Basics of Stochastic Processes

Peter Lukianchenko

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Filtration

Exam 29.10 13-15

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space.

Definition 1

An increasing family of sub-σ-algebras $\{\mathcal{F}_n, n \geq 0\}$ satisfying $\mathcal{F}_0 \subseteq \mathcal{F}_1 \ldots \subseteq \mathcal{F}$ is called *filtration*.

Denote $\mathcal{F}_\infty := \sigma\left(\bigcup_n \mathcal{F}_n\right) \subseteq \mathcal{F}$.

Intuitive idea: The concept of filtration is used to express the growth of information in time. At time n the information about ω consists precisely of the values of all $\mathcal{F}_n$-measurable functions $Z(\omega)$. Usually $\{\mathcal{F}_n\}$ is the natural filtration

$$F_n = \sigma(W_0, W_1, \ldots, W_n)$$

of some random (stochastic) process $\{W_n\}$, and then the information about ω which we have at time n is the current history of the process i.e.

$$(\omega), W_1(\omega), \ldots, W_n(\omega)$$

(from which ω can not be determined uniquely, as a rule).

A Random Process **Adopted** to the Filtration

Definition 2

We say that the random process $X = (X_n : n \geq 0)$ is *adopted* to the filtration $\{\mathcal{F}_n\}$ if for each n, X_n is $\mathcal{F}_n$ -measurable.

Intuitive idea:

If X is adopted, then its value $X_n(\omega)$ is known to us at time n. Usually, $\mathcal{F}_n = \sigma(W_0, W_1, \dots, W_n)$ and $X_n = f_n(W_0, W_1, \dots, W_n)$ for some measurable function f_n.

Martingales

Definition 3

A process X is called *martingale* (relative to $\{F_N\}$) if

1. X is adopted,

2. $E|Xn| < \infty$,

3. $E(X_n|\mathcal{F}_{n-1}) = X_{n-1}$ a. s. $(n \geq 1)$.

- ***Super****martingale* is defined similarly, except that (3) is replaced by

$$E(X_n|\mathcal{F}_{n-1}) \leq X_{n-1} \text{ a. s. } (n \geq 1)$$

- ***Sub****martingale* is defined with (3) replaced by

$$E(X_n|\mathcal{F}_{n-1}) \geq X_{n-1} \text{ a. s. } (n \geq 1)$$

- A supermartingale decreases on average, a submartingale increases on average in time.

Martingales: examples

- *Example 1*. Sums of independent zero-mean RV's.

Let $X_1, X_2, \ldots$ be a sequence of independent RVs with $E|X_k| < \infty, \forall\, k$ and $EX_k = 0, \forall\, k$.

Define $S_0 = 0, \mathcal{F}_0 = \{\emptyset, \Omega\}$ and

$$S_n = \sum_{i=1}^{n} X_i\,, n \geq 1;\;\; \mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n), n \geq 1.$$

Then $S = (S_n : n \geq 0)$ is martingale relative to filtration $\{F_n\}$ (show that!).

$$S_n = S_{n-1} + X_n$$

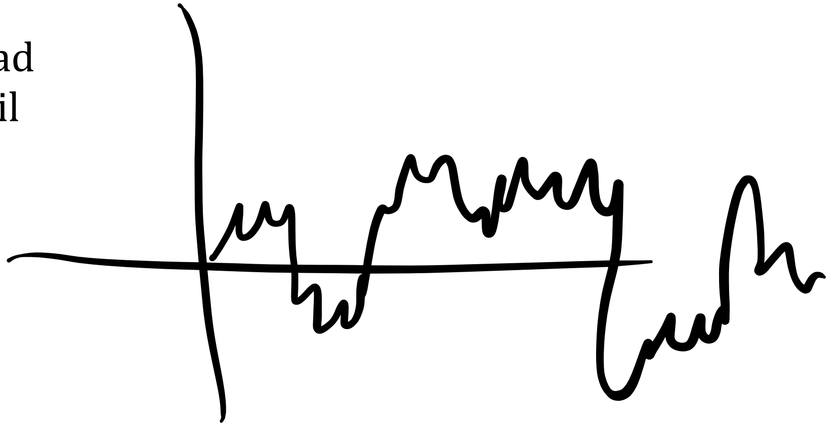
$$E[S_n \mid \mathcal{F}_{n-1}] = \underbrace{E[S_{n-1} \mid \mathcal{F}_{n-1}]}_{= S_{n-1}} + \underbrace{E[X_n \mid \mathcal{F}_{n-1}]}_{= E[X_n] = 0}$$

