; Fundamental theorem of numerical analysis)

For a consistent scheme, stability is equivalent to convergence, provided the problem is linear and well-posed.

Ito's Lemma

$$\frac{\partial}{\partial t}u(t_{i+1}, x_j) = \frac{u_{i+1,j} - u_{i,j}}{\Delta t} + \mathcal{O}(\Delta t)$$

can be seen as forward or backward difference quotient, leading to

$$\frac{\overline{u}_{i+1,j} - \overline{u}_{i,j}}{\Delta t} = \frac{\overline{u}_{i,j+1} - 2\overline{u}_{i,j} + \overline{u}_{i,j-1}}{\Delta x^2} \quad \text{or} \quad \frac{\overline{u}_{i+1,j} - \overline{u}_{i,j}}{\Delta t} = \frac{\overline{u}_{i+1,j+1} - 2\overline{u}_{i+1,j} + \overline{u}_{i+1,j-1}}{\Delta x^2}$$

Instead, take the mean of the right hand sides:

Crank–Nicolson scheme

Define $\overline{u}_{i+1,:}$ as solution of the system
$\overline{u}_{i+1,j} - \overline{u}_{i,j} = \frac{\Delta t}{2\Delta x^2}\left(\overline{u}_{i,j+1} - 2\overline{u}_{i,j} + \overline{u}_{i,j-1} + \overline{u}_{i+1,j+1} - 2\overline{u}_{i+1,j} + \overline{u}_{i+1,j-1}\right).$

- System of equations of the form $A\overline{u}_{i+1,:} = B\overline{u}_{i,:}$.
- Unconditionally stable and converges with error $\mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2)$.

Ito's Lemma

$$\left(\frac{\partial}{\partial \tau}\widetilde{v}(\tau,y) - \frac{\partial^2}{\partial y^2}\widetilde{v}(\tau,y)\right)\left(\widetilde{v}(\tau,y) - g(\tau,y)\right) = 0, \quad \frac{\partial}{\partial \tau}\widetilde{v}(\tau,y) - \frac{\partial^2}{\partial y^2}\widetilde{v}(\tau,y) \geq 0,$$

$$\widetilde{v}(\tau,y) \geq g(\tau,y), \quad \widetilde{v}(0,y) = g(0,y),$$

$$\widetilde{v}(\tau,y) = g(\tau,y) \text{ for } y \to -\infty, \quad \widetilde{v}(\tau,y) = 0 \text{ for } y \to \infty,$$

- Have to solve linear inequality systems of the form $Aw - b \geq 0$, $w \geq g$, $(aw - b)^{\top}(w - g) = 0$ for the approximate solution w. Projection SOR (Successive over-relaxation algorithm).
- Poor man's algorithm: Use standard FD iterations, but take maximum with payoff function at each iteration step.

Consider, for simplicity, the Poisson equation $\Delta u = f$ on $[0,1]$ with $u(0) = u(1) = 0$.

1. **Variational (weak) formulation**: u is the only element of $V := H_0^1$ such that for every test function $v \in V$:
$$A(u,v) = L(v), \quad A(u,v) := -\int_0^1 u'(x)v'(x)\,\mathrm{d}x, \quad L(v) := \int_0^1 f(x)v(x)\,\mathrm{d}x$$
2. **Projection onto finite dimensional space**: Choose $V_h \subset V$, $\dim V_h < \infty$, $h > 0$, and consider the projected problem $\forall v \in V_h :\ A(u_h, v) = L(v)$, with solution $u_h \in V_h$. E.g.,
$$V_h := \left\{ v \in C([0,1]) \,\middle|\, v|_{[x_i, x_{i+1}]} \text{ affine}, i = 0, \ldots, N,\ v(0) = v(1) = 0 \right\},\ x_i := ih,\ h := \frac{1}{N+1}.$$
3. **Parameterization in terms of basis**: Choose a basis $(\phi_i)_{i=1}^N$ of V_h obtaining the system $A(u_h, \phi_i) = L(\phi_i),\ i = 1, \ldots, N$. E.g., ϕ_i piecewise-linear, $\phi_i(x_j) = \delta_{ij},\ j = 0, \ldots, N+1$.
4. **Solve the system**: For $u_h = \sum_{i=1}^N \xi_i \phi_i$, $\overline{A}\xi = \overline{L}$, where $\overline{A}_{i,j} := A(\phi_i, \phi_j)$, $\overline{L}_i := L(\phi_i)$, $i, j = 1, \ldots, N$.

