

Time Series

Peter Lukianchenko

11 February 2023

Type of Non-Stationary TimeSeries

- Time trend
- Unit root
- Structural break in levels
- Structural break in variance

Dickey Fuller Test

- Model:

$$Y_t = \theta Y_{t-1} + \varepsilon_t$$

- True process: $Y_t = Y_{t-1} + \varepsilon_t$
- The null: $H_0 : \theta = 1$ vs $H_1 : |\theta| < 1$
- Estimate by OLS, form the test statistic:

$$t_n = \frac{\hat{\theta} - 1}{s.e.(\hat{\theta})}$$

- What's the distribution?
- Test with significance level α : Reject H_0 if $t_n < DF_n^\alpha$

Augmented Dickey Fuller Test

- Model:

$$Y_t = c + \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + \varepsilon_t$$

- True process has a unit root: $\theta_1 + \theta_2 = 1$
- Write the equation:

$$Y_t - Y_{t-1} = c + \theta_1 Y_{t-1} - Y_{t-1} + \theta_2 Y_{t-1} + \theta_2 Y_{t-2} - \theta_2 Y_{t-1} + \varepsilon_t$$

$$\Delta Y_t = c + (\theta_1 + \theta_2 - 1) Y_{t-1} - \theta_2 \Delta Y_{t-1} + \varepsilon_t$$

$$\Delta Y_t = c + \theta^* Y_{t-1} + \theta_2^* \Delta Y_{t-1} + \varepsilon_t$$

- Estimate by OLS, form the test statistic $t_n = \frac{\hat{\theta}^*}{s.e.(\hat{\theta}^*)}$
- The same distribution as before
- Test with significance level α : Reject H_0 if $t_n < DF_n^\alpha$

TIME SERIES

- **MASE:** The mean absolute scaled error is independent of the scale of the data, so can be used to compare forecasts across data sets with different scales.

- Non-seasonal time series

$$MASE = \frac{\frac{1}{J} \sum_{i=1}^J |e_i|}{\frac{1}{n-1} \sum_{t=2}^n |Y_t - Y_{t-1}|} \quad \text{MAPE}$$

- Seasonal time series

$$\frac{1}{J} \sum_{i=1}^J |e_i|$$

TIME SERIES

- This is a forecasting procedure based on a simple updating equations to calculate forecasts using the underlying pattern of the series. Not based on ARIMA approach.
- Recent observations are expected to have more power in forecasting values so a model can be constructed that places more weight on recent observations than older observations.

TIME SERIES

- *Smoothed* curve (eliminate up-and-down movement)
- Trend
- Seasonality

TIME SERIES

- 3 periods moving averages

$$\hat{Y}_t = (Y_{t-1} + Y_{t-2} + Y_{t-3})/3$$

- Also, 5 periods MA can be considered.

Period	Actual	3 Quarter MA Forecast	5 Quarter MA forecast
Mar-83	239.3	Missing	Missing
Jun-83	239.8	Missing	Missing
Sep-83	236.1	Missing	Missing
Dec-83	232	238.40	Missing
Mar-84	224.75	235.97	Missing
Jun-84	237.45	230.95	234.39
Sep-84	245.4	231.40	234.02
Dec-84	251.58	235.87	235.14
...		So on..	

TIME SERIES

- One can impose *weights* and use weighted moving averages (WMA).

$$\text{Eg } \hat{Y}_t = 0.6Y_{t-1} + 0.3Y_{t-2} + 0.1Y_{t-3}$$

- How many periods to use is a question; more significant smoothing-out effect with longer lags.
- Peaks and troughs (bottoms) are not predicted.
- Events are being averaged out.
- Since any moving average is serially correlated, any sequence of random numbers could appear to exhibit cyclical fluctuation.

TIME SERIES

- Suppressing short-run fluctuation by smoothing the series
- Weighted averages of all previous values with more weights on recent values
- No trend, No seasonality

TIME SERIES

- Observed time series

$$Y_1, Y_2, \dots, Y_n$$

- The equation for the model is

$$S_t = \alpha Y_{t-1} + (1 - \alpha) S_{t-1}$$

where α : the smoothing parameter, $0 \leq \alpha \leq 1$

Y_t : the value of the observation at time t

S_t : the value of the smoothed obs. at time t .

TIME SERIES

- The equation can also be written as

$$S_t = S_{t-1} + \alpha \underbrace{(Y_{t-1} - S_{t-1})}_{\text{the forecast error}}$$

- Then, the forecast is

$$\begin{aligned} S_{t+1} &= \alpha Y_t + (1 - \alpha) S_t \\ &= S_t + \alpha (Y_t - S_t) \end{aligned}$$

TIME SERIES

- Why Exponential?: For the observed time series $Y_1, Y_2, \dots, Y_n, Y_{n+1}$ can be expressed as a weighted sum of previous observations.

$$\hat{Y}_t(1) = c_0 Y_t + c_1 Y_{t-1} + c_2 Y_{t-2} + \dots$$

where c_i 's are the weights.

