Dig-In:

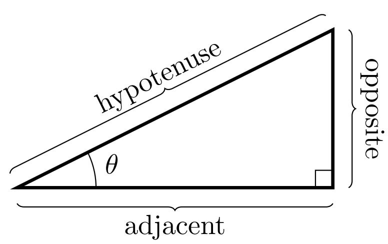
Trigonometric functions

We review trigonometric functions.

What are trigonometric functions?

Definition 1. *A* ***trigonometric function*** *is a function that relates a measure of an angle of a right triangle to a ratio of the triangle's sides.*

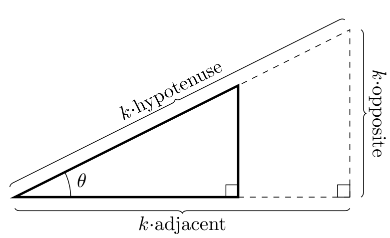
The basic trigonometric functions are cosine and sine. They are called "trigonometric" because they relate measures of angles to measurements of triangles. Given a right triangle



we define

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \text{and} \qquad \sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}.$$

Note, the values of sine and cosine do not depend on the scale of the triangle. Being very explicit, if we scale a triangle by a scale factor k,



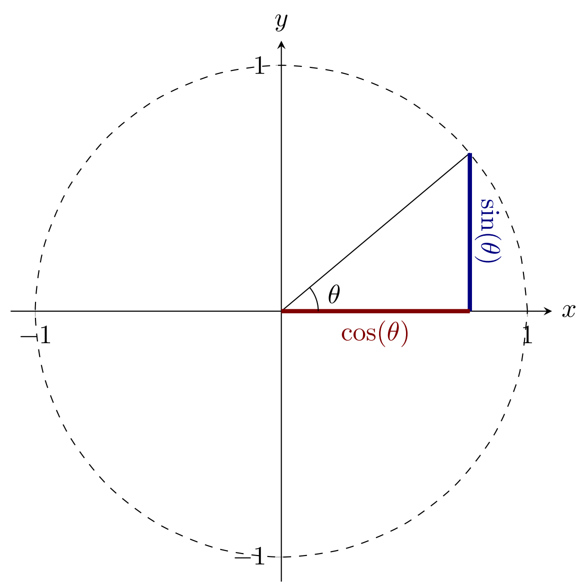
Learning outcomes: Understand the properties of trigonometric functions. Evaluate expressions and solve equations involving trigonometric functions and inverse trigonometric functions. Evaluate limits involving trigonometric functions.

$$\cos(\theta) = \frac{k \cdot \text{adjacent}}{k \cdot \text{hypotenuse}} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

and

$$\sin(\theta) = \frac{k \cdot \text{opposite}}{k \cdot \text{hypotenuse}} = \frac{\text{opposite}}{\text{hypotenuse}}.$$

At this point we could simply assume that whenever we draw a triangle for computing sine and cosine, that the hypotenuse will be 1. We can do this because we are simply scaling the triangle, and as we see above, this makes absolutely no difference when computing sine and cosine. Hence, when the hypotenuse is 1, we find that a convenient way to think about sine and cosine is via the unit circle:



If we consider all possible combinations of ratios of

$$\text{adjacent}, \qquad \text{opposite}, \qquad \text{hypotenuse},$$

(allowing the adjacent and opposite to be negative, as on the unit circle) we obtain all of the trigonometric functions.

Definition 2. *The trigonometric functions are:*

$$\cos(\theta) = \frac{adj}{hyp} \qquad \sin(\theta) = \frac{opp}{hyp}$$
$$\sec(\theta) = \frac{1}{\cos(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)}$$
$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

where the domain of sine and cosine is all real numbers, and the other trigonometric functions are defined precisely when their denominators are nonzero.

Question 1 *Which of the following expressions are equal to* $\sec(\theta)$*?*

Select All Correct Answers:

(a) $\dfrac{1}{\cos(\theta)}$ ✓

(b) $\dfrac{1}{\sin(\theta)}$

(c) $\dfrac{adj}{hyp}$

(d) $\dfrac{hyp}{adj}$ ✓

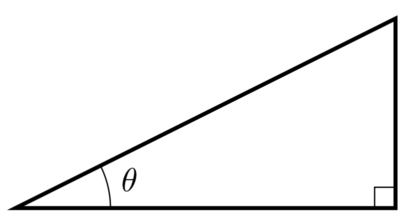
(e) $\dfrac{\tan(\theta)}{\sin(\theta)}$

