Homework 5: Non-CFLs and Turing Machines

Theory of Computing (CSE 30151), Spring 2025

Due: 2025-03-21 5pm

Instructions

- Create a PDF file (or files) containing your solutions. You can write your solutions by hand, but please scan them into a PDF.

- Please name your PDF file(s) as follows to ensure that the graders give you credit for all of your work:

    - If you're making a complete submission, name it *netid*-hw5.pdf, where *netid* is replaced with your NetID.

    - If you're submitting some problems now and want to submit other problems later, name it *netid*-hw5-part123.pdf, where 123 is replaced with the problem number(s) you are submitting at this time.

- Submit your PDF file(s) in Canvas.

Problems (10 points each)

1. **Non-closure properties of CFLs**

    (a) [Exercise 2.2a] Use the languages

    $$A = \{\mathsf{a}^m\mathsf{b}^n\mathsf{c}^n \mid m, n \geq 0\}$$
 $$B = \{\mathsf{a}^n\mathsf{b}^n\mathsf{c}^m \mid m, n \geq 0\}$$

    to prove that the class of context-free languages is *not* closed under intersection.

    (b) [Exercise 2.2b] Use Problem 1a and DeMorgan's law to prove that the class of context-free languages is *not* closed under complementation.

2. **There and back again.** Imagine a robot turtle that you can give instructions `u` (go up 1 cm), `d` (go down 1 cm), `l` (go left 1 cm), `r` (go right 1 cm). A program is a string of instructions.

Let C be the set of programs that make the turtle return to its starting point. For example, `uurdrdll` is in C, as shown in this picture:
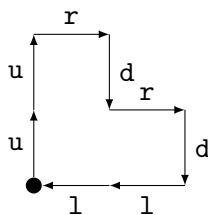
(a) Prove that C is not context-free.

(b) Write a **formal description** of a Turing machine that decides C.

3. **Turing closure properties.** Let $\Sigma = \{0, 1\}$. Recall in HW2 we defined

$$\text{STRETCH}(w_1 w_2 \cdots w_n) = w_1 w_1 w_2 w_2 \cdots w_{n-1} w_{n-1} w_n w_n$$

for any string $w = w_1 \cdots w_n \in \Sigma^*$. This induces an operation on languages,

$$\text{STRETCH}(L) = \{\text{STRETCH}(w) \mid w \in L\}.$$

(a) Write an **implementation-level** description of a Turing machine S that, on input $v \in \Sigma^*$, decides whether $v = \text{STRETCH}(u)$ for some u. Moreover, if S accepts v, then when it halts, the contents of the tape should be u. For example, if the input is `001100`, S should accept and the final contents of the tape should be `010`. But if the input is `001101`, S should reject.

(b) Prove that if L is a Turing-decidable language over Σ, then $\text{STRETCH}(L)$ is also Turing-decidable. You should let M be a Turing machine that decides L, then use your answer to 3a to give an **implementation-level** description of a Turing machine that decides $\text{STRETCH}(L)$. One of the lines of your description can be "Simulate M."

(c) Prove that if L is a Turing-recognizable language, then $\text{STRETCH}(L)$ is also Turing-recognizable.