On the Uniqueness of Solutions to the Burgers Equation in Sobolev Spaces

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The Burgers Cauchy Problem

Let $u, \omega \in C(I, H^s(\mathbb{T})),\ s > 3/2$ be two solutions to the Cauchy-problem

$$\partial_t u = -u\partial_x u \tag{0.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R} \tag{0.2}$$

with common initial data. Let $v = u - w$; then v solves the Cauchy-problem

$$\partial_t v = -\frac{1}{2}\partial_x[v(u+w)], \tag{0.3}$$

$$v(x,0) = 0. \tag{0.4}$$

Applying D^σ to both sides of (0.3), then multiplying both sides by $D^\sigma v$ and integrating, we obtain

An Energy Estimate for the Difference of Solutions

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{H^\sigma(\mathbb{T})} = -\frac{1}{2}\int_{\mathbb{T}} D^\sigma \partial_x [v(u+w)] \cdot D^\sigma v \; dx. \qquad (0.5)$$

Unfortunately, it will not be enough to simply use Cauchy-Schwartz, the Sobolev Imbedding Theorem, or the algebra property of Sobolev Spaces to estimate the right hand side of this equation. We require more heavy machinery.

Rough Estimates

Proceeding, we rewrite

$$\begin{aligned} |i| &= \Big| -\frac{1}{2}\int_{\mathbb{T}} [D^\sigma \partial_x,\ u+w]\, v \cdot D^\sigma v \ dx \\ &\quad - \frac{1}{2}\int_{\mathbb{T}} (u+w) D^\sigma \partial_x v \cdot D^\sigma v \ dx \Big| \\ &\leq \left| -\frac{1}{2}\int_{\mathbb{T}} [D^\sigma \partial_x,\ u+w]\, v \cdot D^\sigma v \ dx \right| \\ &\quad + \left| \frac{1}{2}\int_{\mathbb{T}} (u+w) D^\sigma \partial_x v \cdot D^\sigma v \ dx \right|. \end{aligned} \tag{0.6}$$

Rough Estimates

Observe that by integrating by parts and applying Cauchy-Schwartz we have

$$\left|\frac{1}{2}\int_{\mathbb{T}}(u+w)D^{\sigma}\partial_x v\cdot D^{\sigma}v\ dx\right| \lesssim \|\partial_x(u+w)\|_{L^{\infty}(\mathbb{T})}\|v\|^2_{H^{\sigma}(\mathbb{T})}. \tag{0.7}$$

Furthermore,

$$\left|-\frac{1}{2}\int_{\mathbb{T}}[D^{\sigma}\partial_x,\ u+w]v\cdot D^{\sigma}v\ dx\right| \lesssim \|[D^{\sigma}\partial_x,\ u+w]v\|_{L^2(\mathbb{T})}\|v\|_{H^{\sigma}(\mathbb{T})} \lesssim \|u+w\|_{H^{\rho}(\mathbb{T})}\|v\|^2_{H^{\sigma}(\mathbb{T})} \tag{0.8}$$

where the last step follows from the following commutator estimate:

Commutator Estimates (Sharp Estimates)

Theorem

If $\rho > 3/2$ *and* $0 \leq \sigma + 1 \leq \rho$*, then*

$$\|[D^\sigma \partial_x, f]v\|_{L^2} \leq C\|f\|_{H^\rho}\|v\|_{H^\sigma}. \tag{0.9}$$

A proof of a more general version of Theorem 1 can be found in a survey article by Michael Taylor on commutator estimates. We note that the proof relies heavily on the Kato-Ponce commutator estimate.

An ODE From a PDE

Recallling (0.5), combining our previous estimates and applying the Sobolev Imbedding Theorem, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{H^\sigma(\mathbb{T})} \lesssim \|u+w\|_{H^\rho(\mathbb{T})}\|v\|^2_{H^\sigma(\mathbb{T})}. \tag{0.10}$$

Gronwall's Inequality

By Gronwall's inequality, (0.10) gives

$$\|v\|_{H^\sigma(\mathbb{T})} \lesssim e^{\int_0^t \|u+w\|_{H^\rho}} \|v_0\|_{H^\sigma(\mathbb{T})}, \qquad |t| \leq T. \tag{0.11}$$

Since $v_0 = 0$ and $\|u + w\|_{H^\rho} \leq \|u + w\|_{H^s(\mathbb{T})} < \infty$ for $|t| \leq T$, we conclude from (0.11) that solutions to the HR i.v.p. with initial data in $H^s(\mathbb{T})$ are unique for $s > 3/2$.