Home Assignment 1

1. Consider the Markov chain with the transition matrix

$$P = \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix}.$$

   The hedgehog starts at the first state and moves randomly according to transition matrix P.

   (a) Draw the graph of this chain.

   (b) What is the probability that the hedgehog will be in state 2 after 3 moves?

   (c) What is the stationary distribution of this chain?

2. Consider iid sequence X_1, X_2, ... of uniform on $[0; 10]$ random variables. Find the following probability limits:

$$L_1 = \operatorname{plim} \frac{X_1 + X_2 + \ldots + X_n}{2n}, \ L_2 = \operatorname{plim} \frac{X_1^2 + X_2^2 + \ldots + X_n^2}{X_1 + X_2 + \ldots + X_n}, \ L_3 = \operatorname{plim}(X_1 \cdot X_2 \cdot \ldots \cdot X_n)^{1/n}.$$

   Hint: maybe there is a function that can transform the product L_3 into the sum? you are free to use any probability limit property.

3. Consider iid sequence X_1, X_2, ... of uniform on $[0; 10]$ random variables.

   (a) Find the probability $\mathbb{P}(|\max\{X_1, X_2, \ldots, X_n\} - 10| > \varepsilon)$.

   (b) Find the probability limit $\operatorname{plim} \max\{X_1, X_2, \ldots, X_n\}$ by definition.

4. Joe Biden throws a die until six or five appears. For every throw he pays 0.1 dollars, but at the end he receives the result of the last throw in dollars.

   (a) What is the expected payoff of Joe?

   (b) Assume now that Joe can stop the game at every moment of time.
   What is the maximal expected payoff and the corresponding strategy?

5. Ilya Muromets stands before the first stone. There are three roads behind the stone. And every road ends with a new stone. And there are three new roads behind every new stone. And so on. Every road is guarded with one-third probability by a three headed dragon Zmei Gorynich. Yes, there are infinitely many Zmeis Gorynichs.

   (a) What is the probability that Ilya will never meet Zmei Gorynich if Ilya chooses a road at random?

   (b) What is the probability that Ilya will meet Zmei Gorynich after passing by even number of stones if Ilya chooses a road at random?

   (c) What is the probability that **there exists** at least one Eternal Peaceful Path without Zmei Gorynich?

Deadline: **2022-10-02, 21:00**.

Home Assignment 2

Hereinafter (W_t) is a standard Wiener process.

1. Some questions about Wiener process!
   (a) Find $\mathbb{E}(W_7 \mid W_5)$, $\mathbb{V}\mathrm{ar}(W_7 \mid W_5)$, $\mathbb{E}(W_7 W_6 \mid W_5)$.
   (b) Find $\mathbb{E}(W_5 \mid W_7)$, $\mathbb{V}\mathrm{ar}(W_5 \mid W_7)$.
2. Using Ito's lemma find dX and the corresponding full form.
   (a) $X_t = W_t^6 \cos t$.
   (b) $X_t = Y_t^3 + t^2 Y_t$ where $dY_t = W_t^2 dW_t + tW_t dt$.
3. Consider two independent Wiener processes A_t and B_t. Check whether these processes are Wiener processes:
   (a) $X_t = (A_t + B_t)/2$.
   (b) $Y_t = (A_t + B_t)/\sqrt{2}$.
4. Consider $I_t = \int_0^t W_u^2 u^2 du$. Find $\mathbb{E}(I_t)$, $\mathbb{V}\mathrm{ar}(I_t)$ and $\mathbb{C}\mathrm{ov}(I_t, W_t)$.
5. Find limits in L^2 of the following sequences for $n \to \infty$:
   (a)
   $$S_n = \sum_{i=1}^{n} (t/n) \left(W(it/n) - W((i-1)t/n) \right).$$
   (b)
   $$T_n = \sum_{i=1}^{n} \left(W(it/n) - W((i-1)t/n) \right)^5.$$
6. (bonus) Let's split the time segment $[0;10]$ into $n = 10^5$ sub-segments of equal lentgh. Let Δ_i be equal to the corresponding increment of Wiener process, $\Delta_i = W(10i/n) - W(10(i-1)/n)$.
   (a) What is the distribution of Δ_i?
   (b) Using any open source software simulate five approximate trajectories of Wiener process and plot them on the same plot. You can generate Δ_i and find a cumulative sum.
   (c) Now simulate $n_{sim} = 10^4$ trajectories but do not plot them. Using these trajectories estimate the probability $p = \mathbb{P}(\max_{t \in [0;10]} W_t > 7)$.

   Do not forget to provide your code.

Deadline: **2022-12-06, 21:00**.

Home Assignment 3

1. Let $X_t = 42 + t^2 W_t^3 + tW_t^2 + \int_0^t 3W_u du + \int_0^t W_u^3 dW_u$.

   (a) Find dX_t.

   (b) Is X_t a martingale? Is $Y_t = X_t - \mathbb{E}(X_t)$ a martingale?

   Hint: only the binary answer for (b) is not sufficient but the argument is very-very short if you solve (a).

2. Consider two-period binomial tree model without dividents. Initial stock price is $S_0 = 200$, in each period the stock price is multiplied by $u = 1.15$ or by $d = 0.75$. One period interest rate is $r = 0.05$.

   (a) Find the risk-neutral probability.

   (b) Price the following binary option: at time $T = 2$ you get 100$ if $S_1 > 200$ and nothing otherwise.

   (c) Price the following chooser option: at $t = 1$ the owner of the option decides whether the option is call or put. The strike price is $K = 200$ and expiry date is $T = 2$.

