Homework 1: Strings and languages

Theory of Computing (CSE 30151), Spring 2023

Due: 2023-01-27 5:00pm

Instructions

- Create a PDF file (or files) containing your solutions. You can write your solutions by hand, but please scan them into a PDF.
- Please name your PDF file(s) as follows to ensure that the graders give you credit for all of your work:
  - If you're making a complete submission, name it `netid-hw1.pdf`, where *`netid`* is replaced with your NetID.
  - If you're submitting some problems now and want to submit other problems later, name it `netid-hw1-123.pdf`, where `123` is replaced with the problem number(s) you are submitting at this time.
- Submit your PDF file(s) in Canvas.

Problems (10 points each)

1. **Proof practice.**

   (a) Convert this paragraph proof to a statement–reason proof. Please be sure to write which statement(s) each statement depends on.

   > To show: If s is a string, every substring of a substring of s is a substring of s.
   >
   > Proof: Let y be a substring of s, that is, $s = xyz$ for some x, z; and let v be a substring of y, that is, $y = uvw$ for some u, w. Then $s = xuvwz$, so v is a substring of s.

   (b) Convert this statement–reason proof to a paragraph proof.

   > To show: If w is a string, every prefix of a suffix of w is a suffix of a prefix of w.

1. v is a suffix of w	Given
2. y is a prefix of v	Given
3. $\exists x$ s.t. $xv = w$	(1), def. suffix
4. $\exists z$ s.t. $yz = v$	(2), def. prefix
5. $xyz = w$	(3), (4), substitution
6. xy is a prefix of w	(5), def. prefix
7. y is a suffix of xy	(6), def. suffix

2. **String homomorphisms.** If Σ and Γ are finite alphabets, define a *string homomorphism* to be a function $\phi : \Sigma^* \to \Gamma^*$ that has the property that for any $u, v \in \Sigma^*$, $\phi(uv) = \phi(u)\,\phi(v)$.

   For example, the function $\phi : \{\mathtt{0}, \ldots, \mathtt{9}, \mathtt{A}, \ldots, \mathtt{F}\}^* \to \{\mathtt{0}, \mathtt{1}\}^*$ that converts a hexadecimal number with $n \geq 0$ digits into a binary number with $4n$ bits is a string homomorphism:

$$\begin{aligned}\phi(\varepsilon) &= \varepsilon \\ \phi(\mathtt{0}) &= \mathtt{0000} \\ \phi(\mathtt{A}) &= \mathtt{1010} \\ \phi(\mathtt{CAB}) &= \mathtt{110010101011}\end{aligned}$$

   Intuitively, a string homomorphism does a "search and replace" where each symbol is replaced with a (possibly empty) string. Prove this more formally: that is, prove that if ϕ is a string homomorphism, then for any $w = w_1 \cdots w_n$ (where $n \geq 0$ and $w_j \in \Sigma$ for $1 \leq j \leq n$), we have

$$\phi(w) = \phi(w_1) \cdots \phi(w_n). \qquad (*)$$

   Use induction on n.

   (a) State and prove the base case ($n = 0$).

   (b) Assume that (*) is true for $n = i$ and prove (*) for $n = i + 1$.

   You may assume the following facts about strings:

   Added on 2023-01-23

   - For all $x \in \Sigma^*$, $x\,\varepsilon = x$ and $\varepsilon\,x = x$.
   - For all $x, y, z \in \Sigma^*$, if $xz = yz$ then $x = y$.
   - For all $x, y, z \in \Sigma^*$, if $xy = xz$ then $y = z$.

3. **Finite and cofinite.** Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$. Define FINITE to be the set of all finite languages over Σ, and let

$$\mathsf{coFINITE} = \{L \mid \overline{L} \in \mathsf{FINITE}\}$$

   (where, for any language L over Σ, $\overline{L}$ is the complement of L, that is, $\Sigma^* \setminus L$). For example, Σ^* is in $\mathsf{coFINITE}$ because its complement is $\emptyset$, which is finite.

(Please think carefully about this definition, and note that $\mathsf{coFINITE}$ isn't the same thing as $\overline{\mathsf{FINITE}}$.)

(a) Are there any languages over Σ in $\mathsf{FINITE} \cap \mathsf{coFINITE}$? Prove your answer.

(b) Are there any languages over Σ that are *not* in $\mathsf{FINITE} \cup \mathsf{coFINITE}$? Prove your answer.