exam 2c, 3b

2c

2a) $\boxed{X_t = \exp(-2W_t - 2t)}$ $\rightarrow X_0 = 1$

$$dX_t =$$

$$= -2 \cdot \exp(-2W_t - 2t) \cdot dW_t$$

$$\boxed{-2 \cdot \exp(-2W_t - 2t) \cdot dt + \frac{1}{2}(-2)^2 \cdot \exp(-2W_t - 2t)(dW_t)^2}$$

$$dX_t = -2 \exp(-2W_t - 2t) dW_t$$

$$X_t = X_0 + \boxed{\int_0^t (-2) \exp(-2W_u - 2u) dW_u}$$

$X_0 = 1$

$$\int_0^t (-2) \exp(-2W_u - 2u) dW_u = X_t - X_0$$

2(c)

$$\int_0^t \exp(-2W_u - 2u) dW_u = \frac{X_t - X_0}{-2}$$

$\exp(-2W_t - 2t)$ $\quad 1$

(3в)

$$\mathrm{Cov}(W_4 W_5, W_5 W_6) = E(W_4 \cdot W_5^2 W_6) - E(W_4 W_5) \cdot E(W_5 W_6)$$

$$E(W_4 \cdot W_5) = E(W_4 \cdot (W_4 + X)) =$$

$X = W_5 - W_4$

$$= E(W_4^2) + E(W_4 \cdot X) =$$

X не зав от W_4

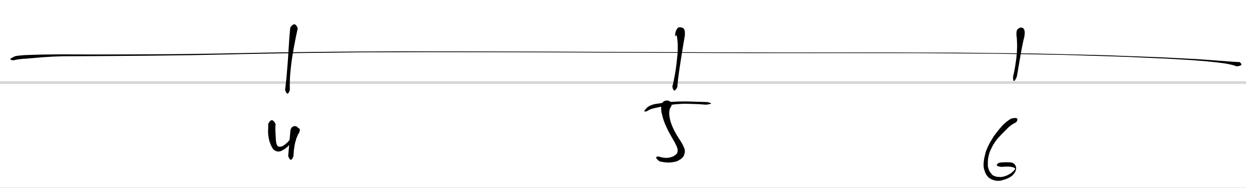
$$= 4 + E(W_4) \cdot E(X) = 4 + 0$$

$$E(W_5 \cdot W_6) = E(W_5 \cdot (\underbrace{W_5}_{\text{тек.}} + \underbrace{(W_6 - W_5)}_{\text{прир}})) =$$

$$= E(W_5^2) + E(W_5) \cdot E(W_6 - W_5) = 5 + 0 \cdot 0 = 5$$

$$E(W_4 \cdot W_5^2 \cdot W_6) = E\left[W_4 \cdot W_5^2 \cdot (W_5 + \overbrace{(W_6 - W_5)}^{Y})\right] =$$

Y не зав от прошлого!



$$= E(W_4 \cdot W_5^3) + E(W_4 \cdot W_5^2 \cdot Y) =$$

$$E(W_4 W_5^2) \cdot \underbrace{E(W_6 - W_5)}_{0}$$

$$E(W_4 \cdot W_5^3) = E(W_4 \cdot (W_4 + X)^3) =$$

$$= E(W_4 \cdot (W_4^3 + 3X \cdot W_4^2 + 3X^2 \cdot W_4 + X^3)) =$$

$$= E(W_4^4) + 3\underbrace{E(X)}_{0} \cdot E(W_4^3) + 3E(X^2) \cdot E(W_4^2) + \underbrace{E(W_4)}_{0} \cdot E(X^3) = \ldots$$