$$E[S_n \mid \mathcal{F}_{n-1}] = S_{n-1}$$

Martingales: Fair and Unfair Games

Let $X_n - X_{n-1}$ be your net winnings per unit stake in game n ($n \geq 1$) in a series of games, played at times $n = 1, 2, \ldots$. There is no game at time 0. A simple example is obtained by a series of coin tosses where the outcome of the toss at time k is

$$\Delta_k = \begin{cases} +1, \text{if head} \\ -1, \text{if tail} \end{cases}$$



and $X_n = \sum_{k=1}^{n} \Delta_k$.

In the martingale case

- (a) $E(X_n - X_{n-1} | \mathcal{F}_{n-1}) = 0$, (game series is fair)

and in the supermartingale case

- (b) $E(X_n - X_{n-1} | \mathcal{F}_{n-1}) \leq 0$, (game series is unfavorable to you).
- **Note** that we have the case (a) if the coin is symmetric, and case (b) if —1 is more probable than +1.

Predictable Process

Definition 4

A process $C = (C_n: n \geq 0)$is called *predictable*, if C_n, is $\mathcal{F}_{n-1}$-measurable $(n \geq 1)$.

- One can think about C_n as your stake on game n, You have to decide on the value of C, based on the history up to (and including) time $n - 1$. This is the intuitive meaning of the predictable character of C. Your winnings on game n are $C_n \cdot (X_n — X_{n-1})$ and your total winnings up to time n are

$$Y_n = \sum_{i=1}^{n} C_i(X_i - X_{i-1}) =: (C \bullet X)_n \qquad (2)$$

- Note that $Y_0 = (C \bullet X)_0 \coloneqq 0 \; and \; Y_n - Yn - 1 = Cn(X_n - x_n - 1)$

Itô Doublin

Martingale

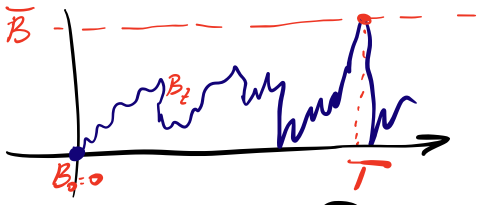
Theorem 2.1. *There exists a probability distribution over the set of continuous functions* $B : \mathbb{R} \to \mathbb{R}$ *satisfying the following conditions:*

(i) $B(0) = 0$.

(ii) ***(stationary)*** *for all* $0 \leq s < t$, *the distribution of* $B(t) - B(s)$ *is the normal distribution with mean* 0 *and variance* $t - s$, *and*

(iii) ***(independent increment)*** *the random variables* $B(t_i) - B(s_i)$ *are mutually independent if the intervals* $[s_i, t_i]$ *are nonoverlapping.*

Properties



$P(T \leq \tau) - ?$

MARKOV'S INEQ.

Let Z be a non-negative random variable. Then,

$$P(Z > t) \leq \frac{\mathbb{E}[Z]}{t}, \quad t > 0.$$

Properties

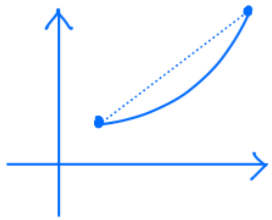
$$f(B_t, W_t)$$

$$df$$

JENSEN'S INEQ.