- Giving more weights to the recent observations, we can use the geometric weights (decreasing by a constant ratio for every unit increase in lag).

$$\Rightarrow c_i = \alpha(1 - \alpha)^i; i = 0, 1, \dots; 0 \leq \alpha \leq 1.$$

TIME SERIES

$$\hat{Y}_t(1) = \alpha(1-\alpha)^0 Y_t + \alpha(1-\alpha)^1 Y_{t-1} + \alpha(1-\alpha)^2 Y_{t-2} + \dots$$

$$\hat{Y}_t(1) = \alpha Y_t + (1-\alpha)\hat{Y}_{t-1}(1)$$

\downarrow
 S_{t+1}

\downarrow
 S_t

TIME SERIES

- **Remarks on α** (smoothing parameter).
 - Choose α between 0 and 1.
 - If $\alpha = 1$, it becomes a naive model; if α is close to 1, more weights are put on recent values. The model fully utilizes forecast errors.
 - If α is close to 0, distant values are given weights comparable to recent values. Choose α close to 0 when there are big random variations in the data.
 - α is often selected as to minimize the MSE.

TIME SERIES

- **Remarks on α** (smoothing parameter).
 - In empirical works, $0.05 \leq \alpha \leq 0.3$ commonly used. Values close to 1 are used rarely.
 - Numerical Minimization Process:
 - Take different α values ranging between 0 and 1.
 - Calculate 1-step-ahead forecast errors for each α .
 - Calculate MSE for each case.
 - Choose α which has the min MSE.

$$e_t = Y_t - S_t \Rightarrow \min \sum_{t=1}^n e_t^2 \Rightarrow \alpha$$

TIME SERIES

- Introduce a Trend factor to the simple exponential smoothing method
- Trend, but still **no** seasonality

SES + Trend = DES

- Two equations are needed now to handle the trend.

$$S_t = \alpha Y_{t-1} + (1 - \alpha)(S_{t-1} + T_{t-1}), 0 \leq \alpha \leq 1$$

$$T_t = \gamma(S_t - S_{t-1}) + (1 - \gamma)T_{t-1}, 0 \leq \gamma \leq 1$$



TIME SERIES HOLT

- Two parameters :
 - α = smoothing parameter
 - γ = trend coefficient
- h -step ahead forecast at time t is

$$\hat{Y}_t(h) = S_t + hT_t$$

Current level Current slope

- Trend prediction is added in the h -step ahead forecast.

TIME SERIES HOLT

- Now, we have two updated equations. The first smoothing equation adjusts S_t directly for the trend of the previous period T_{t-1} by adding it to the last smoothed value S_{t-1} . This helps to bring S_t to the appropriate base of the current value. The second smoothing equation updates the trend which is expressed as the difference between last two values.

TIME SERIES

- Initial value problem:

- S_1 is set to Y_1
- $T_1 = Y_2 - Y_1$ or $(Y_n - Y_1)/(n-1)$

α and γ can be chosen as

the value between $0.02 < \alpha, \gamma < 0.2$

or by minimizing the MSE as in SES.

HOLT-WINTER'S EXPONENTIAL SMOOTHING

- Introduce both Trend and Seasonality factors
- Seasonality can be added additively or multiplicatively.
- Model (multiplicative):

$$S_t = \alpha \frac{Y_{t-1}}{I_{t-s}} + (1 - \alpha)(S_{t-1} - T_{t-1})$$

$$T_t = \gamma(S_t - S_{t-1}) + (1 - \gamma)T_{t-1}$$

$$I_t = \delta \frac{Y_t}{S_t} + (1 - \delta)I_{t-s}$$

HOLT-WINTER'S EXPONENTIAL SMOOTHING

- h-step ahead forecast

$$\hat{Y}_t(h) = (S_t + hT_t)I_{t+h-s}$$

- Seasonal factor is multiplied in the h-step ahead forecast

α, γ and δ can be chosen as




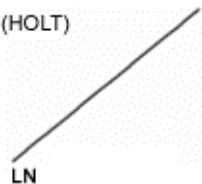
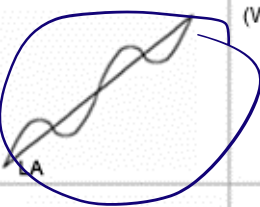
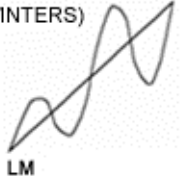



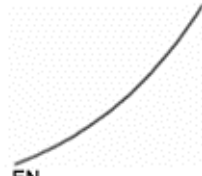
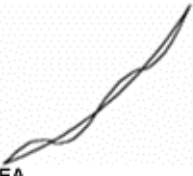
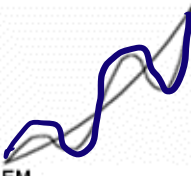
the value between $0.02 < \alpha, \gamma, \delta < 0.2$

or by minimizing the MSE as in SES.

HOLT-WINTER'S EXPONENTIAL SMOOTHING

- Note that, if a computer program selects 0 for γ and δ , this does not mean that there is no trend or seasonality.
- For Simple Exponential Smoothing, a level weight near zero implies that simple differencing of the time series may be appropriate.
- For Holt Exponential Smoothing, a level weight near zero implies that the smoothed trend is constant and that an ARIMA model with deterministic trend may be a more appropriate model.
- For Winters Method and Seasonal Exponential Smoothing, a seasonal weight near one implies that a nonseasonal model may be more appropriate and a seasonal weight near zero implies that deterministic seasonal factors may be present.

HOLT-WINTER'S EXPONENTIAL SMOOTHING

	Nonseasonal	Additive Seasonal	Multiplicative Seasonal
Constant Level	(SIMPLE)  NN	 NA	 NM
Linear Trend	(HOLT)  LN	 LA	(WINTERS)  LM
Damped Trend (0.95)	 DN	 DA	 DM
Exponential Trend (1.05)	 EN	 EA	 EM