(y_t) $\quad (y_t)_{t=0}^{\infty} \quad y_0, y_1, y_2, y_3 \dots$

$(y_t)_{t=-\infty}^{\infty} \quad \dots y_{-2}, y_{-1}, y_0, y_1, y_2 \dots$

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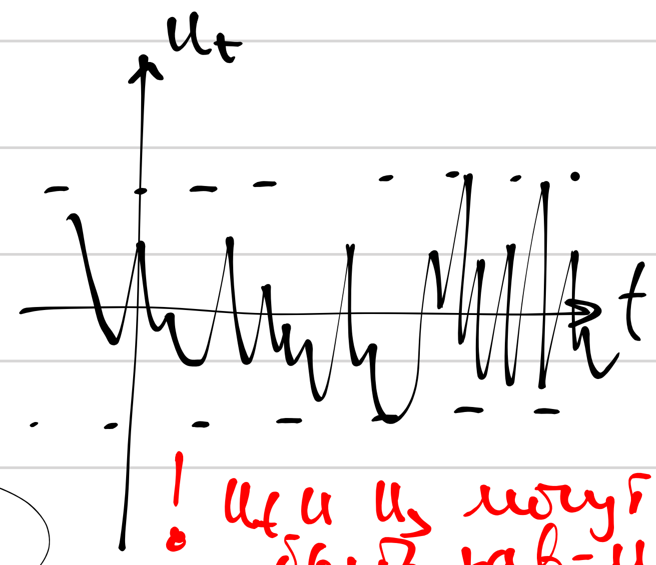
def white noise (белый шум)

$(u_t) \sim$ б. шум:

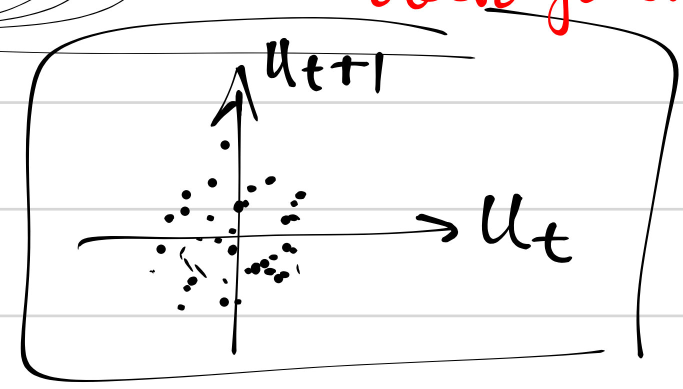
для $\forall t$: $E(u_t)=0$

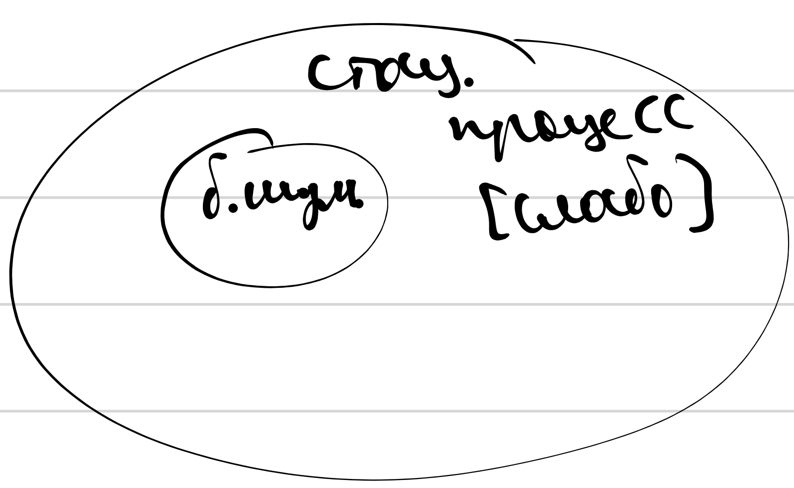
$Var(u_t)=\sigma^2$

для $\forall t\neq s$ $Cov(u_t, u_s)=0$



! u_t и u_s могут быть зав-мы





def. (y_t) – [weak] stationary [слабо] стац-ый

если

$\forall t$ $\quad E(y_t)=\mu$

$Var(y_t)=\gamma_0$

$\forall t, k$ $\quad Cov(y_t, y_{t+k})=\gamma_k$

в частности, требуется чтобы:

$\gamma_4 = Cov(y_1, y_5) = Cov(y_2, y_6) = Cov(y_{100}, y_{104}) = \dots$

могут отлич.

