Time Series

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Type of Non-Stationary TimeSeries

- Time trend
- Unit root
- Structural break in levels
- Structural break in variance

Dickey Fuller Test

- Model:

$$Y_t = \theta Y_{t-1} + \varepsilon_t$$

- True process: $Y_t = Y_{t-1} + \varepsilon_t$
- The null: $H_0 : \theta = 1$ vs $H_1 : |\theta| < 1$
- Estimate by OLS, form the test statistic:

$$t_n = \frac{\hat{\theta} - 1}{s.e.(\hat{\theta})}$$

- What's the distribution?
- Test with significance level α: Reject H_0 if $t_n < DF_n^{\alpha}$

Augmented Dickey Fuller Test

- Model:

$$Y_t = c + \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + \varepsilon_t$$

- True process has a unit root: $\theta_1 + \theta_2 = 1$
- Write the equation:

$$\begin{aligned} Y_t - Y_{t-1} &= c + \theta_1 Y_{t-1} - Y_{t-1} + \theta_2 Y_{t-1} + \theta_2 Y_{t-2} - \theta_2 Y_{t-1} + \varepsilon_t \\ \Delta Y_t &= c + (\theta_1 + \theta_2 - 1) Y_{t-1} - \theta_2 \Delta Y_{t-1} + \varepsilon_t \\ \Delta Y_t &= c + \theta^* Y_{t-1} + \theta_2^* \Delta Y_{t-1} + \varepsilon_t \end{aligned}$$

- Estimate by OLS, form the test statistic $t_n = \frac{\hat{\theta}^*}{s.e.(\hat{\theta}^*)}$
- The same distribution as before
- Test with significance level α: Reject H_0 if $t_n < DF_n^{\alpha}$

TIME SERIES

- **MASE:** The mean absolute scaled error is independent of the scale of the data, so can be used to compare forecasts across data sets with different scales.
- Non-seasonal time series

$$MASE = \frac{\frac{1}{J}\sum_{i=1}^{J}|e_i|}{\frac{1}{n-1}\sum_{t=2}^{n}|Y_t - Y_{t-1}|}$$

MAPE

- Seasonal time series

$$\frac{1}{J}\sum_{i=1}^{J}|e_i|$$

TIME SERIES

- This is a forecasting procedure based on a simple updating equations to calculate forecasts using the underlying pattern of the series. Not based on ARIMA approach.
- Recent observations are expected to have more power in forecasting values so a model can be constructed that places more weight on recent observations than older observations.

TIME SERIES

- *Smoothed* curve (eliminate up-and-down movement)
- Trend
- Seasonality

TIME SERIES

- 3 periods moving averages

$$\hat{Y}_t = (Y_{t-1} + Y_{t-2} + Y_{t-3})/3$$

- Also, 5 periods MA can be considered.

Period	Actual	3 Quarter MA Forecast	5 Quarter MA forecast
Mar-83	**239.3**	Missing	Missing
Jun-83	**239.8**	Missing	Missing
Sep-83	**236.1**	Missing	Missing
Dec-83	232	**238.40**	Missing
Mar-84	224.75	235.97	Missing
Jun-84	237.45	230.95	**234.39**
Sep-84	245.4	231.40	234.02
Dec-84	251.58	235.87	235.14
…		So on..	

TIME SERIES

- One can impose *weights* and use weighted moving averages (WMA).

  Eg $\hat{Y}_t = 0.6Y_{t-1} + 0.3Y_{t-2} + 0.1Y_{t-2}$

- How many periods to use is a question; more significant smoothing-out effect with longer lags.
- Peaks and troughs (bottoms) are not predicted.
- Events are being averaged out.
- Since any moving average is serially correlated, any sequence of random numbers could appear to exhibit cyclical fluctuation.

TIME SERIES

- Suppressing short-run fluctuation by smoothing the series
- Weighted averages of all previous values with more weights on recent values
- <u>No trend, No seasonality</u>

TIME SERIES

- Observed time series

$$Y_1,\ Y_2,\ \ldots,\ Y_n$$

- The equation for the model is

$$S_t = \alpha Y_{t-1} + (1-\alpha) S_{t-1}$$

where α: the smoothing parameter, $0 \le \alpha \le 1$

Y_t: the value of the observation at time t

S_t: the value of the smoothed obs. at time t.

TIME SERIES

- The equation can also be written as

$$S_t = S_{t-1} + \alpha \underbrace{\left(Y_{t-1} - S_{t-1}\right)}_{the\ forecast\ error}$$

- Then, the forecast is

$$\begin{aligned} S_{t+1} &= \alpha Y_t + (1-\alpha) S_t \\ &= S_t + \alpha\left(Y_t - S_t\right) \end{aligned}$$

TIME SERIES

- Why Exponential?: For the observed time series $Y_1, Y_2, \ldots, Y_n$, Y_{n+1} can be expressed as a weighted sum of previous observations.

$$\hat{Y}_t(1) = c_0 Y_t + c_1 Y_{t-1} + c_2 Y_{t-2} + \cdots$$

where c_i's are the weights.