(f) $\dfrac{1}{\sin(\theta)\cdot\cot(\theta)}$

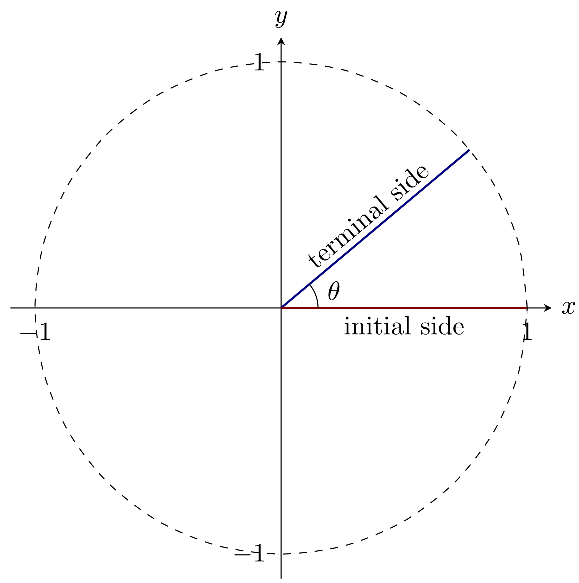
Feedback (attempt): *Note,* $\dfrac{\tan(\theta)}{\sin(\theta)} \neq \sec(\theta)$*, and* $\dfrac{1}{\sin(\theta)\cdot\cot(\theta)} \neq \sec(\theta)$ *since they differ when* $\theta = 0$*.*

Not all angles come from triangles.

Given a right triangle like



the angle θ cannot exceed $\dfrac{\pi}{2}$ radians. That means to talk about trigonometric functions for *other* angles, we need to be able to describe the trigonometric functions a little more generally. To do this, we use the unit circle from the previous section. Given an angle θ, we construct the angle with initial side along the positive x-axis and vertex at the origin.



As the angle θ grows larger and larger, the terminal side of that angle spins around the circle. The trigonometric functions of the angle θ are defined in terms of the terminal side.

Definition 3. *Suppose (x, y) is the point at which the terminal side of the angle with measure θ intersects the unit circle. The the trigonometric functions are defined to be*

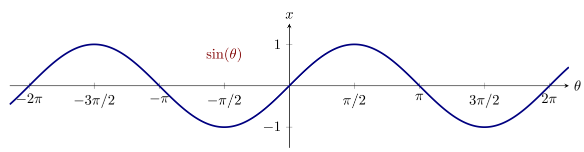
$$\cos(\theta) = x \qquad \sin(\theta) = y$$
$$\sec(\theta) = \frac{1}{x} \qquad \csc(\theta) = \frac{1}{y}$$
$$\tan(\theta) = \frac{y}{x} \qquad \cot(\theta) = \frac{x}{y}$$

provided these values exist.

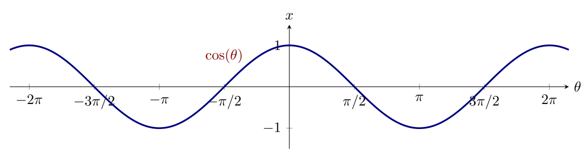
From the picture, you see that this agrees with what you know about trigonometry for triangles, but it allows us to extend the definition of sine and cosine to all real numbers, instead of only the interval $\left(0, \frac{\pi}{2}\right)$

Graphs

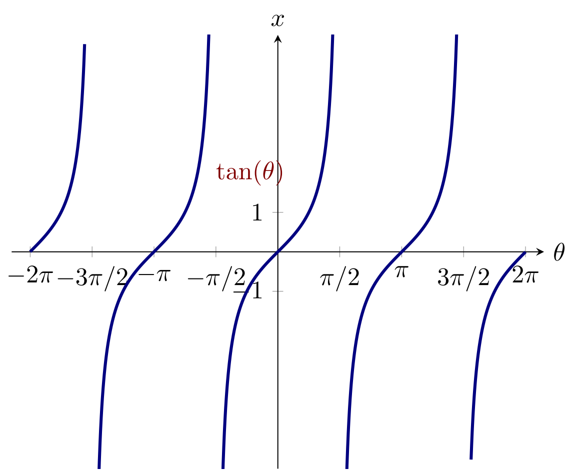
As a reminder, we include the graphs here.



