Finitary Coalgebraic Bisimulation

or they look the same so far, and I can't wait forever

Erik Wennstrom

Department of Informatics
Indiana University, Bloomington

9th Annual Graduate Student Conference in Logic, 2008

Outline

1. Bisimulation
   - Without Coalgebras
   - With Coalgebras

2. Equivalence
   - Behavioral Equivalence
   - Finitary Equivalence

Directed Graphs

A bisimulation on a directed graph $(V, \to)$ is a relation $\sim \in V \times V$ such that:

- (zig) For all $x_1 \sim y_1$, if $x_1 \to x_2$, then there is a y_2 with $y_1 \to y_2$ and $x_2 \sim y_2$
- (zag) For all $x_1 \sim y_1$, if $y_1 \to y_2$, then there is an x_2 with $x_1 \to x_2$ and $x_2 \sim y_2$

Directed Graphs

A bisimulation on a directed graph $(V, \rightarrow)$ is a relation $\sim \in V \times V$ such that:

- (zig) For all $x_1 \sim y_1$, if $x_1 \rightarrow x_2$, then there is a y_2 with $y_1 \rightarrow y_2$ and $x_2 \sim y_2$
- (zag) For all $x_1 \sim y_1$, if $y_1 \rightarrow y_2$, then there is an x_2 with $x_1 \rightarrow x_2$ and $x_2 \sim y_2$

Probabilistic Transition Systems

- A bisimulation of PTS (probabilistic transition system) $(S, \Pr)$ is a relation $\sim \in S \times S$ such that for all $x \sim y$ and any equivalence class $[z]$ of $\sim$, $\Pr(x, [z]) = \Pr(y, [z])$.
- In other words, any two bisimilar states x and y are indistinguishable with respect to the probability of ending up in some other set $[z]$ of bisimilar states.
- This definition does not include terminating states or other distinguishing factors between states, so technically, any relation is a bisimulation, but adding such extra info isn't very difficult.

Probabilistic Transition Systems

- A bisimulation of PTS (probabilistic transition system) (S, Pr) is a relation $\sim \in S \times S$ such that for all $x \sim y$ and any equivalence class $[z]$ of $\sim$, $\mathrm{Pr}(x, [z]) = \mathrm{Pr}(y, [z])$.
- In other words, any two bisimilar states x and y are indistinguishable with respect to the probability of ending up in some other set $[z]$ of bisimilar states.
- This definition does not include terminating states or other distinguishing factors between states, so technically, any relation is a bisimulation, but adding such extra info isn't very difficult.

Probabilistic Transition Systems

- A bisimulation of PTS (probabilistic transition system) (S, Pr) is a relation $\sim \in S \times S$ such that for all $x \sim y$ and any equivalence class $[z]$ of $\sim$, $\mathrm{Pr}(x, [z]) = \mathrm{Pr}(y, [z])$.
- In other words, any two bisimilar states x and y are indistinguishable with respect to the probability of ending up in some other set $[z]$ of bisimilar states.
- This definition does not include terminating states or other distinguishing factors between states, so technically, any relation is a bisimulation, but adding such extra info isn't very difficult.

Definition of Coalgebras

- For any functor $F : \mathrm{Set} \to \mathrm{Set}$, an F-coalgebra is a carrier set A and a structure map $\alpha : A \to FA$.
  - A directed graph $(V, \to)$ is a coalgebra for the finite power set functor $\mathcal{P}$ where the carrier set is V and the structure map sends a vertex to the set of adjacent vertices.
  - A PTS (S, Pr) is a coalgebra for the finite probability distribution functor $\mathcal{D}$ where the carrier set is S and the structure map sends a state x to the probability distribution of possible transitions $Pr(x,)$.*
- An F-coalgebra morphism $f : (A, \alpha) \to (B, \beta)$ is a map $f : A \to B$ that plays nice with the structure maps (i.e., $\beta \circ f = Ff \circ \alpha$).

Definition of Coalgebras

- For any functor $F : \mathrm{Set} \to \mathrm{Set}$, an F-coalgebra is a carrier set A and a structure map $\alpha : A \to FA$.
  - A directed graph $(V, \to)$ is a coalgebra for the finite power set functor $\mathcal{P}$ where the carrier set is V and the structure map sends a vertex to the set of adjacent vertices.
  - A PTS (S, Pr) is a coalgebra for the finite probability distribution functor $\mathcal{D}$ where the carrier set is S and the structure map sends a state x to the probability distribution of possible transitions $Pr(x,)$.*
- An F-coalgebra morphism $f : (A, \alpha) \to (B, \beta)$ is a map $f : A \to B$ that plays nice with the structure maps (i.e., $\beta \circ f = Ff \circ \alpha$).

