# Wiener process.

## $(W_t)$ is a Wiener process

1. $W_0 = 0$
2. Independent increments.

$t_0 \quad t_1 \quad \cdots\cdots \quad t_n$

like Poisson process

$$\left.\begin{matrix} W(t_1) - W(t_0) \\ W(t_2) - W(t_1) \\ \vdots \\ W(t_n) - W(t_{n-1}) \end{matrix}\right\} \text{ these r.vs are independ}$$

3. $W(t) - W(s) \sim N(0; t-s) \quad (t \geq s)$

   in P.P. $N_t - N_s \sim Poiss(\lambda(t-s))$

4. $P($ traject of $W_t$ is continuous $) = 1.$

   in P.P. $P($traj has jumps$) = 1$

**Ex 1** $(W_t)$ – is Wiener process.

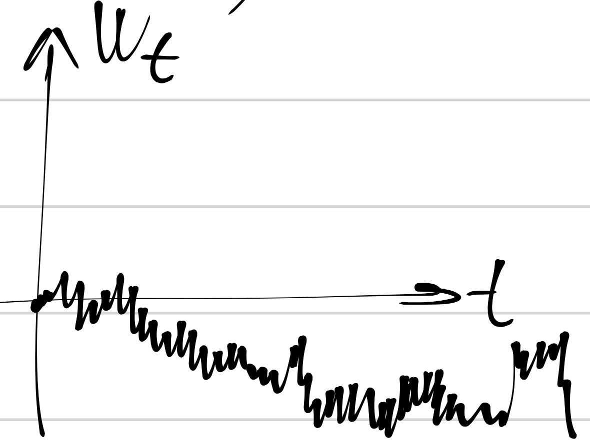
a) $E(W_{10})$, $Var(W_{15})$ !!

b) $cov(W_{10}, W_{15})$

c) $E(W_5 \cdot W_7)$

d) $E(W_7 | W_5)$, $Var(W_7 | W_5)$

e) $P(W_7 > 2 | W_5)$




a) $W_{10} = W_{10} - W_0 \sim N(0; 10)$

assumpt assumpt

$W_{10} \sim N(0; 10) \quad E(W_{10}) = 0 \quad Var(W_{10}) = 10$

trick: $\boxed{\text{future} = \text{current value} + \text{increment}}$

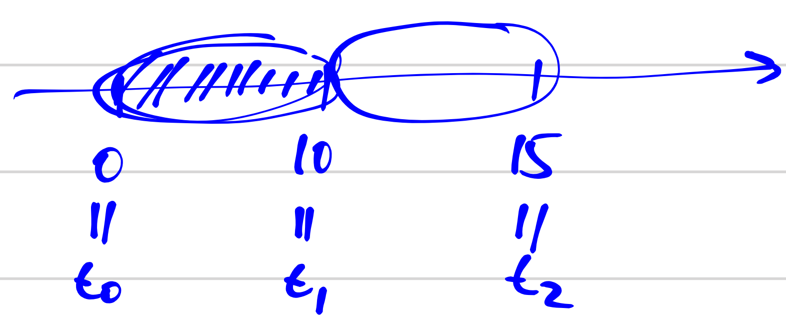
b) $\mathrm{Cov}(W_{10}, W_{15}) =$

$$\left\{ W_{15} = W_{10} + (W_{15} - W_{10}) \right\}$$

$$= \mathrm{Cov}\big(W_{10}, W_{10} + (W_{15} - W_{10})\big) =$$

$W_0 = 0$

$$= \underbrace{\mathrm{Cov}(W_{10}, W_{10})}_{= \mathrm{Var}(W_{10}) = 10} + \underbrace{\mathrm{Cov}(W_{10}, W_{15} - W_{10})}_{= \mathrm{Cov}(W_{10} - W_0,\, W_{15} - W_{10}) = 0} = 10 + 0$$