Let Z be a random variable and let $\psi : \mathbb{R} \to \mathbb{R}$ be a convex function. Then,

$$\mathbb{E}[\psi(Z)] \geq \psi(\mathbb{E}[Z])$$



Properties

CAUCHY SCHWARZ

Suppose X and Y are r.v.s with $\mathbb{E}[X^2] < \infty$, $\mathbb{E}[Y^2] < \infty$. Then,

$$Cov(X,Y) = E(X \cdot Y) - E(X)E(Y)$$

$$E(X)E(Y) + Cov(X,Y) = \mathbb{E}[XY] \leq \left(\mathbb{E}[X^2]\right)^{1/2}\left(\mathbb{E}[Y^2]\right)^{1/2}$$

Hölder's Inequality:

$$\mathbb{E}[|XY|] \leq \left(\mathbb{E}[|X|^p]\right)^{1/p}\left(\mathbb{E}[|Y|^q]\right)^{1/q}$$

for $p, q \in [1, \infty]$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$.

Process

A stochastic process $\{X_t, t \in T\}$ is well-defined or specified (for purposes of this course) when we specify the following.

(i) the state space S

(ii) the time index T.

(iii) all finite-dimensional joint distributions, that is, the joint distributions of all vectors of the type

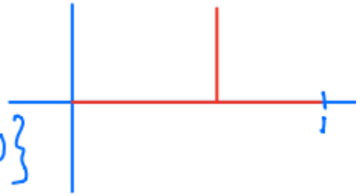
$$(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$$

where $t_1, t_2, \ldots, t_n \in T$, $n \in \mathbb{N}$.

Process

(i), (ii), (iii) may not always suffice.
The issue is with (iii).

<u>Example</u>



$$X_t := \begin{cases} 0 & t \in [0,1] \setminus \{U\} \\ 1 & U = t \end{cases}$$

where $U \sim$ Uniform $(0,1)$ r.v.

$$Y_t := 0 \quad \forall\, t \in [0,1]$$

We can check that X_t and Y_t have the same finite dimensional distributions

However, $P(X_t \leq \frac{1}{2}\ \forall t) = 0$

$P(Y_t \leq \frac{1}{2}\ \forall t) = 1$

y_t - non-st. OR mean $\neq 0$

↳ $z_t = \Delta y_t = y_t - y_{t-1}$

↳ diff. of 2^{nd}-order

$w_t = \Delta z_t = z_t - z_{t-1}$

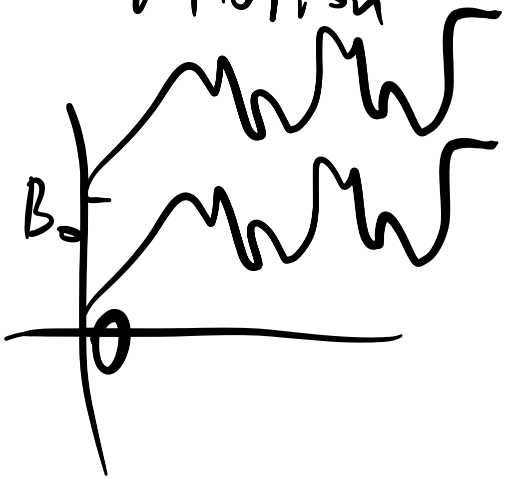
$= \Delta^2 y_t$

Process

$s<t \quad 0: E[B_t - B_s \mid \mathcal{F}_s] = E[B_t \mid \mathcal{F}_s] - \underbrace{E[B_s \mid \mathcal{F}_s]}_{= B_s}$

Wiener

Brownian Motion



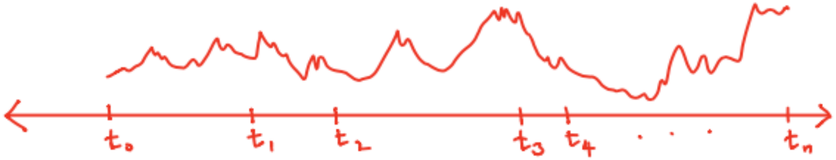
Particular stochastic processes are specified for a context based on whether specific properties are satisfied. Let's look at some.