Definition of Coalgebras

- For any functor $F : \mathrm{Set} \to \mathrm{Set}$, an F-coalgebra is a carrier set A and a structure map $\alpha : A \to FA$.
  - A directed graph $(V, \to)$ is a coalgebra for the finite power set functor $\mathcal{P}$ where the carrier set is V and the structure map sends a vertex to the set of adjacent vertices.
  - A PTS (S, Pr) is a coalgebra for the finite probability distribution functor $\mathcal{D}$ where the carrier set is S and the structure map sends a state x to the probability distribution of possible transitions $Pr(x,)$.*
- An F-coalgebra morphism $f : (A, \alpha) \to (B, \beta)$ is a map $f : A \to B$ that plays nice with the structure maps (i.e., $\beta \circ f = Ff \circ \alpha$).

Definition of Coalgebras

- For any functor $F : \mathrm{Set} \to \mathrm{Set}$, an F-coalgebra is a carrier set A and a structure map $\alpha : A \to FA$.
  - A directed graph $(V, \to)$ is a coalgebra for the finite power set functor $\mathcal{P}$ where the carrier set is V and the structure map sends a vertex to the set of adjacent vertices.
  - A PTS (S, Pr) is a coalgebra for the finite probability distribution functor $\mathcal{D}$ where the carrier set is S and the structure map sends a state x to the probability distribution of possible transitions $Pr(x,)$.*
- An F-coalgebra morphism $f : (A, \alpha) \to (B, \beta)$ is a map $f : A \to B$ that plays nice with the structure maps (i.e., $\beta \circ f = Ff \circ \alpha$).

Definition of Coalgebraic Bisimulation

- For any F-coalgebras (A, α) and (B, β), a relation $R \in A \times B$ is called a bisimulation if we can turn it into a coalgebra (R, ρ) where the projection maps $\pi_1 : R$ and $\pi_2 : R \rightarrow B$ are coalgebra morphisms.
- This generalizes the other definitions of bisimulation that I just gave (and many others!)

Definition of Coalgebraic Bisimulation

- For any F-coalgebras (A, α) and (B, β), a relation $R \in A \times B$ is called a bisimulation if we can turn it into a coalgebra (R, ρ) where the projection maps $\pi_1 : R$ and $\pi_2 : R \to B$ are coalgebra morphisms.
- This generalizes the other definitions of bisimulation that I just gave (and many others!)

Definition of Behavioral Equivalence

- Kurz and Pattinson think of the states in transition systems as processes and, using these same coalgebraic formulations, define behavioral equivalence:
- Two states (or "processes") $a \in A$ and $b \in B$ are behaviorally equivalent $a \equiv b$ when they can be mapped to the same state in some other coalgebra (C, γ) by coalgebra morphisms $f : A \to C$ and $g : B \to C$.

Definition of Behavioral Equivalence

- Kurz and Pattinson think of the states in transition systems as processes and, using these same coalgebraic formulations, define behavioral equivalence:
- Two states (or "processes") $a \in A$ and $b \in B$ are behaviorally equivalent $a \equiv b$ when they can be mapped to the same state in some other coalgebra (C, γ) by coalgebra morphisms $f : A \to C$ and $g : B \to C$.

Behavioral Equivalence and Bisimulation

- Bisimulation and behavioral equivalence are very closely related.
- If there is a bisimulation R that identifies two states ($a \sim_R b$), then they are behaviorally equivalent ($a \equiv b$).
- If the functor F preserves weak pullbacks*, then the converse is also true.

Behavioral Equivalence and Bisimulation

- Bisimulation and behavioral equivalence are very closely related.
- If there is a bisimulation R that identifies two states ($a \sim_R b$), then they are behaviorally equivalent ($a \equiv b$).
- If the functor F preserves weak pullbacks*, then the converse is also true.

Behavioral Equivalence and Bisimulation

- Bisimulation and behavioral equivalence are very closely related.
- If there is a bisimulation R that identifies two states ($a \sim_R b$), then they are behaviorally equivalent ($a \equiv b$).
- If the functor F preserves weak pullbacks*, then the converse is also true.