$\gamma_7 = Cov(y_3, y_{10}) = Cov(y_8, y_{15}) = \dots$

def. у стац-го процесса:

автоковариационная ф-ция $\gamma_k = Cov(y_t, y_{t+k})$

автокорреляционная ф-ция $\rho_k = Corr(y_t, y_{t+k})$

Упр. Для каждого процесса найдите: $E(y_t)$, $Var(y_t)$, $Cov(y_t, y_{t+k})$, стац-н ли?

если стац-н, найдите ρ_k?

$a_t = u_t + 2u_{t-1} + 7$

$b_t = u_t + \boxed{6t}$ ← нестац! $E(6t) = 6t$

$c_t = u_1 + u_2 + u_3 + \dots + u_t + 7$

$\boxed{(u_t) - \text{б. шум с } Var(u_t) = \sigma^2}$

$\Downarrow$

$E(u_t) = 0$

$Cov(u_t, u_s) = 0$ [при $t \neq s$]

[пункт E]?

$E(a_t) = 0 + 2 \cdot 0 + 7$

$E(b_t) = 0 + 6t$

$E(c_t) = 0 + \dots + 0 + 7$

[пункт Var]

$Var(a_t) = Var(u_t + 2u_{t-1} + 7) = \underbrace{Var(u_t)}_{\sigma^2} + \underbrace{Var(2u_{t-1})}_{4\sigma^2} = 5\sigma^2$

$Var(b_t) = Var(u_t + 6t) = Var(u_t) = \sigma^2$

$Var(c_t) = Var(u_1 + u_2 + u_3 + \dots + u_t + 7) =$

$= \sigma^2 + \sigma^2 \dots + \sigma^2 = t \cdot \sigma^2$

уже: (b_t) – не стац, (c_t) – не стац.

хотим искать $Cov(y_t, y_s)$?

$Cov(y_t, y_{t+1})$, $Cov(y_t, y_{t+2})$, $Cov(y_t, y_{t+3})$, ...

$Cov(a_t, a_{t+1}) = Cov(u_t + 2u_{t-1} + 7, u_{t+1} + 2u_t + 7) =$

[ищем совп. индексы]

$= Cov(u_t, 2u_t) = 2Cov(u_t, u_t) = 2\sigma^2 = \gamma_1$

$Cov(a_t, a_{t+2}) = Cov(u_t + 2u_{t-1} + 7, u_{t+2} + 2u_{t+1} + 7) =$

$= 0 = \gamma_2$

$Cov(a_t, a_{t+3}) = \gamma_3 = Cov(u_t + 2u_{t-1} + 7, u_{t+3} + 2u_{t+2} + 7) =$

$= 0$

для (a_t) $\quad \mathrm{Cov}(a_t, a_{t+k}) = \begin{cases} 5\sigma^2 & \text{при } k=0 \\ 2\sigma^2 & \text{при } k=1 \\ 0 & \text{при } k \geq 2 \end{cases}$

$$\mathrm{Cov}(a_t, a_s) = \begin{cases} 5\sigma^2, & \text{если } t=s \\ 2\sigma^2, & \text{если } |t-s|=1 \\ 0, & \text{если } |t-s| \geq 2 \end{cases}$$

б.шум

$$b_t = u_t + 6t$$

$$\mathrm{Cov}(b_t, b_{t+1}) = \mathrm{Cov}(u_t + 6t, u_{t+1} + 6(t+1)) =$$
$$= 0$$

$$\mathrm{Cov}(b_t, b_{t+2}) = \mathrm{Cov}(u_t + 6t, u_{t+2} + 6(t+2)) = 0$$

$$\mathrm{Cov}(b_t, b_{t+k}) = \begin{cases} \sigma^2 & \text{при } k=0 \\ 0 & \text{при } k \geq 1 \end{cases}$$

$$\mathrm{Cov}(b_t, b_s) = \begin{cases} \sigma^2 & \text{при } t=s \\ 0 & \text{при } t \neq s \end{cases}$$

c_t $\quad \mathrm{Cov}(c_t, c_{t+1}) =$

$$= \mathrm{Cov}(u_1 + u_2 + u_3 + \ldots + u_t + 7, u_1 + u_2 + u_3 + \ldots + u_t + u_{t+1} + 7)$$