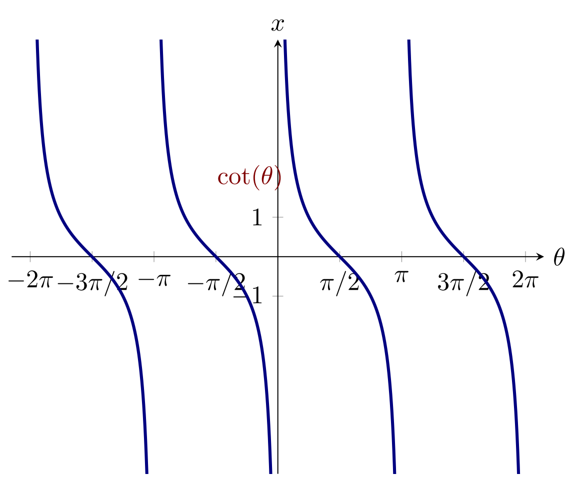
x
1
sin(θ)
θ
−2π
−3π/2
−π
−π/2
π/2
π
3π/2
2π
−1



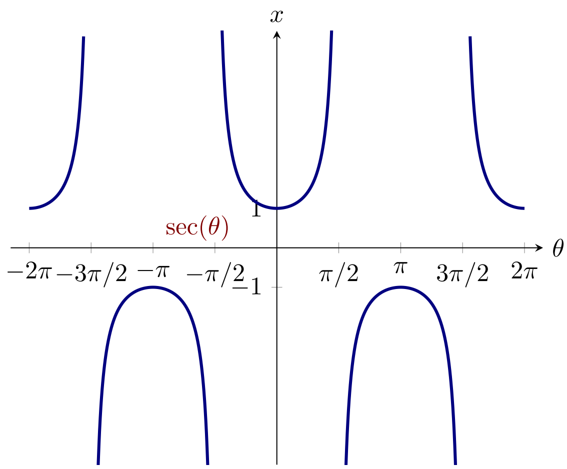
x
1
cos(θ)
θ
−2π
−3π/2
−π
−π/2
π/2
π
3π/2
2π
−1

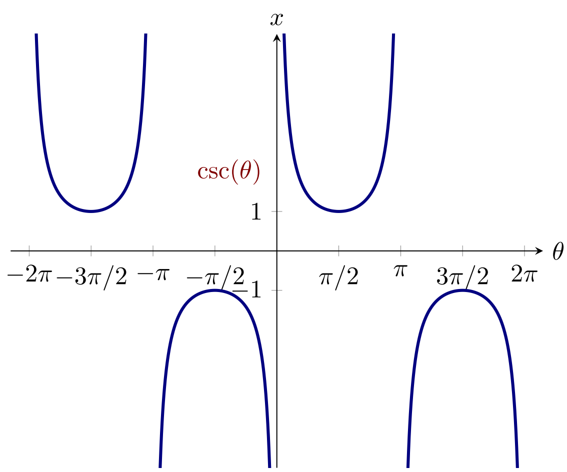


x
tan(θ)
1
θ
−2π
−3π/2
−π
−π/2
π/2
π
3π/2
2π
−1



x
cot(θ)
1
θ
−2π
−3π/2
−π
−π/2
π/2
π
3π/2
2π
−1

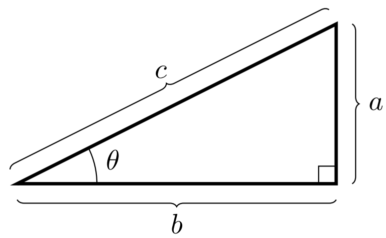




The power of the Pythagorean Theorem

The Pythagorean Theorem is probably the most famous theorem in all of mathematics.

Theorem 1 (Pythagorean Theorem)**.** *Given a right triangle:*



We have that:

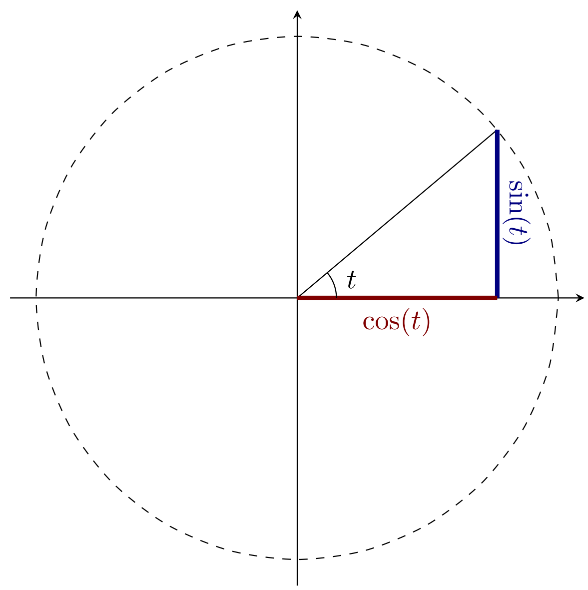
$$a^2 + b^2 = c^2$$

The Pythagorean Theorem gives several key trigonometric identities.