(i) Independent Increments
(ii) Stationary Increments
(iii) Martingale
(iv) Markov
(v) Stationarity

$E[B_t \mid \mathcal{F}_s] = B_s$

Process

P1. Independent Increments (e.g., Brownian motion, Poisson process, random walks)



$\{X_t, t \in T\}$ is said to have independent increments if the increments exhibited on disjoint intervals are independent, that is,

for $t_0 < t_1 < t_2 < \cdots t_n \in T$,

$$X_{t_1} - X_{t_0}, \; X_{t_2} - X_{t_1}, \; \ldots, \; X_{t_n} - X_{t_{n-1}}$$

are independent.

If t_0 is the smallest element in T, then

$$X_{t_0}, \; X_{t_1} - X_{t_0}, \; \ldots, \; X_{t_n} - X_{t_{n-1}}$$

are independent.

P2. Stationary Increments (e.g., homogeneous Poisson process)

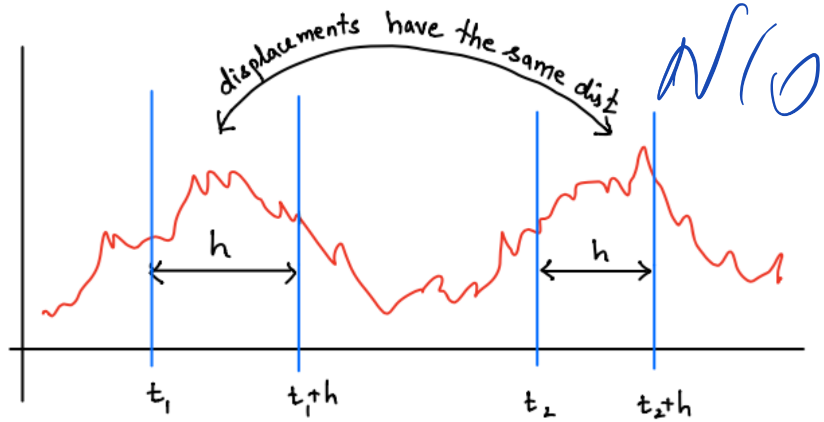
$\{X_t, t \in T\}$ is said to have stationary increments if the distribution of

$X_{t_1+h} - X_{t_1}$ and $X_{t_2+h} - X_{t_2}$

depends only on h (for any $t_1, t_2 \in T$), that is,

$$P(X_{t_1+h} - X_{t_1} \in A) = P(X_{t_2+h} - X_{t_2} \in A)$$

$\mathcal{N}(0, h)$

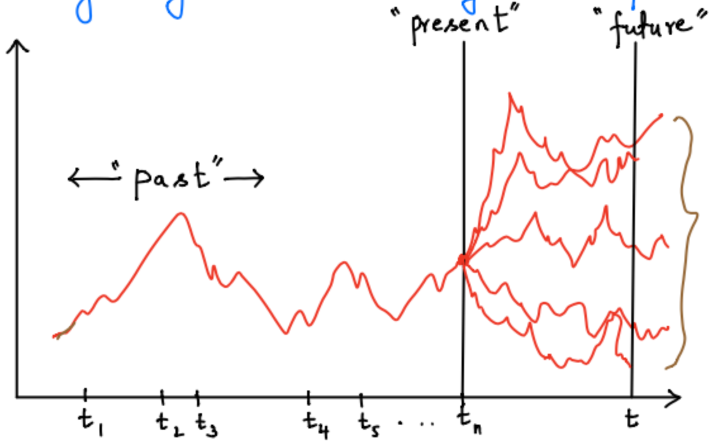


P3. Martingale (e.g., Brownian motion with zero drift, random walk with zero drift)

$\{X_t, t \in T\}$ is said to be a martingale if X_t is a "fair game" that is, $\mathbb{E}[|X_t|] < \infty \ \forall t$ and for any $t_1 < t_2 < \cdots < t_n \leq t \in T$

$$\mathbb{E}\left[X_t \,\middle|\, X_s = a_s\ s \in \{t_1, t_2, \ldots t_n\} \right] = a_{t_n}.$$

On average, you add nothing in the future!