Finitary Modal Logic

- Since Kripke models are essentially just directed graphs*, coalgebras have a lot to say about modal logic.
- However, since n-bisimulation does not yet have a coalgebraic generalization*, we can't easily connect coalgebras to finitary modal logics.
- n-bisimulation seems appropriate for systems inspired by practical concerns (like probabilistic systems) as it describes when processes have the same behavior over a fixed number of steps.

Finitary Modal Logic

- Since Kripke models are essentially just directed graphs*, coalgebras have a lot to say about modal logic.
- However, since n-bisimulation does not yet have a coalgebraic generalization*, we can't easily connect coalgebras to finitary modal logics.
- n-bisimulation seems appropriate for systems inspired by practical concerns (like probabilistic systems) as it describes when processes have the same behavior over a fixed number of steps.

Finitary Modal Logic

- Since Kripke models are essentially just directed graphs*, coalgebras have a lot to say about modal logic.
- However, since n-bisimulation does not yet have a coalgebraic generalization*, we can't easily connect coalgebras to finitary modal logics.
- n-bisimulation seems appropriate for systems inspired by practical concerns (like probabilistic systems) as it describes when processes have the same behavior over a fixed number of steps.

n-Bisimulation

For a Kripke model $(W, \to)$, n-bisimulation is defined inductively. A sequence of binary relations$(R_0, R_1, \ldots, R_n)$ is an n-bisimulation if:

- if $v \sim_{R_0} w$, then a and b satisfy the same atomic propositions,
- (zig) if $v_1 \sim_{R_{n+1}} w_1$ and $v_1 \to v_2$, then there is a w_2 such that $v_2 \sim_{R_n} w_2$ and $w_1 \to w_2$, and
- (zag) if $v_1 \sim_{R_{n+1}} w_1$ and $w_1 \to w_2$, then there is a v_2 such that $v_2 \sim_{R_n} w_2$ and $w_1 \to w_2$,

n-Bisimulation

For a Kripke model $(W, \rightarrow)$, n-bisimulation is defined inductively. A sequence of binary relations$(R_0, R_1, \ldots, R_n)$ is an n-bisimulation if:

- if $v \sim_{R_0} w$, then a and b satisfy the same atomic propositions,
- (zig) if $v_1 \sim_{R_{n+1}} w_1$ and $v_1 \rightarrow v_2$, then there is a w_2 such that $v_2 \sim_{R_n} w_2$ and $w_1 \rightarrow w_2$, and
- (zag) if $v_1 \sim_{R_{n+1}} w_1$ and $w_1 \rightarrow w_2$, then there is a v_2 such that $v_2 \sim_{R_n} w_2$ and $w_1 \rightarrow w_2$,

n-Bisimulation

For a Kripke model $(W, \to)$, n-bisimulation is defined inductively. A sequence of binary relations$(R_0, R_1, \ldots, R_n)$ is an n-bisimulation if:

- if $v \sim_{R_0} w$, then a and b satisfy the same atomic propositions,
- (zig) if $v_1 \sim_{R_{n+1}} w_1$ and $v_1 \to v_2$, then there is a w_2 such that $v_2 \sim_{R_n} w_2$ and $w_1 \to w_2$, and
- (zag) if $v_1 \sim_{R_{n+1}} w_1$ and $w_1 \to w_2$, then there is a v_2 such that $v_2 \sim_{R_n} w_2$ and $w_1 \to w_2$,

n-Behavioral Equivalence

- As of yet, there is no notion of coalgebraic n-bisimulation*, but
- Kurz and Pattinson do give a definition of n-behavioral equivalence, based on the terminal sequence of a functor and utilizing maps between coalgebras that are not quite coalgebra morphisms.

n-Behavioral Equivalence

- As of yet, there is no notion of coalgebraic n-bisimulation*, but
- Kurz and Pattinson do give a definition of n-behavioral equivalence, based on the terminal sequence of a functor and utilizing maps between coalgebras that are not quite coalgebra morphisms.

Summary

- Coalgebras are great for bisimulation!
- Can we get coalgebraic n-bisimulation? Do we need the full power of n-behavioral equivalence?

Summary

- Coalgebras are great for bisimulation!
- Can we get coalgebraic n-bisimulation? Do we need the full power of n-behavioral equivalence?

For Further Reading I

A. Kurz and D. Pattinson
Definability, Canonical Models, Compactness for Finitary Coalgebraic Modal Logic.
Electronic Notes in Theoretical Computer Science, 65 No. 1, 2002.

E.P. de Vink and J.J.M.M. Rutten
Bisimulation for probabilistic transition systems: a coalgebraic approach
Software Engineering, SEN-R9825 October 31, 1998.