$$= t\sigma^2$$

$$\mathrm{Cov}(c_t, c_{t+2}) =$$
$$= \mathrm{cov}(u_1 + u_2 + \ldots + u_t + 7, u_1 + u_2 + u_3 + \ldots + u_t + u_{t+1} + u_{t+2} + 7)$$
$$= t\sigma^2$$

$$\mathrm{Cov}(c_t, c_{t+k}) = t \cdot \sigma^2 \text{ при } k \geq 0$$

$$\mathrm{Cov}(c_t, c_s) = \min(t, s) \cdot \sigma^2$$

$$\boxed{\begin{aligned} &\mathrm{Cov}(c_5, c_7) = 5 \cdot \sigma^2 \\ &\mathrm{Cov}(c_9, c_6) = \\ &= \mathrm{Cov}(c_6, c_9) = 6\sigma^2 \end{aligned}}$$

$(b_t), (c_t)$ – не стац.

$E(a_t) = 7 \quad Var(a_t) = 5\sigma^2 \quad Cov(a_t, a_s) =$

$$= \begin{cases} 5\sigma^2 & |t-s| = 0 \\ 2\sigma^2 & |t-s| = 1 \\ 0 & |t-s| \geq 2 \end{cases}$$

(a_t) – стац-ый.

ρ_k автокоррел. ф-ция для (a_t).

$\rho_0 = Corr(a_t, a_t) = 1$

$$\rho_1 = Corr(a_t, a_{t+1}) = \frac{Cov(a_t, a_{t+1})}{\sqrt{Var(a_t) \cdot Var(a_{t+1})}} =$$

$$= \frac{Cov(a_t, a_{t+1})}{\sqrt{Var(a_t) \cdot Var(a_t)}} = \frac{Cov(a_t, a_{t+1})}{Var(a_t)} = \frac{2\sigma^2}{5\sigma^2} = \frac{2}{5}$$

$\rho_2 = 0 \quad \rho_3 = 0 \quad \rho_4 = 0 \; \dots\dots$

$\boxed{\text{2010-2021 ср10 №6}}$

$P(x_t = 0) = P(x_t = 1) = \frac{1}{2} \quad \forall t \rightarrow E(x_t) = \frac{1}{2}$

$u_t \sim N(0;1)$

$x_1, x_2, x_3 \dots \quad u_1, u_2, u_3 \dots$ не зав.

$z_t = x_t \cdot (1 - x_{t-2}) \cdot u_t$

a) $E(z_t)$, $Var(z_t)$, $Cov(z_t, z_s)$

б) явл-ся ли (z_t) стац-м?

(z_t) б. шумом?

$$E(z_t) = E(x_t \cdot (1 - x_{t-2}) \cdot u_t) = \underbrace{E(x_t)}_{= \frac{1}{2}} \cdot \underbrace{E(1 - x_t)}_{= \frac{1}{2}} \cdot \underbrace{E(u_t)}_{= 0}$$

$$Var(z_t) = E(z_t^2) - (\underbrace{E(z_t)}_{0})^2 = E(z_t^2) =$$

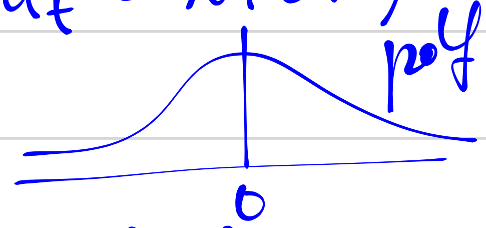
$$= E(x_t^2 \cdot (1-x_{t-2})^2 \cdot u_t^2) \overset{\uparrow}{=} \underbrace{E(x_t^2)}_{1/2} \cdot \underbrace{E((1-x_{t-2})^2)}_{1/2} \cdot \underbrace{E(u_t^2)}_{1}$$