P4. Markov Process

$\{X_t, t \in T\}$ is said to be Markov if the "future depends only on the present, but not how we got to the present."

$$P(X_t \in A \mid X_{t_1} = x_1, X_{t_2} = x_2, \dots, X_{t_n} = x_n)$$

$$= P(X_t \in A \mid X_{t_n} = x_n)$$

whenever $t > t_n > t_{n-1} \cdots > t_1$ and appropriate sets A.

A Markov process with a countable state space is called a Markov chain.

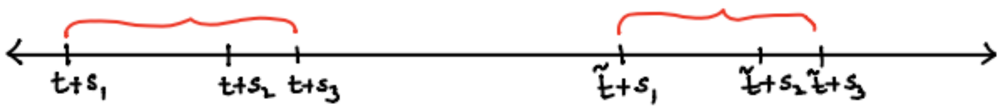
A Markov process with continous sample paths $\{X_t, t \in [0, \infty)\}$ is called a diffusion

P5. Stationary Processes

A stochastic process is said to be strictly stationary if the joint distribution of

$$\left(X_{t+s_1}, X_{t+s_2}, \ldots, X_{t+s_n} \right)$$

is the same for all t and arbitrary selection of $s_1, s_2, \ldots, s_n$.



A stochastic process is said to be covariance stationary if it has finite second moments and

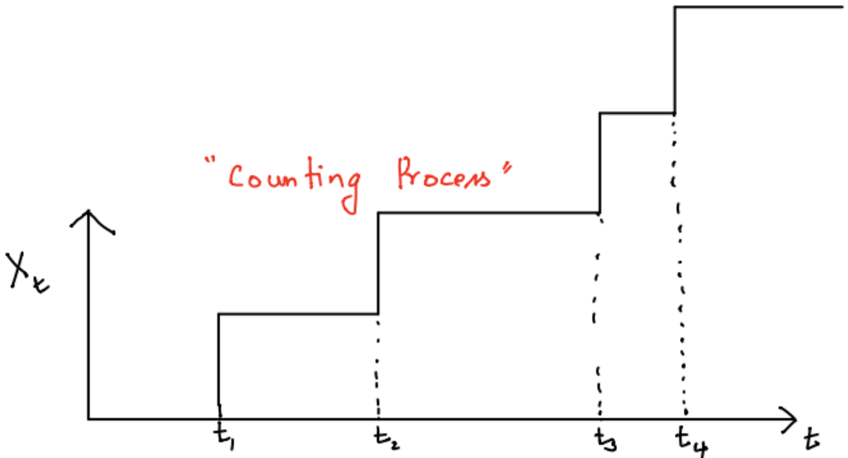
$$\mathrm{Cov}\left(X_{t+h}, X_t \right)$$

depends only on h for all t.

Process

<u>Example I.</u> Poisson Process.

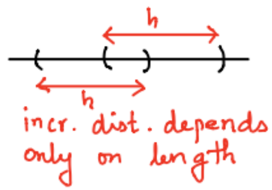
$\{X_t,\ t \geq 0\}$, $X_t \in \mathbb{N}$. $X_0 = 0$



Specified by three postulates :

(a) Independent Increments

(b) Stationary Increments



(c) $P(X_{t+h} - X_t \geq 1) = \lambda h + o(h)$, $\lambda > 0$

$P(X_{t+h} - X_t \geq 2) = o(h)$.

Simple Random Walk

Let $Y_1, Y_2, \cdots$ be i.i.d. random variables such that $Y_i = \pm 1$ with equal probability. Let $X_0 = 0$ and

$$X_k = Y_1 + \cdots + Y_k,$$

for all $k \geq 1$. This gives a probability distribution over the sequences $\{X_0, X_1, \cdots, \}$, and thus defines a discrete time stochastic process. This process is known as the *one-dimensional simple random walk*, which we conveniently refer to as *random walk* from now on.

$\int_a^b f(x)\,dx$

$\int_a^b f(x)\,dF(x) = \int_a^b f(x)\cdot p(x)\,dx$

$\int_a^b X_t\,dW_t$ $W_t \sim$ Wiener process

Itô

Ch. #

Shrève