!! c) $E(W_5 \cdot W_7)$

d) $E(W_7 \mid W_5)$, $\mathrm{Var}(W_7 \mid W_5)$

e) $P(W_7 > 2 \mid W_5)$

$$\underbrace{\mathrm{Cov}(W_5, W_7)}_{5} = E(W_5 \cdot W_7) - \underbrace{E(W_5)}_{0} \cdot \underbrace{E(W_7)}_{0}$$

a) $W_5 \sim N(0; 5)$

$E(W_t) = 0$

$\mathrm{Var}(W_t) = t$

$\mathrm{Cov}(W_t, W_s) = \min(t, s)$

d) $E(W_7 \mid W_5) = E(W_5 + (W_7 - W_5) \mid W_5) =$

$$= W_5 + E(\underbrace{W_7 - W_5}_{\text{indep}} \mid W_5 - \underbrace{W_0}_{0}) = W_5 + \underbrace{E(W_7 - W_5)}_{0} = W_5$$

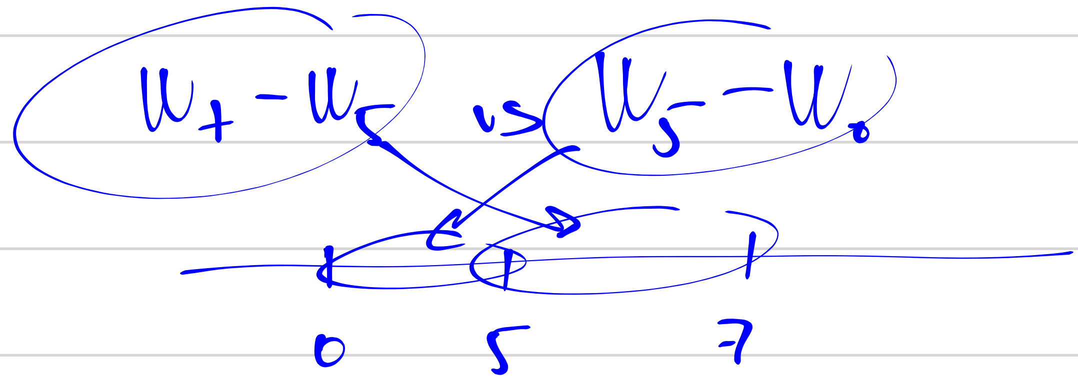
$$\underset{\substack{\| \\ 7}}{Var(W_7)} \neq Var(W_7 \mid W_5) = Var(\boxed{W_5} + \boxed{(W_7 - W_5)} \mid W_5) =$$

we know it

$$= Var(\underbrace{W_7 - W_5}_{\text{L indep}} \mid W_5) = Var(\underbrace{W_7 - W_5}_{\sim N(0;7-5)}) = 2$$

$$Var(S + R) = Var(R) \quad [\text{uncond}]$$

$$Var(S + R \mid S) = Var(R \mid S) \quad [\text{cond}]$$




$$E(W_7 \mid W_5) = W_5$$

$$Var(W_7 \mid W_5) = 2$$

e) $P(W_7 > 2 \mid W_5) =$

$$= P\left(\frac{W_7 - W_5}{\sqrt{2}} > \frac{2 - W_5}{\sqrt{2}} \,\middle|\, W_5\right) = *$$

$$\left(\frac{W_7 - W_5}{\sqrt{2}} \,\middle|\, W_5\right) \sim N(0;1)$$

$F(t) =$ cdf of $N(0:1) = P(N(0;1) \leq t)$ $\qquad F(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du$

$$* = P\left(\frac{W_5 - W_7}{\sqrt{2}} < \frac{W_5 - 2}{\sqrt{2}} \,\middle|\, W_5\right) =$$

$$= F\left(\frac{W_5 - 2}{\sqrt{2}}\right)$$

Ex 2 $(W_t)$ - Wiener Process.

a) is $(W_t)$ a martingale

b) $\boxed{\text{find } \alpha \text{ such that } Y_t = W_t^2 + \alpha t \text{ is a martingale}}$