Theorem 2 (Pythagorean Identities)**.** *The following hold:*

$$\cos^2\theta + \sin^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad \cot^2\theta + 1 = \csc^2\theta$$

Explanation. *From the unit circle we can see*



via the Pythagorean Theorem that

$$\cos^2(t) + \sin^2(t) = 1.$$

If we divide this expression by $\underset{\text{given}}{\boxed{\cos^2(t)}}$ *we obtain*

$$1 + \tan^2(t) = \sec^2(t)$$

and if we divide $\cos^2(t) + \sin^2(t) = 1$ *by* $\underbrace{\boxed{\sin^2(t)}}_{\text{given}}$ *we obtain*

$$\cot^2(t) + 1 = \csc^2(t).$$

Limits involving trigonometric functions

Back when we introduced continuity we mentioned that each trigonometric function is continuous on its domain.

Example 1. *Compute the limit:*

$$\lim_{\theta\to 2\pi/3} \theta \tan\theta.$$

Explanation. *The multiplicative limit law allows us to split this into*

$$\left(\lim_{\theta\to 2\pi/3} \theta\right)\left(\lim_{\theta\to 2\pi/3} \tan\theta\right).$$

The function $f(\theta) = \theta$ *is continuous everywhere, so* $\lim_{\theta\to 2\pi/3} \theta = \underbrace{\boxed{\frac{2\pi}{3}}}_{\text{given}}$. *Since* $\frac{2\pi}{3}$ *is in the domain of* $\tan\theta$, *we have* $\lim_{\theta\to 2\pi/3} \tan\theta = \tan\left(\underbrace{\boxed{\frac{2\pi}{3}}}_{\text{given}}\right) = -\sqrt{3}$. *Putting these together we find*

$$\lim_{\theta\to 2\pi/3} \theta \tan\theta = -\frac{2\pi\sqrt{3}}{3}.$$

Example 2. *Compute the limit:*

$$\lim_{\theta\to \pi^-} \cot\theta.$$

Explanation. *Recall that* $\cot\theta = \frac{\cos\theta}{\sin\theta}$, *so that*

$$\lim_{\theta\to\pi^-} \cot\theta = \lim_{\theta\to\pi^-} \frac{\cos\theta}{\sin\theta}.$$

Since sine and cosine are continuous, $\lim_{\theta\to\pi^-} \cos\theta = \cos\left(\underbrace{\boxed{\pi}}_{\text{given}}\right) = \underbrace{\boxed{-1}}_{\text{given}}$ *and* $\lim_{\theta\to\pi^-} \sin\theta = \sin\left(\underbrace{\boxed{\pi}}_{\text{given}}\right) = \underbrace{\boxed{0}}_{\text{given}}$. *That is,* $\lim_{\theta\to\pi^-} \cot\theta$ *is of the form* $\frac{\#}{\mathbf{0}}$.

The numerator is negative for θ *near* π. *From the graph of* $\sin\theta$, *we know that the denominator is negative and approaching* 0 *as* $\theta \to \pi^-$. *That means*

$$\lim_{\theta\to\pi^-} \cot\theta = -\infty.$$

Question 2 *Compute the limit:*

$$\lim_{\theta \to \frac{\pi}{12}} \frac{2\cos(4\theta)\sin(6\theta)}{\theta} = \boxed{\frac{12}{\pi}}$$

Trigonometric equations

Frequently we are in the situation of having to determine precisely which angles satisfy a particular equation. The most basic example is probably like this one.

Example 3. *Solve the equation:*

$$\sin\theta = -\frac{1}{2}.$$

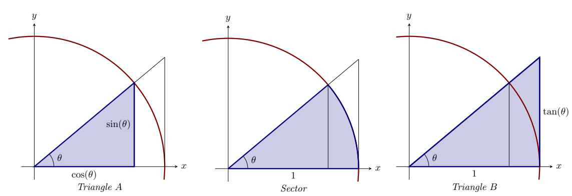
Explanation. *We'll start by finding the reference angle.*

We'll end with a couple very involved limits where the Squeeze Theorem makes a surprising return.