- Giving more weights to the recent observations, we can use the geometric weights (decreasing by a constant ratio for every unit increase in lag).

$$\Rightarrow c_i = \alpha(1-\alpha)^i ; i = 0,1,\ldots ; 0 \le \alpha \le 1.$$

TIME SERIES

$$\hat{Y}_t(1) = \alpha(1-\alpha)^0 Y_t + \alpha(1-\alpha)^1 Y_{t-1} + \alpha(1-\alpha)^2 Y_{t-2} + \cdots$$

$$\hat{Y}_t(1) = \alpha Y_t + (1-\alpha)\hat{Y}_{t-1}(1)$$

S_{t+1} $\qquad\qquad$ S_t

TIME SERIES

- **Remarks on α** (smoothing parameter).
  - Choose α between 0 and 1.
  - If $\alpha = 1$, it becomes a naive model; if α is close to 1, more weights are put on recent values. The model fully utilizes forecast errors.
  - If α is close to 0, distant values are given weights comparable to recent values. Choose α close to 0 when there are big random variations in the data.
  - α is often selected as to minimize the MSE.

TIME SERIES

- **Remarks on α** (smoothing parameter).
  - In empirical works, $0.05 \le \alpha \le 0.3$ commonly used. Values close to 1 are used rarely.
  - Numerical Minimization Process:
    - Take different α values ranging between 0 and 1.
    - Calculate 1-step-ahead forecast errors for each α.
    - Calculate MSE for each case.
    - Choose α which has the min MSE.

$$e_t = Y_t - S_t \Rightarrow min \sum_{t=1}^{n} e_t^2 \Rightarrow \alpha$$

TIME SERIES

- Introduce a Trend factor to the simple exponential smoothing method
- Trend, but still **no** seasonality

SES + Trend = DES

- Two equations are needed now to handle the trend.

$$S_t = \alpha Y_{t-1} + (1-\alpha)(S_{t-1} + T_{t-1}), 0 \le \alpha \le 1$$
$$T_t = \gamma(S_t - S_{t-1}) + (1-\gamma)T_{t-1}, 0 \le \gamma \le 1$$

TIME SERIES HOLT

- Two parameters :

  α = smoothing parameter

  γ = trend coefficient

- h-step ahead forecast at time t is

$$\hat{Y}_t(h) = \underbrace{S_t}_{\text{Current level}} + \underbrace{hT_t}_{\text{Current slope}}$$

- Trend prediction is added in the h-step ahead forecast.

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TIME SERIES HOLT

- Now, we have two updated equations. The first smoothing equation adjusts S_t directly for the trend of the previous period T_{t-1} by adding it to the last smoothed value S_{t-1}. This helps to bring S_t to the appropriate base of the current value. The second smoothing equation updates the trend which is expressed as the difference between last two values.

TIME SERIES

- Initial value problem:
  - S_1 is set to Y_1
  - $T_1 = Y_2 - Y_1$ or $(Y_n - Y_1)/(n-1)$

α and γ can be chosen as

the value between $0.02 < \alpha, \gamma < 0.2$

or by minimizing the MSE as in SES.

HOLT-WINTER'S EXPONENTIAL SMOOTHING

- Introduce both Trend and Seasonality factors
- Seasonality can be added additively or multiplicatively.
- Model (multiplicative):

$$S_t = \alpha \frac{Y_{t-1}}{I_{t-s}} + (1-\alpha)(S_{t-1} - T_{t-1})$$

$$T_t = \gamma(S_t - S_{t-1}) + (1-\gamma)T_{t-1}$$

$$I_t = \delta \frac{Y_t}{S_t} + (1-\delta)I_{t-s}$$

HOLT-WINTER'S EXPONENTIAL SMOOTHING

- h-step ahead forecast

$$\hat{Y}_t(h) = (S_t + hT_t)I_{t+h-s}$$

- Seasonal factor is multiplied in the h-step ahead forecast

  α, γ and δ can be chosen as

  the value between $0.02 < \alpha, \gamma, \delta < 0.2$

  or by minimizing the MSE as in SES.

HOLT-WINTER'S EXPONENTIAL SMOOTHING

- Note that, if a computer program selects 0 for γ and δ, this does not mean that there is no trend or seasonality.
- For Simple Exponential Smoothing, a level weight near zero implies that simple differencing of the time series may be appropriate.
- For Holt Exponential Smoothing, a level weight near zero implies that the smoothed trend is constant and that an ARIMA model with deterministic trend may be a more appropriate model.
- For Winters Method and Seasonal Exponential Smoothing, a seasonal weight near one implies that a nonseasonal model may be more appropriate and a seasonal weight near zero implies that deterministic seasonal factors may be present.

HOLT-WINTER'S EXPONENTIAL SMOOTHING