3. In the framework of Black and Scholes model find the price at $t = 0$ of the following two financial assets, $dS_t = \mu S_t dt + \sigma S_t dW_t$ is the share price equation.

   (a) The asset pays you at time T exactly one dollar if $S_T < K$ where K is a constant specified in the contract.

   (b) The asset pays you at time T exactly S_T^2 dollars.

4. Consider the Vasicek interest rate model,

$$dR_t = 5(0.06 - R_t)\,dt + 3\,dW_t, \quad R_0 = 0.07.$$

   Here R_t is the interest rate.

   (a) Using the substitution $Y_t = e^{at} R_t$ find the solution of the stochastic differential equation. Start by finding dY_t.

   (b) Find $\mathbb{E}(R_t)$ and $\mathbb{V}\text{ar}(R_t)$.

   (c) Which value in this model would you call long-term equilibrium rate and why?

   Hint: you may have integrals in you expression for R_t, but no R_t.

5. Let X_t be the exchange rate measured in roubles per dollar. We suppose that $dX_t = \mu X_t dt + \sigma X_t dW_t$.

   Consider the inverse exchange rate $Y_t = 1/X_t$ measured in dollars per rouble.

   Write the stochastic differential equation for dY_t. The equation may contain Y_t and constants, but not X_t.

Deadline: **2022-12-18, 21:00**.

Home Assignment 4: stationarity, white noise, MA model

1. The process (u_t) is a white noise. Consider the processes $a_t = (1+L)^3 u_t$ and $b_t = t^2 + 6t + (1-2L)^2 u_t$.

   (a) Write explicit expression of these processes without lag L operator.

   (b) Check whether these processes are stationary.

   (c) For stationary processes find the autocorrelation function.

   (d) Check whether these processes are white noises.

   (e) If the process is $MA(k)$ process with respect to (u_t) then find the value of k.

2. The process (y_t) is stationary with $\gamma_k = \mathbb{C}\mathrm{ov}(y_t, y_{t+k})$. Consider the process $b_t = 4y_t - 3y_{t-1} + 18$.

   (a) Find new covariances $\theta_k = \mathbb{C}\mathrm{ov}(b_t, b_{t+k})$ in terms of old covariances (γ_j).

   (b) Is (b_t) stationary?

3. Provide an example of two dependent processes (a_t) and (b_t) such that each of them is stationary, but their sum is not stationary.

4. Consider three variables (y_1, y_2, y_3) that are jointly normal

$$y \sim \mathcal{N}\left(\begin{pmatrix} 5 \\ 6 \\ 11 \end{pmatrix}; \begin{pmatrix} 9 & 0 & -1 \\ 0 & 4 & 1 \\ -1 & 1 & 4 \end{pmatrix}\right).$$

   Find $\mathbb{C}\mathrm{orr}(y_1, y_2)$ and $\mathrm{p}\mathbb{C}\mathrm{orr}(y_1, y_2; y_3)$.

5. (bonus) Variables u_1 and u_2 are independent $\mathcal{N}(0;1)$. Consider the process $y_t = 7 + u_1 \cos(\pi t/2) + u_2 \sin(\pi t/2)$.

   (a) Find $\mathbb{E}(y_t)$, $\mathbb{V}\mathrm{ar}(y_t)$, $\gamma_k = \mathbb{C}\mathrm{ov}(y_t, y_{t+k})$.

   (b) Is (y_t) stationary? Is it a white noise process?

   (c) Your know that $y_{100} = 0.2023$. What is your best prediction for y_{104}? What about predictive interval?

   (d) Is this process $MA(\infty)$ process with respect to *some* white noise, not necessary (u_t)?

Deadline: **2023-02-12, 21:00**.

Home Assignment 5: recurrence equations, AR-model, ACF/PACF

1. Consider the recurrence equation $y_t = 4 + 10y_{t-1} + u_t$ where (u_t) is the white noise.
   (a) Find two non-stationary solution.
   (b) Find one stationary solution. Does this solution have $MA(\infty)$ form with respect to (u_t)?

2. Consider two equations (A) $y_t = 4 + 0.6y_{y-1} + 0.2y_{t-2} + u_t$ and (B) $y_t = 3 + y_{t-1} + 6y_{t-2} + u_t$.
   (a) How many non-stationary solutions does each equation have?
   (b) How many stationary solutions does each equation have?
   (c) How many stationary solutions that are $MA(\infty)$ with respect to (u_t) does each equation have?

3. For $MA(2)$ process $y_t = 5 + u_t + 3u_{t-2}$ find all values of the autocorrelation function ρ_k and first two values of the partial autocorrelation function ϕ_{kk}.

4. For stationary $AR(1)$ process with equation $y_t = 5 + 0.3y_{t-1} + u_t$ find all values of the autocorrelation function ρ_k and all values of the partial autocorrelation function ϕ_{kk}.

5. For stationary $AR(2)$ process with equation $y_t = 5 + 0.3y_{t-1} - 0.02y_{t-2} + u_t$ find first two values of the autocorrelation function ρ_k and all values of the partial autocorrelation function ϕ_{kk}.

6. (bonus) Consider the process
   $$y_t = \frac{1 - 0.7F}{1 - 0.7L} u_t,$$
   where (u_t) is a white noise and F is the forward operator.
   (a) Write explicit expression for (y_t) without lag nor forward operator.
   (b) Is (y_t) a white noise?

Deadline: **2023-02-19, 21:00**.