незав. по совок-сти

$$\overset{0/1}{x_t} \cdot x_t = x_t$$

$$(1-x_{t-2})^2 = 1 - x_{t-2}$$

$$u_t \sim N(0;1)$$



$$E(u_t) = 0$$

$$Var(u_t) = E(u_t^2) = 1$$

$$Var(z_t) = \frac{1}{4}$$

$$Cov(z_t, z_s) \overset{?}{=}$$

$$Cov(z_t, z_{t+1}) = Cov(x_t \cdot (1-x_{t-2}) \cdot u_t,\ x_{t+1} \cdot (1-x_{t-1}) \cdot u_{t+1})$$

$$= 0$$

нет совп-х индексов

$$Cov(z_t, z_{t+2}) = Cov(x_t(1-x_{t-2})u_t,\ x_{t+2} \cdot (1-x_t) \cdot u_{t+2}) =$$

$$= E(z_t \cdot z_{t+2}) - \underbrace{E(z_t)}_{=0} \cdot \underbrace{E(z_{t+2})}_{=0} =$$

зав-мы!

$$= E\left[x_t \cdot (1-x_{t-2}) \cdot u_t \cdot x_{t+2} \cdot (1-x_t) \cdot u_{t+2} \right] =$$

$$x_t \cdot (1-x_t) \overset{?}{=} 0 \quad (\text{при } x_t \in \{0,1\})$$

$$= E(0) = 0$$

$$Cov(z_t, z_{t+3}) = Cov(\underbrace{x_t(1-x_{t-2})\cdot u_t}, \underbrace{x_{t+3}\cdot(1-x_{t+1})\cdot u_{t+3}}) = 0$$

нет совп. индексов

$$\begin{array}{cccccccc}
u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & \ldots \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \ldots \\
z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & \ldots
\end{array}$$

$$\boxed{\begin{array}{ll} E(z_t)=0 & Var(z_t)=\frac{1}{4} \\ & Cov(z_t, z_{t+k})=0 \quad \text{при } k\geq 1 \end{array}}$$

(z_t) – б. шум.

(z_t) – стац-ый.

в) найдите самые короткие предсказательные интервалы с вер-стью покрытия $\geq 95\%$ для z_{101} и z_{102}, если $z_{100}=2.3$.

в1) $P(z_{101}\in[a;b] \mid z_{100}=2.3)\geq 0.95$?

в2) $P(z_{102}\in[a;b] \mid z_{100}=2.3)\geq 0.95$?

$z_{100}=2.3 = X_{100}\cdot(1-X_{98})\cdot u_{100}$

$X_{100}=1$

$X_{98}=0$

$u_{100}=2.3$

и никакой инфо про ост. в-ые (X_t) и (u_t)

$z_{102}=X_{102}\cdot(1-X_{100})\cdot u_{102}=0$

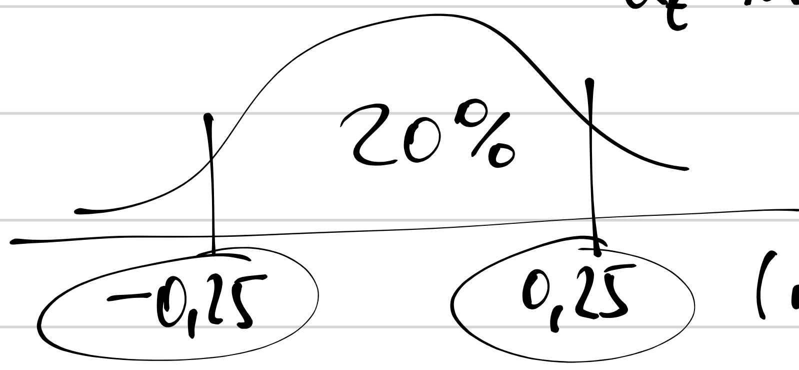
$P(z_{102}\in[0;0] \mid z_{100}=2.3)=1\geq 0.95$

$z_{101}=X_{101}\cdot(1-X_{99})\cdot u_{101}$ $\qquad P(z_{101}=0)\overset{?}{=}\frac{3}{4}$

$P(z_{101}\in[a;b])\geq 0.95$

75% $[0;0]$ $P(z_{101} \in [0;0]) = 0{,}75$

$u_t \sim N(0;1)$



(по табл / комп / телефон)

Ответ: $P(z_{101} \in [-0{,}25; 0{,}25]) = 0{,}95$