Example 4. *Compute:*

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}$$

Explanation. *To compute this limit, use the Squeeze Theorem. First note that we only need to examine* $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ *and for the present time, we'll assume that* θ *is positive. Consider the diagrams below:*



From our diagrams above we see that

$$\textit{Area of Triangle A} \leq \textit{Area of Sector} \leq \textit{Area of Triangle B}$$

and computing these areas we find

$$\frac{\cos(\theta)\sin(\theta)}{2} \leq \frac{\theta}{2} \leq \frac{\tan(\theta)}{2}.$$

Multiplying through by 2*, and recalling that* $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$ *we obtain*

$$\cos(\theta)\sin(\theta) \le \theta \le \frac{\sin(\theta)}{\cos(\theta)}.$$

Dividing through by $\sin(\theta)$ *and taking the reciprocals (reversing the inequalities), we find*

$$\cos(\theta) \le \frac{\sin(\theta)}{\theta} \le \frac{1}{\cos(\theta)}.$$

Note, $\cos(-\theta) = \cos(\theta)$ *and* $\dfrac{\sin(-\theta)}{-\theta} = \dfrac{\sin(\theta)}{\theta}$*, so these inequalities hold for all* $\theta \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$*. Additionally, we know*

$$\lim_{\theta\to 0}\cos(\theta) = \underset{\text{given}}{\boxed{1}} = \lim_{\theta\to 0}\frac{1}{\cos(\theta)},$$

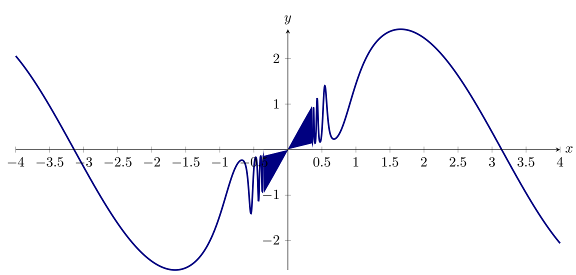
and so we conclude by the Squeeze Theorem, $\displaystyle\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = \underset{\text{given}}{\boxed{1}}$.

When solving a problem with the Squeeze Theorem, one must write a sort of mathematical poem. You have to tell your friendly reader exactly which functions you are using to "squeeze-out" your limit.

Example 5. *Compute:*

$$\lim_{x\to 0}\left(\sin(x)e^{\cos\left(\frac{1}{x^3}\right)}\right)$$

Explanation. *Let's graph this function to see what's going on:*



The function $\sin(x)e^{\cos\left(\frac{1}{x^3}\right)}$ *has two factors:*

$$\overbrace{\sin(x)}^{\textit{goes to zero as } x\to 0}\cdot\underbrace{e^{\cos\left(\frac{1}{x^3}\right)}}_{\textit{bounded between } e^{-1} \textit{ and } e}$$

Hence we have that when $x > 0$

$$\sin(x)\underbrace{\boxed{e^{-1}}}_{\text{given}} \leq \sin(x)e^{\cos\left(\frac{1}{x^3}\right)} \leq \sin(x)\underbrace{\boxed{e}}_{\text{given}}$$

and we see

$$\lim_{x\to 0^+} \sin(x)\underbrace{\boxed{e^{-1}}}_{\text{given}} = \underbrace{\boxed{0}}_{\text{given}} = \lim_{x\to 0^+} \sin(x)\underbrace{\boxed{e}}_{\text{given}}$$

and so by the Squeeze theorem,

$$\lim_{x\to 0^+} \left(\sin(x)e^{\cos\left(\frac{1}{x^3}\right)}\right) = \underbrace{\boxed{0}}_{\text{given}}.$$

In a similar fashion, when $x < 0$,

$$\sin(x)\underbrace{\boxed{e}}_{\text{given}} \leq \sin(x)e^{\cos\left(\frac{1}{x^3}\right)} \leq \sin(x)\underbrace{\boxed{e^{-1}}}_{\text{given}}$$

and so

$$\lim_{x\to 0^-} \sin(x)\underbrace{\boxed{e}}_{\text{given}} = \underbrace{\boxed{0}}_{\text{given}} = \lim_{x\to 0^-} \sin(x)\underbrace{\boxed{e^{-1}}}_{\text{given}},$$

and again by the Squeeze Theorem $\lim_{x\to 0^-} \left(\sin(x)e^{\cos\left(\frac{1}{x^3}\right)}\right) = 0$. *Hence we see that*

$$\lim_{x\to 0} \left(\sin(x)e^{\cos\left(\frac{1}{x^3}\right)}\right) = \underbrace{\boxed{0}}_{\text{given}}.$$