Ito's ~~Lemma~~

$$S_0(1+r) = \pi \cdot u \cdot S_0 + (1-\pi)\, d S_0$$

$$1+r = \pi \cdot u + d - \pi \cdot d$$

$$\pi(u-d) = 1+r-d$$

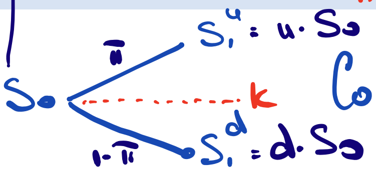
$$\pi = \frac{1+r-d}{u-d}$$

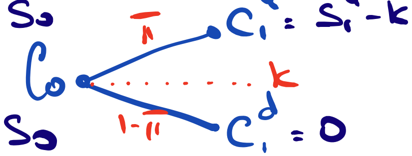
Risk Neutral Prob. lity

$$B_0 \longrightarrow B_1 = (1+r) B_0$$

B_t - bond's price at time t

AIM



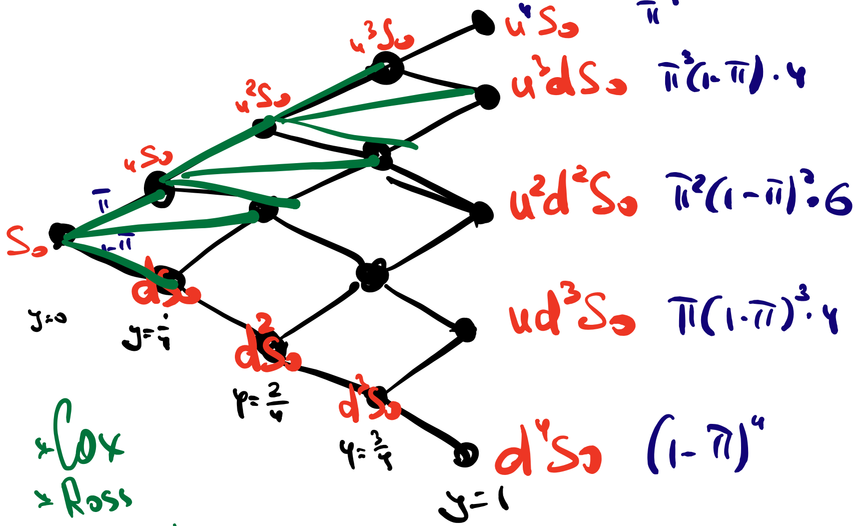


$$P_t = \Delta \cdot S_t - B_1 \Rightarrow \begin{cases} \Delta \cdot S_1^u - B_1 = S_1^u - K \\ \Delta \cdot S_1^d - B_1 = 0 \end{cases} \rightarrow \Delta (S_1^u - S_1^d) = S_1^u - K$$

$$\Delta = \frac{u \cdot S_0 - K}{(u-d) S_0} = \frac{u - K/S_0}{u-d}$$

self-financing portfolio

$$C_0 = \frac{E^*(\text{payoff})}{1+r_f} = \frac{\pi \cdot C_1^u + (1-\pi) C_1^d}{1+r_f} = \frac{\frac{1+r-d}{u-d} C_1^u + \frac{u-1-r}{u-d} \cdot C_1^d}{1+r_f}$$



* Cox
* Ross
* Rubinstein

$$\Sigma \dots \quad n \to +\infty$$

$$C_0 = \frac{E^*(\text{payoff})}{1+r_f} \xrightarrow[n\to\infty]{} S_0 N(d_1) - K e^{-r(T-t)} N(d_2)$$

Thank you for your attention!

See next week!