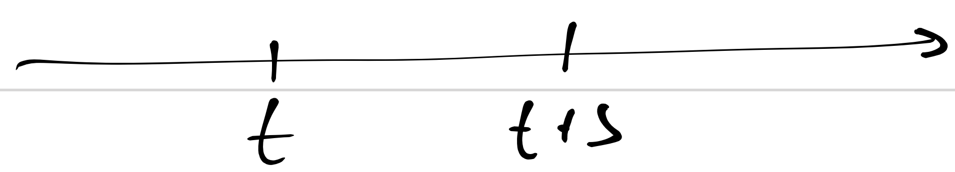
c) find $\beta$ such that $R_t = \exp(W_t + \beta t)$ is a martingale.

def $(M_t)$ is a martingale: $\boxed{E(M_{t+s} \mid \mathcal{F}_t) = M_t}$

discr. $\mathcal{F}_t = \sigma(M_1, M_2, \ldots, M_t)$

cont. $\mathcal{F}_t = \sigma(M_s, s \in [0:t])$ natural filtration.

a) $E(\underbrace{W_{t+s}}_{\text{future}} \mid \mathcal{F}_t) = E(\underbrace{W_t}_{\text{curr.}} + \underbrace{(W_{t+s} - W_t)}_{\text{increment}} \mid \mathcal{F}_t) =$




$= W_t + E(\underbrace{W_{t+s} - W_t \mid \mathcal{F}_t}_{\text{indep}}) = W_t + E(W_{t+s} - W_t) =$

$= W_t + 0 = W_t$

$(W_t)$ is a mart

Axioms: $W_t - W_s \sim N(0; t-s)$

b) $\alpha$? $\boxed{E(Y_{t+s} \mid \mathcal{F}_t) = Y_t}$ $\to \ldots\ldots \to \alpha = -1$

$E(W_{t+s}^2 + \alpha \cdot (t+s) \mid \mathcal{F}_t) = W_t^2 + \alpha \cdot t$

$W_{t+s} = \underbrace{W_t}_{\text{is known given } \mathcal{F}_t} + \underbrace{(W_{t+s} - W_t)}_{\text{indep from } \mathcal{F}_t}$

$\mathcal{A} = \sigma(W_{t+s} - W_t)$

$A \in \mathcal{A}$ $\quad B \in \mathcal{F}_t$ $\quad P(A \cap B) = P(A) \cdot P(B)$

$E(Y_{t+s} | \mathcal{F}_t) = Y_t$

$$E\left((W_t + \Delta)^2 + \alpha(t+s) \,\middle|\, \mathcal{F}_t\right) = \underbrace{W_t^2 + \alpha t}_{\text{const}} \quad Y_t$$

$\Delta = W_{t+s} - W_t$

$E(W_{t+s}) = 0$

$E(W_t) = 0$

$$E\left(\underbrace{W_t^2}_{\text{known}} + 2\Delta \underbrace{W_t}_{\text{known}} + \Delta^2 + \alpha t + \alpha s \,\middle|\, \mathcal{F}_t\right) = W_t^2 + \alpha t \quad Y_t$$

$$\alpha t + \alpha s + W_t^2 + 2W_t \cdot E(\Delta | \mathcal{F}_t) + E(\Delta^2 | \mathcal{F}_t) = W_t^2 + \alpha t$$

$\Delta$ is indep of $\mathcal{F}_t$

$$\alpha s + 2W_t \cdot E(\Delta) + E(\Delta^2) = 0$$

$$\Delta \sim N(0; t+s-t) \sim N(0; s)$$

assumpt

$E(\Delta) = 0$

$Var(\Delta) = s$

$E(\Delta^2)$

$$\alpha \cdot s + 2 \cdot W_t \cdot 0 + s = 0$$

$$\alpha = -1$$

Conclusion: $Y_t = W_t^2 - t$ is a martingale

c) $\beta$! $R_t = \exp(W_t + \beta t)$ is a martingale

$$E(R_{t+s} | \mathcal{F}_t) = R_t$$

$$W_{t+s} = W_t + (W_{t+s} - W_t)$$

$$E\left(\exp(W_{t+s} + \beta t + \beta s) \,\middle|\, \mathcal{F}_t\right) = \exp(W_t + \beta t)$$

