

# Homework 9: NP-completeness

CSE 30151 Spring 2016

Due 2016/04/26

## Instructions

Please note that you will **lose one point** if you don't follow these instructions.

- You can prepare your solutions however you like, but you must submit them as a single PDF file.
- Please name your PDF `netid-hw9.pdf`, where `netid` is replaced with your NetID, or `netid-hw9-1234.pdf`, where 1234 is replaced with the problems you are submitting.
- If you use the same name twice, only the most recent version will be graded!
- Submit your PDF file in Sakai. Don't forget to click the Submit (or Resubmit) button!

## Problems

A *two-dimensional TM* (2dTM) has, in place of a tape, a sheet of graph paper that has a top and left edge but extends infinitely to the right and down. Initially the head is at the upper left corner. The transitions are as in a standard TM, except that there are four directions: L (left), R (right), U (up), and D (down).

1. Describe how to simulate a 2dTM using a TM that has one or more one-dimensional tapes, in three parts:
  - (a) Describe how a two-dimensional sheet with a head can be represented as one or more one-dimensional tapes with heads. You can use the construction we sketched in class or your own.
  - (b) Simulating reading and writing symbols is presumably trivial, but if not, give implementation-level descriptions.
  - (c) Give implementation-level descriptions of how to simulate moving left, right, up, and down.

2. Let  $n$  be the length of the input string. Choose a  $k$  and show that if a computation takes  $t(n)$  steps in a 2dTM, it takes, in the worst case,  $O((t(n))^k)$  steps in a standard TM (single one-way infinite tape). Your algorithm doesn't have to be optimal and your analysis doesn't have to be tight.
3. [Problem 7.39 in 2nd ed., 7.41 in 3rd ed.] In the proof of the Cook-Levin theorem, a window is a  $2 \times 3$  rectangle of cells. Show why the proof would have failed if we had used  $2 \times 2$  windows instead. Your answer should include:
  - (a) An example NTM  $N$ .
  - (b) Two configurations of  $N$  (call them upper and lower) such that all of the following hold:
    - The upper configuration is legal.
    - The lower configuration either is illegal or doesn't follow the upper configuration according to  $N$ 's rules.
    - Every window of the two configurations (when the upper is stacked on top of the lower) is legal according to  $N$ 's rules.
4. In class we saw a puzzle called TetraVex that is NP-complete.<sup>1</sup> In this problem, we'll prove the NP-completeness of a simpler version of the puzzle:
  - You are given a set of tiles, each with a north, south, east, and west label. You can make as many copies as you want of each tile. You can't rotate the tiles.
  - You are also given a rectangular frame with labels.
  - The object is to fill the frame with tiles such that all abutting labels match.

The proof is by reduction from 3SAT. Let  $x_1, \dots, x_n$  be a set of variables, and let  $\phi$  be a formula in 3CNF,

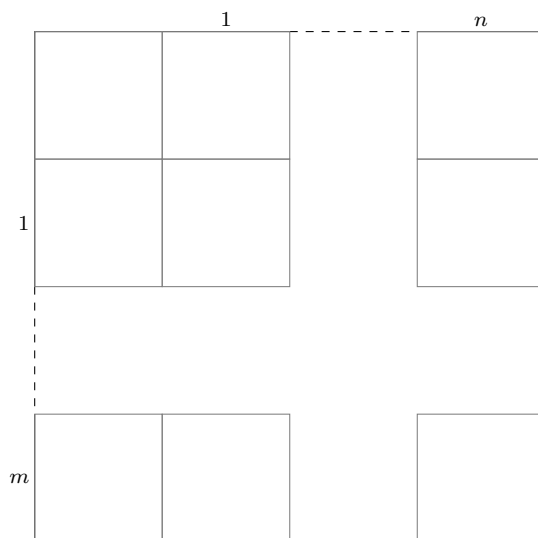
$$\phi = (\phi_{11} \vee \phi_{12} \vee \phi_{13}) \wedge \cdots \wedge (\phi_{m1} \vee \phi_{m2} \vee \phi_{m3})$$

where each  $\phi_{i1}, \phi_{i2}, \phi_{i3}$  is either  $x_j$  or  $\overline{x_j}$  for some  $j$ .

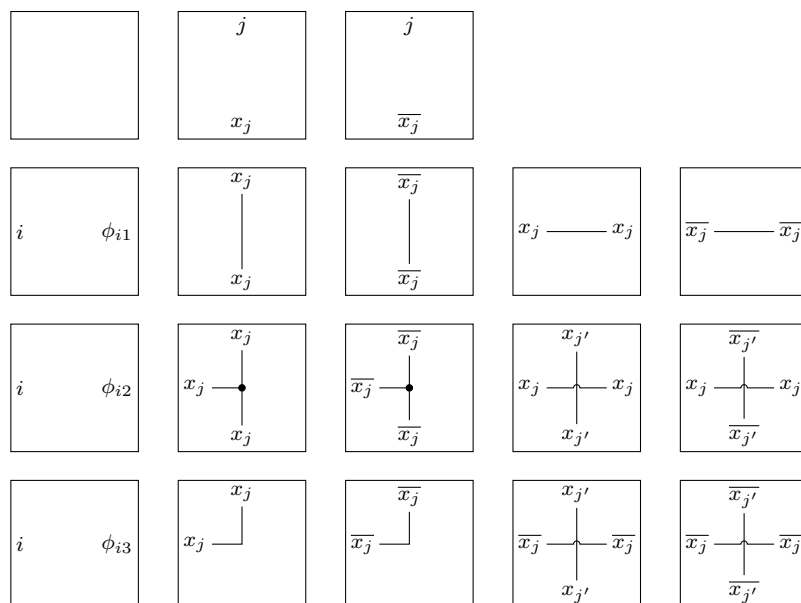
---

<sup>1</sup><http://arxiv.org/pdf/0903.1147v1.pdf>

Define a function  $f$  that converts  $\phi$  into an instance of the puzzle as follows. The frame is:



The tiles are, for all  $i \in \{1, \dots, m\}$  and  $j, j' \in \{1, \dots, n\}$ :



Notes:

- Each  $\phi_{i1}, \phi_{i2}, \phi_{i3}$  is equal to either  $x_j$  or  $\overline{x_j}$  for some  $j$ .
- Blank edges only match other blank edges.
- The “wires” drawn in the interior of some tiles are merely suggestive; they don’t affect matching at all.

Prove the following statements. Your answers don’t have to be very formal, but try to make them clear.

- (a) The set of solvable puzzles is in NP.
- (b)  $f$  is computable in polynomial time.
- (c) If  $\phi$  is satisfiable, then the corresponding puzzle  $f(\phi)$  is solvable.
- (d) If the puzzle  $f(\phi)$  is solvable, then  $\phi$  is satisfiable.

For the formula  $\phi = (x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee x_2)$ , the frame would be

		1	2
1			
2			
3			

And the tiles would be

1	1	2	2
$x_1$	$\overline{x_1}$	$x_2$	$\overline{x_2}$
1	2	3	
$x_1$	$\overline{x_1}$	$\overline{x_1}$	
1	2	3	
$x_2$	$\overline{x_2}$	$x_2$	