$$E\left(\exp(W_t) \cdot \exp(W_{t+s} - W_t) \cdot \exp(\beta t) \cdot \exp(\beta s) \,\middle|\, \mathcal{F}_t\right) =$$

$$= \exp(W_t) \cdot \exp(\beta t)$$

take out known RVs

indep

$$\exp(W_t) \cdot \exp(\beta t) \cdot \exp(\beta s) \cdot E\left(\exp(W_{t+s} - W_t) \,\middle|\, \mathcal{F}_t\right) =$$

$$= \exp(W_t) \cdot \exp(\beta t)$$

$$\exp(\beta s)\cdot E\left(\exp\left(W_{t+s}-W_t\right)\right) = 1$$

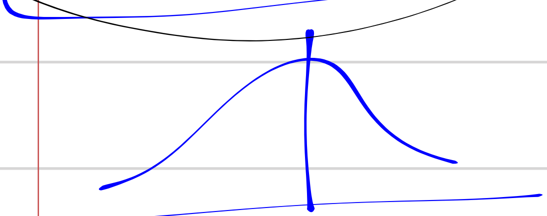
Assumpt: $W_{t+s}-W_t \sim N(0;t+s-t) \sim N(0;s)$

$$\frac{W_{t+s}-W_t-0}{\sqrt{s}} \sim N(0;1) \qquad z\sim N(0;1)$$

$$\boxed{\exp(\beta s)\cdot E\left(\exp(\sqrt{s}\cdot z)\right) = 1} \qquad \exp\left(\frac{s}{2}\right)$$

$$E(z^4)=\int_{-\infty}^{\infty} z^4\cdot pdf(z)\,dz \qquad E\left(\exp(\sqrt{s}\cdot z)\right) = \int_{-\infty}^{\infty}\exp(\sqrt{s}\cdot z)\cdot pdf(z)\,dz =$$

$$\boxed{\int_{-\infty}^{\infty} pdf(z)\,dz = 1} \qquad \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)$$



$$=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\cdot\exp\left(-\frac{z^2}{2}+\frac{2\sqrt{s}\cdot z}{2}\right)dz =$$

$$=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\Big(\underbrace{-\frac{1}{2}\left(z^2+2\sqrt{s}\cdot z+s\right)}_{\text{full square}}+\frac{s}{2}\Big)\,dz =$$

$$=\exp\left(+\frac{s}{2}\right)\cdot\underbrace{\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(z+\sqrt{s})^2\right)dz}_{\text{pdf of } N(-\sqrt{s};1)} =$$

integr. under some pdf

$$=\exp\left(\frac{s}{2}\right)$$

$$E(R_{t+s}\mid\mathcal{F}_t)=R_t \iff \exp(\beta s)\cdot\exp\left(\frac{s}{2}\right)=1 \qquad \beta=-\frac{1}{2}$$

Conclusion $R_t=\exp\left(W_t-\frac{t}{2}\right)$ is martingale.

# Time inversion $\ddot{\smile}$

Theorem: If $(W_t)$ is a Wiener process then

$$Y_t = \begin{cases} t \cdot W(1/t) & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$ is

also a Wiener process.


future of $W_t$ (after $t=1$) $\longleftrightarrow$ prehist of $Y_t$ (before $t=1$)

$$Y_5 = 5 \cdot W\left(\tfrac{1}{5}\right)$$
$$Y_{1/4} = \tfrac{1}{4} \cdot W(4)$$
$$Y_1 = 1 \cdot W(1)$$

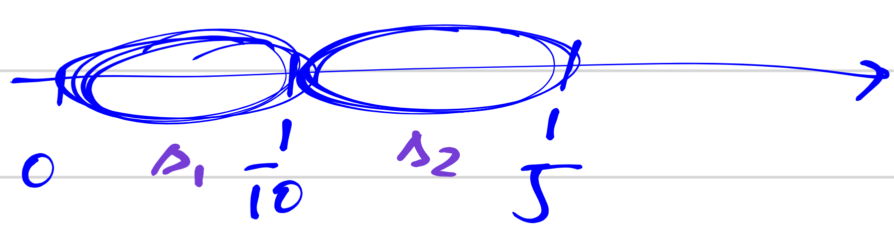
(part of the proof) 1) $Y_0 \stackrel{?}{=} 0$ $\ddot{\smile}$

2) . . .

3)

4) . . .

$$Y_{10} - Y_5 = \underbrace{10 \cdot W(1/10) - 5 \cdot W(1/5)}_{} \stackrel{?}{=} *$$




$$W\left(\tfrac{1}{5}\right) = W\left(\tfrac{1}{10}\right) + \Delta_2 \qquad \Delta_2 = W\left(\tfrac{1}{5}\right) - W\left(\tfrac{1}{10}\right)$$

$W_t - W_s \sim N(0; t-s)$

$N(0; \frac{1}{5} - \frac{1}{10})$

$$* = 10 \cdot W\left(\tfrac{1}{10}\right) - 5\left(W\left(\tfrac{1}{10}\right) + \Delta_2\right) =$$
$$= 5W\left(\tfrac{1}{10}\right) - 5\Delta = 5\left(W\left(\tfrac{1}{10}\right) - \underbrace{\left(W\left(\tfrac{1}{5}\right) - W\left(\tfrac{1}{10}\right)\right)}_{\Delta_2}\right)$$

indep

$\underbrace{W\left(\tfrac{1}{10}\right) - W_0}_{\Delta_1}$

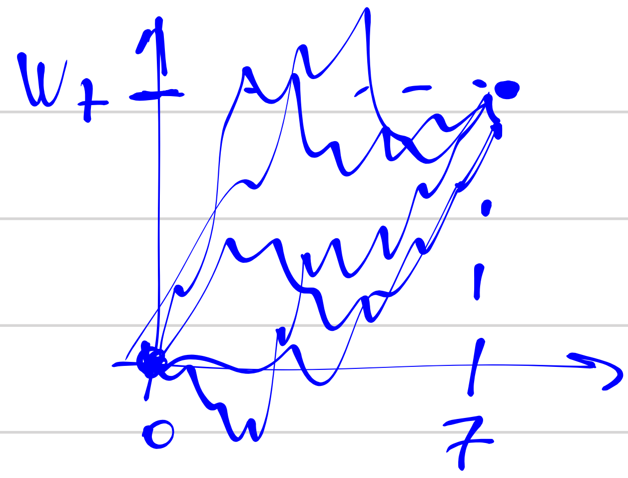
$Y_t - Y_s \sim N(0; t-s)$

$E(Y_{10} - Y_5) = 0$

$Y_{10} - Y_5 \sim N(0; 5)$

$$Var(Y_{10} - Y_5) = 25 \cdot Var(\Delta_1 + \Delta_2)$$
$$= 25 \cdot (Var(\Delta_1) + Var(\Delta_2))$$

Ex. $E(W_5 | W_7)$ ? =




$E(W_7 | W_5) = W_5$

mart

$$Y_t = t \cdot W(1/t) \Longleftrightarrow \frac{1}{t} \cdot Y_t = W(1/t)$$

$$1/t = s$$

$$W(s) = s \cdot Y_{1/s}$$

$$= E(5 \cdot Y_{1/5} \mid 7 \cdot Y_{1/7}) =$$

$(Y_t)$ - Wiener process $\qquad \sigma(R) = \sigma(7R)$

$$= 5\, E(Y_{1/5} \mid Y_{1/7}) = 5 \cdot Y_{1/7} =$$

$$E(W_5 / W_7) = 5 \cdot \frac{1}{7} \cdot W_7 = \frac{5}{7} \cdot W_7$$

$\frac{5}{7} \cdot W_7 = E(W_5 | W_7)$